Inclination lemmas with dominated convergence

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Introduction

Inclination lemmas help to study dynamical systems in the neighborhood of a saddle point. The content of the first version, the λ-lemma of Palis [3, 4], may be rephrased as follows. Suppose $g: U \to E$ is a C^1-diffeomorphism in a finite-dimensional space E with $g(0) = 0$ which is hyperbolic at $x = 0$, i.e. no eigenvalue of $T = Dg(0)$ lies on the unit circle. Suppose also that g is normalized so that the local stable and unstable manifolds of $x = 0$ are contained in the stable and unstable linear spaces Q and P of T, respectively. Define the slope of a vector $t \in E$ with components $pt \in P$, $0 \neq qt \in Q$ as $s(t) = |pt|/|qt|$, and consider a transversal H to P: Backward iterates $H_k = g^{-k}(H)$ converge to Q as k tends to infinity, and slopes of tangent vectors $t \in T_x H_k$, $t \neq 0$, tend to zero uniformly with respect to $x \in H_k$.

It is not hard to extend Palis' result to diffeomorphisms in arbitrary Banach spaces, see [3] and Proposition 10.4 in [2]. Recently Hale and Lin [1] obtained inclination lemmas also for maps which are not necessarily reversible.

With an application to functional differential equations in mind, we shall consider this most general situation, too. The aim of the present note is to show how one can improve uniform convergence for slopes of tangents by estimates of the speed of convergence.

For C^2-maps we derive geometric convergence

$$\sup\{s(t): 0 \neq t \in T_x H_k, x \in H_k\} \leqq \text{const} \cdot \beta^{-k}$$

for all $k \in N$, with some $\beta > 1$, and in special cases, in particular for $\dim P = 1$, the much sharper pointwise estimate

$$(t, x) \quad s(t) \leqq \text{const} \cdot |px|$$

for all $t \in T_x H_k \setminus \{0\}$, $x \in H_k$, $k \in N$ – see Lemma 2.1.

Estimate (t, x) is the key to an investigation of stability in a nonlocal bifurcation problem which we now briefly describe. Consider a state variable on a circle, with one attractive rest point and a delayed reaction to deviations. A simple differential equation for such a system is $\dot{y}(t) = f(y(t - \alpha))$, or equivalently

$$\dot{x}(t) = \alpha f(x(t - 1))$$

where $f: R \to R$ is periodic, with zeros $A < 0 < B$, period $B - A$, and with $0 < f$ in $(A, 0)$, $f < 0$ in $(0, B)$. Under additional hypotheses on f, there is a heteroclinic solution from A to B at some critical parameter α_0 which bifurcates to solutions representing periodic rotation around the circle, for $\alpha > \alpha_0$. Existence was proved in [7], but questions of uniqueness and stability for the bifurcating "periodic solutions of the second kind" remained open.

Estimate (t, x) makes it possible to give a positive answer. Details will be found in [9].

For earlier results on nonlocal bifurcation, from homoclinic to periodic solutions of ordinary differential equations in R^n, we refer to Šilnikov's papers [6, 5]. Lemma 3.3 in [5] on certain diffeomorphisms given by an O.D.E. appears to be related to the somewhat later λ-lemma of Palis. Estimate (3.5) in [5] may be compared to geometric convergence in Lemma 2.1 below.

Lemma 2.1 is stated in a way which yields uniform estimates also in case of parameterized maps, convenient for the application to bifurcation problems. On the other hand, it is restricted to a simplified situation where

a) slopes of tangents to the given set H are already small and
b) the linearized map T is an expansion on P and a contraction on Q.

These hypotheses will not present essential difficulties in applications but have the advantage to allow a relatively short proof concentrated on the basic estimates which lead to dominated convergence.

At the end we formulate a generalization of the result on geometric convergence which holds without hypothesis a) and with b) replaced by the more natural assumption that $Dg(0)$ is hyperbolic, see Lemma 3.1. For reasons of length we do not give a proof but refer to the preprint [8]. Lemma 3.1 will not be used in the study of nonlocal bifurcation mentioned above.

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1. Preliminaries

I. Let E be a Banach space. We begin with C^2-maps $\tilde{g}: \tilde{U} \to E$ with

$(\tilde{U})$ $\quad 0 \in \tilde{U}, \quad \tilde{U} \subset E$ open, $\quad \tilde{g}(0) = 0$.

