JUSTUS-LIEBIG-
UNIVERSITÄT
GIESSEN

Mathematisches Institut
Fachbereich 07
Justus-Liebig-Universität Gießen

Exceptional twin buildings of type $\tilde{\mathrm{C}}_2$

A contribution to the classification of twin buildings

Dissertation

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Katharina Wendlandt

Betreuer: Prof. Dr. Bernhard M. Mühlherr

ABSTRACT

We provide a classification of affine twin buildings of type $\tilde{\mathsf{C}}_2$ having at least one exceptional residue with one exception in the case E_7 in characteristic two. Relying on the main results of [TW], [W09] and [MPW] we settle the uniqueness. The existence part is settled by refining and adapting the theory of descent in buildings developed in [MPW] to our specific situation.

KURZFASSUNG

Wir klassifizieren diejenigen affinen Zwillingsgebäude vom Typ $\tilde{\mathsf{C}}_2$, die mindestens ein Residuum vom Ausnahmetyp enthalten. Ein noch offenes Problem bildet der Spezialfall E_7 in Charakteristik zwei. Die Eindeutigkeit basiert auf den Hauptresultaten in [TW], [W09] und [MPW]. Der Existenzbeweis beruht auf einer Weiterentwicklung der Theorie über *descent in buildings* in [MPW].

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INTRODUCTION

Buildings have been introduced by J. Tits in order to provide a unified geometric framework for understanding semisimple complex Lie groups and, later, semisimple algebraic groups over an arbitrary field. The definition evolved gradually during the 1950s and 1960s and reached a mature form in about 1965. At that time, Tits thought of a building as a simplicial complex with a family of subcomplexes called *apartments*, subject to a few axioms. Each apartment is made up of *chambers*, which are the top-dimensional simplices. In the more 'modern' approach, introduced by Tits in [Ti81], one forgets about all simplices except the chambers. The definition is recast entirely in terms of objects called *chamber systems*.

One of the most important results in the theory of buildings is the classification of irreducible spherical buildings of rank at least 3 in [Ti74]. At the heart of the classification is the famous Theorem 4.1.2 of [Ti74] which states that every 'local isomorphism' from one thick irreducible spherical building to another extends to an isomorphism of the buildings in question. Meanwhile, there is a simplified proof in [TW] which makes use of the classification of Moufang polygons. Irreducible spherical buildings of rank 2 are called generalized polygons. Generalized polygons themselves are too numerous to classify, but it was observed that (as a consequence of 4.16 of [Ti74]) every thick irreducible spherical building of rank at least 3 as well as every irreducible residue of such a building satisfies the *Moufang condition*. As a consequence, every thick irreducible spherical building of higher rank is an amalgamation, in a certain sense, of Moufang polygons. According to [TW], Moufang n-gons exist only for $n = 3, 4, 6$ or 8 and there are six families of Moufang quadrangles. The family of Moufang quadrangles of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8 together with the family of Moufang quadrangles of type F_4 constitutes the *exceptional Moufang quadrangles*.

About 30 years ago, M. Ronan and J. Tits defined a new class of buildings which generalize spherical buildings in a natural way, namely the class of twin buildings. The motivation of their definition is provided by the theory of Kac-Moody groups: Twin buildings are naturally associated to 'groups of Kac-Moody type' in the same way that spherical buildings are associated to algebraic groups. The main idea of a twin building is that a twin building

consists of a pair (Δ_+, Δ_-) of two buildings of the same type together with a symmetric relation between the chambers of the two different buildings which has properties similar to the opposition relation on the chambers of a spherical building. In this way, the twin building behaves in many respects like a spherical building, whereas the individual buildings Δ_+ and Δ_- are generally not spherical.

In view of the classification of spherical buildings it is natural to ask whether it is possible to classify higher rank twin buildings. A large part of [Ti92] deals with this question. As a first observation, it turns out that such a classification seems only to be feasible under the additional assumption that the entries in the corresponding Coxeter matrices are all finite. The classification program described in [Ti92] is based on the conjecture that there is a bijective correspondence between twin buildings of a given type Π and certain Moufang foundations of type Π for each 2-spherical Coxeter diagram Π.

In [Ti92], J. Tits conjectures that one can classify all 2-spherical twin buildings if one has a classification of all rank 3 twin buildings. This conjecture is true under the assumption that all rank 3 residues are spherical, due to unpublished results in [BM]. This is of course a severe restriction. In his Habilitationsschrift [MHab], B. Mühlherr reduces the proof of the general conjecture to the verification that each 2-spherical twin building of rank 3 can be constructed via Galois descent. Most of the rank 3 twin buildings can be handled with the methods established in [MHab] and [M99]. However, there are some exceptions which need to be considered separately. The verification for twin buildings of type $\tilde{\mathsf{A}}_2$ and those of triangle type 443 are a consequence of [WDis]. The only serious case left is the case of affine twin buildings of type $\tilde{\mathsf{C}}_2$. It turns out that, if all panels of the twin building contain at least four chambers, the halves of such a twin building are so-called *Bruhat-Tits buildings*, i.e. these are affine buildings whose building at infinity is Moufang. There are unpublished partial results about the classification of twin buildings of type $\tilde{\mathsf{C}}_2$ by B. Mühlherr and H. Van Maldeghem in [MvM20]. They treat the case in which each residue is a classical (i.e. non-exceptional) Moufang quadrangle. The goal of this thesis is a classification of all affine twin buildings of type $\tilde{\mathsf{C}}_2$ having at least one exceptional Moufang quadrangle as a residue. Except for one specific case in characteristic 2 having a residue of type E_7, we provide a complete solution of this problem.

A generalization of Tits' extension theorem by B. Mühlherr and M. Ronan in [MR] states that almost all twin buildings, just as spherical buildings, are uniquely determined by their 'local structure' or, more precisely, by the rank 2 neighbourhood of one of its chambers which can be thought of an amalgamation of Moufang polygons. As a first step along the classification of the exceptional $\tilde{\mathsf{C}}_2$- twin buildings we determine those amalgamations of

two Moufang quadrangles which are candidates for being the local structure of a twin building. For this, we extensively make use of the classification of Moufang polygons in [TW] and the classification of Bruhat-Tits buildings of type $\tilde{\mathsf{C}}_2$ in [W09] and [MPW, part 2]. Another ingredient which is needed is a property of possible residues called (Ind). This property is satisfied in almost all of our cases. Once we have carved out all possible candidates, we finally have to prove that these can, in fact, be realized as the local structure of a twin building. We will show the existence of such twin buildings by giving an explicit construction of those as fixed point structures of certain automorphism groups in higher rank twin buildings. This relies on the work of B. Mühlherr, H. Petersson and R. Weiss in [MPW]. Their main result about the descent in buildings states that (under minimal and clearly necessary conditions) the set of residues of a building Δ stabilized by an arbitrary subgroup of Aut(Δ) form a thick building. In [MW], B. Mühlherr and R. Weiss apply the theory of descent in buildings to give elementary constructions of the exceptional Moufang quadrangles as the fixed point building of a Galois involution of a higher rank building. We will use their descriptions and extend them in a suitable way.

In part II of this thesis we provide the combinatorial properties of the incidence geometries associated with the spherical buildings in question and give elementary constructions of certain local automorphisms (which will later on appear as restrictions of the automorphisms used by B. Mühlherr and R. Weiss in [MW]).

Part III is dedicated to the generalization of the theory of descent in buildings to a theory of descent in twin buildings. In order to combine descent theory and the extension theorem we have to adjust the notion of a foundation of a building. This requires some careful analysis of the extension theorem and descent theory. We examine the interplay between the Γ-residues of the different halves of the twin building and derive a slight variation of the famous extension theorem of B. Mühlherr and M. Ronan which provides a pair of opposite Γ-chambers. This is necessary since, even though the theory of descent in buildings ensures that the fixed point structure of each half of the twin building is itself a building, we need a codistance function between the chambers of these two fixed point buildings in order to form a twin building. As is shown, the existence of a suitable pair of opposite Γ-chambers provides the existence of such a codistance function.

Part IV deals with the determination of possible candidates for being the local structure of $\tilde{\mathsf{C}}_2$-twin buildings having an exceptional residue. Using the results of part II and part III we give an explicit construction of all $\tilde{\mathsf{C}}_2$-twin buildings as fixed point structures in higher rank twin buildings whose local structure coincides with the candidates of part IV.

In this way we obtain an almost complete classification of all exceptional twin buildings of type $\tilde{\mathsf{C}}_2$ except in the one specific case mentioned above.

Part I

Preliminaries

Chapter 1

Parameter systems

In this chapter we fix notation which will be used throughout and recall definitions and results from the literature which play an essential role in what follows.

Vector spaces

1.1 Definition
A *skew field* is a triple $(\mathbb{K}, +, \cdot)$ such that the following hold:

(SF1) The pair $(\mathbb{K}, +)$ is a commutative group.

(SF2) The pair $(\mathbb{K}, \cdot)$ is a group.

(SF3) For all $r, s, t \in \mathbb{K}$ we have

$$r \cdot (s + t) = r \cdot s + r \cdot t \text{ and } (r + s) \cdot t = r \cdot t + r \cdot s.$$

A commutative skew field is called *field*.

1.2 Definition
Let $\mathbb{K}, \mathbb{K}'$ be skew fields. An *(anti-)isomorphism* is an additive bijective transformation $\sigma \colon \mathbb{K} \to \mathbb{K}'$ such that for all $s, t \in \mathbb{K}$:

$$\sigma(st) = \sigma(s)\sigma(t) \quad (\text{respectively } \sigma(st) = \sigma(t)\sigma(s)).$$

1.3 Definition
A *right vector space* is a pair $(V, \mathbb{K})$ consisting of a commutative group $(V, +)$ and a skew field $\mathbb{K}$ together with a scalar multiplication $\cdot \colon V \times \mathbb{K} \to V$ satisfying

$$\forall\, v \in V : v \cdot 1_{\mathbb{K}} = v, \quad \forall\, v \in V, s, t, \in \mathbb{K} : (v \cdot s) \cdot t = v \cdot (st)$$

and

$$\forall\, v, w \in V, s, t \in \mathbb{K} : (v + w) \cdot s = v \cdot s + w \cdot s, \quad v \cdot (s + t) = v \cdot s + v \cdot t.$$

If $(V, \mathbb{K})$ is a vector space, we also say that V is *a $\mathbb{K}$-vector space* or that V *is a vector space over* $\mathbb{K}$. If $\mathbb{K}$ is a field we consider vector spaces over $\mathbb{K}$ as left vector spaces.

1.4 Definition
Two vector spaces $(V, \mathbb{K})$ and $(V', \mathbb{K}')$ are *isomorphic* if there is a pair (φ, ϕ) of isomorphisms $\phi \colon \mathbb{K} \to \mathbb{K}'$ and $\varphi \colon V \to V'$ of skew fields and groups, respectively, satisfying

$$\forall\, s \in \mathbb{K}, v \in V : \varphi(v \cdot s) = \varphi(v) \cdot \phi(s).$$

If $(\varphi, \phi) \colon (V, \mathbb{K}) \to (V', \mathbb{K}')$ is an isomorphism of vector spaces, we also say that φ is a *ϕ-semi-linear* isomorphism.

Sesquilinear forms

Let $\mathbb{K}$ be a skew field and let V be a vector space over $\mathbb{K}$.

1.5 Definition
Let $\sigma \colon \mathbb{K} \to \mathbb{K}$ be an anti-automorphism. A *σ-sesquilinear form* is a bi-additive transformation $f \colon V \times V \to \mathbb{K}$ such that

$$\forall\, v, w \in V, s, t \in \mathbb{K} : f(v \cdot s, w \cdot t) = \sigma(s)\, f(v, w)\, t.$$

A σ-sesquilinear form $f \colon V \times V \to \mathbb{K}$ is called *non-degenerate*, if for any $0_V \neq v \in V$ there exists a vector $0_V \neq w \in V$ such that $f(v, w) \neq 0_{\mathbb{K}}$ and vice versa.

1.6 Proposition
Let $\sigma \colon \mathbb{K} \to \mathbb{K}$ be an anti-automorphism and let $f \colon V \times V \to \mathbb{K}$ be a non-degenerate σ-sesquilinear form. The following two conditions are equivalent:

(i) f is *reflexive*, i.e. for all $v, w \in V$, the relation

$$f(v, w) = 0_{\mathbb{K}} \Leftrightarrow f(w, v) = 0_{\mathbb{K}}$$

holds.

(ii) f is *(σ, ε)-hermitian*, i.e. there exists $\varepsilon \in \mathbb{K}$ such that

$$f(w, v) = \sigma(f(v, w))\varepsilon$$

for all $v, w \in V$. Note that $\varepsilon \neq 0_{\mathbb{K}}$ since f is presumed to be non-degenerate.

Proof This is [Ue, 4.5.8]. □

1.7 Definition
Let $\mathrm{id}_{\mathbb{K}} \neq \sigma \colon \mathbb{K} \to \mathbb{K}$ be an anti-automorphism such that $\sigma^2 = \mathrm{id}_{\mathbb{K}}$. A $(\sigma, -1_{\mathbb{K}})$-hermitian form f is also called *skew-hermitian* (with respect to σ).

1.8 Lemma
Let $f \colon V \times V \to \mathbb{K}$ be a (σ, ε)-hermitian sesquilinear form and suppose that $\sigma^2 = \mathrm{id}_{\mathbb{K}}$ and $\varepsilon^2 = 1_{\mathbb{K}}$. Then the following hold:

(a) $\sigma(\varepsilon) = \varepsilon$ and

(b) $\varepsilon \in Z(\mathbb{K})$.

Proof By [Ue, 4.5.10(a)] we have $\varepsilon = \varepsilon^{-1} = \sigma(\varepsilon)$. Due to [Ue, 4.5.10(c)] we have $\lambda = \sigma^2(\lambda) = \varepsilon \lambda \varepsilon$ for any $\lambda \in \mathbb{K}$. □

Quadratic spaces

1.9 Definition
A *quadratic space* is a triple $\Lambda := (\mathbb{K}, V, Q)$ such that

(i) $\mathbb{K}$ is a commutative field,

(ii) V is a vector space over $\mathbb{K}$ and

(iii) $Q \colon V \to \mathbb{K}$ is a quadratic form, i.e. the map $f_Q \colon V \times V \to \mathbb{K}$ defined by
$$f_Q(v, w) := Q(v + w) - Q(v) - Q(w)$$
is bilinear and for all $v \in V, \lambda \in \mathbb{K}$ we have $Q(\lambda v) = \lambda^2 Q(v)$.

Clearly, if $U \leq_{\mathbb{K}} V$ is a subspace of the $\mathbb{K}$-vector space V, the triple $(\mathbb{K}, U, Q|_U)$ is a quadratic space and we will call it a *quadratic subspace of Λ*.

1.10 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ and $\Lambda' = (\mathbb{K}, V', Q')$ be quadratic spaces over the same field $\mathbb{K}$. The *orthogonal sum* $\Lambda \oplus \Lambda'$ of Λ and Λ' is the quadratic space $(\mathbb{K}, V \oplus V', Q \oplus Q')$, where the quadratic form $Q \oplus Q'$ is given by
$$Q \oplus Q'(v, v') = Q(v) + Q'(v')$$
for all $v \in V$ and all $v' \in V'$. Note that $f_{Q \oplus Q'}((v, 0_{V'}), (0_V, v')) = 0_{\mathbb{K}}$ for all $v \in V$ and all $v' \in V'$.

1.11 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space and let $\mathbb{E}$ be an extension field of $\mathbb{K}$.
We form the tensor product $V_{\mathbb{E}} = V \otimes \mathbb{E}$ and endow $V_{\mathbb{E}}$ with the structure
of an $\mathbb{E}$-vector space in the usual way, i.e. $s \cdot (v \otimes t) = (v \otimes st)$ for all $v \in V$
and all $s, t \in \mathbb{E}$.
As is shown, for example, in [J69, 1.7-1.8], there exists a unique quadratic
form $Q_{\mathbb{E}}$ on $V_{\mathbb{E}}$ over $\mathbb{E}$ such that

$$Q_{\mathbb{E}}(v \otimes s) = Q(v)s^2 \text{ and } f_{Q_{\mathbb{E}}}(u \otimes s, v \otimes t) = f_Q(u, v)st$$

for all $u, v \in V$ and $s, t \in \mathbb{K}$. We will call the quadratic space $\Lambda_{\mathbb{E}} = (\mathbb{E}, V_{\mathbb{E}}, Q_{\mathbb{E}})$
the *scalar extension of* Λ from $\mathbb{K}$ to $\mathbb{E}$.

1.12 Definition
Let $\Lambda := (\mathbb{K}, V, Q)$ be a quadratic space. We set

$$\mathrm{Def}(\Lambda) := \{v \in V \mid f_Q(v, w) = 0_{\mathbb{K}} \ \forall \ w \in V\}$$

and

$$\mathrm{Rad}(\Lambda) := \{v \in \mathrm{Def}(\Lambda) \mid Q(v) = 0_{\mathbb{K}}\}.$$

The quadratic space Λ is

◇ *non-degenerate* if $\mathrm{Def}(\Lambda) = \{0_V\}$ and it is

◇ *regular* if $\mathrm{Rad}(\Lambda) = \{0_V\}$.

1.13 Remark
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space and let $U \leq_{\mathbb{K}} V$ be a subspace of V.
We define the *orthogonal complement* of U by

$$U^{\perp} := \{v \in V \mid f_Q(u, v) = 0_{\mathbb{K}} \ \forall \ u \in U\}.$$

This is a subspace of V and we set $(\mathbb{K}, U, Q|_U)^{\perp} := (\mathbb{K}, U^{\perp}, Q|_{U^{\perp}})$.
If $\dim_{\mathbb{K}}(V) < \infty$ and $(\mathbb{K}, U, Q|_U)$ is non-degenerate, [EKM, 1.6] yields that
$\Lambda = (\mathbb{K}, U, Q|_U) \oplus (\mathbb{K}, U, Q|_U)^{\perp}$.

1.14 Definition
Let $\Lambda := (\mathbb{K}, V, Q)$ be a quadratic space.

◇ A vector $0_V \neq v \in V$ is called *isotropic* if $Q(v) = 0_{\mathbb{K}}$.

◇ A subspace $U \leq_{\mathbb{K}} V$ is called *anisotropic* if $Q(v) \neq 0_{\mathbb{K}}$ for all $0_V \neq v \in U$.
   It is called *isotropic* otherwise. If $Q|_U \equiv 0$ the subspace $U \leq_{\mathbb{K}} V$ is called
   totally isotropic.

◇ The quadratic space Λ is called *(an)isotropic* if V is (an)isotropic.

⋄ The quadratic space Λ is called *proper*, if it is anisotropic and if the associated bilinear form f_Q is not identically zero.

⋄ The *Witt index* of Λ is the maximal dimension of a totally isotropic subspace.

1.15 Definition
Let $\Lambda := (\mathbb{K}, V, Q)$ be a quadratic space and let $0_{\mathbb{K}} \neq t \in \mathbb{K}$. The triple $\Lambda_t := (\mathbb{K}, V, tQ)$, where $tQ \colon V \to \mathbb{K}$ is given by $(tQ)(v) := tQ(v)$, is a quadratic space, called the *t-translate of Λ*.

1.16 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ and $\Lambda' = (\mathbb{K}', V', Q')$ be two quadratic spaces.

⋄ An *isometry from Λ to Λ'* is an isomorphism of vector spaces

$$(\phi, \varphi) \colon (\mathbb{K}, V) \to (\mathbb{K}', V') \text{ such that } Q' \circ \varphi = \phi \circ Q.$$

⋄ A *similarity from Λ to Λ'* is an isometry from Λ onto Λ'_t for some $0_{\mathbb{K}'} \neq t \in \mathbb{K}'$.

1.17 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space. A *hyperbolic pair* of Λ is a pair of elements $v, w \in V$ such that $Q(v) = Q(w) = 0_{\mathbb{K}}$ and $f_Q(v, w) = 1_{\mathbb{K}}$.
For each hyperbolic pair (v, w) of Λ we denote by $\mathbb{H}(v, w)$ the corresponding quadratic (sub)space $(\mathbb{K}, \langle v, w \rangle, Q|_{\langle v, w \rangle})$.

If (v, w) is a hyperbolic pair of a quadratic space Λ then, in view of 1.13, $\Lambda = \mathbb{H}(v, w) \oplus \mathbb{H}(v, w)^{\perp}$.

1.18 Lemma
Let $\Lambda = (\mathbb{K}, V, Q)$ be a regular quadratic space of Witt index $k \geq 1$. Then there exists a hyperbolic pair (v, w) of Λ.

Proof   This follows from [EKM, 7.13]   □

1.19 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space. A *norm splitting* of Λ is a triple $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ such that

(i) $\mathbb{E}/\mathbb{K}$ is a separable quadratic extension,

(ii) $\cdot$ is a scalar multiplication from $\mathbb{E} \times V$ to V extending the scalar multiplication from $\mathbb{K} \times V$ to V and

(iii) $\{v_1, \ldots, v_d\}$ is a basis of V over $\mathbb{E}$ (with respect to $\cdot$) and

$$Q\left(\sum_{i=1}^{d} t_i \cdot v_i\right) = \sum_{i=1}^{d} s_i N(t_i)$$

for all $t_1, \ldots, t_d \in \mathbb{E}$, where $s_i = Q(v_i)$ for all $1 \le i \le d$ and N is the norm of the extension $\mathbb{E}/\mathbb{K}$.

The elements $s_1, \ldots, s_d \in \mathbb{K}$ are called the *constants* of the norm splitting.

Note that the definition of a norm splitting requires the vector space to be finite-dimensional. This will be sufficient for our purposes.

Hyperbolic quadratic spaces

1.20 Definition
A quadratic space Λ is called *hyperbolic* if there exist finitely many hyperbolic pairs $(v_1, w_1), \ldots, (v_n, w_n)$ of Λ such that

$$\Lambda = \mathbb{H}(v_1, w_1) \oplus \cdots \oplus \mathbb{H}(v_n, w_n).$$

Note that, by definition, a hyperbolic quadratic space is finite dimensional and non-degenerate (and thus regular).

1.21 Remark
Let $\Lambda = (\mathbb{K}, V, Q)$ be a hyperbolic quadratic space. Let $(v_1, w_1), \ldots, (v_n, w_n)$ be hyperbolic pairs of Λ such that $\Lambda = \bigoplus_{i=1}^{n} \mathbb{H}(v_i, w_i)$. Then $\dim_{\mathbb{K}}(V) = 2n$ and for any $x = \sum_{i=1}^{n} \lambda_i v_i + \mu_i w_i \in V$ we have $Q(x) = \sum_{i=1}^{n} \lambda_i \mu_i$.

1.22 Lemma
Let N be the norm of a quadratic extension $\mathbb{E}/\mathbb{K}$ and let $\Lambda = (\mathbb{K}, \mathbb{E}, N)$ be the associated quadratic space. The scalar extension $\Lambda_{\mathbb{E}}$ is a hyperbolic quadratic space.

Proof Let $a \in \mathbb{E}\backslash\mathbb{K}$. By [MPW, 2.12], the element $1 \otimes a - a \otimes 1 \in \mathbb{E} \otimes \mathbb{E}$ is isotropic. The assertion now follows from 1.18. $\qquad\square$

1.23 Definition
A quadratic space $\Lambda = (\mathbb{K}, V, Q)$ is called *pseudo-split* if it can be written as $\Lambda = \Lambda' \oplus \Lambda''$, where Λ' is a hyperbolic quadratic space and Λ'' an anisotropic quadratic space whose associated bilinear form is identically zero.

Quadratic spaces of type $\mathsf{E}_6, \mathsf{E}_7$ and E_8

1.24 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space. Then

  (i) Λ is *of type* E_6 if Q is anisotropic, $\dim_{\mathbb{K}}(V) = 6$ and Λ has a norm splitting.

 (ii) Λ is a quadratic space *of type* E_7 if Q is anisotropic, $\dim_{\mathbb{K}}(V) = 8$ and Λ has a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_4\})$ with constants $s_1, \ldots, s_4$ such that $s_1 \cdots s_4 \notin N(\mathbb{E})$.

(iii) Λ is a quadratic space *of type* E_8 if Q is anisotropic, $\dim_{\mathbb{K}}(V) = 12$ and Λ has a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_6\})$ with constants $s_1, \ldots, s_6$ such that $-s_1 \cdots s_6 \in N(\mathbb{E})$.

1.25 Remark
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_k for some $k \in \{6, 7, 8\}$ and let $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ be a norm splitting of Λ. Since Λ is anisotropic, $s_i \neq 0_{\mathbb{K}}$ for all $1 \leq i \leq d$ and hence $\mathrm{Def}(\Lambda) = \{0_V\}$.

1.26 Lemma
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8. For any norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ of Λ the quadratic space $\Lambda_{\mathbb{E}}$ is pseudo-split.

