



Radial and cylindrical symmetry of solutions to the Cahn–Hilliard equation

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Received: 4 February 2019 / Accepted: 31 January 2020 / Published online: 23 March 2020
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Abstract

The paper is devoted to the classification of entire solutions to the Cahn–Hilliard equation $-\Delta u = u - u^3 - \delta$ in \mathbb{R}^N , with particular interest in those solutions whose nodal set is either bounded or contained in a cylinder. The aim is to prove either radial or cylindrical symmetry, under suitable hypothesis.

Mathematics Subject Classification 35B10 · 35B06

1 Introduction

We consider the entire equation

$$-\Delta u = f(u) - \delta \text{ in } \mathbb{R}^N, \quad (1.1)$$

with $f(u) := u - u^3$ and $\delta \in \mathbb{R}$. This equation has a variational characterisation, indeed, if we consider it on a domain $\Omega \subset \mathbb{R}^N$, it arises as the Euler equation of the Ginzburg–Landau functional

$$E(u, \Omega) := \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx, \quad W(t) := \frac{(1-t^2)^2}{4}, \quad (1.2)$$

under the mass constraint

$$\frac{1}{|\Omega|} \int_{\Omega} u dx = m, \quad m \in (-1, 1), \quad (1.3)$$

Communicated by M. Del Pino.

The author was partially supported by Fondecyt postdoctoral research Grant 3170111 and Fondo Basal AFB170001 CMM-Chile. Moreover, the author is particularly grateful to his PhD advisor Alberto Farina for his crucial comments and remarks about this paper.

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which gives rise to the Lagrange multiplier δ . The interest in the minimisers u of $E(\cdot, p, \Omega)$ arises from the phase transitions theory. In other words, if two different fluids are mixed in a container Ω , the number $u(x)$ represents the density of one of the two at x , in an equilibrium configuration. Here we take $\delta \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}})$, so that the polynomial $f_\delta(t) := t - t^3 - \delta$ admits exactly 3 real roots

$$z_1(\delta) < -1/\sqrt{3} < z_2(\delta) < 1/\sqrt{3} < z_3(\delta),$$

with $z_2(\delta)$ satisfying $\delta z_2(\delta) \geq 0$. The main results of the paper deal with symmetry properties of entire solutions to the Cahn–Hilliard equation (1.1).

Theorem 1 *Let $N \geq 2$, $\delta \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}})$ and let u_δ be a solution to (1.1) such that*

$$u_\delta > z_2(\delta) \text{ outside a ball } B_R \subset \mathbb{R}^N. \tag{1.4}$$

- (1) *If $\delta \in (-\frac{2}{3\sqrt{3}}, 0]$, then $u \equiv z_3(\delta)$.*
- (2) *If $\delta \in (0, \frac{2}{3\sqrt{3}})$, then u_δ is radially symmetric (not necessarily constant).*

We note that, for $\delta > 0$, nontrivial bubble solutions are known to exist. This is an important difference with the case $\delta \leq 0$. Moreover, we will see that the zero level set of radial solutions is non empty. In particular, we have the following Corollary.

Corollary 2 *Let $\delta \in (0, \frac{2}{3\sqrt{3}})$ and let u_δ be a non constant solution to (1.1) such that $u_\delta > z_2(\delta)$ outside a ball B_R . Then the nodal set of u_δ is a sphere.*

This result agrees with the variational theory, which studies the asymptotic behaviour of the scaled functionals

$$E_\varepsilon(u, \Omega) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) dx \tag{1.5}$$

as $\varepsilon \rightarrow 0$. For instance, Modica proved that, if ε_k is a sequence of positive numbers tending to 0 and u_{ε_k} is a sequence of minimisers of $E_{\varepsilon_k}(\cdot, p, \Omega)$ under the constraint (1.3) such that $u_{\varepsilon_k} \rightarrow u_0$ in $L^1(\Omega)$, then $u_0(x) \in \{\pm 1\}$ for almost every $x \in \Omega$, and the boundary in Ω of the set $E := \{x \in \Omega : u_0(x) = 1\}$ has minimal perimeter among all subsets $F \subset \Omega$ such that $|F| = |E|$, where $|\cdot|$ denotes the volume (see [15], Theorem 1). Further Γ -convergence results relating $E_\varepsilon(\cdot, p, \Omega)$ to the perimeter can be found in [16]. Therefore, given a family $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ of minimisers under the constraint (1.3), their nodal set is expected to be close to a compact Alexandrov-embedded constant mean curvature surface, at least for ε small. Corollary 2, together with a scaling argument, shows that, for ε small enough, the nodal set of any entire solution to

$$-\varepsilon \Delta u = \varepsilon^{-1}(u - u^3) - \ell, \quad \ell > 0, \tag{1.6}$$

in \mathbb{R}^N such that $u > z_2(\varepsilon \ell)$ outside a ball is actually a sphere, which is known to be the unique compact Alexandrov-embedded constant mean surface in \mathbb{R}^N (see [1]).