Let $T := D\tilde{g}(0)$ be an expansion-contraction. That is,

$$E = P \oplus Q \text{ with } T\text{-invariant subspaces } P \text{ and } Q, \tag{1}$$

$$(\alpha\beta\gamma) \quad |Tqx| \leqq \alpha\,|qx|, \quad \beta\,|px| \leqq |Tpx| \leqq \gamma\,|px| \quad \text{for all } x \in E$$

with contants $\alpha < 1, \beta > 1, \gamma \geqq \beta$. p and q denote the projections of E onto P and Q given by (1).

In addition, we assume

$$\tilde{g}(\tilde{U} \cap P) \subset P, \quad \tilde{g}(\tilde{U} \cap Q) \subset Q. \tag{2}$$

In applications, this can be arranged by a change of coordinates which uses local stable and unstable manifolds.

For the remainder term $R \in C^2(\tilde{U}, E), \tilde{g} = T + R$, and for the partial derivatives D_p and D_q with respect to the decomposition $E = P \oplus Q$, we find

$$(R) \qquad R(0) = 0,\ DR(0) = 0,\ R(\tilde{U} \cap P) \subset P,\ R(\tilde{U} \cap Q) \subset Q,$$
$$D_p qR(x) = 0 \quad \text{for all } x \in \tilde{U} \cap P,$$
$$D_q pR(x) = 0 \quad \text{for all } x \in \tilde{U} \cap Q.$$

II. For applications of Lemma 2.1 to bifurcation problems where p and q may depend on parameters it is convenient not to consider the map(s) $\tilde{g}$ but restrictions g to sets $U \subset E$ with

$$(U) \quad U \subset \tilde{U} \text{ open, } 0 \in U,\ px + sqx \in \tilde{U},\ spx + qx \in \tilde{U} \quad \text{for all } x \in U \text{ and } s \in [0, 1].$$

Let $c > 0$. Set $\alpha_c := \alpha + c$, $\beta_c := \beta - c$, $\gamma_c := \gamma + c$. If

$$(c) \quad |DpR(x)| \leqq c \quad \text{and} \quad |DqR(x)| \leqq c \quad \text{for all } x \in \tilde{U},$$

then

$$(g, c) \quad |qg(x)| \leqq \alpha_c\,|qx|,\ \beta_c\,|px| \leqq |pg(x)| \leqq \gamma_c\,|px| \quad \text{for all } x \in U.$$

Proof. For $x \in U$, (U) implies that the straight line from x to px is contained in $\tilde{U}$. We have $|qg(x)| = |Tqx + qR(x)|$, and $|qR(x)| = |qR(x) - qR(px)| \leqq c\,|x - px| = c\,|qx|$, by (c) and (U) and the mean value theorem. $(\alpha\beta\gamma)$ now shows the first inequality. The others follow in the same way.

Remark. Property (c) yields $|D_i jR(x)| \leqq c$ on U, for $i, j \in \{p, q\}$.

III. Let $m > 0$, $\bar{c} > 0$ be given with

$$(m) \quad |D(D_q pR)(x)| \leqq m \quad \text{for all } x \in \tilde{U},$$

$$(\bar{c}, m) \quad |px| \leqq \bar{c}/m \quad \text{for all } x \in U.$$

Then $|D_q pR(x)| \leqq m\,|px| \leqq \bar{c} \quad$ for all $x \in U$.

Proof. The straight line from $x \in U$ to qx is contained in $\tilde{U}$ so that the mean value theorem gives $|D_q pR(x)| = |D_q pR(x) - D_q pR(qx)| \leqq m\,|x - qx| = m\,|px|$.

IV. We shall consider tangents to preimages of some set $H \subset U$ under g. These preimages, and the set H itself, are not necessarily manifolds. A vector $t \in E$ is called tangent to a set S in E at a point $x \in S$ if there is a differentiable curve $\phi\colon (-1, 1) \to E$ with $\phi(0) = x$, $\phi((-1, 1)) \subset S$ and $t = D\phi(0)(1)$. $T_x S$ denotes the set of all tangents to S at x. As usual, the hypotheses $S \subset W$, $\psi\colon W \to E$ differentiable, $\psi(S) \subset S'$, $t \in T_x S$ altogether imply $D\psi(x)\, t \in T_{\psi(x)} S'$.

Preimages of a set $S \subset E$ with respect to a map $\psi\colon W \to E$, $W \subset E$, are defined by $S_0 := S$, $S_{k+1} := \psi^{-1}(S_k)$ for all $k \in N_0$.