Proof By [TW, 12.10] the quadratic space Λ is isomorphic to the quadratic space $(\mathbb{K}, \mathbb{E}^d, s_1 N \oplus \cdots \oplus s_d N)$. The assertion follows from lemma 1.22. $\square$

1.27 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_k for $k \in \{6, 7, 8\}$, choose $0_V \neq \varepsilon \in V$ and replace Q by $Q(\varepsilon)^{-1} Q$ (so $Q(\varepsilon) = 1_{\mathbb{K}}$). Let $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ be a norm splitting as in 1.24 and let X_0 be as in [TW, 13.9] (thus, X_0 is a vector space of dimension 2^{k-3} over $\mathbb{K}$). Let $g \colon X_0 \times X_0 \to \mathbb{K}$ be defined as in [TW, 13.26] and set $S := X_0 \times \mathbb{K}$. For $(a, s), (b, t) \in S$ we define

$$(a, s) \cdot (b, t) := (a + b, s + t + g(a, b)).$$

The pair $(S, \cdot)$ is a (non-abelian) group.

Quadratic spaces of type F_4

1.28 Definition
A quadratic space $\Lambda = (\mathbb{K}, V, Q)$ is *of type* F_4 if $\mathrm{char}(\mathbb{K}) = 2$ and the following hold:

  (i) Λ is anisotropic,

(ii) $Q(\mathrm{Def}(\Lambda))/Q(\rho)$ is a subfield of $\mathbb{K}$ for some $0_V \neq \rho \in \mathrm{Def}(\Lambda)$ and

(iii) for some complement S_0 of $\mathrm{Def}(\Lambda)$ in V, the corresponding quadratic subspace $(\mathbb{K}, S_0, Q|_{S_0})$ has a norm splitting $(\mathbb{E}, \cdot, \{v_1, v_2\})$ with constants $s_1, s_2 \in \mathbb{K}$ such that $s_1 s_2 \in Q(\mathrm{Def}(\Lambda))/Q(\rho)$.

1.29 Lemma
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type F_4, let $0_V \neq \rho \in \mathrm{Def}(\Lambda)$ and let $\mathbb{F}$ be the subfield of $\mathbb{K}$ as in 1.28(ii).

(a) The field $\mathbb{F}$ is independent of the choice of the element $0_V \neq \rho \in \mathrm{Def}(\Lambda)$.

(b) $\mathbb{K}^2 \subseteq \mathbb{F} \subseteq \mathbb{K}$

Proof This follows from [TW, 14.2 and 14.4]. $\square$

1.30 Lemma
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type F_4, let S_0 be a complement of $\mathrm{Def}(\Lambda)$ in V and let $(\mathbb{E}, \cdot, \{v_1, v_2\})$ be a norm splitting of the quadratic subspace $(\mathbb{K}, S_0, Q|_{S_0})$. Then the quadratic space $\Lambda_{\mathbb{E}}$ is pseudo-split.

Proof By [TW, 12.10] the quadratic space $(\mathbb{K}, S_0, Q|_{S_0})$ is isomorphic to the quadratic space $(\mathbb{K}, \mathbb{E}^2, s_1 N \oplus s_2 N)$. It follows from 1.22 that $(\mathbb{K}, S_0, Q|_{S_0})_{\mathbb{E}}$ is a hyperbolic quadratic space. Since the bilinear form associated to $\mathrm{Def}(\Lambda)$ is identically zero, proposition [MPW, 2.29] gives that the scalar extension $(\mathbb{K}, \mathrm{Def}(\Lambda), Q|_{\mathrm{Def}(\Lambda)})_{\mathbb{E}}$ is anisotropic. We conclude that $\Lambda_{\mathbb{E}}$ is pseudo-split, since for all $u \otimes s, v \otimes t \in \mathrm{Def}(\Lambda) \otimes \mathbb{E}$ we have

$$f_{Q_{\mathbb{E}}}(u \otimes s, v \otimes t) = f_Q(u, v)st = 0_{\mathbb{E}}.$$

$\square$

1.31 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type F_4, let S_0 be a complement of $\mathrm{Def}(\Lambda)$ in V and let $(\mathbb{E}, \cdot, \{v_1, v_2\})$ be a norm splitting of $(\mathbb{K}, S_0, Q|_{S_0})$ with constants $s_1, s_2 \in \mathbb{K}$. Let $\mathbb{F}$ be the subfield of $\mathbb{K}$ as in 1.28(ii). Let $\mathbb{D}$ denote the composite field $\mathbb{E}\mathbb{F}$ and set $X := \mathbb{D} \oplus \mathbb{D}$. We define a quadratic form $\hat{Q}$ on the $\mathbb{F}$-vector space $\hat{V} := X \oplus \mathbb{K}$ via

$$\hat{Q}(x, y, t) := s_1 s_2 N(x) + s_1^{-1} s_2 N(y) + t^2$$

for all $(x, y, t) \in \hat{V}$, where N denotes the norm of the extension $\mathbb{D}/\mathbb{F}$.
The quadratic space $\hat{\Lambda} = (\mathbb{F}, \hat{V}, \hat{Q})$ is called a *dual of* Λ.

1.32 Remark
Let Λ be a quadratic space of type F_4, let S_0 be a complement of $\mathrm{Def}(\Lambda)$ in V and let $(\mathbb{E}, \cdot, \{v_1, v_2\})$ be a norm splitting of $(\mathbb{K}, S_0, Q|_{S_0})$ with constants $s_1, s_2 \in \mathbb{K}$. Let $\hat{\Lambda}$ be the quadratic space constructed in 1.31 with respect to these data.

(a) By [TW, 14.13], $\hat{\Lambda}$ is a quadratic space of type F_4.

(b) Even though $\hat{\Lambda}$ appears to depend not only on the quadratic space Λ but also on the choice of some $0_V \neq \rho \in \mathrm{Def}(\Lambda)$, some complement S_0 of $\mathrm{Def}(\Lambda)$ in V and a norm splitting $(\mathbb{E}, \cdot, \{v_1, v_2\})$ of $(\mathbb{K}, S_0, Q|_{S_0})$, it follows from [TW, 28.44] that, up to similarity, this quadratic space is, in fact, independent of these choices.
Thus, we will refer to $\hat{\Lambda}$ as *the* dual of Λ in the following, without emphasizing the chosen element $0_V \neq \rho \in \mathrm{Def}(Q)$, the complement S_0 of $\mathrm{Def}(Q)$ in V or the norm splitting $(\mathbb{E}, \cdot, \{v_1, v_2\})$.

(c) Applying the recipe for the dual to $\hat{\Lambda}$, we find that the dual of $\hat{\Lambda}$ is similar to the original quadratic space Λ.

(d) According to [TW, 14.25] there exist quadratic spaces of type F_4 which are isomorphic to its dual space. We call such spaces *self-dual*.

Semi-linear similitudes

1.33 Definition
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space with $\dim_{\mathbb{K}}(V) > 0$. An additive bijection $\tau \colon V \to V$ is a *semi-linear similitude* of Λ if there exist an automorphism $\sigma \in \mathrm{Aut}(\mathbb{K})$ and $\mu \in \mathbb{K}$ such that τ is σ-semi-linear and

$$Q(\tau(v)) = \mu \; \sigma(Q(v))$$

holds for all $v \in V$.
We also say that τ is a σ-semi-linear μ-similitude.
We denote the set of all semi-linear similitudes of Λ by $\Gamma O(\Lambda)$.

The following lemma will be needed in section 8.2.

1.34 Lemma
Let $\Lambda = (\mathbb{K}, V, Q)$ be a regular quadratic space of Witt index $n \geq 1$ and let $\tau \in \Gamma O(\Lambda)$ be a σ-semi-linear μ-similitude such that $\tau^2 = c \cdot \mathrm{id}_V$ for some $0 \neq c \in \mathbb{K}$. Then the following hold:

(a) $\sigma^2 = \mathrm{id}_{\mathbb{K}}$,

(b) $c^2 = \mu \; \sigma(\mu)$,

(c) $c \in \mathrm{Fix}(\sigma)$.

Proof

(a) Choose any $0 \neq v \in V$. For any $\lambda \in \mathbb{K}$ we have

$$c\lambda \cdot v = \tau^2(\lambda \cdot v) = \sigma^2(\lambda) \cdot \tau^2(v) = \sigma^2(\lambda)c \cdot v,$$

which implies that $\sigma^2(\lambda) = \lambda$.

(b) Let (v, w) be a hyperbolic pair of Λ (which exists by 1.18). Then

$$\begin{aligned}
c^2 &= f_Q(cv, cw) = f_Q(\tau^2(v), \tau^2(w)) \\
&= Q(\tau^2(v + w)) - Q(\tau^2(v)) - Q(\tau^2(w)) \\
&= \mu \ \sigma(Q(\tau(v + w))) = \mu \ \sigma(\mu \cdot Q(v + w)) \\
&= \mu \ \sigma(\mu) \ \sigma(Q(v + w)) \\
&= \mu \ \sigma(\mu) \ \sigma(f_Q(v, w) + Q(v) + Q(w)) = \mu \ \sigma(\mu).
\end{aligned}$$

(c) Let $0 \neq v \in V$ be any vector and calculate

$$c \cdot \tau(v) = \tau^2(\tau(v)) = \tau(\tau^2(v)) = \tau(c \cdot v) = \sigma(c) \cdot \tau(v).$$

$\square$

Pseudo-quadratic spaces

1.35 Definition
A *(right) pseudo-quadratic space* is a quintuple $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$, where

(i) $\mathbb{K}$ is a skew-field, σ is an involutory anti-automorphism of $\mathbb{K}$ and $\mathbb{K}_0$ is an additive subgroup of $\mathbb{K}$ such that

$$1_{\mathbb{K}} \in \mathbb{K}_0, \ \{a + \sigma(a) \mid a \in \mathbb{K}\} \subseteq \mathbb{K}_0 \subseteq \mathrm{Fix}(\sigma), \ \forall a \in \mathbb{K} : \sigma(a)\mathbb{K}_0 a \subseteq \mathbb{K}_0,$$

(ii) V is a right vector space over $\mathbb{K}$ and

(iii) Q is a pseudo-quadratic form on V with respect to σ, i.e. there is a skew-hermitian form f on V such that the following hold:

(PS1) $\forall a, b \in V : Q(a + b) \equiv Q(a) + Q(b) + f(a, b) \mod \mathbb{K}_0$,

(PS2) $\forall a \in V, t \in \mathbb{K} : Q(at) \equiv \sigma(t)Q(a)t \mod \mathbb{K}_0$.

If, in addition

(PS3) $Q(a) \equiv 0_{\mathbb{K}} \mod \mathbb{K}_0$ only for $a = 0_V$,

then Ξ is called *anisotropic* with respect to $\mathbb{K}_0$.

1.36 Definition
A pseudo-quadratic space $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$ is called *proper* if $\sigma \neq \mathrm{id}_{\mathbb{K}}$, $V \neq \{0_V\}$ and if the associated skew-hermitian form f is non-degenerate.

1.37 Definition
Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$ be a pseudo-quadratic space. We set

$$T := T(\Xi) := \{(a, t) \in V \times \mathbb{K} \mid Q(a) - t \in \mathbb{K}_0\}$$

and for $(a, t), (b, s) \in T$ we define

$$(a, t) \cdot (b, s) := (a + b, t + s + f(b, a)).$$

The pair $(T, \cdot)$ is a group with $(a, t)^{-1} = (-a, -\sigma(t))$ for each $(a, t) \in T$ and $Z(T) = \{(0_V, t) \mid t \in \mathbb{K}_0\} \simeq \mathbb{K}_0$.

1.38 Definition
Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$ be a pseudo-quadratic space, let $0_{\mathbb{K}} \neq \gamma \in \mathbb{K}_0$, set $\hat{\mathbb{K}}_0 := \gamma \mathbb{K}_0$ and define $\hat{\sigma} \colon \mathbb{K} \to \mathbb{K}$ by $\hat{\sigma}(t) := \gamma \sigma(t) \gamma^{-1}$. Let $\hat{Q} \colon V \to \mathbb{K}$ be defined by $\hat{Q}(a) = \gamma Q(a)$ for all $a \in V$. Then $\hat{\Xi} := (\mathbb{K}, \hat{\mathbb{K}}_0, \hat{\sigma}, V, \hat{Q})$ is a pseudo-quadratic space, called the *translate of Ξ with respect to γ*.

1.39 Definition
Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$ and $\Xi' = (\mathbb{K}', \mathbb{K}_0', \sigma', V', Q')$ be pseudo-quadratic spaces.

- An *isomorphism* from Ξ onto Ξ' is an isomorphism of right vector spaces $(\varphi, \phi) \colon (V, \mathbb{K}) \to (V', \mathbb{K}')$ such that

$$\phi(\mathbb{K}_0) = \mathbb{K}_0', \quad \phi \circ \sigma = \sigma' \circ \phi, \quad \phi \circ Q \equiv Q' \circ \varphi \mod \mathbb{K}_0'.$$

- A *similarity* from Ξ onto Ξ' is an isomorphism from Ξ onto Ξ'_γ for some $0_{\mathbb{K}'} \neq \gamma \in \mathbb{K}_0'$.

Chapter 2

Buildings

Coxeter systems

Let I be a non-empty set with $|I| < \infty$.

2.1 Definition
A *Coxeter matrix* is a symmetric array $[m_{ij}]$ with index set I and entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ij} \geq 2$ if i and j in I are distinct and $m_{ij} = 1$ if they are not.
The *Coxeter diagram* of a Coxeter matrix $[m_{ij}]$ is the graph with vertex set I and edge set consisting of all unordered pairs $\{i,j\}$ such that $m_{ij} \geq 3$ together with the labeling which assigns the label m_{ij} to the edge $\{i,j\}$. The label $m_{ij} = 3$ is usually suppressed and the label $m_{ij} = 4$ is often represented by a double edge connecting i and j.
A Coxeter diagram Π is called *irreducible* if its underlying graph is connected.

2.2 Definition
Let $[m_{ij}]$ be a Coxeter matrix with index set I and let Π denote the corresponding Coxeter diagram. The *Coxeter group of type* Π is the group

$$W := \langle S := \{s_i \mid i \in I\} \mid \{(s_i s_j)^{m_{ij}} = 1 \mid i,j \in I, m_{ij} < \infty\}\rangle.$$

The pair (W, S) will be called the *Coxeter system of type* Π. The diagram Π is called *spherical* if $|W| < \infty$.

2.3 Notation
The following Coxeter diagrams will arise several times throughout this thesis. We fix the following labeling of the vertices which follows [B]:

A_n

$\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!-\!-\!\bullet\!\!-\!\!\bullet$
1   2   3 $n-1$ n

C_n

1   2   3 $n-2$ $n-1$ n

$\tilde{C}_n$

0   1   2 $n-2$ $n-1$ n

D_n

$n-1$

1   2   3 $n-3$ $n-2$

n

E_6

1   3   4   5   6

2

$\tilde{E}_6$

1   3   4   5   6

2

0

E_7

1   3   4   5   6   7

2

$\tilde{E}_7$

0   1   3   4   5   6   7

2

E_8

1   3   4   5   6   7   8

2

$\tilde{E}_8$

1   3   4   5   6   7   8   0

2

F_4

1   2   3   4

$\tilde{F}_4$

0   1   2   3   4
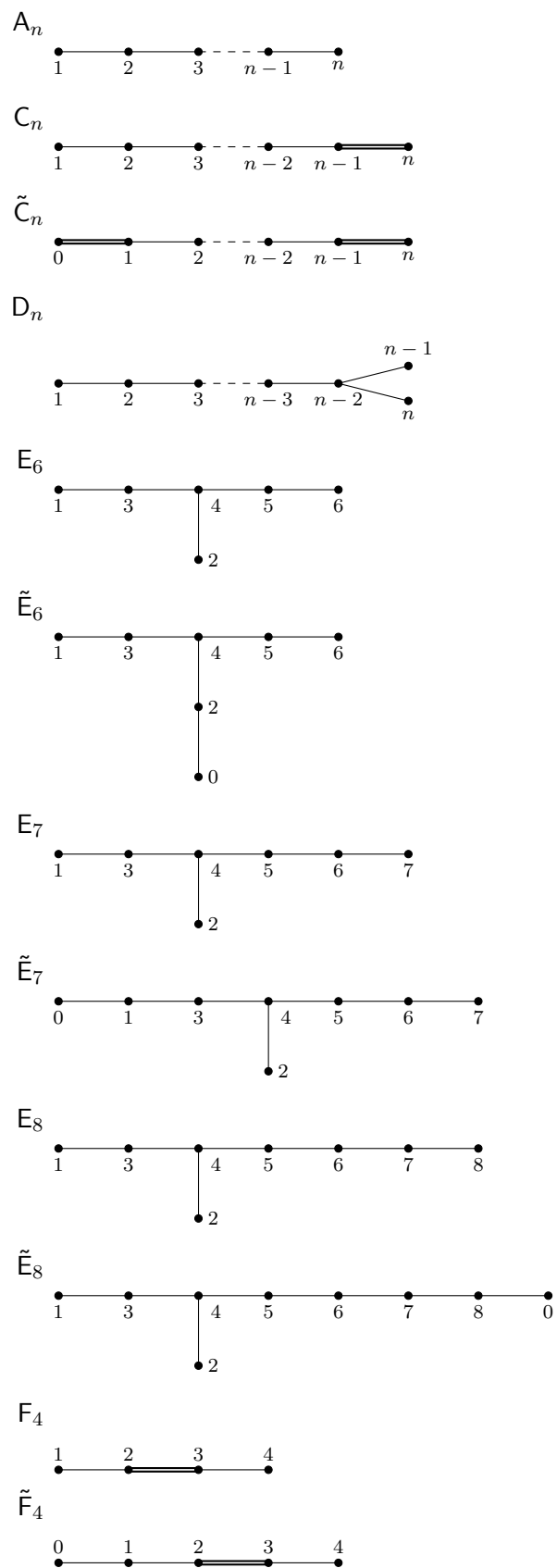
2.4 Definition

Let (W, S) be a Coxeter system. For $w \in W$ we define

$$\ell_S(w) := \min\{k \in \mathbb{N} \mid \exists t_1, \ldots, t_k \in S : w = t_1 \cdots t_k\}$$

and call $\ell_S \colon W \to \mathbb{N}$ the *length function on W with respect to S.*

2.5 Definition

Let Π be a Coxeter diagram and let (W, S) be the corresponding Coxeter system.

(a) For any subset $J \subseteq S$, we denote by W_J the subgroup $\langle J \rangle$ generated by J.

(b) We let Π_J denote the subdiagram of Π whose vertex set is J and where to vertices $i, j \in J$ are joined by an m_{ij}-labeled edge if they are joined by an m_{ij}-labeled edge in Π.

(c) A subset $J \subseteq S$ is called spherical, if the subgroup W_J is finite.

(d) A subset $J \subseteq S$ is called irreducible, if the diagram Π_J is connected.

2.6 Remark

Let Π be a Coxeter diagram and let (W, S) be a Coxeter system of type Π. For any subset $J \subseteq S$ the pair (W_J, J) is a Coxeter system of type Π_J. Moreover, by [AB, 2.14], $\ell_J(w) = \ell_S(w)$ for any $w \in W_J$.

2.7 Definition

Let (W, S) and (W', S') be Coxeter systems. An isomorphism of Coxeter systems is a group isomorphism $\sigma \colon W \to W'$ such that $\sigma(S) = S'$.

We identify isomorphisms between Coxeter systems with isomorphisms between their Coxeter diagrams. In particular, if (W, S) is a Coxeter system of type Π we will think of $\mathrm{Aut}(W, S)$ and $\mathrm{Aut}(\Pi)$ as being the same.

2.8 Proposition

Let (W, S) be a Coxeter system and let $J, K \subseteq S$. For $w \in W$ we set $J^{\pm}(w) := \{s \in S \mid \ell_S(ws) = \ell_S(w) \pm 1\}$.

(a) Every double coset $W_J w W_K \in W_J / W \backslash W_K$ has a unique element of minimal length. We denote this element by $\min\{W_J w W_K\}$.

(b) Let $w_1 := \min\{W_J w W_K\}$. Every element $w \in W_J w_1 W_K$ can be written as $w = w' w_1 w''$ with $w' \in W_J$ and $w'' \in W_K$ such that $\ell_S(w) = \ell_S(w') + \ell_S(w_1) + \ell_S(w'')$.

(c) If $w \in W$ and $J \subseteq J^+(w)$, then $w = \min\{w W_J\}$ and for all $v \in W_J$ we have $\ell_S(wv) = \ell_S(w) + \ell_S(v)$.

(d) If $J \subseteq S$ is spherical, then W_J has a unique element of maximal length. We denote this longest element by r_J. Since $\ell_S(r_J^{-1}) = \ell_S(r_J)$, the element r_J must have order two.

(e) If $w \in W$ and $J \subset J^-(w)$, then J is spherical, $w = \min\{wW_J\}\ r_J$ and $\ell_S(w) = \ell_S(\min\{wW_J\}) + \ell_S(r_J)$. Moreover, for any $u \in W_J$ we have $\ell_S(w) = \ell_S(wu) + \ell_S(u)$. In particular, w is the unique element of maximal length in wW_J and will be denoted by $\max\{wW_J\}$.

Proof   This follows from [AB, 2.23] and [MPW, 19.8].   □

2.9 Remark

Let Π be a Coxeter diagram and let (W, S) be the corresponding Coxeter system. Let $J \subseteq S$ be spherical. The map $s \mapsto r_J s r_J$ is an automorphism of the subdiagram Π_J. We denote this map by op_J. The map op_J stabilizes every connected component of Π_J and acts non-trivially on a given connected component if and only if it is isomorphic to the Coxeter diagram $\mathsf{A}_n(n \geq 2)$, $\mathsf{D}_n(n \geq 5 \text{ odd})$, E_6 or $\mathsf{I}_2(n)(n \geq 5)$.

If Π is a connected spherical or affine Coxeter diagram with $|\mathrm{Aut}(\Pi)| > 2$, then Π is isomorphic to the Coxeter diagram $\tilde{\mathsf{A}}_n(n \geq 2), \mathsf{D}_4, \tilde{\mathsf{D}}_n(n \geq 4)$ or $\tilde{\mathsf{E}}_6$. Thus, if Π is a connected spherical or affine Coxeter diagram not in this short list, then $|\mathrm{Aut}(\Pi)| \leq 2$.

2.10 Lemma

Let (W, S) be a Coxeter system and suppose that there exists $s \in S$ such that $st = ts$ for all $t \in S$. Then

(a) If $w \in W$ and $t_1, \ldots, t_n \in S$ are such that $w = t_1 \cdots t_n$, where $n = \ell_S(w)$, then $t_i = s$ for at most one $1 \leq i \leq n$.

(b) Suppose that W is finite and let $J := S \backslash \{s\}$. Then $r_S = r_J\, s = s\, r_J$.

Proof

(a) Suppose that there exist $1 \leq i < j \leq n$ such that $t_i = s = t_j$. As $st = ts$ for all $t \in S$

$$w = sst_1 \cdots t_{i-1}t_{i+1} \cdots t_{j-1}t_{j+1} \cdots t_n = t_1 \cdots t_{i-1}t_{i+1} \cdots t_{j-1}t_{j+1} \cdots t_n.$$

But this is a contradiction to the fact that $\ell_S(w) = n$.

(b) By definition $S \subseteq J^-(r_S)$ and hence $J \subseteq J^-(r_S)$. By 2.8(e) we thus have $r_S = \min\{r_S W_J\}r_J$ and $\ell_S(r_S) = \ell_S(\min\{r_S W_J\}) + \ell_S(r_J)$.

On the other hand, by part (a), $r_S = sw$ for some $w \in W_J$ and $\ell_S(r_S) = \ell_S(sw) = \ell_S(w) + 1$. Now

$$\ell_S(r_S) - \ell_S(\min\{r_S W_J\}) = \ell_S(r_J) \geq \ell_S(w) = \ell_S(r_S) - 1.$$

It follows that $\ell_S(\min\{r_S W_J\}) = 1$ and thus $\ell_S(r_J) = \ell_S(w)$. By 2.8(d) it follows that $r_J = w$.

$$\square$$

Tits indices

2.11 Definition
A *Tits index* is a triple $\mathbf{T} = (\Pi, \Theta, A)$ consisting of

(i) a Coxeter diagram Π,

(ii) a subgroup $\Theta \leq \mathrm{Aut}(W, S)$, where (W, S) is the Coxeter system corresponding to Π, and

(iii) a proper subset $A \subsetneq S$ stabilized by Θ such that for each $s \in S \backslash A$ the subset $J_s := \Theta(s) \cup A$ is spherical and $r_{J_s} A r_{J_s} = A$, where $\Theta(s)$ denotes the Θ-orbit containing s.

2.12 Definition
Let $\mathbf{T} = (\Pi, \Theta, A)$ be a Tits index. The Coxeter system (W, S) (equivalently the Coxeter diagram Π) is called the *absolute type* of $\mathbf{T}$.

