A Theorem on the Amplitudes of Periodic Solutions of Differential Delay Equations with Applications to Bifurcation

HANS-OTTO WALTHER

Mathematisches Institut der Universität München, Theresienstr. 39, D 8000 München 2

1. INTRODUCTION

Since the work of G. S. Jones [10] on the equation

$$y'(t) = -\alpha y(t-1)[1 + y(t)] \tag{1}$$

with real parameter α, several methods were developed to prove the existence of periodic solutions of differential delay equations; see, for example, the papers of Grafton [7], Nussbaum [14, 15], Kaplan and Yorke [11–13], and Chow [3]. Bifurcation from the trivial solution was studied, notably by Chafee [2], Nussbaum [16–18], Cushing [6], and Chow and Mallet-Paret [4, 5].

We consider the equation

$$x'(t) = -\alpha f(x(t-1)) \tag{2}$$

with parameter $\alpha > 0$. We assume that f is a continuous real function defined on R with property

(H) $\xi f(\xi) > 0$ for $\xi \neq 0$, f bounded from below, f differentiable at $\xi = 0$, $f'(0) = 1$.

These equations generalize Eq. (1) in a certain sense: With regard to solutions $y > -1$, Eq. (1) is equivalent to Eq. (2) with $f(\xi) = e^{\xi} - 1$ by the transformation $x = \log(1 + y)$.

In [16], Nussbaum showed that there is a continuum of (initial values of) nontrivial periodic solutions of Eq. (2) which bifurcates from the trivial solution at $\alpha = \pi/2$. The instability of the linear equation

$$z'(t) = -\alpha z(t-1) \tag{3}$$

implies that for every $\alpha > \pi/2$ there is a periodic solution of Eq. (2) which belongs to the continuum. For $\alpha \leqslant \pi/2$, the zero solution of Eq. (3) is stable—see Theorem 5 in [20] and Chap. 4 in [1]. In this case, Nussbaum's result provides no information about nontrivial periodic solutions in the continuum.

In this paper we give sufficient conditions on the function f in Eq. (2) such that the continuum given by Nussbaum's result leaves the bifurcation point at $\alpha = \pi/2$ in the direction of decreasing α (Theorem 3). This means in particular that there are equations of type (2) which have nontrivial periodic solutions also in the case of asymptotically stable linearization $\alpha < \pi/2$.

We shall derive Theorem 3 with the aid of Theorem 2, which is our central result. Theorem 2 establishes a relation between the slope of the function αf on the right side of Eq. (2) and the amplitudes of periodic solutions. It will be proved by a variant of a method invented by Kaplan and Yorke in order to show the existence and certain stability properties of periodic solutions [12, 13].

In addition, Theorem 2 leads to global results on the location of the continuum of periodic solutions given by Nussbaum. Corollary 1 below permits an application of this type to the well-studied example (1).

Theorem 3 does not apply to Eq. (1). Chow and Mallet-Paret who extended the local theory of Hopf bifurcation to functional differential equations proved that Hopf bifurcation to the right takes place if the parameter α in Eq. (1) passes the critical value $\pi/2$ [4]. Their work should also imply local results similar to our Theorem 3. However, the technique of averaging, which is used to determine the direction of Hopf bifurcation, requires additional smoothness properties of the function f in Eq. (2).

2. Results

We shall deal with a special kind of periodic solutions.

Definition. A differentiable periodic function $x: R \to R$ with period $p > 0$ is said to be slowly oscillating if x has zeros z_1, $z_2 > z_1$ such that $x' < 0$ in $[z_1, z_1 + 1)$, $0 < x'$ in $(z_1 + 1, z_2 + 1)$ and $x' < 0$ in $(z_2 + 1, z_1 + p]$.