Suppose that the C^2-map $g\colon \tilde{U} \to E$ and $U \subset \tilde{U}$ satisfy (U), (1) and $(\alpha\beta\gamma)$ for $T = D\tilde{g}(0)$, (2), (U), (c) and $1 < \beta_c = \beta - c$. Let $H \subset U$ and $x \in H_k$, the k-th preimage with respect to g, $k \in N$. Then $x_j \in H_j$ for the $k+1$ points given by $x_k := x$, $x_{j-1} := g(x_j)$ for $j = k, \ldots, 1$; and (g, c) yields

$$\gamma_c^{-j}\,|px_{k-j}| \leqq |px| \leqq \beta_c^{-j}\,|px_{k-j}| \qquad \text{for all } j = 0, \ldots, k.$$

2. Dominated convergence for slopes of tangents

In order to formulate a result which yields uniform bounds in parameterized problems with families of maps $\tilde{g}$, we introduce the set $B \subset R^7$ of vectors $b = (\alpha, \beta, \gamma, c, m, p_1, p_2)$ with $\alpha, \beta, \gamma, c, m$ positive, p_1 and p_2 nonnegative and

$$(B) \quad (\alpha_c + c =)\ \alpha + 2c < 1 < \beta - c (= \beta_c), \qquad p_1 \leqq p_2.$$

For $b \in B$ given we set $\bar{\beta}_c := (\beta_c + 1)/2 \in (1, \beta_c)$ and

$$\bar{c} := \min\left\{c, \frac{\beta_c - 1}{\beta_c + 1}, \frac{\beta_c - 1}{\beta_c + 1} \cdot [\beta_c - \bar{\beta}_c]\right\} > 0.$$

Lemma 2.1. Let $b \in B$. There is a constant $c_b > 0$ such that for all C^2-maps $\tilde{g}$: $\tilde{U} \to E$ and all sets $U \subset \tilde{U}$ with $(\tilde{U})$, (1) and $(\alpha\beta\gamma)$ for $T = D\tilde{g}(0)$, (2), (U), (c), (m), $(\bar{c}, m)$ and for every set $H \subset U$ with

$$\left.\begin{array}{ll} (p) & p_1 \leqq |px| \leqq p_2 \\ ((\bar{c})) & qt \neq 0 \text{ and } s(t) \leqq \bar{c} \end{array}\right\} \quad \text{for all } x \in H,\ t \in T_x H \backslash \{0\},$$

we have:

$$(p, k) \quad p_1 \gamma_c^{-k} \leqq |px| \leqq p_2 \beta_c^{-k},$$

$$(t, \beta_c) \qquad qt \neq 0 \quad \text{and} \quad s(t) \leqq c_b \beta_c^{-k},$$

and in case

$$(*) \qquad 0 < p_1 \quad \text{and} \quad (\alpha + 3c)\, \gamma_c/\beta_c \leqq 1,$$

$$(t, x) \qquad qt \neq 0 \quad \text{and} \quad s(t) \leqq c_b\, |px|$$

for all preimages H_k, $k \in N_0$, of H under g, and all $x \in H_k$, $t \in T_x H_k \setminus \{0\}$.

Remarks. Conditions (1) and $(\alpha\beta\gamma)$ say that $E = P \oplus Q$, with $T = D\tilde{g}(0)$ an expansion on P and a contraction on Q. Condition (2) means that $\tilde{g}$ is normalized, with local stable and unstable manifolds contained in Q and P, respectively.

Typical applications will start with maps $\tilde{\tilde{g}}: \tilde{\tilde{U}} \to E$ which are given and satisfy $\tilde{\tilde{g}}(0) = 0$, (1), $(\alpha\beta\gamma)$, $\alpha < 1 < \beta \leqq \gamma$, and (2) with $\tilde{\tilde{U}}$ instead of $\tilde{U}$. Then a small constant $c > 0$ is chosen so that (B) holds. It is now easy to find a constant m and neighborhoods $\tilde{U}$, U of 0 so that the remaining conditions in Lemma 2.1 on $\tilde{g} := \tilde{\tilde{g}} \mid \tilde{U}$ and $\tilde{U}$, U are satisfied for all vectors $(\alpha, \beta, \gamma, c, m, \cdot, \cdot) \in B$. In this sense conditions (c), (m), $(\bar{c}, m)$, (U) are not restrictive.

Inequality (p) and $((\bar{c}))$ express a weak kind of transversality of H and P. We do not assume that H is smooth, and we do not exclude cases with $\dim H < \dim Q$, $H \cap Q \neq \emptyset$, $H \cap P = \{0\}$. However, hypothesis $((\bar{c}))$ requires that slopes of tangent vectors to H are "sufficiently small".