2.13 Remark
Let $\mathbf{T} = (\Pi, \Theta, A)$ be a Tits index. Let $J \subseteq S$ be a Θ-invariant subset of S such that A is a proper subset of J. Then the triple

$$\mathbf{T}_J := (\Pi_J, \Theta_J, A)$$

is a Tits index, where Θ_J denotes the subgroup of $\mathrm{Aut}(\Pi_J)$ induced by Θ.

2.14 Definition
Let $\mathbf{T} = (\Pi, \Theta, A)$ be a Tits index and let (W, S) be the Coxeter system corresponding to Π. For each $s \in S \backslash A$ we define

$$\tilde{s} := r_{J_s} r_A, \quad \tilde{S} := \{\tilde{s} \mid s \in S \backslash A\} \text{ and } \tilde{W} := \langle \tilde{s} \mid \tilde{s} \in \tilde{S} \rangle.$$

Note that the element $\tilde{s}$ depends only on the orbit $\Theta(s)$, not on the element s itself.

2.15 Remark
Let $\mathbf{T} = (\Pi, \Theta, A)$ be a Tits index and let $\tilde{S}$ and $\tilde{W}$ be as in 2.14. Let $w \in \tilde{W}$ and let $\tilde{s}_1, \ldots, \tilde{s}_k \in \tilde{S}$ such that $w = \tilde{s}_1 \cdots \tilde{s}_k$. We say that the tuple $(\tilde{s}_1, \ldots, \tilde{s}_k)$ is a *compatible representation* if $\ell_S(w) = \sum_{i=1}^k \ell_S(\tilde{s}_i)$.
According to [MPW, 20.18], every element of $\tilde{W}$ has a compatible representation and by [MPW, 20.24], if $(\tilde{s}_1', \ldots, \tilde{s}_n')$ is another compatible representation of w, then $k = n$. Thus we may define $\tilde{\ell}(w)$ to be the length of any compatible representation of w.

2.16 Proposition
Let $\mathbf{T} = (\Pi, \Theta, A)$ be a Tits index and let (W, S) be the Coxeter system corresponding to Π. Let $\tilde{S}, \tilde{W}$ and $\tilde{\ell}$ be as in 2.14 and 2.15. Then

(a) The pair $(\tilde{W}, \tilde{S})$ is a Coxeter system and the length on $\tilde{W}$ with respect to $\tilde{S}$ is given by $\tilde{\ell}$.

(b) (W, S) is spherical if and only if $(\tilde{W}, \tilde{S})$ is spherical. If (W, S) is irreducible/ affine, then $(\tilde{W}, \tilde{S})$ is also irreducible/ affine.

Proof This follows from [MPW, 20.32, 20.35(i), 20.40 and 20.43].   □

2.17 Definition
Let $\mathbf{T} = (\Pi, \Theta, A)$ be a Tits index. Let $\tilde{S}$ and $\tilde{W}$ be as in 2.14. The Coxeter system $(\tilde{S}, \tilde{W})$ (or equivalently the corresponding Coxeter diagram $\tilde{\Pi}$) is called the *relative type* of $\mathbf{T}$.

Buildings

We fix a Coxeter system (W, S) and let $\ell := \ell_S$ be the length function on W with respect to S. Let Π be the corresponding Coxeter diagram.

2.18 Definition
A building of type (W, S) is a pair $\Delta = (\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$, whose elements are called *chambers*, together with a map $\delta \colon \mathcal{C} \times \mathcal{C} \to W$, called the *Weyl distance function*, such that for all $c, d \in \mathcal{C}$ the following three conditions hold:

(WD1) $\delta(c, d) = 1_W$ if and only if $c = d$.

(WD2) If $\delta(c, d) = w$ and $c' \in \mathcal{C}$ satisfies $\delta(c', c) = s \in S$, then $\delta(c', d) = sw$ or $\delta(c', d) = w$. If, in addition, $\ell(sw) = \ell(w) + 1$, then $\delta(c', d) = sw$.

(WD3) If $\delta(c, d) = w$, then for any $s \in S$ there is a chamber $c' \in \mathcal{C}$ such that $\delta(c', c) = s$ and $\delta(c', d) = sw$.

The building Δ is *spherical*, if the set S is spherical, or, equivalently, if W is finite. It is *irreducible*, if S is irreducible, or, equivalently, if the Coxeter diagram Π is connected.

When we refer to the *type* of a building Δ, we mean either the corresponding Coxeter system (W, S) or, equivalently, the corresponding Coxeter diagram Π.

2.19 Remark

Note that there is more than one approach to buildings. Originally, buildings were defined as chamber complexes (cf., for instance, [Ti74]) . The definition of a building as a chamber system was introduced in [Ti81] and a slight variation of it was taken as the definition of a building in the books by Ronan [R] and Weiss [W03]. This approach is closely related to our definition of a building. They define the Weyl distance using *galleries*. Conversely, the algebraic properties of our Weyl distance function δ given in 2.18 enable us to define adjacency and galleries (cf., for instance, [AB, 5.15 and 5.16]).

The reason, why we introduce buildings as an abstract system $(\mathcal{C}, \delta)$ subject to axioms characterizing the Weyl distance function δ, is, that twin buildings are defined in a similar way. For a discussion of the equivalence between these points of view, see, for example, [Ti81, 2.2], [AB, 5.93] and [AB, 5.23].

2.20 Definition

Let $\Delta = (\mathcal{C}, \delta)$ and $\Delta' = (\mathcal{C}', \delta')$ be buildings of type Π. Let $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{X}' \subseteq \mathcal{C}'$. A bijective mapping $\varphi \colon \mathcal{X} \to \mathcal{X}'$ is called an

(i) *isomorphism* if there exists $\sigma \in \mathrm{Aut}(\Pi)$ such that for all $c, d \in \mathcal{X}$ the following holds:
$$\delta'(\varphi(c), \varphi(d)) = \sigma(\delta(c, d)).$$

In this case we also call φ a σ-*isometry*.

(ii) *isometry* if it is an isomorphism and $\sigma = \mathrm{id}_W$.

As usual, an automorphism of a building $\Delta = (\mathcal{C}, \delta)$ is an isomorphism from $\mathcal{C}$ onto $\mathcal{C}$. We denote the corresponding group by $\mathrm{Aut}(\Delta)$.

2.21 Definition

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S).

$\diamond$ Given $J \subseteq S$ and $c \in \mathcal{C}$ the set
$$\mathcal{R}_J(c) := \{d \in \mathcal{C} \mid \delta(c, d) \in W_J\}$$

is called the *J-residue of Δ containing the chamber c*. Note that $\mathcal{R}_J(c) = \mathcal{R}_J(d)$ if $\delta(c, d) \in W_J$.

⋄ A *residue* of Δ is a subset $\mathcal{R} \subseteq \mathcal{C}$ such that $\mathcal{R} = \mathcal{R}_J(c)$ for some $J \subseteq S$ and $c \in \mathcal{C}$. The set J is called the *type of* $\mathcal{R}$ and it is denoted by $\mathrm{Typ}(\mathcal{R})$. The number $r = |J|$ is the *rank of* $\mathcal{R}$ and it is denoted by $rk(\mathcal{R})$.

⋄ A residue of rank 1 is called *panel*. Single chambers are residues of rank 0 and type $\emptyset$.

⋄ The building Δ is *thick*, if every panel of Δ contains at least three chambers. It is called *thin*, if every panel of Δ contains exactly two chambers.

Note that the intersection of two residues $\mathcal{R}$ and $\mathcal{T}$ of a building Δ is either empty or a residue of type $\mathrm{Typ}(\mathcal{R}) \cap \mathrm{Typ}(\mathcal{T})$.

2.22 Lemma
Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S), let $\mathcal{R} \subseteq \mathcal{C}$ be a J-residue and let $\mathcal{T} \subseteq \mathcal{C}$ be a K-residue of Δ.

(a) Let $c \in \mathcal{R}$ and $d \in \mathcal{T}$ be chambers and set $w := \delta(c, d)$. Then $\delta(\mathcal{R}, \mathcal{T}) := \{\delta(x, y) \mid x \in \mathcal{R}, y \in \mathcal{T}\} = W_J w W_K$.

(b) The pair $(\mathcal{R}, \delta|_{\mathcal{R} \times \mathcal{R}})$ is a building of type (W_J, J).

Proof   This follows from [AB, 5.29 and 5.30].   □

2.23 Example
If we define $\delta_W \colon W \times W \to W$ by $\delta_W(w_1, w_2) := w_1^{-1} w_2$, then the pair (W, δ_W) is a thin building of type (W, S). We call (W, δ_W) the *standard thin building* of type (W, S).

Projections

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S).

2.24 Definition
Let $c, d \in \mathcal{C}$ be chambers. We define

$$\mathrm{dist}(c, d) := \ell(\delta(c, d)).$$

Note that, since inverting is an automorphism of the Coxeter system (W, S), $\mathrm{dist}(c, d) = \mathrm{dist}(d, c)$ for all $c, d \in \mathcal{C}$.

2.25 Definition
Let $\mathcal{R}$ be a residue of Δ and let $c \in \mathcal{C}$ be a chamber. Due to [AB, 5.34] there exists a unique chamber $d \in \mathcal{R}$ such that $\delta(d, c) = \min\{\delta(\mathcal{R}, c)\}$. This chamber d has the following properties:

(i) $\delta(x, c) = \delta(x, d)\, \delta(d, c)$ for all chambers $x \in \mathcal{R}$.

(ii) $\mathrm{dist}(x, c) = \mathrm{dist}(x, d) + \mathrm{dist}(d, c)$ for all chambers $x \in \mathcal{R}$.

This unique chamber is called the *projection of c onto $\mathcal{R}$* and is denoted by $\mathrm{proj}_{\mathcal{R}}(c)$.

2.26 Definition
Let $\mathcal{R}$ and $\mathcal{T}$ be residues of Δ.

(a) We set $\mathrm{proj}_{\mathcal{R}}(\mathcal{T}) := \{\mathrm{proj}_{\mathcal{R}}(c) \mid c \in \mathcal{T}\}$ and call it the *projection of $\mathcal{T}$ onto $\mathcal{R}$*. Note that, by [AB, 5.36(2)], the set $\mathrm{proj}_{\mathcal{R}}(\mathcal{T})$ is again a residue of Δ.

(b) The residues $\mathcal{R}$ and $\mathcal{T}$ are *parallel*, if $\mathcal{T} = \mathrm{proj}_{\mathcal{T}}(\mathcal{R})$ and $\mathcal{R} = \mathrm{proj}_{\mathcal{R}}(\mathcal{T})$.

2.27 Lemma
Let $\mathcal{R}$ and $\mathcal{T}$ be parallel residues of Δ.

(a) The restriction of $\mathrm{proj}_{\mathcal{R}}$ to $\mathcal{T}$ is an isomorphism from $\mathcal{T}$ to $\mathcal{R}$, the restriction of $\mathrm{proj}_{\mathcal{T}}$ to $\mathcal{R}$ is an isomorphism from $\mathcal{R}$ to $\mathcal{T}$ and these two isomorphisms are inverses of each other.

(b) The element $w := \delta(c, \mathrm{proj}_{\mathcal{T}}(c))$ is independent of the choice of $c \in \mathcal{R}$.

Proof This follows from [MPW, 21.10(i) and (ii)]. $\square$

2.28 Lemma
Let $\mathcal{R}, \mathcal{T}$ be residues of Δ.

(a) If $\mathcal{T} \subseteq \mathcal{R}$ we have $\mathrm{proj}_{\mathcal{T}}(c) = \mathrm{proj}_{\mathcal{T}}(\mathrm{proj}_{\mathcal{R}}(c))$ for all $c \in \mathcal{C}$.

(b) If $\mathcal{R} \cap \mathcal{T} \neq \emptyset$, then $\mathrm{proj}_{\mathcal{R}}(\mathcal{T}) = \mathcal{R} \cap \mathcal{T}$.

Proof Part (a) is [MPW, 21.6(iii)]. For part (b) note that, if $\mathcal{R} \cap \mathcal{T} \neq \emptyset$, we have $\min\{\delta(\mathcal{R}, \mathcal{T})\} = 1_W$. The assertion now follows from [AB, 5.36(1)], since $\mathrm{proj}_{\mathcal{R}}(\mathcal{T}) = \{c \in \mathcal{R} \mid 1_W \in \delta(c, \mathcal{T})\}$. $\square$

Apartments

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S).

2.29 Definition
   (a) Let $\mathcal{M}$ be a nonempty subset of $\mathcal{C}$. If $(\mathcal{M}, \delta|_{\mathcal{M} \times \mathcal{M}})$ is a building of type (W, S), then it is called a *subbuilding* of Δ.

   (b) A thin subbuilding of Δ is called an *apartment* of Δ.

   (c) Let Σ be an apartment of Δ. A *root of* Σ is a subset $\alpha \subset \Sigma$ such that $\alpha = \{c \in \Sigma \mid \text{dist}(c, x) < \text{dist}(c, y)\}$ for some ordered pair (x, y) of chambers such that $\delta(x, y) \in S$.

   (d) A *root of* Δ is a root of some apartment of Δ.

2.30 Lemma
Let Σ be an apartment of Δ and let $\mathcal{R}$ be a residue of Δ such that $\mathcal{R} \cap \Sigma \neq \emptyset$. Then $\text{proj}_{\mathcal{R}}(c) \in \Sigma$ for each chamber $c \in \Sigma$.

Proof According to [W03, 8.9], apartments are convex. The assumption now follows from [AB, 5.45]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

2.31 Lemma
Let c, d, e be chambers of Δ. Then they are contained in a common apartment of Δ if and only if

$$\delta(c, e) = \delta(c, d)\delta(d, e).$$

Proof The only if part is [AB, 5.55].
Conversely, let (W, δ_W) be the standard thin building of type (W, S) (cf. 2.23) and consider the map $\beta \colon \{c, d, e\} \to \{1, \delta(c, d), \delta(c, e)\} \subseteq W$ defined by $\beta(x) := \delta(c, x)$. If $\delta(c, e) = \delta(c, d)\delta(d, e)$ holds, it is an isometry. According to [AB, 5.73], any subset of $\mathcal{C}$ that is isometric to a subset of W is contained in a common apartment.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Moufang spherical buildings

An important concept along the classification of spherical buildings is the *Moufang property*: Thick, spherical buildings with the Moufang property have turned out to be classifiable (cf. [TW]). Roughly speaking, the Moufang property ensures that Δ has a great deal of symmetry. A remarkable theorem of Tits says that *every* thick, irreducible, spherical building of rank at least 3 has the Moufang property.

2.32 Definition

Let Δ be a spherical building of rank at least two, let α be a root of Δ, and let U_α denote the subgroup of $\text{Aut}(\Delta)$ consisting of all elements which act trivially on every panel $\mathcal{P}$ of Δ with $|\mathcal{P} \cap \alpha| = 2$. The subgroup U_α is called the *root group* associated with the root α.

2.33 Definition

Let Δ be a spherical building of rank at least two. Then Δ has the *Moufang property* if for each root α of Δ, the root group U_α acts transitively on the set of all apartments of Δ containing α.

2.34 Remark

Note that in the definition sphericity is not used. But it turns out that this definition is to weak in the non-spherical case. We will discuss this later in chapter 10 of part II.

2.35 Theorem

Every thick irreducible spherical building of rank at least three has the Moufang property.

Proof This is [W03, 11.6]. □

In light of 2.35, we will call a spherical building *Moufang* if it is thick, irreducible, has rank at least two and satisfies the Moufang condition.

2.36 Theorem

Every irreducible residue of rank at least two of a Moufang spherical building is a Moufang spherical building.

Proof This is [W03, 11.8]. □

Descent in Buildings

In this chapter we assemble the results of [MPW] on descent in buildings that we will require.

Throughout, let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) and let Π be the corresponding Coxeter diagram.

2.37 Definition

Let Γ be a subgroup of $\text{Aut}(\Delta)$.

    ⋄ A Γ-*residue* is a residue of Δ stabilized by Γ.

    ⋄ A Γ-*chamber* is a Γ-residue which is minimal with respect to inclusion.

⋄ A Γ-*panel* is a Γ-residue $\mathcal{P}$ such that for some Γ-chamber C, $\mathcal{P}$ is minimal in the set of all Γ-residues containing C properly.

⋄ The group Γ is called *isotropic* if there exist Γ-residues other than Δ itself.

⋄ The group Γ is called *spherical* if there exist spherical Γ-chambers.

2.38 Notation
Suppose that $\Gamma \leq \mathrm{Aut}(\Delta)$ is isotropic and let Θ denote the subgroup of $\mathrm{Aut}(W, S)$ induced by Γ. We denote by $\mathcal{C}^\Gamma$ the set of all Γ-chambers of Δ. By [MPW, 22.3(iii)], any two Γ-chambers $C, D \in \mathcal{C}^\Gamma$ are parallel and $\delta(c, \mathrm{proj}_D(c)) \in \mathrm{Fix}(\Theta)$ for all $c \in C$. We define $\bar{\delta} \colon \mathcal{C}^\Gamma \times \mathcal{C}^\Gamma \to \mathrm{Fix}(\Theta)$ by $\bar{\delta}(C, D) := \delta(c, \mathrm{proj}_D(c))$ for any $c \in C$.

2.39 Definition
A subgroup $\Gamma \leq \mathrm{Aut}(\Delta)$ is a *descent group of* Δ if it is isotropic and if each Γ-panel contains at least three Γ-chambers.

2.40 Theorem
Let Γ be a spherical descent group of Δ. Then all Γ-chambers are of the same type $A \subseteq S$, the triple $\mathbf{T} = (\Pi, \Theta, A)$ is a Tits index and the pair $\Delta^\Gamma = (\mathcal{C}^\Gamma, \bar{\delta})$ is a thick building of type $(\tilde{W}, \tilde{S})$, where $\mathcal{C}^\Gamma$ and $\bar{\delta}$ are as in 2.38 and $(\tilde{W}, \tilde{S})$ is the relative type of $\mathbf{T}$.

Proof This is [MPW, 22.25].   □

2.41 Theorem
Suppose that Δ is a Moufang spherical building and let Γ be a descent group of Δ. Let Δ^Γ be the fixed point building as in 2.40 and let k be the rank of the building Δ^Γ.

(i) If $k \geq 2$, then Δ^Γ satisfies the Moufang condition.

(ii) If $k = 1$, then there exists a Moufang structure $\mathbb{M}$ as defined in [MPW, 24.6] such that the pair $(\Delta^\Gamma, \mathbb{M})$ is a Moufang set as defined in 9.1.1.

Proof This is [MPW, 24.31].   □

2.42 Theorem
Suppose that Δ is spherical and let Γ be isotropic. If there exists a Γ-chamber C such that every Γ-panel containing C contains at least three Γ-chambers the following hold:

(a) $\mathbf{T} = (\Pi, \Theta, \mathrm{Typ}(C))$ is a Tits index.

(b) $\text{Typ}(D) = \text{Typ}(C)$ for all Γ-chambers D.

(c) $\bar{\delta}(C, D) \in \tilde{W}$, where $(\tilde{W}, \tilde{S})$ denotes the relative type of the Tits index **T**.

(d) If each Γ-panel contains at least two Γ-chambers, then $\Delta^{\Gamma} := (\mathcal{C}^{\Gamma}, \bar{\delta})$ is a building of type $(\tilde{W}, \tilde{S})$.

(e) If there exists a Γ-chamber C_1 such that $\bar{\delta}(C, C_1)$ equals the longest element in the relative type of **T**, then Γ is a descent group.

Proof   Part (a) follows from [MPW, 22.37(i)]. Parts (b)-(d) follow from [MPW, 22.14(i)-(iii)], knowing that $\mathbf{T} = (\Pi, \Theta, \text{Typ}(C))$ is a Tits index. Part (e) is [MPW, 22.37(ii)]. $\square$

Twin Buildings

In this section we will give a brief introduction to twin buildings and assemble some basic properties. We fix a Coxeter diagram Π with corresponding Coxeter system (W, S) and let $\ell\colon W \to \mathbb{N}$ denote the length function on W with respect to S.

2.43 Definition
A twin building of type (W, S) is a triple $(\Delta_+, \Delta_-, \delta_*)$ consisting of two buildings $\Delta_+ = (\mathcal{C}_+, \delta_+)$ and $\Delta_- = (\mathcal{C}_-, \delta_-)$ of type (W, S) together with a *codistance function*

$$\delta_*\colon (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$$

satisfying the following conditions for each $\varepsilon \in \{+, -\}$, any $c \in \mathcal{C}_\varepsilon$ and any $d \in \mathcal{C}_{-\varepsilon}$, where $w := \delta_*(c, d)$:

(Tw1) $\delta_*(c, d) = \delta_*(d, c)^{-1}$.

(Tw2) If $c' \in \mathcal{C}_\varepsilon$ satisfies $\delta_\varepsilon(c', c) = s$ with $s \in S$ and $\ell(sw) < \ell(w)$, then $\delta_*(c', d) = sw$.

(Tw3) For any $s \in S$, there exists a chamber $c' \in \mathcal{C}_\varepsilon$ with $\delta_\varepsilon(c', c) = s$ and $\delta_*(c', d) = sw$.

2.44 Definition
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S).

$\diamond$ Δ is called *thick* if each of the buildings Δ_+ and Δ_- is thick.

⋄ A residue (panel) of Δ is a residue (panel) of one of the buildings Δ_+ or Δ_-.

⋄ Δ is called *2-spherical*, if each subset $J \subseteq S$ with $|J| = 2$ is spherical.

2.45 Definition

Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S) and let $\varepsilon \in \{+, -\}$.

⋄ We define the *numerical codistance* between chambers $c \in \mathcal{C}_\varepsilon$ and $d \in \mathcal{C}_{-\varepsilon}$ by

$$\mathrm{dist}(c, d) := \ell(\delta_*(c, d)).$$

⋄ Two chambers $c \in \mathcal{C}_\varepsilon$ and $d \in \mathcal{C}_{-\varepsilon}$ are *opposite*, if $\mathrm{dist}_*(c, d) = 0$ or, equivalently, if $\delta_*(c, d) = 1_W$.

2.46 Definition

A *twin apartment* of a twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ is a pair $\Sigma = (\Sigma_+, \Sigma_-)$ such that Σ_+ is an apartment of Δ_+, Σ_- is an apartment of Δ_- and every chamber in $\Sigma_+ \cup \Sigma_-$ is opposite (as defined in 2.45) precisely one other chamber in $\Sigma_+ \cup \Sigma_-$.

2.47 Remark

Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment of a twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$. We define the *opposition involution* op_Σ to be the map which associates to each chamber $c \in \Sigma_+ \cup \Sigma_-$ the unique chamber $c' := \mathrm{op}_\Sigma(c) \in \Sigma_+ \cup \Sigma_-$ satisfying $\delta_*(c, c') = 1_W$.
According to [AB, 5.173(1)] for each $\varepsilon \in \{+, -\}$ the map $\mathrm{op}_\Sigma \colon \Sigma_\varepsilon \to \Sigma_{-\varepsilon}$ is an isometry.

If Δ is a 2-spherical twin building satisfying the following connectivity condition, then, by [MR, 1.4], the local structure of Δ determines the global structure:

2.48 Definition

A twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ satisfies *(co)* if for every $\varepsilon \in \{+, -\}$ and every chamber $c \in \mathcal{C}_\varepsilon$, the set c^{op} of chambers opposite c is a gallery-connected subset of $\mathcal{C}_{-\varepsilon}$.

2.49 Remark

Almost all thick, irreducible , 2-spherical twin buildings of rank at least 3 satisfy (co). More precisely:

(a) According to [MR, 1.5], Δ satisfies (co) if every rank 2 residue satisfies (co).

(b) According to [AvM], there are only four Moufang spherical buildings
of rank 2 that do not satisfy (co), namely the buildings associated to
the finite groups $\mathrm{Sp}(\mathbb{F}_2), \mathrm{G}_2(\mathbb{F}_2), \mathrm{G}_2(\mathbb{F}_3)$ and ${}^2\mathrm{F}_4(\mathbb{F}_2)$.
In particular, the unique building of type B_2 which does not satisfy
(co) is the unique example with three chambers per panel.