After that, we set

$$C_R := \{(x', x_N) \in \mathbb{R}^N : |x'| < R\}$$

and we consider solutions satisfying

$$u_\delta > z_2(\delta) \text{ outside a cylinder } C_R \subset \mathbb{R}^N. \tag{1.7}$$

The aim is to study their symmetry properties and their asymptotic behaviour as $\delta \rightarrow 0$, with particular interest in solutions which have one periodicity direction.

Theorem 3 Let $\{u_\delta\}_{\delta \in (0, \frac{2}{3\sqrt{3}})}$ be a family of non constant solutions to (1.1) in \mathbb{R}^N , with $N \geq 2$. Assume furthermore that u_δ is periodic in x_N and, for any $\delta \in (0, \frac{2}{3\sqrt{3}})$, there exists $R(\delta) > 0$ such that (1.7) is true. Then

- (1) $z_1(\delta) < u_\delta(x) < z_3(\delta)$, for any $x \in \mathbb{R}^N$.
- (2) u_δ is radially symmetric in x' .
- (3) $u_\delta \rightarrow -1$ as $\delta \rightarrow 0$ uniformly on compact subsets of \mathbb{R}^N .

In view of the aforementioned Γ -convergence results, given a solution u to (1.6) satisfying (1.7), with $\delta = \varepsilon\ell$, we expect its nodal set to be close to an Alexandrov-embedded constant mean curvature surface which is contained in a cylinder. This kind of surfaces are fully classified, at least the ones which are embedded in \mathbb{R}^3 , in fact it is known that the unique examples are the sphere and Delaunay *unduloids*, that is a family of non compact revolution surfaces obtained by rotating a periodic curve around a fixed axis in \mathbb{R}^3 , which can be taken to be the x_3 -axis, parametrised by a real number $\tau \in (0, 1)$. We will denote the period of D_τ by T_τ . For a detailed introduction of Delaunay surfaces, we refer to [12,14]. For any $\tau \in (0, 1)$, Kowalczyk and Hernandez [11] constructed a family $\{u_{\tau,\varepsilon}\}_{\varepsilon \in (0,\varepsilon_0)}$ of solutions to (1.6) in \mathbb{R}^3 , with $\ell = \ell_\varepsilon$ depending on ε , such that

- (1) ℓ_ε is positive and bounded uniformly in ε .
- (2) $u_{\tau,\varepsilon}$ is radially symmetric in x' .
- (3) $u_{\tau,\varepsilon}(x) \rightarrow \pm 1$ as $\varepsilon \rightarrow 0$, uniformly on compact subsets of Ω_τ^\pm , where Ω_τ^\pm denote the exterior and the interior of the Delaunay surface D_τ , respectively.
- (4) $u_{\tau,\varepsilon}(x', x_3) \rightarrow z_3(\varepsilon\ell_\varepsilon)$ as $|x'| \rightarrow \infty$, uniformly in x_3 .
- (5) $u_{\tau,\varepsilon}$ is periodic in x_3 of period T_τ .

We observe that the solutions $u_{\varepsilon,\tau}$ constructed in [11] are actually negative outside a cylinder, however, in order to obtain the aforementioned family, thanks to the oddness of f , it is enough to replace them with $-u_{\varepsilon,\tau}$. An interesting question is uniqueness. In other words, we are interested in the following question.

Question 4 (Uniqueness) Let $\varepsilon_0 > 0$, $\tau \in (0, 1)$ and let v be a non constant solution to (1.6) in \mathbb{R}^3 with $\ell = \ell_\varepsilon$, for $\varepsilon \in (0, \varepsilon_0)$. Assume in addition that

- ℓ_ε is bounded uniformly in ε .
- v is periodic in x_3 , with period T_τ .
- $v > z_2(\varepsilon\ell_\varepsilon)$ outside a ball B_R .

Is it true that $v = u_{\varepsilon,\tau}$, at least if ε_0 is small enough?

This would be the counterpart of Corollary 2 for periodic solutions. For now we are not able to give a full answer to this question. However Theorem 3 is a first step in this direction, since it proves that any family $\{v_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ of such solutions has to share many properties with the family $\{u_{\tau,\varepsilon}\}_{\varepsilon \in (0,\varepsilon_0)}$ constructed by Hernandez and Kowalczyk. For instance, for ε small, v_ε has to satisfy (1), (2), (3) and the scaled functions $v_\varepsilon(\varepsilon x)$ tend to -1 uniformly on compact subsets of \mathbb{R}^N as $\varepsilon \rightarrow 0$.

The plan of the paper is the following. In Sect. 2 we will state some quite general results, of which the Theorems stated in the introduction are consequences. Section 3 is devoted to the proofs. It is divided into three subsections, dedicated to prove global boundedness, radial symmetry and the asymptotic behaviour for δ small respectively.

2 Some relevant results

In this section we state some results that are proved in Sect. 3. First we prove boundedness of solutions, which holds irrespectively of the sign of δ .

Proposition 5 *Let $\delta \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}})$ and let $u_\delta \in L^3_{loc}(\mathbb{R}^N)$ be a distributional solution to the Cahn–Hilliard equation (1.1). Then*

$$z_1(\delta) \leq u_\delta(x) \leq z_3(\delta)$$

a. e. in \mathbb{R}^N .