Part IV of section 1 shows that (t, x) is a better estimate than (t, β_c).

Inequality $(*)$ is easily verified in applications with $\dim P = 1$: If a C^2-map $\tilde{\tilde{g}}: \tilde{\tilde{U}} \to E$ is given with $\tilde{\tilde{g}}(0) = 0$, (1) and $(\alpha\beta\gamma)$, $\alpha < 1 < \beta \leqq \gamma$, and (2) with $\tilde{\tilde{U}}$ instead of $\tilde{U}$ then $\dim P = 1$ allows to choose $\beta = \gamma$, and we have $\alpha\gamma/\beta = \alpha < 1$ so that $(*)$ holds for small c.

Finally, note that without hypothesis $p_1 > 0$ estimate (t, x) will not be true: Consider $b \in B$ with $(*)$, a direct sum $E = P \oplus Q$ of nontrivial Banach spaces, $H := (-\varepsilon, \varepsilon)\, t$ with $t \in E$, $0 < \varepsilon\, |pt| \leqq p_2$, $qt \neq 0 \neq pt$, $s(t) \leqq \bar{c}$. Then $t \in T_0 H_0$, $0 < s(t)$, $p0 = 0$.

Proof of Lemma 2.1. Let $b \in B$ and $\tilde{g}$, U and H be given, satisfying the hypotheses of Lemma 2.1. We consider the restricted map $g: U \to E$ and preimages H_k of $H \subset U$ under g. Part IV of section 1 implies (p, k).

I. Let $x \in U$, $t \in E$. Set $u := Dg(x)\, t$. Part III of section 1 yields

$$|pu| = |pDg(x)\, t| = |Tpt + DpR(x)\,[pt + qt]| \geqq \beta\, |pt| - c\, |pt| - \bar{c}\, |qt|$$
$$= \beta_c\, |pt| - \bar{c}\, |qt|.$$

In addition,

$$|qu| \leqq \cdots \leqq \alpha_c\, |qt| + c\, |pt|.$$

II. We show

$$(A_k) \quad qt \neq 0 \text{ and } s(t) \leqq \bar{c} \sum_{i}{}_0^k\, \beta_c^{-i} \text{ for all } k \in N_0,\ x \in H_k,\ t \in T_x H_k \setminus \{0\}.$$

II.1. Hypothesis (($\bar{c}$)) gives (A_0).

II.2. Suppose (A_k) for some $k \in N_0$. Let $x \in H_{k+1}$, $t \in T_x H_{k+1} \setminus \{0\}$. Set $u := Dg(x)\, t \in T_{g(x)} H_k$.

II.2.1. Proof of $qt \neq 0$: Assume $qt = 0$. By I, $|pu| \geqq \beta_c |pt| - 0 > 0$. From ($A_k$), $qu \neq 0$. Again by I, $0 < |qu| \leqq \alpha_c |qt| + c\,|pt| = c\,|pt|$. Hence $1 < \beta_c/c \leqq s(u) \leqq \bar{c} \sum_{i\,0}^{k} \bar{\beta}_c^{-i} \leqq \bar{c}/(1 - \bar{\beta}_c^{-1}) = \bar{c}(\beta_c + 1)/(\beta_c - 1)$ which contradicts $\bar{c} \leqq (\beta_c - 1)/(\beta_c + 1)$.

II.2.2. In case $u = 0$, we find $0 = |pu| \geqq \beta_c |pt| - \bar{c}\,|qt|$, $s(t) \leqq \bar{c}/\beta_c \leqq \bar{c} \sum_{i\,0}^{k+1} \bar{\beta}_c^{-i}$.

II.2.3. In case $u \neq 0$, (A_k) gives $qu \neq 0$ and $s(u) \leqq \bar{c} \sum_{i\,0}^{k} \bar{\beta}_c^{-i} \leqq \bar{c}/(1 - \bar{\beta}_c^{-1}) = \bar{c}(\beta_c + 1)/(\beta_c - 1)$. From I we obtain

$$(\beta_c - c\,s(u))\, s(t) \leqq \bar{c} + \alpha_c s(u).$$

Hence

$$\left(\beta_c - c\bar{c}\,\frac{\beta_c + 1}{\beta_c - 1}\right) s(t) \leqq \bar{c} + s(u).$$

The definition of $\bar{c}$ implies

$$\bar{\beta}_c s(t) \leqq \bar{c} + s(u).$$

Using (A_k), we get the desired estimate for (A_{k+1}).