We shall need the following lemma in section 9.2:

2.50 Lemma
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S) and let $\varepsilon \in \{+, -\}$.
For all $c \in \mathcal{C}_\varepsilon$ and $d, e \in \mathcal{C}_{-\varepsilon}$ we have

$$\mathrm{dist}(c, d) \geq \mathrm{dist}(c, e) - \mathrm{dist}(e, d).$$

Proof We proceed by induction on $l := \mathrm{dist}(d, e)$.
$l = 0$: Then $e = d$ and the assertion is trivially true.
$l > 1$: Then $e \neq d$ and hence $w := \delta_{-\varepsilon}(e, d) \neq 1_W$. Let $s \in S$ such that
$\ell(sw) = \ell(w) - 1$. By (WD3) there exists a chamber $c' \in \mathcal{C}_{-\varepsilon}$ such that
$\delta_{-\varepsilon}(c', e) = s$ and $\delta_{-\varepsilon}(c', d) = sw$. Thus

$$\mathrm{dist}(d, c') = \ell(sw) = \ell(w) - 1 = l - 1 < l = \mathrm{dist}(d, e)$$

and the induction hypothesis gives $\mathrm{dist}(c, d) \geq \mathrm{dist}(c, c') - \mathrm{dist}(c', d)$.
Let $v := \delta_*(e, c)$. By [AB, 5.139(1)] we have $\delta_*(c', c) \in \{v, sv\}$.
Consequently,

$$\mathrm{dist}(c, d) \geq \mathrm{dist}(c, c') - \mathrm{dist}(c', d) \geq \mathrm{dist}(c, e) - 1 - (\mathrm{dist}(e, d) - 1).$$

$\square$

Part II

Spherical buildings, geometries and extensions of automorphisms

Chapter 3

Point-line-spaces related to buildings

3.1 Point-line-spaces

3.1.1 Definition

A *point-line-space* is a pair $\mathcal{S} = (P, L)$ consisting of a set P (whose elements will be called *points*) and a subset $L \subseteq \mathcal{P}_2(P) := \{X \in 2^P \mid |X| \geq 2\}$ (whose elements will be called *lines*).

3.1.2 Definition

Let $\mathcal{S} = (P, L)$ be a point-line-space.

(a) Two points of $\mathcal{S}$ are said to be *collinear* if there exists a line of $\mathcal{S}$ containing both or if they are equal.

(b) A point p and a line l are said to be *incident*, if the point p is contained in the line l.

(c) For any point $p \in P$ we define $p^\perp := \{q \in P \mid p \text{ and } q \text{ are collinear}\}$ and for any subset $X \subseteq P$ we set $X^\perp := \bigcap_{x \in X} x^\perp$.

(d) A subset $X \subseteq P$ is called *singular* if $X \subseteq X^\perp$.

3.1.3 Definition

A subset $U \subseteq P$ is called a *subspace of* $\mathcal{S}$ if the relation $|l \cap U| \geq 2$ implies $l \subseteq U$ for each line $l \in L$.

3.1.4 Definition

Let $\mathcal{S} = (P, L)$ be a point-line-space.

(a) A *hyperplane* of $\mathcal{S}$ is a proper subspace h of $\mathcal{S}$, such that any line of $\mathcal{S}$ has at least one point in common with h.

(b) For any subspace U of $\mathcal{S}$ we have a point-line-structure $\mathcal{S}_U := (U, L_U)$ on U induced from $\mathcal{S}$, where $L_U = \{l \in L \mid l \subseteq U\}$.

(c) A subspace U of $\mathcal{S}$ is *singular*, if the set U is singular.

3.1.5 Definition
Let $\mathcal{S} = (P, L)$ and $\mathcal{S}' = (P', L')$ be point-line-spaces. A map $\alpha \colon P \to P'$ is a *homomorphism of point-line-spaces* if for each line l of $\mathcal{S}$ there exists a line l' of $\mathcal{S}'$ such that $\alpha(l) \subseteq l'$. An *isomorphism of point-line spaces* is a bijective homomorphism α such that the inverse map α^{-1} is also a homomorphism. An *automorphism* is defined in the obvious way.

3.1.6 Lemma
Let $\mathcal{S} = (P, L)$ be a point-line-space.

(a) The intersection of any family of (singular) subspaces of $\mathcal{S}$ is a (singular) subspace of $\mathcal{S}$.

(b) Any set $X \subseteq P$ of pairwise collinear points of $\mathcal{S}$ generates a singular subspace

$$\langle X \rangle := \bigcap U \mid U \text{ singular subspace}, X \subseteq U.$$

It is the smallest singular subspace of $\mathcal{S}$ containing the set X.

Proof   This follows from [Ue, 4.2.1] and [Ue, 4.2.3]. $\square$
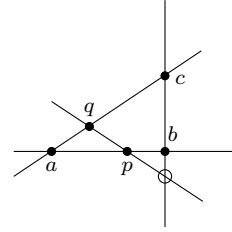
3.2 Projective Spaces

3.2.1 Definition
A *projective space* is a point-line-space $\mathcal{S} = (P, L)$ having at least two distinct lines such that the following axioms are satisfied:

(PS_1) Any two points of $\mathcal{S}$ are collinear and each line is uniquely determined by two of its points.

(PS_2) For any 5-tuple of pairwise distinct points $a, b, c, p, q \in P$ such that a, b, p and a, c, q are collinear on distinct lines, the line through b and c and the line through p and q have a common point.

It readily follows from the definition that a subspace of a projective space is singular. A projective space $\mathcal{S}$ is said to be *thick* if each line is incident with at least three points.

3.2.2 Lemma
Let $\mathcal{S} = (P, L)$ be a projective space and let U be a subspace of $\mathcal{S}$ such that there are at least two lines in U. Then the point-line-space $\mathcal{S}_U$ is a projective space.

Proof This is [Ue, 1.4.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2.3 Definition
Let $\mathcal{S} = (P, L)$ be a projective space. We put

$$\mathrm{rk}(\mathcal{S}) := \min\{|X| \mid X \subseteq P, \langle X \rangle = P\} - 1.$$

This number will be called *the rank of $\mathcal{S}$*. If U is a subspace of $\mathcal{S}$ we put

$$\dim(U) := \mathrm{rk}(\mathcal{S}_U).$$

3.2.4 Remark
Let $\mathbb{K}$ be a skew field and let V be a left vector space over $\mathbb{K}$ of positive dimension. We denote by $\mathcal{V}(V)$ the set of all subspaces of V. For each $X \in \mathcal{V}(V)$, we define

$$\mathcal{P}(X) := \{U \leq_{\mathbb{K}} X \mid \dim_{\mathbb{K}}(U) = 1\}$$

and

$$\mathcal{L}(X) := \{\mathcal{P}(U) \mid \dim_{\mathbb{K}}(U) = 2\}.$$

The pair $\mathbf{P}(V) = (\mathcal{P}(V), \mathcal{L}(V))$ is a projective space, called *the projective space associated with V*.

The next theorem is known as *The First Fundamental Theorem for Projective Spaces*:

3.2.5 Theorem
Let $\mathcal{S} = (P, L)$ be a projective space of finite rank $n \geq 3$. Then there exists a skew field $\mathbb{K}$ and a left vector space V over $\mathbb{K}$ with $\dim_{\mathbb{K}}(V) = n + 1$ such that $\mathcal{S} \simeq \mathbf{P}(V)$.

Proof This follows from [Ue, 2.6.1 and 3.7.18]. $\square$

Polarities

3.2.6 Definition
Let $\mathcal{S} = (P, L)$ be a projective space of finite rank n and let δ be a bijective transformation of the set of points of $\mathcal{S}$ onto the set of hyperplanes of $\mathcal{S}$. Then δ is a *duality* of $\mathcal{S}$ if for any three collinear points $p, q, r \in P$ the hyperplanes $\delta(p), \delta(q)$ and $\delta(r)$ meet in a common subspace of codimension 2, that is

$$\delta(p) \cap \delta(q) = \delta(p) \cap \delta(r) = \delta(q) \cap \delta(r) = \delta(p) \cap \delta(q) \cap \delta(r).$$

According to [Ue, 4.4.4] dualities act on subspaces as follows:

3.2.7 Proposition
Let $\mathcal{S} = (P, L)$ be a projective space of finite rank n and let δ be a duality of $\mathcal{S}$. For any subspace U of $\mathcal{S}$ we have

$$\delta(U) = \bigcap_{x \in U} \delta(x).$$

3.2.8 Definition
Let $\mathcal{S} = (P, L)$ be a projective space of finite rank $n \geq 2$ and let π be a bijective transformation of the set of points of $\mathcal{S}$ onto the set of hyperplanes of $\mathcal{S}$. Then π is a *polarity* of $\mathcal{S}$ if for any two points $p, q \in P$ the relation $p \in \pi(q)$ implies the relation $q \in \pi(p)$.

Equivalently well, a polarity of a projective space $\mathcal{S}$ is a duality of $\mathcal{S}$ of order 2.

We will use the following theorem of Birkhoff and von Neumann which claims that polarities of projective spaces over vector spaces are induced by sesquilinear forms. A proof of this theorem can be found, for example, in [Ue, 5.11].

3.2.9 Theorem (Birkhoff, von Neumann)
Let $\mathbb{K}$ be a skew field and let V be a left vector space over $\mathbb{K}$. Let π be a polarity of the projective space $\mathbf{P}(V)$. Then there exists a (σ, ε)-hermitian sesquilinear form f on V with $\sigma^2 = \mathrm{id}_{\mathbb{K}}$ and $\varepsilon \in \{\pm 1_{\mathbb{K}}\}$ such that

$$\pi(\langle v \rangle) = \{w \in v \mid f(v, w) = 0_{\mathbb{K}}\}$$

for all $v \in V$.

Given a polarity π of a projective space $\mathbf{P}(V)$, in general, there exist more than one reflexive sesquilinear form $f \colon V \times V \to \mathbb{K}$ inducing π. This is due to the fact that the polarity π is defined on the 1-dimensional subspaces of V, whereas f is defined on the elements of V.

3.2.10 Definition
Let $\mathcal{S} = (P, L)$ be a projective space and let π be a polarity of $\mathcal{S}$.

(a) A point $p \in P$ is called *absolute with respect to π* if $p \in \pi(p)$.

(b) A subspace U of $\mathcal{S}$ is called *absolute with respect to π* if $U \subseteq \pi(U)$.

3.2.11 Theorem
Let $\mathcal{S} = (P, L)$ be a projective space and let π be a polarity of $\mathcal{S}$.

(a) If U is a subspace of $\mathcal{S}$ which is absolute with respect to π, every point of U is absolute with respect to π.

(b) Let p and q be points of $\mathcal{S}$ which are absolute with respect to π. Then the line through p and q is absolute with respect to π if and only if $q \in \pi(p)$.
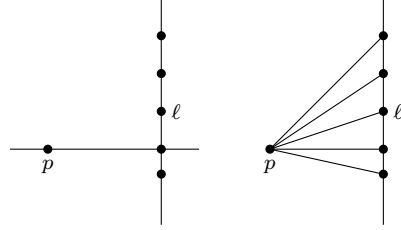
Proof This is [Ue, 4.4.6]. □

3.3   Polar Spaces

3.3.1 Definition
A point-line space $\mathcal{S} = (P, L)$ is called a *polar space* if the following axiom is satisfied:

(P_1) Let l be a line and let p be a point not on l. Then either there exists exactly one point on l collinear with p or p is collinear with all points on l.

A polar space $\mathcal{S} = (P, L)$ will be called *non-degenerate* if

(P_2) for every point p of $\mathcal{S}$ there exists a point q of $\mathcal{S}$ such that p and q are non-collinear.

3.3.2 Definition
Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space.

(a) We say that $\mathcal{S}$ is of finite rank n if there exists a natural number n such that for every chain $\emptyset \neq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_r$ of singular subspaces the relation $r \leq n$ holds and if there is at least one chain of length n.

(b) A singular subspace U of $\mathcal{S}$ has finite rank n if there exists a natural number n such that for every chain $\emptyset \neq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_r = U$ of proper singular subspaces the relation $r \leq n$ holds and if there is at least one chain of length n.

(c) A polar space $\mathcal{S}$ of finite rank n is called *thick* if every line contains at least three points and if every singular subspace of rank $n - 1$ is contained in at least three maximal subspaces.

(d) A *hyperbolic pair of $\mathcal{S}$* is a pair of non-empty singular subspaces U and V of $\mathcal{S}$ such that $U \cap V^{\perp} = \emptyset = V \cap U^{\perp}$.

3.3.3 Lemma
Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space.

(a) Each point is incident with at least two lines.

(b) For any set $X \subseteq P$, the set $X^{\perp}$ is a subspace of $\mathcal{S}$.

(c) If $\mathcal{S}$ is of finite rank 2, only the first case of axiom (P_1) occurs, i.e. $\mathcal{S}$ satisfies the stronger condition

(P_1') Let l be a line and let p be a point not on l. Then there exists a unique point x on l such that x and p are collinear.

(d) Let $P' := L$ and $L' := P$ and define $l \in P'$ and $p \in L'$ to be incident if and only if they are incident in $\mathcal{S}$. If $\mathcal{S}$ is of finite rank 2, the pair $\mathcal{S}' := (P', L')$ is a non-degenerate polar space of rank 2.

Proof

(a) Let $p \in P$ be a point and let $l \in L$ be a line. First suppose that p is not incident with l. If all points on l are collinear with p, p is incident with a line through each point on l. Since there are at least two points on l, the assertion follows. Otherwise there exists a unique point q on l such that p and q are collinear. Choose a point z which is non-collinear with q and let h be the unique line through z and a point of l. As p is collinear with at least one point on h, the assertion follows.

No suppose that p is contained in l and let q be a point non-collinear with p. There exists a unique point z on l which is collinear with q. Let x be point non-collinear with z. Let h be the line through x and a unique point on the line through q and z. Then p is collinear with at least one point on h.

(b) Let $p \in X$ be a point and let $l \in L$ be a line such that $|l \cap p^\perp| \geq 2$. Then p is collinear with all points on l and hence $l \subseteq p^\perp$. Now let $l \in L$ be a line such that $|l \cap X^\perp| \geq 2$. Since $X^\perp \subseteq x^\perp$ for all $x \in X$ the assertion follows.

(c) This follows from [Ue, 4.2.20].

(d) This is [Ti74, 7.2.8]

$\square$

3.3.4 Proposition
Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space of finite rank 2 and suppose that there exists a line $l \in L$ which is incident with at least three points and a point $p \in P$ which is incident with at least three lines. Then $\mathcal{S}$ is thick.

Proof According to Let l and g be two disjoint lines. Then each point on l is collinear with a unique point on g and vice versa. Hence there is a bijection between the point sets of two disjoint lines. In particular, two disjoint lines have the same number of points.

Now let $p \in P$ be a point which is incident with at least three lines. Let l and g be two lines which are not incident with p. We show that l and g have the same number of points. Let $L_p := \{l \in L \mid p \in l\}$. By assumption we have $|L_p| \geq 3$.

Assume that there are $h \neq h' \in L_p$ such that $l \cap h \neq \emptyset \neq l \cap h'$ and let x and x' denote the corresponding intersection points. Note that, since $l \notin L_p$, we have $x \neq p \neq x'$. But then the point p is collinear with two points on l which is a contradiction to (P_1').
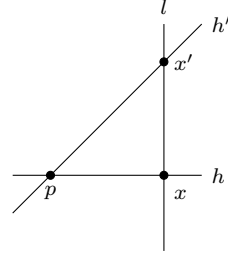
Hence l and g can intersect at most one line through p each, there exist at least one line $h'' \in L_p$ which is disjoint to both, l and g. Hence l, g and h'' have the same number of points.
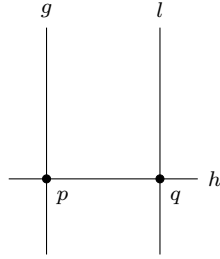
Now let h, g be two lines through p and let q be a point on h different from p. Let l be a line through q different from h. Then l is a line not through p (since $h \neq l$) and $g \cap l = \emptyset$ since otherwise the intersection point would be collinear with two points on h.

We have seen that each line not through p has the same number of points and moreover, each line through p has the same number of points as a line not through p. We conclude that each line is incident with the same number of points. As there is at least one line which is incident with at least three points, all lines are incident with at least three points.

We now set $P' := L$ and $L' := P$ and $\mathcal{S}' := (P', L')$. According to 3.3.3(c) $\mathcal{S}'$ is a non-degenerate polar space of rank 2. By assumption there is a line of $\mathcal{S}$ which is incident with at least three points of $\mathcal{S}$. Whence there is a point of $\mathcal{S}'$ which is incident with at least three lines of $\mathcal{S}'$. We now use the previous considerations to obtain that every line of $\mathcal{S}'$ is incident with at least three points. Thus every point of $\mathcal{S}$ is incident with at least three lines of $\mathcal{S}$. □

3.3.5 Lemma
Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space of finite rank $n \geq 2$. Then the following hold:

(a) Any two points of $\mathcal{S}$ are incident with at most one line of $\mathcal{S}$.

(b) For any singular subspace U of $\mathcal{S}$ containing at least two lines the point-line-space $\mathcal{S}_U$ is a projective space.

(c) If $p, q \in P$ are non-collinear, then $\mathcal{S}_{pq} := (p^\perp \cap q^\perp, L_{p^\perp \cap q^\perp})$ is a non-degenerate polar space.

(d) Let (U, V) be a hyperbolic pair of $\mathcal{S}$. Then $rk(U) = rk(V)$ and the point-line-space $\mathcal{S}_{(U,V)} := (U^\perp \cap V^\perp, L_{U^\perp \cap V^\perp})$ is a non-degenerate polar space.

Proof

(a) This is [BC, 7.4.11].

(b) This is [BC, 7.4.13(iv)].

(c) This is [BC, 7.4.8]

(d) Induction on the rank of U using part (c).

□

3.3.6 Remark
Let $\mathcal{S}$ be a non-degenerate polar space and l and g be two lines such that (l, g) is a hyperbolic pair of $\mathcal{S}$. Let x be a point on l and suppose that x is collinear with all points on g. Then $x \in l \cap g^\perp = \emptyset$, a contradiction. Hence each point on l is collinear with a unique point on g and vice versa.

3.3.7 Lemma
Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space, let U be a singular subspace of $\mathcal{S}$ of finite rank $k \geq 2$ and let p be a point which is not collinear with all points of U. Then $U \cap p^\perp$ is a singular subspace of $\mathcal{S}$ of rank $k - 1$.

Proof Note that, since U contains at least one line, the intersection $U \cap p^\perp$ is non-empty and hence $l := \mathrm{rk}(U \cap p^\perp) \geq 1$.

According to 3.3.3(b), the set $p^\perp$ is a subspace of $\mathcal{S}$. Thus, in view of 3.1.6, the intersection $U \cap p^\perp$ is a subspace of $\mathcal{S}$ which is singular since it is contained in the singular set U. Let $\emptyset \neq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_{l-1} \subsetneq U_l = U \cap p^\perp$ be a maximal chain of singular subspaces. As $U \cap p^\perp \subsetneq U$ we conclude that $l \leq k - 1$.

We show by induction on k: If U is a singular subspace of $\mathcal{S}$ of rank k such that $U \not\subseteq p^\perp$, then $\mathrm{rk}(U \cap p^\perp) \geq k - 1$.
If $k = 2$ (P_1) implies that $U \cap p^\perp$ consists of a single point (since p is not collinear with all points on U).
Suppose that $k > 2$ and let $\emptyset \neq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_{k-1} \subsetneq U_k = U$ be a maximal chain of singular subspaces. If $U_{k-1} \subseteq p^\perp$, then $U_{k-1} \cap p^\perp = U_{k-1}$ and hence $\mathrm{rk}(U \cap p^\perp) \geq k - 1$. If $U_{k-1} \not\subseteq p^\perp$ we may apply the induction

hypothesis and obtain $\operatorname{rk}(U_{k-1} \cap p^\perp) \geq k - 2$. Let $x \in U \backslash U_{k-1}$ be a point. If x is collinear with p, then $\operatorname{rk}(U \cap p^\perp) > \operatorname{rk}(U_{k-1} \cap p^\perp) \geq k - 2$. Thus, assume that x is non-collinear with p and choose a point $y \in U_{k-1}$ which is also non-collinear with p. As U is singular, the points x and y lie on a common line h. Let z be the unique point on h which is collinear with p. As U_{k-1} is a subspace, $z \notin U_{k-1}$ (otherwise $h \subseteq U_{k-1}$ which is impossible since $x \notin U_{k-1}$). Thus, $U_{k-1} \cap p^\perp \subsetneq U \cap p^\perp$ and $l \geq k - 1$.

We obtain $k - 1 \leq l \leq k - 1$.

$\square$

The geometry of totally isotropic subspaces of a vector space with a suitable form give the most familiar examples of polar spaces. These spaces are called *embeddable*.

3.3.8 Proposition
Let $\mathcal{S} = (P, L)$ be a projective space and let π be a polarity of $\mathcal{S}$ such that there exists at least one absolute line with respect to π. The absolute points and the absolute lines with respect to π define a polar space.

Proof   This is [Ue, 4.4.7]. $\square$

3.3.9 Remark
Let $\mathcal{S} = (P, L)$ be a projective space and let π be a polarity of $\mathcal{S}$ such that there exists at least one absolute line with respect to π.

  (a) The polar space of proposition 3.3.8 is called *the polar space defined by π* and it is denoted by $\mathcal{S}_\pi$.

  (b) A point and a line of $\mathcal{S}_\pi$ are incident in $\mathcal{S}_\pi$ if they are incident in $\mathcal{S}$.

  (c) Two points p and q of $\mathcal{S}_\pi$ are collinear if and only if $p \in \pi(q)$.

3.3.10 Remark
As a special case we obtain the following:
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of Witt index $k \geq 2$. Then the associated bilinear form f_Q is a $(\operatorname{id}_\mathbb{K}, 1_\mathbb{K})$-hermitian sesquilinear form on V. We denote by $\mathcal{V}(\Lambda)$ all non-trivial totally isotropic subspaces of V. We set

$$\mathcal{P}(\Lambda) := \{ U \in \mathcal{V}(\Lambda) \mid \dim_\mathbb{K}(U) = 1 \}$$

and

$$\mathcal{L}(\Lambda) := \{ \mathcal{P}(U) \mid U \in \mathcal{V}(\Lambda), \dim_\mathbb{K}(U) = 2 \}.$$

The pair $\mathbf{P}(\Lambda) := (\mathcal{P}(\Lambda), \mathcal{L}(\Lambda))$ is a polar space of finite rank k. It is called *the polar space associated with Λ*.

3.3.11 Remark

(a) If the quadratic space Λ is regular, the associated polar space $\mathbf{P}(\Lambda)$ is non-degenerate: Indeed, by 1.18 there exists a hyperbolic pair (v, w) of Λ. The corresponding line $\langle v, w \rangle$ of the polar space $\mathbf{P}(\Lambda)$ is not totally-isotropic, since

$$Q(v + w) = f_Q(v, w) - Q(v) - Q(w) = f_Q(v, w) = 1 \neq 0.$$

(b) The totally isotropic subspaces of V of dimension d are in one-to-one correspondence with the singular subspaces of $\mathbf{P}(\Lambda)$ of rank d.

The next result shows that each semi-linear similitude of the quadratic space Λ induces an automorphism of the associated polar space $\mathbf{P}(\Lambda)$. If the dimension of V is „high enough", every automorphism of the polar space is induced by a semi-linear similitude.

3.3.12 Proposition

Let $\Lambda = (\mathbb{K}, V, Q)$ be a regular quadratic space of Witt index $k \geq 1$.

(a) There is a natural homomorphism $\varphi_\Lambda \colon \Gamma O(\Lambda) \to \mathrm{Aut}(\mathbf{P}(\Lambda))$ from the set of all semi-linear similitudes of Λ onto the set of all automorphisms of the polar space $\mathbf{P}(\Lambda)$. Its kernel is $\mathrm{HT}(V) = \{\lambda \,\mathrm{id}_V \mid \lambda \in \mathbb{K}\}$. The map φ_Λ is defined by

$$\varphi_\Lambda(\alpha)(\langle v \rangle_{\mathbb{K}}) := \langle \alpha(v) \rangle_{\mathbb{K}}$$

for all $\alpha \in \Gamma O(\Lambda)$ and $v \in V$.

(b) If $\dim_{\mathbb{K}}(V) \geq 5$ and $k \geq 2$, the homomorphism φ_Λ is surjective.

Proof   This is [MPW, 2.38]. $\square$

3.3.13 Remark

Let $\Lambda = (\mathbb{K}, V, Q)$ be a regular quadratic space of Witt index $k \geq 3$ and let (v, w) be a hyperbolic pair of Λ. Let $\mathcal{S} := \mathbf{P}(\Lambda)$ be the polar space associated with Λ and let $p := \langle v \rangle$ and $q := \langle w \rangle$. Furthermore, we set $H := \mathbb{H}(v, w)^\perp$ and $\Lambda' := (\mathbb{K}, H, Q|_H)$. Now

> x is a point of $\mathbf{P}(\Lambda')$
>
> $\Leftrightarrow \exists\, u \in \mathbb{H}(v, w)^\perp$ isotropic such that $x = \langle u \rangle$
>
> $\Leftrightarrow \exists\, u \in V$ isotropic such that $f_Q(v, u) = 0 = f_Q(w, u), \langle u \rangle = x$
>
> $\Leftrightarrow x \in p^\perp \cap q^\perp$

shows that the point sets of $\mathbf{P}(\Lambda')$ and $\mathcal{S}_{pq}$ coincide. As two points of $\mathbf{P}(\Lambda)$ (respectively $\mathcal{S}_{pq}$) are collinear if and only if they are collinear in $\mathcal{S}$, we conclude that $\mathcal{S}_{pq} = \mathbf{P}(\Lambda')$.

Polar Spaces of type D

3.3.14 Definition
Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space.

(a) A *submaximal* singular subspace of $\mathcal{S}$ is a proper hyperplane in a maximal singular subspace.

