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The Morse Property of Limit Functions Appearing in Mean Field Equations on Surfaces with Boundary

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Abstract

In this paper, we study the Morse property for functions related to limit functions of mean field equations on a smooth, compact surface Σ with boundary $\partial\Sigma$. Given a Riemannian metric g on Σ we consider functions of the form

$$f_g(x) := \sum_{i=1}^{m} \sigma_i^2 R^g(x_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \sigma_i\sigma_j G^g(x_i,x_j) + h(x_1,\dots,x_m),$$

where $\sigma_i \neq 0$ for $i = 1,\dots,m$, G^g is the Green function of the Laplace-Beltrami operator on (Σ, g) with Neumann boundary conditions, R^g is the corresponding Robin function, and $h \in \mathcal{C}^2(\Sigma^m, \mathbb{R})$ is arbitrary. We prove that for any Riemannian metric g, there exists a metric $\widetilde{g}$ which is arbitrarily close to g and in the conformal class of g such that $f_{\widetilde{g}}$ is a Morse function. Furthermore we show that, if all $\sigma_i > 0$, then the set of Riemannian metrics for which f_g is a Morse function is open and dense in the set of all Riemannian metrics.

Keywords Green function · Robin function · Morse property · Transversality theorem

Mathematics Subject Classification 35J08 · 35J25 · 53C21 · 58J60

1 Introduction and Main Results

Let Σ be a smooth, compact surface with boundary $\partial\Sigma$. For a Riemannian metric g on Σ let $G^g : \Sigma \times \Sigma \to \mathbb{R}$ be the Green function for the Laplace-Beltrami operator

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$-\Delta_g$ with Neumann boundary conditions and mean value 0; i.e. for each $\xi \in \Sigma$ the function $G^g(\cdot, \xi)$ is the unique solution of

$$\begin{cases} -\Delta_g G^g(\cdot, \xi) = \delta_\xi - \dfrac{1}{|\Sigma|_g} & \text{in } \mathring{\Sigma} := \Sigma \setminus \partial\Sigma, \\ \partial_{\nu_g} G^g(\cdot, \xi) = 0 & \text{on } \partial\Sigma, \\ \displaystyle\int_\Sigma G^g(\cdot, \xi)\, dv_g = 0. \end{cases}$$

Here ν_g is the unit outward normal on $\partial\Sigma$, dv_g is the area element of Σ, $|\Sigma|_g = \int_\Sigma dv_g$ and δ_ξ is the Dirac distribution on Σ concentrated at $\xi \in \Sigma$. Setting $\kappa(\xi) = 2\pi$ for $\xi \in \mathring{\Sigma}$ and $\kappa(\xi) = \pi$ for $\xi \in \partial\Sigma$, the Robin function $R^g : \Sigma \to \mathbb{R}$ is defined by

$$R^g(\xi) := \lim_{x \to \xi} \left(G^g(x, \xi) + \frac{1}{\kappa(\xi)} \log d_g(x, \xi) \right),$$

where d_g denotes the distance with respect to the metric g. Given integers $m \geq l \geq 0$ with $m \geq 1$ and real numbers $\sigma_i \neq 0$, $i = 1, \ldots, m$, we set

$$X := \mathring{\Sigma}^l \times (\partial\Sigma)^{m-l} \qquad \text{and} \qquad \Delta_X := \{x \in X : x_i = x_j \text{ for some } i \neq j\},$$

and define, for an arbitrary given function $h \in \mathcal{C}^2(\Sigma^m, \mathbb{R})$:

$$f_g : X \setminus \Delta_X \to \mathbb{R}, \quad f_g(x) = \sum_{i=1}^m \sigma_i^2 R^g(x_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^m \sigma_i \sigma_j G^g(x_i, x_j) + h(x_1, \ldots, x_m). \tag{1.1}$$

We will prove that f_g is a Morse function for a “generic” metric g on Σ, in a sense that will be made precise below.

Functions of the form (1.1) play a crucial role in understanding the blow-up behavior of solutions of mean field equations like

$$\begin{cases} -\Delta_g u = \lambda \left(\frac{Ve^u}{\int_\Sigma Ve^u\, dv_g} - \frac{1}{|\Sigma|_g} \right) & \text{in } \mathring{\Sigma}, \\ \partial_{\nu_g} u = 0 & \text{on } \partial\Sigma, \\ \int_\Sigma u\, dv_g = 0. \end{cases} \tag{1.2}$$

For a compact surface without boundary it has been proven in [7] that a nondegenerate or more generally, an isolated stable critical point $x = (x_1, \ldots, x_m)$ of f_g, with $l = m$ and $h(x) = 2\sum_{i=1}^m \sigma_i \log V(x_i)$, gives rise to solutions (λ, u) of (1.2) with λ close to $8\pi m$ and such that u blows up as $\lambda \to 8\pi m$ precisely at $x_1, \ldots, x_m \in \Sigma$. Similar results have been obtained in the case of mean field equations on a bounded domain in $\mathbb{R}^2$ with Dirichlet boundary conditions in [8, 13], and for solutions of Gelfand's problem and steady states of the Keller–Segel system (see [2, 5, 6, 10] for instance).

Now we state our main results. We fix an integer $k \geq 0$ and $0 < \alpha < 1$, and let $\mathrm{Riem}^{k+2,\alpha}(\Sigma)$ be the space of Riemannian metrics of class $\mathcal{C}^{k+2,\alpha}$ on Σ, i.e. the space of symmetric and positive definite sections $\Sigma \to (T\Sigma \otimes T\Sigma)^*$ of class $\mathcal{C}^{k+2,\alpha}$.

Theorem 1.1 *If $\sigma_i \neq 0$ for $i = 1, \dots, m$ and $\sigma_1 + \cdots + \sigma_m \neq 0$ then for any $g \in \mathrm{Riem}^{k+2,\alpha}(\Sigma)$ the set*

$$\mathcal{M}_g^{k+2,\alpha}(\Sigma) := \{\psi \in \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+) : f_{\psi g} \textit{ is a Morse function }\}$$

is a residual subset of $\mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$.

As a consequence of Theorem 1.1 for any Riemannian metric g on Σ, there exists a metric $\widetilde{g}$ which is arbitrarily close to g and conformal to g, and such that $f_{\widetilde{g}}$ is a Morse function.

Theorem 1.2 *If $\sigma_i > 0$ for all $i = 1, \dots, m$ then the set*

$$\mathrm{Riem}^{k+2,\alpha}_{Morse}(\Sigma) := \{g \in \mathrm{Riem}^{k+2,\alpha}(\Sigma) : f_g \textit{ is a Morse function}\}$$

is an open and dense subset of $\mathrm{Riem}^{k+2,\alpha}(\Sigma)$.

Remark 1.3 In [1] the authors considered functions f_g of the form (1.1) on a surface Σ without boundary and with $h(x_1, \dots, x_m) = \sum_{i=1}^m \log V(x_i)$ where $V \in \mathcal{C}^2(\Sigma, \mathbb{R}_+)$. This is motivated by the mean field equation (1.2). They proved that f_g is a Morse function for V in an open and dense subset of $\mathcal{C}^2(\Sigma, \mathbb{R}_+)$. Also related is the paper [3], which deals with functions like f_g where G is the Green function of the Dirichlet Laplace operator on a bounded smooth domain $\Omega \subset \mathbb{R}^n$. In [3], it is proved that f_g is a Morse function for a generic domain Ω. The present paper seems to be the first investigating the Morse property of f_g as in (1.1) as a function of the Riemannian metric.