Remark 6 • Using Proposition 5, standard elliptic estimates (see [10], Theorem 8.8 and Corollary 6.3) and a bootstrap argument, it is possible to show that any distributional solution $u \in L^3_{loc}(\mathbb{R}^3)$ is actually in $C^\infty(\mathbb{R}^N)$. This parallels the regularity result proved in [6] for the Allen–Cahn equation.

- It follows from the strong maximum principle that either u_δ is constant, and in this case it has to be either $z_1(\delta)$, or $z_2(\delta)$ or $z_3(\delta)$, or it satisfies $z_1(\delta) < u_\delta < z_3(\delta)$ in \mathbb{R}^N .

We observe that Proposition 5 and Remark 6 prove point (1) of Theorem 3, which is actually true for any non constant entire solution. After that, we rule out the case $\delta \leq 0$, in which only constant solutions are allowed.

Proposition 7 *Let u_δ be a solution to (1.1) in \mathbb{R}^N , with $-\frac{2}{3\sqrt{3}} < \delta \leq 0$ such that $u_\delta > z_2(\delta)$ outside a stripe $\{x \in \mathbb{R}^N : |x_N| < L\}$. Then $u_\delta \equiv z_3(\delta)$.*

We stress that the latter result proves point (1) of Theorem 1 and agrees with the sign of δ obtained by Hernández and Kowalczyk in [11]. Using boundedness and the famous result by Gidas et al. [9], or Theorem 2 of [7], which relies on the moving planes method, we can prove this symmetry result.

Proposition 8 *Let $\delta \in (0, 2/3\sqrt{3})$ and let u_δ be a non constant solution to (1.1) such that $u_\delta > z_2(\delta)$ outside a ball B_R , for some $R > 0$. Then*

- u_δ is radially symmetric, that is, up to a translation, $u_\delta(x) = w_\delta(|x|)$.
- u_δ is radially increasing, in the sense that $(\nabla u_\delta(x), x) > 0$, for any $x \in \mathbb{R}^N \setminus \{0\}$.

Proposition 8 proves point (2) of Theorem 1. More precisely, it is known that, for $\delta \in (0, \frac{2}{3\sqrt{3}})$, the problem

$$\begin{cases} -\Delta v_\delta = v_\delta - v_\delta^3 - \delta & \text{in } \mathbb{R}^N \\ v_\delta(0) = \min_{\mathbb{R}^N} v_\delta, v_\delta < z_3(\delta), v_\delta(x) \rightarrow z_3(\delta) & \text{as } |x| \rightarrow \infty \end{cases} \quad (2.1)$$

admits a unique solution which is radially symmetric (see [4,17,18]), that is $v_\delta(x) = w_\delta(|x|)$. In view of this fact, we can actually prove the following classification result.

Proposition 9 *Let $\delta \in (0, \frac{2}{3\sqrt{3}})$ and let u_δ be a non constant solution to (1.1) such that $u_\delta > z_2(\delta)$ outside a ball B_R . Then, up to a translation, $u_\delta = v_\delta$.*

In the sequel, we will use the notation $W_\delta(t) := W(t) + \delta t$.

Remark 10 It is possible to see that, for any $\delta \in (0, \frac{2}{3\sqrt{3}})$, there exists $R(\delta) > 0$ such that $w_\delta(R(\delta)) = 0$. In fact, the energy

$$E_\delta(r) := \frac{1}{2}(w'_\delta)^2 - W_\delta(w_\delta)$$

is strictly decreasing, since

$$\frac{d}{dr}E(r) = w'_\delta(w''_\delta - W'_\delta(w_\delta)) = -\frac{N-1}{r}(w'_\delta)^2 < 0, \quad \forall r > 0.$$

Thus, using that, by Proposition 8, v_δ is decreasing,

$$-W_\delta(w_\delta(0)) = E_\delta(0) > 0, \tag{2.2}$$

which yields that $w_\delta(0) < 0$.

In particular, in view of Remark 10, which yields that the nodal set of v_δ is neither empty nor a singleton, Corollary 2 is true.

Considering solutions that are approaching a positive limit just with respect to $N - 1$ variables, we can prove the following.

Proposition 11 *Let $\delta \in (0, 2/3\sqrt{3})$ and let u_δ be a non constant solution to (1.1) such that $u_\delta > z_2(\delta)$ outside a cylinder C_R , for some $R > 0$. If u_δ is periodic in x_N , then*

- u_δ is radially symmetric in x' , that is, up to a translation, $u_\delta(x) = w_\delta(|x'|, x_N)$.
- u_δ is radially increasing, in the sense that $(\nabla u_\delta(x), (x', 0)) > 0$, for any $x = (x', x_N) \in \mathbb{R}^N \setminus \{0\}$.

We note that this proves point (2) of Theorem 3. Even in this case, our result agrees with the construction of [11], where the authors prove the existence of a family of solutions fulfilling the symmetries of the Delaunay surface D_τ , hence, in particular they are periodic in x_N , radially symmetric and radially increasing in x' . Here we show that *any* periodic solution has to be radially symmetric and radially increasing in x' . Finally, in order to prove point (3) of Theorem 3, we need the following result, which shows that the phase transition has to be complete.

