Irregular Shearlet Frames

Inaugural-Dissertation
zur Erlangung des Doktorgrades der Naturwissenschaften
(**Dr. rer. nat.**)

eingereicht beim

der Justus-Liebig-Universität Gießen
Fachbereich Mathematik und Informatik, Physik, Geographie

vorgelegt von

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Gießen, 2009

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To my beloved Parents

Zusammenfassung

Die Themen dieser Dissertation sind die notwendigen und die hinreichenden Bedingungen für die Frameeigenschaft in $L^2(\mathbb{R}^2)$ bei irregulren Shearlet-Systemen. Zu diesem Zweck führen wir die Konzepte der Dichte irregulärer Shearlet-Systeme ein, die als geeignete Ansätze zum Erkennen der Verbindung zwischen der Geometrie diskreter, mit den irregulren Shearlet-Systemen verbundener Mengen (parametrisiert durch Raum/Scherung/Skalierung) einerseits, und ihren Frame-Eigenschaften andererseits dienen. Um im Bezug auf die Dichte die notwendigen Bedingungen fr die Existenz irregulärer Shearlet-Systeme herzuleiten, benutzen wir die sogenannte homogene Approximationseigenschaft (homogeneous approximation property HAP), die von Gabor und Wavelet-Frames erfllt wird, und die wir für die irregulären Shearlet-Frames herleiten. Dann benutzen wir Folgerungen aus der HAP fr die irregulären Shearlet-Frames, um notwendige Dichtebedingungen zu bekommen, die bewirken, daßdie Systeme Frames im $L^2(\mathbb{R}^2)$ sind. Wir geben Bedingungen fr die Zeit/Skala/Scherungs Parameter für die notwendigen Bedingungen an, ebenso fr die einzelne erzeugende Funktion, damit die dazugehörigen irregulären Shearlet-Systeme Frames in $L^2(\mathbb{R}^2)$ sind. Wir geben weiterhin eine Reihe von Konstruktionsbeispielen für die Shearlet-Frames an. Wir schließen mit einer Untersuchung der Stabilitäteigenschaften der irregulären Shearlet-Frames.

Abstract

This thesis discusses the necessary and sufficient conditions for irregular shearlet systems to be frames for $L^2(\mathbb{R}^2)$. For this purpose, the notions of densities for irregular shearlet systems are introduced, and they are used as efficient tools for observing the connection between the geometry of discrete sets of space-scale-shear parameters associated with irregular shearlet systems and their frames properties. In order to derive the necessary conditions for the existence of irregular shearlet frames in terms of the densities, we employ the Homogeneous Approximation Property (HAP) which is satisfied by Gabor and wavelet frames to obtain the HAP for irregular shearlet frames. We then use the consequence of the HAP for irregular shearlet frames to establish necessary density conditions for irregular shearlet systems to be frames for $L^2(\mathbb{R}^2)$. For sufficient conditions, we specify conditions of the time-scale-shear parameters and the single generating function, so that the associated irregular shearlet systems are frames for $L^2(\mathbb{R}^2)$. Additionally, we provide several examples of constructions of shearlet frames. Finally, we study the stability issue on irregular shearlet frames.

Acknowledgements

I would like to express my sincere gratitude to all those people who made these four and a half years a rewarding experience, and without them this thesis would not be possible. First of all, I would like to thank my advisor Prof. Dr. Gitta Kutyniok of Institute of Mathematics, Universiät Osnabrück. Her enthusiasm and dedication toward mathematics are an inspiration in my academic life, and her guidance, encouragement and support have meant to me. Then I would like to express my gratitude to my co-advisor Prof. Dr. Martin Buhmann of Institute of Mathematics, Justus-Liebig- Universität Gießen for his valuable suggestions and the productive discussion.

I would also like to thank Dr. Wang-Q Lim of Institute of Mathematics, Universiät Osnabrück for his asserted comments and useful suggestions and discussions which significantly improved the quality of this thesis.

I want to express my sincere appreciation to Prince of Songkla University for offering the scholarship which enabled me to continue my advanced studies in Germany. I am indebted to my colleagues in Department of Mathematics at Prince of Songkla University for supporting my higher study in Germany and taking on the extra workload in my long absence.

I am thankful to Prof. Dr. Eckart Schulz of Institute of Mathematics at Suranaree University of Technology, Thailand for his kindness, suggestions and discussions during my visit to Nakhon Ratchasima, Thailand.

A special thank must go to Thai monastery (Wat Pha Phuritattaram), Thai Buddhist society and Thai students in Gießen for their moral support and warm hospitality during my years of study in Gießen.

I am also thankful all of my friends for their valuable friendship and support during these all years.

Last but not least I want to express my deepest appreciation to my parents and my brother who always believe in me and support me with love.

Contents

Chapter 1

Introduction

Wavelet theory originated from signal theory which has the goal to represent functions that are local in time and frequency. The classical method used in signal theory is the *Fourier transform* which transforms a signal f in the time domain to another function $\hat{f}$ in the frequency domain. However, the Fourier transform provides information on the frequency content over the whole duration of the signal, but it does not tell what frequencies occur at a specific time. To overcome this disadvantage, Gabor [35] modified the Fourier transform by multiplying the signal with a translated window function ψ inside the integral,

$$\mathcal{G}_\psi f(\omega, b) = \int_{\mathbb{R}} f(t)\overline{\psi(t-b)}e^{-2i\pi\omega t}dt. \tag{1.1}$$

This method is known as the *continuous Gabor transform.* One can shift this window to any point t in time by means of the translation parameter b. In general, the window function ψ should be smooth and resemble a characteristic function closely, so that f is windowed by the shifted support of ψ. For this reason, its transform provides information about the frequency decomposition of f on that time window. However, this method still has a drawback in that the size of the window is fixed, and since this fixed window is used for all frequencies in the transformation, there is a limit as to how well the signal can be localized in time.

In 1984, Grossmann and Morlet [42] defined an integral transform which is now often called the *continuous wavelet transform.* It is similar to the continuous Gabor transform but uses the two-parameter family of functions $\psi_{a,b}(t) = |a|^{-1/2}\psi(\frac{t}{a} - b)$, comprising both translations by real numbers b and dilations by positive real numbers a. The *continuous wavelet transform* of $f \in L^2(\mathbb{R})$ is given by

$$\mathcal{W}_\psi f(a,b) = a^{-1/2}\int_{\mathbb{R}} f(t)\overline{\psi\left(\frac{t}{a} - b\right)}dt. \tag{1.2}$$

The continuous wavelet transform can be expressed as an inner product $\mathcal{W}_\psi f(a,b) = \langle f, \psi_{a,b}\rangle$ in $L^2(\mathbb{R})$, and Grossmann and Morlet showed that it is directly related to the theory of group representations. By using Duflo and Moore's theory of square integrable representations, Grossmann and Morlet classified those functions $\psi \in L^2(\mathbb{R})$ which allow for the *reconstruction formula*,

$$f = \frac{1}{c_\psi}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \langle f, \psi_{a,b}\rangle\, \psi_{a,b}\, da\, db, \tag{1.3}$$

as a weak integral in $L^2(\mathbb{R})$ (see Section 2.4 for more details).

However, for practical applications, it is much easier to work with series than with weak integrals. Therefore, instead of reconstructing the function f from its continuous wavelet transform in the sense of the weak integral (1.3), one searches for discrete subsets $\Gamma = \{(a^j, bk) : j, k \in \mathbb{Z}\}$, $a > 1$, $b > 0$ of $\mathbb{R}^+ \times \mathbb{R}$ so that

$$f = \frac{1}{A} \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k} \tag{1.4}$$

with convergence in $L^2(\mathbb{R})$ for some $A > 0$. This leads us to the concepts of *frames* introduced by Duffin and Schaeffer [24]. A frame is a collection of vectors $\{\psi_j\}_{j \in J}$ in a Hilbert space $\mathcal{H}$ such that

$$A||f||^2 \leq \sum_{j \in J} |\langle f, \psi_j \rangle|^2 \leq B||f||^2 \qquad \text{for all } f \in \mathcal{H},$$

with some constants $0 < A \leq B < \infty$. In particular, such a collection of functions $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ in (1.4) is called a *tight frame*. More precisely, tight frames can be realized as a generalization of orthonormal bases (see more details in Section 2.1). In order words, they provide the advantage of allowing redundancy in contrast to orthonormal bases (see, for instance, Benedetto and Fickus [5], Casazza et al. [10] and Goyal, Kovačević and Kelner [37]).

Over the last decade, wavelets and frames have merged together, and had a growing impact on many fields due to their unifying role in mathematical theory as well as their success in applications. Especially, wavelets can provide optimally sparse representation for piecewise smooth functions in one dimension. However, recently it was observed that although wavelets are good at catching point singularities, they do not efficiently detect singularities along curves or surfaces in higher dimensions. In fact, in order to achieve sparse representations for multivariate functions with singularities along curves or surfaces, one needs to use the "directional representation scheme" which is richer in directions and allows more support shapes for its basis elements. This observation inspired several approaches to overcome the limit of wavelets, such as *ridgelets* by Candés and Donoho [7], *curvelets* by Candés and Donoho [8], *contourlets* by Do and Vetterli [23] and recently *shearlets* by Guo et al. [47], Guo and Labate [46], Guo, Kutyniok and Labate [45], Kutyniok and Labate [61], Labate et al. [64], and Easley, Labate and Lim [26].

The curvelet system, in particular, is one of the most successful directional representation system for images and provides nearly optimal approximations for two-dimensional piecewise smooth functions with discontinuities along C^2-curves. However, the construction of curvelets requires a rotation operation and corresponds to a two-dimensional frequency partition based on the polar coordinate. This causes the discrete implementation for discrete data, sampled on a rectangle grid, to be very challenging.

In recent years, Guo et al. [47] introduced the construction of an efficient representation system for multivariate functions, which is called the *shearlet system*. The shearlet system has many similarities to the curvelet system. For example, both of them provide optimally sparse representations for two-dimensional piecewise smooth functions with discontinuities along C^2-curves. However, unlike the curvelet system, the shearlet system is a two-dimensional *affine-like system* in the sense that it is generated by applying translations followed by anisotropic dilation and shearing to a single function $\psi \in L^2(\mathbb{R}^2)$ called the *generating shearlet*. Moreover, Labate et al. showed in [64] that the shearlet system is associated with a generalized Multiresolution Analysis which is a more convenient setting for discrete implementations.

In [61], Kutyniok and Labate introduced the *continuous shearlet transform* of $f \in L^2(\mathbb{R}^2)$ defined by $\mathcal{SH}_\psi f(a, s, t) = \langle f, T_{S_s A_a t} D_{S_s A_a} \psi \rangle$, where D_a and T_b are the dilation and translation operators, respectively. Similar to wavelet system, the shearlet system is a discrete analogue of its continuous counterpart. Guo, Kutyniok and Labate [45], Guo and Labate [46], Kutyniok and Labate [62] and Labate et al. [64] studied the shearlet system $\mathcal{SH}_\psi(\Gamma)$ of the form:

$$\mathcal{SH}_\psi(\Gamma) = \left\{ T_{cS_{bka^{j/2}}A_{a^j}m} D_{S_{bka^{j/2}}A_{a^j}} \psi : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2 \right\},$$

where $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ and $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ are anisotropic dilation and shear matrices, respectively. More precisely, the above shearlet system is obtained by sampling the continuous shearlet transform at the points lying on the following discrete subset Γ of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$,

$$\Gamma = \{(a^j, bka^{j/2}, cS_{bka^{j/2}}A_{a^j}m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \qquad a > 1 \text{ and } b,c > 0.$$

Such a system is called a *regular shearlet system.* It was proved that for a certain choice of the generating shearlet ψ, the regular shearlet system $\mathcal{SH}_\psi(\Gamma)$ constitutes a Parseval frame for $L^2(\mathbb{R}^2)$ (Guo, Kutyniok and Labate [45], Guo and Labate [46] and Labate et al. [64]). However, in many real applications, one requires the samplings step to be fluctuated. That is, the discrete set Γ might be chosen as an arbitrary discrete subset Λ of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$. We call such a shearlet system $\mathcal{SH}_\psi(\Lambda)$ associated with Λ an *irregular shearlet system.* In [62], Kutyniok and Labate considered the construction of irregular shearlet frames associated with an irregular discrete subset $\Lambda = \{(a_j, s_{j,k}, cS_{s_{j,k}}A_{a_j}m)\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$. They proved that under some regularity conditions on ψ, i.e., ψ is band-limited, the irregular shearlet system $\mathcal{SH}_\psi(\Lambda)$ is a frame for $L^2(\mathbb{R}^2)$.

The main objectives of this thesis are to derive necessary and sufficient conditions for the existence of irregular shearlet frames. More precisely, in the first part of this thesis, we introduce the new notion of densities for shearlet systems and use them as efficient tools for deriving necessary conditions for the existence of irregular shearlet frames. In the second part of this thesis, we improve the construction of irregular shearlet frames by Kutyniok and Labate [62] by replacing the assumption on band-limitedness of the generating shearlet ψ with a mild decay assumption on $\hat{\psi}$. We then derive some sufficient conditions on $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$ and $\{s_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathbb{R}$, so that the irregular shearlet system is a frame for $L^2(\mathbb{R}^2)$. Furthermore, we analyze the stability of irregular shearlet frames under the perturbation of the translation parameter.

1.1 Density for Irregular Gabor and Wavelet Systems

The concept of density is well established as a useful tool in studying irregular frames, especially to establish necessary conditions for the existence of general frames. For example, the Density Theorems for irregular Gabor systems introduced by Ramanathan and Steger [68], Gröchenig and Razafinjatovo [41] and Christensen, Deng and Heil [14] provide necessary conditions for an irregular Gabor system to be a frame or a Riesz basis. An irregular Gabor system $\mathcal{G}_g(\Lambda)$ is generated by applying modulations and translations to a function $g \in L^2(\mathbb{R})$ which is sometimes called the *generator*, as follows :

$$\mathcal{G}_g(\Lambda) = \{e^{2\pi ibx} g(t-a) : (a,b) \in \Lambda\},$$

where Λ is a discrete subset of $\mathbb{R}^2$. The authors (Ramanathan and Steger [68], Gröchenig and Razafinjatoro [41] and Christensen, Deng and Heil [14]) used the notions of *upper and lower Beurling density* $D^+(\Lambda)$, $D^-(\Lambda)$ of Λ, which measure the average number of points of Λ lying in unit cubes, to formulate the following Density Theorem.

Theorem 1.1 (Density Theorem for Irregular Gabor Frames). *Let $g \in L^2(\mathbb{R})$ and let $\Lambda \subset \mathbb{R}^2$ be discrete. Then the Gabor system $\mathcal{G}_g(\Lambda)$ has the following properties:*

(a) If $\mathcal{G}_g(\Lambda)$ is a frame for $L^2(\mathbb{R})$, then $1 \le D^-(\Lambda) \le D^+(\Lambda) < \infty$.

(b) If $\mathcal{G}_g(\Lambda)$ is a Riesz basis for $L^2(\mathbb{R})$, then $D^-(\Lambda) = D^+(\Lambda) = 1$.

Theorem 1.1 shows that if $\mathcal{G}_g(\Lambda)$ forms a frame for $L^2(\mathbb{R})$, then the set Λ cannot be locally "too sparse" or "too dense" in the time-frequency domain. The *cut-off density* $D^-(\Lambda) = D^+(\Lambda) = 1$, which separates frames from non-frames, is sometimes called the *Nyquist density*.

Variation of the density theorem were stated and proved by several authors. Ramanathan and Steger [68] introduced the notion of *Homogenous Approximation Property (HAP)* which has been used as an important tool in the proof of the density theorem. Gröchenig and Razafinjatoro [41] employed the HAP and used the fact that Gabor frames fulfill the HAP to show Nyquist density condition for Gabor systems. However, this approach requires some restrictions on the choice of generator g. Without any restrictions on g and Λ, Christensen, Deng and Heil [14] showed the same results for the higher-dimensional case and extended these to multiple generators for the Gabor systems. For the history and development of density theorem for Gabor systems we refer the reader to Heil [50].

Concerning density results for irregular wavelet system, a notion of *affine-Beurling density* for irregular wavelet systems were introduced in parallel by Heil and Kutyniok [51] and Sun and Zhou [74]. In particular, Heil and Kutyniok [51] considered *weighted irregular wavelet systems* of the form

$$\mathcal{W}_\psi(\Lambda, w) = \left\{ w(a,b)^{1/2} D_a T_b \psi(t) = w(a,b)^{1/2} a^{-1/2} \psi\left(\frac{t}{a} - b\right) : (a,b) \in \Lambda \right\},$$

where $\psi \in L^2(\mathbb{R})$ is called a *generating wavelet*, Λ is a discrete subset of $\mathbb{R}^+ \times \mathbb{R}$, and $w : \Lambda \to \mathbb{R}^+$ is a weight function. The authors introduced the notion of *upper and lower weighted affine densities* $D_w^+(\Lambda)$, $D_w^-(\Lambda)$ of Λ. These are used as suitable tools for the study of the geometry of the affine group.

So far, the most widely exploited and studied wavelet system is the *classical wavelet system*

$$\mathcal{W}_\psi(\Gamma) = \{D_{a^j} T_{bk} \psi(t) = a^{-j/2} \psi\left(a^{-j} t - bk\right) : j,k \in \mathbb{Z}\},$$

where $\Gamma = \{(a^j, bk)\}_{j,k\in\mathbb{Z}}$ for $a > 1$ and $b > 0$. It was shown by numerous authors (Daubechies [21], Heil and Walnut [53], Chui and Shi [17], Christensen [11], Frazier et al. [33], etc.) that for a certain $\psi \in L^2(\mathbb{R})$, the classical wavelet system is a frame or even an orthonormal basis for $L^2(\mathbb{R})$ for any $a > 1$ and $b > 0$. From the density point of view, Heil and Kutyniok showed in [51] that the classical wavelet system, which is an unweighted system, even possesses a *uniform affine-Beurling density*, i.e., $D^+(\Gamma) = D^-(\Gamma) = \frac{1}{b \ln a}$.

Notice that the classical wavelet system is not invariant under integer translations. By switching the order of dilation and translation operators in the classical wavelet system, we obtain the shift-invariant system, $\{T_{bk} D_{a^j} \psi\}$, called the *co-affine* system. In [38], Gressman et al. proved that the co-affine system cannot be a frame. In [51], Heil and Kutyniok showed that the lower and upper affine Beurling densities of the discrete set $\Gamma = \{(a^j, bk, cS_k A_{a^j} m)\}_{j,k\in\mathbb{Z}, m\in\mathbb{Z}^2}$ associated with the co-affine system are $D^-(\Gamma) = 0$ and $D^+(\Gamma) = \infty$, respectively.

The *quasi-affine* system, introduced by Ron and Shen [69], is one of the most notable shift-invariant systems. The success of this system comes from the fact that it provides a complete characterization of the frame property of its corresponding affine system. In particular, the quasi-affine system is a weighted system of the form

$$\mathcal{W}_\psi(\Gamma, w) = \{D_{a^j} T_k \psi : j < 0, k \in \mathbb{Z}\} \cup \{a^{j/2} T_k D_{a^j} \psi : j \geq 0, k \in \mathbb{Z}\},$$

with

$$w(j,k) = \begin{cases} 1 & : j < 0, k \in \mathbb{Z} \\ a^{-j} & : j \geq 0, k \in \mathbb{Z} \end{cases},$$

and $\Gamma = \{(a^j, bk) : j < 0, k \in \mathbb{Z}\} \cup \{(a^j, a^{-j}bk) : j \geq 0, k \in \mathbb{Z}\}$.

In particular, quasi-affine systems can be viewed as a special cases of oversampled affine systems. An *oversampled affine system*, introduced by numerous authors (Chui and Shi [16], Gressman et al. [38], Hermandez et al. [55] and Johnson [57]), is a weighted system of the form

$$\mathcal{W}_\psi(\Gamma, w) = \left\{ r_j^{1/2} a^{-j/2} \psi\left(a^{-j}t - \frac{bk}{r_j}\right) : j,k \in \mathbb{Z} \right\}, \quad \text{with } w(j,k) = \frac{1}{r_j},$$

where $r_j > 0$ for any $j \in \mathbb{Z}$ and the corresponding discrete set is $\Gamma = \left\{(a^j, \frac{bk}{r_j})\right\}_{j,k \in \mathbb{Z}}$. In particular, it was shown by Chui and Shi [16] that if the original wavelet system is a frame for $L^2(\mathbb{R})$, then the oversampled affine system is also a frame for $L^2(\mathbb{R})$ with the same frame bounds. Heil and Kutyniok [51] showed that an oversampled affine system possesses a uniform weighted affine-Beurling density, i.e., $D_w^+(\Gamma) = D_w^-(\Gamma) = \frac{1}{b \ln a}$, which coincides with the affine-Beurling density for the classical wavelet system.

From these insightful examples of wavelet systems, Heil and Kutyniok [51] established necessary conditions for the existence of wavelet frames in terms of density in the following theorem. Additionally, it was shown by Daubechies [21] that if the classical wavelet system is a tight frame for a special choice of generating wavelet ψ, then its frame bounds are exactly equal to $\frac{1}{b \ln a}$. The question is now: Does the number $\frac{1}{b \ln a}$ play the role of a Nyquist density for the wavelet system? Studies by Daubechies [21], Balan [3] and recently Kutyniok [59] revealed that there does not exist a Nyquist density for the wavelet system. In [59], Kutyniok derived the relationship between density, frame bounds and the admissibility constant, and then used it to demonstrate why there does not exist a Nyquist density for the wavelet system.

In [51] and [52], Heil and Kutyniok derived the following necessary density conditions for the irregular wavelet system to be a frame:

Theorem 1.2 (Density Theorem for Irregular Wavelet Frames [51], [52]). *Given a nonzero function $\psi \in L^2(\mathbb{R})$, a subset Λ of $\mathbb{R}^+ \times \mathbb{R}$ and a weight function $w : \Lambda \to \mathbb{R}^+$, the following statements hold:*

(a) If $\mathcal{W}_\psi(\Lambda, w)$ possesses an upper frame bound for $L^2(\mathbb{R})$, then $D_w^+(\Lambda) < \infty$.

(b) If $\mathcal{W}_\psi(\Lambda)$ is a frame for $L^2(\mathbb{R})$ and satisfies the HAP, then $D^-(\Lambda) > 0$.

On the other hand, Sun and Zhou [74] introduced another notion of affine-Beurling density with respect to another affine group. However, their approach has to use some special weighted functions in order to derive a uniform density for classical wavelet systems. By employing the isomorphism on the affine group, Kutyniok [60] showed how to obtain a uniform affine-Beurling density for classical wavelet systems without the necessity to add weights. In [74], Sun and Zhou also proved a similar necessary density conditions theorem (Theorem 1.2) for unweighted irregular wavelet systems.

So far density results can only provide necessary conditions for the existence of general irregular frames without any assumptions on the choice of generating function ψ. However, in order to derive sufficient density conditions for the existence of irregular frames, one needs to know not only density conditions of the discrete set Λ but also for which class of functions ψ, the corresponding system forms a frame.

1.2 Construction of Irregular Wavelet Frames

As we have mentioned before, the frame properties of classical wavelet systems in $L^2(\mathbb{R})$ are very well understood. The classical construction of smooth regular tight wavelet frames in one dimension was introduced by Daubechies, Grossman and Meyer [22]. In recent years, the problem of the construction of multivariate wavelet frames has attracted considerable attention. For the regular multivariate case, the wavelet system is generated by a family of functions of the form

$$\mathcal{W}_\psi(\Gamma) = \{|\det A|^{-j/2}\psi(A^{-j}x - bk) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \quad \Gamma = \{(A^j, bk) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where $A \in GL_n(\mathbb{R})$ and $b > 0$. Dai, Larson and Speegle [20], Speegle [70] and Yang and Zhou [80] showed that for a fixed expansive matrix A (all eigenvalues λ of A have absolute values greater than one), the regular wavelet system, generated by a band-limited function $\psi \in L^2(\mathbb{R}^n)$, forms a frame for $L^2(\mathbb{R}^n)$. In [54], [55] and [63], Hermández, Labate, Weiss and Wilson introduced a large class of functions $\psi \in L^2(\mathbb{R}^n)$, which satisfies a certain condition called the locally integrability condition (LIC), and characterized the existence of the regular tight wavelet frame generated by this class of functions.

More generally, the dilation matrix can be chosen to be an arbitrary matrix $A_j \in GL_n(\mathbb{R})$, which yields the irregular wavelet system $\mathcal{W}_\psi(\Lambda)$ defined by,

$$\mathcal{W}_\psi(\Lambda) = \{|\det A_j|^{-1/2}\psi(A_j^{-1}x - bk) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \quad \Lambda = \{(A_j, bk) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

So far, there are two approaches to construct irregular wavelet frames. The first approach is based on the stability of the regular wavelet frame (Gröchenig [39], Zhou and Li [78], Chui and Shi [15], Olsen and Seip [67], Favier and Zalik [27], Christensen and Wenchang [13] and Sun and Zhou [76], [77]). More precisely, stability means the following: For a given regular wavelet system $\{a^{-j/2}\psi(a^{-j}x-bk)\}_{j,k\in\mathbb{Z}}$. The irregular wavelet system $\{a_j^{-1/2}\psi(a_j^{-1}x-bk)\}_{j,k\in\mathbb{Z}}$ ($\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$) is a frame, if a_j is sufficiently close to a^j ("stable") and b is small enough. However, their constructions required rather complicated conditions on ψ or the sequence $\{a_j\}_{j\in\mathbb{Z}}$.

The another approach is obtained by sampling the continuous wavelet transform on an irregular discrete set Λ. In [1] and [2], Aldroubi, Cabrelli and Molter studied the irregular wavelet system $\{|\det A_j|^{-1/2}\psi(A_j^{-1}x - b_{j,k})\}_{j,k\in\mathbb{Z}}$ generated by a band-limited function ψ and arbitrary grids $\{b_{j,k}\}_{j,k\in\mathbb{Z}} \subset \mathbb{R}^n$. Under a certain assumption on $\{b_{j,k}\}_{j,k\in\mathbb{Z}}$, they introduced a general construction of irregular wavelet frames of $L^2(\mathbb{R}^n)$. In [81], Yang and Zhou considered the irregular wavelet system $\{a_j^{n/2}\psi(a_jx - bk)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ generated by a non-bandlimited function ψ, and with arbitrary dilation factors $a_j \in \mathbb{R}^+$. With some simple regularity assumptions on the sequence $\{a_j\}_{j\in\mathbb{Z}}$, the authors derived sufficient conditions for the irregular wavelet system to be a frame for $L^2(\mathbb{R}^n)$:

Theorem 1.3 ([81]). *Let $\{a_j\}_{j\in\mathbb{Z}}$ be an increasing sequence of positive numbers, and let $\psi \in L^2(\mathbb{R}^n)$. Then the following conditions hold:*

(i) There exists a constant $L > 0$ such that

$$L \leq \sum_{j\in\mathbb{Z}} |\hat{\psi}(a_j^{-1}\xi)|^2 \qquad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\}.$$

(ii) There exists constants $C > 0$, $\alpha > 0$ and $\beta > n$ such that

$$|\hat{\psi}(\xi)| \leq C \min\{|\xi|^{\alpha}, |\xi|^{-\beta}\} \qquad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then the irregular wavelet system $\{a_j^{n/2}\psi(a_j x - bk)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is a frame for $L^2(\mathbb{R}^n)$.

Furthermore, Yang and Zhou [81] also studied the stability of irregular wavelet frames when the translation parameter b and the generating wavelet ψ are perturbed.

1.3 Overview of the Thesis

Since shearlet systems are considered to be two-dimensional affine systems, it is natural to ask whether shearlet systems possess properties similar to wavelet systems. In this thesis, we focuss on the study of frame properties of irregular shearlet systems.

In the first part of this thesis, i.e., Chapter 3 and 4, we extend the density results for one-dimensional wavelet systems by Heil and Kutyniok [51] to shearlet systems. The starting point is a generalized notion of weighted density for shearlet systems. In Chapter 3, we introduce the new notions of weighted density for shearlet systems, depending on the different choice of groups, which we call *shearlet group* $\mathbb{S} = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$. More precisely, we define the notions of densities for shearlet systems associated with four different types of shearlet groups: $\mathbb{S}_1 - \mathbb{S}_4$. We also show some connections between shearlet groups by employing the isomorphism from one to another shearlet group. Then we compute the upper and lower density for the classical shearlet systems associated with each shearlet group $\mathbb{S}_1 - \mathbb{S}_4$. In some examples, we also include the computation of densities of discrete subsets associated with oversampled shearlet systems and the co-shearlet systems. Because of the group isomorphism between shearlet groups, it suffices to study the shearlet system associated with shearlet group $\mathbb{S}_1$. More precisely, in this thesis we mainly study the irregular shearlet system of the form

$$\mathcal{SH}_{1,\psi}(\Lambda) = \{a^{3/4}\psi(S_s A_a x - t) : (a, s, t) \in \Lambda\},$$

where $\psi \in L^2(\mathbb{R}^2)$ and Λ is a discrete subset of $\mathbb{S}_1$. We end Chapter 3 with a converse theorem (Theorem 3.24) for the density associated with $\mathbb{S}_1$. That is, for given certain finite numbers $\alpha \geq \beta > 0$, we show for which discrete subset Λ of $\mathbb{S}_1$ the corresponding upper and lower densities are equal to α and β, respectively.

Inspired by the density theorem for irregular wavelet frames by Heil and Kutyniok [51], in the first part of Chapter 4, we begin with deriving necessary density conditions for the irregular shearlet system to possess an upper frame bound (Theorem 4.1). By adapting the HAP for irregular wavelet frames introduced by Heil and Kutyniok [52] and Sun [73], and the HAP for coherent frames by Gröchenig [40], we show that irregular shearlet frames also possess a HAP (Theorem 4.6). We derive a consequence of the HAP for irregular shearlet frames, the Comparison Theorem (Theorem 4.8), which we then use to establish necessary density conditions for the existence of a lower frame bound for irregular shearlet frames (Theorem 4.9).

In the second part of this thesis (Chapter 5), we study irregular shearlet systems $\mathcal{SH}_{1,\psi}(\Lambda)$ with $\Lambda = \{(a_j, s_k, cm)\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2} \subset \mathbb{S}_1$, $c > 0$. We extend the construction of *band-limited* irregular shearlet frames by Kutyniok and Labate [62] to the case of *non-bandlimited* irregular shearlet frames. In Chapter 5, we show how the construction of irregular wavelet frames $\{a_j^{n/2}\psi(a_j x - bk)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ with the scalar dilations $a_j \in \mathbb{R}^+$ by Yang and Zhou [81] can be adapted to the irregular shearlet systems $\{a_j^{3/4}\psi(S_{s_k}A_{a_j}x - cm)\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ with the composite dilation matrices $S_{s_k}A_{a_j} \in GL_2(\mathbb{R})$. In particular, by replacing the band-limited assumption of ψ with mild decay condition on $\hat{\psi}$, we specify some conditions on $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$ and $\{s_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, so that $\mathcal{SH}_{1,\psi}(\Lambda)$ is a frame for $L^2(\mathbb{R}^2)$ (Theorem 5.4). We also present the construction of regular shearlet frames $\mathcal{SH}_{1,\psi}(a^j, k, cm)$ for $j, k \in \mathbb{Z}$ and $m \in \mathbb{Z}^2$ (Theorem 5.7), and provide some numerical examples to estimate frame bounds. Finally, we study perturbations of the translation parameter c, and show the stability of the associated irregular shearlet frames (Theorem 5.10).

Chapter 2

Basic Background

This chapter is devoted to mathematical concepts used in this thesis. With a few exceptions all results are stated without proof. We begin this chapter with the notion and basic properties of frames. We then recall the definition of Fourier transform in $L^2(\mathbb{R}^n)$ and its well-known properties. Furthermore, we include some basic concepts of group representations. Finally, we review the continuous wavelet transform from the group theoretical point of view.

2.1 Frames

In this section we briefly review the definition of frames and its basic properties. Additional details and proofs can be found in Christensen [11], Duffin and Schaffer [24], Casazza [9] and Han et al. [48].

Definition 2.1 (Definition/Facts)**.** (a) A sequence $\{f_j : j \in J\}$ in a Hilbert space $\mathcal{H}$ is a *frame* if there exist $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$ we have

$$A||f||^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B||f||^2. \tag{2.1}$$

The numbers A and B are called the *upper* and *lower* frame bounds respectively. The frame is *tight* if $A = B$, and if $A = B = 1$, we called $\{f_j : j \in J\}$ a *Parseval frame*.

We remark that any orthonormal basis in a Hilbert space is a Parseval frame. On the other hand, even a Parseval frame needs not be a basis.

(b) Let $\{f_j : j \in J\}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds $0 < A \leq B < \infty$. Then the *frame operator*

$$Sf = \sum_{j\in J} \langle f, f_j\rangle f_j, \qquad \forall\, f \in \mathcal{H}$$

is a bounded, positive and invertible operator on $\mathcal{H}$.
A direct computation yields

$$\langle Sf, f\rangle = \sum_{j\in J} |\langle f, f_j\rangle|^2.$$

This implies

$$A\,\mathrm{Id} \leq S \leq B\,\mathrm{Id},$$

where Id denotes the identity operator on $\mathcal{H}$.

(c) Let $\{f_j : j \in J\}$ be a frame for a Hilbert space H with frame bounds $0 < A \le B < \infty$. A sequence $\{\tilde{f}_j : j \in J\}$ in $\mathcal{H}$ is called the *dual frame* for $\{f_j : j \in J\}$, if it allows reconstruction of $f \in \mathcal{H}$ by

$$f = \sum_{j \in J} \langle f, f_j \rangle \tilde{f}_j, \qquad \forall\, f \in \mathcal{H}.$$

In particular, we can choose $\tilde{f}_j = S^{-1} f_j$ for all $j \in J$, and $\{S^{-1} f_j : j \in J\}$ is called the *canonical dual frame* with lower frame bound $\frac{1}{B}$ and upper frame bound $\frac{1}{A}$. Then we have the *frame expansions*

$$f = \sum_{j \in J} \langle f, S^{-1} f_j \rangle f_j = \sum_{j \in J} \langle f, f_j \rangle S^{-1} f_j, \qquad \forall\, f \in \mathcal{H}.$$

(d) The *trace-class operator* of T on a Hilbert space $\mathcal{H}$, denote it by $\operatorname{tr}(T)$, is the sum $\sum_{j \in J} \langle T e_j, e_j \rangle$, where $\{e_j : j \in J\}$ is any orthonormal basis for $\mathcal{H}$.

Proposition 2.1. *Let $\{f_j : j \in J\}$ be a sequence in a Hilbert space $\mathcal{H}$ which satisfies only the upper frame bound estimate in (2.1). Then we call $\{f_j : j \in J\}$ a* Bessel sequence *and the constant B is a* Bessel bound*, such that*

$$\left\| \sum_{j \in J} c_j f_j \right\|_2^2 \le B \sum_{j \in J} |c_j|^2 \qquad \textit{for any } \{c_j\}_{j \in J} \in l^2(J).$$

The following result is about the estimate of the trace of an operator which is stated in Gröchenig [40].

Proposition 2.2 ([40])**.** *Let T be a positive trace operator on a Hilbert space $\mathcal{H}$ and $\{f_j : j \in J\}$ be a frame with frame bounds $0 < A \le B < \infty$. Then*

$$\frac{1}{B} \sum_{j \in J} \langle T f_j, f_j \rangle \le tr(T) \le \frac{1}{A} \sum_{j \in J} \langle T f_j, f_j \rangle.$$

2.2 The Fourier Transform in Higher Dimensions

We will use the following spaces of continuous functions throughout this thesis.

Definition 2.2. Let $p \in \{0, 1, 2, ...\}$. Then

1. $C^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f$ is p times continuously differentiable$\}$.
2. $C_c^p(\mathbb{R}^n) = \{f \in C^p : f$ has compact support$\}$.
3. $C^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f$ is infinitely differentiable$\}$.
4. $C_c^\infty(\mathbb{R}^n) = \{f \in C^\infty : f$ has compact support$\}$.
5. $\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : f$ and all of its derivative decay rapidly$\}$. The space $\mathcal{S}(\mathbb{R}^n)$ is called the *Schwartz class.*

We recall the definition and the well-know properties of Fourier transform in higher dimensions. Details can be found in Gasquet, Witomski and Ryan [36] and Stein [72].

Definition 2.3. The *Fourier transform* of $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by

$$Ff(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle \xi, x \rangle} dx.$$

Similarly, the *inverse Fourier transform* of $\hat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is given by

$$\overline{F}\hat{f}(x) = f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi.$$

Since F and $\overline{F}$ preserve the L^2-norms, and $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the maps F and $\overline{F}$ extend to unitary operators on $L^2(\mathbb{R}^n)$.

Theorem 2.3 (Plancherel)**.** *(i) The Fourier transform F is a unitary operator on $L^2(\mathbb{R}^n)$.*

(ii) $\overline{F}$ is the inverse operator of F, i.e., $\overline{F}\hat{f} = f$ for all $f \in L^2(\mathbb{R}^n)$.

Theorem 2.4. *The Fourier transform F and its inverse Fourier transform $\overline{F}$ are linear one-to-one maps from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$.*

2.3 Group Theoretical Foundations

In this section, we briefly review the important group-theoretical concepts needed in this thesis. For more information on group-theoretical background we refer to Nachbin [66], Young [82], Folland [31], [32] and Stein [71].

Definition 2.4. A *topological group* is a set G which is both a group and topological space, such that the group operations $(g, h) \mapsto gh$ from $G \times G$ into G and $g \mapsto g^{-1}$ from G into itself are continuous in this topology. Any subgroup of a topological group G becomes a topological group in the relative topology of G.

A *locally compact group* is a topological group whose topology is locally compact and Hausdorff.

Let us consider some examples of topological groups:

1. The set $\mathbb{R}^n$ with its usual topology and with addition as the group operation is a locally compact group.

2. The set $\mathbb{Q}$ of rational numbers, with the subspace topology induced from $\mathbb{R}$ and with addition as the group operation is a topological subgroup of $\mathbb{R}$, but it is not locally compact.

3. $GL_n(\mathbb{R})$, the group of $n \times n$ real-invertible matrices.
   Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{R}$. $M_n(\mathbb{R})$ is a finite dimensional normed linear space, isomorphic to $\mathbb{R}^{n^2}$. Give $GL_n(\mathbb{R})$ the relative topology of $M_n(\mathbb{R})$. Then $GL_n(\mathbb{R})$ is a multiplicative locally compact topological group.

Definition 2.5. Let G be a locally compact group and $\mathcal{H}$ be a Hilbert space. A *representation* π of G on $\mathcal{H}$ is a mapping satisfying:

1. $\pi : G \to \mathcal{U}(\mathcal{H})$. ($\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$),

2. π is a homomorphism: $\pi_{gh} = \pi_g \pi_h$ for all $g, h \in G$,

3. π is continuous with respect to the strong operator topology of $\mathcal{U}(\mathcal{H})$, that is $g \to \pi_g \xi$ is continuous for each $\xi \in \mathcal{H}$.

A representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ is called *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ which are invariant under π_g for each $g \in G$.

Definition 2.6. A representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ is called *square integrable* if

1. π is irreducible,
2. there exists a vector $\psi \in \mathcal{H}\backslash\{0\}$ such that $\int_G |\langle \psi, \pi_g \psi \rangle|^2 d\mu(g) < \infty$ where μ is the left Haar measure on G. That is, the function $g \to \langle \psi, \pi_g \psi \rangle$ is square integrable. Such a vector ψ is called *admissible.*

In dealing with representations of locally compact groups, measures and integrals are important tools.

Definition 2.7. A Borel measure μ on a locally compact group G is called *left translation invariant* or a *left Haar measure* provided that for every continuous compactly supported function f on G and every $h \in G$ we have

$$\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g).$$

A *right Haar measure* γ is defined similarly.

Definition 2.8. Let G be a locally compact group, $\mathcal{H}$ Hilbert space, and $F : G \to \mathcal{H}$ continuous. If there exists a vector $f \in \mathcal{H}$ such that

$$\langle f, h \rangle = \int\limits_G \langle F(g), h \rangle \, d\mu(g), \qquad \forall\, h \in \mathcal{H}$$

then we say that $f = \int\limits_G F(g)\, d\mu(g)$ as a *weak integral* in $\mathcal{H}$.

2.4 The Abstract Wavelet Transform

In the most general sense, wavelets can be defined by group representations as we will explain below. This section is devoted to the continuous wavelet transform in its abstract setting. For more detail we refer to Heil and Walnut [53], Grossmann and Morlet [43], Bernier and Taylor [6], Führ [34] and Louis, Maass and Rieder [65].

Definition 2.9. Let G be a locally compact group whose left Haar measure is μ, and let $\pi_g : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G on a Hilbert space $\mathcal{H}$. Given a vector $\psi \in \mathcal{H}$, the collection $\{\pi_g \psi\}$ of vectors in $\mathcal{H}$ is called the *family of wavelets* generated by ψ, and ψ is called the *mother wavelet.* The mapping W_ψ, taking $f \in \mathcal{H}$ to a continuous function on G defined by

$$(W_\psi f)(g) = \langle f, \pi_g \psi \rangle_{\mathcal{H}} \tag{2.2}$$

is called the *wavelet transform of f with respect to ψ.*

Numerous authors have studied the theory of wavelets from the point of view of square integrable group representations. The following important theorem links the wavelet transform to the theory of square integrable representations:

Theorem 2.5. *(Duflo-Moore [25]): If π is a square integrable representation of a locally compact group G on $\mathcal{H}$, then there exists a unique densely defined operator K on $\mathcal{H}$, self adjoint and positive which satisfies the following:*

i) The set of admissible vectors in $\mathcal{H}$ coincides with the domain of K, that is dom $K = \{\psi \in \mathcal{H} : \psi$ is admissible $\}$.

ii) If ψ is an admissible vector and f is an arbitrary vector in $\mathcal{H}$, then

$$||W_\psi f||_{L^2(G)} = \sqrt{c_\psi}||f||_{\mathcal{H}}$$

where $c_\psi = ||K\psi||^2_{\mathcal{H}}$.

iii) If the group G is unimodular, then K is a multiple of the identity.

Thus, if the representation π is square integrable, then there exists a dense set of vectors ψ in $\mathcal{H}$ such that

$$||W_\psi f||_{L^2(G)} = \sqrt{c_\psi}||f||_{\mathcal{H}}, \qquad \forall f \in \mathcal{H} \tag{2.3}$$

for some constant c_ψ, i.e. the wavelet transform W_ψ associated with ψ is a multiple of an isometry from $\mathcal{H}$ into $L^2(G)$. It turns out that equality (2.3) holds if and only if for all $f \in \mathcal{H}$,

$$f = \frac{1}{c_\psi}\int_G (W_\psi f)(g)\, \pi_g \psi \, d\mu(g) \tag{2.4}$$

as a weak integral in $\mathcal{H}$.

The traditional wavelet transform operates on functions defined on $\mathbb{R}^n$, thus we will choose $\mathcal{H} = L^2(\mathbb{R}^n)$ from now on.

Definition 2.10. Let $G^\sharp$ be the group consisting of pairs $(a, b) \in GL_n(\mathbb{R}) \times \mathbb{R}^n$ together with the group operation

$$(x, y) \cdot (a, b) = (xa, a^{-1}y + b)$$

and the product topology. $G^\sharp$ is called the *affine group*. This kind of group construction is called a *semi-direct product*, and thus $G^\sharp$ is also called the semi-direct product of $GL_n(\mathbb{R})$ and $\mathbb{R}^n$, written $GL_n(\mathbb{R}) \rtimes \mathbb{R}^n$.

If D is a closed subgroup of $GL_n(\mathbb{R})$ then $G = \{(a, b) \in G^\sharp, a \in D, b \in \mathbb{R}^n\}$ is a closed subgroup of $G^\sharp$, and G is the semi-direct product $D \rtimes \mathbb{R}^n$. We call D the *dilation subgroup* of G and $\mathbb{R}^n$ the *translation subgroup* of G.