(b) The polar space $\mathcal{S}$ is called *of type D* if it is of finite rank and if each submaximal singular subspace of $\mathcal{S}$ is contained in precisely two maximal ones.

3.3.15 Remark
Let $\mathcal{S}$ be a polar space of type D of finite rank n. According to [Ti74, 8.4.3], $\mathcal{S}$ is the polar space associated with a hyperbolic quadratic space of dimension $2n$.

Automorphisms

In this section we assemble some basic results about automorphisms of polar spaces. Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space.

3.3.16 Lemma
Let $\alpha \in \mathrm{Aut}(\mathcal{S})$ be an automorphism. If $l \in L$ is a line stabilized by α and $p \in P$ is a point fixed by α which is collinear with a unique point x on l, then x is fixed by α.

Proof Since α preserves collinearity, the points $\alpha(p) = p$ and $\alpha(x)$ are collinear. Since $\alpha(x)$ is incident with l and since there is precisely one point on l which is collinear with p, we conclude that $\alpha(x) = x$. □

3.3.17 Lemma
Let $\alpha \in \mathrm{Aut}(\mathcal{S})$ be an involution and let U and U' be singular subspaces of $\mathcal{S}$ which are stabilized by α. If the subset $U \cup U'$ is singular, then the singular subspace $M := \langle U, U' \rangle$ is α-invariant.

Proof By definition, $M := \langle U, U' \rangle$ is the smallest singular subspace of $\mathcal{S}$ which contains the subset $U \cup U'$. As $U \cup U' = \alpha(U \cup U') \subseteq \alpha(M)$, we conclude that $M \subseteq \alpha(M)$.
Let $x \in \alpha(M)$ be a point and let $y \in M$ such that $\alpha(y) = x$. As α is an involution and since $y \in M \subseteq \alpha(M)$, there exists $z \in M$ such that $y = \alpha(z)$. Now $x = \alpha(y) = \alpha^2(z) = z \in M$ and hence $\alpha(M) \subseteq M$. □

3.3.18 Lemma
Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space of finite rank $n \geq 4$, let $\alpha \in \mathrm{Aut}(\mathcal{S})$ be an involution and let U be a singular subspace of $\mathcal{S}$ with $\mathrm{rk}(U) = 4$ such that $\alpha(U) = U$. Suppose that α does not fix any points of U. Then U contains at least two α-invariant lines of $\mathcal{S}$. If each line of $\mathcal{S}$ is incident with at least three points, then U contains at least three α-invariant lines.

Proof   Let $p \in U$ be any point. By assumption $\alpha(p) \in U$ and $\alpha(p) \neq p$. Since U is singular, the points p and $\alpha(p)$ lie on a common line l which is completely contained in U. Choose a point $z \in U$ which does not lie on l. Again, $z \neq \alpha(z) \in U$ and the points z and $\alpha(z)$ lie on a common line g which is completely contained in U.
Note that $\alpha(l)$ is a line which contains the points $\alpha(p)$ and $\alpha^2(p) = p$. In view of 3.3.5(a), $\alpha(l) = l$. Similarly, $\alpha(g) = g$. Since z is a point on g which is not on l, we conclude that $l \neq g$.
Suppose that l and g intersect in a single point x, i.e. $l \cap g = \{x\}$. Then $\alpha(x) = \alpha(l \cap g) \subseteq \alpha(l) \cap \alpha(g) = l \cap g = \{x\}$. Since there are no fixed points, this is impossible. Assume that each line is incident with at least three points.

Let $h \subseteq U$ denote the line which is incident with the points z and p and choose a point x on h different from z and p. Then $x \neq \alpha(x)$ and x and $\alpha(x)$ lie on a common line which is α-invariant.

$\square$
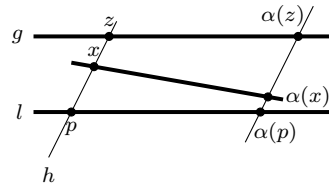
Chapter 4

Spherical buildings and geometries

4.1 Opposites in spherical buildings

Throughout this section let (W, S) be a spherical Coxeter system, let Π be the corresponding Coxeter diagram and let $\Delta = (\mathcal{C}, \delta)$ be a building of type Π.

A finite Coxeter group W always has a unique element of maximal length (cf. 2.8). The existence of a longest element in W leads to the fundamental concept that distinguishes spherical buildings from general buildings:

4.1.1 Definition
  (a) Two chambers $c, d \in \mathcal{C}$ are called *opposite*, if $\delta(c, d) = r_S$.

  (b) Two residues $\mathcal{R}$ and $\mathcal{T}$ are called *opposite*, if for each $c \in \mathcal{R}$ there exists $d \in \mathcal{T}$ such that $\delta(c, d) = r_S$ and vice versa. Equivalently, $\mathcal{R}$ and $\mathcal{T}$ are opposite if there exists a pair of chambers $(c, d) \in \mathcal{R} \times \mathcal{T}$ such that $\delta(c, d) = r_S$ and $\mathrm{Typ}(\mathcal{R}) = \mathrm{op}_S(\mathrm{Typ}(\mathcal{T}))$.

4.1.2 Lemma
Two opposite residues of Δ are parallel.

Proof A proof can be found in [AB, 5.114]. $\square$

4.1.3 Lemma
Let $s \in S$ such that $st = ts$ for all $t \in S$ and set $J := S\backslash\{s\}$.

  (a) If $\mathcal{R}$ is a residue of type J and c is a chamber not contained in $\mathcal{R}$, then $\delta(c, \mathrm{proj}_{\mathcal{R}}(c)) = s$.

(b) Any two residues $\mathcal{R}$ and $\mathcal{T}$ of type J are either equal or opposite.

Proof Choose a chamber $d \in \mathcal{R}$ and let $c \in \mathcal{C} \backslash \mathcal{R}$. Then $w := \delta(c,d) = sw'$ for some $w' \in W_J$ by 2.10(a). Hence

$$\delta(c, \mathrm{proj}_{\mathcal{R}}(c)) = \min\{\delta(c,d)W_J\} = \min\{wW_J\} = s.$$

Let $\mathcal{R}$ and $\mathcal{T}$ be two residues of type J and suppose that $\mathcal{R} \neq \mathcal{T}$. Since the opposition map op_S stabilizes every connected component of Π (cf. 2.9) we have $\mathrm{Typ}(\mathcal{R}) = J = \mathrm{op}_S(J) = \mathrm{op}_S(\mathrm{Typ}(\mathcal{T}))$. Let $c \in \mathcal{R}$ be any chamber. Since $c \notin \mathcal{T}$, part (a) implies that $\delta(c, \mathrm{proj}_{\mathcal{T}}(c)) = s$. Choose a chamber d opposite $\mathrm{proj}_{\mathcal{T}}(c)$ in $\mathcal{T}$. Then, by 2.10(b),

$$\delta(c,d) = \delta(c, \mathrm{proj}_{\mathcal{T}}(c))\ \delta(\mathrm{proj}_{\mathcal{T}}(c),d) = sr_J = r_S.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.1.4 Lemma
Let c and d be opposite chambers of Δ. There exists a unique apartment of Δ containing c and d.

Proof This is [W03, 9.2] $\square$

4.1.5 Lemma
Let $\tau \in \mathrm{Aut}(\Delta)$ be an involution and set $\Gamma := \langle \tau \rangle \leq \mathrm{Aut}(\Delta)$. Let C be a Γ-chamber of Δ and set $A := \mathrm{Typ}(C)$. Then for each chamber $c \in C$ we have $\delta(c, \tau(c)) = r_A$.

Proof By assumption we have $\tau|_C \in \mathrm{Aut}(C)$ and the group $\Gamma_C := \langle \tau|_C \rangle$ stabilizes no proper residues of C. Now, by [MPW, 25.17], each chamber $c \in C$ is opposite to its image $\tau(c)$ in C, i.e. $\delta(c, \tau(c)) = r_A$. $\square$

4.2   Isometries on buildings

Throughout this section let Π be a Coxeter diagram with vertex set I and let (W, S) denote the corresponding Coxeter system. Let $\ell \colon W \to \mathbb{N}$ denote the length function on W with respect to S.

4.2.1 Definition
A subset $X \subseteq 2^S$ is called *essential* if the following conditions are satisfied:

(e1) $S \notin X$,

(e2) $\bigcup_{M \in X} M = S$ and

(e3) for each irreducible subset $J \subseteq S$ with $|J| = 2$ there exists a subset $M \in X$ such that $J \subseteq M$.

4.2.2 Definition
Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S). Given a chamber $c \in \mathcal{C}$ we define

(a) $E_k(c) := \bigcup_{\substack{J \subseteq S \\ |J| \leq k}} \mathcal{R}_J(c)$ for any natural number $k \in \mathbb{N}$.

(b) $E_2^*(c) := \bigcup_{\substack{J \subseteq S \text{ connected} \\ |J| \leq 2}} \mathcal{R}_J(c)$.

(c) $E_X(c) := \bigcup_{J \in X} \mathcal{R}_J(c)$ for any subset $X \subseteq 2^S$.

4.2.3 Proposition
Let Π be the Coxeter diagram $\mathsf{A}_1 \times \mathsf{A}_1$ and let $\Delta = (\mathcal{C}, \delta)$ and $\Delta' = (\mathcal{C}', \delta')$ be thick buildings of type Π. Let $c \in \mathcal{C}$ and $c' \in \mathcal{C}'$ and suppose that $\varphi \colon E_1(c) \to E_1(c')$ is an isometry such that $\varphi(c) = c'$. Then φ extends uniquely to an isometry from Δ onto Δ'.

Proof Let $S = \{s, t\}$ and note that $E_1(c) = \mathcal{P}_s(c) \cup \mathcal{P}_t(c)$. Let $d \in \mathcal{C} \backslash E_1(c)$. Then $\delta(c, d) = st = ts$. We set

$$d_s := \mathrm{proj}_{\mathcal{P}_s(c)}(d) \text{ and } d_t := \mathrm{proj}_{\mathcal{P}_t(c)}(d).$$

Then $d_s, d_t \in E_1(c) \backslash \{c\}$ and $\varphi(d_s), \varphi(d_t) \in E_1(c') \backslash \{c'\}$ are defined. Moreover, for each $r \in S$, $\delta'(\varphi(d_r), c') = \delta(d_r, c) = r$.
Note that, since $c' = \mathrm{proj}_{\mathcal{P}_s(c')}(\varphi(d_t))$, we have

$$\delta'(\varphi(d_s), \varphi(d_t)) = \delta'(\varphi(d_s), c') \ \delta'(c', \varphi(d_t)) = st$$

and whence the chambers $\varphi(d_s)$ and $\varphi(d_t)$ are opposite in Δ'. In particular, the panels $\mathcal{P}_t(\varphi(d_t))$ and $\mathcal{P}_t(\varphi(d_s))$ are opposite and thus parallel (cf. 4.1.2). Hence, $d' = \mathrm{proj}_{\mathcal{P}_t(\varphi(d_s))}(\varphi(d_t))$ is the unique chamber in $\mathcal{P}_t(\varphi(d_s))$ satisfying $\delta'(\varphi(d_t), d') = s$. Similarly, $d'' := \mathrm{proj}_{\mathcal{P}_s(\varphi(d_t))}(\varphi(d_s))$ is the unique chamber in the panel $\mathcal{P}_s(\varphi(d_t))$ satisfying $\delta'(\varphi(d_s), d'') = t$. By construction, $d' \in \mathcal{P}_s(\varphi(d_t)) \cap \mathcal{P}_t(\varphi(d_s))$ and thus $d' = d''$.
This enables us to define

$$\varphi(d) := \mathrm{proj}_{\mathcal{P}_s(\varphi(d_t))}(\varphi(d_s)).$$

In this way we extend φ to a map $\varphi \colon \mathcal{C} \to \mathcal{C}'$.

Let $x \in \mathcal{C}'$. If $x \in E_1(c')$, then, by assumption, there exists a chamber $y \in E_1(c)$ such that $\varphi(y) = x$. Suppose that $x \in \mathcal{C}' \backslash E_1(c')$. Then x is

opposite to the chamber c'. Let $x_s := \mathrm{proj}_{\mathcal{P}_s(c')}(x)$ and $x_t := \mathrm{proj}_{\mathcal{P}_t(c')}(x)$. By assumption there exist chambers $y_s \in \mathcal{P}_s(c)$ and $y_t \in \mathcal{P}_t(c)$ such that $\varphi(y_s) = x_s$ and $\varphi(y_t) = x_t$. Let Σ be the unique apartment of Δ containing the chambers c, y_s and y_t and let $d \in \Sigma$ be the unique chamber which is opposite to c. Then, by 2.30, $y_s = \mathrm{proj}_{\mathcal{P}_s(c)}(d)$ and $y_t = \mathrm{proj}_{\mathcal{P}_t(c)}(d)$. We claim that $\varphi(d) = x$.
Indeed, by definition, $\varphi(d) = \mathrm{proj}_{\mathcal{P}_s(x_t)}(x_s)$ is the unique chamber in $\mathcal{P}_s(x_t)$ satisfying $\delta'(\varphi(d), x_s) = t$. Hence, $\varphi(d) = x$ and we conclude that the map $\varphi \colon \mathcal{C} \to \mathcal{C}'$ is surjective.

It remains to show that φ is an isometry. In view of [AB, 5.61] it suffices to show that for any two chambers $x, y \in \mathcal{C}$ with $\mathrm{dist}(x, y) = 1$ we have $\delta'(\varphi(x), \varphi(y)) = \delta(x, y)$.
Let $x, y \in \mathcal{C}$ be such that $\delta'(x, y) \in S$.

If $x, y \in E_1(c)$ then the assumption on φ gives $\delta'(\varphi(x), \varphi(y)) = \delta(x, y)$.

So suppose that $x \in E_1(c)$ and that $y \notin E_1(c)$ and let $s \in S$ such that $\delta(x, y) = s$. Then y is a chamber opposite to c and since $\mathrm{dist}(x, y) = 1$ we have $x = \mathrm{proj}_{\mathcal{P}_t(c)}(y)$. Let $y_s := \mathrm{proj}_{\mathcal{P}_s(c)}(y)$. By definition we have $\varphi(y) \in \mathcal{P}_s(\varphi(x)) \cap \mathcal{P}_t(\varphi(y_s))$ and hence $\delta'(\varphi(x), \varphi(y)) \in \{1_W, s\}$. Suppose that $\delta'(\varphi(x), \varphi(y)) = 1_W$, i.e. $\varphi(x) = \varphi(y)$. Since φ is a bijection on $E_1(c)$ this fact implies that $\varphi(y) \neq \varphi(y_s)$. Now

$$\delta'(c', \varphi(y)) = \delta'(c', \varphi(x)) = t = \delta'(\varphi(y), \varphi(y_s)).$$

But this implies $\varphi(y_s) \in \mathcal{P}_t(c') \cap \mathcal{P}_s(c') = \{c'\}$, a contradiction. Hence, $\delta'(\varphi(x), \varphi(y)) = s$.

Let $x \neq y$ be chambers both opposite to c and let $s \in S$ such that $\delta(x, y) = s$. Then $d := \mathrm{proj}_{\mathcal{P}_s(x)}(c)$ is the unique chamber in $\mathcal{P}_s(x) = \mathcal{P}_s(y)$ satisfying $\delta(c, d) = t$. In particular, $d \in E_1(c)$. The considerations in the previous case imply that

$$\delta'(\varphi(d), \varphi(x)) = \delta(d, x) = s = \delta(d, y) = \delta'(\varphi(d), \varphi(y)).$$

Hence, as $\varphi(x), \varphi(y) \in \mathcal{P}_s(\varphi(d))$, $\delta'(\varphi(x), \varphi(y)) = s$.

We conclude that φ is an isometry from Δ onto Δ'.

Now let $\psi, \psi' \colon \mathcal{C} \to \mathcal{C}'$ be isometries which coincide on $E_1(c)$. Let $\Sigma \subseteq \mathcal{C}$ be an apartment of Δ containing the chamber c. As $\psi|_\Sigma \colon \Sigma \to \psi(\Sigma)$ is an isometry, we conclude that the set $\psi(\Sigma)$ is isometric to the standard thin building (W, δ_W) of type (W, S) (cf. 2.23). Hence, $\psi(\Sigma)$ is an apartment of Δ'. Similarly, $\psi'(\Sigma)$ is an apartment of Δ'. Let c_s and c_t be the unique chambers of Σ satisfying $\delta(c, c_s) = s$ and $\delta(c, c_t) = t$. Since $c_s, c_t \in E_1(c)$, both apartments $\psi(\Sigma)$ and $\psi'(\Sigma)$ contain the chambers $\psi(c_s)$ and $\psi(c_t)$ which are opposite in Δ'. In view of 4.1.4 we conclude that $\psi(\Sigma) = \psi'(\Sigma)$. In particular, if $d := \mathrm{op}_\Sigma(c)$ is the unique chamber in Σ which is opposite

to c, then $\psi(d) = \psi'(d)$. Now $\psi \circ \psi'^{-1} \colon \mathcal{C} \to \mathcal{C}$ is an isometry which fixes $E_1(c) \cup \{d\}$ pointwise. The rigidity theorem [AB, 5.205] yields that $\psi \circ \psi'^{-1}$ is the identity.   □

4.2.4 Proposition
Let $\Delta = (\mathcal{C}, \delta)$ and $\Delta' = (\mathcal{C}', \delta')$ be thick buildings of type Π. Let $c \in \mathcal{C}$ and $c' \in \mathcal{C}'$ and suppose that $\varphi \colon E_2(c) \to E_2(c')$ is a bijective mapping such that

(i) $\varphi(c) = c'$ and

(ii) for all $J \subseteq S$ with $|J| \leq 2$ the restriction $\varphi|_{\mathcal{R}_J(c)} \colon \mathcal{R}_J(c) \to \mathcal{R}_J(c')$ is an isometry.

Then φ is an isometry.

Proof Let $x, y \in E_2(c)$ and let $J, K \subseteq S$ be subsets with $|J|, |K| \leq 2$ such that $x \in \mathcal{R}_J(c)$ and $y \in \mathcal{R}_K(c)$.

Case 1: $J \subseteq K$ or $K \subseteq J$. Then $x, y \in \mathcal{R}_K(c)$ or $x, y \in \mathcal{R}_J(c)$ and the assertion follows by assumption (ii).

Case 2: $J \cap K = \emptyset$. In this case $\mathcal{R}_J(c) \cap \mathcal{R}_K(c) = \{c\}$ and hence, in view of 2.28(b), $\mathrm{proj}_{\mathcal{R}_K(c)}(x) = c$. Note that $\varphi(x) \in \mathcal{R}_J(c')$ and $\varphi(y) \in \mathcal{R}_K(c')$. Since $\mathcal{R}_J(c') \cap \mathcal{R}_K(c') = \{c'\}$ we conclude that $\mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x)) = c'$. We obtain

$$\begin{aligned}
\delta(x, y) &= \delta(x, \mathrm{proj}_{\mathcal{R}_K(c)}(x)) \; \delta(\mathrm{proj}_{\mathcal{R}_K(c)}(x), y) = \delta(x, c) \; \delta(c, y) \\
&= \delta'(\varphi(x), \varphi(c)) \; \delta'(\varphi(c), \varphi(y)) = \delta(\varphi(x), c') \; \delta'(c', \varphi(y)) \\
&= \delta'(\varphi(x), \mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x))) \; \delta'(\mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x)), \varphi(y)) \\
&= \delta'(\varphi(x), \varphi(y)),
\end{aligned}$$

where we used that φ is an isometry on both, $\mathcal{R}_J(c)$ and $\mathcal{R}_K(c)$.

Case 3: $|J| = 2 = |K|$ and $|J \cap K| = 1$. Let $s \in S$ be such that $J \cap K = \{s\}$. We may assume that neither x nor y is contained in the panel $\mathcal{P}_s(c)$ (otherwise the assertion follows from case 1). Using again 2.28(b), we obtain $\mathrm{proj}_{\mathcal{R}_K(c)}(x) \in \mathcal{P}_s(c)$ and $\mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x)) \in \mathcal{P}_s(c')$. By (ii), φ restricts to an isometry $\mathcal{P}_s(c) \to \mathcal{P}_s(c')$. The map $x \mapsto \varphi(x)$ extends this restriction to an isometry $\mathcal{P}_s(c) \cup \{x\} \to \mathcal{P}_s(c') \cup \{\varphi(x)\}$ (since $\mathcal{P}_s(c) \cup \{x\} \subseteq \mathcal{R}_J(c)$). Now [MR, 4.2] gives that $\varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(x)) = \mathrm{proj}_{\mathcal{P}_s(c')}(\varphi(x))$. Futhermore, by 2.28(a),

$$\mathrm{proj}_{\mathcal{P}_s(c)}(x) = \mathrm{proj}_{\mathcal{P}_s(c)}(\mathrm{proj}_{\mathcal{R}_K(c)}(x)) = \mathrm{proj}_{\mathcal{R}_K(c)}(x),$$

since $\mathrm{proj}_{\mathcal{R}_K(c)}(x) \in \mathcal{P}_s(c)$. Similarly,

$$\mathrm{proj}_{\mathcal{P}_s(c')}(\varphi(x)) = \mathrm{proj}_{\mathcal{P}_s(c')}(\mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x))) = \mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x)),$$

since $\mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x)) \in \mathcal{P}_s(c')$. Hence

$$
\begin{aligned}
\delta(x,y) &= \delta(x, \mathrm{proj}_{\mathcal{R}_K(c)}(x))\, \delta(\mathrm{proj}_{\mathcal{R}_K(c)}(x), y) \\
&= \delta(x, \mathrm{proj}_{\mathcal{P}_s(c)}(x))\, \delta(\mathrm{proj}_{\mathcal{P}_s(c)}(x), y) \\
&= \delta'(\varphi(x), \varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(x)))\, \delta'(\varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(x)), \varphi(y)) \\
&= \delta'(\varphi(x), \mathrm{proj}_{\mathcal{P}_s(c')}(\varphi(x)))\, \delta'(\mathrm{proj}_{\mathcal{P}_s(c')}(\varphi(x)), \varphi(y)) \\
&= \delta'(\varphi(x), \mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x)))\, \delta'(\mathrm{proj}_{\mathcal{R}_K(c')}(\varphi(x)), \varphi(y)) \\
&= \delta'(\varphi(x), \varphi(y)).
\end{aligned}
$$

$\square$

4.2.5 Proposition
Let Π be a spherical Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let $\Delta = (\mathcal{C}, \delta)$ be a building of type Π. Let $\mathcal{R}$ be a residue of type J and let $\mathcal{T}$ be a residue which is opposite to $\mathcal{R}$. Let $\sigma \in \mathrm{Aut}(W, S)$ be given by $\sigma(s) := (r_S r_J)s(r_S r_J)^{-1}$ for all $s \in S$.

(a) The projection map $\mathrm{proj}_{\mathcal{T}}^{\mathcal{R}} \colon \mathcal{R} \to \mathcal{T}$ is a σ-isometry.

(b) The projection maps $\mathrm{proj}_{\mathcal{T}}^{\mathcal{R}}$ and $\mathrm{proj}_{\mathcal{R}}^{\mathcal{T}}$ are mutually inverse.

(c) For any $\varphi \in \mathrm{Aut}(\Delta)$ we have $\varphi \circ \mathrm{proj}_{\mathcal{T}}^{\mathcal{R}} = \mathrm{proj}_{\varphi(\mathcal{T})}^{\varphi(\mathcal{R})} \circ \varphi|_{\mathcal{R}}$.

Proof Parts (a) and (b) of the assertion follow from [AB, 5.116].
Let $\tau \in \mathrm{Aut}(W, S)$ be the accompanying automorphism of φ. Let $c \in \mathcal{R}$ and $d \in \mathcal{T}$ be chambers and set $w := \delta(c, d)$. Then,

$$
\begin{aligned}
\ell(\delta(\varphi(c), \varphi(\mathrm{proj}_{\mathcal{T}}(c)))) &= \ell(\tau(\delta(c, \mathrm{proj}_{\mathcal{T}}(c)))) = \ell(\delta(c, \mathrm{proj}_{\mathcal{T}}(c))) \\
&= \ell(\min\{\delta(c, \mathcal{T})\} = \ell(\min\{w W_{\mathrm{op}_S(J)}\}) \\
&= \ell(\min\{\tau(w) W_{\tau(\mathrm{op}_S(J))}) = \ell(\min\{\delta(\varphi(c), \varphi(\mathcal{T}))\}).
\end{aligned}
$$

e conclude that $\varphi(\mathrm{proj}_{\mathcal{T}}(c)) = \mathrm{proj}_{\varphi(\mathcal{T})}(\varphi(c))$. $\square$

4.3 Building geometries

We freely use the interplay between spherical buildings and flag complexes of geometries as developed in [Ti81, section 1], [Ti74, chapters 6-9] and [BC, chapter 3].

Let I be a set.