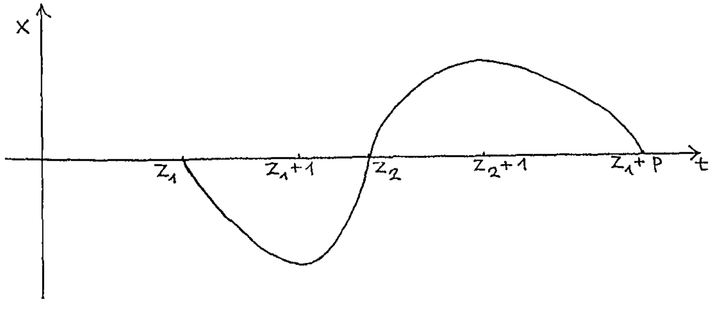

Figure 1

Let C denote the Banach space of continuous functions $\varphi: [-1, 0] \to R$ with supremum-norm, and equip the set $C \times R^+$ with the metric $((\varphi, \alpha), (\psi, \beta)) \mapsto \| \varphi - \psi \| + | \alpha - \beta |$. We restate Nussbaum's Theorem 2.1 in [16] in a slightly different way as

THEOREM 1 (Nussbaum). *Let $f: R \to R$ be a continuous function with property* (H). *Then there exists a closed connected set $P \subset C \times R^+$ with the following properties*:

(i) $(0, \pi/2) \in P$, *and for every $\alpha > \pi/2$ there is a function $\varphi \in C$ with $(\varphi, \alpha) \in P$*,

(ii) $\varphi \neq 0$, *if $(\varphi, \alpha) \in P$ and $\alpha \neq \pi/2$*,

(iii) *if $(\varphi, \alpha) \in P$ and $\varphi \neq 0$, then φ increases on $[-1, 0]$, $\varphi(-1) = 0$, and φ is the restriction of a slowly oscillating periodic function x which satisfies $x'(t) = -\alpha f(x(t-1))$ on R.*

Let us now look for conditions which guarantee that the pairs $(\varphi, \alpha) \in P$ with $\alpha > \pi/2$ (or with $\alpha \geqslant \pi/2$ and $\varphi \neq 0$) have a certain distance from $(0, \alpha)$. It turns out that it is sufficient to ensure that the amplitudes $\min x$ or $\max x$ of the corresponding periodic solutions are bounded away from zero. As an example, consider the function $f = f_1$ with $f_1(\xi) = \xi$ for $\xi \geqslant -1$, $f_1(\xi) = -1$ for $\xi < -1$. Then every solution $x > -1$ of Eq. (2) satisfies Eq. (3), which has no slowly oscillating periodic solution for $\alpha > \pi/2$ (see Theorem 5 and Theorem 6 in [20]). Therefore, $\min x \leqslant -1$ for every slowly oscillating periodic solution of Eq. (2) with $\alpha > \pi/2$.

We may expect a similar behavior with vast amplitudes of periodic solutions if the differential equation is more unstable than in the preceding example. If we interpret Theorem 5 in [20] in the sense that the instability of the linear equation (3) increases with α, then we should conjecture that Eq. (2) becomes more unstable if the slope of the function $g = \alpha f$ is increased. Concerning the amplitudes of periodic solutions, we are led to

THEOREM 2. *Let $g: R \to R$ be a continuous function with $\xi g(\xi) > 0$ for $\xi \neq 0$. Assume that g is differentiable in an interval (a, b), $a < 0 < b$, with $g'(0) \geqslant \pi/2$ and $g'(\xi) > \pi/2$ for $\xi \neq 0$, $a < \xi < b$. It follows that the equation*

$$x'(t) = -g(x(t-1)) \tag{4}$$

has no slowly oscillating periodic solution with $x(R) \subset (a, b)$.

The example $g = \pi/2\, f_1$ shows that the condition $g'(\xi) > \pi/2$ for $\xi \neq 0$ cannot be weakened. We prove Theorem 2 in Section 3.