Note that the Haar measure $d\nu(a, b)$ on G is simply the product of the Haar measure $d\mu(a)$ on D with the Lebesgue measure $d\lambda(b)$ on $\mathbb{R}^n$. In fact, for any $f \in L^1(G)$,

$$\begin{aligned}\int_G f\big((x, y) \cdot (a, b)\big)d\nu(a, b) &= \int_D\int_{\mathbb{R}^n} f\big(xa, a^{-1}y + b\big)d\lambda(b)d\mu(a)\\ &= \int_D\int_{\mathbb{R}^n} f(a, b)d\lambda(b)d\mu(a)\\ &= \int_G f(a, b)d\nu(a, b).\end{aligned}$$

This shows that $d\nu(a,b) = d\lambda(b)d\mu(a)$ is left translation invariant. There is a natural representation π of $G^\sharp$ on $L^2(\mathbb{R}^n)$ given by

$$\pi_{(a,b)}\psi(x) = |\det\ a|^{-1/2}\psi(a^{-1}x - b) \equiv \psi_{a,b}(x) \tag{2.5}$$

for $(a,b) \in G^\sharp$ and $\psi \in L^2(\mathbb{R}^n)$. If G is a subgroup of $G^\sharp$ as in definition 2.10, then the wavelet transform W_ψ induced by ψ and the representation π becomes

$$W_\psi f(a,b) = \langle f, \psi_{a,b}\rangle = \frac{1}{\sqrt{|\det\ a|}}\int_{\mathbb{R}^n} f(x)\overline{\psi(a^{-1}x - b)}dx \tag{2.6}$$

for $f \in L^2(\mathbb{R}^n)$ and $(a,b) \in G$.

Remark 2.1. *We want to find a reconstruction formula for the wavelet transform (2.6). From the discussion following the Duflo-Moore theorem we know that if (2.3) holds then reconstruction formula (2.4) follows, which here becomes*

$$f(x) = \int_D\int_{\mathbb{R}^n} W_\psi f(a,b)\,\psi_{a,b}(x)d\lambda(b)\,d\mu(a) \tag{2.7}$$

in case $c_\psi = 1$. Thus, we need to investigate under what condition (2.3) holds.
Note that by scaling the function ψ we may always assume that $c_\psi = 1$.

Remark 2.2. *When discussing the wavelet transform, one usually makes use of the tools of the Fourier transform. Since the Fourier transform $F : f \to \hat{f}$ constitutes a unitary operator on $L^2(\mathbb{R}^n)$, π induces a representation*

$$\rho = F\pi\overline{F} \tag{2.8}$$

of $G^\sharp$ on $L^2(\mathbb{R}^n)$. Let us compute this representation. For $\widehat{\psi} = F\psi \in L^2(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
\rho_{(a,b)}\widehat{\psi}(\xi) &= F\pi_{(a,b)}\overline{F}(F\psi)(\xi)\\
&= \int_{\mathbb{R}^n} \big(\pi_{(a,b)}\psi\big)(x)e^{-2\pi i\langle\xi,x\rangle}\,dx\\
&= \int_{\mathbb{R}^n} |\det\ a|^{-1/2}\psi(a^{-1}x - b)e^{-2\pi i\langle\xi,x\rangle}\,dx\\
&= \int_{\mathbb{R}^n} |\det\ a|^{1/2}\psi(x - b)e^{-2\pi i\langle\xi,ax\rangle}\,dx\\
&= |\det\ a|^{1/2}e^{-2\pi i\langle\xi,ab\rangle}\int_{\mathbb{R}^n}\psi(x)e^{-2\pi i\langle\xi,ax\rangle}\,dx\\
&= |\det\ a|^{1/2}e^{-2\pi i\langle\xi,ab\rangle}\widehat{\psi}(a^T\xi) \equiv \widehat{\psi}_{a,b}(\xi)
\end{aligned}$$

where elements ξ of $\mathbb{R}^n$ are now written as row vectors, and x are column vectors.
Formula (2.6) for the wavelet transform becomes now

$$\begin{aligned}
(W_\psi f)(a,b) &= \langle f, \pi_{(a,b)}\psi\rangle = \langle \hat{f}, \rho_{(a,b)}\widehat{\psi}\rangle = \langle \hat{f}, \widehat{\psi}_{a,b}\rangle\\
&= |\det\ a|^{1/2}\int_{\mathbb{R}^n}\hat{f}(\xi)\overline{\widehat{\psi}(a^T\xi)}e^{2\pi i\langle\xi,ab\rangle}d\xi.
\end{aligned} \tag{2.9}$$

Chapter 3

Shearlet Groups and their Weighted Density

In this chapter, we will review four definitions of a continuous shearlet transform from a group theoretic point of view and determine the relationship between them. Furthermore, we will give definitions of weighted density for each shearlet system and show their basic properties which will be used in Chapter 4. Finally, we will give special examples of discrete subset of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ and compute their density. We shall mention that many ideas and proofs of this chapter are inspired by Heil and Kutyniok [51] and Kutyniok [58].

3.1 Continuous Shearlet Transform Associated to the Shearlet Group $\mathbb{S}_1$

In this section we show that the continuous shearlet transform is directly related to the theory of group representations as the following results show. In order to define the shearlet group we first require the following lemma.

Lemma 3.1. *Let $\mathbb{S}_1 = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ be equipped with the group multiplication*

$$(a, s, t) \cdot (a', s', t') = (aa', s' + s\sqrt{a'}, t' + S_{s'} A_{a'} t),$$

where $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$, *and* $S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ *for $a > 0$ and $s \in \mathbb{R}^+$. Then $(\mathbb{S}_1, \cdot)$ forms a group.*

Proof. The identity element of $\mathbb{S}_1$ is $e_1 = (1, 0, 0)$, and inverses are given by

$$(a, s, t)^{-1} = \left(\frac{1}{a}, \frac{-s}{\sqrt{a}}, -S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t\right),$$

since

$$(a, s, t) \cdot \left(\frac{1}{a}, \frac{-s}{\sqrt{a}}, -S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t\right) = \left(1, \frac{-s}{\sqrt{a}} + \frac{s}{\sqrt{a}}, -S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t + S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t\right) = (1, 0, 0)$$

and

$$\left(\frac{1}{a}, \frac{-s}{\sqrt{a}}, -S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t\right) \cdot (a, s, t) = \left(1, s - \frac{s}{\sqrt{a}}\sqrt{a}, t - S_s A_a (S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t)\right) = (1, 0, 0)$$

because

$$S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} = A_a^{-1} S_s^{-1}.$$

The fact that the group multiplication is associative can be shown as follows:

$$\begin{aligned}
((a,s,t)\cdot(a',s',t'))\cdot(a'',s'',t'') &= \left(aa',\, s'+s\sqrt{a'},\, t'+S_{s'}A_{a'}t\right)\cdot(a'',s'',t'') \\
&= \left(aa'a'',\, s''+(s'+s\sqrt{a'})\sqrt{a''},\, t''+S_{s''}A_{a''}(t'+S_{s'}A_{a'}t)\right) \\
&= \left(a(a'a''),\, (s''+s'\sqrt{a''})+s\sqrt{a'a''},\, (t''+S_{s''}A_{a''}t')+S_{s''}A_{a''}(S_{s'}A_{a'}t)\right) \\
&= (a,s,t)\cdot\left(a'a'',\, s''+s'\sqrt{a''},\, t''+S_{s''}A_{a''}t'\right) \\
&= (a,s,t)\cdot\left((a',s',t')\cdot(a'',s'',t'')\right),
\end{aligned}$$

where we have used that

$$\begin{aligned}
S_{s''}A_{a''}S_{s'}A_{a'} &= \begin{pmatrix} a'' & s''\sqrt{a''} \\ 0 & \sqrt{a''} \end{pmatrix}\begin{pmatrix} a' & s'\sqrt{a'} \\ 0 & \sqrt{a'} \end{pmatrix} = \begin{pmatrix} a'a'' & s'\sqrt{a'}a''+s''\sqrt{a'a''} \\ 0 & \sqrt{a'a''} \end{pmatrix} \\
&= S_{s''+s'\sqrt{a''}}A_{a'a''}.
\end{aligned}$$

□

Definition 3.1. The set $\mathbb{S}_1 = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ equipped with the multiplication given by

$$(a,s,t)\cdot(a',s',t') = (aa', s'+s\sqrt{a'}, t'+S_{s'}A_{a'}t),$$

is called the *shearlet group*.

Throughout this thesis we shall use the notations $T_t f(x) = f(x-t)$, $t \in \mathbb{R}^2$ and $D_M f(x) = |\det M|^{-1/2} f(M^{-1}x)$, $M \in GL_2(\mathbb{R})$ as the *translation* and *dilation* operator on $L^2(\mathbb{R}^2)$, respectively.

Let $\psi \in L^2(\mathbb{R}^2)$, and define $\sigma_1 : \mathbb{S}_1 \longrightarrow \mathcal{U}(L^2(\mathbb{R}^2))$ by

$$\sigma_1(a,s,t)\psi(x) = D_{A_a^{-1}S_s^{-1}} T_t \psi(x) = a^{3/4}\psi(S_sA_ax - t) =: \psi_{a,s,t}(x), \tag{3.1}$$

for $(a,s,t) \in \mathbb{S}_1$. This is a unitary representation which can be shown as follows:

$$\begin{aligned}
\sigma_1(a,s,t)\left[\sigma_1(a',s',t')\psi(x)\right] &= a^{3/4}\sigma_1(a',s',t')\psi(S_sA_ax-t) \\
&= (aa')^{3/4}\psi\left(S_{s'}A_{a'}(S_sA_ax-t)-t'\right) \\
&= (aa')^{3/4}\psi\left(S_{s'+s\sqrt{a'}}A_{aa'}x-(t'+S_{s'}A_{a'}t)\right) \\
&= \sigma_1\left(aa',\, s'+s\sqrt{a'},\, t'+S_{s'}A_{s'}t\right)\psi(x) \\
&= \sigma_1\left((a,s,t)\cdot(a',s',t')\right)\psi(x).
\end{aligned}$$

Definition 3.2. Let $\sigma_1 : \mathbb{S}_1 \to \mathcal{U}(L^2(\mathbb{R}^2))$ be a unitary representation of $\mathbb{S}_1$ on $L^2(\mathbb{R}^2)$. Given $\psi \in L^2(\mathbb{R}^2)$, the family $\{\sigma_1(a,s,t)\psi : (a,s,t) \in \mathbb{S}_1\}$ of functions in $L^2(\mathbb{R}^2)$ is called *shearlets*. The mapping $\mathcal{SH}_{1,\psi}$ taking $f \in L^2(\mathbb{R}^2)$ to a continuous function on $\mathbb{S}_1$ defined by

$$\mathcal{SH}_{1,\psi}(a,s,t) = \langle f, \sigma_1(a,s,t)\psi\rangle = \langle f, D_{A_a^{-1}S_s^{-1}}T_t\psi\rangle = \langle f, \psi_{a,s,t}\rangle, \tag{3.2}$$

is called the *continuous shearlet transform*.

Remark 3.1. *The first definition of a continuous shearlet transform was introduced by Guo, Kutyniok and Labate [45] and then the studies of Kutyniok and Labate [61] and Dahlke et al. [18], [19], but in a different form as in (3.2). We will show in the subsequent sections that there are other definitions of continuous shearlet transform, depending on the choice of a group multiplication, and show the relationship between these definitions in the sense of group isomorphism.*

Note that the left-Haar measure $d\mu_{\mathbb{S}_1}(a,s,t)$ on $\mathbb{S}_1$ is simply the product of the Haar measure $d\nu(a) = \frac{da}{a}$ on $\mathbb{R}^+$ and the Lebesgue measure $ds\,dt$ on $\mathbb{R}\times\mathbb{R}^2$. We have, for any $f \in L^1(\mathbb{S}_1)$,

$$\begin{aligned}\int\limits_{\mathbb{S}} f((a',s',t')\cdot(a,s,t))\,d\mu_{\mathbb{S}_1}(a,s,t) &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+} f(a'a, s+s'\sqrt{a}, t+S_sA_at')\,d\nu(a)\,ds\,dt \\ &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+} f(a,s,t)\,d\nu(a)\,ds\,dt\end{aligned}$$

This shows that $d\mu_{\mathbb{S}_1}(a,s,t) = \frac{da\,ds\,dt}{a}$ is left translation invariant.

Definition 3.3. We call a function $\psi \in L^2(\mathbb{R}^2)$ *admissible* if it satisfies

$$C_\psi = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+} \frac{|\hat{\psi}(u_1,u_2)|^2}{u_1^2}\,du_1du_2 < \infty, \tag{3.3}$$

and we call the condition (3.3) *the admissibility condition.*

Proposition 3.2. *Let $\psi \in L^2(\mathbb{R}^2)$ and $\psi_{a,s,t}$ be defined as in (3.1). Then the continuous shearlet transform*

$$\mathcal{SH}_{1,\psi} : f \mapsto \mathcal{SH}_{1,\psi}f$$

maps $L^2(\mathbb{R}^2)$ into $\mathcal{H} := L^2\left(\mathbb{S}_1, \frac{da\,ds\,dt}{a}\right)$.
Furthermore,

$$\langle \mathcal{SH}_{1,\psi}f, \mathcal{SH}_{1,\psi}g\rangle_{\mathcal{H}} = C_\psi\langle f,g\rangle_{L^2(\mathbb{R}^2)} \tag{3.4}$$

where

$$C_\psi = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^+} \frac{|\hat{\psi}(u_1,u_2)|^2}{u_1^2}\,du_1du_2 < \infty. \tag{3.5}$$

Remark 3.2. *By the polarization identity, the equality (3.4) is equivalent to*

$$||\mathcal{SH}_{1,\psi}f||_{\mathcal{H}}^2 = C_\psi||f||_2^2 \qquad \textit{for all } f \in L^2(\mathbb{R}^2). \tag{3.6}$$

Proof of Proposition 3.2. It suffices to prove (3.6) which also implies injectivity. By the Plancherel Theorem, for any $f \in L^2(\mathbb{R}^2)$, we have

$$\begin{aligned}||\mathcal{SH}_{1,\psi}(a,s,t)f||_{\mathcal{H}}^2 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \left|\langle \hat{f}, \hat{\psi}_{a,s,t}\rangle\right|^2 \frac{da}{a}\,ds\,dt \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} a^{-3/2}\left|\int_{\mathbb{R}^2} \hat{f}(\xi)\overline{\hat{\psi}(S_s^{-T}A_a^{-1}\xi)}e^{2\pi i\langle\xi, A_a^{-1}S_s^{-1}t\rangle}\,d\xi\right|^2 \frac{da}{a}\,ds\,dt\end{aligned}$$

Now we will set $F_{a,s}(\xi) = \hat{f}(\xi)\overline{\hat{\psi}(S_s^{-T}A_a^{-1}\xi)}$. Since both $\hat{f}, \hat{\psi} \in L^2(\mathbb{R}^2)$, the product is also in $L^1(\mathbb{R}^2)$ by the Cauchy-Schwarz inequality. Inside the absolute value, the integral is the inverse Fourier transform $\check{F}_{a,s}$ of $F_{a,s}$, so that

$$\begin{aligned}
||\mathcal{SH}_{1,\psi}(a,s,t)f||^2_{\mathcal{H}} &= \int_{\mathbb{R}}\int_{\mathbb{R}+}\left[\int_{\mathbb{R}^2} a^{-3/2}\left|\check{F_{as}}(A_a^{-1}S_s^{-T}t)\right|^2 dt\right]\frac{da}{a}\,ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}+}\left[\int_{\mathbb{R}^2}\left|\check{F_{as}}(t)\right|^2 dt\right]\frac{da}{a}\,ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}+}\left[\int_{\mathbb{R}^2}|F_{as}(t)|^2\,dt\right]\frac{da}{a}\,ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}+}\left[\int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\left|\hat{\psi}(S_s^{-T}A_a^{-1}\xi)\right|^2 d\xi\right]\frac{da}{a}\,ds \\
&= \int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\left[\int_{\mathbb{R}+}\int_{\mathbb{R}}\left|\hat{\psi}(S_s^{-T}A_a^{-1}\xi)\right|^2 ds\frac{da}{a}\right]d\xi \\
&= \int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\left[\int_{\mathbb{R}+}\int_{\mathbb{R}}\left|\hat{\psi}\left(\frac{\xi_1}{a}, -\frac{s\xi_1}{a}+\frac{\xi_2}{\sqrt{a}}\right)\right|^2 ds\frac{da}{a}\right]d\xi \\
&= \int_{\mathbb{R}^2}|\hat{f}(\xi)|^2\left[\int_{\mathbb{R}}\int_{\mathbb{R}+}\left|\hat{\psi}\left(\frac{\xi_1}{a}, u_2\right)\right|^2 da\,du_2\right]d\xi \\
&= \int_{\mathbb{R}^2}|\hat{f}(\xi)|^2 d\xi\left[\int_{\mathbb{R}}\int_{\mathbb{R}+}\frac{\left|\hat{\psi}\left(u_1,u_2\right)\right|^2}{u_1^2}du_1\,du_2\right] \\
&= C_\psi||\hat{f}||_2^2 = C_\psi||f||_2^2.
\end{aligned}$$

This proves the proposition. ☐

Proposition 3.3. *(Reconstruction formula for the continuous shearlet transform) Let σ_1 be a unitary representation of $\mathbb{S}_1$ on $L^2(\mathbb{R}^2)$, and let $\psi \in L^2(\mathbb{R}^2)$. Then (3.6) holds if and only if for all $f \in L^2(\mathbb{R}^2)$*

$$f = \frac{1}{C_\psi}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}+}(\mathcal{SH}_{1,\psi}(a,s,t)f)\sigma_1(a,s,t)\psi\,\frac{da\,ds\,dt}{a} \tag{3.7}$$

in a weak sense.

Proof. (⇒) Assume (3.6) holds. By the polarization identity, for all $f \in L^2(\mathbb{R}^2)$ we have equality (3.4). Divide (3.4) by C_ψ and rewrite the inner product in $L^2(\mathbb{S}_1)$ as

$$\begin{aligned}
\langle f, g\rangle &= \frac{1}{C_\psi}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}+}(\mathcal{SH}_{1,\psi}(a,s,t)f)\overline{(\mathcal{SH}_{1,\psi}(a,s,t)g)}\,\frac{da\,ds\,dt}{a} \\
&= \frac{1}{C_\psi}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}+}(\mathcal{SH}_{1,\psi}(a,s,t)f)\overline{\langle g, \sigma_1(a,s,t)\psi\rangle}\,\frac{da\,ds\,dt}{a} \\
&= \frac{1}{C_\psi}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}+}(\mathcal{SH}_{1,\psi}(a,s,t)f)\langle \sigma_1(a,s,t)\psi, g\rangle\,\frac{da\,ds\,dt}{a} \\
&= \frac{1}{C_\psi}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}+}\langle(\mathcal{SH}_{1,\psi}(a,s,t)f)\sigma_1(a,s,t)\psi, g\rangle\,\frac{da\,ds\,dt}{a}.
\end{aligned}$$

This means that

$$f = \frac{1}{C_\psi} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}^+} (\mathcal{SH}_{1,\psi}(a,s,t)f)\sigma_1(a,s,t)\psi \, \frac{da\, ds\, dt}{a}$$

as a weak integral.
($\Leftarrow$) Assume that the reconstruction formula (3.7) holds in a weak sense. By going backwards in the above computation, one easily verifies that (3.6) holds, i.e., $\mathcal{SH}_{1,\psi}$ is an isometry. □

Remark 3.3. *It turns out that the equality (3.4) gives a reconstruction formula if and only if the admissibility condition (3.3) holds.*

3.2 Weighted Density for Shearlet Systems of $\mathbb{S}_1$

It was shown in the previous section that we can reconstruct a function from its shearlet transform in the form of a weak integral, but in general computations, it is much easier to work with series than with weak integrals. That leads to the question of how can we find a discrete subset Λ of $\mathbb{S}_1$ such that the associated family of functions $\{\psi_{a,s,t}\}_{(a,s,t)\in\Lambda}$, called *shearlet system*, is a frame for $L^2(\mathbb{R}^2)$, where $\psi_{a,s,t}$ is defined as (3.1).

Since the study of density of a sequence of time-scale parameters associated with wavelet systems have turned out to be an effective tool for deriving necessary conditions for the existence of an upper and lower frame bound, it is natural to ask whether shearlet systems share similar properties because shearlet systems are indeed an two-dimensional affine-like systems. This will be explored in the next chapter.

In this section we will introduce notions of density for shearlet systems and study their basic properties.

For all $h > 0$, we let Q_h denote a fixed family of neighborhoods of the identity element $e = (1,0,0)$ in $\mathbb{S}_1$

$$Q_h = [e^{-h/2}, e^{h/2}) \times \left[-\frac{h}{2}, \frac{h}{2}\right) \times \left[-\frac{h}{2}, \frac{h}{2}\right)^2.$$

Then, for $(x,y,z) \in \mathbb{S}_1$, we define $Q_h(x,y,z)$ to be the set Q_h left-translated via the group action, so that it is centered at the point (x,y,z), i.e.,

$$\begin{aligned} Q_h(x,y,z) &= (x,y,z)\cdot Q_h \\ &= \left\{(xa, s+y\sqrt{a}, t+S_sA_az) \; : a \in [e^{-h/2}, e^{h/2}), s \in \left[-\frac{h}{2}, \frac{h}{2}\right), t \in \left[-\frac{h}{2}, \frac{h}{2}\right)^2\right\}. \end{aligned}$$

Definition 3.4. Let $X = \{x_i\}_{i\in I}$ be a sequence of elements in $\mathbb{S}_1$.

(i) X is called *Q_h-dense* in $\mathbb{S}_1$, if $\bigcup_{i\in I} x_i \cdot Q_h = \mathbb{S}_1$.

(ii) X is called *separated*, if for some compact neighborhood Q_h we have $x_i\cdot Q_h \cap x_j \cdot Q_h = \emptyset$, $i \neq j$, and called *relatively separated*, if X is a finite union of separated sets.

We choose the left-invariant Haar measure $\mu_{\mathbb{S}_1} = \frac{da}{a}\, ds\, dt$, to define the volume of $Q_h(x,y,z)$:

$$\mu_{\mathbb{S}_1}(Q_h(x,y,z)) = \mu_{\mathbb{S}_1}(Q_h) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \frac{1}{a}\, da\, ds\, dt_1\, dt_2 = h^4.$$

Let Λ be a discrete subset of $\mathbb{S}_1$. For a weight function $w : \Lambda \to \mathbb{R}^+$, we define the weighted number of elements of Λ lying in a subset K of $\mathbb{S}_1$ to be

$$\#_w(K) = \sum_{(a,s,t)\in K} w(a,s,t).$$

Definition 3.5. Let Λ be a discrete subset of $\mathbb{S}_1$. Then *the upper weighted density* of Λ is

$$D_w^+(\Lambda) = \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_1} \frac{\#_w(\Lambda \cap Q_h(x,y,z))}{h^4},$$

and the *lower weighted density* of Λ is

$$D_w^-(\Lambda) = \liminf_{h\to\infty} \inf_{(x,y,z)\in\mathbb{S}_1} \frac{\#_w(\Lambda \cap Q_h(x,y,z))}{h^4}.$$

Remark 3.4. *Intuitively, the density of Λ is a measurement of the average number of points of Λ lying in sets $Q_h(x,y,z)$. Since the points in Λ do not normally spread uniformly, a single definition of density is not enough to capture this behavior appropriately. Therefore, in the definition, we move the sets Q_h via a group multiplication, i.e., $Q_h(x,y,z) = (x,y,z)\cdot Q_h$, in order to find the maximum and minimum average of points of Λ lying in sets $Q_h(x,y,z)$. We then use the upper and lower limits to define the upper and lower density, respectively. In the sequent propositions, we will show how to compute the density of several kinds of discrete subsets Λ of $\mathbb{S}_1$.*

In the next technical lemma, we will show that there exist sets of the form $Q_h(x,y,z)$ which lead to a covering of the shearlet group $\mathbb{S}_1$, but not a disjoint one. However, the number of overlaps of those sets $Q_h(x,y,z)$ is independent of $(x,y,z)\in\mathbb{S}_1$, which can be seen in the second and third parts of the following lemma.

Lemma 3.4. *Let $h > 0$ and $r \geq 1$ be given, and let*

$$X_1 = \{(e^{jh}, he^{-h/4}k, he^{-h/2}m) : j,k\in\mathbb{Z}, m\in\mathbb{Z}^2\}.$$

Then the following statements hold:

1. *X_1 is Q_h-dense in $\mathbb{S}_1$.*
2. *Any set $Q_{rh}(x,y,z)$ intersects at most*

$$N_r := (r+2)(r+1)^3\left[e^{h/2}+\frac{1}{r+1}\right]\left[e^{h}+\frac{1}{r+1}\right]\left[e^{3h/4}+\frac{1}{r+1}\right]$$

   elements in X_1, i.e., X_1 is relatively separated.

3. *Any set $Q_{rh}(x,y,z)$ contains at least*

$$\tilde{N}_r := r(r+1)^3 e^{9h/4}$$

   elements in X_1.

Proof. 1. Fix any $(x, y, z) \in \mathbb{S}_1$. We show that there exist $(a, s, t) \in Q_h$, $j, k \in \mathbb{Z}$, and $m \in \mathbb{Z}^2$ such that

$$\begin{aligned}(x, y, z) &= (ae^{jh}, s + he^{-h/4}k\sqrt{a}, t + he^{-h/2}S_sA_am) \\ &= (e^{jh}, he^{-h/4}k, he^{-h/2}m) \cdot (a, s, t) \in Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m).\end{aligned}$$

In particular,

$$x = ae^{jh} \quad , \quad y = s + he^{-h/4}k\sqrt{a}, \quad \text{and} \quad z = t + he^{-h/2}S_sA_am.$$

These three equalities are equivalent to

$$j = \frac{\ln x}{h} - \frac{\ln a}{h}, \tag{3.8}$$

$$k = \frac{ye^{h/4}}{h\sqrt{a}} - \frac{se^{h/4}}{h\sqrt{a}}, \tag{3.9}$$

$$m_1 = \frac{(z_1 - t_1)e^{h/2}}{ha} - \frac{s(z_2 - t_2)e^{h/2}}{ha}, \tag{3.10}$$

$$m_2 = \frac{(z_2 - t_2)e^{h/2}}{h\sqrt{a}}. \tag{3.11}$$

Now we observe the following:

- Let $a \in [e^{-h/2}, e^{h/2})$. Following from (3.8), we form the interval $\left[\frac{\ln x}{h} - \frac{1}{2}, \frac{\ln x}{h} + \frac{1}{2}\right)$ which contains a unique integer j.
- Take the same number a as above, and $s \in [-\frac{h}{2}, \frac{h}{2})$. Following from (3.9), we form the interval $\left[\frac{ye^{h/4}}{h\sqrt{a}} - \frac{e^{h/4}}{2\sqrt{a}}, \frac{ye^{h/4}}{h\sqrt{a}} + \frac{e^{h/4}}{2\sqrt{a}}\right)$ which contains an integer k.
- Now take a as above, and $t_2 \in [-\frac{h}{2}, \frac{h}{2})$. By (3.11), we form the interval $\left[\frac{z_2e^{h/2}}{h\sqrt{a}} - \frac{e^{h/2}}{2\sqrt{a}}, \frac{z_2e^{h/2}}{h\sqrt{a}} + \frac{e^{h/2}}{2\sqrt{a}}\right)$ which contains an integer m_2.
- Now take a, s and t_2 as above. Using (3.10), we form the interval $\left[\frac{z_1e^{h/2}}{ha} - \frac{sz_2e^{h/2}}{ha} + \frac{st_2e^{h/2}}{ha} - \frac{e^{h/2}}{2a}, \frac{z_1e^{h/2}}{ha} - \frac{sz_2e^{h/2}}{ha} + \frac{st_2e^{h/2}}{ha} + \frac{e^{h/2}}{2a}\right)$ which contains an integer m_1.

Thus $\{Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ is a covering of $\mathbb{S}_1$, i.e., X_1 is Q_h-dense in $\mathbb{S}_1$.
2. Fix $(x, y, z) \in \mathbb{S}_1$, and suppose $(u, v, w) \in Q_{rh}(x, y, z) \cap Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m)$. Then there exist $(a, s, t) \in Q_{rh}$ and $(a', s', t') \in Q_h$ such that

$$\begin{aligned}(u, v, w) &= (x, y, z) \cdot (a, s, t) \\ &= (ax, s + y\sqrt{a}, t + S_sA_az) \in Q_{rh}(x, y, z) \\ \text{and} \quad (u, v, w) &= (e^{jh}, he^{-h/4}k, he^{-h/2}m) \cdot (a', s', t') \\ &= (a'e^{jh}, s' + he^{-h/4}k\sqrt{a'}, t' + he^{-h/2}S_{s'}A_{a'}m) \in Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m).\end{aligned}$$

In particular, $ax = a'e^{jh}$ with $a \in [e^{-rh/2}, e^{rh/2})$ and $a' \in [e^{-h/2}, e^{h/2})$. We have

$$xe^{-h(r+1)/2} \leq e^{jh} \leq xe^{h(r+1)/2}, \tag{3.12}$$

$$\text{and} \quad \frac{\ln x}{h} - \frac{r+1}{2} \leq j \leq \frac{\ln x}{h} + \frac{r+1}{2}.$$

This is satisfied for at most $r+2$ values of j.
Further, $k = \frac{(s-s')e^{h/4}}{h\sqrt{a'}} + \frac{y\sqrt{a}e^{h/4}}{h\sqrt{a'}}$ with $s \in [-\frac{rh}{2}, \frac{rh}{2})$ and $s' \in [-\frac{h}{2}, \frac{h}{2})$, so that

$$\frac{y\sqrt{a}e^{h/4}}{h\sqrt{a'}} - \left(\frac{r+1}{2}\right)e^{h/2} \leq k \leq \frac{y\sqrt{a}e^{h/4}}{h\sqrt{a'}} + \left(\frac{r+1}{2}\right)e^{h/2}. \tag{3.13}$$

For a given value of $a \in [e^{-rh/2}, e^{rh/2})$ and $a' \in [e^{-h/2}, e^{h/2})$, this is satisfied for at most $(r+1)e^{h/2}+1$ values of k.
Furthermore, we have

$$he^{-h/2}m = \underbrace{A_{a'}^{-1}S_{s'}^{-1}S_sA_az}_{=:(C_1,C_2)^T} + A_{a'}^{-1}S_{s'}^{-1}(t-t')$$

$$\begin{pmatrix} he^{-h/2}m_1 \\ he^{-h/2}m_2 \end{pmatrix} = \begin{pmatrix} C_1 + \frac{(t_1-t_1')}{a'} - \frac{s'(t_2-t_2')}{a'} \\ C_2 + \frac{(t_2-t_2')}{\sqrt{a'}} \end{pmatrix},$$

with $t_1', t_2' \in \left[-\frac{h}{2}, \frac{h}{2}\right)$, and $t_1, t_2 \in \left[-\frac{rh}{2}, \frac{rh}{2}\right)$, hence

$$\frac{C_1e^{h/2}}{h} - \frac{s'e^{h/2}(t_2-t_2')}{a'h} - \frac{(r+1)e^h}{2} \leq m_1 \leq \frac{C_1e^{h/2}}{h} - \frac{s'e^{h/2}(t_2-t_2')}{a'h} + \frac{(r+1)e^h}{2} \tag{3.14}$$

$$\frac{C_2e^{h/2}}{h} - \frac{(r+1)e^{3h/4}}{2} \leq m_2 \leq \frac{C_2e^{h/2}}{h} + \frac{(r+1)e^{3h/4}}{2}. \tag{3.15}$$

For a given value of $a \in [e^{-rh/2}, e^{rh/2})$, $a' \in [e^{-h/2}, e^{h/2})$, $s \in [-\frac{rh}{2}, \frac{rh}{2})$ and $s' \in [-\frac{h}{2}, \frac{h}{2})$, (3.14) is satisfied for at most $(r+1)e^h+1$ values of m_1, and (3.15) is satisfied for at most $(r+1)e^{3h/4}+1$ values of m_2. Thus $Q_{rh}(x,y,z)$ can intersect at most

$$(r+2)(r+1)^3\left[e^{h/2} + \frac{1}{r+1}\right]\left[e^h + \frac{1}{r+1}\right]\left[e^{3h/4} + \frac{1}{r+1}\right]$$

sets of the form $Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m)$.
3. There are at least r values of j satisfies (3.12). For a given value of j, there are at least $(r+1)e^{h/2}$ values of k satisfies (3.13). Furthermore, for a given value of j and k, (3.14) is satisfied for at least $(r+1)e^h$ values of m_1, and for a given value of j, (3.15) is satisfied for at least $(r+1)e^{3h/4}$ values of m_2. Thus $Q_{rh}(x,y,z)$ must intersect at least $r(r+1)^3e^{9h/4}$ sets of the form $Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m)$. □

By using the above lemma, the following proposition provides some information on the structure of subsets Λ of the shearlet group $\mathbb{S}_1$, which have finite upper weighted density and positive lower weighted density.

Proposition 3.5. *For $\Lambda \subset \mathbb{S}_1$ and $w : \Lambda \to \mathbb{R}^+$, the following conditions are equivalent:*

1. $D_w^+(\Lambda) < \infty$.

2. *There exists an* $h > 0$ *such that* $\sup_{(x,y,z)\in\mathbb{S}_1} \#_w(\Lambda \cap Q_h(x,y,z)) < \infty$.

Also the following conditions are equivalent

1. $D_w^-(\Lambda) > 0$.

2. *There exists* $h > 0$ *such that* $\inf_{(x,y,z)\in\mathbb{S}_1} \#_w(\Lambda \cap Q_h(x,y,z)) > 0$.

Proof. ($\Rightarrow$) is trivial.
($\Leftarrow$) Suppose there exists $h > 0$ such that $R := \sup_{(x,y,z)\in\mathbb{S}_1} \#_w(\Lambda \cap Q_h(x,y,z)) < \infty$.
For $0 < t < h$, we have $\#_w(\Lambda \cap Q_t(x,y,z)) \leq \#_w(\Lambda \cap Q_h(x,y,z))$ for all $(x,y,z) \in \mathbb{S}_1$.
Hence $\sup_{(x,y,z)\in\mathbb{S}_1} \#_w(\Lambda \cap Q_t(x,y,z)) < R$. If $t \geq h$, assume $t = rh$ where $r \geq 1$. By Lemma 3.4 the box $Q_{rh}(x,y,z)$ is covered by a union of at most N_r sets of the form $Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m)$.
This implies

$$\begin{aligned}\sup_{(x,y,z)\in\mathbb{S}_1} \#_w(\Lambda \cap Q_{rh}(x,y,z)) &\leq N_r \cdot \sup_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2} \#_w(\Lambda \cap Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m)) \\ &\leq N_r \cdot R.\end{aligned}$$

Thus,

$$\begin{aligned}D^+(\Lambda) &\leq \limsup_{r\to\infty} \frac{N_r R}{(rh)^4} \\ &= R \cdot \limsup_{r\to\infty} \frac{(r+2)(r+1)^3}{(rh)^4} \left[e^{h/2} + \frac{1}{r+1}\right] \left[e^{h} + \frac{1}{r+1}\right] \left[e^{3h/4} + \frac{1}{r+1}\right] \\ &= R \cdot \lim_{r\to\infty} \frac{(r+2)(r+1)^3}{(rh)^4} \left[e^{h/2} + \frac{1}{r+1}\right] \left[e^{h} + \frac{1}{r+1}\right] \left[e^{3h/4} + \frac{1}{r+1}\right] \\ &= \frac{e^{9h/4}}{h^4} < \infty.\end{aligned}$$

A similar argument shows the last equivalent conditions. □

3.3 Special Examples of Discrete Subsets of $\mathbb{S}_1$ and their Densities

We start with the example of the well-known class of shearlet systems which was first introduced by Guo, Kutyniok and Labate [45].

3.3.1 Classical Shearlet Systems

Definition 3.6. Let $\psi \in L^2(\mathbb{R}^2)$, $a > 1$, and $b, c > 0$ be given. By sampling the continuous shearlet transform (3.2) on the discrete subset Λ_1 of $\mathbb{S}_1$ of the form

$$\Lambda_1 := \{(a^j, bk, cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \tag{3.16}$$

we obtain the system induced by ψ and Λ_1 of the form

$$\mathcal{SH}_{1,\psi}(\Lambda_1) = \{D_{A_{a^j}^{-1} S_{bk}^{-1}} T_{cm}\psi = a^{3j/4}\psi(S_{bk}A_{a^j} \cdot -cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \tag{3.17}$$

which we call the *classical shearlet systems.*

The plots of Λ_1 corresponding to each coordinate are illustrated in Figure 3.1. Next we will show that $\mathcal{SH}_{1,\psi}(\Lambda_1)$ possesses the uniform density $\frac{1}{bc^2 \ln a}$.

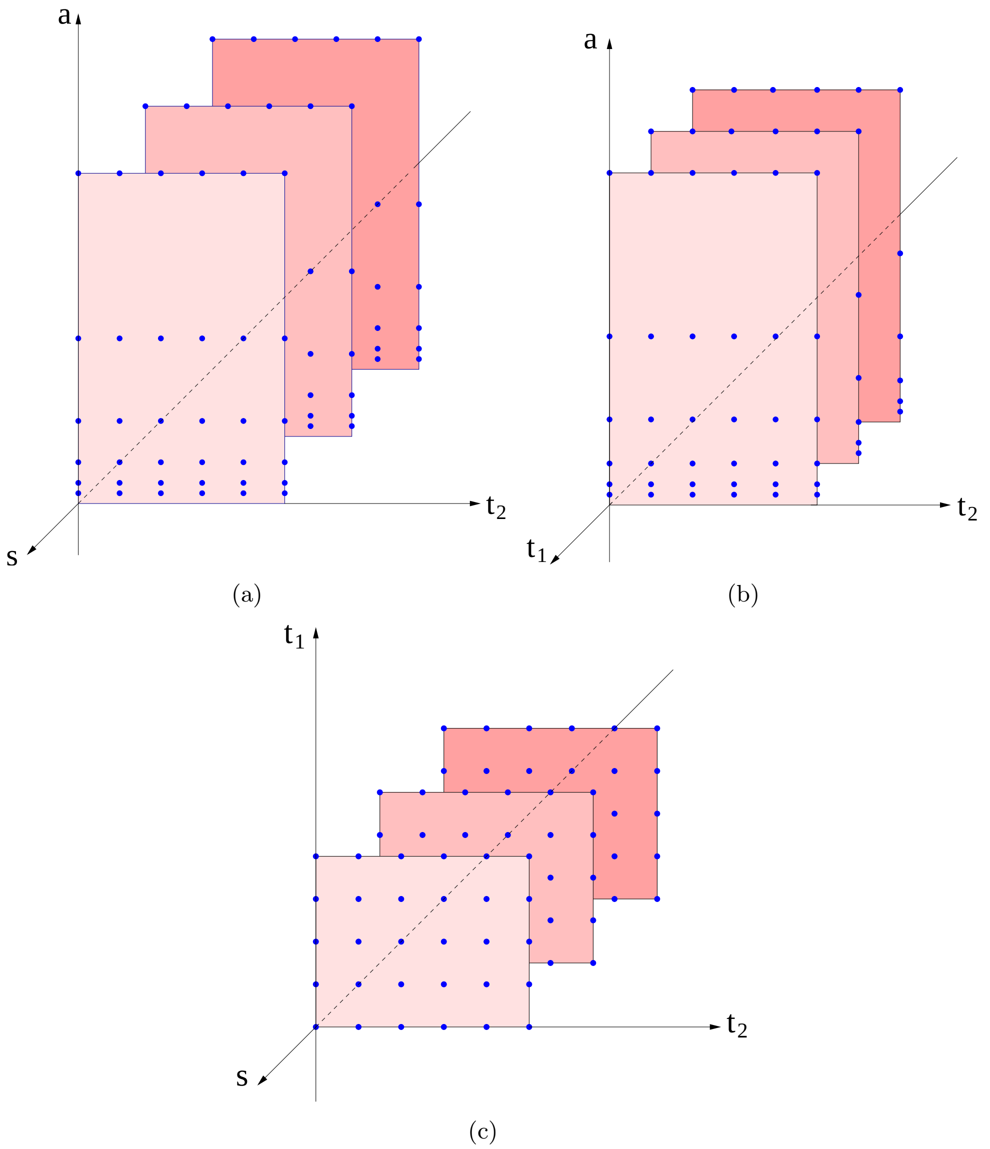

Figure 3.1: The set $\Lambda_1 = \{(4^j, 2k, m_1, m_2)\}_{j,k,m_1,m_2 \in \mathbb{Z}}$ plotted with along one coordinate fixed

Proposition 3.6. *Let* $\Lambda_1 = \{(a^j, bk, cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\} \subset \mathbb{S}_1$ *where* $a > 1$ *and* $b, c > 0$. *Then*

$$D^+(\Lambda_1) = D^-(\Lambda_1) = \frac{1}{bc^2 \ln a}.$$

Proof. Fix $(x, y, z) \in \mathbb{S}_1$. If $(a^j, bk, cm) \in Q_h(x, y, z)$, then

$$\left(\frac{1}{x}, \frac{-y}{\sqrt{x}}, -A_x^{-1} S_y^{-1} z\right) \cdot (a^j, bk, cm) = \left(\frac{a^j}{x}, bk - \frac{y a^{j/2}}{\sqrt{x}}, cm - S_{bk} A_{a^j} A_x^{-1} S_y^{-1} z\right) \in Q_h,$$

where $Q_h = [e^{-h/2}, e^{h/2}) \times \left[-\frac{h}{2}, \frac{h}{2}\right) \times \left[-\frac{h}{2}, \frac{h}{2}\right)^2$.
Now

- the fact that $\frac{a^j}{x} \in [e^{-h/2}, e^{h/2})$ implies

$$\log_a x - \frac{h}{2\ln a} \leq j \leq \log_a x + \frac{h}{2\ln a}.$$

  This is satisfied for at most $\frac{h}{\ln a} + 1$ and at least $\frac{h}{\ln a} - 1$ of values of j.

- Further, $bk - \frac{ya^{j/2}}{\sqrt{x}} \in \left[-\frac{h}{2}, \frac{h}{2}\right)$ implies

$$\frac{ya^{j/2}}{b\sqrt{x}} - \frac{h}{2b} \leq k \leq \frac{ya^{j/2}}{b\sqrt{x}} + \frac{h}{2b}.$$

  For a given value of j, this is satisfied for at most $\frac{h}{b} + 1$ and at least $\frac{h}{b} - 1$ values of k.