4.3.1 Definition
A triple $\mathcal{G} = (X, \tau, *)$ is called an *incidence system* over I if

   (i) X is a set;

  (ii) $*$ is a symmetric and reflexive relation on X;

 (iii) τ is a map from X to I such that for any two elements $x, y \in X$ with $\tau(x) = \tau(y)$, the relation $x * y$ holds if and only if $x = y$.

The elements of X are called the *vertices* of $\mathcal{G}$, the relation $*$ is called *incidence relation* and the image under τ of a vertex is called its *type*.

4.3.2 Definition
Let $\mathcal{G} = (X, *, \tau)$ be an incidence system over I.

  (a) If $A \subseteq X$, we say that A is of *type* $\tau(A)$ and of *rank* $|\tau(A)|$. The *cotype* of A is $I \backslash \tau(A)$ and the *corank* of A is the cardinality of $I \backslash \tau(A)$.

  (b) Given $Y \subset X$, we set $Y^* := \{x \in X \mid x * y \; \forall y \in Y\}$.

  (c) A *flag* of $\mathcal{G}$ is a set of mutually incident elements of $\mathcal{G}$. Flags of $\mathcal{G}$ of type I are called *chambers*.

  (d) We denote the set of all flags of $\mathcal{G}$ by $\mathrm{flag}(\mathcal{G})$.

4.3.3 Definition
An incidence system over I in which every maximal flag is a chamber is called *geometry* over I.

4.3.4 Definition
A geometry $\mathcal{G}$ is *thick* if every flag of type other than I is contained in at least three distinct chambers of $\mathcal{G}$.

4.3.5 Definition
Let $\mathcal{G} = (X, *, \tau)$ and $\mathcal{G}' = (X', *', \tau')$ be geometries over type sets I and I' respectively. A bijection $\alpha \colon X \to X'$ is an *isomorphism* if α and α^{-1} are incidence preserving.

Moreover, if $I = I'$ and $\tau = \tau' \circ \alpha$, then α is called *special*.

4.3.6 Definition
Let $\mathcal{G}$ be a geometry and let $F \in \text{flag}(\mathcal{G})$ be a flag. Set $res(F) := F^* \backslash F$ and let $*_F$ and τ_F be the restrictions of $*$ (resp. τ) to $res(F) \times res(F)$ (resp. $res(F)$). The *residue* of F in $\mathcal{G}$ is the geometry $\mathcal{R}es(F) := (res(F), *_F, \tau_F)$ over the type set $I \backslash \tau(F)$.

4.3.7 Remark
Let Π be a Coxeter diagram with vertex set I and let Δ be a building of type Π. Then Δ can be seen as the flag complex of a geometry $\mathcal{G}$ over the type set I. More precisely: the elements of Δ may be identified with the flags of $\mathcal{G}$. To every residue $\mathcal{R}$ of Δ there corresponds a flag $F_\mathcal{R}$ of $\mathcal{G}$ in such a way that $\mathcal{R} \simeq \text{Flag}(\mathcal{R}es(F_\mathcal{R}))$. In particular, the chambers of Δ correspond to the maximal flags of $\mathcal{G}$. Geometries arising in this way from a building are called *building geometries*.
Let $\mathcal{G}$ be a building geometry. The associated building $\Delta = \text{Flag}(\mathcal{G})$ is thick if and only if $\mathcal{G}$ is thick.

4.3.8 Remark (Building geometries of type A_n)
Let Π be the Coxeter diagram A_n labeled as in 2.3. Let $\mathcal{S} = (P, L)$ be a projective space of finite rank n and let $\mathcal{V}(\mathcal{S})$ be the set of all non-trivial subspaces of $\mathcal{S}$. We define $\tau \colon \mathcal{V}(\mathcal{S}) \to \{1, \dots, n\}$ via $\tau(U) := \dim(U) + 1$ as well as an incidence relation $*$ on $\mathcal{V}(\mathcal{S})$ by putting $U * W$ if and only if $U \subseteq W$ or $W \subseteq U$. The resulting triple $\mathcal{G}(\mathcal{S}) = (\mathcal{V}(\mathcal{S}), *, \tau)$ is a geometry, called *the projective geometry associated with $\mathcal{S}$*. The corresponding flag complex $\text{Flag}(\mathcal{G}(\mathcal{S}))$ is a building of type A_n.

Conversely, if Δ is a building of type A_n, there exists a projective space $\mathcal{S}'$ of finite rank n such that $\Delta \simeq \text{Flag}(\mathcal{G}(\mathcal{S}'))$, where $\mathcal{G}(\mathcal{S}')$ is the projective geometry associated with $\mathcal{S}'$.

An automorphism of the projective space $\mathcal{S}$ induces an automorphism of the geometry $\mathcal{G}(\mathcal{S})$ (and thus an automorphism of the building $\text{Flag}(\mathcal{G}(\mathcal{S}))$). The following is well known: Let $\alpha \in \text{Sym}(\mathcal{V}(\mathcal{S}))$ be an automorphism of $\mathcal{G}(\mathcal{S})$. Then α either preserves inclusion or it reverses inclusion. Moreover, if α preserves inclusion, then there exists a unique automorphism of $\mathcal{S}$ inducing α. If α reverses inclusion, it is induced by a unique duality of $\mathcal{S}$.

Two elements $U, V \in \mathcal{V}(\mathcal{S})$ are said to be *opposite*, if $U \cap V = \emptyset$ and $\langle U, V \rangle = P$. Two flags $F, F' \in \text{flag}(\mathcal{G}(\mathcal{S}))$ are *opposite* if for each $U \in F$ there exists $V \in F'$ such that U and V are opposite and vice versa.
Let $F = \{p, h\}$ and $F' = \{q, h'\}$ be opposite flags each consisting of a point and a hyperplane. Then $\mathcal{G}(\mathcal{R}es(F)) \simeq \mathcal{G}(\mathcal{S}_{h \cap h'})$.

4.3.9 Remark (Building geometries of type C_n)
Let Π be the Coxeter diagram C_n labeled as in 2.3. Let $\mathcal{S} = (P, L)$ be a non-degenerate polar space of finite rank n and let $\mathcal{V}(\mathcal{S})$ be the set of all non-trivial singular subspaces of $\mathcal{S}$. We define $\tau \colon \mathcal{V}(\mathcal{S}) \to \{1, \dots, n\}$ via

$\tau(U) := \mathrm{rk}(U)$ as well as an incidence relation $*$ on $\mathcal{V}(\mathcal{S})$ by putting $U * W$ if and only if $U \subseteq W$ or $W \subseteq U$. The resulting triple $\mathcal{G}(\mathcal{S}) = (\mathcal{V}(\mathcal{S}), *, \tau)$ is a geometry, called *the polar geometry associated with $\mathcal{S}$*. The corresponding flag complex $\mathrm{Flag}(\mathcal{G}(\mathcal{S}))$ is a building of type C_n.

Conversely, if Δ is a building of type C_n, there exists a non-degenerate polar space $\mathcal{S}'$ of finite rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}(\mathcal{S}'))$, where $\mathcal{G}(\mathcal{S}')$ is the polar geometry associated with $\mathcal{S}'$.

An automorphism of the polar space $\mathcal{S}$ induces an automorphism of the geometry $\mathcal{G}(\mathcal{S})$ (and thus an automorphism of the building $\mathrm{Flag}(\mathcal{G}(\mathcal{S}))$). Let $\alpha \in \mathrm{Sym}(\mathcal{V}(\mathcal{S}))$ be an automorphism of $\mathcal{G}(\mathcal{S})$ and suppose that $n \geq 3$. Then α preserves inclusion and there exists a unique automorphism of $\mathcal{S}$ inducing α.

Two elements $U, V \in \mathcal{V}(\mathcal{S})$ are said to be *opposite*, if (U, V) is a hyperbolic pair of $\mathcal{S}$. Two flags $F, F' \in \mathrm{flag}(\mathcal{G}(\mathcal{S}))$ are *opposite* if for each $U \in F$ there exists $V \in F'$ such that U and V are opposite and vice versa.
Let p and q be two non-collinear points of $\mathcal{S}$. Then $\mathcal{G}(\mathcal{R}es(p)) \simeq \mathcal{G}(\mathcal{S}_{pq})$.

4.3.10 Remark (Building geometries of type D_n)
Let Π be the Coxeter diagram D_n labeled as in 2.3. Let $\mathcal{S} = (P, L)$ be a polar space of type D of finite rank n and let $\mathcal{V}_o(\mathcal{S})$ be the set of all non-trivial singular subspaces of $\mathcal{S}$ which are not submaximal. We fix a maximal singular subspace M of $\mathcal{S}$. We define a map $\tau \colon \mathcal{V}_o(\mathcal{S}) \to \{1, \dots, n\}$ via

$$\tau(U) := \begin{cases} i, & 1 \leq \mathrm{rk}(U) = i \leq n - 2 \\ n - 1, & U \cap M \in \mathcal{V}_o(\mathcal{S}) \\ n, & \text{otherwise} \end{cases}$$

as well as an incidence relation $*$ on $\mathcal{V}_o(\mathcal{S})$ by $U * W$ if and only if $U \subseteq W$ or $W \subseteq U$ or if $U \cap W \notin \mathcal{V}_o(\mathcal{S})$. The resulting triple $\mathcal{G}_o(\mathcal{S}) = (\mathcal{V}_o(\mathcal{S}), *, \tau)$ is a geometry, called *the oriflamme geometry associated with $\mathcal{S}$*. The flag complex $\mathrm{Flag}(\mathcal{G}_o(\mathcal{S}))$ is a building of type D_n.

Conversely, if Δ is a building of type D_n, there exists a polar space $\mathcal{S}'$ of type D of finite rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}_o(\mathcal{S}'))$, where $\mathcal{G}_o(\mathcal{S}')$ is the oriflamme geometry associated with $\mathcal{S}'$.

If $n \geq 5$, each $\alpha_o \in \mathrm{Aut}(\mathcal{G}_o(\mathcal{S}))$ is an inclusion-preserving permutation of $\mathcal{V}_o(\mathcal{S})$. Moreover, $\mathrm{Aut}(\mathcal{G}(\mathcal{S}))$ stabilizes the set $\mathcal{V}_o(\mathcal{S})$ and the mapping $\pi \mapsto \pi|_{\mathcal{V}_o(\mathcal{S})}$ is an isomorphism from $\mathrm{Aut}(\mathcal{G}(\mathcal{S}))$ onto $\mathrm{Aut}(\mathcal{G}_o(\mathcal{S}))$.

4.3.11 Remark
It readily follows from the definitions that the automorphism groups of $\mathcal{G}$ and $\mathrm{Flag}(\mathcal{G})$ are the same and that the stabilizer of a subset $X' \subseteq X$ in $\mathrm{Aut}(\mathcal{G})$ corresponds to the stabilizer of $\mathrm{flag}(X')$ in $\mathrm{Aut}(\mathrm{Flag}(\mathcal{G}))$.

If $\mathcal{G}$ is a geometry associated to a projective or polar space $\mathcal{S}$ as described in 4.3.8 and 4.3.9, we identify the automorphisms of $\mathrm{Aut}(\mathcal{S})$ with the automorphisms of $\mathcal{G}$ they induce.

4.3.12 Definition
Let Δ be a building and let $\mathcal{G}$ be a geometry such that $\Delta \simeq \mathrm{Flag}(\mathcal{G})$.

(a) Let F and F' be two flags of $\mathcal{G}$ corresponding to residues $\mathcal{R}$ and $\mathcal{T}$ of Δ. We set $\mathcal{R}' := \mathrm{proj}_{\mathcal{R}}(\mathcal{T})$ and let F'' be the flag corresponding to $\mathcal{R}'$. We define $\mathrm{proj}_F(F') := F'' \backslash F$.

(b) Let F and F' be opposite flags in $\mathcal{G}$. We define the mapping

$$\mathrm{proj}_{F'}^F \colon \mathcal{R}es(F) \to \mathcal{R}es(F') \;\text{ by }\; \mathrm{proj}_{F'}^F := \mathrm{proj}_{F'}\,|_{\mathcal{R}es(F)}.$$

4.3.13 Remark
Let $\mathcal{G}$ be a spherical building geometry and let $F, F' \in \mathrm{flag}(\mathcal{G})$ be opposite flags of $\mathcal{G}$. Then the projection mappings $\mathrm{proj}_{F'}^F$ and $\mathrm{proj}_F^{F'}$ are mutually inverse isomorphisms and they commute with every automorphism $\alpha \in \mathrm{Aut}(\mathcal{G})$, i.e.

$$\mathrm{proj}_{\alpha(F')}^{\alpha(F)} \circ \alpha|_{\mathcal{R}es(F)} = \alpha \circ \mathrm{proj}_{F'}^F \, .$$

Chapter 5

Some specific extensions

5.1 Sesquilinear forms

5.1.1 Notation

Let V be a vector space over a skew field $\mathbb{K}$ of finite dimension $n \geq 3$. Choose linearly independent $v, w \in V$ and let $\mathcal{B} := \{v, w, b_3, \ldots, b_n\}$ be a basis of V. Let $H := \langle b_3, \ldots, b_n \rangle$ and suppose that $f \colon H \times H \to \mathbb{K}$ is a (σ, ε)-hermitian sesquilinear form with $\sigma^2 = \mathrm{id}_{\mathbb{K}}$ and $\varepsilon^2 = 1_{\mathbb{K}}$.

Let $p_H \colon V \to H$ be the projection onto H with respect to the basis $\mathcal{B}$, i.e. for any $x = \sum_{b \in \mathcal{B}} \lambda_b \cdot b \in V$ there is $p_H(x) = \sum_{b \in \mathcal{B} \setminus \{v, w\}} \lambda_b \cdot b$. Similarly we let $\gamma_v \colon V \to \langle v \rangle$ and $\gamma_w \colon V \to \langle w \rangle$ be the projections onto $\langle v \rangle$ and $\langle w \rangle$ with respect to the basis $\mathcal{B}$ respectively, i.e. $\gamma_v(x) = \lambda_v$ and $\gamma_w(x) = \lambda_w$.

To abbreviate notation we write $\bar{\lambda}$ instead of $\sigma(\lambda)$ for all $\lambda \in \mathbb{K}$.

5.1.2 Definition

We define a map $f_{v,w} \colon V \times V \to \mathbb{K}$ via

$$f_{v,w}(x, y) := \varepsilon \gamma_v(x) \overline{\gamma_w(y)} + \gamma_w(x) \overline{\gamma_v(y)} + f(p_H(x), p_H(y))).$$

5.1.3 Proposition

The map $\bar{f} := f_{v,w}$ is a reflexive non-degenerate σ-sesquilinear form on V and $\bar{f}|_H = f$.

Proof Let $x, x', y, y' \in V$. An easy calculation, using that $\gamma_v, \gamma_w \colon V \to \mathbb{K}$ and $p_H \colon V \to H$ are linear, shows that

$$\bar{f}(x + x', y + y') = \bar{f}(x, y) + \bar{f}(x, y') + \bar{f}(x', y) + \bar{f}(x', y').$$

Furthermore, since $\varepsilon \in Z(\mathbb{K})$, for $c, d \in \mathbb{K}$ we have

$$\bar{f}(c \cdot x, d \cdot y) = \varepsilon \gamma_v(cx)\overline{\gamma_w(dy)} + \gamma_w(cx)\overline{\gamma_v(dy)} + f(p_H(cx), p_H(dy))$$
$$= c(\varepsilon \gamma_v(x)\gamma_w(y) + \overline{\gamma_w(x)\gamma_v(y)})\bar{d} + cf(p_H(x), p_H(y))\bar{d}$$
$$= c\bar{f}(x, y)\bar{d}.$$

Hence, $\bar{f}$ is a σ-sesquilinear form on V.

Next we show that $\bar{f}$ is reflexive. For, let $x, y \in V$ such that

$$\bar{f}(x, y) = \varepsilon \gamma_v(x)\overline{\gamma_w(y)} + \gamma_w(x)\overline{\gamma_v(y)} + f(p_H(x), p_H(y))) = 0_{\mathbb{K}}.$$

Multiplying the equation with ε yields

$$0_{\mathbb{K}} = \varepsilon^2 \gamma_v(x)\overline{\gamma_w(y)} + \varepsilon \gamma_w(x)\overline{\gamma_v(y)} + \varepsilon f(p_H(x), p_H(y))).$$

Applying the anti-automorphism σ yields

$$0_{\mathbb{K}} = \overline{\gamma_v(x)\overline{\gamma_w(y)} + \varepsilon \gamma_w(x)\overline{\gamma_v(y)} + \varepsilon f(p_H(x), p_H(y)))}$$
$$= \varepsilon \gamma_v(y)\overline{\gamma_w(x)} + \gamma_w(y)\overline{\gamma_v(x)} + f(p_H(y), p_H(x)) = \bar{f}(y, x).$$

In order to show that $\bar{f}$ is non-degenerate, due to the fact that $\bar{f}$ is reflexive, it suffices to show that for any $x \in V$ there exists a vector $y \in V$ such that $\bar{f}(x, y) \neq 0 - \mathbb{K}$. Since f is non-degenerate it suffices to consider the case that $\gamma_v(x) \neq 0_{\mathbb{K}}$ or $\gamma_w(x) \neq 0_{\mathbb{K}}$. In the first case choose $\beta \in \mathbb{K}\backslash\{\overline{(\varepsilon \gamma_v(x))^{-1}(-\gamma_w(x))}\}$. Then

$$\bar{f}(x, v + \beta \cdot w) = \varepsilon \gamma_v(x)\overline{\gamma_w(v + \beta \cdot w)} + \gamma_w(x)\overline{\gamma_v(v + \beta \cdot w)}$$
$$= \varepsilon \gamma_v(x)\bar{\beta} + \gamma_w(x) \neq 0_{\mathbb{K}}.$$

If $\gamma_v(x) = 0_{\mathbb{K}}$ choose $\nu \in \mathbb{K}\backslash\{0_{\mathbb{K}}\}$. Then

$$\bar{f}(x, \nu \cdot v) = \gamma_w(x)\overline{\gamma_v(\nu \cdot v)} = \gamma_w(x)\bar{\nu} \neq 0_{\mathbb{K}}.$$

$\square$

5.2   Semi-linear similitudes

5.2.1 Notation

Let $\Lambda = (\mathbb{K}, V, Q)$ be a regular quadratic space of finite dimension and Witt index $k \geq 4$. Let (v, w) be a hyperbolic pair of Λ and set $H := \mathbb{H}(v, w)^{\perp}$. Let $\Lambda' := (\mathbb{K}, H, Q|_H)$. Let $\mathcal{B}_H$ be a basis of H and set $\mathcal{B} := \mathcal{B}_H \cup \{v, w\}$. As in the previous section we let $p_H \colon V \to H$, $\gamma_v \colon V \to \langle v \rangle$ and $\gamma_w \colon V \to \langle w \rangle$ be the corresponding projections with respect to the basis $\mathcal{B}$.

Let $\tau \in \Gamma O(\Lambda')$ be a semi-linear similitude such that $\tau^2 \in \mathrm{HT}(H)$. Let $\sigma \in \mathrm{Aut}(\mathbb{K})$ and $\mu \in \mathbb{K}^*$ be the accompanying automorphism and constant of τ and let $c \in \mathbb{K}$ be such that $\tau^2 = c\,\mathrm{id}_H$. Note that, by 1.34(a) we have $\sigma^2 = \mathrm{id}_{\mathbb{K}}$. To abbreviate notation we write $\bar{\lambda}$ instead of $\sigma(\lambda)$ for all $\lambda \in \mathbb{K}$. We suppose that there exist $\lambda_1, \lambda_2 \in \mathbb{K}$ satisfying $\lambda_1 \bar{\lambda}_1 = c = \lambda_2 \bar{\lambda}_2$ and $\lambda_1 \lambda_2 = \mu$.

5.2.2 Definition

We define a map $\tau_{(v,w),(\lambda_1,\lambda_2)} \colon V \to V$ via

$$\tau_{(v,w),(\lambda_1,\lambda_2)}(x) := \lambda_1 \overline{\gamma_v(x)} \cdot v + \lambda_2 \overline{\gamma_w(x)} \cdot w + \tau(p_H(x)).$$

5.2.3 Proposition

The map $\bar{\tau} := \tau_{(v,w),(\lambda_1,\lambda_2)}$ is a σ-semi-linear μ-similitude such that $\bar{\tau}^2 = c\,\mathrm{id}_V$ and $\bar{\tau}|_H = \tau$.

Proof   First notice that for any $x \in V$ we have the following:

$$Q(x) = Q(p_H(x)) + \gamma_v(x)\gamma_w(x). \tag{5.1}$$

Now for all $x, y \in V$ and $\nu \in \mathbb{K}$ we have

$$
\begin{aligned}
\bar{\tau}(x + y) &= \lambda_1 \overline{\gamma_v(x + y)} \cdot v + \lambda_2 \overline{\gamma_w(x + y)} \cdot w + \tau(p_H(x + y)) \\
&= \lambda_1 \overline{\gamma_v(x)} \cdot v + \lambda_2 \overline{\gamma_w(x)} \cdot w + \tau(p_H(x)) + \lambda_1 \overline{\gamma_v(y)} \cdot v \\
&\quad + \lambda_2 \overline{\gamma_w(y)} \cdot w + \tau(p_H(y)) \\
&= \bar{\tau}(x) + \bar{\tau}(y)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\bar{\tau}(\nu x) &= \lambda_1 \overline{\gamma_v(\nu \cdot x)} \cdot v + \lambda_2 \overline{\gamma_w(\nu \cdot x)} \cdot w + \tau(p_H(\nu \cdot x)) \\
&= \bar{\nu}(\lambda_1 \overline{\gamma_v(x)} v + \lambda_2 \overline{\gamma_w(x)} w + \tau(p_H(x)) = \bar{\nu}\bar{\tau}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
Q(\bar{\tau}(x)) &= Q(p_H(\bar{\tau}(x))) + \gamma_v(\bar{\tau}(x))\gamma_w(\bar{\tau}(x)) \\
&= Q(\tau(p_H(x))) + \lambda_1 \overline{\gamma_v(x)} \lambda_2 \overline{\gamma_w(x)} \\
&= \mu \overline{Q(p_H(x))} + \lambda_1 \lambda_2 \overline{\gamma_v(x)\gamma_w(x)} \\
&= \mu (\overline{Q(p_H(x)) + \gamma_v(x)\gamma_w(x)}) = \mu \overline{Q(x)}.
\end{aligned}
$$

Moreover, for any $x \in V$

$$
\begin{aligned}
\bar{\tau}^2(x) &= \bar{\tau}(\lambda_1 \overline{\gamma_v(x)} \cdot v + \lambda_2 \overline{\gamma_w(x)} \cdot w + \tau(p_H(x))) \\
&= \lambda_1 \overline{\lambda_1} \gamma_v(x) \cdot v + \lambda_2 \overline{\lambda_2} \gamma_w(x) \cdot w + \tau^2(p_H(x)) \\
&= c \cdot (\gamma_v(x) \cdot v + \gamma_w(x) \cdot w + \cdot p_H(x)) = c \cdot x.
\end{aligned}
$$

$\square$

5.2.4 Notation
Let $\Lambda = (\mathbb{K}, V, Q)$ be a hyperbolic quadratic space of dimension $2n \geq 10$.
Let $v_1, \ldots, v_n, w_1, \ldots, w_n \in V$ such that $V = \bigoplus_{i=1}^n \mathbb{H}(v_i, w_i)$.
Let $H := \bigoplus_{i=3}^n \mathbb{H}(v_i, w_i)$ and $\Lambda' = (\mathbb{K}, H, Q|_H)$. As in the previous section we let $p_H \colon V \to H$, $\gamma_{v_i} \colon V \to \langle v_i \rangle$ and $\gamma_{w_i} \colon V \to \langle w_i \rangle$ for $i = 1, 2$ be the corresponding projections with respect to the basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$.

Let $\tau \in \Gamma O(\Lambda')$ be a semi-linear similitude such that $\tau^2 = c\,\mathrm{id}_H \in \mathrm{HT}(H)$ for some $c \in \mathbb{K}$. Let $\sigma \in \mathrm{Aut}(\mathbb{K})$ and $\mu \in \mathbb{K}^*$ be the accompanying automorphism and constant of τ. By 1.34, $\sigma^2 = \mathrm{id}_\mathbb{K}$, $\sigma(c) = c$ and $c^2 = \mu\sigma(\mu)$.

We may assume that $f_Q(v_2, w_2) = c^{-1}\mu$.
To abbreviate notation we write $\bar{\lambda}$ instead of $\sigma(\lambda)$ for all $\lambda \in \mathbb{K}$.

5.2.5 Definition
We define a map $\tau_{(v_1,w_1),(v_2,w_2)} \colon V \to V$ via

$$
\tau_{(v_1,w_1),(v_2,w_2)}(x) := \overline{\gamma_{v_2}(x)} c \cdot v_1 + \overline{\gamma_{v_1}(x)} \cdot v_2 + \overline{\gamma_{w_2}(x)} \cdot w_1 + \overline{\gamma_{w_1}(x)} c \cdot w_2 + \tau(p_H(x)).
$$

5.2.6 Proposition
The map $\bar{\tau} := \tau_{(v_1,w_1),(v_2,w_2)}$ is a σ-semi-linear μ-similitude, $\bar{\tau}^2 = c\,\mathrm{id}_V$ and $\bar{\tau}|_H = \tau$.