Proposition 12 *For any $\epsilon > 0$ there exists $\delta_0 \in (0, \frac{2}{3\sqrt{3}})$ such that, for any $\delta \in (0, \delta_0)$ and for any non constant solution u_δ to (1.1) satisfying $\sup_{\mathbb{R}^N} u_\delta = z_3(\delta)$, we have*

$$\inf_{\mathbb{R}^N} u_\delta < -1 + \epsilon. \tag{2.3}$$

This result somehow parallels Lemma 2.5 of [8]. The proof relies on both the moving planes and the sliding method. For a detailed proof of point (3) of Theorem 3, we refer to Sect. 3.

3 The proofs

3.1 Boundedness

In order to prove boundedness for distributional solutions to (1.1), we will rely on a result proved by Brezis [2].

Lemma 13 (Brezis–Kato inequality) *Let $p > 1$ and assume that $v \in L^p_{loc}(\mathbb{R}^k)$ satisfies*

$$-\Delta v + |v|^{p-1}v \leq 0 \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

Then $v \leq 0$ a.e. in \mathbb{R}^N .

Now we prove Proposition 5.

Proof Writing $-f_\delta(t) = (t - z_1(\delta))(t - z_2(\delta))(t - z_3(\delta))$ and setting

$$\begin{aligned} \alpha &:= z_1(\delta) - z_2(\delta) < 0, \\ \beta &:= z_3(\delta) - z_2(\delta) > 0, \\ w &:= u_\delta - z_2(\delta), \end{aligned}$$

we have

$$\Delta w = \Delta u_\delta = (u_\delta - z_1(\delta))(u_\delta - z_2(\delta))(u_\delta - z_3(\delta)) = w(w - \alpha)(w - \beta), \tag{3.1}$$

thus

$$\Delta(w - \beta)^+ = \chi_{\{w > \beta\}} \Delta w = \chi_{\{w > \beta\}} w(w - \alpha)(w - \beta) \geq ((w - \beta)^+)^3,$$

where $\chi_{\{w > \beta\}}$ denotes the characteristic function of the set $\{x \in \mathbb{R}^N : w(x) > \beta\}$. By the Kato–Brezis inequality (see Lemma 13), we have $w \leq \beta$. The same argument applied to $(\alpha - w)^+$ gives the lower bound $w \geq \alpha$. □

Remark 14 A similar argument is used in [5] to prove boundedness for solutions to a class of vectorial equations of the form

$$\Delta u = u P'_n(|u|^2), \quad P_n(t) := \frac{1}{2} \prod_{j=1}^n (t - k_j)^2,$$

with $0 < k_1 < \dots < k_n$. The scalar Allen–Cahn equation is included in this class. Here we prove that a similar result is true for a slightly different non linearity, due to the presence of δ .

Now we can prove Proposition 7, using boundedness and a result of [6] where non-existence of ground states for some special non linearities is proved.

Proof By Lemma 13, $z_1(\delta) \leq u_\delta \leq z_3(\delta)$, in particular, since $\delta \leq 0$, $|z_1(\delta)| \leq z_3(\delta)$, hence $|u_\delta| \leq z_3(\delta)$. By Lemma 15, $u_\delta \rightarrow z_3(\delta)$ as $x_1 \rightarrow \pm\infty$, the limit being uniform in x' . Moreover, setting $f_\delta(t) := f(t) - \delta$, we have

- $f_\delta(t) \geq 0, \forall t \in (0, z_3(\delta))$,
- $f_\delta(t) + f_{-\delta}(t) = -2\delta \geq 0, \forall t \in (0, z_3(\delta))$,
- $f_\delta(t)$ is non increasing in a left neighbourhood of $z_3(\delta)$.

Therefore, by Theorem 4.2 of [6], $u_\delta \equiv z_3(\delta)$. □

3.2 Radial symmetry

The aim of this subsection is to prove Proposition 11. In order to do so, we need some decay at infinity of the solution. From now on, we denote the variables by $x := (x_1, x'') \in \mathbb{R} \times \mathbb{R}^{N-1}$. For $\lambda \in \mathbb{R}$, we set

$$\Sigma_\lambda := \{x \in \mathbb{R}^3 : x_1 < \lambda\}. \tag{3.2}$$

This changing of notation is justified by the fact that several times this section x_N is the periodicity variable, hence we are not allowed to start the moving planes in that direction.