- Furthermore, $cm - S_{bk} A_{a^j} A_x^{-1} S_y^{-1} z = \gamma \in \left[-\frac{h}{2}, \frac{h}{2}\right)^2$, that is

$$m = \underbrace{\frac{1}{c} S_{bk} A_{a^j} A_x^{-1} S_y^{-1} z}_{=:(C_1, C_2)^T} - \frac{1}{c}\gamma,$$

  this implies

$$C - \frac{h}{2c} \leq m_1 \leq C + \frac{h}{2c} \tag{3.18}$$

$$a^{j/2} \frac{z_2}{c\sqrt{x}} - \frac{h}{2c} \leq m_2 \leq a^{j/2} \frac{z_2}{c\sqrt{x}} + \frac{h}{2c}. \tag{3.19}$$

For a given value of j and k, (3.18) is satisfied for at most $\frac{h}{c} + 1$ and at least $\frac{h}{c} - 1$ values of m_1, and for a given value of j, (3.19) is satisfied for at most $\frac{h}{c} + 1$ and at least $\frac{h}{c} - 1$ values of m_2. We compute

$$\left[\frac{h}{\ln a} - 1\right] \left[\frac{h}{b} - 1\right] \left[\frac{h}{c} - 1\right]^2 \leq \#(\Lambda_1 \cap Q_h(x, y, z)) \leq \left[\frac{h}{\ln a} + 1\right] \left[\frac{h}{b} + 1\right] \left[\frac{h}{c} + 1\right]^2 \tag{3.20}$$

Thus,

$$\begin{aligned} D^+(\Lambda_1) &= \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_1} \frac{\#(\Lambda_1 \cap Q_h(x,y,z))}{h^4} \\ &\leq \limsup_{h\to\infty} \frac{h^4}{(bc^2 \ln a)\, h^4} \left[1 + \frac{\ln a}{h}\right] \left[1 + \frac{b}{h}\right] \left[1 + \frac{c}{h}\right]^2 \\ &= \lim_{h\to\infty} \frac{h^4}{(bc^2 \ln a)\, h^4} \left[1 + \frac{\ln a}{h}\right] \left[1 + \frac{b}{h}\right] \left[1 + \frac{c}{h}\right]^2 \\ &= \frac{1}{bc^2 \ln a}, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} D^-(\Lambda_1) &= \liminf_{h\to\infty} \inf_{(x,y,z)\in\mathbb{S}_1} \frac{\#(\Lambda_1\cap Q_h(x,y,z))}{h^4} \\ &\geq \liminf_{h\to\infty} \frac{h^4}{(bc^2\ln a)\,h^4}\left[1-\frac{\ln a}{h}\right]\left[1-\frac{b}{h}\right]\left[1-\frac{c}{h}\right]^2 \\ &= \lim_{h\to\infty} \frac{h^4}{(bc^2\ln a)\,h^4}\left[1-\frac{\ln a}{h}\right]\left[1-\frac{b}{h}\right]\left[1-\frac{c}{h}\right]^2 \\ &= \frac{1}{bc^2\ln a}. \end{aligned} \tag{3.22}$$

Therefore, $\frac{1}{bc^2\ln a} \leq D^-(\Lambda_1) \leq D^+(\Lambda_1) \leq \frac{1}{bc^2\ln a}$ implies $D^+(\Lambda_1) = D^-(\Lambda_1) = \frac{1}{bc^2\ln a}$. □

Remark 3.5. *It can be seen from the proof that for a given $(x,y,z)\in\mathbb{S}_1$, the maximum and minimum number of points $(a^j, bk, cm)\in\Lambda_1$ lying on the set $Q_h(x,y,z)$ differ only by the terms ±1 in each factor, e.g. in (3.20) . This is not significant as each ±1 only contributes a lower power of h in the product in limits (3.21) and (3.22). To make the proof shorter and more transparent, (3.20) can be rewritten by*

$$\#(\Lambda_1\cap Q_h(x,y,z)) = \frac{h}{\ln a}\cdot\frac{h}{b}\cdot\left(\frac{h}{c}\right)^2 + \mathcal{O}(h^3),$$

for large h. Then

$$\begin{aligned} D^+(\Lambda_1) &= \limsup_{h\to\infty}\left[\frac{h^4}{(bc^2\ln a)\,h^4} + \frac{1}{h^4}\mathcal{O}(h^3)\right] \\ &= \lim_{h\to\infty}\left[\frac{h^4}{(bc^2\ln a)\,h^4} + \frac{1}{h^4}\mathcal{O}(h^3)\right] \\ &= \liminf_{h\to\infty}\left[\frac{h^4}{(bc^2\ln a)\,h^4} + \frac{1}{h^4}\mathcal{O}(h^3)\right] \\ &= \frac{1}{bc^2\ln a} = D^-(\Lambda_1). \end{aligned}$$

3.3.2 Oversampled Shearlet Systems

The notion oversampled shearlet systems is obtained similarly to the oversampled affine systems in higher dimensions by Hermández et al. [55], [54]. That is, for a given classical shearlet system $\mathcal{SH}_{1,\psi}(\Lambda) = \{D_{A^{-1}_{a^j}S^{-1}_{bk}}T_{cm}\psi : j,k\in\mathbb{Z}, m\in\mathbb{Z}^2\}$ for $a>1$ and $b,c>0$, the corresponding oversampled shearlet system can be obtained by choosing a larger collection translations instead of the lattice translation set $\{cm\}_{m\in\mathbb{Z}^2}$, in other words, it is defined as $\mathcal{SH}^{R_{j,k}}_{1,\psi}(\Lambda) = \{|\det R_{j,k}|^{-1/2}D_{A^{-1}_{a^j}S^{-1}_{bk}}T_{cR^{-1}_{j,k}m}\psi : j,k\in\mathbb{Z}, m\in\mathbb{Z}^2\}$, for $\{R_{j,k}\}_{j,k\in\mathbb{Z}}\subset GL_2(\mathbb{R})$. More specifically, under the appropriate choice of $\{R_{j,k}\}_{jk\in\mathbb{Z}}$, one can construct the oversampled systems which are shift-invariant. The most notable oversampled shift-invariant systems are the *quasi-affine* systems introduced by Ron and Shen [69]. For more detial on oversampled and quasi affine systems, we refer the reader to Hermández et al. [55], [54] and Ron and Shen [69].

Definition 3.7. Let $\psi\in L^2(\mathbb{R}^2)$, $a>1$, and $b,c>0$ be given, we define the *oversampled shearlet systems* generated by ψ relative to the sequence of matrices $\{R_{j,k}\}_{j,k\in\mathbb{Z}}\subset GL_2(\mathbb{R})$ as the collection of functions the form

$$\mathcal{SH}_{1,\psi}^{R_{j,k}}(\Lambda) = \{|\det R_{j,k}|^{-1/2} D_{A_{a^j}^{-1} S_{bk}^{-1}} T_{cR_{jk}^{-1}m}\psi : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

Remark 3.6. *(1) The oversampled shearlet system is indeed a weighted shearlet system* $\mathcal{SH}_{1,\psi}^{R_{j,k}}(\Lambda)$ *with*

$$\Lambda = \left\{ \left(a^j, bk, cR_{j,k}^{-1}m \right) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2 \right\} \quad \text{and} \;\; w(a^j, bk, cR_{j,k}^{-1}m) = \frac{1}{|\det R_{j,k}|}.$$

(2) The definition of oversampled shearlet systems defined as above is a special case of the notion of oversampled affine systems in higher dimensions introduced by Hermández et al. [55].

In the following, we choose the matrix $R_{j,k}$ to be a diagonal matrix for any $j,k \in \mathbb{Z}$, and show that the corresponding oversampled shearlet systems possess exactly the same uniform weighted density as the classical shearlet systems.

Proposition 3.7. *If* $\mathcal{SH}_{1,\psi}^{R_{j,k}}(\Lambda)$ *is an oversampled shearlet system which* $\{R_{j,k}\}_{j,k\in\mathbb{Z}}$ *is a sequence of diagonal matrices, then* Λ *has the uniform weighted density*

$$D_w^+(\Lambda) = D_w^-(\Lambda) = \frac{1}{bc^2 \ln a}.$$

Proof. Fix $(x,y,z) \in \mathbb{S}_1$. If $(a^j, bk, cR_{j,k}^{-1}m) \in Q_h(x,y,z)$, then

$$\left(\frac{1}{x}, \frac{-y}{\sqrt{x}}, -A_x^{-1}S_y^{-1}z \right) \cdot (a^j, bk, cR_{j,k}^{-1}m) = \left(\frac{a^j}{x}, bk - \frac{ya^{j/2}}{\sqrt{x}}, cR_{j,k}^{-1}m - S_{bk}A_{a^j}A_x^{-1}S_y^{-1}z \right) \in Q_h,$$

where $Q_h = [e^{-h/2}, e^{h/2}) \times \left[-\frac{h}{2}, \frac{h}{2}\right) \times \left[-\frac{h}{2}, \frac{h}{2}\right)^2$.
Similar argument as in Proposition 3.6, this requires

$$\log_a x - \frac{h}{2\ln a} \le j \le \log_a x + \frac{h}{2\ln a}, \tag{3.23}$$

$$\frac{ya^{j/2}}{b\sqrt{x}} - \frac{h}{2b} \le k \le \frac{ya^{j/2}}{b\sqrt{x}} + \frac{h}{2b}, \tag{3.24}$$

and $cR_{j,k}^{-1}m - \underbrace{S_{bk}A_{a^j}A_x^{-1}S_y^{-1}z}_{=:(C_1,C_2)^T} = \gamma \in \left[-\frac{h}{2}, \frac{h}{2}\right)^2$, that is

$$C_1 - \frac{h}{2c}|r_{j,k}^{(1,1)}| \le m_1 \le C_1 + \frac{h}{2c}|r_{j,k}^{(1,1)}| \tag{3.25}$$

$$C_2 - \frac{h}{2c}|r_{j,k}^{(2,2)}| \le m_2 \le C_2 + \frac{h}{2c}|r_{j,k}^{(2,2)}|. \tag{3.26}$$

By the explanation in Remark 3.5, it is enough to observe that for a fixed $(x,y,z) \in \mathbb{S}_1$, (3.23) is satisfied for approximately $\frac{h}{\ln a}$ values of j. For a given value of j, there are approximately $\frac{h}{b}$ values of k satisfying (3.24). Further, for a given value of j,k and m_2, there are about $\frac{h|r_{j,k}^{(1,1)}|}{c}$ and $\frac{h|r_{j,k}^{(2,2)}|}{c}$ values of $m_1, m_2 \in \mathbb{Z}$ satisfying (3.25) and (3.26), respectively. We compute

$$\#_w(\Lambda \cap Q_h(x,y,z)) = \frac{1}{|\det R_{j,k}|} \frac{h}{\ln a} \cdot \frac{h}{b} \cdot \frac{h|r_{j,k}^{(1,1)}|}{c} \cdot \frac{h|r_{j,k}^{(2,2)}|}{c} + \mathcal{O}(h^4).$$

Thus,

$$\begin{aligned} D_w^+(\Lambda) &= \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_1} \frac{\#_w(\Lambda\cap Q_h(x,y,z))}{h^4} \\ &= \limsup_{h\to\infty} \left[\frac{h^4 |r_{j,k}^{(1,1)} \cdot r_{j,k}^{(2,2)}|}{bc^2 \ln a |\det R_{j,k}|} + \frac{1}{h^4}\mathcal{O}(h^4) \right] \\ &= \lim_{h\to\infty} \left[\frac{h^4}{bc^2 \ln a} + \frac{1}{h^4}\mathcal{O}(h^4) \right] \\ &= \frac{1}{bc^2 \ln a}. \end{aligned}$$

A similar argument shows $D_w^-(\Lambda) = \frac{1}{bc^2 \ln a}$. □

In the following results, we consider the oversampled shearlet systems associated to the sequence of *shear matrices* $\{S_{bk}\}_{k\in\mathbb{Z}}$, for $b > 0$. More precisely, we study the following unweighted shearlet systems:

$$\mathcal{SH}_{1,\psi}^{S_{bk}}(\Lambda) = \{\psi_{j,k,m} = D_{A_{a^j}^{-1} S_{bk}^{-1}} T_{cS_{bk}m}\psi : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\},$$

where $\Lambda = \{(a^j, bk, cS_{bk}m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$. Similar computations show that this oversampled shearlet systems also possess a uniform density.

Proposition 3.8. *If* $\mathcal{SH}_{1,\psi}^{R_{j,k}}(\Lambda)$ *is an oversampled shearlet system with* $R_{j,k} = S_{bk}$*, for all* $j,k \in \mathbb{Z}$ *and* $b > 0$*, then* Λ *has uniform density*

$$D^+(\Lambda) = D^-(\Lambda) = \frac{1}{bc^2 \ln a}.$$

Proof. Fix $(x,y,z) \in \mathbb{S}_1$. If $(a^j, bk, cS_{bk}m) \in Q_h(x,y,z)$, then

$$\left(\frac{1}{x}, \frac{-y}{\sqrt{x}}, -A_x^{-1}S_y^{-1}z\right) \cdot (a^j, bk, cm) = \left(\frac{a^j}{x}, bk - \frac{ya^{j/2}}{\sqrt{x}}, cm - S_{bk}A_{a^j}A_x^{-1}S_y^{-1}z\right) \in Q_h,$$

where $Q_h = [e^{-h/2}, e^{h/2}) \times \left[-\frac{h}{2}, \frac{h}{2}\right) \times \left[-\frac{h}{2}, \frac{h}{2}\right)^2$.
By using the same argument as in Proposition 3.6 , this requires

$$\log_a x - \frac{h}{2\ln a} \le j \le \log_a x + \frac{h}{2\ln a}, \tag{3.27}$$

$$\frac{ya^{j/2}}{b\sqrt{x}} - \frac{h}{2b} \le k \le \frac{ya^{j/2}}{b\sqrt{x}} + \frac{h}{2b}, \tag{3.28}$$

and $cS_{bk}m - \underbrace{S_{bk}A_{a^j}A_x^{-1}S_y^{-1}z}_{=:(C_1,C_2)^T} = \gamma \in \left[-\frac{h}{2}, \frac{h}{2}\right)^2$, that is

$$\frac{C_2}{c} - \frac{h}{2c} \le m_2 \le \frac{C_2}{c} + \frac{h}{2c} \tag{3.29}$$

$$\frac{C_1}{c} - bkm_2 - \frac{h}{2c} \le m_1 \le \frac{C_1}{c} - bkm_2 + \frac{h}{2c}. \tag{3.30}$$

For a fixed $(x, y, z) \in \mathbb{S}_1$, (3.27) is satisfied for approximately $\frac{h}{\ln a}$ values of j. For a given value of j, there are approximately $\frac{h}{b}$ values of k satisfying (3.28), and $\frac{h}{c}$ values of m_2 satisfying (3.29). Furthermore, for a given value of j, k and m_2, there are about $\frac{h}{c}$ values of m_1 satisfying (3.30). We compute

$$\#(\Lambda \cap Q_h(x, y, z)) = \frac{h}{\ln a} \cdot \frac{h}{b} \cdot \left(\frac{h}{c}\right)^2 + \mathcal{O}(h^3).$$

Thus,

$$\begin{aligned} D^+(\Lambda) &= \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_1} \frac{\#(\Lambda \cap Q_h(x, y, z))}{h^4} \\ &= \lim_{h\to\infty} \frac{h^4}{(bc^2 \ln a)\, h^4} = \frac{1}{bc^2 \ln a}. \end{aligned}$$

A similar argument shows $D^-(\Lambda) = \frac{1}{bc^2 \ln a}$. This completes the proof. □

3.3.3 Co-Shearlet Systems

Since the classical shearlet systems are a family of functions written as $\{D_{A_{a^j}^{-1} S_{bk}^{-1}} T_{cm}\psi : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$, by switching the operators we obtain the new systems, called the *co-shearlet* systems.

Definition 3.8. Let $\psi \in L^2(\mathbb{R}^2)$, $a > 1$, and $b, c > 0$ be given. The systems of the form

$$\{T_{cm} D_{A_{a^j}^{-1} S_{bk}^{-1}} \psi = a^{3j/4}\psi(S_{bk} A_{a^j}(\cdot - cm)) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\},$$

are called the *co-shearlet systems*.

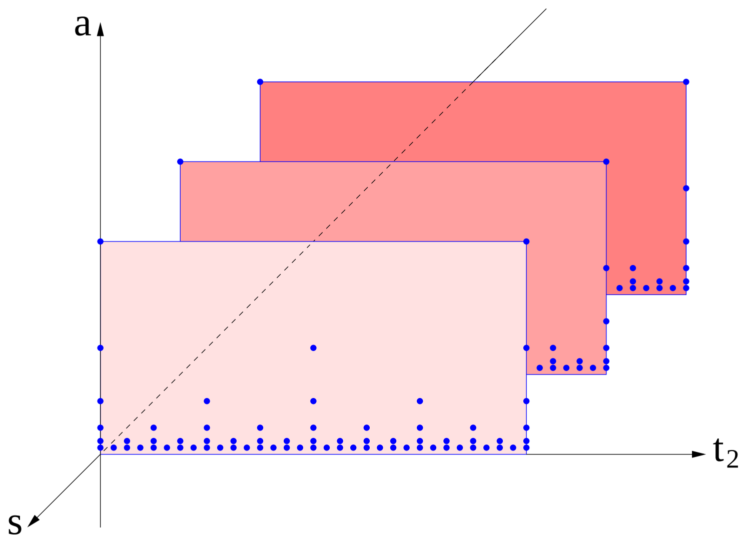

Figure 3.2: The set $\Lambda = \{(4^j, k, 4^j m_1 + 2^j m_2, 2^j m_2)\}_{j,k,m_1,m_2\in\mathbb{Z}}$ corresponding to the coordinates $(a, s, t_2) = \{(4^j, k, 2^j m_2)\}_{j,k,m_2\in\mathbb{Z}}$.

We note that the co-shearlet systems are unweighted systems, and equal to the systems $\{D_{A_{a^j}^{-1}S_{bk}^{-1}}T_{cS_{bk}A_{a^j}m}\psi : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$. In this case, the corresponding set Λ of co-shearlet systems is the set $\{(a^j, bk, cS_{bk}A_{a^j}m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$.

Figure 3.2 gives a sketch of the set Λ with $a = 4$ and $b, c = 1$ for the co-shearlet systems. As compared to the set Λ_1 for the classical shearlet systems, illustrated in Figure 3.1, the points in Λ are not uniform in distance, and are more densely located on the t_2-axis. For this reason, we have the following results which show that the co-shearlet systems can possess neither an upper nor lower frame bound.

Proposition 3.9. *Fix $a > 1$, $b, c > 0$, and set $\Lambda = \{(a^j, bk, cS_{bk}A_{a^j}m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$. Then*

$$D^+(\Lambda) = \infty \quad \text{and} \quad D^-(\Lambda) = 0.$$

Proof. Fix $(x,y,z) \in \mathbb{S}$. If $(a^j, bk, cS_{bk}A_{a^j}m) \in Q_h(x,y,z)$, then

$$\left(\frac{1}{x}, \frac{-y}{\sqrt{x}}, -A_x^{-1}S_y^{-1}z\right)\cdot(a^j, bk, cS_{bk}A_{a^j}m) = \left(\frac{a^j}{x}, bk - \frac{ya^{j/2}}{\sqrt{x}}, cS_{bk}A_{a^j}m - S_{bk}A_{a^j}A_x^{-1}S_y^{-1}z\right) \in Q_h.$$

This requires the following conditions:

- $\frac{a^j}{x} \in [e^{-h/2}, e^{h/2})$ implies

$$\log_a x - \frac{h}{2\ln a} \le j \le \log_a x + \frac{h}{2\ln a}.$$

- $bk - \frac{ya^{j/2}}{\sqrt{x}} \in \left[-\frac{h}{2}, \frac{h}{2}\right)$ implies

$$\frac{ya^{j/2}}{b\sqrt{x}} - \frac{h}{2b} \le k \le \frac{ya^{j/2}}{b\sqrt{x}} + \frac{h}{2b}. \tag{3.31}$$

- $cS_{bk}A_{a^j}m - S_{bk}A_{a^j}A_x^{-1}S_y^{-1}z = \gamma \in \left[-\frac{h}{2}, \frac{h}{2}\right)^2$ implies

$$m = \frac{1}{c}\left[A^{-j}S_k^{-1}\gamma + \underbrace{A_x^{-1}S_y^{-1}z}_{=:(C_1,C_2)^T}\right].$$

  We have

$$\begin{aligned} m_1 &= C_1 + \frac{a^{-j}\gamma_1}{c} - \frac{bka^{-j}\gamma_2}{c}, \\ m_2 &= C_2 + \frac{a^{-j/2}}{c}\gamma_2, \end{aligned}$$

  where $\gamma_1, \gamma_2 \in \left[-\frac{h}{2}, \frac{h}{2}\right)$. Hence

$$C_1 - a^{-j}(1 + b|k|)\frac{h}{2c} \le m_1 \le C_1 + a^{-j}(1 + b|k|)\frac{h}{2c}, \tag{3.32}$$

$$C_2 - \frac{a^{-j/2}h}{2c} \le m_2 \le C_2 + \frac{a^{-j/2}h}{2c}. \tag{3.33}$$

It is enough to show that for a fixed $(x, y, z) \in \mathbb{S}_1$ and for a given value of j and k, there are approximately $a^{-j}(1 + b|k|)\frac{h}{c}$ values of m_1 satisfying (3.32) and $\frac{a^{-j/2}h}{c}$ values of m_2 satisfying (3.33). We compute

$$\#(\Lambda \cap Q_h(x, y, z)) = \sum_{j=\lfloor \log_a x - \frac{h}{4\ln a} \rfloor}^{\lceil \log_a x + \frac{h}{4\ln a} \rceil} \sum_{k=\lfloor \frac{ya^{j/2}}{b\sqrt{x}} - \frac{h}{2b} \rfloor}^{\lceil \frac{ya^{j/2}}{b\sqrt{x}} + \frac{h}{2b} \rceil} \left[a^{-j}(1 + b|k|)\frac{h}{c} \right] \left[\frac{a^{-j/2}h}{2c} \right] + \mathcal{O}(h^2).$$

By changing x, we can make this quantity arbitrary small or large. For instance, it can be seen from Figure 3.2 that if x lies very close to the t_2-axis, this quantity is very large. On the other hand, this quantity is close to zero when x is far from the t_2-axis. Hence we can conclude that $D^-(\Lambda) = 0$ and $D^+(\Lambda) = \infty$. □

3.4 Other Shearlet Groups and their Weighted Densities

In Heil and Kutyniok [51], Sun and Zhou [74] and Kutyniok [58], the density notion for the affine group can be defined in many different way, depending on the choice of group multiplication. In this section, we will study the other three definitions of shearlet groups and define a density notion for each shearlet group.

3.4.1 Shearlet Group $\mathbb{S}_2$

Let $\mathbb{S}_2$ be the group consisting of $(a, s, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ together with the group multiplication

$$(a, s, t) \star (a', s', t') = (aa', s' + s\sqrt{a'}, t + A_a^{-1}S_s^{-1}t').$$

The identity element of $\mathbb{S}_2$ is $(1, 0, 0)$, thus $(a, s, t)^{-1} = \left(\frac{1}{a}, -\frac{s}{\sqrt{a}}, -S_sA_at\right)$.

Similar to the definition of shearlet group $\mathbb{S}_1$ defined in Section 3.2, there is a unitary representation σ_2 of $\mathbb{S}_2$ on $L^2(\mathbb{R}^2)$ given by

$$\sigma_2(a, s, t)\psi(x) = T_tD_{A_a^{-1}S_s^{-1}}\psi(x) = a^{3/4}\psi(S_sA_a(x - t))$$

for $(a, s, t) \in \mathbb{S}_2$ and $\psi \in L^2(\mathbb{R}^2)$. Then the continuous shearlet transform $\mathcal{SH}_{2,\psi}$ induced by ψ and the unitary representation σ_2 becomes

$$\mathcal{SH}_{2,\psi}f = \langle f, \sigma_2(a, s, t)\psi \rangle = \langle f, T_tD_{A_a^{-1}S_s^{-1}}\psi \rangle, \tag{3.34}$$

for $f \in L^2(\mathbb{R}^2)$.

Remark 3.7. *(***Connection between $\mathbb{S}_1$ and $\mathbb{S}_2$ and between $\mathcal{SH}_1$ and $\mathcal{SH}_2$***):*
Let $\Phi_{1,2} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ be defined by

$$\Phi_{1,2}(x, y, z) = \left(x, y, A_x^{-1}S_y^{-1}z\right). \tag{3.35}$$

Now we prove that $\Phi_{1,2}$ is a group isomorphism. Since $A_a^{-1}S_s^{-1}A_x^{-1}S_y^{-1} = A_{ax}^{-1}S_{y+s\sqrt{x}}^{-1}$, it follows from (3.35) that

$$\begin{aligned}
\Phi_{1,2}((a, s, t) \cdot (x, y, z)) &= \Phi_{1,2}(ax, y + s\sqrt{x}, z + S_yA_xt) \\
&= \left(ax, y + s\sqrt{x}, A_{ax}^{-1}S_{y+s\sqrt{x}}^{-1}(z + S_yA_xt)\right) \\
&= \left(a, s, A_a^{-1}S_s^{-1}t\right) \star (x, y, A_x^{-1}S_y^{-1}) \\
&= \Phi_{1,2}(a, s, t) \star \Phi_{1,2}(x, y, z).
\end{aligned}$$

Since $\Phi_{1,2}$ is bijection, this shows $\Phi_{1,2}$ is a group isomorphism from $\mathbb{S}_1$ to $\mathbb{S}_2$.

Let Λ be a subset of $\mathbb{S}_1$, it follows from the above isomorphism that

$$\mathcal{SH}_{1,\psi}(\Lambda) = \mathcal{SH}_{2,\psi}(\Phi_{1,2}(\Lambda)). \tag{3.36}$$

In the following the definition of density of a discrete subset Λ of $\mathbb{S}_2$ can be similarly defined as before. Recall that the set of neighborhoods of the identity $\{Q_h : h > 0\}$ was defined by $Q_h = [e^{-h/2}, e^{h/2}) \times [-\frac{h}{2}, \frac{h}{2}) \times [-\frac{h}{2}, \frac{h}{2})^2$, for $h > 0$. For $(x, y, z) \in \mathbb{S}_2$ we define

$$Q_h(x, y, z) = (x, y, z) \star Q_h = \{(xa, s + y\sqrt{t}, z + A_x^{-1} S_y^{-1} t) : (a, s, t) \in Q_h\}.$$

Since the left-invariant Haar measure for the group $\mathbb{S}_2$ is given by $d\mu_{\mathbb{S}_2} = \sqrt{a}\, da\, ds\, dt$, we can compute the volume of $Q_h(x, y, z)$ by

$$\mu_{\mathbb{S}_2}(Q_h) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \sqrt{a}\, da\, ds\, dt = \frac{2h^3}{3}(e^{3h/4} - e^{-3h/4}).$$

Let Λ be a subset of $\mathbb{S}_2$ with associated a weight function $w : \Lambda \to \mathbb{R}^+$. Then *the upper and lower weighted densities* of Λ are defined, respectively, by

$$D_w^+(\Lambda) = \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_2} \frac{\#_w(\Lambda \cap Q_h(x, y, z))}{\frac{2h^3}{3}(e^{3h/4} - e^{-3h/4})},$$
$$D_w^-(\Lambda) = \liminf_{h\to\infty} \inf_{(x,y,z)\in\mathbb{S}_2} \frac{\#_w(\Lambda \cap Q_h(x, y, z))}{\frac{2h^3}{3}(e^{3h/4} - e^{-3h/4})}.$$

As already shown in Proposition 3.5, the following proposition will describe how a subset Λ of $\mathbb{S}_2$ possesses finite upper density and positive lower density.

Proposition 3.10. *For $\Lambda \subset \mathbb{S}_2$ the following conditions are equivalent.*

1. *$D_w^+(\Lambda) < \infty$.*
2. *There exists $h > 0$ such that $\sup_{(x,y,z)\in\mathbb{S}_2} \#_w(\Lambda \cap Q_h(x, y, z)) < \infty$.*

Also the following conditions are equivalent:

1. *$D_w^-(\Lambda) > 0$.*
2. *There exists $h > 0$ such that $\inf_{(x,y,z)\in\mathbb{S}_2} \#_w(\Lambda \cap Q_h(x, y, z)) > 0$.*

Proof. The proof is similar to the Proposition 3.5 and stated in Appendix B. □

Lemma 3.11. *Given $a > 1$, $h > 0$ and $\alpha \neq 0$. Then the following statement holds true*

$$(1) \qquad \frac{x^\alpha a^\alpha \left(e^{\frac{\alpha h}{2}} - e^{-\frac{\alpha h}{2}}\right)}{a^\alpha - 1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a}\rceil}^{\lceil \log_a x + \frac{h}{2\ln a}\rceil} a^{\alpha j} < \frac{x^\alpha \left(a^{2\alpha} e^{\frac{\alpha h}{2}} - e^{-\frac{\alpha h}{2}}\right)}{a^\alpha - 1}, \tag{3.37}$$

$$(2) \qquad \frac{x^\alpha \left(e^{\frac{\alpha h}{2}} - a^\alpha e^{-\frac{\alpha h}{2}}\right)}{a^\alpha - 1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a}\rceil}^{\lfloor \log_a x + \frac{h}{2\ln a}\rfloor} a^{\alpha j} \leq \frac{x^\alpha a^\alpha \left(e^{\frac{\alpha h}{2}} - a^{-\alpha} e^{-\frac{\alpha h}{2}}\right)}{a^\alpha - 1}, \tag{3.38}$$

for any $x \in \mathbb{R}^+$.

Proof. The proof is given in the Appendix A □

Recall that $\Lambda_1 = \{(a^j, bk, cm) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ is the discrete set associated to the classical shearlet systems $\mathcal{SH}_{1,\psi}(\Lambda_1)$. Then it follows from the isomorphism (3.35) that

$$\begin{aligned}\Phi_{1,2}(\Lambda_1) &= \{\Phi_{1,2}(a^j, bk, cm) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}\\ &= \{(a^j, bk, cA_{a^j}^{-1} S_{bk}^{-1} m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\} =: \Lambda_2.\end{aligned}$$

It is obviously that Λ_2 is a discrete subset of $\mathbb{S}_2$.

In order to obtain the classical shearlet systems for this case, we sample the continuous shearlet transform (3.34) on the discrete subset Λ_2 of $\mathbb{S}_2$ defined as above. Then we obtain the following systems:

$$\mathcal{SH}_{2,\psi}(\Lambda_2) := \{T_{cA_{a^j}^{-1} S_{bk}^{-1} m} D_{A_{a_j}^{-1} S_{bk}^{-1}} \psi(x) = a^{3j/4} \psi(S_{bk} A_{a^j} x - cm) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

Then we also call $\mathcal{SH}_{2,\psi}(\Lambda_2)$ the *classical shearlet systems*, which indeed coincides with $\mathcal{SH}_{1,\psi}(\Lambda_1)$.

However, the following results show that the system $\mathcal{SH}_{2,\psi}(\Lambda_2)$ does not possess a uniform density in contrast to the system $\mathcal{SH}_{1,\psi}(\Lambda_1)$ which possesses a uniform density $D^+(\Lambda_1) = D^-(\Lambda_1) = \frac{1}{bc^2 \ln a}$.

Proposition 3.12. *Let* $\Lambda_2 = \{(a^j, bk, cA_{a^j}^{-1} S_{bk}^{-1} m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ *where* $a > 1, b > 0, c > 0$. *Then*

$$\frac{3}{2bc^2(a^{3/2} - 1)} < D^-(\Lambda_2) \leq \frac{3a^{3/2}}{2bc^2(a^{3/2} - 1)} < D^+(\Lambda_2) < \frac{3a^3}{2bc^2(a^{3/2} - 1)}.$$

Proof. Fix $(x, y, z) \in \mathbb{S}_2$. If $(a^j, bk, cA_{a^j}^{-1} S_{bk}^{-1} m) \in Q_h(x, y, z)$, then

$$\begin{aligned}&\left(\frac{1}{x}, -\frac{y}{\sqrt{x}}, -S_y A_x z\right) \star (a^j, bk, cA_{a^j}^{-1} S_{bk}^{-1} m)\\ &\qquad = \left(\frac{a^j}{x}, bk - \frac{y}{\sqrt{x}} a^{j/2}, -S_y A_x z + cA_{\frac{1}{x}}^{-1} S_{-\frac{y}{\sqrt{x}}}^{-1} A_{a^j}^{-1} S_{bk}^{-1} m\right) \in Q_h.\end{aligned}$$

This requires the following conditions

$$\log_a x - \frac{h}{2\ln a} \leq j \leq \log_a x + \frac{h}{2\ln a}.$$

$$\frac{ya^{j/2}}{b\sqrt{x}} - \frac{h}{2b} \leq k \leq \frac{ya^{j/2}}{b\sqrt{x}} + \frac{h}{2b}. \tag{3.39}$$

$$C_2 \frac{a^{j/2}}{c\sqrt{x}} - \frac{ha^{j/2}}{2c\sqrt{x}} \leq m_2 \leq C_2 \frac{a^{j/2}}{c\sqrt{x}} + \frac{ha^{j/2}}{2c\sqrt{x}} \tag{3.40}$$

$$C_1 \frac{a^j}{x} - (y\sqrt{x} a^{j/2} - bkx) \frac{m_2}{cx} - \frac{ha^j}{2cx} \leq m_1 \leq C_1 \frac{a^j}{x} - (y\sqrt{x} a^{j/2} - bkx) \frac{m_2}{cx} + \frac{ha^j}{2cx}. \tag{3.41}$$

For a fixed $(x, y, z) \in \mathbb{S}_2$, and for a given value of j, there are approximately $\frac{h}{b}$ values of k satisfying (3.39) and there are approximately $\frac{ha^{j/2}}{c\sqrt{x}}$ values of m_2 satisfying (3.40). Further, for given values of j, k and m_2, there are approximately $\frac{ha^j}{cx}$ values of m_1 satisfying (3.41). We compute

$$\sup_{(x,y,z)\in\mathbb{S}_2} \#(\Lambda_2 \cap Q_h(x,y,z)) = \sup_{(x,y,z)\in\mathbb{S}_2} \sum_{j=\lceil \log_a x - \frac{h}{2\ln a}\rceil}^{\lceil \log_a x + \frac{h}{2\ln a}\rceil} \left(\frac{h}{b}\right)\left(\frac{ha^{j/2}}{c\sqrt{x}}\right)\left(\frac{ha^j}{cx}\right) + \mathcal{O}(h^3 e^{3h/4}), \tag{3.42}$$

$$\inf_{(x,y,z)\in\mathbb{S}_2} \#(\Lambda_2 \cap Q_h(x,y,z)) \leq \inf_{(x,y,z)\in\mathbb{S}_2} \sum_{j=\lceil \log_a x - \frac{h}{2\ln a}\rceil}^{\lfloor \log_a x + \frac{h}{2\ln a}\rfloor} \left(\frac{h}{b}\right)\left(\frac{ha^{j/2}}{c\sqrt{x}}\right)\left(\frac{ha^j}{cx}\right) + \mathcal{O}(h^3 e^{3h/4}). \tag{3.43}$$

By using Lemma 3.11, we obtain

$$x^{3/2}\frac{a^{3/2}(e^{3h/4} - e^{-3h/4})}{a^{3/2}-1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a}\rceil}^{\lceil \log_a x + \frac{h}{2\ln a}\rceil} a^{3j/2} < x^{3/2}\frac{(a^3 e^{3h/4} - e^{-3h/4})}{a^{3/2}-1}, \tag{3.44}$$

$$x^{3/2}\frac{(e^{3h/4} - a^{3/2}e^{-3h/4})}{a^{3/2}-1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a}\rceil}^{\lfloor \log_a x + \frac{h}{2\ln a}\rfloor} a^{3j/2} \leq a^{3/2}x^{3/2}\frac{(e^{3h/4} - a^{-3/2}e^{-3h/4})}{a^{3/2}-1}. \tag{3.45}$$

Therefore, it follows from (3.42) and (3.44) that

$$\begin{aligned}
&\lim_{h\to\infty}\left[\frac{h^3}{\frac{2}{3}bc^2h^3(e^{3h/4}-e^{-3h/4})}\cdot\frac{a^{3/2}(e^{3h/4}-e^{-3h/4})}{a^{3/2}-1} + \frac{\mathcal{O}(h^3e^{3h/4})}{\frac{2}{3}h^3(e^{3h/4}-e^{-3h/4})}\right] \\
&\quad < \; D^+(\Lambda_2) = \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_2} \frac{\#(\Lambda_2\cap Q_h(x,y,z))}{\frac{2}{3}h^3(e^{3h/4}-e^{-3h/4})} \\
&\quad < \lim_{h\to\infty}\left[\frac{h^3}{\frac{2}{3}bc^2h^3(e^{3h/4}-e^{-3h/4})}\cdot\frac{(a^3e^{3h/4}-e^{-3h/4})}{a^{3/2}-1} + \frac{\mathcal{O}(h^3e^{3h/4})}{\frac{2}{3}h^3(e^{3h/4}-e^{-3h/4})}\right].
\end{aligned}$$

Hence

$$\frac{3a^{3/2}}{2bc^2(a^{3/2}-1)} < D^+(\Lambda_2) < \frac{3a^3}{2bc^2(a^{3/2}-1)}.$$

Similarly, by (3.43) and (3.45), we obtain

$$\frac{3}{2bc^2(a^{3/2}-1)} < D^-(\Lambda_2) \leq \frac{3a^{3/2}}{2bc^2(a^{3/2}-1)}.$$

This completes the proof. □

Remark 3.8. *(***Uniform Density***) : The above results show that the density for classical shearlet systems $\mathcal{SH}_{2,\psi}(\Lambda_2)$ cannot possess a uniform density such as the density of classical shearlet systems $\mathcal{SH}_{1,\psi}(\Lambda_1)$ did in Proposition 3.6, although the systems coincide with each other. The reason for this behavior will be explained later.*

In order to obtain a uniform density for the systems $\mathcal{SH}_{2,\psi}(\Lambda_2)$, we will use the group isomorphism $\Phi_{1,2}$ connection between $\mathbb{S}_1$ and $\mathbb{S}_2$ and choose the sets $(\widetilde{Q}_h)_{h>0}$ to be another sequence of boxes in $\mathbb{S}_2$. So now we choose the sequence of neighborhoods of the identity in $\mathbb{S}_2$ by

$$\widetilde{Q}_h = \Phi_{1,2}(Q_h) = \left\{ \left(a, s, A_a^{-1} S_s^{-1} t\right) : (a,s,t) \in Q_h \right\},$$

where $Q_h = [e^{-\frac{h}{2}}, e^{\frac{h}{2}}) \times [-\frac{h}{2}, \frac{h}{2}) \times [-\frac{h}{2}, \frac{h}{2})^2$, and let $\widetilde{Q}_h(x,y,z) = (x,y,z) \star \widetilde{Q}_h$ for $(x,y,z) \in \mathbb{S}_2$. Then the following lemma shows that this choice of neighborhood gives the desired property.

Proposition 3.13. *Let $a > 1, b > 0, c > 0$ and define $\Lambda_2 = \{(a^j, bk, cA_{a^j}^{-1} S_{bk}^{-1} m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$. Then*

$$\limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_2} \frac{\#(\Lambda_2 \cap \widetilde{Q}_h(x,y,z))}{\mu_{\mathbb{S}_2}(\widetilde{Q}_h)} = \liminf_{h\to\infty} \inf_{(x,y,z)\in\mathbb{S}_2} \frac{\#(\Lambda_2 \cap \widetilde{Q}_h(x,y,z))}{\mu_{\mathbb{S}_2}(\widetilde{Q}_h)} = \frac{1}{bc^2 \ln a}.$$

Proof. First we investigate that

$$\mu_{\mathbb{S}_2}(\widetilde{Q}_h) = \mu_{\mathbb{S}_2}(\Phi_{1,2}(Q_h)) = \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2\sqrt{a}}}^{\frac{h}{2\sqrt{a}}} \int_{\frac{(-h-st_2)}{2a}}^{\frac{(h-st_2)}{2a}} \sqrt{a}\, dt_1\, dt_2\, ds\, da = h^4 = \mu_{\mathbb{S}_1}(Q_h).$$

Since $\Phi_{1,2}$ is an isomorphism, we obtain

$$\begin{aligned}
\limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_2} \frac{\#(\Lambda_2 \cap \widetilde{Q}_h(x,y,z))}{\mu_{\mathbb{S}_2}(\widetilde{Q}_h)} &= \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_2} \frac{\#(\Phi_{1,2}(\Phi_{1,2}^{-1}(\Lambda_2)) \cap (x,y,z) \star \Phi_{1,2}(Q_h))}{h^4} \\
&= \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_1} \frac{\#(\Phi_{1,2}^{-1}(\Lambda_2) \cap Q_h(x,y,z))}{h^4} \\
&= D^+(\Phi_{1,2}^{-1}(\Lambda_2)) \\
&= D^+(\Lambda_1) = \frac{1}{bc^2 \ln a}.
\end{aligned}$$

The remaining claim can be proved by the same argument. □

Remark 3.9. ***(Dependency on the set Q_h)*** *As we have shown in the previous lemma, by choosing a different choices of sets of sequence Q_h of neighborhoods of the identity in $\mathbb{S}_2$ leads to different values of the associated densities, and also changes a non-uniform density into a uniform density.*

3.4.2 Shearlet Group $\mathbb{S}_3$

Let $\mathbb{S}_3$ be the group consisting of $(a,s,t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ together with the group multiplication

$$(a,s,t) * (a',s',t') = (aa', s + s'\sqrt{a}, t + S_s A_a t').$$

The identity element of $\mathbb{S}_3$ is $(1,0,0)$, thus $(a,s,t)^{-1} = \left(\frac{1}{a}, -\frac{s}{\sqrt{a}}, -A_a^{-1} S_s^{-1} t\right)$.

Similar to the definition of the shearlet group $\mathbb{S}_1$ defined in Section 3.2, there is a unitary representation σ_3 of $\mathbb{S}_3$ on $L^2(\mathbb{R}^2)$ given by

$$\sigma_3(a,s,t)\psi(x) = T_t D_{S_a A_a} \psi(x) = a^{-3/4} \psi(A_a^{-1} S_s^{-1}(x - t))$$

for $(a,s,t)\in\mathbb{S}_3$ and $\psi\in L^2(\mathbb{R}^2)$. Then the continuous shearlet transform $\mathcal{SH}_{3,\psi}$ induced by ψ and the unitary representation σ_3 becomes

$$\mathcal{SH}_{3,\psi}f(a,s,t)=\langle f,\sigma_3(a,s,t)\psi\rangle=\langle f,T_tD_{S_aA_a}\psi\rangle, \tag{3.46}$$

for $f\in L^2(\mathbb{R}^2)$ and $(a,s,t)\in\mathbb{S}_3$.

Remark 3.10. *(***Connection between $\mathbb{S}_1$ and $\mathbb{S}_3$ and between $\mathcal{SH}_1$ and $\mathcal{SH}_3$***):*
Let $\Phi_{1,3}:\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}^2$ be defined by

$$\Phi_{1,3}(x,y,z)=\left(\frac{1}{x},-\frac{y}{\sqrt{x}},S_{-\frac{y}{\sqrt{x}}}A_{\frac{1}{x}}z\right). \tag{3.47}$$

We note that

$$S_{\frac{-(y+s\sqrt{x})}{\sqrt{ax}}}A_{\frac{1}{ax}}=S_{-\frac{s}{\sqrt{a}}}A_{\frac{1}{a}}S_{-\frac{y}{\sqrt{x}}}A_{\frac{1}{x}}\quad \text{and}\quad S_{\frac{-(y+s\sqrt{x})}{\sqrt{ax}}}A_{\frac{1}{ax}}S_yA_x=S_{-\frac{s}{\sqrt{a}}}A_{\frac{1}{a}}.$$

Now we prove that $\Phi_{1,3}$ is a group isomorphism. From (3.47), it follows that:

$$\begin{aligned}\Phi_{1,3}((a,s,t)\cdot(x,y,z))&=\Phi_{1,3}(ax,y+s\sqrt{x},z+S_yA_xt)\\&=\left(\frac{1}{ax},\frac{-(y+s\sqrt{x})}{\sqrt{ax}},S_{\frac{-(y+s\sqrt{x})}{\sqrt{ax}}}A_{\frac{1}{ax}}(z+S_yA_xt)\right)\\&=\left(\frac{1}{ax},-\frac{y}{\sqrt{ax}}-\frac{s}{\sqrt{a}},S_{-\frac{s}{\sqrt{a}}}A_{\frac{1}{a}}S_{-\frac{y}{\sqrt{x}}}A_{\frac{1}{x}}z+S_{-\frac{s}{\sqrt{a}}}A_{\frac{1}{a}}t\right)\\&=\left(\frac{1}{a},-\frac{s}{\sqrt{a}},S_{-\frac{s}{\sqrt{a}}}A_{\frac{1}{a}}t\right)*\left(\frac{1}{x},-\frac{y}{\sqrt{x}},S_{-\frac{y}{\sqrt{x}}}A_{\frac{1}{x}}z\right)\\&=\Phi_{1,3}(a,s,t)*\Phi_{1,3}(x,y,z).\end{aligned}$$

Since $\Phi_{1,3}$ is bijection, this shows $\Phi_{1,3}$ is a group isomorphism from $\mathbb{S}_1$ to $\mathbb{S}_3$.