Proof For all $x, y \in V$ and $\lambda \in \mathbb{K}$ we have

$$
\begin{aligned}
\bar{\tau}(x + y) &= \overline{\gamma_{v_2}(x+y)} c \cdot v_1 + \overline{\gamma_{v_1}(x+y)} \cdot v_2 + \overline{\gamma_{w_2}(x+y)} \cdot w_1 \\
&\quad + \overline{\gamma_{w_1}(x+y)} c \cdot w_2 + \tau(p_H(x+y)) \\
&= \bar{\tau}(x) + \bar{\tau}(y)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\bar{\tau}(\lambda \cdot x) &= \overline{\gamma_{v_2}(\lambda \cdot x)} c \cdot v_1 + \overline{\gamma_{v_1}(\lambda \cdot x)} \cdot v_2 + \overline{\gamma_{w_2}(\lambda \cdot x)} \cdot w_1 + \overline{\gamma_{w_1}(\lambda \cdot x)} c \cdot w_2 \\
&\quad + \tau(p_H(\lambda \cdot x)) \\
&= \bar{\lambda}(\bar{\tau}(x)).
\end{aligned}
$$

Note that

$$
\begin{aligned}
Q(x) &= Q(\gamma_{v_1}(x) \cdot v_1 + \gamma_{v_2}(x) \cdot v_2 + \gamma_{w_1}(x) \cdot w_1 + \gamma_{w_2}(x) \cdot w_2 + p_H(x)) \\
&= f_Q(\gamma_{v_1}(x) \cdot v_1 + \gamma_{v_2}(x) \cdot v_2 + \gamma_{w_1}(x) \cdot w_1 + \gamma_{w_2}(x) \cdot w_2, p_H(x)) \\
&\quad + Q(\gamma_{v_1}(x) \cdot v_1 + \gamma_{v_2}(x) \cdot v_2 + \gamma_{w_1}(x) \cdot w_1 + \gamma_{w_2}(x) \cdot w_2) \\
&\quad + Q(p_H(x)) \\
&= f_Q(\gamma_{v_1}(x) \cdot v_1 + \gamma_{w_1}(x) \cdot w_1, \gamma_{v_2}(x) \cdot v_2 + \gamma_{w_2}(x) \cdot w_2) \\
&\quad + Q(\gamma_{v_1}(x) \cdot v_1 + \gamma_{w_1}(x) \cdot w_1) + Q(\gamma_{v_2}(x) \cdot v_2 + \gamma_{w_2}(x) \cdot w_2) \\
&\quad + Q(p_H(x)) \\
&= f_Q(\gamma_{v_1}(x) \cdot v_1, \gamma_{w_1}(x) \cdot w_1) + f_Q(\gamma_{v_2}(x) \cdot v_2, \gamma_{w_2}(x) \cdot w_2) \\
&\quad + Q(p_H(x)) \\
&= \gamma_{v_1}(x)\gamma_{w_1}(x) + c^{-1}\mu\gamma_{v_2}(x)\gamma_{w_2}(x) + Q(p_H(x)).
\end{aligned}
$$

Using this it follows that

$$
\begin{aligned}
Q(\bar{\tau}(x)) &= Q(\overline{\gamma_{v_2}(x)}c \cdot v_1 + \overline{\gamma_{v_1}(x)} \cdot v_2 + \overline{\gamma_{w_2}(x)} \cdot w_1 + \overline{\gamma_{w_1}(x)}c \cdot w_2 \\
&\quad + \tau(p_H(x))) \\
&= f_Q(\overline{\gamma_{v_2}(x)}c \cdot v_1 + \overline{\gamma_{v_1}(x)} \cdot v_2 + \overline{\gamma_{w_2}(x)} \cdot w_1 + \overline{\gamma_{w_1}(x)}c \cdot w_2, \\
&\quad \tau(p_H(x))) + Q(\tau(p_H(x))) \\
&\quad + Q(\overline{\gamma_{v_2}(x)}c \cdot v_1 + \overline{\gamma_{v_1}(x)} \cdot v_2 + \overline{\gamma_{w_2}(x)} \cdot w_1 + \overline{\gamma_{w_1}(x)}c \cdot w_2) \\
&= f_Q(\overline{\gamma_{v_2}(x)}c \cdot v_1 + \overline{\gamma_{w_2}(x)} \cdot w_1, \overline{\gamma_{v_1}(x)} \cdot v_2 + \overline{\gamma_{w_1}(x)}c \cdot w_2) \\
&\quad + Q(\overline{\gamma_{v_2}(x)}c \cdot v_1 + \overline{\gamma_{w_2}(x)} \cdot w_1) \\
&\quad + Q(\overline{\gamma_{v_1}(x)} \cdot v_2 + \overline{\gamma_{w_1}(x)}c \cdot w_2) + \mu\overline{Q(p_H(x))} \\
&= f_Q(\overline{\gamma_{v_2}(x)}c \cdot v_1, \overline{\gamma_{w_2}(x)} \cdot w_1) + f_Q(\overline{\gamma_{v_1}(x)} \cdot v_2, \overline{\gamma_{w_1}(x)}c \cdot w_2) \\
&\quad + \mu\overline{Q(p_H(x))} \\
&= c\overline{\gamma_{v_2}(x)\gamma_{w_2}(x)} + \mu\overline{\gamma_{v_1}(x)\gamma_{w_1}(x)} + \mu\overline{Q(p_H(x))} \\
&= \mu(\overline{\mu c^{-1}\gamma_{v_2}(x)\gamma_{w_2}(x) + \gamma_{v_1}(x)\gamma_{w_1}(x) + Q(p_H(x))}) = \mu\overline{Q(x)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\bar{\tau}^2(x) &= \bar{\tau}(\overline{\gamma_{v_2}(x)}c \cdot v_1 + \overline{\gamma_{v_1}(x)} \cdot v_2 + \overline{\gamma_{w_2}(x)} \cdot w_1 + \overline{\gamma_{w_1}(x)}c \cdot w_2 \\
&\quad + \tau(p_H(x))) \\
&= \gamma_{v_1}(x)c \cdot v_1 + \gamma_{v_2}(x)\bar{c} \cdot v_2 + \gamma_{w_1}(x)\bar{c} \cdot w_1 + \gamma_{w_2}(x)\bar{c} \cdot w_2 \\
&\quad + \tau^2(p_H(x)) \\
&= c\left(\gamma_{v_1}(x) \cdot v_1 + \gamma_{v_2}(x) \cdot v_2 + \gamma_{w_1}(x) \cdot w_1 + \gamma_{w_2}(x) \cdot w_2 + p_H(x)\right) \\
&= c \cdot x.
\end{aligned}
$$

<div align="right">□</div>

5.3   On projective spaces

Throughout this section let $\mathcal{S} = \mathbf{P}(V)$ be a projective space of dimension $d \geq 5$. Let $F = \{p, h\} \in \mathrm{flag}(\mathcal{G}(\mathcal{S}))$ be a flag of the projective geometry associated with $\mathcal{S}$ consisting of a point p and a hyperplane h. Let $F' = \{q, h'\}$ be a flag opposite to F and let $H \leq_{\mathbb{K}} V$ be such that $\mathcal{S}' := \mathcal{S}_{h \cap h'} = \mathbf{P}(H)$. Let $v, w \in V$ be such that $p = \langle v \rangle$ and $q = \langle w \rangle$.

We choose a Basis $\mathcal{B}_H$ of H and let $\mathcal{B} := \mathcal{B}_H \cup \{v, w\}$. Let $p_H \colon V \to H$ be the projection onto H with respect to the basis $\mathcal{B}$, let $\gamma_v \colon V \to \langle v \rangle$ and $\gamma_w \colon V \to \langle w \rangle$ be the projections onto $\langle v \rangle$ and $\langle w \rangle$ with respect to the basis $\mathcal{B}$ respectively.

We suppose that there is a polarity π of $\mathcal{S}'$ such that there exist at least three points of $\mathcal{S}'$ which are absolute with respect to π such that there are no absolute subspaces of dimension higher than 0. According to 3.2.9 there exists a reflexive sesquilinear form f on H inducing the polarity π.

Let $\bar{f} := f_{v,w}$ be the reflexive non-degenerate sesquilinear form on V defined in 5.1.2 and let $\bar{\pi}$ be the polarity of $\mathcal{S}$ induced by $\bar{f}$.

5.3.1 Lemma
A point $z = \langle x \rangle$ of $\mathcal{S}$ is contained in $\bar{\pi}(p)$ if and only if $\gamma_w(x) = 0_{\mathbb{K}}$. It is contained in $\bar{\pi}(q)$ if and only if $\gamma_v(x) = 0_{\mathbb{K}}$.

Proof

Let $x \in V$ and let $z = \langle x \rangle$ be the corresponding point of $\mathcal{S}$. Then

$$
\begin{aligned}
z \in \bar{\pi}(p) &\Leftrightarrow \bar{f}(v, x) = 0_{\mathbb{K}} \\
&\Leftrightarrow \varepsilon \gamma_v(v) \overline{\gamma_w}(x) + \gamma_w(v) \overline{\gamma_v}(x) + f(p_H(v), p_H(x)) = 0_{\mathbb{K}} \\
&\Leftrightarrow \varepsilon \overline{\gamma_w}(x) = 0_{\mathbb{K}} \Leftrightarrow \gamma_w(x) = 0_{\mathbb{K}}
\end{aligned}
$$

where the last equivalence follows since $\varepsilon \neq 0_{\mathbb{K}}$. Similarly we have

$$
z \in \bar{\pi}(q) \Leftrightarrow \gamma_v(x) = 0_{\mathbb{K}}.
$$

$\square$

5.3.2 Corollary
(a) The points p and q are absolute with respect to $\bar{\pi}$.

(b) The unique line through p and q is not absolute.

(c) If z is a point of $\mathcal{S}'$ which is absolute with respect to π, it is also absolute with respect to $\bar{\pi}$ and the unique line of $\mathcal{S}$ through p and z is absolute.

(d) $\bar{\pi}(p) = h$.

Proof In view of 3.2.11(b), parts (a)-(c) follow from 5.3.1. By 5.3.1 we have $\bar{\pi}(p) = \mathbf{P}(\mathrm{Ker}(\gamma_w))$ is a $(d-1)$-dimensional projective space. Since $\langle H, v \rangle \subseteq \mathrm{Ker}(\gamma_w)$ we conclude that $h = \mathbf{P}(\langle H, v \rangle) \subseteq \bar{\pi}(p)$. Since $\bar{\pi}(p)$ is a hyperplane, equality holds.   □

5.3.3 Lemma
For any point z of $\mathcal{S}'$ we have $\bar{\pi}(z) = \langle \pi(z), p, q \rangle$.

Proof Let $x \in H$ such that $z = \langle x \rangle$. Since $\bar{f}$ is an extension of f we have by definition

$$\pi(z) = \{ y \in H \mid f(x, y) = 0_{\mathbb{K}} \} \subseteq \{ u \in V \mid \bar{f}(x, u) = 0_{\mathbb{K}} \} = \bar{\pi}(z).$$

Since p and q are contained in $\bar{\pi}(z)$ by 5.3.1 and since $\dim(\pi(z)) = d - 3$ while $\dim(\bar{\pi}(z)) = d - 1$ we conclude that $\bar{\pi}(z) = \langle \pi(z), p, q \rangle$.   □

5.3.4 Lemma
Let U be a subspace of $\mathcal{S}$ which is absolute with respect to $\bar{\pi}$. Then U is a point or a line.

Proof Let U be a subspace of $\mathcal{S}$ which is absolute with respect to $\bar{\pi}$ and suppose that $\dim(U) \geq 1$.

Assume that $p \in U$. Then $p \in U \subseteq \bar{\pi}(U) \subseteq \bar{\pi}(p)$ an hence $U \in res(\{p, \bar{\pi}(p)\})$. Thus $U' := U \cap \bar{\pi}(q)$ is a subspace of $\mathcal{S}'$ with $\dim(U') = \dim(U) - 1$.
By definition $\pi(U') = \{ x \in H \mid f(y, x) = 0_{\mathbb{K}} \ \forall y \in U' \}$. Let $u, u' \in U' \subseteq U$. Since U is presumed to be absolute with respect to $\bar{\pi}$ we have

$$0_{\mathbb{K}} = \bar{f}(u, u') = f(u, u')$$

and hence $U' \subseteq \pi(U')$, i.e. U' is absolute with respect to π. Our assumption yields that U' has to be a point. Hence, since $\dim(U) = \dim(U') + 1$, U is a line.

Now suppose that $p \notin U$. Then $U \subseteq \bar{\pi}(U) \not\subseteq \bar{\pi}(p)$. Let $z \in U$ be a point which is not contained in $\bar{\pi}(p)$. Note that z is absolute with respect to $\bar{\pi}$ by 3.2.11. Since $z \notin \bar{\pi}(p)$ the two flags $F_p := \{p, \bar{\pi}(p)\}$ and $F_z := \{z, \bar{\pi}(z)\}$ are opposite, stabilized by $\bar{\pi}$ and, by construction, $U \in res(F_z)$.
Let $U' := \mathrm{proj}_{F_p}^{F_z}(U)$. Then, applying 4.3.13 and using the fact that $U \subseteq \bar{\pi}(U)$, we obtain

$$U' = \mathrm{proj}_{F_p}^{F_z}(U) \subseteq \mathrm{proj}_{F_p}^{F_z}(\bar{\pi}(U)) = \bar{\pi}(\mathrm{proj}_{F_p}^{F_z}(U)) = \bar{\pi}(U').$$

Thus, U' is a subspace containing p which is absolute with respect to $\bar{\pi}$. In view of the first part of this proof, U' is a line and the assertion follows since $1 = \dim(U') = \dim(U)$. $\qquad\square$

5.3.5 Lemma
There exist at least three lines containing p which are absolute with respect to $\bar{\pi}$.

Proof By our assumption there are at least three points z_1, z_2, z_3 of $\mathcal{S}'$ which are absolute with respect to π. For each $1 \leq i \leq 3$ the subspace $\langle p, x_i \rangle$ is a line containing p. In view of 5.3.2 each of these lines is absolute with respect to $\bar{\pi}$. $\qquad\square$

5.3.6 Proposition
The polar space $\mathcal{S}_{\bar{\pi}}$ is thick, non-degenerate and of finite rank $n = 2$.

Proof

It follows from 5.3.4 that $\mathrm{rk}(\mathcal{S}_{\bar{\pi}}) = 2$.

Consider the points p and q. Since $p \notin \bar{\pi}(q)$, p and q are non-collinear.
Now let z be any point of $\mathcal{S}$ which is absolute with respect to $\bar{\pi}$. If z is non-collinear with p or with q there is nothing to do. So suppose that z is collinear with both, p and q. By 5.3.1, $z = \langle x \rangle$ for some $x \in H$ and it is absolute with respect to π. By assumption there exists a second point z' of $\mathcal{S}'$ which is absolute with respect to π. The line through z and z' is not absolute with respect to π. Hence z' is a point of $\mathcal{S}$ which is absolute with respect to $\bar{\pi}$ and non-collinear with z.

By 5.3.5 there exists an absolute point which is incident with at least three absolute lines. Let l be an absolute line. Since l is incident with at least three points and all these points are absolute by 3.2.11, it follows from 3.3.4 that $\mathcal{S}_{\bar{\pi}}$ is thick. $\qquad\square$

5.3.7 Corollary
Let $\mathcal{S}$ be a projective space of finite dimension $d \geq 3$, let F and F' be opposite flags of the associated projective geometry $\mathcal{G}(\mathcal{S})$ each consisting of a point and a hyperplane. Let π be a polarity of the projective space associated to the geometry $\mathcal{R}es(F)$ and suppose that there are at least three points which are absolute with respect to π but no higher dimensional absolute subspaces. Then π can be extended to a polarity $\bar{\pi}$ of $\mathcal{S}$ stabilizing the flag F such that the polar space defined by $\bar{\pi}$ is thick, non-degenerate and of finite rank 2.

5.4   On polar spaces I

Throughout this section let $\mathcal{S} = \mathbf{P}(\Lambda)$ be a polar space defined over a finite-dimensional regular quadratic space $\Lambda = (\mathbb{K}, V, Q)$ of Witt index $k \geq 4$.

Let p, q be two non-collinear points of $\mathcal{S}$, let $v, w \in V$ be such that $\langle v \rangle = p$ and $\langle w \rangle = q$. Set $H := \mathbb{H}(v, w)^{\perp}$ and $\Lambda' := (\mathbb{K}, H, Q|_H)$. With this setup we have $\mathcal{S}_{pq} = \mathbf{P}(\Lambda')$ (cf. 3.3.13).

Suppose that there is an involution $\alpha \in \mathrm{Aut}(\mathcal{S}_{pq})$ which fixes at least three pairwise non-collinear points but no higher rank singular subspaces. By 3.3.12(b) there exist $\sigma \in \mathrm{Aut}(\mathbb{K})$ and $\mu \in \mathbb{K}$ such that α is induced by a σ-semi-linear μ-similitude $\tau \in \Gamma O(\Lambda')$.

Since α is an involution, $\tau^2 \in \mathrm{Ker}(\varphi_{\Lambda'}) = \mathrm{HT}(H)$. In particular, there exists $c \in \mathbb{K}$ such that $\tau^2 = c\,\mathrm{id}_H$.

Let x, y be two non-collinear points of $\mathcal{S}_{pq}$ such that $\alpha(x) = x$ and $\alpha(y) = y$ and let $u_1, u_2 \in H$ such that $\langle u_1 \rangle = x$ and $\langle u_2 \rangle = y$. We may assume that u_1 and u_2 were chosen in such a way that $f_Q(u_1, u_2) = 1$. Let $\lambda_1, \lambda_2 \in \mathbb{K}$ such that $\tau(u_i) = \lambda_i u_i$ for $i = 1, 2$. Then

$$cu_i = \tau^2(u_i) = \lambda_i \sigma(\lambda_i) u_i$$

and whence $\lambda_i \sigma(\lambda_i) = c$ for $i = 1, 2$. Moreover,

$$\begin{aligned}
\lambda_1 \lambda_2 &= f_Q(\lambda_1 u_1, \lambda_2 u_2) = f_Q(\tau(u_1), \tau(u_2)) \\
&= Q(\tau(u_1) + \tau(u_2)) - Q(\tau(u_1)) - Q(\tau(u_2)) \\
&= Q(\tau(u_1 + u_2)) = \mu\sigma(Q(u_1 + u_2)) = \mu\sigma(f_Q(u_1, u_2)) = \mu,
\end{aligned}$$

since u_1 and u_2 are isotropic.

Let $\bar{\tau} := \tau_{(v,w),(\lambda_1,\lambda_2)} \in \Gamma O(\Lambda)$ be the σ-semi-linear μ-similitude defined in 5.2.2 and let $\varphi := \varphi_{\Lambda}(\bar{\tau}) \in \mathrm{Aut}(\mathcal{S})$ be the automorphism induced by $\bar{\tau}$.

5.4.1 Proposition
The automorphism $\varphi \in \mathrm{Aut}(\mathcal{S})$ has the following properties:

(a) φ is an involution.

(b) φ fixes the points p and q.

(c) φ stabilizes three lines through p.

(d) The line through p and x is incident with three fixed points.

(e) If U is a singular subspace of $\mathcal{S}$ which is stabilized by φ, then U is a point or a line.

(f) Each line of $\mathcal{S}$ which is stabilized by φ is incident with at least two fixed points.

Proof

(a) By 3.3.12(a) we have

$$\mathrm{id}_{\mathcal{S}} = \varphi_\Lambda(c\,\mathrm{id}_V) = \varphi_\Lambda(\bar{\tau}^2) = \varphi_\Lambda(\bar{\tau})^2.$$

(b) By definition we have

$$\varphi(p) = \langle \bar{\tau}(v) \rangle = \langle \lambda_1 v \rangle = \langle v \rangle = p$$

and similarly, since $\bar{\tau}(w) = \lambda_2 w$, $\varphi(q) = q$.

(c) By assumption there exist three points $x_1, x_2, x_3 \in \mathcal{S}_{pq}$ which are fixed by α and hence by φ. For $i = 1, 2, 3$ let l_i be the unique line through p and x_i. Since x_i is non-collinear with x_j for all $1 \le i \ne j \le 3$ the lines l_i are pairwise distinct. Now $\varphi(l_i)$ is a line through the points $\varphi(p) = p$ and $\varphi(x_i) = \alpha(x_i) = x_i$, hence $\varphi(l_i) = l_i$.

(d) Recall that $x = \langle u_1 \rangle$ and $\bar{\tau}(u_1) = \tau(u_1) = \lambda_1 u_1$. By definition we have $p = \langle v \rangle$ and $\bar{\tau}(v) = \lambda_1 v$. The unique line l through p and x is given by $l = \langle u_1, v \rangle$. Consider the point $z := \langle u_1 + v \rangle$. Then z is incident with l and fixed by φ since $\bar{\tau}(u_1 + v) = \lambda_1(u_1 + v)$.

(e) Let U be a singular subspace of $\mathcal{S}$ which is stabilized by φ and suppose that $k := \mathrm{rk}(U) \ge 3$.
If $U \subseteq p^\perp \cap q^\perp$, the assumption yields $\mathrm{rk}(U) = 1$, a contradiction. So $U \not\subseteq q^\perp$ or $U \not\subseteq p^\perp$. Suppose that $U \not\subseteq q^\perp$ but $U \subseteq p^\perp$. Then, by 3.3.7, $U' := U \cap q^\perp$ is a singular subspace of $\mathcal{S}$ of rank $k - 1 \ge 2$. By construction, U' is a subspace of $\mathcal{S}_{pq}$ which is stabilized by α and hence $\mathrm{rk}(U') \le 1$. Again, this is impossible. As the same contradiction arises in the case $U \not\subseteq p^\perp$ and $U \subseteq q^\perp$ we conclude that $U \not\subseteq p^\perp \cup q^\perp$. Let $x \in U$ be a point such that x and p are non-collinear. Note that $U \subseteq x^\perp$ and that $\varphi(x) \in U$ is a point non-collinear with $\varphi(p) = p$. Let $U' := \mathrm{proj}_p^x(U)$. Then, in view of 4.2.5 and 4.3.12,

$$\varphi(U') = \varphi(\mathrm{proj}_p^x(U)) = \mathrm{proj}_p^{\varphi(x)}(\varphi(U)) = \mathrm{proj}_p^{\varphi(x)}(U) = U'.$$

Hence, U' is a singular subspace of rank k satisfying $U' \subseteq p^\perp$ which is stabilized by φ. The considerations above imply that such a subspace does not exist.

(f) Let l be a line of $\mathcal{S}$ which is stabilized by φ.
First assume that l is a line through p. Then q is collinear with a unique point on l and by 3.3.16 this point is a fixed point. Similarly each line stabilized by φ through q is incident with two fixed points. Suppose that l is not incident with p nor with q. Let z and z' be

the unique point on l which is collinear with p and q respectively. If $z \neq z'$ then the assertion follows since z and z' are fixed by φ by 3.3.16. Otherwise $z = z'$ is a fixed point of $\mathcal{S}_{pq}$ and we may choose another fixed point $y \in \mathcal{S}_{pq}$. Since z and y cannot be collinear (otherwise there would exist a line in $\mathcal{S}_{pq}$ which is stabilized by α) there is a unique point on l collinear with y and this point is fixed according to 3.3.16.

$\square$

We have seen that there are points and lines which are stabilized by φ. Moreover, as each such line is incident with at least two fixed points, we are able to make the following definition:

5.4.2 Definition
We define a point-line space $\mathcal{S}_\varphi = (P_\varphi, L_\varphi)$ via

$$P_\varphi := \{p \in P \mid \varphi(p) = p\}$$

and

$$L_\varphi := \{l \in L \mid \varphi(l) = l\}.$$

5.4.3 Proposition
The point-line space $\mathcal{S}_\varphi$ is a thick, non-degenerate polar space of rank 2.

Proof Let l be a line of $\mathcal{S}_\varphi$ and let p be a point of $\mathcal{S}_\varphi$ not on l. Since l is a line of $\mathcal{S}$ and p is a point of $\mathcal{S}$ axiom (P_1) gives that p is either incident with all points on l or with precisely one point on l. In particular, in the first case, p is collinear with all fixed points on l. If p is collinear with a unique point on l, this point needs to be a fixed point by 3.3.16. Thus (P_1) is satisfied.

Recall that the points p and q are non-collinear. So let x be a point of $\mathcal{S}_\varphi$ collinear with p and q. Then x is a point of $\mathcal{S}_{pq}$ which is fixed by α. Since there are at least three points in $\mathcal{S}_{pq}$ which are fixed by α and since all such points are pairwise non-collinear, there is a fixed point non-collinear with x. It follows from 5.4.1(b) that $\mathcal{S}_\varphi$ is of rank 2.