Lemma 15 *Let u_δ be a solution to (1.1). Assume furthermore that $u_\delta > z_2(\delta)$ in the half-space $\mathbb{R}^N \setminus \Sigma_\lambda$, for some $\lambda \in \mathbb{R}$. Then*

$$u(x_1, x'') \rightarrow z_3(\delta), \text{ as } x_1 \rightarrow \infty, \text{ uniformly in } x'' \tag{3.3}$$

Proof The statement is trivial if u_δ is constant (see Remark 6), hence we can assume that it is non constant. We apply Lemma 2.3 of [6] to $w := u_\delta - z_2(\delta)$ in the half space $\mathbb{R}^N \setminus \Sigma_\lambda$, where, by Lemma 13, $0 < w < \beta$. This is possible since the non linearity $g(t) := -t(t - \alpha)(t - \beta)$ is positive in $(0, \beta)$ and $g'(0) > 0$. We recall that the constants α and β are defined in the Proof of Proposition 5. The conclusion is that

$$w(x_1, x'') \rightarrow \beta \text{ as } x_1 \rightarrow \infty,$$

and the limit is uniform in the other variables. □

Using the fact that $f'(z_3(\delta)) < 0$, we can actually prove a better result about the decay rate of $z_3(\delta) - u_\delta$.

Lemma 16 *Let u_δ be a solution to (1.1) such that $u_\delta > z_2(\delta)$ in the half space $\mathbb{R}^N \setminus \Sigma_\lambda$, for some $\lambda \in \mathbb{R}$. Then, for any $\gamma \in (0, \sqrt{-f'(z_3(\delta))})$, there exists a constant $C(\gamma) > 0$, depending on γ , such that*

$$0 < z_3(\delta) - u_\delta(x_1, x'') \leq C(\gamma)e^{-\gamma x_1}, \quad \forall x = (x_1, x'') \in \mathbb{R}^N \setminus \Sigma_\lambda. \tag{3.4}$$

Proof We compare the bounded function $v := z_3(\delta) - u_\delta$ with the barrier $\mu e^{-\gamma x_1}$, for $\gamma \in (0, \sqrt{-f'(z_3(\delta))})$, in the half-space $\mathbb{R}^N \setminus \Sigma_M$, with $M > 0$ large enough. In fact, on $\partial(\mathbb{R}^N \setminus \Sigma_M)$, we have

$$v(x) \leq \|v\|_{L^\infty(\mathbb{R}^N)} \leq \mu e^{-\gamma M},$$

provided $\mu \geq \|v\|_{L^\infty(\mathbb{R}^N)} e^{\gamma M}$. Note that here we use the fact that $v \in L^\infty$, which is true by Lemma 13. Moreover, setting $h_\delta(v) := -f_\delta(z_3(\delta) - v)$, we have $h_\delta(0) = -f_\delta(z_3(\delta)) = 0$ and $h'_\delta(0) = f'(z_3(\delta)) < 0$, thus

$$(-\Delta + \gamma^2)(v - \mu e^{\gamma x_1}) = h_\delta(v) + \gamma^2 v \leq 0$$

in $\mathbb{R}^N \setminus \Sigma_M$ if M is large enough, since, by Lemma 15, $z_3(\delta) - v$ is decaying as $x_1 \rightarrow \infty$, uniformly with respect to x'' . Thus, by the maximum principle for possibly unbounded domains (see Lemma 2.1 of [3]), we conclude that (3.4) is true in $\mathbb{R}^N \setminus \Sigma_M$. Changing, if necessary, the constant $C(\gamma)$, the required inequality is fulfilled in the whole space. □

Now we prove Proposition 8

Proof By Proposition 5, $z_1(\delta) < u_\delta < z_3(\delta)$ and, by Remark 6, u_δ is smooth. By Lemma 15, it converges to $z_3(\delta)$ as $|x| \rightarrow \infty$, therefore, by the famous symmetry result by [9], or by Theorem 2 of [7], we conclude that u_δ is radially symmetric and radially decreasing. □

Now we prove Proposition 9.

Proof Since, by Proposition 8, u_δ is radially symmetric and radially decreasing, then, up to translation, we have $u_\delta(0) = \min_{\mathbb{R}^N} u_\delta$. Since, by Lemma 15, $u_\delta(x) \rightarrow z_3(\delta)$ as $|x| \rightarrow \infty$, then it solves (2.1), therefore, by uniqueness, $u_\delta = v_\delta$. □

In order to prove Proposition 11, we need to apply Theorem 2 of [7], which we recall, for the reader’s convenience.

Theorem 17 ([7]) *Let $v > 0$ be a bounded entire solution to*

$$-\Delta v = g(v)$$

in \mathbb{R}^N , with $g \in C^1(\mathbb{R})$ such that $g'(s) \leq 0$ in $(0, \eta)$, for some $\eta > 0$. Writing $x = (y, z) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$, we assume that

- $v(y, z) \rightarrow 0$ as $|y| \rightarrow \infty$, uniformly in z .
- v is periodic in z .

Then v is radially symmetric in y , that is, up to a translation, $v(y, z) = w(|y|, z)$, and radially decreasing in y , that is $\partial_{y_j} v(y, z) < 0$ for any $x = (y, z) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$ with $y \neq 0$.

Proof By Proposition 5, $z_1(\delta) < u_\delta < z_3(\delta)$ and, by Remark 6, u_δ is smooth. By Lemma 15, it converges to $z_3(\delta)$ as $|x'| \rightarrow \infty$, uniformly in x_N . Since u_δ is periodic, in order to conclude that it is radially symmetric in x' and radially decreasing, it is enough to apply Theorem 17 to $v := z_3(\delta) - u_\delta$. □

3.3 The asymptotic behaviour for δ small

First we show that if a solution lies between $1/\sqrt{3}$ and $z_3(\delta)$, then it is constant. This is proved by the moving planes method.