THEOREM 3. *Let $f: R \to R$ be a continuous function with property* (H) *which is differentiable in an interval $(-a, a)$, $a > 0$, with $f'(\xi) \geqslant 1$ for $|\xi| < a$.*

(i) *Then there is a neighborhood U of $(0, \pi/2)$ in $C \times R^+$ with $\alpha \leqslant \pi/2$ for every pair (φ, α) in the nonempty set $(P \cap U)\backslash\{(0, \alpha) \mid \alpha > 0\}$.*

(ii) *If in addition $f'(\xi) > 1$ for $0 < |\xi| < a$, then there is a neighborhood U of $(0, \pi/2)$ in $C \times R^+$ with $\alpha < \pi/2$ for every pair (φ, α) in the nonempty set $(P \cap U)\backslash\{(0, \alpha) \mid \alpha > 0\}$.*

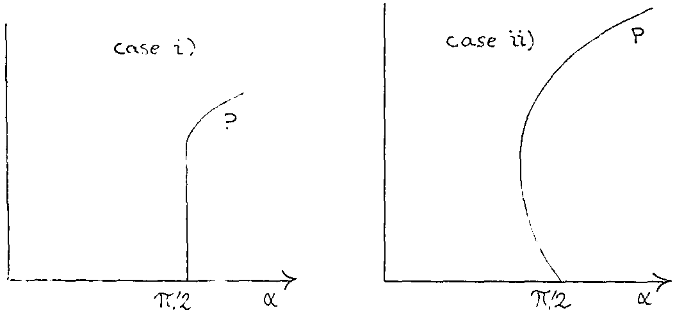

FIGURE 2

Proof of Theorem 3. (i) Let $(\varphi, \alpha) \in P$, $\pi/2 < \alpha \leqslant 3$. Then $\varphi = x \mid [-1, 0]$ with a slowly oscillating periodic solution x of Eq. (2). Theorem 2 for $g = \alpha f$ gives $a < \max x = \varphi(0)$ or $x(z_1 + 1) = \min x \leqslant -a < 0$, since x is slowly oscillating. $\varphi(0) = \max x \leqslant a$ and $x(z_1 + 1) \leqslant -a$ imply

$$-a \geqslant x(z_1 + 1) - x(z_1) = \int_{z_1}^{z_1+1} x'(s)\, ds = -\alpha \int_{z_1}^{z_1+1} f(x(s-1))\, ds$$

$$\geqslant -\alpha \max f \mid [0, \max x] \geqslant -3 \max f \mid [0, \max x] = -3f(\varphi(0)),$$

since $f' > 0$ on $[0, a)$. We infer the existence of $\delta > 0$ with $\| \varphi \| = \varphi(0) > \delta$ for every $(\varphi, \alpha) \in P$ with $\pi/2 < \alpha \leqslant 3$. Set $U := \{\varphi \in C \mid \| \varphi \| < \delta\} \times \{(0, 3)\}$. Obviously, $\alpha \leqslant \pi/2$ for every $(\varphi, \alpha) \in P \cap U$. Theorem 1 implies that $(P \cap U) \backslash \{(0, \alpha) \mid \alpha > 0\}$ is nonempty.

(ii) As above, Theorem 2 yields the existence of $\delta > 0$ with $\| \varphi \| > \delta$ for $(\varphi, \alpha) \in P$, $\pi/2 \leqslant \alpha \leqslant 2$, $\varphi \neq 0$. Set $U := \{\varphi \in C \mid \| \varphi \| < \delta\} \times \{(0, 2)\}$. The conclusion follows as in (i).

Now let us consider Eq. (1) which is equivalent to a simple population growth model (see G. E. Hutchinson [8]). Theorem 2 yields upper bounds for the (negative) minima of slowly oscillating periodic solutions.

COROLLARY 1. *For every slowly oscillating periodic solution y of Eq.* (1) *with $\alpha > \pi/2$, we have* $-1 < \min y \leqslant \pi/2\alpha - 1$.

Proof. Let y be such a solution. Theorem 1 in [20] gives $y > -1$. Set $x := \log(y + 1)$. Then $x'(t) = -\alpha(e^{x(t-1)} - 1)$. Our Theorem 2 with $a := \log \pi/2\alpha$, $b := \infty$, $g(\xi) := \alpha(e^{\xi} - 1)$ and $\alpha > \pi/2$ implies $\min x \leqslant \log \pi/2\alpha$; hence, $-1 < \min y \leqslant \pi/2\alpha - 1$.

Corollary 1 may be used with the estimates of Jones [9] to describe the location of the branch of slowly oscillating periodic solutions in $C \times R^+$.