III. Note that II implies $qt \neq 0$ and $s(t) \leqq 1$ for all $k \in N_0$, $x \in H_k$, $t \in T_x H_k \setminus \{0\}$.

IV. If $x \in U$, $0 \neq t \in E$, $Dg(x)\, t = 0$, $qt \neq 0$ then $s(t) \leqq m \beta_c^{-1} |px|$.

Proof. An estimate as in I yields $0 = |pDg(x)\, t| \geqq \beta_c |pt| - |DpR(x)\, qt| \geqq \beta_c |pt| - m\,|px|\,|qt|$, with III (section 1).

V. If $k \in N$, $x \in H_k$, $0 \neq t \in T_x H_k$, $0 \neq Dg(x)\, t =: u$ then $qt \neq 0 \neq qu$ and $s(t) \leqq \dfrac{\alpha_c + c}{\beta_c}\, s(u) + \dfrac{m}{\beta_c}\, |px|$.

Proof. We obtain $u \in T_{g(x)} H_{k-1} \setminus \{0\}$. By III, $qu \neq 0$, $qt \neq 0$, $s(t) \leqq 1$. Estimates as in I give $|pu| \geqq \beta_c |pt| - m\,|px|\,|qt|$, $|qu| \leqq \alpha_c |qt| + c\,|pt|$. Hence

$$s(u) \geqq \frac{\beta_c s(t) - m\,|px|}{\alpha_c + c\,s(t)}, \qquad (\alpha_c + c1)\, s(u) \geqq \beta_c s(t) - m\,|px|.$$

VI. We show $s(t) \leqq \tilde{c}_b \beta_c^{-k}$, with

$$\tilde{c}_b := 1 + \frac{2m}{\beta_c}\, p_2\, \frac{1}{1 - (\alpha + 2c)}, \qquad \text{for all } k \in N_0;\ x \in H_k,\ t \in T_x H_k \setminus \{0\}.$$

VI.1. Note $qt \neq 0$ and $s(t) \leqq \bar{c} < 1 \leqq \tilde{c}_b$ for all $x \in H$, $t \in T_x H \setminus \{0\}$.

VI.2.1. Let $k \in N$, $x \in H_k$, $0 \neq t \in T_x H_k$. Consider the points and tangent vectors defined by $x_k := x$, $x_{i-1} := g(x_i)$, $t_k := t$, $t_{i-1} := Dg(x_i)\, t_i$ for $i = k, \ldots, 1$. Then $x_i \in H_i$ and $t_i \in T_{x_i} H_i$ for $i = k, \ldots, 0$. By III, $qt \neq 0$.

VI.2.2. Assume $t_i \neq 0$ for $i = k, k-1, \ldots, k-j \geqq 0$ with $1 \leqq j \leqq k$, $j \in N$. Then $qt_i \neq 0$ for these indices, see III. We prove

$$(S_\iota) \quad s(t_{k-j+\iota}) \leqq \beta_c^{-\iota} [(\alpha_c + c)^\iota s(t_{k-j}) + m\,|p x_{k-j+1}| \textstyle\sum_{\kappa 0}^{\iota-1} (\alpha_c + c)^\kappa]$$
$$\text{for } \iota = 1, \ldots, j:$$

For $\iota = 1$, V gives $s(t_{k-j+1}) \leqq \dfrac{\alpha_c + c}{\beta_c} s(t_{k-j}) + \dfrac{m}{\beta_c} |p x_{k-j+1}|\, 1$. Suppose (S_ι) holds for some $\iota \in \{1, \ldots, j-1\}$. Part V gives

$$\begin{aligned}
s(t_{k-j+\iota+1}) &\leqq \frac{\alpha_c + c}{\beta_c} s(t_{k-j+\iota}) + \frac{m}{\beta_c} |p x_{k-j+\iota+1}| \\
&\leqq \frac{\alpha_c + c}{\beta_c} s(t_{k-j+\iota}) + \frac{m}{\beta_c} \beta_c^{-\iota} |p x_{k-j+1}| \qquad \text{(IV, section 1)} \\
&\leqq \beta_c^{-\iota-1} (\alpha_c + c)^{\iota+1} s(t_{k-j}) + \beta_c^{-\iota-1} m\, |p x_{k-j+1}| \\
&\quad \times \textstyle\sum_{\kappa 1}^{\iota-1+1} (\alpha_c + c)^\kappa + m \beta_c^{-\iota-1} |p x_{k-j+1}| \qquad \text{(with } (A_\iota)) \\
&= \beta_c^{-(\iota+1)} [(\alpha_c + c)^{\iota+1} s(t_{k-j}) + m\, |p x_{k-j+1}| \textstyle\sum_{\kappa 0}^{\iota+1-1} (\alpha_c + c)^\kappa].
\end{aligned}$$