2 Preliminaries

Riemann surfaces are locally conformally flat, so there exist isothermal coordinates where the metric is conformal to the Euclidean metric (see [4, 11, 15]). We modify the isothermal coordinates applied in [7, 9, 16] to a Riemann surface with boundary. For $\xi \in \Sigma$ there exists a local chart $y_\xi : \Sigma \supset U(\xi) \to B^\xi \subset \mathbb{R}^2$ transforming g to $e^{\varphi_\xi \circ y_\xi} \langle \cdot, \cdot \rangle_{\mathbb{R}^2}$. We may assume that $y_\xi(\xi) = 0$. For $\xi \in \mathring{\Sigma}$ we may also assume that $\overline{U(\xi)} \subset \mathring{\Sigma}$ and that the image of y_ξ is a disc $B^\xi := \{y \in \mathbb{R}^2 : |y| < 2r_\xi\}$ of radius $2r_\xi > 0$. For $\xi \in \partial\Sigma$, by Lemma 4 in [16], there exists an isothermal coordinate system $\big(U(\xi), y_\xi\big)$ near ξ such that the image of y_ξ is a half disk $B^+_{2r_\xi} := \{y \in \mathbb{R}^2 : |y| < 2r_\xi, y_2 \geq 0\}$ of radius $2r_\xi > 0$ and $y_\xi\,(U(\xi) \cap \partial\Sigma) = B^+_{2r_\xi} \cap \partial\mathbb{R}^2_+$. For any

$x \in y_\xi^{-1}\left(B_{2r_\xi}^+ \cap \partial\mathbb{R}_+^2\right)$, we have

$$\left(y_\xi\right)_* (\nu_g(x)) = -\exp\left(-\frac{\varphi_\xi(y)}{2}\right) \frac{\partial}{\partial y_2}\bigg|_{y=y_\xi(x)}. \tag{2.1}$$

In this case, we take $B^\xi = B_{2r_\xi}^+$. For $\xi \in \Sigma$ and $0 < r \leq 2r_\xi$ we set

$$B_r^\xi := B^\xi \cap \{y \in \mathbb{R}^2 : |y| < r\} \quad \text{and} \quad U_r(\xi) := y_\xi^{-1}(B_r^\xi).$$

Recall that $\varphi_\xi : B^\xi \to \mathbb{R}$ is related to K, the Gaussian curvature of Σ, by the equation

$$-\Delta\varphi_\xi(y) = 2K\left(y_\xi^{-1}(y)\right)e^{\varphi_\xi(y)} \quad \text{for all } y \in B^\xi.$$

We can assume that y_ξ and φ_ξ depend smoothly on ξ, and that $\varphi_\xi(0) = 0$ and $\nabla\varphi_\xi(0) = 0$.

Observe that we have for $\zeta \in U(\xi)$,

$$\lim_{x\to\zeta} \frac{d_g(x,\zeta)}{\left|\varphi_\xi(x) - \varphi_\xi(\zeta)\right|} = e^{\frac{1}{2}\varphi_\xi(\zeta)},$$

which implies

$$R^g(\zeta) = \lim_{x\to\zeta}\left(G^g(x,\zeta) + \frac{1}{\kappa(\zeta)}\log\left|y_\xi(x) - y_\xi(\zeta)\right|\right) + \frac{1}{2\kappa(\zeta)}\varphi_\xi\left(y_\xi(\zeta)\right), \tag{2.2}$$

and in particular, using $\varphi_\xi\left(y_\xi(\xi)\right) = \varphi_\xi(0) = 0$:

$$R^g(\xi) = \lim_{x\to\xi}\left(G^g(x,\xi) + \frac{1}{\kappa(\xi)}\log\left|y_\xi(x)\right|\right).$$

Now we construct a regular part $H^g(x,\xi)$ of the Green function $G^g(x,\xi)$ such that $H^g(\xi,\xi) = R^g(\xi)$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ be such that

$$\chi(s) = \begin{cases} 1 & \text{if } |s| \leq 1, \\ 0 & \text{if } |s| \geq 2. \end{cases}$$

For $\xi \in \mathring{\Sigma}$ we choose $\delta_\xi = \min\{\frac{1}{2}r_\xi, \frac{1}{2}\operatorname{dist}(x,\partial\Sigma)\}$, and for $\xi \in \partial\Sigma$ we set $\delta_\xi := \frac{1}{2}r_\xi$. Next we define the cut-off function $\chi_\xi \in \mathcal{C}^\infty(\Sigma, [0,1])$ by

$$\chi_\xi(x) := \begin{cases} \chi\left(|y_\xi(x)|/\delta_\xi\right) & \text{if } x \in U(\xi) \\ 0 & \text{if } x \in \Sigma \setminus U(\xi) \end{cases}.$$

Then, for $\xi \in \Sigma$ the function $H^g_\xi := H^g(\cdot, \xi) : \Sigma \to \mathbb{R}$ is defined to be the unique solution of the Neumann problem

$$\begin{cases} \Delta_g H^g_\xi = \dfrac{1}{\kappa(\xi)}(\Delta_g \chi_\xi)\log\left|y_\xi\right| + \dfrac{2}{\kappa(\xi)}\left\langle \nabla^g \chi_\xi, \nabla^g \log|y_\xi|\right\rangle_g + \dfrac{1}{|\Sigma|_g} & \text{in } \mathring{\Sigma}, \\ \partial_{\nu_g} H^g_\xi = \dfrac{1}{\kappa(\xi)}\left(\partial_{\nu_g}\chi_\xi\right)\log|y_\xi| + \dfrac{1}{\kappa(\xi)}\chi_\xi \partial_{\nu_g}\log|y_\xi| & \text{on } \partial\Sigma \\ \displaystyle\int_\Sigma H^g_\xi dv_g = \dfrac{1}{\kappa(\xi)}\int_\Sigma \chi_\xi \log|y_\xi|\, dv_g. \end{cases} \tag{2.3}$$

Lemma 2.1 *For $g \in \mathrm{Riem}^{k+2,\alpha}(\Sigma)$ the function H^g is of class $\mathcal{C}^{k+3,\alpha}$ in any compact subsets of $\Sigma \times \mathring{\Sigma}$ and in $\Sigma \times \partial\Sigma$. Moreover, it satisfies*

$$G^g(x,\xi) = -\frac{1}{\kappa(\xi)}\chi_\xi(x)\cdot\log\left|y_\xi(x)\right| + H^g(x,\xi)$$

and $H^g(\xi,\xi) = R^g(\xi)$. Consequently R^g is of class $\mathcal{C}^{k+3,\alpha}$ in any compact subsets of $\mathring{\Sigma}$ and in $\partial\Sigma$.

Proof First we observe that $e^{\varphi_\xi} \in \mathcal{C}^{k+2,\alpha}(B^\xi, \mathbb{R})$ because $g \in \mathrm{Riem}^{k+2,\alpha}(\Sigma)$. For $\xi \in \mathring{\Sigma}$, by the choice of δ_ξ we have that $-\Delta_g H^g_\xi$ is of class $\mathcal{C}^{k+2,\alpha}$ in Σ and $\partial_{\nu_g} H^g(x,\xi) \equiv 0$ on $\partial\Sigma$. Now the Schauder estimate for the Neumann problem (see [14, 16], for instance) implies that the solution of (2.3) uniquely exists in $\mathcal{C}^{k+4,\alpha}(\Sigma,\mathbb{R})$.

For $x \in U(\xi)\cap\partial\Sigma$, setting $y = y_\xi(x)$ we have:

$$\partial_{\nu_g}\log|y_\xi(x)| \overset{(2.1)}{=} -e^{-\frac{1}{2}\varphi_\xi(y)}\frac{\partial}{\partial y_2}\log|y| = -e^{-\frac{1}{2}\varphi_\xi(y)}\frac{y_2}{|y|^2} \equiv 0.$$

Clearly, $\partial_{\nu_g}\chi(|y_\xi|(x)) = 0$ for $x \in \partial\Sigma \cap U_{\delta_\xi}(\xi)$. It follows that $\partial_{\nu_g} H^g_\xi$ is of class $\mathcal{C}^{k+2,\alpha}$ on $\partial\Sigma$. Moreover $\Delta_g H^g_\xi$ is of class $\mathcal{C}^{k+2,\alpha}$ in Σ. Consequently (2.3) has a unique solution $H^g_\xi \in \mathcal{C}^{k+3,\alpha}(\Sigma,\mathbb{R})$ by the Schauder estimates.

Finally H^g_ξ is uniformly bounded in $\mathcal{C}^{k+3,\alpha}$ for ξ in any compact subsets of $\mathring{\Sigma}$ and in $\partial\Sigma$. Therefore $H^g(\xi,\xi)$ is in $\mathcal{C}^{k+3,\alpha}$ in any compact subsets of $\mathring{\Sigma}$ and in $\partial\Sigma$, and $H^g(\xi,\xi) = R^g(\xi)$ by (2.2). □

For $g \in \mathrm{Riem}^{k+2,\alpha}(\Sigma)$ we now consider the map

$$\mathcal{H}^g : \Sigma\times\Sigma\times\mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}_+) \to \mathbb{R}, \quad \mathcal{H}^g(x,\xi,\psi) := H^{\psi g}(x,\xi).$$

Proposition 1 *The map $\mathcal{H}^g$ is $\mathcal{C}^1$ in $\Sigma\times\mathring{\Sigma}\times\mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}_+)$ and in $\Sigma\times\partial\Sigma\times\mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}_+)$. Moreover, we have*

$$D_\psi\mathcal{H}^g(x,\xi,1)[\theta] = -\frac{1}{|\Sigma|_g}\int_\Sigma (G^g(z,x) + G^g(z,\xi))\theta(z)dv_g(z), \tag{2.4}$$

for any $\theta \in \mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R})$.