Because the result of Chow and Mallet-Paret on the direction of bifurcation is local, the problem of periodic solutions of Eq. (1) for $\alpha \leqslant \pi/2$ remains open.

E. M. Wright's Theorem 3 in [20] and the author's Corollary 3 in [19] imply that nonconstant periodic solutions are impossible for α in an open interval $(0, \alpha_0)$ with $3/2 < \alpha_0 \leqslant \pi/2$.

3

The proof of Theorem 2 is based on a technique used by Kaplan and Yorke to prove the important Lemma 3.4 in [13]. Compare also [12]. In that lemma, Kaplan and Yorke gave sufficient conditions such that the trajectories (x, x') and (y, y') in R^2 of two solutions x and y of a differential delay equation cannot intersect.

Proof of Theorem 2. Assume that x is a slowly oscillating periodic solution of Eq. (4) with period p such that $x(R) \subset (a, b)$. We derive a contradiction.

Set $I := [z_1, z_1 + p]$. The trajectory $X: I \ni t \mapsto (x(t), x'(t)) \in R^2$ is a simple closed curve with $(0, 0) \notin X(I)$. There is a number $c > 0$ such that the ellipse $E(c) := \{c(\sin \pi t/2, \pi/2 \cos \pi t/2) \mid 2 \leqslant t \leqslant 6\}$ has a point in common with $X(I)$, while $X(I) \cap E(c') = \varnothing$ for $0 \leqslant c' < c$. We have $E(c) = Y([2, 6])$ for the trajectory $Y: [2, 6] \ni t \mapsto (y(t), y'(t)) \in R^2$ of the solution $y: t \mapsto c \sin \pi t/2$ of Eq. (3) with $\alpha = \pi/2$. Obviously, y is a slowly oscillating periodic function.

We may assume that there is a point $S = (s_1, s_2) \in X(I) \cap E(c)$ with $s_2 > 0$ or with $s_2 = 0$ and $s_1 > 0$. The proof in the other case is the same, with some signs reversed.

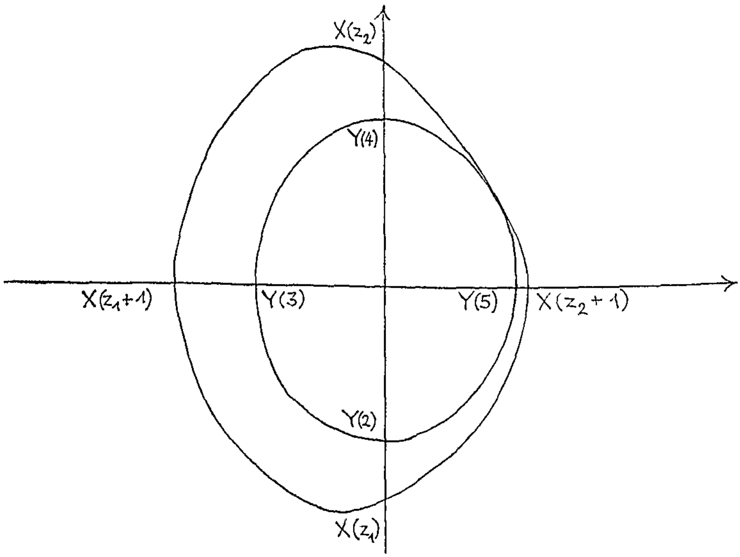

FIGURE 3

We have $X(t_x) = S = Y(t_y)$ with $z_1 + 1 < t_x \leqslant z_2 + 1$ and $3 < t_y \leqslant 5$. It will be sufficient to discuss the cases

(i) $z_1 + 1 < t_x < z_1 + 2$ and

(ii) $z_1 + 2 \leqslant t_x \leqslant z_2 + 1$ and $X(t) \notin E(c)$ for $z_1 + 1 < t < t_x$.

First, we have

(a) $x(t_x - 1) = 0 = y(t_y - 1)$ or $y(t_y - 1) < x(t_x - 1) < 0$.