VI.2.3. Suppose $t_i \neq 0$ for all $i = k, \ldots, 0$. Then VI.2.2 with $j = k$ and (S_j) imply

$$s(t) \leqq \beta_c^{-k} \left[1\,\bar{c} + m \beta_c^{-1} p_2 \frac{1}{1 - (\alpha_c + c)} \right] \leqq \tilde{c}_b \beta_c^{-k}$$

(with $\alpha + 2c < 1$, $|p x_1| \leqq \beta_c^{-1} |p x_0|$, $x_0 \in H$, $s(t_0) \leqq \bar{c} \leqq c < 1$).

VI.2.4. Suppose there exists $j \in \{0, \ldots, k-1\}$ with $t_i \neq 0$ for $i = k, \ldots, k-j$ and with $t_{k-j-1} = 0$. By IV, $s(t_{k-j}) \leqq m \beta_c^{-1} |p x_{k-j}|$. In case $1 \leqq j$, (S_j) implies

$$\begin{aligned}
s(t) &\leqq \beta_c^{-j} [(\alpha_c + c)^j s(t_{k-j}) + m\, |p x_{k-j+1}| \textstyle\sum_{\kappa 0}^{j-1} (\alpha_c + c)^\kappa] \\
&\leqq \beta_c^{-j} \left[m \beta_c^{-1} |p x_{k-j}| + m\, |p x_{k-j+1}| \frac{1}{1 - (\alpha_c + c)} \right].
\end{aligned}$$

By IV (section 1), the last term is majorized by

$$\begin{aligned}
&\beta_c^{-j} \left[m \beta_c^{-1} \beta_c^{-(k-j)} |p x_0| + m \beta_c^{-(k-j+1)} |p x_0| \frac{1}{1 - (\alpha_c + c)} \right] \\
&\leqq \beta_c^{-k} \left[m \beta_c^{-1} p_2 + m \beta_c^{-1} p_2 \frac{1}{1 - (\alpha_c + c)} \right] \leqq \tilde{c}_b \beta_c^{-k}.
\end{aligned}$$

In case $j = 0$ we use IV and find

$$s(t) \leqq m\beta_c^{-1}|px| \leqq m\beta_c^{-1}\beta_c^{-k}p_2 \qquad \text{(with IV, section 1)}$$
$$\leqq \tilde{c}_b \beta_c^{-k}.$$

VII. We prepare the proof of (t, x). Set $c_1 := c/m$, $c_2 := \beta_c c_1 > c_1$. Claim: $k \in N$, $x \in H_k$, $0 \neq t \in T_x H_k$, $0 \neq Dg(x)\,t =: u$, $qu \neq 0$ and

$$|pg(x)| \leqq c_2 s(u)$$

imply $qt \neq 0$ and

$$s(t) \leqq \frac{\alpha_c + 2c}{\beta_c} s(u).$$

Proof. We have $0 \neq u \in T_{g(x)} H_{k-1}$. By III, $qt \neq 0$. Use V and

$$\beta_c |px| \leqq |pg(x)| \leqq c_2 s(u) = \beta_c (c/m)\, s(u).$$

VIII. Suppose $b \in B$ satisfies $(*)$ and $p_1 > 0$. Let $k \in N_0$, $x \in H_k$, $t \in T_x H_k \backslash \{0\}$.

VIII.1. If $k = 0$ then $qt \neq 0$ and $s(t) \leqq \bar{c} \leqq \bar{c} p_1^{-1} |px|$.

VIII.2. Assume $k \in N$. We consider sequences of points $x_i \in H_i$ and tangent vectors $t_i \in T_{x_i} H_i$, $i = k, \ldots, 0$, as in VI.2.1. Because of III, $qt_k = qt \neq 0$.