Proof For $g \in \mathrm{Riem}^{k+2,\alpha}(\Sigma)$ and $\psi \in \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$ we clearly have $\psi g \in \mathrm{Riem}^{k+2,\alpha}(\Sigma)$. By a direct calculation we obtain the following equations for $H_\xi^{\psi g} - H_\xi^g$:

$$\begin{cases} -\Delta_g(H_\xi^{\psi g} - H_\xi^g) = \dfrac{1}{|\Sigma|_g} - \dfrac{\psi}{|\Sigma|_{\psi g}} & \text{in } \mathring{\Sigma}, \\ \partial_{\nu_g}(H_\xi^{\psi g} - H_\xi^g) = 0 & \text{on } \partial\Sigma, \\ \displaystyle\int_\Sigma (H_\xi^{\psi g} - H_\xi^g)\, dv_g = -\int_\Sigma G_\xi^{\psi g}(\psi - 1) dv_g. \end{cases}$$

An expansion of $|\Sigma|_{\psi g} = \int_\Sigma dv_{\psi g}$ yields:

$$-\Delta_g(H_\xi^{\psi g} - H_\xi^g) = \frac{1-\psi}{|\Sigma|_g} + \frac{\int_\Sigma (\psi - 1) dv_g}{|\Sigma|_g^2} + \mathcal{O}(\|\psi - 1\|^2_{\mathcal{C}^{k+2,\alpha}}) \quad \text{as } \|\psi - 1\|_{\mathcal{C}^{k+2,\alpha}} \to 0.$$

Recall that $G_\xi^{\psi g} = -\frac{4}{\sigma(\xi)}\chi(|y_\xi|)\log|y_\xi| + H_\xi^{\psi g}$, so the representation formula gives:

$$\begin{aligned} H_\xi^{\psi g}(x) - H_\xi^g(x) = &\frac{1}{|\Sigma|_g}\int_\Sigma G^{\psi g}(z,\xi)(\psi(z) - 1)\, dv_g(z) \\ &- \frac{1}{|\Sigma|_{\psi g}}\int_\Sigma G^g(z,x)(\psi(z) - 1)\, dv_g(z). \end{aligned} \tag{2.5}$$

By standard elliptic estimates (see [14, 16]) there exists a constant C such that

$$\left\| H_\xi^{\psi g} - H_\xi^g \right\|_{\mathcal{C}^{k+4,\alpha}} \le C \cdot \|\psi - 1\|_{\mathcal{C}^{k+2,\alpha}},$$

thus

$$H_\xi^{\psi g} \to H_\xi^g \quad \text{in } \mathcal{C}^{k+4,\alpha} \text{ as } \psi \to 1 \text{ in } \mathcal{C}^{k+2,\alpha}. \tag{2.6}$$

According to the construction of $H^g(x,\xi)$, the convergence in (2.6) is uniform for ξ in any compact subset of $\mathring{\Sigma}$, and in $\partial\Sigma$. It follows that $\mathcal{H}_g(x,\xi,\cdot)$ is continuous at 1, uniformly for $x \in \Sigma$ and ξ in any compact subsets of $\mathring{\Sigma}$ or $\xi \in \partial\Sigma$. Using $\mathcal{H}_g(x,\xi,\psi) = \mathcal{H}_{\psi g}(x,\xi,1)$ we see that $\mathcal{H}_g(x,\xi,\cdot)$ is continuous at every $\psi \in \mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}_+)$.

Next we prove that $\mathcal{H}^g(x,\xi,\psi)$ is $\mathcal{C}^1$ with respect to ψ. We fix $\theta \in \mathcal{C}^{2+k,\alpha}(\Sigma,\mathbb{R})$ and consider the metric $(1+t\theta)g$ with t sufficiently small so that $1 + t\theta > 0$. Then

$$\begin{aligned} w_\xi^t(x) &:= \frac{1}{t}\big(H^{(1+t\theta)g}(x,\xi) - H^g(x,\xi)\big) \\ &= \frac{1}{t}\big(\mathcal{H}^g(x,\xi,1+t\theta) - \mathcal{H}^g(x,\xi,1)\big) \end{aligned} \tag{2.7}$$

satisfies the following equations as $t \to 0$:

$$\begin{cases} -\Delta_g w_\xi^t = \dfrac{1}{|\Sigma|_g^2}\displaystyle\int_\Sigma \theta\, dv_g - \dfrac{\theta}{|\Sigma|_g} + \mathcal{O}(t) & \text{in } \mathring{\Sigma}, \\ \partial_{\nu_g} w_t(x,\xi) = 0 & x \in \partial\Sigma, \\ \displaystyle\int_\Sigma w_\xi^t\, dv_g = -\int_\Sigma G_\xi^{(1+t\theta)g} \cdot \theta\, dv_g \end{cases} \tag{2.8}$$

where $\mathcal{O}(t)$ is defined with respect to the $\mathcal{C}^{k+2,\alpha}$-norm. Applying the standard elliptic estimates, w_ξ^t converges as $t \to 0$ to some function $w_\xi^0(x) = D_\psi \mathcal{H}^g(x,\xi,1)[\theta]$ in $\mathcal{C}^{k+4,\alpha}(\Sigma,\mathbb{R})$. Moreover, w_ξ^0 satisfies the equations

$$\begin{cases} -\Delta_g w_\xi^0 = \dfrac{1}{|\Sigma|_g^2}\displaystyle\int_\Sigma \theta\, dv_g - \dfrac{\theta}{|\Sigma|_g} & \text{in } \mathring{\Sigma}, \\ \partial_{\nu_g} w_\xi^0 = 0 & \text{on } \partial\Sigma, \\ \displaystyle\int_\Sigma w_\xi^0\, dv_g = -\int_\Sigma G_\xi^g \cdot \theta\, dv_g. \end{cases} \tag{2.9}$$

The representation formula now implies (2.4):

$$D_\psi \mathcal{H}^g(x,\xi,1)[\theta] = w_\xi^0(x) = -\frac{1}{|\Sigma|_g}\int_\Sigma \big(G^g(z,x) + G^g(z,\xi)\big)\cdot \theta(z)\, dv_g(z).$$

Replacing g by ψg and θ by $\frac{\theta}{\psi}$, we obtain the derivative of $\mathcal{H}_g(x,\xi,\psi)$ for arbitrary $\psi \in \mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}_+)$:

$$\begin{aligned} D_\psi \mathcal{H}^g(x,\xi,\psi)[\theta] &= \lim_{t\to 0}\frac{1}{t}\big(H^{(\psi+t\theta)g}(x,\xi) - H^{\psi g}(x,\xi)\big) \\ &= -\frac{1}{|\Sigma|_{\psi g}}\int_\Sigma \big(G^{\psi g}(z,x) + G^{\psi g}(z,\xi)\big)\cdot\frac{\theta(z)}{\psi(z)} dv_{\psi g}(z) \\ &= -\frac{1}{|\Sigma|_{\psi g}}\int_\Sigma \big(G^{\psi g}(z,x) + G^{\psi g}(z,\xi)\big)\cdot\theta(z) dv_g(z). \end{aligned} \tag{2.10}$$

In order to see that $\mathcal{H}_g(x,\xi,\psi)$ is continuously Fréchet differentiable with respect to ψ it is sufficient to prove that $D_\psi\mathcal{H}_g(x,\xi,\psi)$ is continuous in ψ as a linear operator on $\mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}_+)$. For $\psi_1,\psi_2 \in \mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}_+)$ there holds

$$\begin{aligned} &\big|D_\psi\mathcal{H}_g(x,\xi,\psi_1)[\theta] - D_\psi\mathcal{H}_g(x,\xi,\psi_2)[\theta]\big| \\ &= \bigg|\frac{1}{|\Sigma|_{\psi_1 g}}\int_\Sigma \big(G^{\psi_1 g}(z,x) + G^{\psi_1 g}(z,\xi)\big)\,\theta(z)dv_g(z) \\ &\quad -\frac{1}{|\Sigma|_{\psi_2 g}}\int_\Sigma \big(G^{\psi_2 g}(z,x) + G^{\psi_2 g}(z,\xi)\big)\,\theta(z)dv_g(z)\bigg| \\ &\le C\cdot\|\theta\|_{\mathcal{C}^{k+2,\alpha}}\cdot\|\psi_1-\psi_2\|_{\mathcal{C}^{k+2,\alpha}} \end{aligned}$$

where we applied (2.10); here $C > 0$ is a constant. Therefore $\mathcal{H}^g$ is $\mathcal{C}^1$ in $\Sigma \times \mathring{\Sigma} \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$ and in $\Sigma \times \partial\Sigma \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$. $\square$

3 Proof of Theorem 1.1

The proof is based on the following transversality theorem [12, Theorem 5.4].