Proof. $x'(t_x) = y'(t_y)$ gives $g(x(t_x - 1)) = \pi/2y(t_y - 1)$. By $z_1 < t_x - 1 \leqslant z_2$, $x(t_x - 1) \leqslant 0$. Hence, $x(t_x - 1) = 0 = y(t_y - 1)$ or, by $x(R) \subset (a, b)$ and $g'(\xi) > \pi/2$ for $a < \xi < 0$, $y(t_y - 1) < x(t_x - 1) < 0$.

(b) $x(z_1 + 1) \leqslant y(3) < 0$.

Proof. $x'(z_1 + 1) = 0 = y'(3)$ and $y(3) < x(z_1 + 1) < 0$ would imply $X(z_1 + 1) \in E(c')$ with $0 < c' < c$.

(c) By $x(z_2) = 0$ and by $x' > 0$ on $(z_1 + 1, z_2]$, (b) implies the existence of exactly one $\tilde{t} \in [z_1 + 1, z_2)$ with $x(\tilde{t}) = y(3)$.

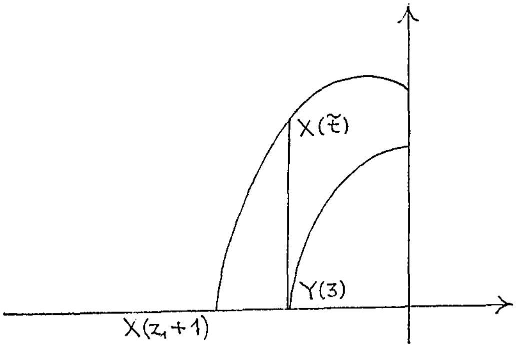

FIGURE 4

We have $t_x > \tilde{t}$.

Proof. $z_1 + 1 < t_x < \tilde{t}$ gives $x(t_x) < x(\tilde{t}) = y(3) = \min y \mid [3, 5] \leqslant y(t_y)$, a contradiction. $t_x = \tilde{t}$ gives $y(t_y) = x(t_x) = x(\tilde{t}) = y(3) = \min y \mid [2, 6]$, therefore $y'(t_y) = 0$; hence, $x'(t_x) = y'(t_y) = 0$, a contradiction to $z_1 + 1 < t_x = \tilde{t} < z_2$.

(d) We want to describe the fact that the trajectory X is outside the trajectory Y near S in terms of the second components x' and y'. To this end we introduce a parameter transformation which equates the first components. Set $p := x \mid (\tilde{t}, t_x)$ and $q := y \mid (3, t_y)$. Both functions map their interval of definition onto the interval $(x(\tilde{t}), x(t_x)) = (y(3), y(t_y))$, with a positive derivative. Therefore, $T := q^{-1} \circ p$ defines a mapping from $(\tilde{t}, t_x)$ onto $(3, t_y)$ with derivative $T' = x'/y' \circ T > 0$ and with $x(t) = y(T(t))$ for $\tilde{t} < t < t_x$.

We have $x'(t) \geqslant y'(T(t))$ for $\tilde{t} < t < t_x$.

Proof. $0 \leqslant x'(t) < y'(T(t))$ and $x(t) = y(T(t))$ imply $X(t) \in E(c')$ with $0 < c' < c$, a contradiction.

(e) The case (ii): First, we have $t_x - 1 \geqslant \tilde{t}$.

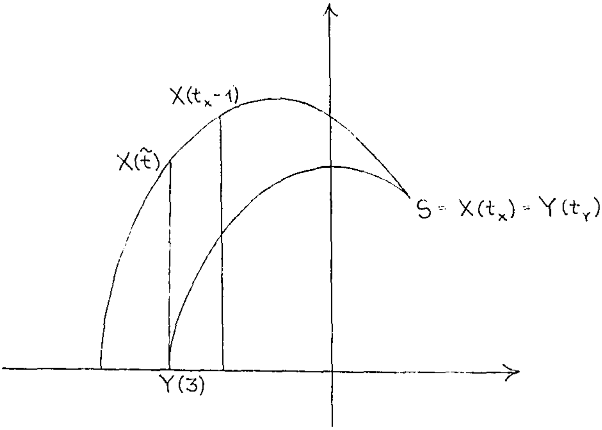

FIGURE 5

Proof. By (a), $x(\tilde{t}) = y(3) = \min y \mid [2, 4] \leqslant y(t_y - 1) \leqslant x(t_x - 1)$, and the assertion follows from $z_1 + 1 \leqslant t_x - 1 \leqslant z_2$ and from $x' > 0$ on $(z_1 + 1, z_2]$.