Let Λ be a discrete subset of $\mathbb{S}_1$, it follows from the above isomorphism that

$$\mathcal{SH}_{1,\psi}(\Lambda)=\mathcal{SH}_{3,\psi}(\Phi_{1,3}(\Lambda)), \tag{3.48}$$

We remark that the continuous shearlet transform (3.46) is the same as defined by Kutyniok and Labate [61], [62] and Dahlke et al. [18], [19]. Similarly, the discretization of continuous shearlet transform was achieved by sampling this continuous shearlet transform (3.46) on the discrete subset Λ_3 of $\mathbb{S}_3$ of the form

$$\Lambda_3:=\{(a^j,bka^{j/2},cS_{bka^{j/2}}A_{a^j}m):j,k\in\mathbb{Z},m\in\mathbb{Z}^2\},$$

where $a>1$ and $b,c>0$. Then we obtain the system:

$$\begin{aligned}\mathcal{SH}_{3,\psi}(\Lambda_3)&:=\{T_{cS_{bka^{j/2}}A_{a^j}m}D_{S_{bka^{j/2}}A_{a^j}}\psi(x)=D_{A_{a^j}S_{-bk}}T_{cm}\psi(x):j,k\in\mathbb{Z},m\in\mathbb{Z}^2\}\\&=\{a^{-3j/4}\psi(S_{bk}A_{a^{-j}}x-cm):j,k\in\mathbb{Z},m\in\mathbb{Z}^2\},\end{aligned}$$

which coincides with the classical shearlet systems $\mathcal{SH}_{1,\psi}(\Lambda_1)$ defined by (3.17). This is because of the isomorphism:

$$\begin{aligned}\Phi_{1,3}(\Lambda_1)&=\{\Phi_{1,3}(a^j,bk,cm):j,k\in\mathbb{Z},m\in\mathbb{Z}^2\}\\&=\{(a^{-j},-bka^{-j/2},cS_{-bka^{-j/2}}A_{a^{-j}}m):j,k\in\mathbb{Z},m\in\mathbb{Z}^2\}\\&=\Lambda_3.\end{aligned}$$

In Subsection 3.3.1, we showed that the associated set Λ_1 of $\mathcal{SH}_{1,\psi}(\Lambda_1)$ possesses the uniform density $\frac{1}{bc^2 \ln a}$. A natural question is whether the associated set Λ_3 of $\mathcal{SH}_{3,\psi}(\Lambda_3)$ does possess a uniform density when considering the other group multiplications.

To answer this question, we first introduce a notion of density for a discrete subset of $\mathbb{S}_3$. Recall the set $Q_h = [e^{-h/2}, e^{h/2}) \times [-\frac{h}{2}, \frac{h}{2}) \times [-\frac{h}{2}, \frac{h}{2})^2, h > 0$. For $(x, y, z) \in \mathbb{S}_3$, we define

$$Q_h(x, y, z) = (x, y, z) * Q_h = \{(xa, y + s\sqrt{x}, z + S_y A_x t) : (a, s, t) \in Q_h\}.$$

Since the left-invariant Haar measure for the group $\mathbb{S}_3$ is given by $d\mu_{\mathbb{S}_3} = \frac{da}{a^3}\, ds\, dt$, we can compute the volume of $Q_h(x, y, z)$ by

$$\mu_{\mathbb{S}_3}(Q_h) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \frac{da}{a^3}\, ds\, dt = h^3(e^h - e^{-h}).$$

Let Λ be a discrete subset of $\mathbb{S}_3$ with associated weight function $w : \Lambda \to \mathbb{R}^+$. Then *the upper and lower weighted density* of $\mathbb{S}_3$ is defined, respectively, by

$$D_w^+(\Lambda) = \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_3} \frac{\#_w(\Lambda \cap Q_h(x, y, z))}{h^3(e^h - e^{-h})},$$

$$D_w^-(\Lambda) = \liminf_{h\to\infty} \inf_{(x,y,z)\in\mathbb{S}_3} \frac{\#_w(\Lambda \cap Q_h(x, y, z))}{h^3(e^h - e^{-h})}.$$

The following proposition is analogous to Proposition 3.5. It describes how a subset Λ of $\mathbb{S}_3$ possesses finite upper density and positive lower density.

Proposition 3.14. *For $\Lambda \subset \mathbb{S}_3$ the following conditions are equivalent.*

1. *$D_w^+(\Lambda) < \infty$.*
2. *There exists $h > 0$ such that $\sup_{(x,y,z)\in\mathbb{S}_3} \#_w(\Lambda \cap Q_h(x, y, z)) < \infty$.*

Also the following conditions are equivalent

1. *$D_w^-(\Lambda) > 0$.*
2. *There exists $h > 0$ such that $\inf_{(x,y,z)\in\mathbb{S}_3} \#_w(\Lambda \cap Q_h(x, y, z)) > 0$.*

Proof. The proof is identical to that of Proposition 3.5. We leave the proof to the Appendix C. ◻

Proposition 3.15. *Let $\Lambda_3 = \{(a^j, bka^{j/2}, cS_{bka^{j/2}} A_{a^j} m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ where $a > 1$ and $b, c > 0$. Then*

$$\frac{1}{bc^2(a^2 - 1)} < D^-(\Lambda_3) \le \frac{a^2}{bc^2(a^2 - 1)} < D^+(\Lambda_3) < \frac{a^4}{bc^2(a^2 - 1)}.$$

Proof. Fix $(x, y, z) \in \mathbb{S}_3$. If $(a^j, bka^{j/2}, cS_{bka^{j/2}} A_{a^j} m) \in Q_h(x, y, z)$, then

$$\left(\frac{1}{x}, -\frac{y}{\sqrt{x}}, -A_x^{-1} S_y^{-1} z\right) * (a^{2j}, bka^{j/2}, cS_{bka^{j/2}} A_{a^j} m)$$
$$= \left(\frac{a^j}{x}, -\frac{y}{\sqrt{x}} + \frac{bka^{j/2}}{\sqrt{x}}, -A_x^{-1} S_y^{-1} z + cS_{-\frac{y}{\sqrt{x}}} A_{\frac{1}{x}} S_{bka^{j/2}} A_{a^j} m\right) \in Q_h.$$

This requires the following conditions

$$\log_a x - \frac{h}{2\ln a} \leq j \leq \log_a x + \frac{h}{2\ln a}. \tag{3.49}$$

$$\frac{ya^{-j/2}}{b} - \frac{h\sqrt{x}a^{-j/2}}{2b} \leq k \leq \frac{ya^{-j/2}}{b} + \frac{h\sqrt{x}a^{-j/2}}{2b}. \tag{3.50}$$

$$\frac{z_2 a^{-j/2}}{c} - \frac{h\sqrt{x}a^{-j/2}}{2c} \leq m_2 \leq \frac{z_2 a^{-j/2}}{c} + \frac{h\sqrt{x}a^{-j/2}}{2c}. \tag{3.51}$$

$$-\left(\frac{yz_2 - z_1}{c} + (bka^j - ya^{j/2})m_2\right) a^{-j} - \frac{hxa^{-j}}{2c} \leq m_1$$
$$\leq -\left(\frac{yz_2 - z_1}{c} + (bka^j - ya^j)m_2\right) a^{-j} + \frac{hxa^{-j}}{2c}. \tag{3.52}$$

It suffices to investigate that for a fixed $(x, y, z) \in \mathbb{S}_3$, and for a given value of j, there are approximately $\frac{h\sqrt{x}}{b}a^{-j/2}$ values of k satisfying (3.50) and there are approximately $\frac{h\sqrt{x}}{c}a^{-j/2}$ values of m_2 satisfying (3.51). Furthermore, for given values of j, k and m_2, there are approximately $\frac{hx}{c}a^{-j}$ values of m_1 satisfying (3.52). By using (3.37) in Lemma 3.11, we obtain

$$\sup_{(x,y,z)\in\mathbb{S}_3} \#(\Lambda_3 \cap Q_h(x, y, z)) = \sup_{(x,y,z)\in\mathbb{S}_3} \sum_{j=\lceil \log_a x + \frac{h}{2\ln a}\rceil}^{\lceil \log_a x - \frac{h}{2\ln a}\rceil} \left(\frac{h^3x^2}{bc^2}a^{-2j}\right) + \mathcal{O}(h^3e^h)$$
$$< \frac{h^3a^4(e^h - a^{-6}e^{-h})}{bc^2(a^2-1)} + \mathcal{O}(h^3e^h),$$

and $\sup_{(x,y,z)\in\mathbb{S}_3} \#(\Lambda_3 \cap Q_h(x, y, z)) > \frac{h^3a^2(e^h - a^{-2}e^{-h})}{bc^2(a^2-1)} + \mathcal{O}(h^3e^h)$.

Therefore

$$\lim_{h\to\infty}\left[\frac{h^3a^2(e^h - a^{-2}e^{-h})}{bc^2(a^2-1)} \cdot \frac{1}{h^3(e^h - e^{-h})} + \frac{\mathcal{O}(h^3e^h)}{h^3(e^h - e^{-h})}\right]$$
$$< \quad D^+(\Lambda_3) = \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_3} \frac{\#(\Lambda_3 \cap Q_h(x, y, z))}{h^3(e^h - e^{-h})}$$
$$< \lim_{h\to\infty}\left[\frac{h^3a^4(e^h - a^{-6}e^{-h})}{bc^2(a^2-1)} \cdot \frac{1}{h^3(e^h - e^{-h})} + \frac{\mathcal{O}(h^3e^h)}{h^3(e^h - e^{-h})}\right].$$

Hence

$$\frac{a^2}{bc^2(a^2-1)} < D^+(\Lambda_3) < \frac{a^4}{bc^2(a^2-1)}.$$

A similar argument shows $\frac{1}{bc^2(a^2-1)} < D^-(\Lambda_3) \leq \frac{a^2}{bc^2(a^2-1)}$. □

3.4.3 Shearlet Group $\mathbb{S}_4$

Let $\mathbb{S}_4$ be the group consisting of $(a, s, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ together with the group multiplication

$$(a, s, t) \otimes (a', s', t') = (aa', s + s'\sqrt{a}, t' + A_{a'}^{-1}S_{s'}^{-1}t)$$

The identity element of $\mathbb{S}_4$ is $(1, 0, 0)$, thus $(a, s, t)^{-1} = \left(\frac{1}{a}, -\frac{s}{\sqrt{a}}, -A_{\frac{1}{a}}^{-1}S_{s\sqrt{a}}^{-1}t\right)$.

Similarly, there is a unitary representation σ_4 of $\mathbb{S}_4$ on $L^2(\mathbb{R}^2)$ given by

$$\sigma_4(a,s,t)\psi(x) = D_{S_sA_s}T_t\psi(x) = a^{-3/4}\psi(A_a^{-1}S_s^{-1}x - t)$$

for $(a,s,t) \in \mathbb{S}_4$ and $\psi \in L^2(\mathbb{R}^2)$. Then the continuous shearlet transform $\mathcal{SH}_{4,\psi}$ induced by ψ and the unitary representation σ_4 becomes

$$\mathcal{SH}_{4,\psi}f = \langle f, \sigma_4(a,s,t)\psi\rangle = \langle f, D_{S_sA_s}T_t\psi\rangle, \tag{3.53}$$

for $f \in L^2(\mathbb{R}^2)$ and $(a,s,t) \in \mathbb{S}_4$.

Remark 3.11. *(***Connection between $\mathbb{S}_1$ and $\mathbb{S}_4$ and between $\mathcal{SH}_1$ and $\mathcal{SH}_4$***):* *Let $\Phi_{1,4} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ be defined by*

$$\Phi_{1,4}(x,y,z) = \left(\frac{1}{x}, -\frac{y}{\sqrt{x}}, z\right). \tag{3.54}$$

Next, we will show that $\Phi_{1,4}$ is a group isomorphism. From (3.54), it follows that:

$$\begin{aligned}
\Phi_{1,4}((a,s,t)\cdot(x,y,z)) &= \Phi_{1,4}(ax, y + s\sqrt{x}, z + S_yA_xt)\\
&= \left(\frac{1}{ax}, \frac{-(y+s\sqrt{x})}{\sqrt{ax}}, z + S_yA_xt\right)\\
&= \left(\frac{1}{ax}, -\frac{y}{\sqrt{ax}} - \frac{s}{\sqrt{a}}, z + A_{\frac{1}{x}}^{-1}S_{-\frac{y}{\sqrt{x}}}t\right)\\
&= \left(\frac{1}{a}, -\frac{s}{\sqrt{a}}, t\right) \otimes \left(\frac{1}{x}, -\frac{y}{\sqrt{x}}, z\right)\\
&= \Phi_{1,4}(a,s,t) \otimes \Phi_{1,4}(x,y,z).
\end{aligned}$$

Since $\Phi_{1,4}$ is bijection, $\Phi_{1,4} : \mathbb{S}_1 \longrightarrow \mathbb{S}_4$ is a group isomorphism. Let Λ be a discrete subset of $\mathbb{S}_1$, it follows from the above isomorphism that

$$\mathcal{SH}_{1,\psi}(\Lambda) = \mathcal{SH}_{4,\psi}(\Phi_{1,4}(\Lambda)), \tag{3.55}$$

In order to define the density of a discrete subset of $\mathbb{S}_4$, we require the following ingredients. For $(x,y,z) \in \mathbb{S}_4$ we define

$$Q_h(x,y,z) = (x,y,z) \otimes Q_h = \{(xa, y + s\sqrt{x}, t + A_a^{-1}S_s^{-1}z) : (a,s,t) \in Q_h\},$$

where $Q_h = [e^{-h/2}, e^{h/2}) \times [-\frac{h}{2}, \frac{h}{2}) \times [-\frac{h}{2}, \frac{h}{2})^2$, $h > 0$.

Since the left-invariant Haar measure for the group $\mathbb{S}_4$ is given by $d\mu_{\mathbb{S}_4} = \frac{da}{a^{3/2}}\, ds\, dt$, we can compute the volume of $Q_h(x,y,z)$ by

$$\mu_{\mathbb{S}_4}(Q_h) = \int_{-\frac{h}{2}}^{\frac{h}{2}}, \int_{-\frac{h}{2}}^{\frac{h}{2}}, \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{e^{-h/2}}^{e^{h/2}} \frac{da}{a^{3/2}}\, ds\, dt = 2h^3(e^{h/4} - e^{-h/4}).$$

Let Λ be a discrete subset of $\mathbb{S}_4$ with associated weight function $w : \Lambda \to \mathbb{R}^+$. Then the *upper and lower weighted densities* of Λ are defined, respectively, by

$$D_w^+(\Lambda) = \limsup_{h\to\infty} \sup_{(x,y,z)\in\mathbb{S}_4} \frac{\#_w(\Lambda \cap Q_h(x,y,z))}{2h^3(e^{h/4} - e^{-h/4})},$$

$$D_w^-(\Lambda) = \liminf_{h\to\infty} \inf_{(x,y,z)\in\mathbb{S}_4} \frac{\#_w(\Lambda \cap Q_h(x,y,z))}{2h^3(e^{h/4} - e^{-h/4})}.$$

Similar results as in Proposition 3.5 in Section 3.2 hold for subsets Λ of $\mathbb{S}_4$ as follow.

Proposition 3.16. *For $\Lambda \subset \mathbb{S}_4$ the following conditions are equivalent.*

1. $D_w^+(\Lambda) < \infty$.
2. *There exists $h > 0$ such that* $\sup_{(x,y,z)\in\mathbb{S}_4} \#_w(\Lambda \cap Q_h(x,y,z)) < \infty$.

Also the following conditions are equivalent

1. $D_w^-(\Lambda) > 0$.
2. *There exists $h > 0$ such that* $\inf_{(x,y,z)\in\mathbb{S}_4} \#_w(\Lambda \cap Q_h(x,y,z)) > 0$.

Proof. The proof is analogous to the proof of Proposition 3.5. We give the proof of this proposition in the Appendix D. □

In order to obtain the classical shearlet system for this case, we use the same procedure as for the previous two cases.

Recall that $\Lambda_1 = \{(a^j, bk, cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ is the discrete set associated to $\mathcal{SH}_{1,\psi}(\Lambda_1)$. Then it follows from the isomorphism (3.54)

$$\begin{aligned}\Phi_{1,4}(\Lambda_1) &= \{\Phi_{1,4}(a^j, bk, cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}\\ &= \{(a^{-j}, -bka^{-j/2}, cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\} =: \Lambda_4.\end{aligned}$$

Now, by sampling the continuous shearlet transform (3.53) on the discrete subset Λ_4 of $\mathbb{S}_4$ defined as above, the classical shearlet system $\mathcal{SH}_{4,\psi}(\Lambda_4)$ is now defined by

$$\begin{aligned}\mathcal{SH}_{4,\psi}(\Lambda_4) := \{D_{S_{-bka^{-j/2}}A_{a^{-j}}} T_{cm}\psi(x) &= a^{3j/4}\psi(A_{a^{-j}}^{-1} S_{-bka^{-j/2}}^{-1} x - cm)\\ &= a^{3j/4}\psi(S_{bk}A_{a^j}x - cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\},\end{aligned}$$

which coincides with $\mathcal{SH}_{1,\psi}(\Lambda_1)$ as well as $\mathcal{SH}_{2,\psi}(\Lambda_2)$ and $\mathcal{SH}_{3,\psi}(\Lambda_3)$.

The following results show that the density for $\mathcal{SH}_{4,\psi}(\Lambda_4)$ is not a uniform. However, by a similar argument as in Remark 3.8, using the group isomorphism connection between $\mathbb{S}_1$ and $\mathbb{S}_4$, we can obtain a uniform density for $\mathcal{SH}_{4,\psi}(\Lambda_4)$.

Proposition 3.17. *Let $\Lambda_4 = \{(a^j, bka^{j/2}, cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\} \subset \mathbb{S}_4$ where $a > 1$ and $b, c > 0$. Then*

$$\frac{1}{2bc^2(\sqrt{a}-1)} < D^-(\Lambda_4) \le \frac{\sqrt{a}}{2bc^2(\sqrt{a}-1)} < D^+(\Lambda_4) < \frac{a}{2bc^2(\sqrt{a}-1)}.$$

Proof. Fix $(x,y,z) \in \mathbb{S}_4$. If $(a^j, bka^{j/2}, cm) \in Q_h(x,y,z)$, then

$$\left(\frac{1}{x}, \frac{-y}{\sqrt{x}}, -A_{\frac{1}{x}}^{-1} S_{y\sqrt{x}}^{-1} z\right) \otimes (a^j, bka^{j/2}, cm) = \left(\frac{a^j}{x}, -\frac{y}{\sqrt{x}} + \frac{bka^j}{\sqrt{x}}, cm - \underbrace{A_{a^j}^{-1} S_{bka^{j/2}}^{-1} A_{1/x}^{-1} S_{y\sqrt{x}}^{-1} z}_{=:(C_1,C_2)^T}\right) \in Q_h.$$

This requires the following conditions

$$\log_a x - \frac{h}{2\ln a} \le j \le \log_a x + \frac{h}{2\ln a}. \tag{3.56}$$

$$\frac{ya^{-j/2}}{b} - \frac{h\sqrt{x}a^{-j/2}}{2b} \le k \le \frac{ya^{-j/2}}{b} + \frac{h\sqrt{x}a^{-j/2}}{2b}. \tag{3.57}$$

$$C_1 - \frac{h}{2} \le cm_1 \le C_1 + \frac{h}{2} \tag{3.58}$$

$$C_2 - \frac{h}{2} \le cm_2 \le C_2 + \frac{h}{2}. \tag{3.59}$$

It suffices to observe that for a given j, there are approximately $\frac{h\sqrt{x}a^{-j/2}}{b}$ values of k satisfying the condition (3.57). Furthermore for a given j, k, there are approximately $(\frac{h}{c})^2$ values of m_1 and m_2 satisfying condition (3.58) and (3.59). Using (3.38) in Lemma 3.11, we have

$$\begin{aligned}\inf_{(x,y,z)\in\mathbb{S}_4} \#(\Lambda_4 \cap Q_h(x,y,z)) &= \inf_{(x,y,z)\in\mathbb{S}_4} \sum_{j=\lceil \log_a x - \frac{h}{2\ln a}\rceil}^{\lfloor \log_a x + \frac{h}{2\ln a}\rfloor} \frac{h\sqrt{x}a^{-j/2}}{bc^2} + \mathcal{O}(h^3 e^{h/4}) \\ &\le \frac{h^3\sqrt{a}(e^{\frac{h}{4}} - e^{-\frac{h}{4}})}{bc^2(\sqrt{a}-1)} + \mathcal{O}(h^3 e^{h/4}),\end{aligned}$$

and $\inf_{(x,y,z)\in\mathbb{S}_4} \#(\Lambda_4 \cap Q_h(x,y,z)) > \frac{h^3(e^{\frac{h}{4}} - e^{-\frac{h}{4}})}{bc^2(\sqrt{a}-1)} + \mathcal{O}(h^3 e^{h/4})$.

Therefore

$$\begin{aligned}&\lim_{h\to\infty}\left[\frac{h^3(e^{\frac{h}{4}} - e^{-\frac{h}{4}})}{2bc^2(\sqrt{a}-1)h^3(e^{\frac{h}{4}} - e^{-\frac{h}{4}})} + \frac{\mathcal{O}(h^3 e^{h/4})}{2h^3(e^{\frac{h}{4}} - e^{-\frac{h}{4}})}\right] \\ &\qquad < \ D^-(\Lambda_4) = \liminf_{h\to\infty} \inf_{(x,y,z)\in\mathbb{S}_4} \frac{\#(\Lambda \cap Q_h(x,y,z))}{2h^3(e^{h/4} - e^{-h/4})} \\ &\qquad \le \lim_{h\to\infty}\left[\frac{h^3\sqrt{a}(e^{\frac{h}{4}} - e^{-\frac{h}{4}})}{2bc^2(\sqrt{a}-1)h^3(e^{\frac{h}{4}} - e^{-\frac{h}{4}})} + \frac{\mathcal{O}(h^3 e^{h/4})}{2h^3(e^{\frac{h}{4}} - e^{-\frac{h}{4}})}\right].\end{aligned}$$

Hence

$$\frac{1}{2bc^2(\sqrt{a}-1)} < D^-(\Lambda_4) \le \frac{\sqrt{a}}{2bc^2(\sqrt{a}-1)}.$$

A similar argument shows $\frac{\sqrt{a}}{2bc^2(\sqrt{a}-1)} < D^+(\Lambda_4) < \frac{a}{2bc^2(\sqrt{a}-1)}$. □

Remark 3.12. *As we have seen in Section 3.2 and Section 3.4, the definition of density for shearlet systems can be defined in different ways depending on the different choices of group multiplications. The classical shearlet systems associated with these four shearlet groups $\mathbb{S}_1 - \mathbb{S}_4$ possess different values of density, even all systems are coincided. This is because a different group multiplication leads to different behavior of the points and also different covering sets in each group. This behaves almost the same as affine-Beurling density for wavelet systems.*

As we have mentioned, the shearlet systems are two-dimensional affine-like systems. More specifically, the classical shearlet systems associated with $\mathbb{S}_3$ and $\mathbb{S}_4$ all correspond to those classical wavelet systems introduced by Sun and Zhou [74] and by Heil and Kutyniok [51], respectively. For *the classical wavelet systems, it was shown that the systems introduced by Heil and Kutyniok [51] possess a uniform density, in contrast to the systems introduced by Sun and Zhou [74]. However this does not hold for their associated shearlet systems. In other words, for classical shearlet systems, there is only the classical shearlet systems associated with $\mathbb{S}_1$ which possess a uniform density.*

3.5 Range of Density

We end this chapter by finding the range of density of a subset of $\mathbb{S}_1$. That is, for any prescribed finite $\alpha \geq \beta > 0$, does there exist some $\Gamma \subset \mathbb{S}_1$ such that $D^+(\Gamma) = \alpha$ and $D^-(\Gamma) = \beta$?

In order to answer this question, we need to study some basic properties of subsets Γ of $\mathbb{S}_1$ of the form $\Gamma = \widetilde{\Lambda} \times T$ where $\widetilde{\Lambda} \subset \mathbb{R}^+ \times \mathbb{R}$ and $T \subset \mathbb{R}^2$.

First we need to define densities for subsets of $\mathbb{R}^+ \times \mathbb{R}$ and subsets of $\mathbb{R}^2$.

Lemma 3.18. *The set $\mathbb{R}^+ \times \mathbb{R}$ equipped with the group multiplication*

$$(a, s) * (a', s') = (aa', s' + s\sqrt{a'})$$

forms a group, which we denote by $\mathbb{D}$.

Proof. The identity element of $\mathbb{S}$ is $e_1 = (1, 0)$, and inverses are given by

$$(a, s)^{-1} = \left(\frac{1}{a}, \frac{-s}{\sqrt{a}}\right),$$

since

$$(a, s) * \left(\frac{1}{a}, \frac{-s}{\sqrt{a}}\right) = \left(1, \frac{-s}{\sqrt{a}} + \frac{s}{\sqrt{a}}\right) = (1, 0),$$

and

$$\left(\frac{1}{a}, \frac{-s}{\sqrt{a}}\right) * (a, s) = \left(1, s - \frac{s}{\sqrt{a}}\sqrt{a}\right) = (1, 0).$$

The associativity can be shown as follows:

$$\begin{aligned}
((a, s) * (a', s')) * (a'', s'') &= \left(aa',\, s' + s\sqrt{a'}\right) * (a'', s'') \\
&= \left(aa'a'',\, s'' + (s' + s\sqrt{a'})\sqrt{a''}\right) \\
&= \left(a(a'a''),\, (s'' + s'\sqrt{a''}) + s\sqrt{a'a''}\right) \\
&= (a, s) * \left(a'a'',\, s'' + s'\sqrt{a''}\right) \\
&= (a, s) * \left((a', s') * (a'', s'')\right).
\end{aligned}$$

□

For all $h > 0$, let $K_h = [e^{-h/2}, e^{h/2}) \times \left[-\frac{h}{2}, \frac{h}{2}\right)$. Then for $(x, y) \in \mathbb{R}^+ \times \mathbb{R}$, we define $K_h(x, y)$ by

$$K_h(x, y) = (x, y) * K_h = \left\{(xa, s + y\sqrt{a}) \;: a \in [e^{-h/2}, e^{h/2}), s \in \left[-\frac{h}{2}, \frac{h}{2}\right)\right\}.$$

We choose the left-invariant Haar measure $\mu_{\mathbb{D}} = \frac{da}{a}ds$ to define the volume of $K_h(x,y)$:

$$\mu_{\mathbb{D}}(K_h(x,y)) = \mu_{\mathbb{D}}(K_h) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{e^{-\frac{h}{2}}}^{e^{\frac{h}{2}}} \frac{1}{a}\, da\, ds = h^2.$$

Let $\widetilde{\Lambda}$ be a subset of $\mathbb{D}$. Then *the upper density* of $\widetilde{\Lambda}$ is

$$D^+(\widetilde{\Lambda}) = \limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{D}} \frac{\#(\widetilde{\Lambda}\cap K_h(x,y))}{h^2},$$

and the *lower density* of $\widetilde{\Lambda}$ is

$$D^-(\widetilde{\Lambda}) = \liminf_{h\to\infty} \inf_{(x,y)\in\mathbb{D}} \frac{\#(\widetilde{\Lambda}\cap K_h(x,y))}{h^2}.$$

The following lemma and proposition can be proven similar to Lemma 3.4 and Proposition 3.5.

Lemma 3.19. *Let $h > 0$ and $r \geq 1$ be given. Then the following statements are true:*

1. *$\{K_h(e^{jh}, he^{-h/4}k) : j,k \in \mathbb{Z}\}$ is a covering of $\mathbb{D}$.*
2. *Any set $K_{rh}(x,y)$ intersects at most $\widetilde{N}_r := (r+2)\cdot\left[(r+1)e^{h/4}+1\right]$ sets of the form $K_h(e^{jh}, he^{-h/4}k)$.*
3. *Any set $K_{rh}(x,y)$ contains at least $\tilde{N}_r := (r+1)^2 e^{h/4}$ sets of the form $K_h(e^{jh}, he^{-h/4}k)$.*

Proposition 3.20. *If $\widetilde{\Lambda} \subset \mathbb{D}$, then the following conditions are equivalent.*

1. *$D^+(\widetilde{\Lambda}) < \infty$.*
2. *There exists $h > 0$ such that $\sup_{(x,y)\in\mathbb{D}} \#(\Lambda \cap K_h(x,y)) < \infty$.*

Also the following conditions are equivalent

1. *$D^-(\widetilde{\Lambda}) > 0$.*
2. *There exists $h > 0$ such that $\inf_{(x,y)\in\mathbb{D}} \#(\widetilde{\Lambda} \cap K_h(x,y)) > 0$.*

Lemma 3.21. *Let $\widetilde{\Lambda} = \{(a^j, bk) : j,k \in \mathbb{Z}\} \subset \mathbb{D}$ where $a > 1$ and $b > 0$. Then*

$$D^+(\widetilde{\Lambda}) = D^-(\widetilde{\Lambda}) = \frac{1}{b\ln a}.$$

Proof. Fix $(x,y) \in \mathbb{D}$. If $(a^j, bk) \in K_h(x,y)$, then

$$\left(\frac{1}{x}, \frac{-y}{\sqrt{x}}\right) * (a^j, bk) = \left(\frac{a^j}{x}, bk - \frac{ya^{j/2}}{\sqrt{x}}\right) \in K_h,$$

where $K_h = [e^{-h/2}, e^{h/2}) \times \left[-\frac{h}{2}, \frac{h}{2}\right)$.
Hence

- $\frac{a^j}{x} \in [e^{-h/2}, e^{h/2})$ implies

$$\log_a x - \frac{h}{2\ln a} \leq j < \log_a x + \frac{h}{2\ln a}. \tag{3.60}$$

- $bk - \frac{ya^{j/2}}{\sqrt{x}} \in \left[-\frac{h}{2}, \frac{h}{2}\right)$ implies

$$\frac{ya^{j/2}}{b\sqrt{x}} - \frac{h}{2b} \leq k < \frac{ya^{j/2}}{b\sqrt{x}} + \frac{h}{2b}. \tag{3.61}$$

For a fixed $(x, y) \in \mathbb{D}$ and $a > 1$, there are approximatively $\frac{h}{\ln a}$ values of j satisfying (3.60) and $\frac{h}{b}$ values of k satisfying (3.61). We compute

$$\#(\widetilde{\Lambda} \cap K_h(x, y)) = \frac{h^2}{b\ln a} + \mathcal{O}(h^2).$$

Thus,

$$\begin{aligned} D^+(\widetilde{\Lambda}) &= \limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{D}} \frac{\#(\widetilde{\Lambda} \cap K_h(x, y))}{h^2} \\ &= \limsup_{h\to\infty} \sup_{(x,y)\in\mathbb{D}} \frac{1}{h^2}\left[\frac{h^2}{b\ln a} + \mathcal{O}(h^2)\right] = \frac{1}{b\ln a}. \end{aligned}$$

A similar argument shows $D^-(\widetilde{\Lambda}) = \frac{1}{b\ln a}$. □

We recall the definition of Beurling density. For a subset T of $\mathbb{R}^2$, the *upper Beurling density* of T is

$$D^+(T) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^2} \frac{\#\left(T \cap \widetilde{K}_h(x)\right)}{h^2},$$

and the *lower Beurling density* of T is

$$D^-(T) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^2} \frac{\#\left(T \cap \widetilde{K}_h(x)\right)}{h^2},$$

where $\widetilde{K}_h(x) := x + \left[-\frac{h}{2}, \frac{h}{2}\right)^2$ is a square centered at x with side lengths h. The following lemma shows Beurling density for lattices on $\mathbb{R}^2$.

Lemma 3.22. *Let $\alpha \geq \beta > 0, h > 0$ be given, and let T be a subset of $\mathbb{R}^2$ of the form:*

$$T = \left\{\frac{1}{\sqrt{\alpha}}(m_1, m_2) : m_1 \in \mathbb{Z}^+ \cup \{0\}, m_2 \in \mathbb{Z}\right\} \cup \left\{\frac{1}{\sqrt{\beta}}(m_1, m_2) : m_1 \in \mathbb{Z}^-, m_2 \in \mathbb{Z}\right\}.$$

Then $D^+(T) = \alpha$, $D^-(T) = \beta$.

Proof. If $h > \frac{1}{\sqrt{\beta}} \geq \frac{1}{\sqrt{\alpha}}$, then at most $\lceil h\sqrt{\alpha}\rceil^2$ and at least $[\lceil h\sqrt{\beta}\rceil - 1]^2$ points of T lie in the set $\widetilde{K}_h(x)$. Therefore $D^+(T) = \alpha$ and $D^-(T) = \beta$. □

In the following lemma, we now adapt a result shown in Kutyniok [59] to show that the density of $\Gamma \subset \mathbb{S}_1$, where Γ is of the form $\widetilde{\Lambda} \times T$, can be computed directly from the densities of subsets $\widetilde{\Lambda}$ of $\mathbb{R}^+ \times \mathbb{R}$ and T of $\mathbb{R}^2$.

Lemma 3.23. *Let $\widetilde{\Lambda} \subset \mathbb{D}$ and $T \subset \mathbb{R}$. If T possesses an upper Beurling density $D^+(T)$ and a lower Beurling density $D^-(T)$, then $D^-(\widetilde{\Lambda} \times T) = D^-(\widetilde{\Lambda}) D^-(T)$ and $D^+(\widetilde{\Lambda} \times T) = D^+(\widetilde{\Lambda}) D^+(T)$.*

Proof. Assume that T possesses a lower Beurling density $D^-(T)$. Then, for any fixed $\epsilon > 0$, there exists $h_0 > 0$ with

$$\left| \frac{\#(T \cap x + \left[-\frac{h}{2}, \frac{h}{2}\right)^2}{h^2} - D^-(T) \right| < \epsilon\,, \qquad \text{for all } x \in \mathbb{R}^2,\ h \geq h_0. \tag{3.62}$$

For each $(x, y, z) \in \mathbb{S}_1$, we consider

$$\begin{aligned} \#(\Gamma \cap Q_h(x,y,z)) &= \#(\Gamma \cap \{(xa, s + y\sqrt{a}, t + S_s A_a z) : (a,s,t) \in Q_h\}) \\ &= \sum_{(a',s') \in \widetilde{\Lambda} \cap K_h(x,y)} \# \left(t \in \left[-\frac{h}{2}, \frac{h}{2}\right)^2 : t + S_{s' - y\sqrt{\frac{a'}{x}}} A_{\frac{a'}{x}} z \in T \right). \end{aligned}$$

Rewriting the right hand side,

$$\begin{aligned} &\sum_{(a',s') \in \widetilde{\Lambda} \cap K_h(x,y)} \# \left(t \in \left[-\frac{h}{2}, \frac{h}{2}\right)^2 : t + S_{s' - y\sqrt{\frac{a'}{x}}} A_{\frac{a'}{x}} z \in T \right) \\ &\qquad = \sum_{(a',s') \in \widetilde{\Lambda} \cap K_h(x,y)} \# \left(T \cap S_{s' - y\sqrt{\frac{a'}{x}}} A_{\frac{a'}{x}} z + \left[-\frac{h}{2}, \frac{h}{2}\right)^2 \right). \end{aligned}$$

By dividing by h^4, taking the infimum over all $(x, y, z) \in \mathbb{S}_1$, and together with (3.62), we get

$$\begin{aligned} \inf_{(x,y) \in \mathbb{D}} \frac{\#(\widetilde{\Lambda} \cap K_h(x,y))}{h^2} (D^-(T) - \epsilon) &\leq \inf_{(x,y,z) \in \mathbb{S}_1} \frac{\#(\Gamma \cap Q_h(x,y,z))}{h^4} \\ &\leq \inf_{(x,y) \in \mathbb{D}} \frac{\#(\widetilde{\Lambda} \cap K_h(x,y))}{h^2} (D^-(T) + \epsilon) \end{aligned}$$

for all $h \geq h_0$. Then apply the $\liminf$ as $h \to \infty$ and noting that we can choose ϵ arbitrary small, hence

$$D^-(\widetilde{\Lambda} \times T) = D^-(\widetilde{\Lambda}) D^-(T).$$

The second claim can be proved in a similar way. □

We now can state the converse result as follow:

Theorem 3.24. *Let $\alpha \geq \beta > 0$ be given and let $\Gamma = \widetilde{\Lambda} \times T$ be a subset of $\mathbb{S}_1$, where $\widetilde{\Lambda} = \left\{(e^{\frac{j}{b}}, bk) : j,k \in \mathbb{Z}\right\}$ is a subset of $\mathbb{D}$, for $b > 0$ and $T = \left\{\frac{1}{\sqrt{\alpha}}(m_1, m_2) : m_1 \in \mathbb{Z}^+ \cup \{0\}, m_2 \in \mathbb{Z}\right\} \cup \left\{\frac{1}{\sqrt{\beta}}(m_1, m_2) : m_1 \in \mathbb{Z}^-, m_2 \in \mathbb{Z}\right\}$ is a subset of $\mathbb{R}^2$. Then*

$$D^+(\Gamma) = \alpha \quad \textit{and} \quad D^-(\Gamma) = \beta.$$

Proof. It follows from Lemma 3.21 that $\widetilde{\Lambda}$ possesses a uniform density, and it is easy to compute that $D^+(\widetilde{\Lambda}) = D^-(\widetilde{\Lambda}) = 1$. It follows from Lemma 3.23 that

$$D^+(\Gamma) = D^+(\widetilde{\Lambda})D^+(T) = \alpha, \quad \text{and} \quad D^-(\Gamma) = D^-(\widetilde{\Lambda})D^-(T) = \beta.$$

This proves the theorem. □

Chapter 4

Existence of Irregular Shearlet Frames

The density theorem has been known as one of the most efficient tools to derive necessary conditions for the existence of frames. In this chapter, we use the notion of density for shearlet systems associated with the shearlet group $\mathbb{S}_1$, introduced in Section 3.2, to derive necessary density conditions for an irregular shearlet system $\mathcal{SH}_{1,\psi}(\Lambda)$ to be a frame. We begin this chapter by deriving necessary density conditions for the existence of an upper frame bound. Then we discuss how the Homogenous Approximation Property (HAP) for irregular shearlet frames leads to necessary density conditions for the existence of a lower frame bound. We would mention that many ideas and proofs in this chapter are inspired by Heil and Kutyniok [51], [52] and Gröchenig [40].

4.1 Existence of an Upper Frame Bound

In this section we restrict our attention to the necessary density conditions for the shearlet system to possess an upper frame bound, i.e., to form a Bessel sequence. The following theorem is analogous to part (a) of Theorem 1 by Heil and Kutyniok [51].

Theorem 4.1. *With the notations used in Section 3.2, let $\psi \in L^2(\mathbb{R}^2)$ be a nonzero function and Λ be a discrete subset of $\mathbb{S}_1$. If $\mathcal{SH}_{1,\psi}(\Lambda)$ possesses an upper frame bound for $L^2(\mathbb{R}^2)$, then $D^+(\Lambda) < \infty$.*

Proof. Suppose that $D^+(\Lambda) = \infty$. We now show that $\mathcal{SH}_{1,\psi}(\Lambda)$ does not possess an upper frame bound. More precisely, we prove that for each $N > 0$ there exists some $g \in L^2(\mathbb{R}^2)$ such that $\sum\limits_{(a,s,t)\in\Lambda} |\langle g, \psi_{a,s,t}\rangle|^2 > N$.

Choose any $\eta \in L^2(\mathbb{R}^2) \setminus \{0\}$ with $||\eta||_2 = 1$. Since the shearlet transform is continuous, there exist $h > 0$ and $(a, s, t) \in Q_h(p, q, r)$ such that

$$\delta = \inf_{(a,s,t)\in Q_h(p,q,r)} |\mathcal{SH}_{1,\psi}\eta(a, s, t)| > 0.$$

Choose $N > 0$. Since $D^+(\Lambda) = \infty$, it follows from Proposition 3.5 that there exists some $(x, y, z) \in \mathbb{S}_1$ such that $\#(\Lambda \cap Q_h(x, y, z)) \geq N$. Define

$$g := \sigma_1((x, y, z) \cdot (p, q, r)^{-1})\eta.$$

Note that $g \in L^2(\mathbb{R}^2)$ and $||g||_2 = ||\eta||_2 = 1$. Also observe that we have

$$(a,s,t) \in Q_h(x,y,z) \Longrightarrow (p,q,r)\cdot(x,y,z)^{-1}\cdot(a,s,t) \in Q_h(p,q,r).$$

Therefore we can compute that

$$\begin{aligned}\sum_{(a,s,t)\in\Lambda} |\langle g, \sigma_1(a,s,t)\psi\rangle|^2 &\geq \sum_{(a,s,t)\in\Lambda\cap Q_h(x,y,z)} |\langle \sigma_1((x,y,z)\cdot(p,q,r)^{-1})\eta, \sigma_1(a,s,t)\psi\rangle|^2 \\ &= \sum_{(a,s,t)\in\Lambda\cap Q_h(x,y,z)} |\langle \eta, \sigma_1((p,q,r)\cdot(x,y,z)^{-1}\cdot(a,s,t))\psi\rangle|^2 \\ &= \sum_{(a,s,t)\in\Lambda\cap Q_h(x,y,z)} |\mathcal{SH}_{1,\psi}\eta(\underbrace{(p,q,r)\cdot(x,y,z)^{-1}\cdot(a,s,t))}_{\in Q_h(p,q,r)}|^2 \\ &\geq \#(\Lambda\cap Q_h(x,y,z))\delta^2 \geq N\delta^2.\end{aligned}$$

Since $||g||_2 = 1$, this shows that $\mathcal{SH}_{1,\psi}\eta(a,s,t)$ cannot possess an upper frame bound. □

4.2 Homogeneous Approximation Property for Irregular Shearlet Frames

In this section, we will show that irregular shearlet frames, associated with a natural class of generating shearlets $\mathcal{B}_0$, possess the analogue of the HAP. As a consequence of the HAP, we obtain necessary density conditions for the existence of a lower frame bound for irregular shearlet systems in the subsequent section.

First of all, in Subsetion 4.2.1, we explain how to define the amalgam space on the shearlet group $\mathbb{S}_1$, and how the natural class of generating shearlets $\mathcal{B}_0$ can be established. Then we discuss the HAP and the comparison theorem for shearlet frames in Subsection 4.2.2 and Subsection 4.2.3, respectively.

4.2.1 Amalgam Spaces on the Shearlet Group $\mathbb{S}_1$

Given $h > 0$, recall that $X_1 = \{(e^{jh}, he^{-h/4}k, he^{-h/2}m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$, defined as in Lemma 3.4, is relatively separated in $\mathbb{S}_1$. Set $B_{j,k,m} = B_{j,k,m}(h) = Q_h(e^{jh}, he^{-h/4}k, he^{-h/2}m)$, where Q_h is a compact neighborhood of e in $\mathbb{S}_1$ defined as in the previous chapter.

Now, for each $1 \leq p < \infty$, the amalgam spaces $W_{\mathbb{S}_1}(L^\infty, L^p)$ on the shearlet group $\mathbb{S}_1$ can be defined as a mixed-norm consisting of functions $f : \mathbb{S}_1 \to \mathbb{C}$ as follows:

Definition 4.1. For $1 \leq p < \infty$, the *amalgam space on the shearlet group* $\mathbb{S}_1$ is defined by

$$W_{\mathbb{S}_1}(L^\infty, L^p) = \left\{ f \in L^p(\mathbb{S}_1) : ||f||_{W_{\mathbb{S}_1}(L^\infty,L^p)} := \left(\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} ||f\cdot\chi_{B_{j,k,m}}||_\infty^p \right)^{1/p} < \infty \right\}. \quad (4.1)$$

For the extensive references to amalgam spaces, we refer to Heil [49] and Feichtinger and Gröchenig [28]. Analogously, we denote the amalgam space $W_{\mathbb{S}_1}(C, L^p)$ as the closure of the subspace of $W_{\mathbb{S}_1}(L^\infty, L^p)$ consisting of continuous functions in $W_{\mathbb{S}_1}(L^\infty, L^p)$. Furthermore, for $1 \leq p \leq q < \infty$, we have $W_{\mathbb{S}_1}(L^\infty, L^p) \subset W_{\mathbb{S}_1}(L^\infty, L^q)$.