VIII.2.1. Suppose there exists $j \in \{1, \ldots, k\}$ such that for all $i \in \{k, \ldots, k - j + 1\}$,

$$t_i \neq 0 \quad \text{and} \quad |px_i| \leqq c_2 s(t_i)$$

while $t_{k-j} \neq 0$ and $c_2 s(t_{k-j}) < |px_{k-j}|$. Then $s(t_k) \leqq c_3 |px_k|$ where

$$c_3 := \left(\frac{\alpha_c + c}{\beta_c} \frac{1}{c_2} + \frac{m}{\beta_c^2}\right) \gamma_c.$$

Proof. (By III, $qt_i \neq 0$, and $s(t_i)$ is defined, for $i = k, \ldots, k - j$.) If $j = 1$ then

$$\begin{aligned}
s(t_k) &\leqq (\alpha_c + c)\, \beta_c^{-1} s(t_{k-1}) + m\beta_c^{-1} |px_k| && \text{(see V)} \\
&\leqq (\alpha_c + c)\, \beta_c^{-1} c_2^{-1} |px_{k-1}| + m\beta_c^{-1} |px| && \text{(hypothesis for } t_{k-j}) \\
&\leqq (\alpha_c + c)\, \beta_c^{-1} c_2^{-1} \gamma_c |px_k| + m\beta_c^{-1} |px_k| && \text{(with IV, section 1)} \\
&\leqq c_3 |px_k| && \text{(with } 1 < \gamma_c/\beta_c).
\end{aligned}$$

If $j \geqq 2$ (and $k \geqq 2$) then $|pq(x_i)| = |px_{i-1}| \leqq c_2 s(t_{i-1}) = c_2 s(Dg(x_i)\, t_i)$ for $i = k, \ldots, k - j + 2$, and VII implies

$$s(t_k) \leqq \left(\frac{\alpha_c + 2c}{\beta_c}\right)^{j-1} s(t_{k-j+1}).$$

By V,

$$\begin{aligned} s(t_{k-j+1}) &\leqq (\alpha_c + c)\, \beta_c^{-1}\, s(t_{k-j}) + m\beta_c^{-1}\, |px_{k-j+1}| \\ &\leqq ((\alpha_c + c)\, \beta_c^{-1} c_2^{-1} + m\beta_c^{-2})\, |px_{k-j}| \\ &\quad \text{(with } (g, c)\text{, hypothesis for } i = k - j) \\ &\leqq (\ldots)\, \gamma_c^j\, |px_k| \quad \text{(with IV, section 1).} \end{aligned}$$

Together,

$$s(t_k) \leqq (\ldots)\, \gamma_c\, [(\alpha_c + 2c)\, \gamma_c \beta_c^{-1}]^{j-1}\, |px_k|,$$

and (*) implies the assertion.

VIII.2.2. The case $t_{k-1} = 0$: Part III and $t_k \neq 0$ give $qt_k \neq 0$. Part IV yields $s(t) \leqq m\beta_c^{-1}\, |px|$.

VIII.2.3. In case $t_{k-1} \neq 0$ and $s(t) < c_2^{-1}\, |px|$ there is nothing to prove.

VIII.2.4. If $t_i \neq 0$ (and $qt_i \neq 0$) and $|px_i| \leqq c_2\, s(t_i)$ for $i = k, \ldots, 0$ then VII implies $s(t) = s(t_k) \leqq ((\alpha_c + 2c)/\beta_c)^k\, s(t_0)$. With

$$s(t_0) \leqq \bar{c} \leqq \bar{c} p_1^{-1}\, |px_0| \leqq \bar{c} p_1^{-1} \gamma_c^k\, |px_k| \quad \text{(IV, section 1),}$$

we obtain

$$s(t) \leqq \bar{c} p_1^{-1}\, [(\alpha_c + 2c)\, \gamma_c/\beta_c]^k\, |px| \leqq \bar{c} p_1^{-1}\, |px| \quad \text{(with (*)).}$$

VIII.2.5. Suppose $t_{k-1} \neq 0$ and $|px_k| \leqq c_2\, s(t_k)$, and that there is a smallest integer $j \in \{1, \ldots, k\}$ such that the statement "$t_{k-j} \neq 0$ and $qt_{k-j} \neq 0$ and $|px_{k-j}| \leqq c_2\, s(t_{k-j})$" is false.

VIII.2.5.1. Suppose in addition $t_{k-j} = 0$. We have

$$s(t_k) \leqq ((\alpha_c + 2c)/\beta_c)^{j-1}\, s(t_{k-j+1}) \quad \text{(see VII)}$$

and

$$s(t_{k-j+1}) \leqq m\beta_c^{-1}\, |px_{k-j+1}| \quad \text{(see IV).}$$

With IV (section 1),

$$s(t) = s(t_k) \leqq m\beta_c^{-1}\, [(\alpha_c + 2c)\, \gamma_c/\beta_c]^{j-1}\, |px_k| \leqq m\beta_c^{-1}\, |px| \quad \text{(see (*)).}$$

VIII.2.5.2. Suppose $t_{k-j} \neq 0$. Then $qt_{k-j} \neq 0$, by III. It follows that $c_2\, s(t_{k-j}) < |px_{k-j}|$. This allows to use VIII.2.1:

$$s(t_k) \leqq c_3\, |px_k|.$$

VIII.2.6. Altogether,

$$s(t) \leqq \hat{c}_b\, |px|$$

with $\hat{c}_b = \bar{c} p_1^{-1} + m\beta_c^{-1} + c_2^{-1} + c_3$.