Therefore, T maps $(t_x - 1, t_x)$ onto an open interval (t^*, t_y) with $t^* \geqslant 3$. We have $T'(t) > 1$ for $t_x - 1 < t < t_x$.

Proof. $0 < x'(t) \leqslant y'(T(t))$ and $x(t) = y(T(t))$ imply $X(t) \in E(c')$ with $0 < c' \leqslant c$ which is impossible in case (ii) for $t \in (t_x - 1, t_x) \subset (z_1 + 1, t_x)$. Hence, $x'(t) > y'(T(t))$, $T'(t) = x'(t)/y'(T(t)) > 1$.

We obtain a contradiction to (a):

$$1 = \int_{t_x-1}^{t_x} 1\, dt < \int_{t_x-1}^{t_x} T(t)\, dt = \int_{t^*}^{t_y} 1\, dt = t_y - t^*$$

yields $t^* < t_y - 1 < t_y$ or $t_y - 1 = T(t)$ with $t \in (t_x - 1, t_x)$. Hence, $y(t_y - 1) = x(t) > x(t_x - 1)$, a contradiction.

(f) The case (i). First we use the transformation T to derive an estimate from the fact that X approaches Y near S from outside.

We have $y'(t_y) = x'(t_x) \neq 0$ in case (i), since $z_1 + 1 < t_x < z_1 + 2 < z_2 + 1$. So T may be defined on $(\tilde{t}, t_x + \epsilon)$ for some $\epsilon > 0$ in the same way as in (d). We obtain $T(t_x) = t_y$ and $T'(t_x) = 1$, by $T' = x'/y' \circ T$ and $x'(t_x) = y'(t_y) = y'(T(t_x))$.

Consider the difference $d := x' - y' \circ T$. $d(t_x) = 0$, (c) and $t < t_x$ give $0 \geqslant [d(t) - d(t_x)]/t - t_x$. Hence, $0 \geqslant d'(t_x) = x''(t_x) - y''(T(t_x))\, T'(t_x) = x''(t_x) - y''(t_y) = -g'(x(t_x - 1))\, x'(t_x - 1) + \pi/2y'(t_y - 1)$. By $x'(t_x - 1) < 0$ (since $z_1 < t_x - 1 < z_1 + 1$) and by $g'(x(t_x - 1)) \geqslant \pi/2$, we arrive at $0 > x'(t_x - 1) \geqslant y'(t_y - 1)$.

On the other hand, we can show $y'(t_y - 1) > x'(t_x - 1)$. This is obvious if $y'(t_y - 1) \geqslant 0$. Let $y'(t_y - 1) < 0$. Then $2 \leqslant t_y - 1 < 3$. By (a) and by $z_1 < t_x - 1 < z_1 + 1$, $y(3) < y(t_y - 1) < x(t_x - 1) < 0 = y(2)$. Hence, $x(t_x - 1) = y(t^*)$ with $2 < t^* < 3$. $y' < 0$ on $(2, 3)$ and $y(t_y - 1) < x(t_x - 1) = y(t^*)$ imply $2 < t^* < t_y - 1 < 3$. By our special choice of the comparison solution y, we infer $y'(t_y - 1) > y'(t^*)$.

In addition, $y'(t^*) \geqslant x'(t_x - 1)$.

Proof. $y'(t^*) < x'(t_x - 1) < 0$ and $y(t^*) = x(t_x - 1)$ yield $X(t_x - 1) \in E(c')$ with $0 < c' < c$.