Theorem 3.1 *Let M, Ψ, N be Banach manifolds of class $\mathcal{C}^r$ for some $r \in \mathbb{N}$, let $\mathcal{D} \subset M \times \Psi$ be open, let $\mathcal{F} : \mathcal{D} \to N$ be a $\mathcal{C}^r$ map, and fix a point $z \in N$. Assume for each $(y, \psi) \in \mathcal{F}^{-1}(z)$ that:*

(1) *$D_y\mathcal{F}(y, \psi) : T_yM \to T_zN$ is semi-Fredholm with index $< r$;*
(2) *$D\mathcal{F}(y, \psi) : T_yM \times T_\psi\Psi \to T_zN$ is surjective;*
(3) *$\mathcal{F}^{-1}(z) \to \Psi$, $(y, \psi) \mapsto \psi$, is σ-proper.*

Then

$$\Psi_{reg} := \{\psi \in \Psi : z \text{ is a regular value of } \mathcal{F}(\cdot, \psi)\}$$

is a residual subset of Ψ.

Proof of Theorem 1.1 We set $\Psi := \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$ and consider the functions $f_{\psi g} : X \setminus \Delta_X \to \mathbb{R}$ for $\psi \in \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$ first in a local isothermal chart. Let $y_{\xi_i} : \Sigma \supset U(\xi_i) \to B^{\xi_i} \subset \mathbb{R}^2$ be isothermal charts of Σ as in Sect. 2 with $\xi_1, \ldots, \xi_l \in \mathring{\Sigma}$, $\xi_{l+1}, \ldots, \xi_m \in \partial\Sigma$. For simplicity of notation we set $Y_i := y_{\xi_i} : U_i := U(\xi_i) \to B_i := B^{\xi_i} \subset \mathbb{R}^2$ for $i = 1, \ldots, l$, and $Y_i := \pi_1 \circ y_{\xi_i} : U_i := U(\xi_i) \cap \partial\Sigma \to B_i \subset \mathbb{R}$ for $i = l+1, \ldots, m$; here $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first component. For $i = l+1, \ldots, m$ we thus have $y_{\xi_i}(x) = (Y_i(x), 0)$. Then

$$\begin{aligned} Y &= Y_1 \times \cdots \times Y_m : X \supset U := U_1 \times \cdots \times U_m \to \mathbb{R}^{l+m}, \quad (x_1, \ldots, x_m) \\ &\mapsto \big(Y_1(x_1), \ldots, Y_m(x_m)\big), \end{aligned}$$

is a chart of X. Set $M = N := \mathbb{R}^{l+m}$ and $V := Y(U \cap X\backslash\Delta_X) \subset \mathbb{R}^{l+m} = M$ so that $\mathcal{D} := V \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$ is an open subset of $\mathbb{R}^{l+m} \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+)$.

We will apply Theorem 3.1 to prove that $0 \in \mathbb{R}^{l+m}$ is a regular value of

$$\nabla\big(f_{\psi g} \circ Y^{-1}\big) : \mathbb{R}^{l+m} \supset V = Y(U \cap X \setminus \Delta_X) \to \mathbb{R}^{l+m}$$

for ψ in a residual subset $\Psi_U \subset \Psi$. This implies that the restriction of $f_{\psi g}$ to $U \cap X\backslash\Delta_X$ is a Morse function for $\psi \in \Psi_U$. Then Theorem 1.1 follows because X is covered by finitely many neighborhoods U as above and because the intersection of finitely many residual sets is a residual set.

It remains to prove that the map

$$\mathcal{F}_g : \mathbb{R}^{l+m} \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+) \supset \mathcal{D} \to \mathbb{R}^{l+m}, \quad (y, \psi) \mapsto \nabla\big(f_{\psi g} \circ Y^{-1}\big)(y),$$

satisfies the assumptions of Theorem 3.1 with $r = 1$. Concerning the differentiability it is clear that $\mathcal{F}_g$ is $\mathcal{C}^1$ as a function of $y \in V$. In order to see that $\mathcal{F}_g$ is also $\mathcal{C}^1$ in ψ, by Lemma 2.1 it is sufficient to prove that $\nabla_y \left(H^{\psi g} \left(Y_i^{-1}(\cdot), Y_j^{-1}(\cdot) \right) \right)$ is $\mathcal{C}^1$ in ψ. We recall that w_ξ^t is defined by (2.7). Applying the representation formula of w_ξ^t and Lebesgue's dominated convergence theorem, we have for $(y, \psi) \in \mathcal{D}$ and $\theta \in \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R})$:

$$\begin{aligned}
&D_\psi\big|_{\psi=1} \nabla_y H^{\psi g}(Y_i^{-1}(y_i), Y_j^{-1}(y_j))[\theta] \\
&= \lim_{t\to 0} \nabla_y \left(w_\xi^t(x)\Big|_{x=Y_i^{-1}(y_i),\xi=Y_j^{-1}(y_j)} \right) \\
&\overset{(2.5)}{=} \lim_{t\to 0} \left(-\frac{1}{|\Sigma|_g} \int_\Sigma \nabla_y G^{(1+t\theta)g}_{Y_j^{-1}(y_j)} \theta \, dv_g \right. \\
&\quad \left. + \int_\Sigma \frac{1}{t} \left(\frac{1}{|\Sigma|_g} - \frac{1+t\theta}{|\Sigma|_{(1+t\theta)g}} \right) \nabla_y G^g_{Y_i^{-1}(y_i)} dv_g \right) \\
&= -\frac{1}{|\Sigma|_g} \int_\Sigma \nabla_y \big(G^g(z, Y_i^{-1}(y_i)) + G^g(z, Y_j^{-1}(y_j)) \big) \cdot \theta(z) \, dv_g(z) \\
&= \nabla_y D_\psi \mathcal{H}^g(Y_i^{-1}(y_i), Y_j^{-1}(y_j), 1)[\theta],
\end{aligned}$$

where we used Proposition 1. Since $D_\psi \mathcal{F}_g(y, \psi)[\theta] = D_\psi \mathcal{F}_{\psi g}(y, 1) \left[\frac{\theta}{\psi} \right]$ we obtain

$$D_\psi \mathcal{F}_g(y, \psi)[\theta] = -\frac{1}{|\Sigma|_{\psi g}} \int_\Sigma \nabla_y \big(G^{\psi g}(z, Y_i^{-1}(y_i)) + G^{\psi g}(z, Y_j^{-1}(y_j)) \big) \cdot \theta(z) \, dv_g(z).$$

As in the proof of Proposition 1 we deduce that $\mathcal{F}_g(y, \psi)$ is $\mathcal{C}^1$ on U.

Now we need to check the assumptions (1)-(3) of Theorem 3.1. Obviously $D_y \mathcal{F}_g(y, \psi) : \mathbb{R}^{l+m} \to \mathbb{R}^{l+m}$ is a Fredholm operator of index $0 < 1$, hence (1) holds. Also, (3) is easy to prove: For $j \in \mathbb{N}$ the set

$$M_j := \left\{ Y(x) \in \mathbb{R}^{l+m} : x \in U,\ d_g\big(x, \Delta_X \cup \partial U\big) \geq 2^{-j} \right\} \subset Y(U) \subset \mathbb{R}^{l+m}$$

is compact as a continuous image of a compact set. Therefore the map

$$M_j \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+) \to \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+), \quad (y, \psi) \mapsto \psi,$$

is proper, hence its restriction to $\mathcal{F}_g^{-1}(0) \cap \big(M_j \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+) \big)$ is proper. Since $V = \bigcup_{j=1}^\infty M_j$, so $\mathcal{D} = \bigcup_{j=1}^\infty \big(M_j \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+) \big)$ it follows that the map

$$\mathcal{F}_g^{-1}(0) = \bigcup_{j=1}^\infty \Big(\mathcal{F}_g^{-1}(0) \cap \big(M_j \times \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+) \big) \Big) \to \mathcal{C}^{k+2,\alpha}(\Sigma, \mathbb{R}_+), \quad (y, \psi) \mapsto \psi,$$

is σ-proper.