Lemma 18 *Let $\delta \in [0, 2/3\sqrt{3})$ and let u_δ be a solution to (1.1) in \mathbb{R}^N such that $u_\delta(x) \geq 1/\sqrt{3}$, for any $x \in \mathbb{R}^N$. Then $u_\delta \equiv z_3(\delta)$.*

Proof We set $v := z_3(\delta) - u_\delta$. Setting, for any $\lambda \in \mathbb{R}$, $v_\lambda(x) := v(2\lambda - x_1, x'')$, we have

$$v - v_\lambda \geq 0 \text{ in } \Sigma_\lambda, \text{ for any } \lambda \in \mathbb{R}. \tag{3.5}$$

In order to prove this fact, we assume by contradiction that there exists $\lambda \in \mathbb{R}$ such that the open set $\Omega_\lambda := \{x \in \Sigma_\lambda : v - v_\lambda < 0\}$ is nonempty, and we observe that, in any connected component ω of Ω_λ we have

$$\begin{cases} -\Delta(v - v_\lambda) = h_\delta(v) - h_\delta(v_\lambda) < 0 & \text{in } \omega, \\ v - v_\lambda = 0 & \text{on } \partial\omega, \end{cases}$$

due to the strict monotonicity of f_δ in $[1/\sqrt{3}, 1)$ (for the definition of h_δ , see the Proof of Lemma 16). As a consequence, by the maximum principle for possibly unbounded domains, we have $v - v_\lambda \leq 0$ in ω , a contradiction.

By (3.5), we have $\partial_{x_1} v \leq 0$ in \mathbb{R}^N . The same argument applied to $\tilde{v}(x) := v(-x_1, x')$ implies that also \tilde{v} satisfies (3.5), hence $\partial_{x_1} v \geq 0$ in \mathbb{R}^N , thus $\partial_{x_1} v \equiv 0$. Composing v with any rotation of \mathbb{R}^N , we conclude that v is a constant solution to (1.1), thus $v \equiv 0$. □

Given the double well potential $W(t) = \frac{(1-t^2)^2}{4}$, $0 < \alpha < W(1/\sqrt{3}) = \frac{1}{9}$ and $\delta \in (0, 2/3\sqrt{3})$, we set

$$\mu(\delta) := \max\{\mu < 0 : W_\delta(\mu) = \alpha\}.$$

Moreover, we take a smooth cutoff function $\chi : \mathbb{R} \rightarrow [0, 1]$ such that $\chi = 1$ in $(-\infty, -1)$ and $\chi = 0$ in $(0, \infty)$ and we set

$$\tilde{W}_\delta := \chi_\delta \alpha + (1 - \chi_\delta) W_\delta, \quad \chi_\delta(t) := \chi(-t/\mu(\delta)). \tag{3.6}$$

We will denote $\tilde{W} := \tilde{W}_0$. It is possible to see that \tilde{W}_δ enjoys the following properties:

$$\tilde{W}_\delta \rightarrow \tilde{W}, \quad \text{as } \delta \rightarrow 0, \text{ uniformly on compact subsets of } \mathbb{R}, \tag{3.7}$$

$$\tilde{W}_\delta(r) = W_\delta(r), \quad \text{for any } r \geq 0 \text{ and } \delta \in [0, 2/3\sqrt{3}], \tag{3.8}$$

and

$$\inf_{(-\infty, 0]} \tilde{W} = \alpha. \tag{3.9}$$

In the sequel, we will be interested in a solution to

$$\begin{cases} -\Delta \beta_{R,\delta} + \tilde{W}_\delta(\beta_{R,\delta}) = 0 & \text{in } B_R, \\ \beta_{R,\delta} = z_1(\delta) & \text{on } \partial B_R, \end{cases} \tag{3.10}$$

for $\delta \geq 0$ small enough and R large. This will be used as a barrier in the Proof of Proposition 12, which relies on a sliding method. This can be obtained in a variational technique, by minimising the functional

$$J_{R,\delta}(v) := \int_{B_R} \left(\frac{1}{2} |\nabla v|^2 + \tilde{W}_\delta(v) \right) dx. \tag{3.11}$$

among all $H^1(B_R)$ functions with trace $z_1(\delta)$ on ∂B_R . The case $\delta = 0$ is treated in Lemma 2.4 of [8].