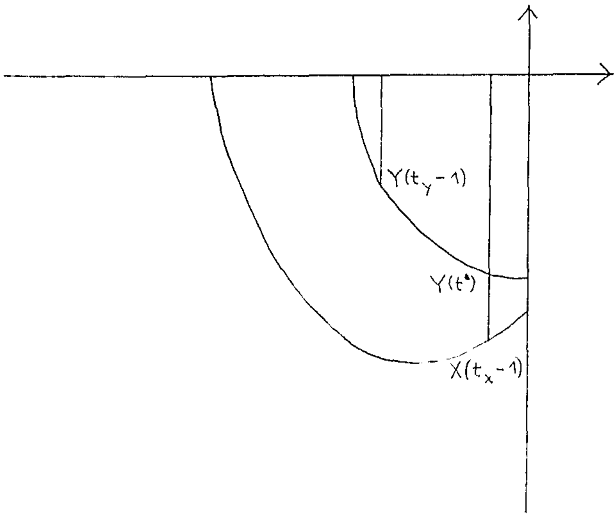

FIGURE 6

With this last assertion, we obtain $y'(t_y - 1) > x'(t_x - 1)$, and the proof of Theorem 2 is complete.

REFERENCES

1. R. BELLMAN AND K. COOKE, "Differential-Difference Equations," Academic Press, New York, 1963.

2. N. N. CHAFEE, A bifurcation problem for a functional differential equation of finitely retarded type, *J. Math. Anal. Appl.* **35** (1971), 312–348.
3. S. N. CHOW, Existence of periodic solutions of autonomous functional differential equations, *J. Differential Equations* **15** (1974), 350–378.
4. S. N. CHOW AND J. MALLET-PARET, Integral averaging and bifurcation, *J. Differential Equations* **26** (1977), 112–159.
5. S. N. CHOW AND J. MALLET-PARET, Fuller's index and global Hopf's bifurcation, to appear.
6. J. M. CUSHING, Periodic solutions of Volterra's population equation with hereditary effects, *SIAM J. Appl. Math.* **31** (1976), 251–261.
7. R. B. GRAFTON, A periodicity theorem for autonomous functional differential equations, *J. Differential Equations* **6** (1969), 87–109.
8. G. E. HUTCHINSON, Circular causal systems in ecology, *Ann. N. Y. Acad. Sci.* **50** (1948), 221–246.
9. G. S. JONES, On the nonlinear differential difference equation $f'(x) = -\alpha f(x - 1) \times \{1 + f(x)\}$, *J. Math. Anal. Appl.* **4** (1962), 440–469.
10. G. S. JONES, The existence of periodic solutions of $f'(x) = -\alpha f(x - 1)\{1 + f(x)\}$, *J. Math. Anal. Appl.* **5** (1962), 435–450.
11. J. L. KAPLAN AND J. A. YORKE, Ordinary differential equations which yield periodic solutions of differential delay equations, *J. Math. Anal. Appl.* **48** (1974), 317–324.
12. J. L. KAPLAN AND J. A. YORKE, On the stability of a periodic solution of a differential delay equation, *SIAM J. Math. Anal.* **6** (1975), 268–282.
13. J. L. KAPLAN AND J. A. YORKE, On the nonlinear differential delay equation $x'(t) = -f(x(t), x(t - 1))$, *J. Differential Equations* **23** (1977), 293–314.
14. R. D. NUSSBAUM, Periodic solutions of some nonlinear autonomous functional differential equations, *Ann. Mat. Pura Appl.* **101** (1974), 263–338.
15. R. D. NUSSBAUM, Periodic solutions of some nonlinear autonomous functional differential equations, II, *J. Differential Equations* **14** (1973), 360–394.
16. R. D. NUSSBAUM, A global bifurcation theorem with applications to functional differential equations, *J. Functional Analysis* **19** (1975), 319–338.
17. R. D. NUSSBAUM, Global bifurcation of periodic solutions of some autonomous functional differential equations, *J. Math. Anal. Appl.* **55** (1976), 699–725.
18. R. D. NUSSBAUM, A Hopf global bifurcation theorem for retarded functional differential equations, to appear.
19. H. O. WALTHER, Stability of attractivity regions for autonomous functional differential equations, *Manuscripta Math.* **15** (1975), 349–363.
20. E. M. WRIGHT, A non-linear difference-differential equation, *J. Reine Angew. Math.* **194** (1955), 66–87.