Finally we prove the surjectivity of the derivative $D\mathcal{F}_g(y,\psi) : \mathbb{R}^{l+m} \times \mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}) \to \mathbb{R}^{l+m}$ at a point $(y,\psi) \in \mathcal{F}_g^{-1}(0)$. In fact, we shall prove that $D_\psi\mathcal{F}_g(y,\psi) : \mathcal{C}^{k+2,\alpha}(\Sigma,\mathbb{R}) \to \mathbb{R}^{l+m}$ is onto. Since

$$D_\psi\mathcal{F}_g(y,\psi)[\theta] = D_\psi\mathcal{F}_{\psi g}(y,1)\,[\theta/\psi] \qquad \text{for } \theta \in \mathcal{C}^{2+k,\alpha}(\Sigma,\mathbb{R})$$

it is sufficient to consider the case $\psi \equiv 1$. We observe for $\theta \in \mathcal{C}^{2+k,\alpha}(\Sigma,\mathbb{R})$:

$$D_\psi\big|_{\psi=1}\nabla_y G^{\psi g}(Y_i^{-1}(y_i),Y_j^{-1}(y_j))[\theta] = D_\psi\big|_{\psi=1}\nabla_y H^{\psi g}(Y_i^{-1}(y_i),Y_j^{-1}(y_j))[\theta].$$

Now Proposition 1 yields for $\theta \in \mathcal{C}^{2+k,\alpha}(\Sigma,\mathbb{R})$ with $\mathrm{supp}(\theta) \subset \Sigma\backslash\{Y_1^{-1}(y_1),\dots,Y_m^{-1}(y_m)\}$:

$$\begin{aligned}
D_\psi\mathcal{F}_g(y,1)[\theta] &= \frac{d}{dt}\bigg|_{t=0}\Bigg(\sum_{i=1}^m \sigma_i^2\nabla_y R^{(1+t\theta)g}(Y_i^{-1}(y_i)) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^m \sigma_i\sigma_j\nabla_y G^{(1+t\theta)g}(Y_i^{-1}(y_i),Y_j^{-1}(y_j)) + \nabla_y\big(h\circ Y^{-1}\big)(y)\Bigg) \\
&= -\frac{1}{|\Sigma|_g}\int_\Sigma\Bigg(\sum_{i=1}^m 2\sigma_i^2\nabla_y G^g(z,Y_i^{-1}(y_i)) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^m \sigma_i\sigma_j\big(\nabla_y G^g(z,Y_i^{-1}(y_i)) + \nabla_y G^g(z,Y_j^{-1}(y_j))\big)\Bigg)\theta(z)\,dv_g(z).
\end{aligned}$$

Consider an element $u = (u_1,\cdots,u_m) \in \mathbb{R}^{l+m}$ with $u_1,\dots,u_l \in \mathbb{R}^2$, $u_{l+1},\dots,u_m \in \mathbb{R}$, that is orthogonal to the range of $D_\psi\mathcal{F}_g(y,1)$, i.e. it satisfies for every $\theta \in \mathcal{C}^{2+k,\alpha}(\Sigma,\mathbb{R})$ with $\mathrm{supp}(\theta) \subset \Sigma\backslash\{Y_1^{-1}(y_1),\dots,Y_m^{-1}(y_m)\}$:

$$\begin{aligned}
0 &= \big\langle u, D_\psi\mathcal{F}_g(y,1)[\theta]\big\rangle = \sum_{i=1}^m\big\langle u_i,(D_\psi\mathcal{F}_g(y,1)[\theta])_i\big\rangle \\
&= -\sum_{i=1}^m\frac{1}{|\Sigma|_g}\Bigg(2\sigma_i^2 + 2\sum_{\substack{j=1\\ j\neq i}}^m\sigma_i\sigma_j\Bigg)\int_\Sigma\big\langle u_i,\nabla_{y_i}G^g(z,Y_i^{-1}(y_i))\big\rangle\cdot\theta(z)\,dv_g(z)
\end{aligned}$$

This implies, using $\sum_{j=1}^m\sigma_j \neq 0$:

$$\sum_{i=1}^m\sigma_i\big\langle u_i,\nabla_{y_i}G^g(z,Y_i^{-1}(y_i))\big\rangle = 0 \qquad \text{for } z \in \Sigma\setminus\big\{Y_1^{-1}(y_1),\dots,Y_m^{-1}(y_m)\big\}. \tag{3.1}$$

Setting $\kappa_i = 2\pi$ for $i = 1, \dots, l$ and $\kappa_i = \pi$ for $i = l + 1, \dots, m$ we have

$$G^g(z, Y_i^{-1}(y_i)) = H^g\big(z, Y_i^{-1}(y_i)\big) - \frac{1}{\kappa_i}\chi\big(4|Y_i(z) - y_i|/\delta_{\xi_i}\big)\log|Y_i(z) - y_i|.$$

Now we define $z_i(t) := Y_i^{-1}(y_i + tu_i)$ for $i \in \{1, \dots, m\}$ and observe that

$$\nabla_{y_j} G^g\big(z_i(t), Y_j^{-1}(y_j)\big) = \mathcal{O}(1) \qquad \text{as } t \to 0 \text{ for } j \neq i$$

whereas

$$\big\langle u_i, \nabla_{y_i} G^g(z_i(t), Y_i^{-1}(y_i))\big\rangle \to \infty \qquad \text{as } t \to 0.$$

Equation (3.1) implies $u_i = 0$ because $\sigma_i \neq 0$. This holds for all i, hence $u = 0$ and $D_\psi \mathcal{F}_g(y, \psi)$ must be onto.

4 Proof of Theorem 1.2

Theorem 1.2 follows easily from the following lemma.

Lemma 4.1 *If* $\sigma_i > 0$ *for* $i = 1, \dots, m$ *then* $\lim\limits_{x \to \partial(X \setminus \Delta_X)} |\nabla^g f_g(x)|_g = \infty$.

Proof of Theorem 1.2 Lemma 4.1 implies that the set of critical points of f_g is compact. It follows that if f_g is a Morse function, then it has only finitely many critical points, and any $\mathcal{C}^2$-perturbation of f_g is also a Morse function. Therefore $\mathrm{Riem}^{k+2,\alpha}_{Morse}(\Sigma)$ is an open subset of $\mathrm{Riem}^{k+2,\alpha}(\Sigma)$. By Theorem 1.1 it is also a dense subset.

Proof of Lemma 4.1 Consider a sequence $x^n \in X \setminus \Delta_X$ such that

$$x^n \to x^\infty \in \partial(X \setminus \Delta_X) = \Delta_X \cup \big((\partial\Sigma)^l \times (\partial\Sigma)^{m-l}\big) \subset \Sigma^l \times (\partial\Sigma)^{m-l}.$$

CASE 1. $x^\infty \in \Delta_X \subset \mathring{\Sigma}^l \times (\partial\Sigma)^{m-l}$.

Then, there exists $\xi \in \Sigma$ such that the set $I := \big\{i \in \{1, \dots, m\} : x_i^\infty = \xi\big\}$ contains at least two elements. Moreover, if $\xi \in \mathring{\Sigma}$, then $I \subset \{1, \cdots, l\}$; if $\xi \in \partial\Sigma$, then $I \subset \{l + 1, \cdots, m\}$. In either case, we have that

$$\big|\nabla^g_{x_i} R^g(x_i^n)\big| = \mathcal{O}(1) \text{ and } \big|\nabla^g_{x_i} G^g(x_i^n, x_j^n)\big| = \mathcal{O}(1) \quad \text{as } n \to +\infty,$$

for $i \in I$ and $j \notin I$, and $\big|\nabla^g_{x_i} H^g(x_i^n, x_j^n)\big| = \mathcal{O}(1)$ for $i, j \in I$. Let $y_\xi : U(\xi) \to B^\xi \subset \mathbb{R}^2$ be the isothermal chart with $y_\xi(\xi) = 0$ introduced in Sect. 2. Then for $i \in I$, setting $y_j^n := y_\xi(x_j^n) \in \mathbb{R}^2$ for $j \in I$ and assuming $y_i^n \neq 0$, we obtain as $n \to \infty$:

$$\begin{aligned}
\left|\nabla^g f_g(x^n)\right|_g &\geq \left|\nabla^g_{x_i} f_g(x^n)\right|_g = 2\left|\nabla^g_{x_i} \sum_{j\in I\setminus\{i\}} \sigma_i\sigma_j G^g(x_i^n, x_j^n)\right|_g + \mathcal{O}(1)\\
&= \frac{2}{\kappa(\xi)}\left|\nabla^g_{x_i} \sum_{j\in I\setminus\{i\}} \sigma_i\sigma_j \log\left|y_\xi(x_i^n) - y_\xi(x_j^n)\right|\right|_g + \mathcal{O}(1)\\
&\geq \frac{2c}{\kappa(\xi)}\left|\nabla_{y_i} \sum_{j\in I\setminus\{i\}} \sigma_i\sigma_j \log\left|y_i^n - y_j^n\right|\right| + \mathcal{O}(1)\\
&= \frac{2c}{\kappa(\xi)}\left|\sum_{j\in I\setminus\{i\}} \sigma_i\sigma_j \frac{y_i^n - y_j^n}{|y_i^n - y_j^n|^2}\right| + \mathcal{O}(1)\\
&\geq \frac{2c}{\kappa(\xi)}\left|\sum_{j\in I\setminus\{i\}} \sigma_i\sigma_j \left\langle \frac{y_i^n - y_j^n}{|y_i^n - y_j^n|^2}, \frac{y_i^n}{|y_i^n|}\right\rangle\right| + \mathcal{O}(1)
\end{aligned}$$

The second inequality is a consequence of the fact that there exists $c > 0$ such that for any function $\mathrm{g} : U(\xi) \to \mathbb{R}$:

$$\left|\nabla^g_x \mathrm{g}(x)\right|_g \geq c\left|\nabla\left(\mathrm{g} \circ y_\xi^{-1}\right)\left(y_\xi(x)\right)\right| \quad \text{for } x \in U(\xi).$$