Lemma 19 *Let $\delta_0 > 0$ be so small that $W_\delta(z_3(\delta)) < \alpha/2$, for any $\delta \in [0, \delta_0)$. Then, For any $R > 0$ and $\delta \in [0, \delta_0)$, there exists a minimiser $\beta_{R,\delta} \in C^2(B_R)$ of (3.11) among all functions with trace $z_1(\delta)$ on ∂B_R . Moreover, there exists $R_0 > 0$ such that, for any $R \geq R_0$ and for any $\delta \in [0, \delta_0)$,*

•

$$z_1(\delta) < \beta_{R,\delta}(x) < z_3(\delta), \quad \forall x \in B_R, \tag{3.12}$$

•

$$\sup_{B_R} \beta_{R,\delta} > \frac{1}{\sqrt{3}}, \tag{3.13}$$

• *there exists a solution β_R of (3.10) with $\delta = 0$ such that*

$$\sup_{B_R} \beta_{R,\delta} \rightarrow \sup_{B_R} \beta_R \in \left[\frac{1}{\sqrt{3}}, 1 \right) \text{ as } \delta \rightarrow 0. \tag{3.14}$$

Proof Existence follows from coercivity and weak lower semi continuity. By the fact that $\tilde{W}_\delta \equiv \alpha$ in $(-\infty, \mu(\delta))$ and (3.8), we can see the minimiser actually has to satisfy $z_1(\delta) \leq \beta_{R,\delta} \leq z_3(\delta)$, thus, due to the strong maximum principle, either (3.12) holds or $\beta_{R,\delta} \equiv z_1(\delta)$.

Now we prove (3.13), which, in particular, shows that $\beta_{R,\delta} > z_1(\delta)$ in B_R , at least for $R \geq R_0$. In order to do so, we assume that there exists a sequence $R_k \rightarrow \infty$ and a sequence $\delta_k \in [0, \delta_0)$ such that

$$\sup_{x \in \mathbb{R}^N} \beta_{R_k, \delta_k} \leq \frac{1}{\sqrt{3}}.$$

It follows that, on the one hand

$$J_{R_k, \delta_k}(\beta_{R_k, \delta_k}) \geq \alpha \omega_N R_k^N, \tag{3.15}$$

where ω_N denotes the surface of S^{N-1} . On the other hand, if, for $R > 1$ and $\delta \in [0, \delta_0)$, we take $w_{R, \delta}$ to be equal to $z_1(\delta)$ on ∂B_R and to $z_3(\delta)$ in B_{R-1} with $|\nabla w_{R, \delta}|$ bounded uniformly in δ , we can see that there exists a constant $C > 0$ such that, for k large enough,

$$J_{R_k, \delta_k}(w_{R_k, \delta_k}) \leq C R_k^{N-1} + W_{\delta_k}(z_3(\delta_k)) \omega_N R_k^N < \alpha \omega_N R_k^N, \tag{3.16}$$

since $\delta_k \in [0, \delta_0)$, hence $W_{\delta_k}(z_3(\delta_k)) < \alpha/2$. This contradicts the minimality of β_{R_k, δ_k} .

Finally we prove (3.14). In the forthcoming argument, $R > 0$ will always be arbitrary but fixed. We observe that, since $\beta_{R, \delta}$ is bounded uniformly in $R > 0$ and $\delta > 0$, then any sequence $\delta_k \rightarrow 0$ admits a subsequence, that we still denote by δ_k , such that β_{R, δ_k} converges in $C^2(B_R)$ to a solution β_R to

$$-\Delta \beta_R + \tilde{W}(\beta_R) = 0 \quad \text{in } B_R$$

satisfying $\beta_R = -1$ on ∂B_R . Since the convergence is uniform and (3.12) holds, then

$$\sup_{B_R} \beta_{R, \delta} \rightarrow \sup_{B_R} \beta_R \in [-1, 1]$$

as $\delta \rightarrow 0$. Moreover, by (3.13) and the strong maximum principle, $\sup_{B_R} \beta_R \in [\frac{1}{\sqrt{3}}, 1)$. \square

Now we can prove Proposition 12.

Proof It is enough to prove that, if there exists a sequence $\delta_k \rightarrow 0$, a sequence u_{δ_k} of solutions to (1.1) and $v > -1$ such that

$$\inf_{\mathbb{R}^N} u_{\delta_k} \geq v, \tag{3.17}$$

then there exists a subsequence $\delta_{k'}$ such that $u_{\delta_{k'}} \equiv z_3(\delta_{k'})$.

Claim For any $\varepsilon > 0$ and $\rho > 0$, there exists a subsequence, which we still denote by u_{δ_k} , and a sequence $x^k \in \mathbb{R}^N$ such that

$$u_{\delta_k}(x) > 1 - \varepsilon, \quad \forall x \in B_\rho(x^k). \tag{3.18}$$

Since $\sup_{\mathbb{R}^N} u_{\delta_k} = z_3(\delta_k)$, there exists $x^k \in \mathbb{R}^N$ such that

$$z_3(\delta_k) - u_{\delta_k}(x^k) < 1/k. \tag{3.19}$$

Therefore the sequence $u^k(x) := u_{\delta_k}(x + x^k)$ admits a subsequence converging, in $C^2_{loc}(\mathbb{R}^N)$, to a solution u^∞ to the Allen–Cahn equation

$$-\Delta u^\infty = f(u^\infty), \quad \text{in } \mathbb{R}^N. \tag{3.20}$$

By (3.19), we can see that $u^\infty(0) = 1$, thus $u^\infty \equiv 1$. As a consequence, for any $\varepsilon > 0$ (small) and $\rho > 0$, there exists a subsequence (still denoted by u^k) such that

$$\|u^k - 1\|_{L^\infty(B_\rho)} < \varepsilon, \quad \forall k$$

hence the claim is true.