3. A result for hyperbolic maps

A C^1 – map $g: V \to E$ on an open set V in a Banach space E with $g(0) = 0$ is called hyperbolic (at $x = 0$) if $\sigma \cap S^1 = \emptyset$ for the spectrum σ of $T = Dg(0)$. Then $E = P \oplus Q$ with closed T-invariant subspaces P and Q so that the induced maps $T_p: P \to P$, $T_Q: Q \to Q$ have spectra $\sigma_P := \{z \in \sigma: |z| > 1\}$ and $\sigma_Q := \{z \in \sigma: |z| < 1\}$, respectively. Let p and q denote the projections onto P and Q defined by the decomposition of E.

Lemma 3.1. Let a C^2-map $g: V \to E$ be given, hyperbolic at $x = 0$ with $\sigma_P \neq \emptyset \neq \sigma_Q$ and $g(V \cap P) \subset P$, $g(V \cap Q) \subset Q$. Then there exist open neighborhoods $W \subset U \subset V$ of $x = 0$ with the following property. For every C^1-submanifold $H \subset W$ with $H \cap P = \{x_0\}$, with $T_{x_0} H \oplus P = E$ and such that

(*H*) for every $\varepsilon > 0$ there exists $\delta > 0$ with $|px - x_0| < \varepsilon$ for all $x \in H$ with $|qx| < \delta$,

there are constants $\tilde{c} > 0$, $h \in (0, 1)$ and an integer $l \in N$ such that for every preimage H_k of H under the restriction $g|U$ with $k \geqq l$,

$$|px| \leqq \tilde{c} \cdot h^k,$$

$$qt \neq 0 \quad \text{and} \quad s(t) \leqq \tilde{c} \cdot h^k$$

for all $x \in H_k$, $t \in T_x H_k \setminus \{0\}$.

For a proof, see [8]. Hypothesis (*H*) is satisfied whenever H is the graph of a C^1-map of an open subset of Q into P. Note that the unstable space may be infinite-dimensional, and that $H \cap Q \neq \emptyset$ or $x_0 = 0$ are allowed.

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Summary

Inclination lemmas (λ-lemmas) serve as tools for the investigation of dynamical systems in the neighborhood of saddle points. They assert convergence of inclinations (slopes of tangents) when the system shifts a given transversal to the stable or unstable manifold towards equilibrium.

We derive estimates of the speed of this convergence, for preimages $g^{-k}(H)$, $k \in N$, of a transversal H to the unstable manifold. g is assumed to be a C^2-map in a Banach space, not necessarily reversible, with a saddle point $x = 0$ and already normalized so that local stable and unstable manifolds are contained in linear spaces.

The estimates are needed particularly in a study of nonlocal bifurcation – from heteroclinic to periodic solutions of the second kind – for parameterized functional differential equations

$$\dot{x}(t) = ah(x(t-1))$$

which describe phase-locked loops

Zusammenfassung

Neigungslemmata (λ-Lemmata) dienen zur Untersuchung dynamischer Systeme in der Nähe von Sattelpunkten. Sie garantieren Konvergenz von Neigungen (Tangentensteigungen), wenn das System gegebene Transversalen zur stabilen oder instabilen Mannigfaltigkeit zum Gleichgewicht hin transportiert.

Wir leiten Abschätzungen der Geschwindigkeit dieser Konvergenz her, für Urbilder $g^{-k}(H)$, $k \in N$, einer Transversalen H zur instabilen Mannigfaltigkeit. g ist dabei eine C^2-Abbildung in einem Banachraum, nicht notwendig umkehrbar, mit Sattelpunkt $x = 0$ und schon normalisiert, so daß lokale stabile und instabile Mannigfaltigkeit in linearen Räumen liegen. Die Abschätzungen werden insbesondere zu einer Untersuchung nichtlokaler Verzweigung – von heteroklinen zu periodischen Lösungen zweiter Art – für parametrisierte Funktionaldifferentialgleichungen

$$\dot{x}(t) = ah(x(t-1))$$

benötigt, die PLL-Schaltungen beschreiben.