Now for $n \in \mathbb{N}$ we choose $i(n) \in I$ with $|y_{i(n)}^n| \geq |y_j^n|$ for all $j \in I$. This implies $y_{i(n)}^n \neq 0$ and

$$\left\langle y_{i(n)}^n - y_j^n, y_{i(n)}^n\right\rangle \geq \frac{1}{2}\left|y_{i(n)}^n - y_j^n\right|^2 > 0 \quad \text{for } j \in I \setminus \{i(n)\},$$

hence

$$\left\langle \frac{y_{i(n)}^n - y_j^n}{|y_{i(n)}^n - y_j^n|^2}, \frac{y_{i(n)}^n}{|y_{i(n)}^n|}\right\rangle \geq \frac{1}{2|y_{i(n)}^n|} \quad \text{for } j \in I \setminus \{i(n)\}.$$

As a consequence we obtain, using $\sigma_i\sigma_j > 0$ for all $i, j \in I$:

$$\begin{aligned}
\left|\nabla^g f_g(x^n)\right|_g \; &\geq \; \left|\nabla^g_{x_{i(n)}} f_g(x^n)\right|_g \geq \frac{c}{\kappa(\xi)}\pi \sum_{j\in I\setminus\{i(n)\}} \sigma_{i(n)}\sigma_j \frac{1}{|y_{i(n)}^n|} + \mathcal{O}(1)\\
&\to \infty \quad \text{as } n \to \infty.
\end{aligned}$$

Case 2. $x^\infty \notin \Delta_X$. Then there exists $i \in \{1, ..., l\}$ such that $x_i^n \to \xi$ for some $\xi \in \partial\Sigma$.

We fix an isothermal chart $(y_\xi, U(\xi))$ around ξ as introduced in Sect. 2. For any $\zeta \in U_{r_\xi}(\xi)$, we decompose $G^g(\cdot, \zeta)$ as follows:

$$G^g(\cdot,\zeta) = \tilde{H}^g(\cdot,\zeta) - \frac{1}{\kappa(\zeta)}\chi(4|y_\xi(\cdot)) - y_\xi(\zeta)|/r_\xi)\log|y_\xi(\cdot) - y_\xi(\zeta)|. \quad (4.1)$$

Equation (2.2) implies $R^g(\zeta) = \tilde{H}^g(\zeta,\zeta) + \frac{1}{2\kappa(\zeta)}\varphi_\xi\big(y_\xi(\zeta)\big)$. Let ∂_1, ∂_2 be the standard basis of $\mathbb{R}^2$ and define $\partial_{\zeta_1}, \partial_{\zeta_2} \in T_\zeta\Sigma$ be the corresponding basis of the tangent space of Σ at $\zeta \in U(\xi)$.

Fix $\zeta \in U_{r_\xi}(\xi) \cap \mathring{\Sigma}$ and set $\delta := y_\xi(\zeta)_2 > 0$. The representation formula for $\tilde{H}^g$ yields:

$$\begin{aligned}
\partial_{\zeta_1}\tilde{H}^g(\eta,\zeta)|_{\eta=\zeta} &= \frac{1}{|\Sigma|_g}\int_\Sigma \partial_{\zeta_1}\tilde{H}^g(\cdot,\zeta)dv_g - \int_\Sigma G^g(\cdot,\zeta)\Delta_g\partial_{\zeta_1}\tilde{H}^g(\cdot,\zeta)dv_g \\
&\quad + \int_{\partial\Sigma} G^g(\cdot,\zeta)\partial_{\nu_g}\partial_{\zeta_1}\tilde{H}^g(\cdot,\zeta)ds_g \\
&= \frac{1}{2\pi}\int_{\partial\Sigma} G^g(\zeta,x)\partial_{\nu_g}\partial_{\zeta_1} \\
&\quad \times \big(\chi\left(4|y_\xi(x) - y_\xi(\zeta)|/r_\xi\right)\log|y_\xi(x) - y_\xi(\zeta)|\big)\,ds_g(x) + \mathcal{O}(1) \\
&= -\frac{1}{2\pi}\int_{B^\xi\cap\partial\mathbb{R}^2_+} G^g\left(\zeta, y_\xi^{-1}(y)\right)\chi(4|y - y_\xi(\zeta)|/r_\xi) \\
&\quad \times \frac{2(y_1 - y_\xi(\zeta)_1)(y_2 - y_\xi(\zeta)_2)}{|y - y_\xi(\zeta)|^4}dy + \mathcal{O}(1) \\
&= -\frac{1}{2\pi\delta}\int_{-\varepsilon/\delta}^{\varepsilon/\delta} G^g\left(\zeta, y_\xi^{-1}\big((\delta s, 0) + (y_\xi(\zeta)_1, 0)\big)\right) \\
&\quad \times \frac{2s}{(s^2+1)^2}ds + \mathcal{O}(1). \qquad (4.2)
\end{aligned}$$

as $\delta \to 0$. Here $\varepsilon \in (0, \frac{r_\xi}{16})$ is chosen sufficiently small. Decomposing G^g as in (4.1), we deduce for $\zeta \in U(\xi)$ with $y_\xi(\zeta)_2 < \frac{r_\xi}{16}$:

$$\begin{aligned}
G^g\left(\zeta, y_\xi^{-1}\big((\delta s, 0) + (y_\xi(\zeta)_1, 0)\big)\right) &= \tilde{H}^g\left(\zeta, y_\xi^{-1}\big((\delta s, 0) + (y_\xi(\zeta)_1, 0)\big)\right) \\
&\quad - \frac{1}{\pi}\log|(\delta s, -\delta)|,
\end{aligned}$$

Now we apply the mean value theorem for $\tilde{H}^g$ and obtain as $\delta s \to 0$:

$$\begin{aligned}
\tilde{H}^g\big(\zeta, y_\xi^{-1}\big((\delta s, 0) + (y_\xi(\zeta)_1, 0)\big)\big) &= \tilde{H}^g\left(\zeta, y_\xi^{-1}\big(y_\xi(\zeta)_1, 0\big)\right) \\
&\quad + \mathcal{O}\big(|\delta s| \sup_{x\in\partial\Sigma}\|\nabla_x\tilde{H}^g(\cdot,x)\|_{\mathcal{C}(\Sigma)}\big) \quad (4.3)
\end{aligned}$$

This implies as $\delta \to 0$:

$$\begin{aligned}
&\left|-\frac{1}{2\pi\delta}\int_{-\varepsilon/\delta}^{\varepsilon/\delta} G^g\big(\zeta, y_\xi^{-1}\big((\delta s, 0) + (y_\xi(\zeta)_1, 0)\big)\frac{2s}{(s^2+1)^2}ds\right| \\
&\le \left|-\frac{1}{2\pi\delta}\tilde{H}^g\big(\zeta, y_\xi^{-1}\big(y_\xi(\zeta)_1, 0\big)\big)\int_{|s|\le\varepsilon/\delta}\frac{2s}{(s^2+1)^2}ds\right| \\
&\quad + \left|\mathcal{O}\left(\int_{|s|\le\varepsilon/\delta}\frac{2s^2}{(s^2+1)^2}ds\right) + \frac{1}{2\pi^2\delta}\int_{|s|\le\varepsilon/\delta}\log(\delta\sqrt{s^2+1})\frac{2s}{(s^2+1)^2}ds\right| \\
&\le \mathcal{O}(1).
\end{aligned}$$

Now (4.2) yields $\partial_{\zeta_1}\tilde{H}^g(\zeta,\zeta) = \mathcal{O}(1)$ for $\zeta \in U_{r_\xi}(\xi) \cap \partial\Sigma$. Consequently, for $\zeta \in U_{r_\xi}(\xi)$ with $y_\xi(\zeta)_2 < \frac{r_\xi}{16}$, we have proven that:

$$\partial_{\zeta_1}R^g(\zeta) = \mathcal{O}(1) \qquad \text{as } |y_\xi(\zeta)| \to 0. \tag{4.4}$$