In order to prove our result, we first observe that, by (3.13), for δ_0 small as in Lemma 19 and $\delta \in (0, \delta_0)$, there exists $R > 0$ and a solution $\beta_{R, \delta}$ to (3.10) such that

$$\sup_{B_R} \beta_{R, \delta} > \frac{1}{\sqrt{3}}, \quad \forall \delta \in (0, \delta_0). \tag{3.21}$$

Moreover, by (3.14), there exists a solution β_R to

$$-\Delta\beta_R + \tilde{W}(\beta_R) = 0 \text{ in } B_R, \quad \beta_R = -1 \text{ on } \partial B_R$$

and $\delta_1 = \delta_1(R) > 0$ such that, for any $\delta \in (0, \delta_1)$, we have

$$\sup_{B_R} \beta_{R,\delta} < \frac{\sup_{B_R} \beta_R + 1}{2} < 1, \quad \forall \delta \in (0, \delta_1). \tag{3.22}$$

As a consequence, for any $\delta \in (0, \bar{\delta})$, where $\bar{\delta} = \bar{\delta}(R) := \min\{\delta_0, \delta_1(R)\}$, we get

$$\frac{1}{\sqrt{3}} < \sup_{B_R} \beta_{R,\delta} < \frac{\sup_{B_R} \beta_R + 1}{2} < 1. \tag{3.23}$$

Now, applying the claim with $\rho = R$ and

$$\varepsilon := 1 - \frac{\sup_{B_R} \beta_R + 1}{2},$$

we can prove the existence of a subsequence, still denoted by u_{δ_k} , and a sequence x^k in \mathbb{R}^N such that

$$u_{\delta_k}(x) > 1 - \varepsilon > \sup_{B_R} \beta_{R,\delta_k} \geq \beta_{R,\delta_k}(x - x^k), \quad \forall x \in B_R(x^k), \forall k.$$

Sliding β_{R,δ_k} , with $k \geq k_0$ fixed, we get the lower bound

$$u_{\delta_k}(x) > 1 - \varepsilon > \frac{1}{\sqrt{3}}, \quad \forall x \in \mathbb{R}^N, \forall k \geq k_0.$$

In conclusion, by Lemma 18, $u_{\delta_k} \equiv z_3(\delta_k)$. □

Proposition 20 *Let $\delta \in (0, 2/3\sqrt{3})$ and let $\{u_\delta\}_{\delta \in (0, \frac{2}{3\sqrt{3}})}$ be a family of non constant solutions to (1.1) in \mathbb{R}^N such that*

- *for any $\delta \in (0, 2/3\sqrt{3})$ there exists $R(\delta) > 0$ such that $u_\delta > z_2(\delta)$ outside the cylinder $C_{R(\delta)}$.*
- *u_δ is periodic in x_N .*

Then

$$u_\delta \rightarrow -1 \text{ as } \delta \rightarrow 0, \text{ uniformly on compact subsets of } \mathbb{R}^N. \tag{3.24}$$

and

$$R(\delta) \rightarrow \infty \text{ as } \delta \rightarrow 0. \tag{3.25}$$

Remark 21 We note that point (3) of Theorem 3 is a consequence of Proposition 20.

Proof By Lemma 13, the family u_δ is uniformly bounded, hence any sequence $\delta_k \rightarrow 0$ admits a subsequence, that we still denote by δ_k , such that u_{δ_k} converges in $C_{loc}^2(\mathbb{R}^N)$ to a solution u^∞ to the Allen–Cahn equation (3.20). Since u_δ are all non constant solutions, then, by Proposition 12, we have

$$\inf_{\mathbb{R}^N} u_\delta \rightarrow -1, \quad \text{as } \delta \rightarrow 0. \tag{3.26}$$

By periodicity and Theorem 11, we know that, for δ small, u_δ is radially symmetric in x' and, up to a translation,

$$\inf_{\mathbb{R}^N} u_\delta = u_\delta(0),$$

hence, passing to the limit, we get

$$u^\infty(0) = \lim_{k \rightarrow \infty} u_{\delta_k}(0) = \lim_{k \rightarrow \infty} \inf_{\mathbb{R}^N} u_{\delta_k} = -1,$$

which yields that $u^\infty \equiv -1$, thus (3.24) holds.

In order to prove (3.25), we assume by contradiction that there exists $\bar{R} > 0$ and a sequence $\delta_k \rightarrow 0$ such that $R(\delta_k) \leq \bar{R}$. By (3.24), $u_{\delta_k} \rightarrow -1$ uniformly in $B_{\frac{R}{2}}^{N-1} \times [-1, 1]$, thus, for k large enough,

$$u_{\delta_k}(x'_k, 0) < -\frac{1}{2} < z_2(\delta_k)$$

if, for instance, $x'_k = (2R(\delta_k), 0) \in \mathbb{R} \times \mathbb{R}^{N-2}$, which contradicts the fact that u_{δ_k} is radially increasing. \square

Acknowledgements Open Access funding provided by Projekt DEAL.

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