In general, the natural class of generating shearlets would be the space $\widetilde{B}_0$ comprising of all functions ψ such that $\mathcal{SH}_{1,\psi}\psi \in W_{\mathbb{S}_1}(C, L^1)$. Unfortunately, our class of generating shearlets $\mathcal{B}_0$ defined in the following theorem is smaller, but still contained in $\widetilde{B}_0$.

Theorem 4.2. *Let $\mathcal{B}_0$ denote the space of Schwartz-functions which satisfy*

(i) $|\psi(x)| \leq \frac{C}{(1+||x||_\infty^2)^\alpha}$, *where* $C > 0$ *and* $\alpha > \frac{3}{2}$,

(ii) $\operatorname{supp} \hat{\psi} \in \{[-a_1, -a_0] \cup [a_0, a_1]\} \times [-b, b]$, $0 < a_0 < a_1$ *and* $b > 0$ *with*

$$\left|\hat{\psi}(\xi)\right| \leq \frac{\xi_1^{2\beta}}{(1+||\xi||_\infty^2)^{2\beta}}, \qquad \text{where} \quad \beta > 4\alpha + 1.$$

Then

1. $\mathcal{B}_0 \subset \widetilde{B}_0$ *where* $\widetilde{B}_0 = \{\psi \in L^2(\mathbb{R}^2) : \mathcal{SH}_{1,\psi}\psi \in W_{\mathbb{S}_1}(C, L^1)\}$.
2. *If* $f, \psi \in \mathcal{B}_0$, *then* $\mathcal{SH}_{1,\psi}f \in W_{\mathbb{S}_1}(C, L^1)$.

Remark 4.1. *It is obvious that the space $\mathcal{B}_0$ is dense in $L^2(\mathbb{R}^2)$ and each element in $\mathcal{B}_0$ is admissible.*

In order to prove this theorem, we require the following preliminary lemmas. They are analogous to Lemma 4.5 and Lemma 4.6 of Dahlke et al. [19].

Lemma 4.3. *For $\alpha > 1$, let $|f(x)| \leq \frac{C}{(1+||x||_\infty^2)^\alpha}$ and $|\psi(x)| \leq \frac{C}{(1+||x||_\infty^2)^\alpha}$. Then the shearlet transform fulfills*

$$|\mathcal{SH}_{1,\psi}f(a,s,t)| \leq Ca^{3/4}\frac{\max\{1,d^2\}}{\left[1+\left\|\frac{A_a^{-1}S_s^{-1}t}{\max\{1,d\}}\right\|_\infty^2\right]^{\alpha-1/2}}, \qquad \forall\,(a,s,t) \in \mathbb{S}_1$$

where $d^2 = \left(1+\frac{|s|}{\sqrt{a}}\right)^2 \max\left\{\frac{1}{a^2}, \frac{1}{a}\right\}$.

Proof. Recall the group isomorphism between $\mathbb{S}_1$ and $\mathbb{S}_3$ in Remark 3.10:

$$\mathcal{SH}_{1,\psi}f(a,s,t) = \mathcal{SH}_{3,\psi}f\left(\frac{1}{a}, -\frac{s}{\sqrt{a}}, S_{-\frac{s}{\sqrt{a}}}A_{\frac{1}{a}}t\right). \tag{4.2}$$

By changing variables, the proof follows from (4.2) and Lemma 4.5 of Dahlke et al. [19], which shows that

$$|\mathcal{SH}_{3,\psi}f(a,s,t)| \leq Ca^{-3/4}\frac{\max\{1,d^2\}}{\left[1+\left\|\frac{t}{\max\{1,d\}}\right\|_\infty^2\right]^{\alpha-1/2}}$$

where $d^2 = (1+|s|^2)\max\{a^2, a\}$. □

Lemma 4.4. *Let* $\operatorname{supp}\hat{\psi} \in \{[-a_1, -a_0] \cup [a_0, a_1]\} \times [-b, b]$ *where* $0 < a_0 < a_1$ *and* $b > 0$ *and* $\hat{f}$ *satisfies*

$$|\hat{f}(\xi)| \leq \frac{\xi^{2\beta}}{(1+||\xi||_\infty^2)^{2\beta}}, \qquad \beta > 0.$$

Then the following estimate holds true:

$$|\mathcal{SH}_{1,\psi}f(a,s,t)| \leq Ca^{3/4}\frac{a^{3\beta/2}}{(1+a^2)^\beta(\sqrt{a}+|s|)^\beta}, \qquad \forall\,(a,s,t) \in \mathbb{S}_1.$$

Proof. It was shown by Dahlke et al. [19] in Lemma 4.6 that under the same assumption, $\mathcal{SH}_{3,\psi}f(a,s,t)$ is bounded by

$$|\mathcal{SH}_{3,\psi}f(a,s,t)| \leq Ca^{-3/4}\frac{a^{\beta}}{(1+a^2)^{\beta}} \cdot \frac{1}{(1+|s|)^{\beta}}.$$

By (4.2) and changing variables, this complete the proof of this lemma. □

Now we can give the proof of Theorem 4.2.

Proof of Theorem 4.2. To prove this, it is enough to prove that for any $f,\psi \in \mathcal{B}_0$, $\mathcal{SH}_{1,\psi}f \in W_{\mathbb{S}_1}(C,L^1)$. Suppose that $f,\psi \in \mathcal{B}_0$. In particular, we have:

(i) ψ and f are Schwartz functions,

(ii) $\exists\, C>0$ and $\alpha>2$ such that $|f(x)|, |\psi(x)| \leq \frac{C}{(1+||x||_{\infty}^2)^{\alpha}}$,

(iii) supp $\hat{\psi} \in \{[-a_1,-a_0]\cup[a_0,a_1]\} \times [-b,b]$, and $\hat{f}$ satisfies

$$|\hat{f}(\xi)| \leq \frac{\xi_1^{2\beta}}{(1+||\xi||_{\infty}^2)^{2\beta}}, \quad \text{where } \beta > 4\alpha+1.$$

Furthermore, we obtain from (ii) that

$$\int_{\mathbb{R}^2}(1+||x||_{\infty})|\psi(x)|\,dx \leq C\int_{\mathbb{R}^2}\frac{(1+||x||_{\infty})}{(1+||x||_{\infty}^2)^{\alpha}}\,dx < \infty.$$

By Lemma 4.3 and Lemma 4.4, we obtain that there exists $C_1>0$ such that

$$|\mathcal{SH}_{1,\psi}f(a,s,t)|^2 \leq C_1a^{3/2}\frac{\max\{1,d^2\}}{\left[1+\left\|\frac{A_a^{-1}S_s^{-1}t}{\max\{1,d\}}\right\|_{\infty}^2\right]^{\alpha-1/2}} \cdot \frac{a^{3\beta/2}}{(1+a^2)^{\beta}(\sqrt{a}+|s|)^{\beta}} \tag{4.3}$$

for all $(a,s,t)\in\mathbb{S}_1$, where $\beta>4\alpha+1$ and $\alpha>\frac{3}{2}$.

Now, set $h=1$, and $B_{j,k,m}=B_{j,k,m}(1)=Q_1(e^j,ke^{-1/4},e^{-1/2}m)$. Suppose that $(a,s,t)\in B_{j,k,m}$ for some $j,k\in\mathbb{Z}$ and $m\in\mathbb{Z}^2$. Hence

$$\begin{aligned}(a,s,t) &= (e^j,ke^{-1/4},e^{-1/2}m)\cdot(x,y,z)\\ &= (xe^j, y+ke^{-1/4}\sqrt{x}, z+e^{-1/2}S_yA_xm),\end{aligned}$$

for some $(x,y,z)\in Q_1=[e^{-1/2},e^{1/2})\times\left[-\frac{1}{2},\frac{1}{2}\right)\times\left[-\frac{1}{2},\frac{1}{2}\right)^2$.
Therefore

- $e^{j-1/2}\leq a\leq e^{j+1/2}$,
- $\frac{|k|}{\sqrt{e}}-\frac{1}{2}\leq|s|\leq|k|+\frac{1}{2}$,
- $e^{-1/2}||S_yA_xm||_{\infty}-\frac{1}{2}\leq||t||_{\infty}\leq e^{-1/2}||S_yA_xm||_{\infty}+\frac{1}{2}$.

By above, we have

$$\begin{aligned}||t||_\infty &\geq e^{-1/2}||S_yA_xm||_\infty - \frac{1}{2} \geq e^{-1/2}\frac{||m||_\infty}{||S_y^{-1}||_\infty||A_x^{-1}||_\infty} - \frac{1}{2}\\ &= e^{-1/2}\frac{||m||_\infty}{(1+|y|)\max\left\{\frac{1}{x}, \frac{1}{\sqrt{x}}\right\}} - \frac{1}{2} \geq \frac{2||m||_\infty}{3e} - \frac{1}{2}\\ &\geq \frac{||m||_\infty}{3e}, \qquad\qquad \text{for any } ||m||_\infty > \frac{3e}{2}.\end{aligned}$$

Next, let

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2}\left\|\mathcal{SH}_{1,\psi}f\cdot\chi_{B_{j,k,m}}\right\|_\infty = S_1+S_2+S_3+S_4,$$

where

$$\begin{aligned}S_1 &= \sum_{j=-\infty}^{0}\sum_{k\in\mathbb{Z}}\sum_{\{m\in\mathbb{Z}^2:||m||_\infty>\frac{3e}{2}\}}\left\|\mathcal{SH}_{1,\psi}f\cdot\chi_{B_{j,k,m}}\right\|_\infty,\\ S_2 &= \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\{m\in\mathbb{Z}^2:||m||_\infty>\frac{3e}{2}\}}\left\|\mathcal{SH}_{1,\psi}f\cdot\chi_{B_{j,k,m}}\right\|_\infty,\\ S_3 &= \sum_{j=-\infty}^{0}\sum_{k\in\mathbb{Z}}\sum_{\{m\in\mathbb{Z}^2:||m||_\infty\leq\frac{3e}{2}\}}\left\|\mathcal{SH}_{1,\psi}f\cdot\chi_{B_{j,k,m}}\right\|_\infty,\\ S_4 &= \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\{m\in\mathbb{Z}^2:||m||_\infty\leq\frac{3e}{2}\}}\left\|\mathcal{SH}_{1,\psi}f\cdot\chi_{B_{j,k,m}}\right\|_\infty.\end{aligned}$$

In order to prove that $\mathcal{SH}_{1,\psi}f \in W_{\mathbb{S}_1}(C, L^1)$, we will first show that each S_1, S_2, S_3 and S_4 are finite, and this implies that $\mathcal{SH}_{1,\psi}f \in W_{\mathbb{S}_1}(L^\infty, L^1)$. Since $\mathcal{SH}_{1,\psi}f$ is continuous, this completes the proof.

Estimate S_1: Suppose that $(a,s,t) \in B_{j,k,m}$ with $j \leq 0$, $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^2, ||m||_\infty > \frac{3e}{2}$. Then $a \leq 1$, $d^2 = \left(1+\frac{|s|}{\sqrt{a}}\right)^2\max\left\{\frac{1}{a^2}, \frac{1}{a}\right\} = \frac{(\sqrt{a}+|s|)^2}{a^3} > 1$, and

$$\begin{aligned}1+\left\|\frac{A_a^{-1}S_s^{-1}t}{\max\{1,d\}}\right\|_\infty^2 &\geq 1+\frac{||t||_\infty^2}{d^2||A_a||_\infty^2||S_s||_\infty^2} \geq 1+\frac{a^3\,\|m\|_\infty^2}{9e^2(\sqrt{a}+|s|)^2a(1+|s|)^2} \quad \left[\because\ ||t||_\infty \geq \frac{||m||_\infty}{3e}\right]\\ &\geq 1+\frac{a^2||m||_\infty^2}{9e^2(1+|s|)^4} \qquad [\because\ a\leq 1].\end{aligned}$$

It follows from (4.3) that

$$\begin{aligned}|\mathcal{SH}_{1,\psi}f(a,s,t)|^2 &\leq C_1a^{3/2}\frac{(\sqrt{a}+|s|)^2}{a^3\left[1+\frac{a^2||m||_\infty^2}{9e^2(1+|s|)^4}\right]^{\alpha-1/2}}\cdot\frac{a^{3\beta/2}}{(1+a^2)^\beta(\sqrt{a}+|s|)^\beta}\\ &\leq C_1a^{3/2}\frac{(1+|s|)^{4\alpha-2}}{a^{2\alpha+2}\left[\frac{9e^2(1+|s|)^4}{a^2}+||m||_\infty^2\right]^{\alpha-1/2}}\cdot\frac{a^{3\beta/2}}{a^{\frac{\beta}{2}-1}\left(1+\frac{|s|}{\sqrt{a}}\right)^{\beta-2}}\end{aligned}$$

$$\le C_1 \frac{a^{\beta-2\alpha+\frac{1}{2}}}{||m||_\infty^{2\alpha-1}} \cdot \frac{1}{(1+|s|)^{\beta-4\alpha}} \qquad [\because\ a \le 1]$$
$$\le C_1 \frac{e^{(\beta-2\alpha+\frac{1}{2})j/2}}{||m||_\infty^{2\alpha-1}} \cdot \frac{1}{(\sqrt{e}+2|k|)^{\beta-4\alpha}} \qquad \left[\because\ |s| \ge \frac{|k|}{\sqrt{e}} - \frac{1}{2}\right].$$

That is

$$|\mathcal{SH}_{1,\psi} f(a,s,t)| \ \le\ C_1 \frac{e^{(\beta-2\alpha+\frac{1}{2})j/4}}{||m||_\infty^{\alpha-\frac{1}{2}}} \cdot \frac{1}{(\sqrt{e}+2|k|)^{(\beta-4\alpha)/2}}.$$

Therefore

$$S_1 = \sum_{j=-\infty}^{0} \sum_{k\in\mathbb{Z}} \sum_{\{m\in\mathbb{Z}^2 : ||m||_\infty > \frac{3e}{2}\}} \left\| \mathcal{SH}_{1,\psi} f \cdot \chi_{B_{j,k,m}} \right\|_\infty$$
$$\le C_1 \sum_{j=-\infty}^{0} e^{(\beta-2\alpha+\frac{1}{2})j/4} \sum_{k\in\mathbb{Z}} \frac{1}{(\sqrt{e}+2|k|)^{(\beta-4\alpha)/2}} \sum_{\{m\in\mathbb{Z}^2 : ||m||_\infty > \frac{3e}{2}\}} ||m||_\infty^{-(\alpha-\frac{1}{2})} < \infty.$$

Estimate S_2: Suppose that $(a,s,t) \in B_{j,k,m}$ with $j > 0, k \in \mathbb{Z}$, $m \in \mathbb{Z}^2$ and $||m||_\infty > \frac{3e}{2}$. Then $a > 1$ and $d^2 = \frac{(\sqrt{a}+|s|)^2}{a^2}$. If $d^2 > 1$, then

$$1 + \left\| \frac{A_a^{-1} S_s^{-1} t}{\max\{1,d\}} \right\|_\infty^2 \ge 1 + \frac{\|t\|^2}{d^2 ||A_a||_\infty^2 ||S_s||_\infty^2} \ge 1 + \frac{a^2 \|m\|^2}{9e^2(\sqrt{a}+|s|)^2 a^2 (1+|s|)^2}$$
$$\ge 1 + \frac{||m||_\infty^2}{9e^2(\sqrt{a}+|s|)^4} \qquad [\because\ a > 1].$$

By (4.3), we have

$$|\mathcal{SH}_{1,\psi} f(a,s,t)|^2 \le C_1 a^{3/2} \frac{(\sqrt{a}+|s|)^2}{a^2 \left[1 + \frac{||m||_\infty^2}{9e^2(\sqrt{a}+|s|)^4}\right]^{\alpha-1/2}} \cdot \frac{a^{3\beta/2}}{(1+a^2)^\beta (\sqrt{a}+|s|)^\beta}$$
$$\le C_1 \frac{a^{-(\beta+1)/2} (\sqrt{a}+|s|)^{4\alpha}}{[9e^2(\sqrt{a}+|s|)^4 + ||m||_\infty^2]^{\alpha-1/2}} \cdot \frac{1}{(\sqrt{a}+|s|)^\beta}$$
$$\le C_1 \frac{a^{-(\beta+1)/2}}{||m||_\infty^{2\alpha-1}} \cdot \frac{1}{(1+|s|)^{\beta-4\alpha}} \qquad [\because\ a > 1]$$
$$\le C_1 \frac{e^{-(\beta+1)j/2}}{||m||_\infty^{2\alpha-1}} \cdot \frac{1}{(\sqrt{e}+2|k|)^{\beta-4\alpha}}.$$

Hence

$$S_2 = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2, ||m||_\infty > \frac{3e}{2}} \left\| \mathcal{SH}_{1,\psi} f \cdot \chi_{B_{j,k,m}} \right\|_\infty$$
$$\le C_1 \sum_{j=1}^{\infty} e^{-(\beta+1)j/4} \sum_{k\in\mathbb{Z}} \frac{1}{(\sqrt{e}+2|k|)^{(\beta-4\alpha)/2}} \sum_{m\in\mathbb{Z}^2} ||m||_\infty^{-(\alpha-\frac{1}{2})} < \infty.$$

On the other hand, if $d^2 \leq 1$, then

$$1+\left\|\frac{A_a^{-1}S_s^{-1}t}{\max\{1,d\}}\right\|_\infty^2 \geq 1+\frac{\|t\|^2}{a^2(1+|s|)^2} \geq 1+\frac{||m||_\infty^2}{9e^2a^2(\sqrt{a}+|s|)^2} \qquad [\because\ a>1].$$

Then we have from (4.3) that

$$\begin{aligned}
|\mathcal{SH}_{1,\psi}f(a,s,t)|^2 &\leq C_1 a^{3/2}\frac{1}{\left[1+\frac{||m||_\infty^2}{9e^2a^2(\sqrt{a}+|s|)^4}\right]^{\alpha-1/2}}\cdot\frac{a^{3\beta/2}}{(1+a^2)^\beta(\sqrt{a}+|s|)^\beta}\\
&\leq C_1\frac{a^{-(\beta-4\alpha-1)/2}(\sqrt{a}+|s|)^{4\alpha-2}}{[9e^2a^2(\sqrt{a}+|s|)^4+||m||_\infty^2]^{\alpha-1/2}}\cdot\frac{1}{(\sqrt{a}+|s|)^\beta}\\
&\leq C_1\frac{a^{-(\beta-4\alpha-1)/2}}{||m||_\infty^{2\alpha-1}}\cdot\frac{1}{(1+|s|)^{\beta-4\alpha+2}} \qquad [\because\ a>1]\\
&\leq C_1\frac{e^{-(\beta-4\alpha-1)j/2}}{||m||_\infty^{2\alpha-1}}\cdot\frac{1}{(\sqrt{e}+2|k|)^{\beta-4\alpha+2}}.
\end{aligned}$$

Hence

$$S_2 \leq C_1\sum_{j=1}^{\infty}e^{-(\beta-4\alpha-1)j/4}\sum_{k\in\mathbb{Z}}\frac{1}{(\sqrt{e}+2|k|)^{(\beta-4\alpha+2)/2}}\sum_{\{m\in\mathbb{Z}^2:||m||_\infty>\frac{3e}{2}\}}||m||_\infty^{-(\alpha-\frac{1}{2})}<\infty.$$

Estimate S_3: Suppose that $(a,s,t)\in B_{j,k,m}$ with $j\leq 0$ (i.e., $a\leq 1$), $k\in\mathbb{Z}$, $m\in\mathbb{Z}^2$ and $||m||_\infty<\frac{3e}{2}$. It follows from Lemma 4.4 that

$$\begin{aligned}
|\mathcal{SH}_{1,\psi}f(a,s,t)| &\leq Ca^{3/4}\cdot\frac{a^{3\beta/2}}{a^{\frac{\beta}{2}}(1+a^2)^\beta\left(1+\frac{|s|}{\sqrt{a}}\right)^\beta}\\
&\leq C\frac{a^{\beta+\frac{3}{4}}}{(1+|s|)^\beta}\\
&\leq C\frac{e^{(\beta+\frac{3}{4})j/2}}{(\sqrt{e}+2|k|)^\beta}.
\end{aligned}$$

Hence

$$\begin{aligned}
S_3 &= \sum_{j=-\infty}^{0}\sum_{k\in\mathbb{Z}}\sum_{\{m\in\mathbb{Z}^2:||m||_\infty\leq\frac{3e}{2}\}}\left\|\mathcal{SH}_{1,\psi}f\cdot\chi_{B_{j,k,m}}\right\|_\infty\\
&\leq C\,(3e+1)^2\sum_{j=-\infty}^{0}e^{(\beta+\frac{3}{4})j/2}\sum_{k\in\mathbb{Z}}\frac{1}{(\sqrt{e}+2|k|)^\beta}<\infty.
\end{aligned}$$

Estimate S_4: Suppose that $(a,s,t)\in B_{j,k,m}$ with $j>0$ (i.e., $a>1$), $k\in\mathbb{Z}$, $m\in\mathbb{Z}^2$ and

$||m||_\infty < \frac{3e}{2}$. It follows from Lemma 4.4 that

$$\begin{aligned} |\mathcal{SH}_{1,\psi}f(a,s,t)| &\le Ca^{3/4}\cdot \frac{a^{3\beta/2}}{a^{2\beta}(a^{-2}+1)^\beta\left(\sqrt{a}+|s|\right)^\beta} \\ &\le Ca^{-\frac{\beta}{2}+\frac{3}{4}}\cdot\frac{1}{(1+|s|)^\beta} \\ &\le C\frac{e^{-(\frac{\beta}{2}-\frac{3}{4})j}}{(\sqrt{e}+2|k|)^\beta}. \end{aligned}$$

Hence

$$\begin{aligned} S_4 &= \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\{m\in\mathbb{Z}^2:||m||_\infty\le\frac{3e}{2}\}}\left\|\mathcal{SH}_{1,\psi}f\cdot\chi_{B_{j,k,m}}\right\|_\infty \\ &\le C\,(3e+1)^2\sum_{j=1}^{\infty}e^{-(\frac{\beta}{2}-\frac{3}{4})j}\sum_{k\in\mathbb{Z}}\frac{1}{(\sqrt{e}+2|k|)^\beta}<\infty. \end{aligned}$$

This completes the proof. □

4.2.2 Homogeneous Approximation Property for Irregular Shearlet Frames

In the following definition, we give two types of definitions of the HAP for shearlet group $\mathbb{S}_1$, the Weak and Strong HAP, and show that shearlet frames associated with a generating function ψ belonging to our class $\mathcal{B}_0$ fulfill the Strong HAP. We begin with the following definitions for the Weak and Strong HAP which are analogous to those used by Balan et al. [4] for a general class of frames associated with a discrete Abelian group, and by Heil and Kutyniok [52] for wavelet frames.

Definition 4.2. Let $\psi\in L^2(\mathbb{R}^2)$ and $\Lambda\subset\mathbb{S}_1$ such that $\mathcal{SH}_{1,\psi}(\Lambda)=\{\sigma_1(a,s,t)\psi:(a,s,t)\in\Lambda\}$ is a shearlet frame for $L^2(\mathbb{R}^2)$. Also, assume that $\{\widetilde{\psi}_{a,s,t}:(a,s,t)\in\Lambda\}$ is its dual frame.
For each $h>0$ and $(p,q,r)\in\mathbb{S}_1$, define a space

$$W(h,(p,q,r)) = \operatorname{span}\{\widetilde{\psi}_{a,s,t}:(a,s,t)\in(p,q,r)Q_h\cap\Lambda\}.$$

(a) A frame $\mathcal{SH}_{1,\psi}(\Lambda)$ is said to possess the *Weak HAP* if for each $f\in L^2(\mathbb{R}^2)$, and

$$\forall\,\epsilon>0,\ \exists\,R=R(f,\epsilon)>0\ \text{ such that }\ \forall\,(p,q,r)\in\mathbb{S}_1$$

$$\text{dist}\,(\sigma_1(p,q,r)f, W(R,(p,q,r)))<\epsilon. \tag{4.4}$$

(b) A frame $\mathcal{SH}_{1,\psi}(\Lambda)$ is said to possess the *Strong HAP* if for each $f\in L^2(\mathbb{R}^2)$, and

$$\forall\,\epsilon>0,\ \exists\,R=R(f,\epsilon)>0\ \text{ such that }\ \forall\,(p,q,r)\in\mathbb{S}_1$$

$$\left\|\sigma_1(p,q,r)f\ -\sum_{(a,s,t)\in(p,q,r)Q_R\cap\Lambda}\langle\sigma_1(p,q,r)f,\sigma_1(a,s,t)\psi\rangle\widetilde{\psi}_{a,s,t}\right\|_2<\epsilon. \tag{4.5}$$

Remark 4.2. *1. It follows from Theorem 4.1 that if $\mathcal{SH}_{1,\psi}(\Lambda)$ is a frame for $L^2(\mathbb{R}^2)$, then $D^+(\Lambda) < \infty$. This means that there are only finitely many points of Λ contained in each box $(p,q,r)Q_h$, therefore $W(h,(p,q,r))$ is a finite dimensional space.*
2. It is obvious that the function $\sum_{(a,s,t)\in(p,q,r)Q_R\cap\Lambda} \langle\sigma_1(p,q,r)f, \sigma_1(a,s,t)\psi\rangle\widetilde{\psi}_{a,s,t}$ is an element of $W(R,(p,q,r))$, therefore the Strong HAP implies the Weak HAP.

Before we prove that $\mathcal{SH}_{1,\psi}(\Lambda)$ with $\psi \in \mathcal{B}_0$ satisfies the Strong HAP, we require the following technical lemma.

Lemma 4.5. *Let $\psi, g \in \mathcal{B}_0$ and $\Lambda \subset \mathbb{S}_1$. Furthermore, let $\mathcal{SH}_{1,\psi}(\Lambda)$ be a frame for $L^2(\mathbb{R}^2)$. Given $\epsilon, \delta > 0$. Then there exists some $R = R(g,\epsilon) > 0$ such that*

$$\sum_{(x,y,z)\in(p,q,r)^{-1}\Lambda\setminus Q_R} |\mathcal{SH}_{1,\psi}g(x,y,z)|^2 \leq \epsilon,$$

for each $(p,q,r) \in \mathbb{S}_1$, where $Q_R = [e^{-\frac{R}{2}}, e^{\frac{R}{2}}) \times \left[-\frac{R}{2}, \frac{R}{2}\right) \times \left[-\frac{R}{2}, \frac{R}{2}\right)^2$.

Proof. Let $\delta > 0$ and $R' > 1$ be given, and define $R = \left(1+\frac{\delta}{2}\right)^2 e^{\delta} R' + \delta\left(1+\frac{\delta}{2}\right)^2 e^{\delta} + \delta$.

First, we claim that

$$Q_\delta \setminus Q_R \neq \emptyset \Longrightarrow Q_\delta(p,q,r) \cap Q_{R'} = \emptyset, \tag{4.6}$$

for any $(p,q,r) \in \mathbb{S}_1$. Suppose that there exist $(a,s,t) \in Q_\delta(p,q,r) \setminus Q_R$. We need to show that if $(x,y,z) \in Q_\delta$, then

$$(p,q,r)\cdot(x,y,z) = (px, y + q\sqrt{x}, z + S_y A_x r) \notin Q_{R'}.$$

Notice that

$$(a,s,t) \notin Q_R, \tag{4.7}$$

$$(p,q,r)^{-1}\cdot(a,s,t) = \left(\frac{a}{p}, s - q\sqrt{\frac{a}{p}}, t - S_s A_a S_{-\frac{q}{\sqrt{a}}} A_{\frac{1}{p}} r\right) \in Q_\delta, \tag{4.8}$$

$$(x,y,z) \in Q_\delta. \tag{4.9}$$

First, suppose that $a > e^{\frac{R}{2}}$, then it follows from (4.7)-(4.9) that

$$px = \frac{p}{a}(ax) \geq e^{-\frac{\delta}{2}} e^{\frac{R}{2}} e^{-\frac{\delta}{2}} = e^{\frac{R}{2}-\delta} \geq e^{\frac{R'}{2}}.$$

Similarly, if $a < e^{-\frac{R}{2}}$, then $px < e^{-\frac{R'}{2}}$.
Next, we consider the term

$$y + q\sqrt{x} = y - \left(s - q\sqrt{\frac{a}{p}}\right)\sqrt{\frac{p}{a}}\sqrt{x} + s\sqrt{\frac{p}{a}}\sqrt{x}.$$

For the case $s \geq \frac{R}{2}$, we have

$$y + q\sqrt{x} \geq -\frac{\delta}{2} - \frac{\delta}{2} e^{\frac{\delta}{4}} e^{\frac{\delta}{4}} + \frac{R}{2} e^{-\frac{\delta}{4}} e^{-\frac{\delta}{4}} = \frac{R}{2} e^{-\frac{\delta}{2}} - \frac{\delta}{2} e^{\frac{\delta}{2}} - \frac{\delta}{2} > \frac{R'}{2}.$$

Similarly, if $s < \frac{R}{2}$, then $y + q\sqrt{x} < -\frac{R'}{2}$.
Finally, we want to show that $||z + S_y A_x r||_\infty > \frac{R'}{2}$. Since we have from (4.8) that $||t - S_s A_a S_{-\frac{q}{\sqrt{p}}} A_{\frac{1}{p}} r||_\infty \leq \frac{\delta}{2}$ and from (4.7) that $||t||_\infty > \frac{R}{2}$, we obtain

$$||S_s A_a S_{-\frac{q}{\sqrt{p}}} A_{\frac{1}{p}} r||_\infty \geq ||t||_\infty - ||t - S_s A_a S_{-\frac{q}{\sqrt{p}}} A_{\frac{1}{p}} r||_\infty \geq \frac{R-\delta}{2}.$$

Hence

$$||r||_\infty \geq \frac{(R-\delta)}{2\left\|S_{s-q\sqrt{\frac{a}{p}}}\right\|_\infty ||A_{\frac{a}{p}}||_\infty} = \frac{(R-\delta)}{2\left(1+\left|s-q\sqrt{\frac{a}{p}}\right|\right)\max\left\{\frac{a}{p},\sqrt{\frac{a}{p}}\right\}} \geq \frac{(R-\delta)}{2e^{\frac{\delta}{2}}\left(1+\frac{\delta}{2}\right)},$$

the last inequality is obtained by (4.8). This implies that

$$||S_y A_x r||_\infty \geq \frac{||r||_\infty}{||S_y^{-1}||_\infty ||A_x^{-1}||_\infty} = \frac{||r||_\infty}{(1+|y|)\max\left\{\frac{1}{x},\frac{1}{\sqrt{x}}\right\}} \geq \frac{R-\delta}{2e^{\delta}\left(1+\frac{\delta}{2}\right)^2} = \frac{R'+\delta}{2}.$$

It follows from (4.9) that $||z||_\infty \leq \frac{\delta}{2}$. Then we have

$$||z + S_y A_x r||_\infty \geq ||S_y A_x r||_\infty - ||z||_\infty \geq \frac{R'}{2}.$$

Thus, we conclude that $(p,q,r)\cdot(x,y,z) = (px, y+q\sqrt{x}, z+S_yA_xr) \notin Q_{R'}$. This completes the claim (4.6).

Next, let $\epsilon > 0$ be given. Since $\mathcal{SH}_{1,\psi}(\Lambda)$ possesses an upper frame bound and by Theorem 4.1, we have $D^+(\Lambda) < \infty$. Then it follows from Proposition 3.5 that

$$M = \sup_{(x,y,z)\in\mathbb{S}_1} \#(\Lambda \cap Q_h(x,y,z)) < \infty.$$

We also have for any $(p,q,r) \in \mathbb{S}_1$ that

$$\begin{aligned}\sup_{(x,y,z)\in\mathbb{S}_1} \#((p,q,r)^{-1}\Lambda \cap Q_h(x,y,z)) &= \sup_{(x,y,z)\in\mathbb{S}_1} \#\left(\Lambda \cap Q_h\left((p,q,r)^{-1}\cdot(x,y,z)\right)\right) \\ &\leq M < \infty. \end{aligned} \tag{4.10}$$

The sets $B_{j,k,m} = B_{j,k,m}(\delta)$ for some $\delta > 0$, given in the beginning of this section, is generated from relative separated sets. It follows from Lemma 3.4 that each point $(x,y,z) \in (p,q,r)^{-1}\Lambda\backslash Q_R$ must lie in some $B_{j,k,m}$, and the claim (4.6) established that this is true when $B_{j,k,m} \cap Q_{R'} = \emptyset$. Next, let us define

$$J := \{(j,k,m) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^2 : B_{j,k,m} \cap Q_{R'} = \emptyset\}.$$

Again, Lemma 3.4 implies that there are no element $B_{i,l,n}$ of $\{B_{j,k,m}\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ intersecting more than $24\left(e^{\frac{\delta}{2}}+\frac{1}{2}\right)\left(e^{\delta}+\frac{1}{2}\right)\left(e^{\frac{3\delta}{4}}+\frac{1}{2}\right)$ of the others. Since $\psi, g \in \mathcal{B}_0$, by Theorem 4.2 we have $\mathcal{SH}_{1,\psi}g \in W_{\mathbb{S}_1}(C,L^1) \subset W_{\mathbb{S}_1}(C,L^2)$. It follows from the definition of norm for $W_{\mathbb{S}_1}(C,L^2)$ that if R' is large enough, then

$$\sum_{(j,k,m)\in J} ||\mathcal{SH}_{1,\psi}g\cdot\chi_{B_{j,k,m}}||_\infty^2 < \frac{\epsilon}{M}. \tag{4.11}$$

By (4.10) and (4.11), this yields

$$\sum_{(x,y,z)\in(p,q,r)^{-1}\Lambda\setminus Q_R} |\mathcal{SH}_{1,\psi}g(x,y,z)|^2 \leq M \sum_{(j,k,m)\in J} \|\mathcal{SH}_{1,\psi}g\cdot\chi_{B_{j,k,m}}\|_\infty^2 < \epsilon.$$

This proves the lemma. □

Theorem 4.6. *(HAP) Let $\psi\in\mathcal{B}_0$ and $\Lambda\subset\mathbb{S}_1$ be given. If $\mathcal{SH}_{1,\psi}(\Lambda)$ is a frame for $L^2(\mathbb{R}^2)$ with frame bounds $0<A\leq B<\infty$. Then $\mathcal{SH}_{1,\psi}(\Lambda)$ satisfies the Strong HAP.*

Proof. First, we claim that $\mathcal{SH}_{1,\psi}(\Lambda)$ with $\psi\in\mathcal{B}_0$ fulfills the Strong HAP for any function in $\mathcal{B}_0$, and then extend this to all functions in $L^2(\mathbb{R}^2)$. Now, choosing $g\in\mathcal{B}_0$ and $\epsilon>0$, consider the frame expansion for $\sigma_1(p,q,r)g$:

$$\sigma_1(p,q,r)g = \sum_{(a,s,t)\in\Lambda} \langle\sigma_1(p,q,r)g,\sigma_1(a,s,t)\psi\rangle\widetilde{\psi}_{a,s,t},$$

with $\{\widetilde{\psi}_{a,s,t}:(a,s,t)\in\Lambda\}$ which is a dual frame for $\mathcal{SH}_{1,\psi}(\Lambda)$.
By using the fact that $\{\widetilde{\psi}_{a,s,t}\}_{(a,s,t)\in\Lambda}$ is also a frame with upper frame bound $\frac{1}{A}$, where A is the lower bound of the shearlet frame $\mathcal{SH}_{1,\psi}(\Lambda)$. We can now show the Strong HAP for any $g\in\mathcal{B}_0$:

$$\begin{aligned}
&\|\sigma_1(p,q,r)g - \sum_{(a,s,t)\in Q_R(p,q,r)\cap\Lambda} \langle\sigma_1(p,q,r)g,\sigma_1(a,s,t)\psi\rangle\widetilde{\psi}_{a,s,t}\|_2^2 \\
&\qquad = \left\|\sum_{(a,s,t)\in\Lambda\setminus Q_R(p,q,r)} \langle\sigma_1(p,q,r)g,\sigma_1(a,s,t)\psi\rangle\widetilde{\psi}_{a,s,t}\right\|_2^2 \\
&\qquad \leq \frac{1}{A}\sum_{(a,s,t)\in\Lambda\setminus Q_R(p,q,r)} \left|\langle g,\sigma_1\left((p,q,r)^{-1}\cdot(a,s,t)\right)\psi\rangle\right|^2 \\
&\qquad = \frac{1}{A}\sum_{(a,s,t)\in\Lambda\setminus Q_R(p,q,r)} \left|\mathcal{SH}_{1,\psi}g\left((p,q,r)^{-1}\cdot(a,s,t)\right)\right|^2 \\
&\qquad = \frac{1}{A}\sum_{(x,y,z)\in(p,q,r)^{-1}\Lambda\setminus Q_R} |\mathcal{SH}_{1,\psi}g\,(x,y,z)|^2 \leq \epsilon. \qquad (4.12)
\end{aligned}$$

The last inequality holds because of Lemma 4.5.

Now suppose that f is any function in $L^2(\mathbb{R}^2)$ and choose any $\epsilon>0$. Since $\mathcal{B}_0$ is dense in $L^2(\mathbb{R}^2)$, there exist $g\in\mathcal{B}_0$ such that

$$||f-g||_2 < \frac{\epsilon\sqrt{A}}{3\sqrt{B}}. \qquad (4.13)$$

Set $R=R(f,\epsilon)=R(g,\frac{\epsilon}{3})$. Then for any $(p,q,r)\in\mathbb{S}_1$, we have

$$\begin{aligned}
&||\sigma_1(p,q,r)f - \sum_{(a,s,t)\in Q_R(p,q,r)\cap\Lambda} \langle\sigma_1(p,q,r)f,\sigma_1(a,s,t)\psi\rangle\widetilde{\psi}_{a,s,t}||_2 \\
&\quad \leq ||\sigma_1(p,q,r)f-\sigma_1(p,q,r)g||_2 \\
&\qquad + \left\|\sigma_1(p,q,r)g - \sum_{(a,s,t)\in Q_R(p,q,r)\cap\Lambda} \langle\sigma_1(p,q,r)g,\sigma_1(a,s,t)\psi\rangle\widetilde{\psi}_{a,s,t}\right\|_2
\end{aligned}$$

$$+\left\| \sum_{(a,s,t)\in Q_R(p,q,r)\cap\Lambda} \langle \sigma_1(p,q,r)g - \sigma_1(p,q,r)f, \sigma_1(a,s,t)\psi\rangle \widetilde{\psi}_{a,s,t} \right\|_2 .$$

By (4.13), (4.12), and the frame property of $\{\widetilde{\psi}_{a,s,t}\}_{(a,s,t)\in\Lambda}$, we have that

$$\begin{aligned}
&||\sigma_1(p,q,r)f - \sum_{(a,s,t)\in Q_R(p,q,r)\cap\Lambda} \langle \sigma_1(p,q,r)f, \sigma_1(a,s,t)\psi\rangle \widetilde{\psi}_{a,s,t}||_2 \\
&\leq \frac{\epsilon\sqrt{A}}{3\sqrt{B}} + \frac{\epsilon}{3} + \left[\frac{1}{A}\sum_{(a,s,t)\in Q_R(p,q,r)\cap\Lambda} |\langle \sigma_1(p,q,r)g - \sigma_1(p,q,r)f, \sigma_1(a,s,t)\psi\rangle|^2\right]^{1/2} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \left[\frac{B}{A}\|\sigma_1(p,q,r)g - \sigma_1(p,q,r)f\|_2^2\right]^{1/2} < \epsilon.
\end{aligned}$$

The last inequality follows from $\{\sigma_1(a,s,t)\psi\}_{(a,s,t)\in\Lambda}$ possesses the upper frame bound B. This completes the proof of theorem. □

4.2.3 The Comparison Theorem for Irregular Shearlet Frames

In this subsection we will use the HAP to prove the comparison theorem between the density of shearlet frames and the density of the another shearlet frames.

First, we require the following lemma of trace formula. We omit the proof and refer to Gröchenig [40] or Lemma 2.2 in Chapter 2.

Lemma 4.7 ([40]). *(Trace Formula) Let T be a positive trace-class operator on a Hilbert space $\mathcal{H}$ and $\{h_k\}_{k\in J}$ be a frame with frame bounds $0 < A \leq B < \infty$. Then*

$$\frac{1}{B}\sum_{k\in J}\langle T h_k, h_k\rangle \leq tr(T) \leq \frac{1}{A}\sum_{k\in J}\langle T h_k, h_k\rangle.$$

Let $\psi, \phi \in L^2(\mathbb{R}^2)$, $\Lambda, \Delta \subset \mathbb{S}_1$, and $\mathcal{SH}_{1,\psi}(\Lambda) = \{\sigma_1(a,s,t)\psi : (a,s,t) \in \Lambda\}$ be a *given frame* whose density we want to study. Assume that $\mathcal{SH}_{1,\psi}(\Lambda)$ satisfies the Weak HAP. Furthermore, let $\mathcal{SH}_{1,\phi}(\Delta) = \{\sigma_1(a,s,t)\phi : (a,s,t) \in \Delta\}$ be a *reference shearlet frame* with frame bounds $0 < A \leq B < \infty$. Recall that we denote $\{\widetilde{\psi}_{a,s,t} : (a,s,t) \in \Lambda\}$ the dual frame for $\mathcal{SH}_{1,\psi}(\Lambda)$.

For $h > 0$ and each $(p,q,r) \in \mathbb{S}_1$, we consider the finite-dimensional subspaces:

$$\begin{aligned}
W(h,(p,q,r)) &= \text{span}\,\{\widetilde{\psi}_{a,s,t} : (a,s,t) \in (p,q,r)Q_h \cap \Lambda\} \\
V(h,(p,q,r)) &= \text{span}\,\{\sigma_1(a,s,t)\phi : (a,s,t) \in (p,q,r)Q_h \cap \Delta\}.
\end{aligned}$$

For any fixed $\epsilon > 0$, let $R = R(\phi,\epsilon)$ be the value such that $\mathcal{SH}_{1,\psi}(\Lambda)$ fulfills the Weak HAP, and let $(a,s,t) \in (p,q,r)Q_h$. If $(x,y,z) \in (a,s,t)Q_R \cap \Lambda$, then

$$(x,y,z) \in (a,s,t)Q_R \subset (p,q,r)Q_hQ_R,$$

and since $Q_hQ_R \subset Q_{R+he^{\frac{R}{2}}+Rhe^{\frac{R}{4}}}$, we have $(x,y,z) \in (p,q,r)Q_{R+he^{\frac{R}{2}}+Rhe^{\frac{R}{4}}} \cap \Lambda$, and this implies

$$W(R,(a,s,t)) \subset W(R + he^{\frac{R}{2}} + Rhe^{\frac{R}{4}}, (p,q,r)). \tag{4.14}$$

This is valid for any $(p,q,r) \in \mathbb{S}_1, h > 0$ and $(a,s,t) \in Q_h(p,q,r)$.

Let P_V and P_W be the orthogonal projections of $L^2(\mathbb{R}^2)$ onto $V(h,(p,q,r))$ and $W(R+he^{\frac{R}{2}}+Rhe^{\frac{R}{4}},(p,q,r))$, respectively. Finally, let $T = P_V P_W P_V : V(h,(p,q,r)) \to V(h,(p,q,r))$. Then T is a positive self-adjoint on $L^2(\mathbb{R}^2)$.