The representation formula of $\tilde{H}^g$ yields for $\zeta \in U_{r_\xi}(\xi) \cap \mathring{\Sigma}$ as $\delta \to 0$:

$$\begin{aligned}
\partial_{\zeta_2}\tilde{H}^g(\eta,\zeta)|_{\eta=\zeta} &= \frac{1}{|\Sigma|_g}\int_\Sigma \partial_{\zeta_2}\tilde{H}^g(\cdot,\zeta)dv_g - \int_\Sigma G^g(\cdot,\zeta)\Delta_g\partial_{\zeta_2}\tilde{H}^g(\cdot,\zeta)dv_g \\
&\quad + \int_{\partial\Sigma} G^g(\cdot,\zeta)\partial_{\nu_g}\partial_{\zeta_2}\tilde{H}^g(\cdot,\zeta)ds_g \\
&= \frac{1}{2\pi}\int_{\partial\Sigma} G^g(\zeta,x)\partial_{\nu_g}\partial_{\zeta_2} \\
&\quad \times \big(\chi\left(4|y_\xi(x) - y_\xi(\zeta)|/r_\xi\right)\log|y_\xi(x) - y_\xi(\zeta)|\big)\,ds_g(x) + \mathcal{O}(1) \\
&\le \frac{1}{2\pi}\int_{B^\xi\cap\partial\mathbb{R}^2_+} G^g(\zeta, y_\xi^{-1}(y))\chi(4|y - y_\xi(\zeta)|/r_\xi) \\
&\quad \times \frac{(y_1 - y_\xi(\zeta)_1)^2 - (y_2 - y_\xi(\zeta)_2)^2}{|y - y_\xi(\zeta)|^4}dy + \mathcal{O}(1) \\
&\le \frac{1}{2\pi\delta}\int_{-\varepsilon/\delta}^{\varepsilon/\delta} G^g(\zeta, y_\xi^{-1}((\delta s, 0) + (y_\xi(\zeta)_1, 0))\frac{s^2-1}{(s^2+1)^2}ds + \mathcal{O}(1) \\
&\overset{(4.3)}{\le} \frac{1}{2\pi\delta}\tilde{H}^g(\zeta, y_\xi^{-1}(y_\xi(\zeta)_1, 0))\int_{|s|\le\varepsilon/\delta}\frac{s^2-1}{(s^2+1)^2}ds \\
&\quad + \mathcal{O}\left(1 + \int_{|s|\le\varepsilon/\delta}\frac{|s(s^2-1)|}{(s^2+1)^2}ds\right) \\
&\quad + \frac{\log(\delta^{-1})}{2\pi^2\delta}\int_{|s|\le\varepsilon/\delta}\frac{s^2-1}{(s^2+1)^2}ds \\
&\quad + \frac{1}{2\pi^2\delta}\int_{|s|\le\varepsilon/\delta}\log\left(\frac{1}{\sqrt{s^2+1}}\right)\frac{s^2-1}{(s^2+1)^2}ds + \mathcal{O}(1) \\
&\le \frac{1}{2\pi^2\delta}\int_{|s|\ge 1}\log\left(\frac{\sqrt{s^{-2}+1}}{\sqrt{s^2+1}}\right)\frac{s^2-1}{(s^2+1)^2}ds
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{2\pi^2\delta}\int_{|s|\geq\varepsilon/\delta}\log\left(\frac{1}{\sqrt{s^2+1}}\right)\frac{s^2-1}{(s^2+1)^2}ds\\
&-\frac{\log(\delta^{-1})\varepsilon}{\pi^2(\delta^2+\varepsilon^2)}+\mathcal{O}(\delta^{-1/2})+\mathcal{O}(\log(\delta^{-1})+1)\\
\leq\, &-\frac{1}{2\pi^2\delta}\int_{|s|\geq 1}\log\left(s^2\right)\frac{s^2-1}{(s^2+1)^2}ds+\mathcal{O}(\delta^{-\frac{1}{2}})\leq-\frac{1}{4\pi\delta}.
\end{aligned}$$

The last inequality used the identity

$$\int_{|s|\geq 1}\log(s^2)\frac{s^2-1}{(s^2+1)^2}ds=\pi.$$

From the above estimate we deduce for $\zeta\in\mathring{\Sigma}\cap U_{r_\xi}(\xi)$:

$$\partial_{\zeta_2}R^g(\zeta)\leq-\frac{1}{2\pi|y_\xi(\zeta)_2|}\qquad\text{as } |y_\xi(\zeta)_2|\to 0. \tag{4.5}$$

Now if $I:=\big\{i\in\{1,\dots,m\}:x_i^n\to\xi\big\}$ contains only a single element i then (4.5) yields

$$\begin{aligned}
\big|\nabla^g f_g(x^n)\big|_g &\geq\big|\nabla^g_{x_i}f_g(x^n)\big|_g\geq\big|\sigma_i^2\nabla^g_{x_i}R^g(x_i^n)+2\sigma_i\sum_{j\neq i}\nabla^g_{x_i}G^g(x_i^n,x_j^n)+\nabla^g_{x_i}h(x^n)\big|_g\\
&\geq\sigma_i^2|\nabla^g_{x_i}R^g(x_i^n)|+\mathcal{O}(1)\geq\frac{c\sigma_i^2}{2\pi|y_\xi(x_i^n)_2|}+\mathcal{O}(1)\\
&\to\infty\qquad\text{as } n\to\infty.
\end{aligned}$$

Here $c>0$ is a constant such that for any function $F:U(\xi)\to\mathbb{R}$:

$$\big|\nabla^g_x F(x)\big|_g\geq c\big|\nabla\big(F\circ y_\xi^{-1}\big)\big(y_\xi(x)\big)\big|\quad\text{for } x\in U(\xi). \tag{4.6}$$

Next we consider the case that I contains at least two elements. Then $\xi_i^n\in U(\xi)$ for n large and $i\in I$. By a direct calculation we obtain for $j\in I\setminus\{i\}$, $\iota=1,2$ and $n\to\infty$:

$$\begin{aligned}
\partial_{\zeta_\iota}G^g(x_j^n,\zeta)|_{\zeta=x_i^n}&=\partial_{\zeta_\iota}\tilde{H}^g(x_j^n,\zeta)|_{\zeta=x_i^n}\\
&\quad+\frac{1}{\kappa(x_i^n)}\log\frac{1}{|y_\xi(x_j^n)-y_\xi(x_i^n)|}\partial_{\zeta_\iota}\chi(4|y_\xi(x_j^n)-y_\xi(\zeta)|/r_\xi)|_{\zeta=x_i^n}\\
&\quad+\frac{1}{\kappa(x_i^n)}\chi(4|y_\xi(x_j^n)-y_\xi(x_i^n)|/r_\xi)\partial_{\zeta_\iota}\log\frac{1}{|y_\xi(x_j^n)-y_\xi(\zeta)|}\Bigg|_{\zeta=x_i^n}\\
&=\partial_{\zeta_\iota}\tilde{H}^g(x_j^n,\zeta)|_{\zeta=x_i^n}-\frac{1}{\kappa(x_i^n)}\frac{(y_\xi(x_i^n)-y_\xi(x_j^n))_\iota}{|y_\xi(x_j^n)-y_\xi(x_i^n)|^2}+\mathcal{O}(1).
\end{aligned} \tag{4.7}$$

For $n \in \mathbb{N}$ we set

$$\varrho_n := \max\Big(\big\{y_\xi(x_i^n)_2 : i \in I\big\} \cup \big\{|y_\xi(x_i^n)_1 - y_\xi(x_j^n)_1| : i, j \in I \text{ with } i \neq j\big\}\Big).$$

If there exists $i(n) \in I$ such that $\varrho_n = y_\xi\big(x_{i(n)}^n\big)_2$, then $i(n) \in \{1, \dots, l\}$ satisfies

$$y_\xi(x_{i(n)}^n)_2 - y_\xi(x_j^n)_2 \geq 0 \text{ and } y_\xi(x_{i(n)}^n)_2 \geq |(y_\xi(x_i^n)_1 - y_\xi(x_j^n)_1| \quad \text{for every } i, j \in I.$$

Given $\zeta = x_{i(n)}^n$ and $\eta = x_j^n$ with $j \in I \cap \{1, \dots, l\} \setminus \{i(n)\})$, we will calculate the upper bound of $\partial_{\zeta_2} \tilde{H}^g(\eta, \zeta)$ as $n \to 0$. Setting $a = y_\xi(\eta)_2 > 0$ and $b = y_\xi(\zeta)_1 - y_\xi(\eta)_1$ we have for $|b| \leq \varrho_n$ as $n \to \infty$:

$$\begin{aligned}
\partial_{\zeta_2} \tilde{H}^g(\eta, \zeta) \ &= \frac{1}{|\Sigma|_g} \int_\Sigma \partial_{\zeta_2} \tilde{H}^g(\cdot, \zeta) dv_g - \int_\Sigma G^g(\cdot, \eta) \Delta_g \partial_{\zeta_2} \tilde{H}^g(\cdot, \zeta) dv_g \\
&\quad + \int_{\partial\Sigma} G^g(\cdot, \eta) \partial_{\nu_g} \partial_{\zeta_2} \tilde{H}^g(\cdot, \zeta) ds_g \\
&= \mathcal{O}(1) + \frac{1}{2\pi} \int_{\partial\Sigma} G^g(\eta, x) \partial_{\nu_g} \partial_{\zeta_2} \big(\chi\big(4|y_\xi(x) - y_\xi(\zeta)|/r_\xi\big) \\
&\quad \times \log|y_\xi(x) - y_\xi(\zeta)|\big) ds_g(x) \\
&\leq \mathcal{O}(1) + \frac{1}{2\pi} \int_{B^\xi \cap \partial\mathbb{R}^2_+} G^g(\eta, y_\xi^{-1}(y)) \chi(4|y - y_\xi(\zeta)|/r_\xi) \\
&\quad \times \frac{(y_1 - y_\xi(\zeta)_1)^2 - (y_2 - y_\xi(\zeta)_2)^2}{|y - y_\xi(\zeta)|^4} dy \\
&\leq \mathcal{O}(1) + \frac{1}{2\pi\varrho_n} \int_{-\varepsilon/\varrho_n}^{\varepsilon/\varrho_n} G^g(\eta, y_\xi^{-1}((\varrho_n s, 0) + (y_\xi(\zeta)_1, 0))) \frac{s^2 - 1}{(s^2 + 1)^2} ds \\
&\overset{(4.3)}{\leq} \frac{1}{2\pi\varrho_n} \tilde{H}^g(\eta, y_\xi^{-1}(y_\xi(\zeta)_1, 0)) \int_{|s| \leq \varepsilon/\varrho_n} \frac{s^2 - 1}{(s^2 + 1)^2} ds \\
&\quad + \mathcal{O}\left(1 + \int_{|s| \leq \varepsilon/\varrho_n} \frac{|s(s^2 - 1)|}{(s^2 + 1)^2} ds\right) \\
&\quad - \frac{\log(a)}{2\pi^2\varrho_n} \int_{|s| \leq \varepsilon/\varrho_n} \frac{s^2 - 1}{(s^2 + 1)^2} ds \\
&\quad + \frac{1}{2\pi^2\varrho_n} \int_{|s| \leq \varepsilon/\varrho_n} \log\left(\frac{1}{\sqrt{1 + (\varrho_n s + b)^2/a^2}}\right) \frac{s^2 - 1}{(s^2 + 1)^2} ds \\
&\leq -\frac{1}{2\pi^2\varrho_n} \int_{|s| \geq \varepsilon/\varrho_n} \left(\log(a^{-1}) + \log\left(\frac{1}{\sqrt{1 + (\varrho_n s + b)^2/a^2}}\right)\right) \\
&\quad \times \frac{s^2 - 1}{(s^2 + 1)^2} ds \\
&\quad + \frac{1}{2\pi^2\varrho_n} \int_{|s| \geq 1} \log\left(\frac{\sqrt{1 + (\varrho_n s^{-1} + b)^2/a^2}}{\sqrt{1 + (\varrho_n s + b)^2/a^2}}\right) \frac{s^2 - 1}{(s^2 + 1)^2} ds \\
&\quad + \mathcal{O}\big(1 + \log(\varrho_n^{-1})\big)
\end{aligned}$$

$$\leq \mathcal{O}\big(\log(\varrho_n^{-1})\big)$$

Here we used the inequalities:

$$\log\left(\frac{\sqrt{1+(\varrho_n s^{-1}+b)^2/a^2}}{\sqrt{1+(\varrho_n s+b)^2/a^2}}\right) \leq 0, \text{ for } |b| \leq \varrho_n, |s| \geq 1$$

and

$$0 \leq \log\sqrt{a^2+(\varrho_n s+b)^2}\,\frac{s^2-1}{(s^2+1)^2} \leq C\frac{\varrho_n^{\frac{1}{2}}}{s^{\frac{3}{2}}}$$

for some constant $C > 0$, any $s \in \{s \in \mathbb{R} : |s| \geq \varepsilon/\varrho_n,\ a^2 + (\varrho_n s + b)^2 \geq 1\}$. We notice that we have for $j \in I \cap \{l+1, \dots, m\}$:

$$|\partial_{\zeta_2}\tilde{H}^g(x_j^n,\zeta)|_{\zeta=x^n_{i(n)}}| = |\partial_{\zeta_2}\tilde{H}^g(\zeta,x_j^n)|_{\zeta=x^n_{i(n)}}| \leq \mathcal{O}(1) \qquad \text{as } n \to \infty; \tag{4.8}$$

and for $j \in I \cap \{1, \dots, l\} \setminus \{i(n)\}$, $\varrho_n = y_\xi(x^n_{i(n)})$:

$$\big|\partial_{\zeta_2}\tilde{H}^g(x_j^n,\zeta)|_{\zeta=x^n_{i(n)}}\big| \leq \mathcal{O}\big(\log(\varrho_n^{-1})\big) \qquad \text{as } n \to \infty. \tag{4.9}$$

Now (4.5)-(4.9) imply for $n \to \infty$:

$$\begin{aligned}
|\nabla^g f_g(x^n)|_g &\geq \left|\nabla^g_{x_{i(n)}}\left(\sigma_i^2 R^g(x^n_{i(n)}) + 2\sigma_{i(n)}\sum_{j\neq i(n)}\sigma_j G^g(x^n_{i(n)},x^n_j) + h(x^n)\right)\right|_g \\
&\geq c\left|\partial_{(x_{i(n)})_2}\left(\sigma^2_{i(n)}R^g(\xi^n_{i(n)}) + 2\sigma_{i(n)}\sum_{j\in I\setminus\{i(n)\}}\sigma_j G^g(\xi_{i(n)},\xi_j)\right)\right| + \mathcal{O}(1) \\
&\geq \mathcal{O}\big(\log(\varrho_n^{-1})\big) + \sigma^2_{i(n)}\frac{c}{2\pi\varrho_n} \\
&\to \infty.
\end{aligned}$$

Here $c > 0$ is as in (4.6). If $\varrho_n > \max\{y_\xi(x_i^n) : i \in I\}$, then we take $i(n) \in I$ such that $y_\xi(x^n_{i(n)})_1 = \max\{y_\xi(x_i^n) : i \in I\}$. The proof of Lemma 6 in [16] implies for $x \in U(\xi)$ and $y_\xi^*(x) := (y_\xi(x)_1, -y_\xi(x)_2)$:

$$\left\|G^g(\cdot,x) + \frac{1}{2\pi}\log\big|y_\xi(\cdot)-y_\xi(x)\big| + \frac{1}{2\pi}\log\big|y_\xi(\cdot)-y_\xi^*(x)\big|\right\|_{\mathcal{C}^1(U_{r_\xi}(\xi))} \leq C$$

where $C > 0$ depends only on (Σ, g) and on ξ. Therefore we have for $j \in I \setminus \{i(n)\}$ as $n \to \infty$:

$$\partial_{\zeta_1}G(x_j^n,\zeta)\big|_{\zeta=x^n_{i(n)}} \leq -\frac{1}{2\pi}\frac{y_\xi(x^n_{i(n)})_1 - y_\xi(x^n_j)_1}{|y_\xi(x^n_{i(n)}) - y_\xi(x^n_j)|^2}$$

$$-\frac{1}{2\pi}\frac{y_\xi(x^n_{i(n)})_1 - y_\xi(x^n_j)_1}{|y_\xi(x^n_{i(n)}) - y^*_\xi(x^n_j)|^2} + \mathcal{O}(1). \tag{4.10}$$

We can take $i'(n) \in I\backslash\{i(n)\}$ such that $\varrho_n = y_\xi(x^n_{i(n)})_1 - y_\xi(x^n_{i'(n)})_1$ by the assumption. The inequality (4.10) yields

$$\partial_{\zeta_1} G(x^n_{i'(n)}, \zeta)\big|_{\zeta = x^n_{i(n)}} \le -\frac{1}{4\pi\varrho_n} \qquad \text{as } n \to \infty. \tag{4.11}$$

From (4.4) together with (4.10) and (4.11) we derive the following estimate for the gradient of f_g as $n \to \infty$:

$$\begin{aligned}
|\nabla^g f_g(x^n)|_g &\ge \left|\nabla^g_{x_{i(n)}}\left(\sigma_i^2 R^g(x^n_{i(n)}) + 2\sigma_{i(n)} \sum_{j \ne i(n)} \sigma_j G^g(x^n_{i(n)}, x^n_j) + h(x^n)\right)\right|_g \\
&\ge c\left|\partial_{(x_{i(n)})_1}\left(\sigma^2_{i(n)} R^g(\xi^n_{i(n)}) + 2\sigma_{i(n)} \sum_{j \in I\backslash\{i(n)\}} \sigma_j G^g(\xi_{i(n)}, \xi_j))\right)\right| + \mathcal{O}(1) \\
&\ge \mathcal{O}(1) + \sigma_{i(n)}\sigma\big(i'(n)\big)\frac{c}{2\pi\varrho_n} \\
&\to \infty
\end{aligned}$$

Again $c > 0$ is as in (4.6). This completes the proof of Lemma 4.1.

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