Now, we are ready to state the Comparison Theorem for irregular shearlet frames:

Theorem 4.8. *(Comparison Theorem) With the notations we introduced above, set $C = ||\phi||_2$. Then for each $\epsilon > 0$, we have*

$$\frac{C(C-\epsilon)}{B(e^{\frac{R}{2}}+Re^{\frac{R}{2}})^4} D^-(\Delta) \le D^-(\Lambda) \ \textit{and} \ \frac{C(C-\epsilon)}{B(e^{\frac{R}{2}}+Re^{\frac{R}{2}})^4} D^+(\Delta) \le D^+(\Lambda).$$

Proof. Since $\{\sigma_1(a,s,t)\phi : (a,s,t) \in \Delta\}$ is a frame with frame bounds $0 < A \le B < \infty$, by Lemma 4.7 we have

$$\begin{aligned}
\operatorname{tr}(T) &\ge \frac{1}{B} \sum_{(a,s,t)\in\Delta} \langle T(\sigma_1(a,s,t)\phi), \sigma_1(a,s,t)\phi\rangle \\
&\ge \frac{1}{B} \sum_{(a,s,t)\in(p,q,r)Q_h\cap\Delta} \langle T(\sigma_1(a,s,t)\phi), \sigma_1(a,s,t)\phi\rangle \\
&= \frac{1}{B} \sum_{(a,s,t)\in(p,q,r)Q_h\cap\Delta} \langle P_V P_W P_V(\sigma_1(a,s,t)\phi), \sigma_1(a,s,t)\phi\rangle \\
&= \frac{1}{B} \sum_{(a,s,t)\in(p,q,r)Q_h\cap\Delta} \langle P_W P_V(\sigma_1(a,s,t)\phi), P_V\sigma_1(a,s,t)\phi\rangle \\
&= \frac{1}{B} \sum_{(a,s,t)\in(p,q,r)Q_h\cap\Delta} \langle P_W(\sigma_1(a,s,t)\phi), \sigma_1(a,s,t)\phi\rangle \\
&= \frac{1}{B} \sum_{(a,s,t)\in(p,q,r)Q_h\cap\Delta} \left[\langle\sigma_1(a,s,t)\phi, \sigma_1(a,s,t)\phi\rangle - \langle(P_W - I)(\sigma_1(a,s,t)\phi), \sigma_1(a,s,t)\phi\rangle\right].
\end{aligned}$$

By Cauchy-Schwarz inequality, (4.14), and applying the Weak HAP, we estimate the second inner product by

$$\begin{aligned}
|\langle(P_W - I)(\sigma_1(a,s,t)\phi), \sigma_1(a,s,t)\phi\rangle| &\le \|(P_W - I)\sigma_1(a,s,t)\phi\|_2 \cdot \|\sigma_1(a,s,t)\phi\|_2 \\
&= \operatorname{dist}\left(\sigma_1(a,s,t)\phi, W(R+he^{\frac{R}{2}}+Rhe^{\frac{R}{4}},(p,q,r))\right) \cdot ||\phi||_2 \\
&\le \operatorname{dist}\left(\sigma_1(a,s,t)\phi, W\left(R,(a,s,t)\right)\right) \cdot ||\phi||_2 \\
&\le \epsilon||\phi||_2 \le \epsilon C.
\end{aligned}$$

This yields a lower bound for the trace

$$\begin{aligned}
\operatorname{tr}(T) &\ge \frac{1}{B} \sum_{(a,s,t)\in(p,q,r)Qh\cap\Delta} C(C-\epsilon) \\
&= \frac{C(C-\epsilon)}{B} \#(Q_h(p,q,r)\cap\Delta).
\end{aligned}$$

On the other hand, since T is a product of the orthogonal projections, its eigenvalues are between 0 and 1, and

$$\begin{aligned}\operatorname{tr}(T) \leq \operatorname{rank}(T) &\leq \dim\left(W(R + he^{\frac{R}{2}} + Rhe^{\frac{R}{4}}, (p,q,r))\right) \\ &\leq \#\left((p,q,r)Q_{R+he^{\frac{R}{2}}+Rhe^{\frac{R}{4}}} \cap \Lambda\right).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{C(C-\epsilon)}{B} \cdot \#(Q_h(p,q,r) \cap \Delta) &\leq \#\left(Q_{R+he^{\frac{R}{2}}+Rhe^{\frac{R}{4}}}(p,q,r) \cap \Lambda\right) \\ &\leq \frac{\#\left(Q_{R+he^{\frac{R}{2}}+Rhe^{\frac{R}{4}}}(p,q,r) \cap \Lambda\right)}{(R + he^{\frac{R}{2}} + Rhe^{\frac{R}{4}})^4} \cdot \frac{(R + he^{\frac{R}{2}} + Rhe^{\frac{R}{4}})^4}{h^4}.\end{aligned}$$

Taking the infimum or supremum over all points $(p,q,r) \in \mathbb{S}_1$, and then take lim inf or lim sup as $h \to \infty$, we obtain

$$\frac{C(C-\epsilon)}{B} D^-(\Delta) \leq D^-(\Lambda)(e^{\frac{R}{2}} + Re^{\frac{R}{4}})^4, \text{ and } \frac{C(C-\epsilon)}{B} D^+(\Delta) \leq D^+(\Lambda)(e^{\frac{R}{2}} + Re^{\frac{R}{4}})^4,$$

respectively. Thus

$$\frac{C(C-\epsilon)}{B(e^{\frac{R}{2}} + Re^{\frac{R}{4}})^4} D^-(\Delta) \leq D^-(\Lambda), \text{ and } \frac{C(C-\epsilon)}{B(e^{\frac{R}{2}} + Re^{\frac{R}{4}})^4} D^+(\Delta) \leq D^+(\Lambda).$$

□

4.3 Existence of a Lower Frame Bound

Heil and Kutyniok [51], [52] and Kutyniok [60] gave two approaches to derive necessary density conditions for the existence of a lower frame bound for the unweighted wavelet frames $\mathcal{W}_\psi(\Lambda)$. The first approach (Heil and Kutyniok [51]) was restricted to the sequence of time-scale parameters Λ, without any restriction on the choice of generating wavelets ψ. In this approach, the authors employed the fact that the continuous wavelet transform can be realized as the *Bergman transform* on the upper half plane as one of the important method of the proof. Unfortunately, it is not clear whether the continuous shearlet transform possesses this property. On the other hand, without any restrictions on Λ, in the second approach [52], Heil and Kutyniok applied the HAP and the Comparison Theorem for irregular for irregular wavelet frames to derive necessary density conditions for irregular wavelet systems to possess a lower frame bound.

In the previous section, we have already shown that irregular shearlet frames fulfill the HAP. In this section, by using the consequence of the HAP, the Comparison Theorem, we will establish necessary density conditions for existence of a lower frame bound.

Theorem 4.9. *Let $\psi \in L^2(\mathbb{R}^2)$ and $\Lambda \subset \mathbb{S}_1$ such that the shearlet system $\mathcal{SH}_{1,\psi}(\Lambda)$ is a frame for $L^2(\mathbb{R}^2)$ and satisfies the Weak HAP. Then*

$$D^-(\Lambda) > 0.$$

Proof. Let $\phi \in L^2(\mathbb{R}^2)$ with $C := ||\phi||_2$, and $\Delta = \{(a^j, bk, cm) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ with $a > 1$ and $b, c > 0$ be given. Suppose that the shearlet system $\mathcal{SH}_{1,\phi}(\Delta)$ is a frame with frame bounds $0 < A \leq B < \infty$. We showed in Lemma 3.6 that $\mathcal{SH}_{1,\phi}(\Delta)$ possesses a uniform density, $D(\Delta) = \frac{1}{bc^2 \ln a}$. Therefore, by applying the Comparison Theorem 4.8 to $\mathcal{SH}_{1,\psi}(\Lambda)$ and $\mathcal{SH}_{1,\phi}(\Delta)$, we have for any $0 < \epsilon < C$:

$$D^-(\Lambda) \geq \frac{1}{bc^2 \ln a} \cdot \frac{C(C-\epsilon)}{B(e^{\frac{R}{2}} + Re^{\frac{R}{4}})^4} > 0.$$

□

Chapter 5

Construction of Irregular Shearlet Frames

In previous constructions of shearlet frames by Guo, Kutyniok and Labate [45], Guo and Labate [46], Kutyniok and Labate [62] and Labate et al. [64], shearlet frames are constructed using only a *band-limited* generating function ψ, i.e., $\hat{\psi}$ has compact support.

In this chapter, we introduce an irregular discrete subset Λ of $\mathbb{S}_1 = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ for which the corresponding shearlet system $\mathcal{SH}_{1,\psi}(\Lambda)$ generated by a certain decay function $\psi \in L^2(\mathbb{R}^2)$ forms a frame for $L^2(\mathbb{R}^2)$ and provide explicit estimates for frame bounds. Furthermore, we study the perturbation of the translation parameter.

5.1 Construction of Irregular Shearlet Frames

Let us begin with the definition of irregular shearlet systems as introduced by Kutyniok and Labate [62],

$$\begin{aligned}\mathcal{SH}_{3,\psi}(\Lambda_3) = \{T_{S_{\tilde{s}_{jk}}A_{\tilde{a}_j}c_0m}D_{S_{\tilde{s}_{jk}}A_{\tilde{a}_j}}\psi(x) &= D_{S_{\tilde{s}_{jk}}A_{\tilde{a}_j}}T_{c_0m}\psi(x)\\ &= \tilde{a}_j^{-3/2}\psi(A_{\tilde{a}_j}^{-1}S_{\tilde{s}_{jk}}^{-1}x - c_0m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \quad (5.1)\end{aligned}$$

where $\Lambda_3 = \big\{(\tilde{a}_j, \tilde{s}_{jk}, S_{\tilde{s}_{jk}}A_{\tilde{a}_j}c_0m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}\big\}$, with $\tilde{a}_j \in \mathbb{R}^+, \tilde{s}_{jk} \in \mathbb{R}$ and $c_0 > 0$.

By choosing $\tilde{a}_j = a_j^{-1}$ and $\tilde{s}_{jk} = -\frac{s_k}{\sqrt{a_j}}$, where $s_k \in \mathbb{R}$, for $j,k \in \mathbb{Z}$, we obtain

$$\begin{aligned}\mathcal{SH}_{3,\psi}(\Lambda_3) &= \left\{a_j^{3/2}\psi\left(A_{a_j}S_{\frac{s_k}{\sqrt{a_j}}}x - c_0m\right) = a_j^{3/2}\psi\left(S_{s_k}A_{a_j}x - c_0m\right) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\right\}\\ &= \mathcal{SH}_{1,\psi}(\Lambda),\end{aligned}$$

where $\Lambda = \{(a_j, s_k, c_0m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\} \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$.

Kutyniok and Labate [62] proved that the irregular shearlet system (5.1) forms a frame for $L^2(\mathbb{R}^2)$, provided that ψ is a band-limited function.

In this section, we replace the compact support assumptions on $\hat{\psi}$ by a certain mild decay conditions on $\hat{\psi}$, and derive sufficient conditions for irregular shearlet systems $\mathcal{SH}_{1,\psi}(\Lambda)$:

$$\mathcal{SH}_{1,\psi}(\Lambda) = \{\psi_{j,k,m} = a_j^{3/4}\psi(S_{s_k}A_{a_j}x - c_0m) : j,k \in \mathbb{Z},\, m \in \mathbb{Z}^2\}$$

to be frames for $L^2(\mathbb{R}^2)$. Many ideas and proofs of our work are inspired by Yang and Zhou [81], Kutyniok and Labate [62] and Daubechies [21].

In the sequel, we consider irregular shearlet systems $\mathcal{SH}_{1,\psi}(\Lambda)$ with $\psi \in L^2(\mathbb{R}^2)$ satisfying

$$|\hat{\psi}(\xi_1, \xi_2)| \leq \frac{C_1 \min\{|\xi_1|^{\gamma+\alpha}, |\xi_1|^{\gamma-\beta}\}}{(1+\xi_1^2+\xi_2^2)^{\gamma/2}}, \tag{5.2}$$

where $\alpha > 0, \gamma > 4$, $2\beta > \gamma$, and

$$\Lambda = \{(a_j, s_k, c_0 m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \quad c_0 > 0.$$

We assume that the sequence $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$ is increasing such that for each $\mu \in (0,1)$ there exists $p \in \mathbb{Z}^+$ such that

$$\frac{a_j}{a_{j+p}} \;<\; \mu \qquad \text{for all } j \in \mathbb{Z}. \tag{5.3}$$

We also assume that the sequence $\{s_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ satisfies

$$\sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{[1+(s_k+x)^2]^{\gamma/4}} < C_0 < \infty \qquad \text{for any } \gamma > 4. \tag{5.4}$$

The following lemma gives useful properties of a sequence $\{a_j\}_{j\in\mathbb{Z}}$ satisfying (5.3), and they will be used for subsequent results. We omit its proof and refer to Lemma 2.1 in Yang and Zhou [81].

Lemma 5.1 ([81]). *If $\mu \in (0,1), p \in \mathbb{Z}^+$ and $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$ is an increasing sequence satisfying (5.3), then for all $\alpha > 0$ and $t > 0$*

$$\sum_{a_j \leq t} a_j^{\alpha} \leq p\, t^{\alpha} \frac{1}{1-\mu^{\alpha}}, \qquad \sum_{a_j \geq t} a_j^{-\alpha} \leq p\, t^{-\alpha} \frac{1}{1-\mu^{\alpha}}.$$

In order to derive a sufficient condition for an irregular shearlet system $\mathcal{SH}_{1,\psi}(\Lambda)$ to form a frame, we need the following technical lemmas.

Lemma 5.2. *Retain the same assumption as in (5.2), (5.3) and (5.4). We let*

$$\Gamma(\omega) = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1}\xi)\right| \left|\hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1}\xi + 2\omega)\right| \quad a.e.\ \omega \in \mathbb{R}^2.$$

Then the following estimates hold :

$$(i) \quad M := \quad \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1}\xi)\right|^2 < \infty, \tag{5.5}$$

$$(ii) \quad R := \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma(\frac{1}{c_0}m)\Gamma(-\frac{1}{c_0}m)\right]^{1/2} \leq \; \tilde{C} c_0^{\frac{\gamma}{2}} \qquad \text{for any } c_0 > 0, \tag{5.6}$$

where $\tilde{C} = 2^{\frac{\gamma}{2}} C_0 C_1^2 \left[\left(\frac{p}{1-\mu^{\beta-\frac{\gamma}{2}}} + \frac{p}{1-\mu^{\alpha}}\right) + \frac{p}{1-\mu^{\beta}}\right] \cdot \sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{-(\frac{\gamma}{2}-1)}.$

Proof. First we prove that for any $\xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2$ where $\xi_1 \neq 0$, there exists a positive integer j_0 such that $\left|\frac{\xi_1}{a_{j_0+1}}\right| \leq 1 \leq \left|\frac{\xi_1}{a_{j_0}}\right|$. Since $\frac{a_j}{a_{j+p}} < \mu$ for any $j \in \mathbb{Z}$, it follows from Lemma 5.1 and (5.4) that

$$\begin{aligned} M &= \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1}\xi)\right|^2 = \left(\sum_{j\leq j_0} + \sum_{j\geq j_0+1}\right) \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(a_j^{-1}\xi_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2)\right|^2 \\ &\leq C_1^2 \left[\sum_{j\leq j_0} |a_j^{-1}\xi_1|^{-2\beta} \sum_{k\in\mathbb{Z}} \frac{1}{\left(\frac{1}{(a_j^{-1}\xi_1)^2} + 1 + (-s_k + \sqrt{a_j}\frac{\xi_2}{\xi_1})^2\right)^\gamma} \right. \\ &\quad \left. + \sum_{j\geq j_0+1} |a_j^{-1}\xi_1|^{2\alpha} \sum_{k\in\mathbb{Z}} \frac{1}{\left(\frac{1}{(a_j^{-1}\xi_1)^2} + 1 + (-s_k + \sqrt{a_j}\frac{\xi_2}{\xi_1})^2\right)^\gamma} \right] \\ &\leq C_1^2 \left[\frac{p}{|a_{j_0}^{-1}\xi_1|^{2\beta}(1-\mu^{2\beta})} + \frac{p|a_{j_0+1}^{-1}\xi_1|^{2\alpha}}{(1-\mu^{2\alpha})} \right] \sup_{\widetilde{\xi}\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\left(1 + (-s_k + \widetilde{\xi})^2\right)^\gamma} \\ &\leq C_0 C_1^2 \left[\frac{p}{(1-\mu^{2\beta})} + \frac{p}{(1-\mu^{2\alpha})} \right] < \infty. \end{aligned}$$

This proves the first equality (5.5).

In order to prove the second inequality (5.6), we note that for $(\xi_1, \xi_2)^T \in \mathbb{R}^2, \xi_1 \neq 0$ and $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$, the set $\{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_2 = 0 \text{ and } a_j^{-1}\xi_1 = -2\omega_1 \text{ for some } j \in \mathbb{Z}\}$ has measure zero. Therefore, we may assume that $a_j^{-1}\xi_1 + 2\omega_1 \neq 0$. We estimate $\Gamma(2\omega)$:

$$\begin{aligned} \Gamma(2\omega_1, 2\omega_2) &= \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}\left(a_j^{-1}\xi_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2\right)\right| \\ &\qquad \cdot \left|\hat{\psi}\left(a_j^{-1}\xi_1 + 2\omega_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + 2\omega_2\right)\right| \\ &= \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \left(\sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|<||\omega||_\infty\}} + \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\geq||\omega||_\infty\}} \right) \sum_{k\in\mathbb{Z}} \left|\hat{\psi}\left(a_j^{-1}\xi_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2\right)\right| \\ &\qquad \cdot \left|\hat{\psi}\left(a_j^{-1}\xi_1 + 2\omega_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + 2\omega_2\right)\right| \\ &= \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2}(I_1 + I_2), \end{aligned}$$

where

$$I_1 := \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\leq||\omega||_\infty\}} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}\left(a_j^{-1}\xi_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2\right)\right| \left|\hat{\psi}\left(a_j^{-1}\xi_1 + 2\omega_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + 2\omega_2\right)\right|$$

$$I_2 := \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|>||\omega||_\infty\}} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}\left(a_j^{-1}\xi_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2\right)\right| \left|\hat{\psi}\left(a_j^{-1}\xi_1 + 2\omega_1, -s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + 2\omega_2\right)\right|.$$

Let us first estimate I_1 :

$$I_1 \le C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\le||\omega||_\infty\}} \min\{|a_j^{-1}\xi_1|^\alpha, |a_j^{-1}\xi_1|^{-\beta}\} \cdot \min\left\{\left|a_j^{-1}\xi_1+2\omega_1\right|^\alpha, \left|a_j^{-1}\xi_1+2\omega_1\right|^{-\beta}\right\}$$

$$\cdot \sum_{k\in\mathbb{Z}} \frac{1}{\left[\frac{1}{(a_j^{-1}\xi_1)^2}+1+\left(-s_k+\sqrt{a_j}\frac{\xi_2}{\xi_1}\right)^2\right]^{\frac{\gamma}{2}}} \frac{\left|1+\frac{2\omega_1}{a_j^{-1}\xi_1}\right|^\gamma}{\left[\frac{1}{(a_j^{-1}\xi_1)^2}+\left(1+\frac{2\omega_1}{a_j^{-1}\xi_1}\right)^2+\left(-s_k+\sqrt{a_j}\frac{\xi_2}{\xi_1}+\frac{2\omega_2}{a_j^{-1}\xi_1}\right)^2\right]^{\frac{\gamma}{2}}}. \quad (5.7)$$

Case 1 : $||\omega||_\infty = |\omega_1| \ge |a_j^{-1}\xi_1|$, we have $|a_j^{-1}\xi_1+2\omega_1| \ge |\omega_1|$. Straightforward computation gives

$$\begin{aligned} I_1 &\le C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\le|\omega_1|\}} \min\{|a_j^{-1}\xi_1|^\alpha, |a_j^{-1}\xi_1|^{-\beta}\} \left|a_j^{-1}\xi_1+2\omega_1\right|^{-\beta} \sum_{k\in\mathbb{Z}} \left[1+(-s_k+\sqrt{a_j}\frac{\xi_2}{\xi_1})^2\right]^{-\gamma/2} \\ &\le C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\le|\omega_1|\}} \min\{|a_j^{-1}\xi_1|^\alpha, |a_j^{-1}\xi_1|^{-\beta}\} \frac{1}{|\omega_1|^\beta} \cdot \sup_{\tilde{\xi}\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\left[1+(-s_k+\tilde{\xi})^2\right]^{\gamma/2}} \\ &\le \frac{C_0C_1^2}{||\omega||_\infty^\beta} \sum_{j\in\mathbb{Z}} \min\{|a_j^{-1}\xi_1|^\alpha, |a_j^{-1}\xi_1|^{-\beta}\}. \end{aligned} \quad (5.8)$$

Case 2 : $||\omega||_\infty = |\omega_2| \ge |a_j^{-1}\xi_1|$ and $|a_j^{-1}\xi_1| > |\omega_1|$. One can easily show that for any $x, y, z \in \mathbb{R}, y \ne 0$ and $z \ne -1$, the following inequality holds for some $\epsilon > 1$

$$\frac{(1+z)^2}{(1+x^2)((1+z)^2+(x+y)^2)} \le \frac{\epsilon\max\{1,(1+z)^2\}}{y^2}. \quad (5.9)$$

Therefore, by using (5.9), the term I_1 as in (5.7) can be estimated as follows:

$$\begin{aligned} I_1 &\le C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\le|\omega_2|\}} \min\{|a_j^{-1}\xi_1|^\alpha, |a_j^{-1}\xi_1|^{-\beta}\} \cdot \min\left\{\left|a_j^{-1}\xi_1+2\omega_1\right|^\alpha, \left|a_j^{-1}\xi_1+2\omega_1\right|^{-\beta}\right\} \\ &\quad \cdot \sum_{k\in\mathbb{Z}} \left[1+\left(-s_k+\sqrt{a_j}\frac{\xi_2}{\xi_1}\right)^2\right]^{-\gamma/4} \cdot \frac{\max\left\{1, \left|1+\frac{2\omega_1}{a_j^{-1}\xi_1}\right|^{\gamma/2}\right\}}{\left|\frac{2\omega_2}{a_j^{-1}\xi_1}\right|^{\gamma/2}} \\ &\le C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\le|\omega_2|\}} \min\{|a_j^{-1}\xi_1|^\alpha, |a_j^{-1}\xi_1|^{-\beta}\} \cdot \min\left\{\left|a_j^{-1}\xi_1+2\omega_1\right|^\alpha, \left|a_j^{-1}\xi_1+2\omega_1\right|^{-\beta}\right\} \\ &\quad \cdot \sup_{\tilde{\xi}\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\left[1+(-s_k+\tilde{\xi})^2\right]^{\gamma/4}} \cdot \frac{\max\left\{1, \left|1+\frac{2\omega_1}{a_j^{-1}\xi_1}\right|^{\gamma/2}\right\}}{\left|\frac{2\omega_2}{a_j^{-1}\xi_1}\right|^{\gamma/2}}. \end{aligned}$$

• If $1 \leq \left|1 + \frac{2\omega_1}{a_j^{-1}\xi_1}\right|$, then we have

$$I_1 \leq C_0 C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\leq|\omega_2|\}} \min\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\} \cdot \min\left\{\left|a_j^{-1}\xi_1 + 2\omega_1\right|^{\alpha+\frac{\gamma}{2}}, \left|a_j^{-1}\xi_1 + 2\omega_1\right|^{-\beta+\frac{\gamma}{2}}\right\} \cdot \frac{1}{|2\omega_2|^{\frac{\gamma}{2}}}.$$

For any $\xi \in \mathbb{R} \setminus \{0\}$, α, $\gamma > 4$ and $2\beta > \gamma$, we have that $\min\{|\xi|^{\alpha+\frac{\gamma}{2}}, |\xi|^{-\beta+\frac{\gamma}{2}}\} \leq 1$. Then we obtain

$$I_1 \leq \frac{C_0 C_1^2}{||2\omega||_\infty^{\frac{\gamma}{2}}} \sum_{j\in\mathbb{Z}} \min\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\}. \tag{5.10}$$

• If $1 \geq \left|1 + \frac{2\omega_1}{a_j^{-1}\xi_1}\right|$, then we obtain

$$\begin{aligned} I_1 &\leq C_0 C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|\leq|\omega_2|\}} \min\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\} \cdot \frac{|a_j^{-1}\xi_1|^{\frac{\gamma}{2}}}{|2\omega_2|^{\frac{\gamma}{2}}} \\ &\leq \frac{C_0 C_1^2}{||2\omega||_\infty^{\frac{\gamma}{2}}} \sum_{j\in\mathbb{Z}} \min\left\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\right\} \cdot |a_j^{-1}\xi_1|^{\frac{\gamma}{2}}. \end{aligned} \tag{5.11}$$

Therefore, by (5.8), (5.10) and (5.11), we can conclude that

$$I_1 \leq \frac{C_0 C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}} \sum_{j\in\mathbb{Z}} \min\left\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\right\} \cdot \max\{1, |a_j^{-1}\xi_1|^{\frac{\gamma}{2}}\}.$$

For $\xi_1 \neq 0$, there exists a positive integer j_0 such that $\left|\frac{\xi_1}{a_{j_0+1}}\right| \leq 1 \leq \left|\frac{\xi_1}{a_{j_0}}\right|$.

$$\begin{aligned} I_1 &\leq \frac{C_0 C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}} \sum_{j\in\mathbb{Z}} \min\left\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\right\} \cdot \max\{1, |a_j^{-1}\xi_1|^{\frac{\gamma}{2}}\} \\ &= \frac{C_0 C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}} \left[\sum_{j\leq j_0} + \sum_{j\geq j_0+1}\right] \min\left\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\right\} \cdot \max\{1, |a_j^{-1}\xi_1|^{\frac{\gamma}{2}}\}. \end{aligned}$$

Since $\{a_j\}_{j\in\mathbb{Z}}$ is an increasing sequence, we have $|a_j^{-1}\xi_1| \geq |a_{j_0}^{-1}\xi_1| \geq 1$ for any $j \leq j_0$. On the other hand, for any $j \geq j_0 + 1$ we obtain that $|a_j^{-1}\xi_1| \leq |a_{j_0+1}^{-1}\xi_1| \leq 1$. Therefore by Lemma 5.1, we can complete the estimate of I_1 as follows:

$$\begin{aligned} I_1 &\leq \frac{C_0 C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}} \left[\sum_{j\leq j_0} \min\left\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\right\} \cdot |a_j^{-1}\xi_1|^{\frac{\gamma}{2}} + \sum_{j\geq j_0+1} \min\left\{|a_j^{-1}\xi_1|^{\alpha}, |a_j^{-1}\xi_1|^{-\beta}\right\}\right] \\ &\leq \frac{C_0 C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}} \left[\sum_{j\leq j_0} |a_j^{-1}\xi_1|^{-\beta+\frac{\gamma}{2}} + \sum_{j\geq j_0+1} |a_j^{-1}\xi_1|^{\alpha}\right] \\ &\leq \frac{C_0 C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}} \left[\frac{p}{(1-\mu^{\beta-\frac{\gamma}{2}})} + \frac{p}{(1-\mu^{\alpha})}\right]. \end{aligned} \tag{5.12}$$

Let us now estimate I_2. Again by using Lemma 5.1, we obtain that

$$\begin{aligned}
I_2 &\leq C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|>||\omega||_\infty\}} \min\{|a_j^{-1}\xi_1|^\alpha, |a_j^{-1}\xi_1|^{-\beta}\}\cdot \min\{|a_j^{-1}\xi_1+2\omega_1|^\alpha, |a_j^{-1}\xi_1+2\omega_1|^{-\beta}\} \\
&\quad \cdot \sum_{k\in\mathbb{Z}} \left[\frac{1}{(a_j^{-1}\xi_1)^2} + 1 + \left(-s_k + \sqrt{a_j}\frac{\xi_2}{\xi_1}\right)^2\right]^{-\gamma/2} \\
&\leq C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|>||\omega||_\infty\}} |a_j^{-1}\xi_1|^{-\beta}\cdot \sup_{\tilde{\xi}\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\left[1+(-s_k+\tilde{\xi})^2\right]^{\gamma/2}} \\
&\leq C_0C_1^2 \sum_{\{j\in\mathbb{Z}:|a_j^{-1}\xi_1|>||\omega||_\infty\}} |a_j^{-1}\xi_1|^{-\beta} \\
&\leq \frac{C_0C_1^2}{||\omega||_\infty^\beta}\frac{p}{1-\mu^\beta}.
\end{aligned} \tag{5.13}$$

By (5.12), (5.13) and since $2\beta > \gamma > 0$, we conclude that

$$\begin{aligned}
\Gamma(2\omega_1, 2\omega_2) &= \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2}(I_1+I_2) \\
&\leq \left[\frac{C_0C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}}\left(\frac{p}{1-\mu^{\beta-\frac{\gamma}{2}}} + \frac{p}{1-\mu^\alpha}\right) + \frac{C_0C_1^2}{||\omega||_\infty^\beta}\frac{p}{1-\mu^\beta}\right] \\
&\leq \frac{C_0C_1^2}{||\omega||_\infty^{\frac{\gamma}{2}}}\left[\left(\frac{p}{1-\mu^{\beta-\frac{\gamma}{2}}} + \frac{p}{1-\mu^\alpha}\right) + \frac{p}{1-\mu^\beta}\right].
\end{aligned}$$

Hence, for any $c_0 > 0$, we have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\,\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2} &\leq c_0^{\frac{\gamma}{2}} C_0C_1^2 \left[\left(\frac{p}{1-\mu^{\beta-\frac{\gamma}{2}}} + \frac{p}{1-\mu^\alpha}\right) + \frac{p}{1-\mu^\beta}\right]\cdot \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left\|\frac{m}{2}\right\|_\infty^{-\frac{\gamma}{2}} \\
&\leq 2^{\frac{\gamma}{2}} c_0^{\frac{\gamma}{2}} C_0C_1^2 \left[\left(\frac{p}{1-\mu^{\beta-\frac{\gamma}{2}}} + \frac{p}{1-\mu^\alpha}\right) + \frac{p}{1-\mu^\beta}\right]\cdot \sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{-(\frac{\gamma}{2}-1)} \\
&= \tilde{C}\, c_0^{\frac{\gamma}{2}},
\end{aligned}$$

where $\tilde{C} = 2^{\frac{\gamma}{2}} C_0C_1^2 \left[\left(\frac{p}{1-\mu^{\beta-\frac{\gamma}{2}}} + \frac{p}{1-\mu^\alpha}\right) + \frac{p}{1-\mu^\beta}\right]\cdot \sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{-(\frac{\gamma}{2}-1)}$.

This proves the last inequality. □

Lemma 5.3. *Suppose that $\{a_j\}_{j\in\mathbb{Z}}$, $\{s_k\}_{k\in\mathbb{Z}}$ and $\psi \in L^2(\mathbb{R}^2)$ satisfy the assumptions in Lemma 5.2. Then for all $f \in L^2(\mathbb{R}^2)$,*

$$\begin{aligned}
&\sum_{j,k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle\right|^2 \\
&\quad = \frac{1}{c_0^2}\int_{\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2} \hat{f}(\xi)\, \overline{\hat{f}\left(\xi + \frac{1}{c_0}A_{a_j}S_{s_k}^T m\right)\hat{\psi}\left(S_{s_k}^{-T}A_{a_j}^{-1}\xi\right)}\,\hat{\psi}\left(S_{s_k}^{-T}A_{a_j}^{-1}\xi + \frac{1}{c_0}m\right) d\xi.
\end{aligned}$$

Proof. Let $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$. For any $f \in L^2(\mathbb{R}^2)$, by Plancherel's Theorem we have

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle\right|^2 = \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} a_j^{-3/2} \left| \int_{\mathbb{R}^2} \hat{f}(\xi)\overline{\hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1}\xi)} e^{2\pi i\langle \xi, A_{a_j}^{-1} S_{s_k}^{-1} c_0 m\rangle}\, d\xi \right|^2$$

$$= \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \frac{a_j^{3/2}}{c_0^2} \left| \int_{\mathbb{R}^2} \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T \xi\right) \overline{\hat{\psi}\left(\frac{1}{c_0}\xi\right)} e^{2\pi i\langle \xi, m\rangle}\, d\xi \right|^2$$

$$= \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \frac{a_j^{3/2}}{c_0^2} \left| \sum_{l\in\mathbb{Z}^2} \int_{\Omega+l} \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T \xi\right) \overline{\hat{\psi}\left(\frac{1}{c_0}\xi\right)} e^{2\pi i\langle \xi, m\rangle}\, d\xi \right|^2$$

$$= \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \frac{a_j^{3/2}}{c_0^2} \left| \int_{\Omega} \sum_{l\in\mathbb{Z}^2} \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T (\xi+l)\right) \overline{\hat{\psi}\left(\frac{1}{c_0}(\xi+l)\right)} e^{2\pi i\langle \xi, m\rangle}\, d\xi \right|^2$$

$$= \sum_{j,k\in\mathbb{Z}} \frac{a_j^{3/2}}{c_0^2} \int_{\Omega} \left| \sum_{l\in\mathbb{Z}^2} \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T (\xi+l)\right) \overline{\hat{\psi}\left(\frac{1}{c_0}(\xi+l)\right)} \right|^2 d\xi$$

$$= \sum_{j,k\in\mathbb{Z}} \frac{a_j^{3/2}}{c_0^2} \int_{\Omega} \sum_{m,l\in\mathbb{Z}^2} \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T (\xi+l)\right)$$

$$\cdot \overline{\hat{\psi}\left(\frac{1}{c_0}(\xi+l)\right) \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T (\xi+m)\right)} \hat{\psi}\left(\frac{1}{c_0}(\xi+m)\right) d\xi$$

$$= \sum_{j,k\in\mathbb{Z}} \frac{a_j^{3/2}}{c_0^2} \sum_{l\in\mathbb{Z}^2} \int_{\Omega+l} \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T \xi\right) \overline{\hat{\psi}\left(\frac{1}{c_0}\xi\right)}$$

$$\cdot \sum_{m\in\mathbb{Z}^2} \overline{\hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T (\xi+m-l)\right)} \hat{\psi}\left(\frac{1}{c_0}(\xi+m-l)\right) d\xi$$

$$= \sum_{j,k\in\mathbb{Z}} \frac{a_j^{3/2}}{c_0^2} \int_{\mathbb{R}^2} \sum_{m\in\mathbb{Z}^2} \hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T \xi\right) \overline{\hat{f}\left(\frac{1}{c_0} A_{a_j} S_{s_k}^T (\xi+m)\right) \hat{\psi}\left(\frac{1}{c_0}\xi\right)} \hat{\psi}\left(\frac{1}{c_0}(\xi+m)\right) d\xi.$$

This implies

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle\right|^2$$

$$= \frac{1}{c_0^2} \int_{\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \hat{f}(\xi) \overline{\hat{f}\left(\xi + \frac{1}{c_0} A_{a_j} S_{s_k}^T m\right) \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)} \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m\right) d\xi.$$

This proves the lemma. ◻

We are now ready to state sufficient conditions for an irregular shearlet system to form a frame for $L^2(\mathbb{R}^2)$.

Theorem 5.4. *Suppose that* $\{a_j\}_{j\in\mathbb{Z}}, \{s_k\}_{k\in\mathbb{Z}}$ *and* $\psi \in L^2(\mathbb{R}^2)$ *satisfy assumptions in Lemma 5.2. Furthermore, define* $\Lambda = \{(a_j, s_k, c_0 m) : j,k\in\mathbb{Z}, m\in\mathbb{Z}^2\}$*. If*

$$L := \operatorname*{ess\,inf}_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1}\xi)\right|^2 \; > \; 0, \tag{5.14}$$

and $0 < c_0 < \left(\frac{L}{\tilde{C}}\right)^{-\frac{\gamma}{2}}$*, where* $\tilde{C}$ *is the constant from Lemma 5.2, then* $\mathcal{SH}_{1,\psi}(\Lambda)$ *is a frame for* $L^2(\mathbb{R}^2)$ *with frame bounds* $0 < A \le B < \infty$ *satisfying*

$$A = \frac{1}{c_0^2}[L - R], \qquad \text{and} \qquad B = \frac{1}{c_0^2}[M + R], \tag{5.15}$$

where $R = \tilde{C} c_0^{\frac{\gamma}{2}}$ *and* M *are constants from equations (5.6) and (5.5), respectively.*

Proof. By Lemma 5.3, for any $f \in L^2(\mathbb{R}^2)$, we have

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle\right|^2 = M(f) + R(f), \tag{5.16}$$

where

$$M(f) = \frac{1}{c_0^2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)\right|^2 d\xi,$$

$$R(f) = \frac{1}{c_0^2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \hat{f}(\xi)\, \overline{\hat{f}\left(\xi + \frac{1}{c_0} A_{a_j} S_k^T m\right) \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)} \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m\right) d\xi.$$

By using the Cauchy-Schwarz inequality, we obtain

$$|R(f)| \le \frac{1}{c_0^2} ||\hat{f}||^2 \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2}. \tag{5.17}$$

It follows from (5.16) and (5.17) that

$$\begin{aligned}
&\frac{1}{c_0^2}||\hat{f}||^2 \left[\operatorname*{ess\,inf}_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)\right|^2 - \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2}\right] \\
&\quad \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle\right|^2 \\
&\quad \le \frac{1}{c_0^2}||\hat{f}||^2 \left[\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \left|\hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)\right|^2 + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2}\right].
\end{aligned}$$

Thus, by (5.5), (5.6), we get

$$\frac{1}{c_0^2}||\hat{f}||^2 \left[L - \tilde{C} c_0^{\frac{\gamma}{2}}\right] \;\le\; \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle\right|^2 \;\le\; \frac{1}{c_0^2}||\hat{f}||^2 \left[M + \tilde{C} c_0^{\frac{\gamma}{2}}\right].$$

This proves the theorem. □

Remark 5.1. *The drawback of the construction of irregular shearlet frames in Theorem 5.4 is that it does not provide an explicit estimate for the lower frame bound.*

5.2 Construction of Irregular Shearlet Frames on the Cone

In this section, we are interested in the construction of frames for a space of functions whose Fourier transforms are supported in the horizontal cone

$$C := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \neq 0, \left| \frac{\xi_2}{\xi_1} \right| \leq 1 \right\}.$$

Retain the same assumptions on $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$, $\{s_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ and $\psi \in L^2(\mathbb{R}^2)$ in Lemma 5.2. We now discretize the continuous shearlet transform associated with $\mathbb{S}_1$. By choosing the irregular discrete subset Λ of $\mathbb{S}_1$ of the form

$$\Lambda = \{(a_j, s_k, c_0 m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\},$$

where $|s_k| \leq \sqrt{a_j}$ for all $j, k \in \mathbb{Z}$ and $c_0 > 0$, we obtain the following shearlet systems:

$$\widetilde{\mathcal{SH}}_{1,\psi}(\Lambda) = \{\psi_{j,k,m} = a_j^{3/2} \psi(S_{s_k} A_{a_j} \cdot - c_0 m) : |s_k| \leq \sqrt{a_j}, j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}. \tag{5.18}$$

With some modification of the proof Theorem 5.4, we obtain the following sufficient conditions:

Theorem 5.5. *Suppose that $\{a_j\}_{j\in\mathbb{Z}}, \{s_k\}_{k\in\mathbb{Z}}$ and $\psi \in L^2(\mathbb{R}^2)$ satisfy all assumptions as in Lemma 5.2. Furthermore, define $\Lambda = \{(a_j, s_k, c_0 m) : |s_k| \leq \sqrt{a_j}, j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$. If there exists $\tilde{L} > 0$ such that $0 < c_0 < \left(\frac{\tilde{L}}{\tilde{C}}\right)^{-\frac{\gamma}{2}}$ where $\tilde{C}$ is the constant from Lemma 5.2, and*

$$\tilde{L} \leq \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\leq\sqrt{a_j}\}} \left| \hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1} \xi) \right|^2 \qquad a.e.\ \xi \in C,$$

then $\widetilde{\mathcal{SH}}_{1,\psi}(\Lambda)$ is a frame for $L^2(C)^\vee := \{f \in L^2(\mathbb{R}^2) : supp\, \hat{f} \in C\}$ with frame bounds A and B satisfying

$$0 < \frac{1}{c_0^2}[\tilde{L} - R] \leq A \leq B \leq \frac{1}{c_0^2}[M + R], \tag{5.19}$$

where $R = \tilde{C} c_0^{\frac{\gamma}{2}}$ and M are constants from equations (5.6) and (5.5), respectively.

Proof. Let $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$. For any $f \in L^2(C)^\vee$,we have

$$\sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\leq\sqrt{a_j}\}} \sum_{m\in\mathbb{Z}^2} \left| \langle \hat{f}, \hat{\psi}_{j,k,m} \rangle_{L^2(C)} \right|^2$$

$$= \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\leq\sqrt{a_j}\}} \sum_{m\in\mathbb{Z}^2} a_j^{-3/2} \left| \int_{\mathbb{R}^2} \hat{f}(\xi) \chi_C(\xi) \overline{\hat{\psi}(S_{s_k}^{-T} A_{a_j}^{-1} \xi)} e^{2\pi i \langle \xi, A_{a_j}^{-1} S_{s_k}^{-1} c_0 m \rangle} \, d\xi \right|^2.$$

By arguments similar to the ones in Lemma 5.3, we get

$$\sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\leq\sqrt{a_j}\}} \sum_{m\in\mathbb{Z}^2} \left| \langle \hat{f}, \hat{\psi}_{j,k,m} \rangle_{L^2(C)} \right|^2$$

$$= \frac{1}{c_0^2} \int_C \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\le\sqrt{a_j}\}} \sum_{m\in\mathbb{Z}^2} \hat{f}(\xi)\, \overline{\hat{f}\left(\xi + \frac{1}{c_0} A_{a_j} S_{s_k}^T m\right) \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)} \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m\right) d\xi$$
$$= M(f) + R(f), \tag{5.20}$$

where

$$M(f) = \frac{1}{c_0^2} \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\le\sqrt{a_j}\}} \int_C \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)\right|^2 d\xi,$$

$$R(f) = \frac{1}{c_0^2} \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\le\sqrt{a_j}\}} \int_C \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \hat{f}(\xi)\, \overline{\hat{f}\left(\xi + \frac{1}{c_0} A_{a_j} S_k^T m\right) \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)} \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m\right) d\xi.$$

By using the Cauchy-Schwarz inequality, we obtain

$$|R(f)| \le \frac{1}{c_0^2} ||\hat{f}||_{L^2(C)}^2 \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2}. \tag{5.21}$$

It follows from (5.20) and (5.21) that

$$\frac{1}{c_0^2} ||\hat{f}||_{L^2(C)}^2 \left[\operatorname*{ess\,inf}_{\xi\in\mathbb{R}^2} \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\le\sqrt{a_j}\}} \left|\hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)\right|^2 - \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2}\right]$$
$$\le \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\le\sqrt{a_j}\}} \sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle_{L^2(C)}\right|^2$$
$$\le \frac{1}{c_0^2} ||\hat{f}||_{L^2(C)}^2 \left[\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\le\sqrt{a_j}\}} \left|\hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)\right|^2 + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2}\right].$$

Thus, by (5.5), (5.6), we get

$$\frac{1}{c_0^2} ||\hat{f}||_{L^2(C)}^2 \left[L - \tilde{C} c_0^{\frac{\gamma}{2}}\right] \le \sum_{j\in\mathbb{Z}} \sum_{\{k:|s_k|\le\sqrt{a_j}\}} \sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{\psi}_{j,k,m}\rangle_{L^2(C)}\right|^2 \le \frac{1}{c_0^2} ||\hat{f}||_{L^2(C)}^2 \left[M + \tilde{C} c_0^{\frac{\gamma}{2}}\right].$$

Hence $\widetilde{\mathcal{SH}}_{1,\psi}(\Lambda)$ is a frame for $L^2(C)^\vee$ with the same estimates for the frame bounds as in Theorem 5.4. □

5.3 Construction of Regular Shearlet Frames

In this section, we apply our results from Section 5.1 to derive sufficient conditions for a regular shearlet system $\mathcal{SH}_{1,\psi}(\Gamma)$ to be a frame. In particular, we choose a discrete subset Γ of $\mathbb{S}_1$ as

$$\Gamma = \{(a^j, k, c_0 m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\},$$

where $a > 1$ and $c_0 > 0$. In a fashion similar to previous constructions of shearlet frames by Guo, Kutyniok and Labate [45], Guo and Labate [46], and Labate et al. [64], we choose $\psi \in L^2(\mathbb{R}^2)$ such that

$$\hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right) \qquad \text{a.e. } \xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2,$$

where $\psi_1 \in L^2(\mathbb{R})$ is usually a wavelet, and $\psi_2 \in L^2(\mathbb{R})$.

Furthermore, inspired by Daubechies [21], we show some concrete examples of regular shearlet frames and numerical estimates of the frame bounds of those frames. We start with the following technical lemma.

Lemma 5.6. *For any $x \in \mathbb{R}$ and $\gamma > 0$,*

$$\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} - 1 \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k+x)^2)^\gamma} \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} + 1.$$

Proof. We consider for any $0 \le x \le 1$ (other values of x can be shifted into this range by translating with a suitable integer k). We therefore obtain that

$$\sum_{k\in\mathbb{Z}} \frac{1}{(1+\max\{k^2,(k+1)^2\})^\gamma} \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k+x)^2)^\gamma} \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+\min\{k^2,(k+1)^2\})^\gamma}$$

$$\begin{aligned}
\iff \quad & \sum_{k\in-\mathbb{N}} \frac{1}{(1+k^2)^\gamma} + \sum_{k\in\mathbb{N}\cup\{0\}} \frac{1}{(1+(k+1)^2)^\gamma} \\
&\qquad \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k+x)^2)^\gamma} \le \sum_{k\in-\mathbb{N}} \frac{1}{(1+(k+1)^2)^\gamma} + \sum_{k\in\mathbb{N}\cup\{0\}} \frac{1}{(1+k^2)^\gamma} \\
\iff \quad & \sum_{k\in-\mathbb{N}} \frac{1}{(1+k^2)^\gamma} + \sum_{k\in\mathbb{N}} \frac{1}{(1+k^2)^\gamma} \\
&\qquad \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k+x)^2)^\gamma} \le \sum_{k\in-\mathbb{N}\cup\{0\}} \frac{1}{(1+k^2)^\gamma} + \sum_{k\in\mathbb{N}\cup\{0\}} \frac{1}{(1+k^2)^\gamma} \\
\iff \quad & \sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} - 1 \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+(k+x)^2)^\gamma} \le \sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} + 1.
\end{aligned}$$

□

The analogous sufficient conditions from Theorem 5.4 for a regular shearlet system to form a frame can be stated as follow:

Theorem 5.7. *Let $\Gamma = \{(a^j, k, c_0 m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ with $a > 1$ and $c_0 > 0$, and let $\psi \in L^2(\mathbb{R}^2)$ satisfy*

$$\hat{\psi}(\xi_1,\xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right), \tag{5.22}$$

where $\psi_1, \psi_2 \in L^2(\mathbb{R})$ and ψ_2 satisfies

$$\left|\hat{\psi}_2(\xi)\right| \le \frac{C_2}{(1+\xi^2)^{\gamma/2}} \qquad a.e.\ \xi \in \mathbb{R}, \tag{5.23}$$

with $\gamma > 2$ and $C_2 > 0$. If $\psi_1 \in L^2(\mathbb{R})$ is such that

$$\widetilde{L} := \inf_{1\le|\xi_1|\le a} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_1(a^{-j}\xi_1)\right|^2 > 0,$$

$$\widetilde{M} := \sup_{1\leq|\xi_1|\leq a} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_1(a^{-j}\xi_1)\right|^2 < \infty,$$

and

$$\widetilde{R} := C_2^2 c_0^{\gamma/2} \left[\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} + 1\right] \cdot \sup_{1\leq|\xi_1|\leq a} \sum_{j\in\mathbb{Z}} |\hat{\psi}_1(a^{-j}\xi_1)| \cdot \sum_{m_1\in\mathbb{Z}\setminus\{0\}} \left|\hat{\psi}_1\left(a^{-j}\xi_1 + \frac{m_1}{c_0}\right)\right|$$

$$\cdot \max\left\{|a^{-j}\xi_1|^{\frac{\gamma}{2}}, \left|a^{-j}\xi_1 + \frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\} \cdot \sum_{m_2\in\mathbb{Z}\setminus\{0\}} |m_2|^{-\gamma/2}$$

$$< \ C_0^2 \widetilde{L} \left[\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} - 1\right], \tag{5.24}$$

then the regular shearlet system $\mathcal{SH}_{1,\psi}(\Gamma)$ forms a frame for $L^2(\mathbb{R}^2)$ with frame bounds

$$A = \frac{1}{c_0^2}\left[C_2^2 \widetilde{L}\left[\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} - 1\right] - \widetilde{R}\right], \tag{5.25}$$

$$B = \frac{1}{c_0^2}\left[C_2^2 \widetilde{M}\left[\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} + 1\right] + \widetilde{R}\right]. \tag{5.26}$$

Proof. Similar to the analogous proof of Theorem 5.4, we need only to estimate the terms M, L and R in (5.15). By using Lemma 5.6, we derive

$$\begin{aligned}
M &= \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(S_k^{-T} A_{a^j}^{-1}\xi)\right|^2 \\
&= \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}_1(a^{-j}\xi_1)\right|^2 \left|\hat{\psi}_2\left(-k + a^{j/2}\frac{\xi_2}{\xi_1}\right)\right|^2 \\
&\leq \sup_{1\leq|\xi_1|\leq a} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_1(a^{-j}\xi_1)\right|^2 \cdot \sup_{\tilde{\xi}\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}_2\left(-k + \tilde{\xi}\right)\right|^2 \\
&\leq C_2^2 \sup_{1\leq|\xi_1|\leq a} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_1(a^{-j}\xi_1)\right|^2 \cdot \sup_{\tilde{\xi}\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \left[1 + \left(k+\tilde{\xi}\right)^2\right]^{-\gamma} \\
&\leq C_2^2 \left[\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} + 1\right] \sup_{1\leq|\xi_1|\leq a} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_1(a^{-j}\xi_1)\right|^2 < \infty.
\end{aligned}$$

With a similar argument, we obtain

$$\begin{aligned}
L &= \operatorname*{ess\,inf}_{\xi\in\mathbb{R}^2} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(S_k^{-T} A_{a^j}^{-1}\xi)\right|^2 \\
&\geq C_2^2 \left[\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^\gamma} - 1\right] \inf_{1\leq|\xi_1|\leq a} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_1(a^{-j}\xi_1)\right|^2 > 0.
\end{aligned}$$

Next, we use the same argument of the proof in Lemma 5.2 to estimate the term R by

$$R = \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \left[\Gamma\left(\frac{1}{c_0}m\right)\Gamma\left(-\frac{1}{c_0}m\right)\right]^{1/2}$$

$$
\begin{aligned}
&= \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^2} \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\hat{\psi}_1(a^{-j}\xi_1)| \left| \hat{\psi}_1 \left(a^{-j}\xi_1 + \frac{m_1}{c_0} \right) \right| \\
&\qquad \cdot \left| \hat{\psi}_2 \left(-k + a^{j/2} \frac{\xi_1}{\xi_1} \right) \right| \left| \hat{\psi}_2 \left(\frac{-ka^{-j}\xi_1 + a^{-j/2}\xi_2 + \frac{2m_2}{c_0}}{a^{-j}\xi_1 + \frac{m_1}{c_0}} \right) \right| \\
&\leq c_0^{\gamma/2} C_2^2 \left(\sum_{k \in \mathbb{Z}} \frac{1}{(1+k^2)^{\gamma/4}} + 1 \right) \sup_{1 \leq |\xi_1| \leq a} \sum_{j \in \mathbb{Z}} |\hat{\psi}_1(a^{-j}\xi_1)| \sum_{m_1 \in \mathbb{Z} \setminus \{0\}} \left| \hat{\psi}_1 \left(a^{-j}\xi_1 + \frac{m_1}{c_0} \right) \right| \\
&\qquad \cdot \max \left\{ |a^{-j}\xi_1|^{\frac{\gamma}{2}}, \left| a^{-j}\xi_1 + \frac{m_1}{c_0} \right|^{\frac{\gamma}{2}} \right\} \cdot \sum_{m_2 \in \mathbb{Z} \setminus \{0\}} |m_2|^{-\gamma/2} \\
&= \widetilde{R}.
\end{aligned}
$$

This completes the proof. □

In many traditional constructions of Parseval frames from regular shearlet systems (Guo, Kutyniok and Labate [45], Guo and Labate [46], Kutyniok and Labate [62] and Labate et al. [64]), the authors chose ψ to be a band-limited function. It is easy to show that if $\mathcal{SH}_{1,\psi}(\Gamma)$ is induced by such a band-limited function ψ, then the remainder term R in (5.24) is equal to zero. In this particular case, we obtain band-limited *Parseval* frames:

Example 5.1. Consider the discrete set $\Gamma = \{(2^j, k, c_0 m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$, $c_0 > 0$. Let $\psi \in L^2(\mathbb{R}^2)$ be band-limited and defined by (5.22). Choose $\psi_1, \psi_2 \in L^2(\mathbb{R})$ with $\operatorname{supp} \hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$ satisfying the following properties:

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi)|^2 = 1 \qquad \text{a.e. } \xi \in \mathbb{R}, \tag{5.27}$$

$$\sum_{k \in \mathbb{Z}} |\hat{\psi}_2(\xi + k)|^2 = 1 \qquad \text{a.e. } \xi \in \mathbb{R}. \tag{5.28}$$

It follows from Theorem 5.7 that the remainder term R equals zero. Therefore $\mathcal{SH}_{1,\psi}(\Gamma)$ is a Parseval frame for $L^2(\mathbb{R}^2)$. In particular, there are several choices of ψ_1 and ψ_2 satisfying (5.27) and (5.28), respectively. For instance, ψ_1 may be the Lemarié -Meyer wavelet and ψ_2 is an orthonormal scaling function (see Guo et al. [46] and Daubechies's book [21] for more detail).

In the following examples, we will apply the construction of regular shearlet frames presented in Theorem 5.7 to a non-bandlimited function $\psi_1 \in L^2(\mathbb{R})$ whose Fourier transform $\hat{\psi}$ has a sufficiently fast decay. We should mention that the following example is inspired by the Mexican hat example presented by Daubechies [21].

Example 5.2. Let $\Gamma = \{(2^j, k, c_0 m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$, $c_0 > 0$, and $\psi_1 \in L^2(\mathbb{R})$ be a function of the form

$$\hat{\psi}_1(\xi_1) = \xi_1^2 e^{-\frac{\xi_1^2}{2}} \qquad \text{for all } \xi_1 \in \mathbb{R},$$

and let $\psi_2 \in L^2(\mathbb{R})$ be such that

$$\hat{\psi}_2(\xi) = \frac{1}{(1+\xi^2)^{\frac{\gamma}{2}}}.$$

By (5.22), it follows that

$$\hat{\psi}(\xi_1,\xi_2) = \frac{\xi_1^{2+\gamma} e^{-\frac{\xi_1^2}{2}}}{(\xi_1^2+\xi_2^2)^{\gamma/2}} \qquad \text{a.e. } \xi = (\xi_1,\xi_2)^T \in \mathbb{R}^2, \tag{5.29}$$

with $\gamma > 2$. This function is plotted in Figure 5.1 for choices of $\gamma = 4$. By using the formulas (5.25), (5.26) and (5.24) in Theorem 5.7, we obtain the following estimated frame bounds

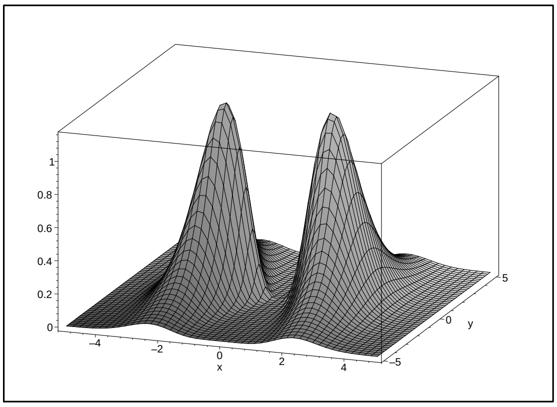

Figure 5.1: The function $\hat{\psi}$ defined by (5.29) with the choice $\gamma = 4$.

$$A = \frac{1}{c_0^2}\left[\left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^{\gamma}} - 1\right)\left(\inf_{1\le|\xi_1|\le 2} \sum_{j\in\mathbb{Z}} (2^{-j}\xi_1)^4 e^{-(2^{-j}\xi_1)^2}\right) - \widetilde{R}\right],$$

$$B = \frac{1}{c_0^2}\left[\left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^{\gamma}} + 1\right)\left(\sup_{1\le|\xi_1|\le 2} \sum_{j\in\mathbb{Z}} (2^{-j}\xi_1)^4 e^{-(2^{-j}\xi_1)^2}\right) + \widetilde{R}\right],$$

where

$$\widetilde{R} = c_0^{\gamma/2}\left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^{\gamma/4}} + 1\right) \sup_{1\le|\xi_1|\le 2} \sum_{j\in\mathbb{Z}} (2^{-j}\xi_1)^2 e^{-\frac{(2^{-j}\xi_1)^2}{2}} \cdot \sum_{m_1\in\mathbb{Z}\setminus\{0\}} \left(2^{-j}\xi_1 + \frac{m_1}{c_0}\right)^2$$
$$\cdot\, e^{-\frac{(2^{-j}\xi_1+\frac{m_1}{c_0})^2}{2}} \max\left\{|2^{-j}\xi_1|^{\frac{\gamma}{2}}, \left|2^{-j}\xi_1 + \frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\} \sum_{m_2\in\mathbb{Z}\setminus\{0\}} \frac{1}{|m_2|^{\gamma/2}}.$$

In Table 5.1 we provide the numerically estimated frame bounds for $\mathcal{SH}_{1,\psi}(\Gamma)$ for choices of parameter $\gamma = 2.1, 4$ and c_0 varying from 0.2 to 0.8 (see Appendix E for the MATLAB code). However, the ratio B/A we obtain are pretty large.

Remark 5.2. *Although we choose the function ψ in Example 5.2 as a function with exponential decay, the ratio B/A still become large. The reason is because* $\left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^{\gamma}} - 1\right) \lll \left(\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^{\gamma}} + 1\right)$ *for large value of parameters γ, and this implies that*

$$\operatorname*{ess\,inf}_{\xi\in\mathbb{R}^2} |\hat{\psi}(S_k^{-T} A_{2^j}^{-1}\xi)|^2 \lll \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} |\hat{\psi}(S_k^{-T} A_{2^j}^{-1}\xi)|^2.$$

Table 5.1: Estimated frame bounds for shearlet frames associated to ψ defined by (5.29).

	$\gamma = 2.1$		
c_0	A	B	B/A
0.2	3.1381	14.4815	4.6147
0.4	0.7845	3.6204	4.6149
0.6	0.3482	1.6095	4.6223

	$\gamma = 4$		
c_0	A	B	B/A
0.2	0.6567	11.7929	17.9578
0.4	0.1641	2.9483	17.9665
0.6	0.0729	1.3104	17.9753

In the following example, we overcome this problem by choosing a function ψ comprising of an orthonormal wavelet ψ_1 with certain decay and vanishing moments, and an orthonormal scaling function ψ_2.

Example 5.3. Let $\Gamma = \{(2^j, k, c_0 m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$, and let $\psi_1 \in L^2(\mathbb{R})$ be a wavelet with a vanishing moment of order α and a decay rate of β, i.e., $|\hat{\psi}_1(\xi_1)| \leq \min\{|\xi_1|^\alpha, |\xi_1|^{-\beta}\}$ for almost all $\xi_1 \in \mathbb{R}$, and satisfying the following

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}_1(2^{-j}\xi)|^2 = 1 \qquad \text{a.e. } \xi \in \mathbb{R}. \tag{5.30}$$

Further, let $\psi_2 \in L^2(\mathbb{R})$ be an orthonormal scaling function with a decay rate γ. That is, $|\hat{\psi}_2(\xi)| \leq \frac{1}{(1+\xi^2)^{\gamma/2}}$, and

$$\sum_{k \in \mathbb{Z}} |\hat{\psi}_2(\xi + k)|^2 = 1 \qquad \text{a.e. } \xi \in \mathbb{R}. \tag{5.31}$$

It can be verified that this regular shearlet system $\mathcal{SH}_{1,\psi}(\Gamma)$ satisfies the hypotheses of Theorem 5.4 with $\mu = \frac{1}{2}$ and $p = 1$. Therefore, we use (5.15) in Theorem 5.4 to estimate the lower and upper frame bounds A, B, and (5.6) in Lemma 5.2 to estimate the remainder term R. Then we have

$$A = \frac{1}{c_0^2}\left[1 - R\right] \quad \text{and} \quad B = \frac{1}{c_0^2}\left[1 + R\right], \tag{5.32}$$

where

$$R \leq (2c_0)^{\frac{\gamma}{2}} \left[\left(\sum_{k \in \mathbb{Z}} \frac{1}{(1+k^2)^{\gamma/4}} + 1 \right) \left(\frac{2^{\beta - \frac{\gamma}{2}}}{2^{\beta - \frac{\gamma}{2}} - 1} + \frac{2^\alpha}{2^\alpha - 1} \right) + \left(\sum_{k \in \mathbb{Z}} \frac{1}{(1+k^2)^{\gamma/2}} + 1 \right) \frac{2^\beta}{2^\beta - 1} \right]$$
$$\cdot \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{|m|^{\frac{\gamma}{2} - 1}}.$$

In Table 5.2, we have the estimates of the frame bounds A and B and the ratio B/A for shearlet systems $\mathcal{SH}_{1,\psi}(\Gamma)$ computed from (5.32). They are dependent on the choices of c_0, α, β and γ. The numbers in the table show that for small values of c_0 and large numbers of α, β and γ the ratio B/A is close to one. This indicates that the frame is close to being tight.

Table 5.2: Estimated frame bounds for shearlet frames associated to ψ defined in Example 5.3.

	$\alpha = 1$,	$\beta = 5$,	$\gamma = 8$
c_0	A	B	B/A
0.1	95.6643	104.3357	1.0906
0.2	7.6574	42.3426	5.5297

	$\alpha = 1$,	$\beta = 7$,	$\gamma = 12$
c_0	A	B	B/A
0.1	99.8524	100.1476	1.0030
0.2	22.6382	27.3618	1.2087

	$\alpha = 1$,	$\beta = 8$,	$\gamma = 12$
c_0	A	B	B/A
0.1	99.8726	100.1274	1.0026
0.2	22.9610	27.0390	1.1776
0.3	0.7889	21.4334	27.1701

	$\alpha = 5$,	$\beta = 8$,	$\gamma = 12$
c_0	A	B	B/A
0.1	99.9017	100.0983	1.0020
0.2	23.4273	26.5727	1.1343
0.3	3.1491	19.0731	6.0567

5.4 Stability of Irregular Shearlet Frames

In this section, we consider perturbations of the translation parameter c_0 for the irregular shearlet frames already considered in the previous subsection. We would mention that many ideas in this section are inspired by Yang and Zhou [81].

First, we begin with the stability of a frame in a Hilbert space. We omit the proof of this results and refer the reader to Christensen and Heil [12] and Favier and Zalik [27].

Proposition 5.8 ([12], [27]). *Let $\{f_i\}$ be a frame in a Hilbert space $\mathcal{H}$ with frame bounds A and B. Assume $\{g_i\} \subset \mathcal{H}$ and $\{f_i - g_i\}$ is a Bessel sequence with bounds $M < A$. Then $\{g_i\}$ is a frame with bounds $A[1 - \sqrt{M/A}]^2$ and $B[1 + \sqrt{M/B}]^2$.*

In the following, we choose $\psi \in L^2(\mathbb{R}^2)$ of the following form :

$$\hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right),$$

where ψ_1 and $\psi_2 \in L^2(\mathbb{R})$ are functions satisfying the following properties :
- $\hat{\psi}_1 \in C(\mathbb{R})$, and
$$\left|\hat{\psi}_1(\xi)\right| \le C_1 \min\{|\xi|^{\alpha}, |\xi|^{-\beta}\} \qquad \text{a.e. } \xi \in \mathbb{R}. \tag{5.33}$$
- $\hat{\psi}_2 \in C(\mathbb{R})$, and
$$\left|\hat{\psi}_2(\xi)\right| \le \frac{C_2}{(1+\xi^2)^{\gamma/2}}, \tag{5.34}$$

with C_1, $C_2 > 0$, $\gamma > 4$ and $2\beta > \gamma$.

Lemma 5.9. *Assume that $\{\psi_{j,k,cm}\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ is a shearlet system induced by $\Lambda = \{(a_j, s_k, cm)\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ where $c > 0$, $\{a_j\}_{j\in\mathbb{Z}}$ and $\{s_k\}_{k\in\mathbb{Z}}$ satisfy assumptions in Lemma 5.2. Let $\phi_{j,k,c_0m} = D_{\frac{c_0}{c}}\psi_{j,k,cm}$ for $j, k \in \mathbb{Z}$, $m \in \mathbb{Z}^2$ and $c_0 > 0$, and let $g = \phi - \psi$. Then the following statement holds:*

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \left|\langle \hat{f}, \hat{g}_{j,k,c_0m}\rangle\right|^2$$

$$\leq \frac{C_0}{c_0^2}||\hat{f}||^2 \sup_{\xi_1\in\mathbb{R}} \sum_{j\in\mathbb{Z}} \sum_{m_1\in\mathbb{Z}} \left[(2|c_0 a_j^{-1}\xi_1| + 2|m_1| + 1) + \sum_{|m_2|>|c_0 a_j^{-1}\xi_1|+|m_1|} \frac{\max\left\{ |a_j^{-1}\xi_1|^{\frac{\gamma}{2}}, \left|a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right|^{\frac{\gamma}{2}} \right\}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}} \right]$$
$$\cdot \left| \left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c} a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right) \right| \left| \left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right) \right|,$$

for any $f \in L^2(\mathbb{R}^2)$.

Proof. Recall the dilation operator $(D_{\frac{c_0}{c}}\varphi)(x) = \left(\frac{c}{c_0}\right)^{1/2} \varphi\left(\frac{c}{c_0}x\right)$ for all $\varphi \in L^2(\mathbb{R}^2)$. Let $\phi = D_{\frac{c_0}{c}}\psi$ and $g = \phi - \psi$. It is easy to see that

$$\phi_{j,k,c_0 m}(x) = (D_{\frac{c_0}{c}}\psi_{j,k,cm})(x) \quad \text{and} \quad \hat{\phi}_{j,k,c_0 m}(\xi) = \left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_{j,k,cm}\left(\frac{c_0}{c}\xi\right).$$

For any $f \in L^2(\mathbb{R}^2)$, it follows from Lemma 5.3 that

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \left| \langle \hat{f}, \hat{g}_{j,k,c_0 m} \rangle \right|^2$$
$$= \frac{1}{c_0^2} \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \int_{\mathbb{R}^2} \hat{f}(\xi) \overline{\hat{f}\left(\xi + \frac{1}{c_0} A_{a_j} S_{s_k}^T m\right) \hat{g}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right)} \hat{g}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m\right) d\xi$$
$$\leq \frac{1}{c_0^2} \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left| \hat{g}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right) \right| \left| \hat{g}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m\right) \right| d\xi$$
$$= \frac{1}{c_0^2} \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left| \left(\frac{c_0}{c}\right) \hat{\psi}\left(\frac{c_0}{c} S_{s_k}^{-T} A_{a_j}^{-1}\xi\right) - \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi\right) \right|$$
$$\cdot \left| \left(\frac{c_0}{c}\right) \hat{\psi}\left(\frac{c_0}{c}(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m)\right) - \hat{\psi}\left(S_{s_k}^{-T} A_{a_j}^{-1}\xi + \frac{1}{c_0} m\right) \right| d\xi$$
$$= \frac{1}{c_0^2} \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left| \left(\frac{c_0}{c}\right) \hat{\psi}_1\left(\frac{c_0}{c} a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right) \right| \left| \hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2}{a_j^{-1}\xi_1}\right) \right|$$
$$\cdot \left| \left(\frac{c_0}{c}\right) \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right) \right| \left| \hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + \frac{m_2}{c_0}}{a_j^{-1}\xi_1 + \frac{m_1}{c_0}}\right) \right| d\xi.$$

By using the Cauchy-Schwarz inequality, it follows that

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \left| \langle \hat{f}, \hat{g}_{j,k,c_0 m} \rangle \right|^2$$
$$\leq \frac{1}{c_0^2}||\hat{f}||^2 \sup_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^2} \left| \hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2}{a_j^{-1}\xi_1}\right) \right| \left| \hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + \frac{m_2}{c_0}}{a_j^{-1}\xi_1 + \frac{m_1}{c_0}}\right) \right|$$
$$\cdot \left| \left(\frac{c_0}{c}\right) \hat{\psi}_1\left(\frac{c_0}{c} a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right) \right| \left| \left(\frac{c_0}{c}\right) \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right) \right|$$
$$= \frac{1}{c_0^2}||\hat{f}||^2 (I_1 + I_2), \tag{5.35}$$

where

$$I_1 := \sup_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \sum_{m_1\in\mathbb{Z}} \sum_{\{m_2:|m_2|\le|c_0 a_j^{-1}\xi_1|+|m_1|\}} \left|\hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2}{a_j^{-1}\xi_1}\right)\right|$$
$$\left|\hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + \frac{m_2}{c_0}}{a_j^{-1}\xi_1 + \frac{m_1}{c_0}}\right)\right| \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|$$
$$\cdot \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right)\right|,$$

$$I_2 := \sup_{\xi\in\mathbb{R}^2} \sum_{j,k\in\mathbb{Z}} \sum_{m_1\in\mathbb{Z}} \sum_{\{m_2:|m_2|>|c_0 a_j^{-1}\xi_1|+|m_1|\}} \left|\hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2}{a_j^{-1}\xi_1}\right)\right|$$
$$\left|\hat{\psi}_2\left(\frac{-s_k a_j^{-1}\xi_1 + a_j^{-1/2}\xi_2 + \frac{m_2}{c_0}}{a_j^{-1}\xi_1 + \frac{m_1}{c_0}}\right)\right| \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|$$
$$\cdot \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right)\right|.$$

Concerning I_1, straightforward computation gives

$$I_1 \le \sup_{\xi\in\mathbb{R}^2} \sum_{j\in\mathbb{Z}} \sum_{m_1\in\mathbb{Z}} \sum_{\{m_2:|m_2|\le|c_0 a_j^{-1}\xi_1|+|m_1|\}} \sum_{k\in\mathbb{Z}} \left[1 + \left(-s_k + \sqrt{a_j}\frac{\xi_2}{\xi_1}\right)^2\right]^{-\frac{\gamma}{2}}$$
$$\cdot \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right| \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right)\right|$$
$$\le C_0 \sup_{\xi_1\in\mathbb{R}} \sum_{j\in\mathbb{Z}} \sum_{m_1\in\mathbb{Z}} \sum_{\{m_2:|m_2|\le|c_0 a_j^{-1}\xi_1|+|m_1|\}} \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|$$
$$\cdot \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right)\right|$$
$$\le C_0 \sup_{\xi_1\in\mathbb{R}} \sum_{j\in\mathbb{Z}} \sum_{m_1\in\mathbb{Z}} \left(2|c_0 a_j^{-1}\xi_1| + 2|m_1| + 1\right) \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|$$
$$\cdot \left|\left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1 + \frac{m_1}{c_0})\right) - \hat{\psi}_1\left(a_j^{-1}\xi_1 + \frac{m_1}{c_0}\right)\right|.$$

Concerning I_2, we apply the inequality (5.9) in Lemma 5.2 to the term I_2 as follows:

$$I_2 \le \sup_{\xi\in\mathbb{R}^2} \sum_{j\in\mathbb{Z}} \sum_{m_1\in\mathbb{Z}} \sum_{|m_2|>|c_0 a_j^{-1}\xi_1|+|m_1|} \sum_{k\in\mathbb{Z}} \left[1 + \left(-s_k + \sqrt{a_j}\frac{\xi_2}{\xi_1}\right)^2\right]^{-\gamma/4}$$
$$\cdot \frac{1}{\left[1 + \left(-s_k + \sqrt{a_j}\frac{\xi_2}{\xi_1}\right)^2\right]^{\gamma/4}} \frac{\left|1 + \frac{m_1}{c_0 a_j^{-1}\xi_1}\right|^{\frac{\gamma}{2}}}{\left[(1 + \frac{m_1}{c_0 a_j^{-1}\xi_1})^2 + \left(-s_k + \sqrt{a_j}\frac{\xi_2}{\xi_1} + \frac{m_2}{c_0 a_j^{-1}\xi_1}\right)^2\right]^{\gamma/4}}$$

$$\cdot\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1+\frac{m_1}{c_0})\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1+\frac{m_1}{c_0}\right)\right|$$

$$\leq C_0\sup_{\xi_1\in\mathbb{R}}\sum_{j\in\mathbb{Z}}\sum_{m_1\in\mathbb{Z}}\sum_{|m_2|>|c_0a_j^{-1}\xi_1|+|m_1|}\frac{\max\left\{|a_j^{-1}\xi_1|^{\frac{\gamma}{2}},\left|a_j^{-1}\xi_1+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}$$

$$\cdot\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1+\frac{m_1}{c_0})\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1+\frac{m_1}{c_0}\right)\right|.$$

This completes the proof of lemma. □

Now we are ready to state the theorem which shows the stability of irregular shearlet frames under a perturbation of the translation parameter $c_0 > 0$.

Theorem 5.10. *Suppose that $\{a_j\}_{j\in\mathbb{Z}},\{s_k\}_{k\in\mathbb{Z}}$ and $\psi\in L^2(\mathbb{R}^2)$ satisfy the same assumptions as in Lemma 5.2. If $\{\psi_{j,k,c_0m,}\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ is a frame for $L^2(\mathbb{R}^2)$ for some $c_0>0$, then there exists $\delta>0$ such that $\{\psi_{j,k,cm}\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ is a frame for $L^2(\mathbb{R}^2)$ for any c with $0<|c-c_0|<\delta$.*

Proof. Let $\phi_{j,k,c_0m}=D_{\frac{c_0}{c}}\psi_{j,k,cm}$, it follows that

$$\hat{\phi}_{j,k,c_0m}(\xi)=\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_{j,k,cm}\left(\frac{c_0}{c}\xi\right).$$

Therefore $\{\psi_{j,k,cm}\}$ is a frame for $L^2(\mathbb{R}^2)$ if and only if $\{\phi_{j,k,c_0m}\}$ is a frame for $L^2(\mathbb{R}^2)$. Now we will prove that $\{\phi_{j,k,c_0m}\}$ is a frame for $L^2(\mathbb{R}^2)$. To this end, we first let $g=\phi-\psi$.

It follows from Lemma 5.9 that

$$\sum_{j,k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2}\left|\langle\hat{f},\hat{g}_{j,k,c_0m}\rangle\right|^2=\sum_{j,k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2}\left|\langle\hat{f},\hat{\phi}_{j,k,c_0m}-\hat{\psi}_{j,k,c_0m}\rangle\right|^2$$

$$\leq\frac{C}{c_0^2}||\hat{f}||^2\sup_{\xi_1\in\mathbb{R}}\sum_{j\in\mathbb{Z}}\sum_{m_1\in\mathbb{Z}}\left[(2|c_0a_j^{-1}\xi_1|+2|m_1|+1)+\sum_{|m_2|>|c_0a_j^{-1}\xi_1|+|m_1|}\frac{\max\left\{|a_j^{-1}\xi_1|^{\frac{\gamma}{2}},\left|a_j^{-1}\xi_1+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right]$$

$$\cdot\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}(a_j^{-1}\xi_1+\frac{m_1}{c_0})\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1+\frac{m_1}{c_0}\right)\right|$$

$$\leq\frac{C}{c_0^2}||\hat{f}||^2\sup_{\tilde{\xi}\in\mathbb{R}}\sum_{m_1\in\mathbb{Z}}\left((2|c_0\tilde{\xi}|+2|m_1|+1)+\sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|}\frac{\max\left\{|\tilde{\xi}|^{\frac{\gamma}{2}},\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right)$$

$$\cdot\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}(\tilde{\xi}+\frac{m_1}{c_0})\right)-\hat{\psi}_1\left(\tilde{\xi}+\frac{m_1}{c_0}\right)\right|\cdot\left[\sup_{\xi_1\in\mathbb{R}}\sum_{j\in\mathbb{Z}}\left|\left(\frac{c_0}{c}\right)^{1/2}\hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|\right].\tag{5.36}$$

Next, for a fixed $c_0>0$, we claim that,

$$\lim_{c\to c_0}\sup_{\tilde{\xi}\in\mathbb{R}}\sum_{m_1\in\mathbb{Z}}\left[(2|c_0\tilde{\xi}|+2|m_1|+1)+\sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|}\frac{\max\left\{|\tilde{\xi}|,\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right]$$

$$\cdot \left| \left(\frac{c_0}{c}\right) \hat{\psi}_1 \left(\frac{c_0}{c}(\tilde{\xi} + \frac{m_1}{c_0})\right) - \hat{\psi}_1 \left(\tilde{\xi} + \frac{m_1}{c_0}\right) \right| = 0. \tag{5.37}$$

Case 1 : $\max\left\{|\tilde{\xi}|^{\frac{\gamma}{2}}, \left|\tilde{\xi} + \frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\} = \left|\tilde{\xi} + \frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}$. Consider

$$\begin{aligned}
&\sum_{m_1\in\mathbb{Z}} \left[(2|c_0\tilde{\xi}| + 2|m_1| + 1) + \sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|} \frac{\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right] \left| \left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(\tilde{\xi}+\frac{m_1}{c_0})\right) - \hat{\psi}_1\left(\tilde{\xi}+\frac{m_1}{c_0}\right)\right| \\
&\le \sum_{m_1\in\mathbb{Z}} \left[(2|c_0\tilde{\xi}| + 2|m_1| + 1) + \sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|} \frac{\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right] \left[\left(\frac{c_0}{c}\right)^{1/2} \left|\frac{c_0}{c}(\tilde{\xi}+\frac{m_1}{c_0})\right|^{-\beta} + \left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{-\beta}\right] \\
&\le \left(\left(\frac{c_0}{c}\right)^{0.5-\beta} + 1\right) \sum_{m_1\in\mathbb{Z}} \left[(2|c_0\tilde{\xi}| + 2|m_1| + 1) + \sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|} \frac{\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right] \left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{-\beta}.
\end{aligned}$$

Now we choose a fixed $c \in \left(\frac{c_0}{2}, \frac{3c_0}{2}\right)$ and $\tilde{\xi} \in \left[\frac{-1}{2c}, \frac{1}{2c}\right]$. Let N_0 be a positive integer such that $\frac{|m_1|}{c_0} \le \frac{|m_1|}{2c_0} - |\tilde{\xi}|$ for any $\tilde{\xi} \in \left[\frac{-1}{2c}, \frac{1}{2c}\right] \subset \left(-\frac{1}{c_0}, \frac{1}{c_0}\right)$ and $m_1 \in \mathbb{Z}$ with $|m_1| \ge N_0$. If $|m_1| > N_0$, then $\frac{1}{4c}|m_1| \le \left|\tilde{\xi} + \frac{m_1}{c_0}\right|$. Thus for any $N \in \mathbb{Z}^+$ with $N > N_0$, we have

$$\begin{aligned}
&\sum_{|m_1|>N} \left[(2|c_0\tilde{\xi}| + 2|m_1| + 1) + \sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|} \frac{\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right] \left| \left(\frac{c_0}{c}\right)^{1/2} \hat{\psi}_1\left(\frac{c_0}{c}(\tilde{\xi}+\frac{m_1}{c_0})\right) - \hat{\psi}_1\left(\tilde{\xi}+\frac{m_1}{c_0}\right)\right| \\
&\le \left(\left(\frac{c_0}{c}\right)^{0.5-\beta} + 1\right) \sum_{|m_1|>N} \left[(3+2|m_1|)\left(\frac{4c}{|m_1|}\right)^{\beta} + \sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|} \frac{\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}-\beta}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right] \\
&\le \left(\left(\frac{c_0}{c}\right)^{0.5-\beta} + 1\right) \sum_{|m_1|>N} \left[(3+2|m_1|)\left(\frac{4c}{|m_1|}\right)^{\beta} + \left(\frac{4c}{|m_1|}\right)^{\beta-\frac{\gamma}{2}} \sum_{m_2\in\mathbb{Z}} \left(\frac{c_0}{m_2}\right)^{\frac{\gamma}{2}}\right] \\
&\le \left(\left(\frac{c_0}{c}\right)^{0.5-\beta} + 1\right) \sum_{|m_1|>N} \left[(3+2|m_1|)\left(\frac{4c}{|m_1|}\right)^{\beta} + \widetilde{C}\frac{1}{|m_1|^{\beta-\frac{\gamma}{2}}}\right],
\end{aligned}$$

where $\widetilde{C} = (4c)^{\beta-\frac{\gamma}{2}} \sum_{m_2\in\mathbb{Z}} \left(\frac{c_0}{m_2}\right)^{\frac{\gamma}{2}} < \infty$ for any $\gamma > 2$.

Therefore for any $c \in \left(\frac{c_0}{2}, \frac{3c_0}{2}\right)$, $N \in \mathbb{Z}^+$ with $N > N_0$,

$$\begin{aligned}
&\sup_{\tilde{\xi}\in\mathbb{R}} \sum_{m_1\in\mathbb{Z}} \left[(2|c_0\tilde{\xi}| + 2|m_1| + 1) + \sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|} \frac{\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right] \left| \left(\frac{c_0}{c}\right)^{\frac{1}{2}} \hat{\psi}_1\left(\frac{c_0}{c}(\tilde{\xi}+\frac{m_1}{c_0})\right) - \hat{\psi}_1\left(\tilde{\xi}+\frac{m_1}{c_0}\right)\right| \\
&\le \sum_{|m_1|\le N} \sup_{\tilde{\xi}\in[-\frac{1}{2c},\frac{1}{2c}]} \left[(3+2|m_1|) + \sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|} \frac{\left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}}{\left|\frac{m_2}{c_0}\right|^{\frac{\gamma}{2}}}\right] \left| \left(\frac{c_0}{c}\right)^{\frac{1}{2}} \hat{\psi}_1\left(\frac{c_0}{c}(\tilde{\xi}+\frac{m_1}{c_0})\right) - \hat{\psi}_1\left(\tilde{\xi}+\frac{m_1}{c_0}\right)\right|
\end{aligned}$$

$$+\sum_{|m_1|>N}\left(\left(\frac{c_0}{c}\right)^{0.5-\beta}+1\right)\left[(3+2|m_1|)\left(\frac{4c}{|m_1|}\right)^{\beta}+\widetilde{C}\frac{1}{|m_1|^{\beta-\frac{\gamma}{2}}}\right]$$

$$\leq \sum_{|m_1|\leq N}\sup_{\tilde{\xi}\in[-\frac{1}{2c},\frac{1}{2c}]}\left[(3+2N)+\sum_{|m_2|>|c_0\tilde{\xi}|+|m_1|}\frac{(1+N)^{\frac{\gamma}{2}}}{|m_2|^{\frac{\gamma}{2}}}\right]\left|\left(\frac{c_0}{c}\right)^{\frac{1}{2}}\hat{\psi}_1\left(\frac{c_0}{c}(\tilde{\xi}+\frac{m_1}{c_0})\right)-\hat{\psi}_1\left(\tilde{\xi}+\frac{m_1}{c_0}\right)\right|$$

$$+\sum_{|m_1|>N}\left(\left(\frac{c_0}{c}\right)^{0.5-\beta}+1\right)\left[(3+2|m_1|)\left(\frac{4c}{|m_1|}\right)^{\beta}+\widetilde{C}\frac{1}{|m_1|^{\beta-\frac{\gamma}{2}}}\right].$$

Since $\hat{\psi}_1$ is uniformly continuous, by letting $c \to c_0$ and $N \to \infty$ respectively, the right-hand side tends to zero.

Case 2 : $\max\left\{|\tilde{\xi}|^{\frac{\gamma}{2}}, \left|\tilde{\xi}+\frac{m_1}{c_0}\right|^{\frac{\gamma}{2}}\right\} = |\tilde{\xi}|^{\frac{\gamma}{2}}$. It can be proven in a similar way, and this completes the proof of the claim (5.37).

Finally, for any $\xi_1 \in \mathbb{R}\setminus\{0\}$, there exists a positive integer j_0 such that $\left|\frac{\xi_1}{a_{j_0+1}}\right| \leq 1 \leq \left|\frac{\xi_1}{a_{j_0}}\right|$. Since $\frac{a_j}{a_{j+p}} < \mu$ for any $j \in \mathbb{Z}$, it follows from Lemma 5.1 that

$$\sum_{j\in\mathbb{Z}}\left|\left(\frac{c_0}{c}\right)\hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right)-\hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right| \leq \sum_{j\in\mathbb{Z}}\left[\,\left|\left(\frac{c_0}{c}\right)\hat{\psi}_1\left(\frac{c_0}{c}a_j^{-1}\xi_1\right)\right|+\left|\hat{\psi}_1\left(a_j^{-1}\xi_1\right)\right|\,\right]$$

$$\leq C_1\left[\left(\frac{c_0}{c}\right)\sum_{j\leq j_0}\left(\frac{c}{c_0}\right)^{\beta}|a_j^{-1}\xi_1|^{-\beta}+\sum_{j\geq j_0+1}\left(\frac{c_0}{c}\right)^{\alpha}|a_j^{-1}\xi_1|^{\alpha}\right]+C_1\left[\sum_{j\leq j_0}|a_j^{-1}\xi_1|^{-\beta}+\sum_{j\geq j_0+1}|a_j^{-1}\xi_1|^{\alpha}\right]$$

$$\leq C_1\left[\left(\frac{c_0}{c}\right)^{1+\alpha}\left(\left(\frac{c}{c_0}\right)^{\beta-\alpha}\frac{p}{1-\mu^{\beta}}+\frac{p}{1-\mu^{\alpha}}\right)+\left(\frac{p}{1-\mu^{\beta}}+\frac{p}{1-\mu^{\alpha}}\right)\right]. \tag{5.38}$$

It follows from (5.36), (5.37) and (5.38) that for any $\varepsilon \in (0,1)$ there exists $\delta > 0$ such that

$$\sum_{j,k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^2}\left|\langle\hat{f},(\hat{\phi}_{j,k,c_0m}-\hat{\psi}_{j,k,c_0m})\rangle\right|^2 \leq \varepsilon||\hat{f}||^2,$$

for any c with $|c-c_0|<\delta$, which shows $\{(\phi-\psi)_{j,k,c_0m}\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ is a Bessel sequence on $L^2(\mathbb{R}^2)$. Hence, by Proposition 5.8, for c sufficiently close to c_0, $\{\phi_{j,k,c_0m}\}_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2}$ is thus a frame for $L^2(\mathbb{R}^2)$. This completes the proof of Theorem 5.10. □

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Appendix

Appendix A

Technical Lemmas

Proof of Lemma 3.11. By using the geometric series, we obtain

$$\sum_{j=l}^{m} a^{\alpha j} = \frac{a^{\alpha(m+1)} - a^{\alpha l}}{a^{\alpha} - 1}.$$

Since $x \leq \lceil x \rceil < x + 1$ and $x - 1 < \lfloor x \rfloor \leq x$, it follows that

$$\begin{aligned}
&\frac{a^{\alpha\left(\log_a x + \frac{h}{2\ln a} + 1\right)} - a^{\alpha\left(\log_a x - \frac{h}{2\ln a} + 1\right)}}{a^{\alpha} - 1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a} \rceil}^{\lceil \log_a x + \frac{h}{2\ln a} \rceil} a^{\alpha j} < \frac{a^{\alpha\left(\log_a x + \frac{h}{2\ln a} + 2\right)} - a^{\alpha\left(\log_a x - \frac{h}{2\ln a}\right)}}{a^{\alpha} - 1} \\
\iff \quad & \frac{a^{\alpha} x^{\alpha} e^{\frac{\alpha h}{2}} - a^{\alpha} x^{\alpha} e^{-\frac{\alpha h}{2}}}{a^{\alpha} - 1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a} \rceil}^{\lceil \log_a x + \frac{h}{2\ln a} \rceil} a^{\alpha j} < \frac{a^{2\alpha} x^{\alpha} e^{\frac{\alpha h}{2}} - x^{\alpha} e^{-\frac{\alpha h}{2}}}{a^{\alpha} - 1} \\
\iff \quad & \frac{x^{\alpha} a^{\alpha} \left(e^{\frac{\alpha h}{2}} - e^{-\frac{\alpha h}{2}}\right)}{a^{\alpha} - 1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a} \rceil}^{\lceil \log_a x + \frac{h}{2\ln a} \rceil} a^{\alpha j} < \frac{x^{\alpha} \left(a^{2\alpha} e^{\frac{\alpha h}{2}} - e^{-\frac{\alpha h}{2}}\right)}{a^{\alpha} - 1}.
\end{aligned}$$

This proves (3.37). A similar argument shows that

$$\begin{aligned}
&\frac{a^{\alpha\left(\log_a x + \frac{h}{2\ln a}\right)} - a^{\alpha\left(\log_a x - \frac{h}{2\ln a} + 1\right)}}{a^{\alpha} - 1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a} \rceil}^{\lfloor \log_a x + \frac{h}{2\ln a} \rfloor} a^{\alpha j} \leq \frac{a^{\alpha\left(\log_a x + \frac{h}{2\ln a} + 1\right)} - a^{\alpha\left(\log_a x - \frac{h}{2\ln a}\right)}}{a^{\alpha} - 1} \\
\iff \quad & \frac{x^{\alpha} \left(e^{\frac{\alpha h}{2}} - a^{\alpha} e^{-\frac{\alpha h}{2}}\right)}{a^{\alpha} - 1} < \sum_{j=\lceil \log_a x - \frac{h}{2\ln a} \rceil}^{\lfloor \log_a x + \frac{h}{2\ln a} \rfloor} a^{\alpha j} \leq \frac{x^{\alpha} a^{\alpha} \left(e^{\frac{\alpha h}{2}} - a^{-\alpha} e^{-\frac{\alpha h}{2}}\right)}{a^{\alpha} - 1}.
\end{aligned}$$

□

Lemma A.1. *Let $h, r > 0$ and $\alpha \neq 0$. The following statement holds true.*

$$\frac{x^{\alpha}\left(e^{\frac{\alpha h(r+3)}{2}} - e^{-\frac{\alpha h(r+1)}{2}}\right)}{e^{\alpha h} - 1} \leq \sum_{j=\lfloor \ln x - \frac{r+1}{2} \rfloor}^{\lceil \ln x + \frac{r+1}{2} \rceil} e^{\alpha h j} < \frac{x^{\alpha}\left(e^{\frac{\alpha h(r+5)}{2}} - e^{-\frac{\alpha h(r-1)}{2}}\right)}{e^{\alpha h} - 1}, \quad \textit{for any } x \in \mathbb{R}^{+}.$$

Proof. Similar as in Lemma 3.11. □

Appendix B

Basic Facts about Density for Shearlet Systems of $\mathbb{S}_2$

Lemma B.1. *Let $h > 0$ and $r \geq 1$ be given, and let*

$$X_2 = \{(e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}}S^{-1}_{he^{-h/4}k}m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

Then the following statements are true.

1. *X_2 is Q_h-dense in $\mathbb{S}_2$.*
2. *Any set $Q_{rh}(x,y,z)$ intersects at most*

$$N_r := r^2\left((r+1)e^{h/2}+1\right)\left[\frac{(e^{\frac{3(r+5)h}{4}} - e^{-\frac{3(r-1)h}{4}})}{e^{3h/2}-1} + \frac{(e^{\frac{(r+5)h}{2}} - e^{-\frac{(r-1)h}{2}})}{r(e^{h}-1)} + \frac{(e^{\frac{(r+5)h}{4}} - e^{-\frac{(r-1)h}{4}})}{r(e^{h/2}-1)} + \frac{(r+2)}{r^2}\right]$$

   elements in X_2.
3. *Any set $Q_{rh}(x,y,z)$ contains at least*

$$\tilde{N}_r := r^2(r+1)e^{h/2}\left[\frac{(e^{\frac{3(r+3)h}{4}} - e^{-\frac{3(r+1)h}{4}})}{e^{3h/2}-1} + \frac{(r+1)}{r^2}\right]$$

   elements in X_2.

Proof. 1. Fix any $(x,y,z) \in \mathbb{S}_2$. We show that there exists $(a,s,t) \in Q_h$, $j,k \in \mathbb{Z}$, and $m \in \mathbb{Z}^2$ such that

$$\begin{aligned}(x,y,z) &= (ae^{jh}, s + he^{-h/4}k\sqrt{a}, A^{-1}_{e^{jh}}S^{-1}_{he^{-h/4}k}(hm+t)) \\ &= (e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}}S^{-1}_{he^{-h/4}k}m) \star (a,s,t) \in Q_h(e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}}S^{-1}_{he^{-h/4}k}m).\end{aligned}$$

In particular,

$$x = ae^{jh} \quad , \quad y = s + he^{-h/4}k\sqrt{a}, \quad \text{and} \quad z = A^{-1}_{e^{jh}}S^{-1}_{he^{-h/4}k}(hm+t).$$

These three equalities are equivalent to

$$j = \frac{\ln x}{h} - \frac{\ln a}{h}, \tag{B.1}$$

$$k = \frac{ye^{h/4}}{h\sqrt{a}} - \frac{se^{h/4}}{h\sqrt{a}}, \tag{B.2}$$

$$m = \underbrace{\frac{1}{h} S_{he^{-h/4}k} A_{e^{jh}} z}_{=:C} - \frac{t}{h}. \tag{B.3}$$

Now we observe the following:

- For any $a \in [e^{-h/2}, e^{h/2})$, by using (B.1), we form the interval $\left[\frac{\ln x}{h} - \frac{1}{2}, \frac{\ln x}{h} + \frac{1}{2}\right)$ which contains a unique integer j.
- Take the same a, and $s \in [-\frac{h}{2}, \frac{h}{2})$. Following from (B.2), we form the interval $\left[\frac{ye^{h/4}}{h\sqrt{a}} - \frac{e^{h/4}}{2\sqrt{a}}, \frac{ye^{h/4}}{h\sqrt{a}} + \frac{e^{h/4}}{2\sqrt{a}}\right)$ which contains an integer k.
- Take the same a as above, and $t_1, t_2 \in [-\frac{h}{2}, \frac{h}{2})$. By using (B.3), we form intervals $\left[C_1 - \frac{h}{2}, C_1 + \frac{h}{2}\right)$ and $\left[C_2 - \frac{h}{2}, C_2 + \frac{h}{2}\right)$ which contain a unique integer m_1 and m_2, respectively.

Thus $\{Q_h(e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}} S^{-1}_{he^{-h/4}k} m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ is a covering of $\mathbb{S}_2$, that means X_2 is Q_h-dense in $\mathbb{S}_2$.

2. Fix $(x, y, z) \in \mathbb{S}_2$, and suppose $(u, v, w) \in Q_{rh}(x, y, z) \cap Q_h(e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}} S^{-1}_{he^{-h/4}k} m)$. Then there exist $(a, s, t) \in Q_{rh}$ and $(a', s', t') \in Q_h$ such that

$$\begin{aligned} (u, v, w) &= (x, y, z) \star (a, s, t) \\ &= (ax, s + y\sqrt{a}, z + A_x^{-1} S_y^{-1} t) \in Q_{rh}(x, y, z), \\ \text{and} \quad (u, v, w) &= (e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}} S^{-1}_{he^{-h/4}k} m) \star (a', s', t') \\ &= (a'e^{jh}, s' + he^{-h/4}k\sqrt{a'}, A^{-1}_{e^{jh}} S^{-1}_{he^{-h/4}k}(hm + t')) \\ &\in Q_h(e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}} S^{-1}_{he^{-h/4}k} m). \end{aligned}$$

In particular, $ax = a'e^{jh}$ with $a \in [e^{-rh/2}, e^{rh/2})$ and $a' \in [e^{-h/2}, e^{h/2})$. We have

$$xe^{-h(r+1)/2} \leq e^{jh} \leq xe^{h(r+1)/2},$$

$$\text{and} \qquad \frac{\ln x}{h} - \frac{r+1}{2} \leq j \leq \frac{\ln x}{h} + \frac{r+1}{2}. \tag{B.4}$$

This is satisfied for at most $r + 2$ values of j.

Further, $k = \frac{(s-s')e^{h/4}}{h\sqrt{a'}} + \frac{y\sqrt{a}e^{h/4}}{h\sqrt{a'}}$ with $s \in [-\frac{rh}{2}, \frac{rh}{2})$ and $s' \in [-\frac{h}{2}, \frac{h}{2})$, so that

$$\frac{y\sqrt{a}e^{h/4}}{h\sqrt{a'}} - \left(\frac{r+1}{2}\right) e^{h/2} \leq k \leq \frac{y\sqrt{a}e^{h/4}}{h\sqrt{a'}} + \left(\frac{r+1}{2}\right) e^{h/2}. \tag{B.5}$$

For a given value of $a \in [e^{-rh/2}, e^{rh/2})$ and $a' \in [e^{-h/2}, e^{h/2})$, this is satisfied for at most $(r + 1)e^{h/2} + 1$ values of k.

Furthermore, we have $z + A_x^{-1} S_y^{-1} t = A^{-1}_{e^{jh}} S^{-1}_{he^{-h/4}k}(hm + t')$, therefore

$$hm = \underbrace{S_{he^{-h/4}k} A_{e^{jh}} z}_{=:C} + S_{he^{-h/4}k} A_{e^{jh}} A_x^{-1} S_y^{-1} t - t'.$$

Using that $S_{he^{-h/4}k}A_{e^{jh}}A_x^{-1}S_y^{-1} = \begin{pmatrix} \frac{e^{jh}}{x} & -\frac{ye^{jh}}{x} + \frac{he^{-h/4}ke^{jh/2}}{\sqrt{x}} \\ 0 & \frac{e^{jh/2}}{\sqrt{x}} \end{pmatrix}$, we obtain

$$hm = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} \frac{e^{jh}}{x}t_1 + \left(-\frac{ye^{jh}}{x} + he^{-h/4}k\frac{e^{jh/2}}{\sqrt{x}}\right)t_2 \\ \frac{e^{jh/2}}{\sqrt{x}}t_2 \end{pmatrix} - \begin{pmatrix} t_1' \\ t_2' \end{pmatrix},$$

where $a' \in [e^{-h/2}, e^{h/2})$, $s', t_1', t_2' \in \left[-\frac{h}{2}, \frac{h}{2}\right)$, and $s, t_1, t_2 \in \left[-\frac{rh}{2}, \frac{rh}{2}\right)$, hence

$$\frac{C_2}{h} - \left(\frac{re^{jh/2}}{2\sqrt{x}} + \frac{1}{2}\right) \le m_2 \le \frac{C_2}{h} + \left(\frac{re^{jh/2}}{2\sqrt{x}} + \frac{1}{2}\right), \tag{B.6}$$

$$\frac{C_1}{h} + \left(-\frac{ye^{jh}}{x} + he^{-h/4}k\frac{e^{jh/2}}{\sqrt{x}}\right)\frac{t_2}{h} - \left(\frac{re^{jh}}{2x} + \frac{1}{2}\right) \le m_1$$
$$\le \frac{C_1}{h} + \left(-\frac{ye^{jh}}{x} + he^{-h/4}k\frac{e^{jh/2}}{\sqrt{x}}\right)\frac{t_2}{h} + \left(\frac{re^{jh}}{2x} + \frac{1}{2}\right). \tag{B.7}$$

For a given value of j, (B.6) is satisfied for at most $\frac{re^{jh/2}}{\sqrt{x}} + 1$ values of m_2 , and for a given value of j and k, (B.7) is satisfied for at most $\frac{re^{jh}}{x} + 1$ values of m_1. By using Lemma A.1, we obtain that $Q_{rh}(x, y, z)$ can intersect at most

$$\sum_{j=\lfloor \frac{\ln x}{h} - \frac{(r+1)}{2} \rfloor}^{\lceil \frac{\ln x}{h} + \frac{(r+1)}{2} \rceil} \left((r+1)e^{h/2} + 1\right)\left(\frac{re^{jh}}{x} + 1\right)\left(\frac{re^{jh/2}}{\sqrt{x}} + 1\right)$$
$$= \left((r+1)e^{h/2} + 1\right) \sum_{j=\lfloor \frac{\ln x}{h} - \frac{(r+1)}{2} \rfloor}^{\lceil \frac{\ln x}{h} + \frac{(r+1)}{2} \rceil} \left[\frac{r^2 e^{3jh/2}}{x^{3/2}} + \frac{re^{jh}}{x} + \frac{re^{jh/2}}{\sqrt{x}} + 1\right]$$
$$\le r^2\left((r+1)e^{h/2} + 1\right)\left[\frac{(e^{\frac{3(r+5)h}{4}} - e^{-\frac{3(r-1)h}{4}})}{e^{3h/2} - 1} + \frac{(e^{\frac{(r+5)h}{2}} - e^{-\frac{(r-1)h}{2}})}{r(e^h - 1)} + \frac{(e^{\frac{(r+5)h}{4}} - e^{-\frac{(r-1)h}{4}})}{r(e^{h/2} - 1)} + \frac{(r+2)}{r^2}\right]$$

sets of the form $Q_h(e^{jh}, he^{-h/4}k, hA_{e^{jh}}^{-1}S_{he^{-h/4}k}^{-1}m)$.

3. There are at least r values of j such that satisfies (B.4). For a give value of j, there are at least $(r+1)e^{h/2}$ values of k satisfies (B.5). Further, for a given value of j, (B.6) is satisfied for at least $\frac{re^{jh/2}}{\sqrt{x}}$ values of m_2, and for a given value of j and k, (B.7) is satisfied for at least $\frac{re^{jh}}{x}$ values of m_1. By Remark A.1, it follows that $Q_{rh}(x, y, z)$ must intersect at least

$$\sum_{j=\lfloor \frac{\ln x}{h} - \frac{(r+1)}{2} \rfloor}^{\lceil \frac{\ln x}{h} + \frac{(r+1)}{2} \rceil} (r+1)e^{h/2}\left(\frac{r^2 e^{3jh/2}}{x^{3/2}}\right) \ge r^2(r+1)e^{h/2}\left[\frac{(e^{\frac{3(r+3)h}{4}} - e^{-\frac{3(r+1)h}{4}})}{e^{3h/2} - 1} + \frac{(r+1)}{r^2}\right]$$

sets of the form $Q_h(e^{jh}, he^{-h/4}k, hA_{e^{jh}}^{-1}S_{he^{-h/4}k}^{-1}m)$. □

Proof of Proposition 3.10. $(\Rightarrow)$ is trivial.
$(\Leftarrow)$ Suppose there exists $h > 0$ such that $R := \sup_{(x,y,z)\in\mathbb{S}_2} \#_w(\Lambda \cap Q_h(x,y,z)) < \infty$.
For $0 < t < h$, we have $\#_w(\Lambda \cap Q_t(x,y,z)) \leq \#_w(\Lambda \cap Q_h(x,y,z))$ for all $(x,y,z) \in \mathbb{S}_2$.
Hence $\sup_{(x,y,z)\in\mathbb{S}_2} \#_w(\Lambda \cap Q_t(x,y,z)) < R$.
If $t \geq h$, assume $t = rh$ where $r \geq 1$. By Lemma B.1 the box $Q_{rh}(x,y,z)$ is covered by a union of at most N_r sets of the form $Q_h(e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}}S^{-1}_{he^{-h/4}k}m)$. This implies

$$\begin{aligned}\sup_{(x,y,z)\in\mathbb{S}_2} \#_w(\Lambda \cap Q_{rh}(x,y,z)) &\leq N_r \cdot \sup_{j,k\in\mathbb{Z},m\in\mathbb{Z}^2} \#_w(\Lambda \cap Q_h(e^{jh}, he^{-h/4}k, hA^{-1}_{e^{jh}}S^{-1}_{he^{-h/4}k}m) \\ &\leq N_r \cdot R.\end{aligned}$$

Thus,

$$\begin{aligned}D^+_w(\Lambda) &\leq \limsup_{r\to\infty} \frac{N_r R}{\frac{2(rh)^3}{3}(e^{3rh/4} - e^{-3rh/4})} \\ &= R \cdot \lim_{r\to\infty} \frac{r^2\left((r+1)e^{h/2}+1\right)}{\frac{2(rh)^3}{3}(e^{3rh/4} - e^{-3rh/4})} \left[\frac{(e^{\frac{3(r+5)h}{4}} - e^{-\frac{3(r-1)h}{4}})}{e^{3h/2}-1} + \frac{(e^{\frac{(r+5)h}{2}} - e^{-\frac{(r-1)h}{2}})}{r(e^h-1)} \right. \\ &\qquad \left. + \frac{(e^{\frac{(r+5)h}{4}} - e^{-\frac{(r-1)h}{4}})}{r(e^{h/2}-1)} + \frac{(r+2)}{r^2} \right] \\ &= \frac{3e^{17h/4}}{2h^3(e^{3h/2}-1)} < \infty.\end{aligned}$$

A similar argument shows the last equivalent conditions. $\square$

Appendix C

Basic Facts about Density for Shearlet Systems of $\mathbb{S}_3$

Lemma C.1. *Let $h > 0$ and $r \geq 1$ be given and let*

$$X_3 = \{(e^{jh}, hke^{jh/2}, hS_{hke^{jh/2}}A_{e^{jh}}m) : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

Then the following statements are true.

1. *X_3 is Q_h-dense in $\mathbb{S}_3$.*

2. *Any set $Q_{rh}(x,y,z)$ intersects at most*

$$N_r := r^3\frac{(e^{(r-1)h} - e^{-(r+5)h})}{e^{2h}-1} + 4r^4\frac{(e^{3(r-1)h/4} - e^{-3(r+5)h/4})}{e^{3h/2}-1}$$
$$+ (2r^2+4r)\frac{(e^{(r-1)h/2} - e^{-(r+5)h/2})}{e^{h}-1} + 8r\frac{(e^{(r-1)h/4} - e^{-(r+5)h/4})}{e^{h/2}-1} + 8(r+1)$$

   elements in X_3.

3. *Any set $Q_{rh}(x,y,z)$ contains at least*

$$\tilde{N}_r := r^3\frac{(e^{(r+1)h} - e^{-(r+3)h})}{e^{2h}-1}$$

   elements in X_3.

Proof. 1. Fix any $(x,y,z) \in \mathbb{S}_3$. We will show that there exist $(a,s,t) \in Q_h$, $j,k \in \mathbb{Z}$ and $m \in \mathbb{Z}^2$ such that

$$\begin{aligned}(x,y,z) &= (e^{jh}, hke^{jh/2}, hkS_{hke^{jh/2}}A_{e^{jh}}m) * (a,s,t)\\ &= (ae^{jh}, hke^{jh/2} + se^{jh/2}, hS_{hke^{jh/2}}A_{e^{jh}}(m+t)) \in Q_h(e^{jh}, hke^{jh/2}, hkS_{hke^{jh/2}}A_{e^{jh}}m).\end{aligned}$$

In particular,

$$x = ae^{jh}, \quad y = (hk+s)e^{jh/2}, \quad z = S_{hke^{jh/2}}A_{e^{jh}}(hm+t).$$

These three equalities are equivalent to

$$j = \frac{\ln a}{h} + \frac{\ln a}{h} \tag{C.1}$$

$$k = \frac{ye^{-jh/2}}{h} - \frac{s}{h} \tag{C.2}$$

$$m = \underbrace{\frac{1}{h} A_{e^{jh}}^{-1} S_{hke^{jh/2}}^{-1} z}_{=:C} - \frac{1}{h} t. \tag{C.3}$$

Now we observe the following

- For any $a \in [e^{-h/2}, e^{h/2})$, by using (C.1), we form the interval $[\frac{\ln x}{h} - \frac{1}{2}, \frac{\ln x}{h} + \frac{1}{2})$ which contains a unique integer j.
- For any $s \in [-\frac{h}{2}, \frac{h}{2})$, by (C.2) we form the interval $[\frac{ye^{-jh/2}}{h} - \frac{1}{2}, \frac{ye^{-jh/2}}{h} + \frac{1}{2})$ which contains a unique integer k.
- Following from (C.3), for any $t \in [-\frac{h}{2}, \frac{h}{2})^2$, we form intervals $[C_1 - \frac{1}{2}, C_1 + \frac{1}{2})$, and $[C_2 - \frac{1}{2}, C_2 + \frac{1}{2})$ which contain unique integers m_1 and m_2, respectively.

Therefore $\{Q_h(e^{jh}, hke^{jh/2}, hS_{hke^{jh/2}} A_{e^{jh}} m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ is a disjoint covering of $\mathbb{S}_3$, i.e, X_3 is Q_h-dense in $\mathbb{S}_3$.
2. Fix any $(x, y, z) \in \mathbb{S}_3$ and suppose $(u, v, w) \in Q_{rh}(x, y, z) \cap Q_h(e^{jh}, hke^{jh/2}, hS_{hke^{jh/2}} A_{e^{jh}} m)$. Then there exist $(a, s, t) \in Q_{rh}$ and $(a', s', t') \in Q_h$ such that

$$\begin{aligned} (u, v, w) &= (x, y, z) * (a, s, t) \\ &= (ax, y + s\sqrt{x}, z + S_y A_x t) \\ \text{and} \qquad (u, v, w) &= (e^{jh}, hke^{jh/2}, hS_{hke^{jh/2}} A_{e^{jh}} m) * (a', s', t') \\ &= (a'e^{jh}, hke^{jh/2} + s'e^{jh/2}, S_{hke^{jh/2}} A_{e^{jh}} (hm + t')). \end{aligned}$$

• In particular, $ax = a'e^{jh}$ with $a \in [e^{-rh/2}, e^{rh/2})$ and $a' \in [e^{-h/2}, e^{h/2})$, we have

$$\frac{\ln x}{h} - \frac{r+1}{2} \leq j \leq \frac{\ln x}{h} + \frac{r+1}{2}.$$

Therefore, this is satisfied for at most $r + 2$ values of j.
• Further, $y + s\sqrt{x} = hke^{jh/2} + s'e^{jh/2}$, i.e., $k = \frac{e^{-jh/2}}{h}(y + s\sqrt{x}) - \frac{s'}{h}$, where $s \in [-\frac{rh}{2}, \frac{rh}{2})$ and $s' \in [-\frac{h}{2}, \frac{h}{2})$. It follows that

$$\frac{ye^{-jh/2}}{h} - \frac{(re^{jh/2}\sqrt{x} + 1)}{2} \leq k \leq \frac{ye^{-jh/2}}{h} + \frac{(re^{jh/2}\sqrt{x} + 1)}{2}.$$

For a given value of j, this is satisfied for at most $re^{jh/2}\sqrt{x} + 2$ values of k.
• Furthermore, $z + S_y A_x t = S_{hke^{jh/2}} A_{e^{jh}} (hm + t')$ with $t_1, t_2 \in [-\frac{rh}{2}, \frac{rh}{2})$ and $t'_1, t'_2 \in [-\frac{h}{2}, \frac{h}{2})$. It follows that

$$m = \underbrace{\frac{1}{h} A_{e^{jh}}^{-1} S_{hke^{jh/2}}^{-1} z}_{=:C} + \frac{1}{h} A_{e^{jh}}^{-1} S_{hke^{jh/2}}^{-1} S_y A_x t - \frac{t'}{h}.$$

Note that $A_{e^{jh}}^{-1} S_{hke^{jh/2}}^{-1} S_y A_x = \begin{pmatrix} xe^{-jh} & y\sqrt{x}e^{-jh} - hke^{-jh/2}\sqrt{x} \\ 0 & \sqrt{x}e^{-jh/2} \end{pmatrix}$. Then we obtain

$$\begin{aligned} m_1 &= C_1 + \frac{xe^{-jh}}{h} t_1 + \frac{\left(y\sqrt{x}e^{-jh} - hke^{-jh/2}\sqrt{x}\right)}{h} t_2 - \frac{t_1'}{h} \\ m_2 &= C_2 + \frac{\sqrt{x}e^{-jh/2}}{h} t_2 - \frac{t_2'}{h}. \end{aligned}$$

Thus,

$$C_2 - \frac{(r\sqrt{x}e^{jh/2} + 1)}{2} \le m_2 \le C_2 + \frac{(r\sqrt{x}e^{jh/2} + 1)}{2}, \tag{C.4}$$

$$\begin{aligned} C_1 + \frac{\left(y\sqrt{x}e^{-jh} - hke^{-jh/2}\sqrt{x}\right)}{h} t_2 - \frac{(rxe^{-jh}+1)}{2} &\le m_1 \\ &\le C_1 + \frac{\left(y\sqrt{x}e^{-jh} - hke^{-jh/2}\sqrt{x}\right)}{h} t_2 + \frac{(rxe^{-jh}+1)}{2}. \end{aligned} \tag{C.5}$$

For a given value of j and k these two inequalities are satisfies at most $r\sqrt{x}e^{-jh/2} + 2$ values of m_2 and at most $rxe^{-jh} + 2$ values of m_1. Hence, by Lemma A.1 $Q_{rh}(x, y, z)$ can intersect at most

$$\begin{aligned} &\sum_{j=\lfloor \frac{\ln x}{h} - \frac{r+1}{2} \rfloor}^{\lceil \frac{\ln x}{h} + \frac{r+1}{2} \rceil} (r\sqrt{x}e^{-jh/2} + 2)^2 \left(rxe^{-jh} + 2\right) \\ &= \sum_{j=\lfloor \frac{\ln x}{h} - \frac{r+1}{2} \rfloor}^{\lceil \frac{\ln x}{h} + \frac{r+1}{2} \rceil} \left[r^3 x^3 e^{-2jh} + (2r^2 + 4r)xe^{-jh} + 4r^2 x^{3/2} e^{-3jh/4} + 8r\sqrt{x}e^{-jh/2} + 8\right] \\ &\le r^3 \frac{(e^{(r-1)h} - e^{-(r+5)h})}{e^{2h} - 1} + 4r^4 \frac{(e^{3(r-1)h/4} - e^{-3(r+5)h/4})}{e^{3h/2} - 1} \\ &\quad + (2r^2 + 4r) \frac{(e^{(r-1)h/2} - e^{-(r+5)h/2})}{e^h - 1} + 8r \frac{(e^{(r-1)h/4} - e^{-(r+5)h/4})}{e^{h/2} - 1} + 8(r + 1) \end{aligned}$$

sets of the form $Q_h(e^{jh}, hke^{jh/2}, hS_{hke^{jh/2}} A_{e^{jh}} m)$.
A similar argument proves the statement 3. $\square$

The following proposition describes how a subset Λ of $\mathbb{S}_3$ possesses finite upper density and positive lower density.

Proof of Proposition 3.14. ($\Rightarrow$) is trivial.
($\Leftarrow$) Suppose there exists $h > 0$ such that $R := \sup_{(x,y,z)\in\mathbb{S}_3} \#_w(\Lambda \cap Q_h(x, y, z)) < \infty$.
For $0 < t < h$, we have $\#_w(\Lambda \cap Q_t(x, y, z)) \le \#_w(\Lambda \cap Q_h(x, y, z))$ for all $(x, y, z) \in \mathbb{S}_3$.
Hence $\sup_{(x,y,z)\in\mathbb{S}_3} \#_w(\Lambda \cap Q_t(x, y, z)) < R$.
If $t \ge h$, assume $t = rh$ where $r \ge 1$. By Lemma C.1 the box $Q_{rh}(x, y, z)$ is covered by a union of at most N_r sets of the form $Q_h(e^{jh}, hke^{jh/2}, hS_{hke^{jh/2}} A_{e^{jh}} m)$. This implies

$$\begin{aligned} \sup_{(x,y,z)\in\mathbb{S}_3} \#_w(\Lambda \cap Q_{rh}(x, y, z)) &\le N_r \cdot \sup_{j,k\in\mathbb{Z}, m\in\mathbb{Z}^2} \#_w(\Lambda \cap Q_h(e^{jh}, hke^{jh/2}, hS_{hke^{jh/2}} A_{e^{jh}} m)) \\ &\le N_r \cdot R. \end{aligned}$$

Thus,

$$\begin{aligned}
D_w^+(\Lambda) &\leq \limsup_{r\to\infty} \frac{N_r R}{(rh)^3(e^{rh}-e^{-rh})} \\
&= \lim_{r\to\infty} \frac{r^3 R}{(rh)^3(e^{rh}-e^{-rh})}\left[\frac{(e^{(r-1)h}-e^{-(r+5)h})}{e^{2h}-1} + 4r^4\frac{(e^{3(r-1)h/4}-e^{-3(r+5)h/4})}{e^{3h/2}-1}\right. \\
&\quad \left. +(2r^2+4r)\frac{(e^{(r-1)h/2}-e^{-(r+5)h/2})}{e^{h}-1} + 8r\frac{(e^{(r-1)h/4}-e^{-(r+5)h/4})}{e^{h/2}-1} + 8(r+1)\right] \\
&= \frac{Re^{-h}}{h^3(e^{2h}-1)} < \infty.
\end{aligned}$$

A similar argument shows the last equivalent conditions. □

Appendix D

Basic Facts about Density for Shearlet Systems of $\mathbb{S}_4$

Lemma D.1. *Let $h > 0$ and $r \geq 1$ be given. Let*

$$X_4 = \{(e^{jh}, hke^{jh/2}, he^{-h/2}m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

Then the following statements hold:

1. *X_4 is Q_h-dense in $\mathbb{S}_4$.*
2. *Any set $Q_{rh}(x, y, z)$ intersects at most*

$$N_r := r(r+1)^2 \left[e^{3h/4} + \frac{1}{(r+1)}\right] \left[e^h + \frac{1}{(r+1)}\right] \left[\frac{(e^{(r+2)h/4} - e^{-(r+1)h/4})}{e^{h/2} - 1} + 1 + \frac{1}{r}\right]$$

   elements in X_4.

3. *Any set $Q_{rh}(x, y, z)$ contains at least*

$$\tilde{N}_r := r(r+1)^2 e^{7h/4} \frac{(e^{(r+2)h/4} - e^{-(r+1)h/4})}{e^{h/2} - 1}$$

   elements in X_4.

Proof. 1. Fix any $(x, y, z) \in \mathbb{S}_4$. We will show that there exist $(a, s, t) \in Q_h$, $j, k \in \mathbb{Z}$ and $m \in \mathbb{Z}^2$ such that

$$\begin{aligned}(x, y, z) &= (e^{jh}, hke^{jh/2}, he^{-h/2}m) \otimes (a, s, t) \\ &= (ae^{jh}, hke^{jh/2} + se^{jh/2}, t + he^{-h/2}A_a^{-1}S_s^{-1}m) \in Q_h(e^{jh}, hke^{jh/2}, he^{-h/2}m).\end{aligned}$$

In particular,

$$x = ae^{jh}, \quad y = (hk + s)e^{jh/2}, \quad z = t + he^{-h/2}A_a^{-1}S_s^{-1}m,$$

are equivalent to

$$j = \frac{\ln a}{h} + \frac{\ln a}{h} \tag{D.1}$$

$$k = \frac{ye^{-jh/2}}{h} - \frac{s}{h} \tag{D.2}$$

$$m = \underbrace{\frac{e^{h/2}}{h} S_s A_a z}_{=:C} - \frac{e^{h/2}}{h} S_s A_a t. \tag{D.3}$$

Now we observe the following

- Following from (D.1), for any $a \in [e^{-h/2}, e^{h/2})$, we form the interval $[\frac{\ln x}{h} - \frac{1}{2}, \frac{\ln x}{h} + \frac{1}{2})$ which contains a unique integer j.
- Take the same number a as above and $s \in [-\frac{h}{2}, \frac{h}{2})$. Following from (D.2), we form the interval $[\frac{ye^{-jh/2}}{h} - \frac{1}{2}, \frac{ye^{-jh/2}}{h} + \frac{1}{2})$ which contains a unique integer k.
- Take a, s as above, and for any $t_2 \in [-\frac{h}{2}, \frac{h}{2})$. By (D.3), we form the interval $\Big[C_2 - \frac{e^{h/2}\sqrt{a}}{2}, C_2 + \frac{e^{h/2}\sqrt{a}}{2}\Big)$ which contains integers m_2. Further, by using (D.3) for any $t_1 \in [-\frac{h}{2}, \frac{h}{2})$, we form the interval $\Big[C_1 - \frac{e^{h/2}s\sqrt{a}t_2}{h} - \frac{e^{h/2}a}{2}, C_1 - \frac{e^{h/2}s\sqrt{a}t_2}{h} + \frac{e^{h/2}a}{2}\Big)$ which contains integers m_1.

Therefore $\{Q_h(e^{jh}, hke^{jh/2}, he^{-h/2}m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}$ is a covering of $\mathbb{S}_4$, i.e., X_4 is Q_h-dense in $\mathbb{S}_4$.
2. Fix any $(x, y, z) \in \mathbb{S}_4$ and suppose $(u, v, w) \in Q_{rh}(x, y, z) \cap Q_h(e^{jh}, hke^{jh/2}, he^{-h/2}m)$. Then there exist $(a, s, t) \in Q_{rh}$ and $(a', s', t') \in Q_h$ such that

$$\begin{aligned} (u, v, w) &= (x, y, z) \otimes (a, s, t) \\ &= (ax, y + s\sqrt{x}, t + A_a^{-1} S_s^{-1} z) \\ \text{and} \qquad (u, v, w) &= (e^{jh}, hke^{jh/2}, he^{-h/2}m) \otimes (a', s', t') \\ &= (a'e^{jh}, hke^{jh/2} + s'e^{jh/2}, t' + he^{-h/2} A_{a'}^{-1} S_{s'}^{-1} m). \end{aligned}$$

- In particular, $ax = a'e^{jh}$ with $a \in [e^{-rh/2}, e^{rh/2})$ and $a' \in [e^{-h/2}, e^{h/2})$, we have

$$\frac{\ln x}{h} - \frac{r+1}{2} \le j \le \frac{\ln x}{h} + \frac{r+1}{2}.$$

  Therefore, this is satisfied for at most $r + 2$ values of j.

- Further, $y + s\sqrt{x} = hke^{jh/2} + s'e^{jh/2}$, i.e., $k = \frac{e^{-jh/2}}{h}(y + s\sqrt{x}) - \frac{s'}{h}$, where $s \in [-\frac{rh}{2}, \frac{rh}{2})$ and $s' \in [-\frac{h}{2}, \frac{h}{2})$. It follows that

$$\frac{ye^{-jh/2}}{h} - \frac{(re^{jh/2}\sqrt{x} + 1)}{2} \le k \le \frac{ye^{-jh/2}}{h} + \frac{(re^{jh/2}\sqrt{x} + 1)}{2}.$$

  For a given value of j, this is satisfied for at most $re^{jh/2}\sqrt{x} + 2$ values of k.

- Furthermore, $t + A_a^{-1}S_s^{-1}z = t' + he^{-h/2}A_{a'}^{-1}S_{s'}^{-1}m$ with $t \in [-\frac{rh}{2}, \frac{rh}{2})^2$ and $t' \in [-\frac{h}{2}, \frac{h}{2})^2$. It follows that

$$m = \underbrace{\frac{e^{h/2}}{h} S_{s'}A_{a'}A_a^{-1}S_s^{-1}z}_{=:C} + \frac{e^{h/2}}{h} S_{s'}A_{a'}(t - t').$$

Then we get

$$m_1 = C_1 + \frac{e^{h/2}}{h}[a'(t_1 - t_1') + s'\sqrt{a'}(t_2 - t_2')]$$
$$m_2 = C_2 + \frac{e^{h/2}\sqrt{a'}(t_2 - t_2')}{h}.$$

Thus,

$$C_2 - \frac{e^{3h/4}(r+1)}{2} \leq m_2 \leq C_2 + \frac{e^{3h/4}(r+1)}{2}, \tag{D.4}$$

and

$$C_1 - \frac{s'\sqrt{a'}e^{h/2}}{h}(t_2 - t_2') - \frac{e^h(r+1)}{2} \leq m_1 \leq C_1 - \frac{s'\sqrt{a'}e^{h/2}}{h}(t_2 - t_2') + \frac{e^h(r+1)}{2}. \tag{D.5}$$

For any fixed $a \in [e^{-rh/2}, e^{rh/2})$, $a' \in [e^{-h/2}, e^{h/2})$, $s \in [-\frac{rh}{2}, \frac{rh}{2})$ and $s' \in [-\frac{h}{2}, \frac{h}{2})$, (D.4) is satisfied for at most $e^{3h/4}(r+1)+1$ values of m_2 and (D.5) is satisfied at most $e^h(r+1)+1$ values of m_1.

By using Lemma A.1 we obtain that $Q_{rh}(x, y, z)$ can intersect at most

$$\sum_{j=\lfloor \frac{\ln x}{h} - \frac{r+1}{2} \rfloor}^{\lceil \frac{\ln x}{h} + \frac{r+1}{2} \rceil} (r\sqrt{x}e^{-jh/2} + 2)(e^h(r+1)+1)(e^{3h/4}(r+1)+1)$$
$$\leq (e^h(r+1)+1)(e^{3h/4}(r+1)+1)\left[r\frac{(e^{(r-1)h/4} - e^{-(r+5)h/4})}{e^{h/2}-1} + (r+1)\right]$$
$$= r(r+1)^2\left[e^{3h/4} + \frac{1}{(r+1)}\right]\left[e^h + \frac{1}{(r+1)}\right]\left[\frac{(e^{(r-1)h/4} - e^{-(r+5)h/4})}{e^{h/2}-1} + 1 + \frac{1}{r}\right]$$

sets of the form $Q_h(e^{jh}, hke^{jh/2}, he^{-h/2}m)$.
A similar argument proves the statement 3. □

The similar results in Proposition 3.5 in Section 3.2 hold for subset Λ of $\mathbb{S}_4$ as follow.

Proof of Proposition 3.16. ($\Rightarrow$) is trivial.
($\Leftarrow$) Suppose there exists $h > 0$ such that $R := \sup_{(x,y,z)\in\mathbb{S}_4} \#_w(\Lambda \cap Q_h(x, y, z)) < \infty$.
For $0 < t < h$, we have $\#_w(\Lambda \cap Q_t(x, y, z)) \leq \#_w(\Lambda \cap Q_h(x, y, z))$ for all $(x, y, z) \in \mathbb{S}_4$.
Hence $\sup_{(x,y,z)\in\mathbb{S}_4} \#_w(\Lambda \cap Q_t(x, y, z)) < R$.
If $t \geq h$, assume $t = rh$ where $r \geq 1$. By Lemma D.1 the box $Q_{rh}(x, y, z)$ is covered by a union of at most N_r sets of the form $Q_h(e^{jh}, hke^{jh/2}, he^{-h/2}m)$. This implies

$$\sup_{(x,y,z)\in\mathbb{S}_4} \#_w(\Lambda \cap Q_{rh}(x, y, z)) \leq N_r \cdot \sup_{j,k\in\mathbb{Z}, m\in\mathbb{Z}^2} \#_w(\Lambda \cap Q_h(e^{jh}, hke^{jh/2}, he^{-h/2}m))$$
$$\leq N_r \cdot R.$$

Thus,

$$
\begin{aligned}
D_w^+(\Lambda) &\leq \limsup_{r\to\infty} \frac{N_r R}{2(rh)^3(e^{rh/4} - e^{-rh/4})} \\
&= \lim_{r\to\infty} \frac{r(r+1)^2 R}{2(rh)^3(e^{rh/4} - e^{-rh/4})} \left[e^{3h/4} + \frac{1}{(r+1)} \right] \left[e^{h} + \frac{1}{(r+1)} \right] \\
&\quad \cdot \left[\frac{(e^{(r-1)h/4} - e^{-(r+5)h/4})}{e^{h/2} - 1} + 1 + \frac{1}{r} \right] \\
&= \frac{R e^{3h/2}}{2h^3(e^{h/2} - 1)} < \infty.
\end{aligned}
$$

A similar argument shows the last equivalent conditions. □

Appendix E

Matlab Code

```
function [z] = ShearMexHat(xi1,xi2,gamma)
    % xi1 < x < xi2 : a fixed interval
    % gamma : the decay rate
        zb = fminbnd(@(xi) maxpsi(xi), xi1, xi2);
        za = fminbnd(@(xi) minpsi(xi), xi1, xi2);
            for i =1:5
                c(i) = i*0.2;
                R(i) = c(i)∧(gamma/2)(eta(gamma/4)+1)*...
                          rest(xi1, xi2, c(i), gamma)*(2*zeta(gamma/2));
                A(i)= 2*pi/((c(i))∧2)*((eta(gamma)-1)* minpsi(za)-rest(xi1, xi2, c(i)));
                B(i)= 2*pi/((c(i))∧2)*(-(eta(gamma)+1)*maxpsi(zb)+ rest(xi1, xi2, c(i)));
                D(i)= B(i)/A(i);
            end
```

% eta(gamma)= $\sum_{k\in\mathbb{Z}} \frac{1}{|1+k^2|^\gamma}$ (we use Maple to compute these values)

% R(i) : Remainder term, A(i) : Upper frame bound, B(i) : Lower frame bound

```
function yb = maxpsi(xi)
```

% yb = $\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} \left|\hat{\psi}(S_k^{-T} A_{2^j}^{-1}\xi)\right|^2$

```
yy=0; N = 20; a=2;
        for m=-N:N
            yy = hpsi(a∧m*xi)+ yy;
        end
        yb = -yy ;
```

```
function ya = minpsi(xi)
```

% ya = $\operatorname*{ess\,inf}_{\xi\in\mathbb{R}^2} \left|\hat{\psi}(S_k^{-T} A_{2^j}^{-1}\xi)\right|^2$

```
yy=0; N = 20; a=2;
        for m=-N:N
            yy = hpsi(a∧m*xi)+ yy;
        end
        ya = yy ;
```

```
function f = hpsi(xi)
% f = |ψ̂(S_k^{-T} A_{2^j}^{-1} ξ)|^2
a=2;
    f = abs(2/sqrt(3)*pi∧(-1/4)*(a*xi)∧(2)*exp(-(a*xi)∧2)/2)∧(2);

function suma = rest(xi1,xi2,c,gamma)
% suma = Σ_{m∈Z^2\{0}} [Γ((1/c_0) m)Γ(-(1/c_0) m)]^{1/2}
        for m1 =1:20
            x(m1) = fminbnd(@(xi) res(xi,m1,c,gamma),xi1,xi2);
            val(m1) = sqrt(res(x(m1),m1,c,gamma)*res(x(m1),m1,-c,gamma));
        end
        suma = sum(val) ;

function f = res(xi,m1,c,gamma)
% f = [Γ((1/c_0) m)Γ(-(1/c_0) m)]^{1/2}
a=2;tq=0; N=10;
        for j=-N:N
            tq = (hpsi(a∧j*xi,m1,c,gamma)+hpsi(a∧j*xi,-m1,c,gamma))+tq;
        end
        f = -tq;

function f = hpsi(xi1,m1,c,gamma)
% f = Γ(ω) = ess sup_{ξ∈R^2} Σ_{j,k∈Z} |ψ̂(S_{s_k}^{-T} A_{a_j}^{-1} ξ)| |ψ̂(S_{s_k}^{-T} A_{a_j}^{-1} ξ + 2ω)|
% M(gamma) = max { |2^{-j} ξ_1|^{γ/2}, |2^{-j} ξ_1 + m_1/c_0|^{γ/2} }
a = 2;
    f = abs((2/sqrt(3)*pi∧(-1/4))∧1*abs(a*xi1)∧(2)* exp(-((a*xi1)∧2)/2)...
    *abs((2/sqrt(3)*pi∧(-1/4))∧1*(abs(a*xi1+2*pi*m1/bb))∧(2)...
    *exp(-((a*xi1 + 2*pi*m1/bb)∧2)/2))*M(gamma);
```