According to 5.4.1 (c) and (d) the point p is incident with three φ-invariant lines and at least one of these lines is incident with at least three fixed points. According to 3.3.4 the polar space $\mathcal{S}_\varphi$ is thick. $\square$

5.4.4 Corollary
Let $\mathcal{S}$ be a non-degenerate polar space defined over a quadratic space of finite rank $n \geq 4$ and let p and q be two non-collinear points of $\mathcal{S}$. Let α be an involutory automorphism of the polar space $\mathcal{S}_{pq}$ and suppose that α fixes at least three points but no singular subspaces of higher rank. Then α can be extended to an involutory automorphism $\bar{\alpha}$ of $\mathcal{S}$ fixing the point p such that the polar space defined by the fixed points and lines of $\bar{\alpha}$ is thick, non-degenerate and of finite rank 2.

5.5   On polar spaces II

Throughout this section let $\mathcal{S}$ be a polar space of type D of finite rank $n \geq 6$ and let $\Lambda = (\mathbb{K}, V, Q)$ be a hyperbolic quadratic space such that $\mathcal{S} \simeq \mathbf{P}(V)$.

Let l, g be two lines of $\mathcal{S}$ such that (l, g) is a hyperbolic pair of $\mathcal{S}$ and let p be a point incident with l. By 3.3.6 there exists a unique point q on g collinear with p. Choose a second point p' on l and let q' be the unique point on g collinear with p'.
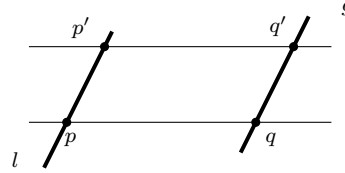
Let $v_1, w_1, v_2, w_2 \in V$ be such that $\langle v_1 \rangle = p$, $\langle v_2 \rangle = p'$, $\langle w_1 \rangle = q'$ and $\langle w_2 \rangle = q$. Since the pair (p, q') is a hyperbolic pair of $\mathcal{S}$, the vectors v_1 and w_1 can be chosen such that (v_1, w_1) is a hyperbolic pair of Λ. Similarly, we may assume that (v_2, w_2) is also a hyperbolic pair of Λ. Set $H := \mathbb{H}(v_1, w_1)^\perp \cap \mathbb{H}(v_2, w_2)^\perp$ and $\Lambda' := (\mathbb{K}, H, Q|_H)$. With this setup we have $\mathcal{S}_{(l,g)} = \mathbf{P}(\Lambda')$ (applying 3.3.13 twice).

Suppose that there is an involution $\alpha \in \mathrm{Aut}(\mathcal{S}_{(l,g)})$ which fixes at least three pairwise opposite lines but no points or higher rank singular subspaces. By 3.3.12(b), there exist $\sigma \in \mathrm{Aut}(\mathbb{K})$ and $\mu \in \mathbb{K}$ such that α is induced by a σ-semi-linear μ-similitude $\tau \in \Gamma O(\Lambda')$.

Since α is an involution, $\tau^2 \in \mathrm{Ker}(\varphi_{\Lambda'}) = \mathrm{HT}(H)$. In particular, there exists $c \in \mathbb{K}$ such that $\tau^2 = c\,\mathrm{id}_H$.

We may assume that $f_Q(v_2, w_2) = c^{-1}\mu$. Let $\bar{\tau} := \tau_{(v_1, w_1), (v_2, w_2)} \in \Gamma O(\Lambda)$ be the σ-semi-linear μ-similitude defined in 5.2.5 and let $\varphi := \varphi_\Lambda(\bar{\tau}) \in \mathrm{Aut}(\mathcal{S})$ be the automorphism induced by $\bar{\tau}$. Note that, by 1.34(c), $\sigma(c) = c$. For reasons of brevity we write $\bar{\lambda}$ instead of $\sigma(\lambda)$ for all $\lambda \in \mathbb{K}$.

5.5.1 Lemma
(a) φ is an involution.

(b) There exists no $\lambda \in \mathbb{K}$ such that $\lambda\bar{\lambda} = c$.

Proof Part (a) is just the same as the proof of 5.4.1(a).
Let h be a line of the polar space $\mathcal{S}_{(l,g)}$ which is stabilized by α and choose a point x on h. By assumption $\alpha(x)$ is a point on h different from x. Let $v \in H$ be isotropic such that $x = \langle v \rangle$. Then $\alpha(x) = \langle \tau(v) \rangle$. Let $\lambda \in \mathbb{K}$ such that $\lambda\bar{\lambda} = c$ and consider the isotropic vector $u := \lambda v + \tau(v) \in H$. Then

$$\tau(u) = \tau(\lambda v) + \tau(\tau(v)) = \bar{\lambda}\tau(v) + c \cdot v = \bar{\lambda}(\tau(v) + \lambda v) = \bar{\lambda}u$$

and the corresponding point $\langle u \rangle$ of $\mathcal{S}_{(l,g)}$ is fixed by α. But such points do not exist.

$\square$

5.5.2 Lemma

The involution φ has the following properties:

(a) φ does not fix any points.

(b) $\varphi(p) = p'$ and $\varphi(q) = q'$.

(c) φ stabilizes the lines l and g.

Proof

(a) Let x be a point of $\mathcal{S}$ and suppose that $\varphi(x) = x$. Let $u \in V$ such that $x = \langle u \rangle$. Then there exists $\lambda \in \mathbb{K}$ such that $\bar{\tau}(u) = \lambda u$. Thus

$$cu = \bar{\tau}^2(u) = \bar{\tau}(\lambda u) = \bar{\lambda}\bar{\tau}(u) = \bar{\lambda}\lambda u,$$

but this is a contradiction to 5.5.1(a).

(b) This follows from the definition of $\bar{\tau}$, since $\bar{\tau}(v_1) = v_2$ and $\bar{\tau}(w_2) = w_1$.

(c) According to (b) we have $\varphi(p) = p'$. Since φ is an involution we also have $\varphi(p') = \varphi^2(p) = p$. Thus, $\varphi(l)$ is a line through p and p'. Whence, $\varphi(l) = l$. Similarly $\varphi(g) = g$.

$\square$

5.5.3 Lemma

Let U be a singular subspace of $\mathcal{S}$ which is stabilized by φ. Then either U is a line or $\mathrm{rk}(U) = 4$.

Proof   According to 5.5.2(a) the automorphism φ does not fix any points of $\mathcal{S}$. Let U be a singular subspace of $\mathcal{S}$ which is stabilized by φ with $k := \mathrm{rk}(U) \geq 3$.

First suppose that $l \subseteq U$. Since $p \in U$ and since q' is non-collinear with p, the set $U_1 := U \cap q'^{\perp}$ is a singular subspace of $\mathcal{S}$ of rank $k - 1 \geq 2$ (cf. 3.3.7). Note that $p' \in U_1$ and that q is a point non-collinear with p'. Applying again 3.3.7, we obtain that $U_2 := U_1 \cap q^{\perp}$ is a singular subspace of $\mathcal{S}$ of rank $k - 2 \geq 1$. By construction, U_2 is a singular subspace of $\mathcal{S}_{(l,g)}$ which is stabilized by α. Hence $\mathrm{rk}(U') = 2$ and thus $k = \mathrm{rk}(U) = 4$.

If $l \nsubseteq U$ we have $|U \cap l| \leq 1$. Suppose that $U \cap l = \{z\}$ for some point z of $\mathcal{S}$. Since U and l are stabilized by φ, the intersection-point z needs to be fixed by φ. But this is impossible and hence $U \cap l = \emptyset$.
If all points of U are collinear with all points on l, consider the singular subspace $M := \langle U, l \rangle$. It contains the line l, is φ-invariant by 3.3.17 and, using

the dimension formula for projective spaces, $\mathrm{rk}(M) \geq 5$. The considerations above show that such a subspace does not exist and we conclude that there exists a point $x \in U$ which is not collinear with all points on l. Let x_l be the unique point on l collinear with x.

As U is φ-invariant, $\varphi(x) \in U$ and since U is singular, the points x and $\varphi(x)$ are collinear. Let h denote the unique line through x and $\varphi(x)$. Since φ preserves collinearity, $\varphi(x)$ is collinear with a unique point on l, namely $\varphi(x_l) \neq x_l$.
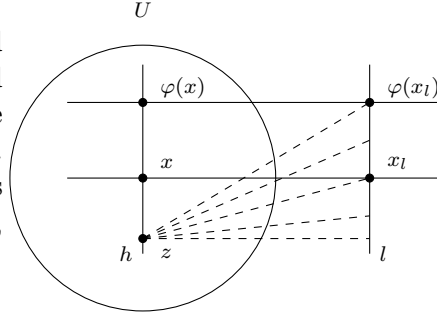
The pair (h, l) is a hyperbolic pair of $\mathcal{S}$: If there exists $z \in h \cap l^\perp$, the point x_l has to be collinear with all points on h, in particular with $\varphi(x)$. But this is a contradiction to the fact that x_l and $\varphi(x)$ are non-collinear. A similar argument shows that $l \cap h^\perp = \emptyset$. Note that $U \in res(h)$ and set $U' := \mathrm{proj}_l^h(U)$. Then, using 4.2.5,

$$\varphi(U') = \varphi(\mathrm{proj}_l^h(U)) = \mathrm{proj}_{\varphi(l)}^{\varphi(h)}(\varphi(U)) = \mathrm{proj}_l^h(U) = U'.$$

Hence, U' is a φ-invariant singular subspace of $\mathcal{S}$ containing the line l. According to the considerations above $\mathrm{rk}(U) = \mathrm{rk}(U') = 4$. $\square$

5.5.4 Lemma
There exist at least three φ-invariant singular subspaces of rank 4 which contain the line l.

Proof By assumption there exist three pairwise opposite lines in $\mathcal{S}_{(l,g)}$ which are stabilized by α and hence by φ. Since each of these lines is contained in $l^\perp$, each such line, together with the line l, spans a singular subspace of $\mathcal{S}$ which is φ-invariant by 3.3.17 and is thus of rank 4 according to 5.5.3. Note that the subspaces are pairwise disjoint, since the lines are assumed to be pairwise opposite. $\square$

We have seen that there are lines and singular subspaces of rank 4 which are stabilized by φ. Moreover, according to 3.3.18, each φ-invariant singular subspace of rank 4 is incident with at least two φ-invariant lines. This enables us to make the following definition:

5.5.5 Definition
We define a point-line-space $\mathcal{S}_\varphi = (P_\varphi, L_\varphi)$ via

$$P_\varphi = \{l \in L \mid \varphi(l) = l\}$$

and
$$L_\varphi := \{U \in \mathcal{V}(\mathcal{S}) \mid rk(U) = 4, \varphi(U) = U\}$$

by claiming that a line $l \in P_\varphi$ is incident with a subspace $U \in L_\varphi$ if and only if $l \subseteq U$. Furthermore, two lines $l, l' \in P_\varphi$ are collinear if and only if the set $l \cup l'$ is singular.

5.5.6 Lemma
Let l and l' be φ-invariant lines of $\mathcal{S}$ and let U be a φ-invariant singular subspace of $\mathcal{S}$ of rank 4.

(a) If $l \neq l'$ we have $l \cap l' = \emptyset$.

(b) If the set $l \cup l'$ is not singular, then the pair (l, l') is a hyperbolic pair.

(c) If $l \not\subseteq U$, then $l \cap U = \emptyset$.

Proof

(a) If $l \neq l'$ the intersection $l \cap l'$ is a single point x. Now

$$\varphi(x) = \varphi(l \cap l') \subseteq \varphi(l) \cap \varphi(l') = l \cap l' = \{x\},$$

which is a contradiction to 5.5.2(a).

(b) Suppose that the set $l \cup l'$ is not singular, in particular, $l \neq l'$. Then there exist $x, x' \in l \cup l'$ such that x and x' are non-collinear. As l and l' are both singular, we may assume that $x \in l$ and $x' \in l'$. Because of (P_1) the point x is collinear with a unique point on l'.
Suppose that there exists a point $y \in l^\perp \cap l'$. Then x and y are collinear. Whence, y is the unique point on l' collinear with x. Since φ preserves collinearity and as l' is φ-invariant, also $\varphi(y)$ is a point on l' collinear with all points on l. In particular, $\varphi(y)$ is collinear with x. Thus, $\varphi(y) = y$. The singular subset $\{y\} \cup l$ spans a singular subspace of $\mathcal{S}$ which is φ-invariant by 3.3.17 and, since $y \notin l$ by (a), this subspace is of rank 3. But this is a contradiction to 5.5.3. Hence $l^\perp \cap l' = \emptyset$.

(c) Suppose that $l \not\subseteq U$. Then $|l \cap U| \leq 1$. Suppose that the intersection $l \cap U$ consists of a single point x of $\mathcal{S}$. Again

$$\varphi(x) = \varphi(l \cap U) \subseteq \varphi(l) \cap \varphi(U) = l \cap U = \{x\}.$$

Because of 5.5.2(a) this is impossible.

$\square$

5.5.7 Proposition
The point-line-space $\mathcal{S}_\varphi$ is a thick, non-degenerate polar space of rank 2.

Proof Let U be a singular subspace of $\mathcal{S}$ which is φ-invariant and of rank 4. Let l be a line stabilized by φ and suppose that U and l are not incident. In view of 5.5.6(c) this implies $l \cap U = \emptyset$.

Furthermore assume that there exists a line $h \subseteq U$ which is φ-invariant such that the set $l \cup h$ is not singular. Choose a point $x \in l$. Since (l, h) is a hyperbolic pair, there exists a unqiue point y on h such that x and y are collinear. In particular, x is not collinear with all points of U. By 3.3.7, therefore, $U_1 := U \cap x^\perp$ is a singular subspace of rank 3. Note that $y \in U_1$ and that $\varphi(x)$ is a point on l which is non-collinear with y (since x is the unique point on l collinear with y). Hence, $U_2 := U_1 \cap \varphi(x)^\perp$ is a singular subspace of rank 2. Moreover, U_2 is φ-invariant: Indeed, let $z \in U_2$. Then, as U is φ-invariant and φ respects collinearity, $\varphi(z) \in U$ and $\varphi(z) \in x^\perp$, i.e. $\varphi(z) \in U_1$. Moreover, as $z \in U_1 \subseteq x^\perp$, we conclude that $\varphi(z) \in \varphi(x)^\perp$ and hence $\varphi(z) \in U_2$. In particular, any point $z \in U_2$ is collinear with both, x and $\varphi(x)$ and hence it is with all points of l. Thus, $U_2 \subseteq l^\perp$ and $U_2 \cup l$ is a singular subset.

Suppose that there are two lines $g, g' \subseteq U$ which are φ-invariant such that the sets $g \cup l$ and $g' \cup l$ are singular. Since the set $g \cup g' \cup l$ is singular, too, the subspace $M := \langle g \cup g' \cup l \rangle$ is a singular subspace of $\mathcal{S}$ which is φ-invariant. Using the dimension formula for projective spaces, we obtain that $\dim(M) \geq 5$, i.e. M is a singular subspace of rank at least 5 which is φ-invariant. This is a contradiction to 5.5.3. Hence there is at most one φ-invariant line $g \subseteq U$ such that $g \cup l$ is singular. We conclude that $\mathcal{S}_\varphi$ is a polar space.

Let h be a φ-invariant line of $\mathcal{S}$. Then either one of the pairs (h, l) or (h, g) is a hyperbolic pair of $\mathcal{S}$ or h is a line of the polar space $\mathcal{S}_{(l,g)}$. In the first case, h is non-collinear with l or g. If $l \subseteq l^\perp \cap g^\perp$ there exists by assumption a line $h' \subseteq l^\perp \cap g^\perp$ which is φ-invariant and non-collinear with h. Thus, $\mathcal{S}_\varphi$ is non-degenerate.

According to 5.5.3, $\mathrm{rk}(\mathcal{S}_\varphi) = 2$ and by 3.3.18, 5.5.4 and 3.3.4, the polar space $\mathcal{S}_\varphi$ is thick. $\qquad\square$

5.5.8 Corollary
Let $\mathcal{S}$ be a quadratic space of type D of finite rank $n \geq 5$ and let l and g be two opposite lines of $\mathcal{S}$. Let α be an involutory automorphism of the polar space $\mathcal{S}_{(l,g)}$ and suppose that α stabilizes at least three pairwise opposite lines but no points or singular subspaces of higher rank. Then α can be extended to an involutory automorphism φ of $\mathcal{S}$ stabilizing the line l such that the polar space defined by the fixed lines and singular subspaces of rank 4 is thick, non-degenerate and of finite rank 2.

5.6   On buildings of type A_n

5.6.1 Proposition

Let Δ be a thick building of type A_n for some $n \geq 3$, let $\Omega \in \mathrm{Aut}(\Delta)$ be an involution such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with corresponding Tits index
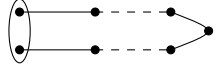
Let $\mathcal{S}$ be a projective space of finite rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}(\mathcal{S}))$. Then there exists a polarity π of $\mathcal{S}$ inducing Ω on Δ and this polarity has the following properties:

(a) If U is a subspace of $\mathcal{S}$ which is absolute with respect to π, then U is a point.

(b) There exist at least three points of $\mathcal{S}$ which are absolute with respect to π.

Proof Let the Coxeter diagram A_n be labeled as in 2.3. Let p, q, r be three collinear points of $\mathcal{S} = (P, L)$, i.e. they lie on a common line $l \in L$ of $\mathcal{S}$. By assumption, $\Omega \in \mathrm{Aut}(\mathcal{G}(\mathcal{S}))$ acts bijectively on the set of subspaces of $\mathcal{S}$, maps points onto hyperplanes and $\Omega^2 = \mathrm{id}_{\mathcal{V}(\mathcal{S})}$.
Moreover, according to [MT, 4.7], $\Omega(\{p\}) \cap \Omega(\{q\}) = \Omega(\langle p, q \rangle) = \Omega(l)$ as well as $\Omega(\{p\}) \cap \Omega(\{r\}) = \Omega(l)$ and $\Omega(\{q\}) \cap \Omega(\{r\}) = \Omega(l)$. Thus, $\Omega(\{p\}) \cap \Omega(\{q\}) \cap \Omega(\{r\}) = \Omega(l)$. We conclude that $\Omega|_P$ is a polarity.

(a) Let U be a subspace of $\mathcal{S}$ which is absolute with respect to π. Then the flag $F := \{U, \pi(U)\} \in \mathrm{flag}(\mathcal{G}(\mathcal{S}))$ is stabilized by Ω and thus the corresponding residue $\mathcal{R}$ of Δ is a Γ-residue of rank $n - 2$. Thus, $\mathrm{Typ}(\mathcal{R}) = \{s_2, \ldots, s_{n-1}\}$ and $\tau(F) = \{1, n\}$. Hence, $\dim(U) = 0$.

(b) By 2.40, there are at least three Γ-chambers of Δ and each Γ-chamber is of type $A := \{s_2, \ldots, s_{n-1}\}$. Let C be a Γ-chamber, let p be a point and h a hyperplane of $\mathcal{S}$ such that C corresponds to the flag $F = \{p, h\} \in \mathrm{flag}(\mathcal{G}(\mathcal{S}))$. As C is Γ-invariant, the corresponding flag F is stabilized by π. We conclude that the point p is absolute with respect to π.
To conclude: Each Γ-chamber of Δ corresponds to a point of $\mathcal{S}$ which is absolute with respect to π.

$\square$

5.6.2 Theorem

Let Δ be a thick building of type A_n for some $n \geq 5$. Let Π be the Coxeter diagram A_n labeled as in 2.3, let (W, S) be the corresponding Coxeter system and let $J := S \backslash \{s_1, s_n\}$. Let Δ_0 be a residue of Δ of type Π_J. Suppose that there is an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with corresponding Tits index
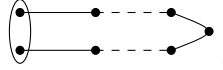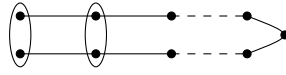
There exists an extension of Ω_0 to an involutory automorphism $\Omega \in \mathrm{Aut}(\Delta)$ such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with Tits index

and such that the fixed point building Δ^Γ is a Moufang quadrangle.

Proof Let $\mathcal{S} = (P, L)$ be a projective space of finite rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}(\mathcal{S}))$ and let $F = \{p, h\} \in \mathrm{flag}(\mathcal{G}(\mathcal{S}))$ be the flag consisting of a point p and a hyperplane h of $\mathcal{S}$ corresponding to the residue Δ_0.
Choose a flag $F' = \{q, h'\}$ which is opposite to F. Then

$$\Delta_0 \simeq \mathrm{Flag}(\mathcal{R}\mathrm{es}(\mathrm{F})) \simeq \mathrm{Flag}(\mathcal{G}(\mathcal{S}_{\mathrm{h} \cap \mathrm{h}'})).$$

According to 5.6.1, there is a polarity π of the projective space $\mathcal{S}_{h \cap h'}$ inducing the involution Ω_0 on Δ_0. Furthermore, there are at least three points of $\mathcal{S}_{h \cap h'}$ which are absolute with respect to π and all subspaces of $\mathcal{S}_{h \cap h'}$ of positive dimension ain't absolute.

By 5.3.7, the polarity π can be extended to a polarity $\bar{\pi}$ of $\mathcal{S}$ in such a way that it stabilizes the flag F and such that the polar space defined by $\bar{\pi}$ is thick, non-degenerate and of finite rank 2. Let Ω be the involutory automorphism of Δ induced by $\bar{\pi}$ and set $\Gamma := \langle \Omega \rangle$. The fixed point structure Δ^Γ is isomorphic to the flag complex $\mathrm{Flag}(\mathcal{G}(\mathcal{S}_{\bar{\pi}}))$, which is a thick building of type C_2. Hence, by 2.42, Γ is a descent group of Δ with Tits index $\mathbf{T} = (\mathsf{A}_n, \langle \mathrm{op}_S \rangle, \{s_3, \cdots, s_{n-2}\})$. As Δ satisfies the Moufang condition, 2.41 implies that Δ^Γ is a Moufang quadrangle.

$\square$

5.7 On buildings of type C_n

5.7.1 Proposition

Let Δ be a building of type C_n for some $n \geq 3$, let $\Omega \in \mathrm{Aut}(\Delta)$ be an involution such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with corresponding Tits index
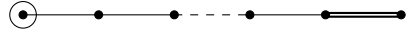
Let $\mathcal{S}$ be a non-degenerate polar space of finite rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}(\mathcal{S}))$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(\mathcal{S})$ inducing Ω on Δ and it has the following properties:

(a) If U is a singular subspace of $\mathcal{S}$ which is α-invariant, then U is a point.

(b) There exist at least three points of $\mathcal{S}$ which are fixed by α. Moreover these points are pairwise non-collinear.

Proof Let Π be the Coxeter diagram C_n labeled as in 2.3. By assumption, $\Omega \in \mathrm{Aut}(\mathcal{G}(\mathcal{S}))$ acts bijectively on the set of singular subspaces $\mathcal{V}(\mathcal{S})$ of $\mathcal{S}$ and $\Omega^2 = \mathrm{id}_{\mathcal{V}(\mathcal{S})}$.
According to [MT, 5.14], Ω preserves inclusion and $\alpha := \Omega|_P \in \mathrm{Aut}(\mathcal{S})$.

(a) Let U be a subspace of $\mathcal{S}$ which is stabilized by α and let $i := \mathrm{rk}(U)$. Then U corresponds to a residue of Δ of type $S \backslash \{s_i\}$ which is stabilized by Ω. We conclude that $s_i = s_1$, i.e. U has rank 1 and thus is a point.

(b) By assumption each Γ-chamber of Δ corresponds to a point of $\mathcal{S}$ which is fixed by α. If two of such points would be collinear, α would stabilize the line through these points. But this is a contradiction to (a).

$\square$

5.7.2 Theorem

Let $\Lambda = (\mathbb{K}, V, Q)$ be a regular but not hyperbolic quadratic space of Witt index $n \geq 4$, let $\mathcal{S} := \mathbf{P}(\Lambda)$ be the associated non-degenerate polar space of rank n and let $\Delta := \mathrm{Flag}(\mathcal{G}(\mathcal{S}))$ be the associated thick spherical building of type C_n. Let Δ_0 be a residue of Δ of type C_{n-1} and suppose that there is an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with corresponding Tits index

There exists an extension of Ω_0 to an involutory automorphism $\Omega \in \mathrm{Aut}(\Delta)$ such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with Tits index

and such that the fixed point building Δ^Γ is a Moufang quadrangle.

Proof

Let p be the point of $\mathcal{S}$ corresponding to the residue Δ_0 and choose a point q of $\mathcal{S}$ which is non-collinear with p. Then

$$\Delta_0 \simeq \mathrm{Flag}(\mathcal{R}\mathrm{es}(p)) \simeq \mathrm{Flag}(\mathcal{G}(\mathcal{S}_{\mathrm{pq}})).$$

According to 5.7.1 there is an involutory automorphism $\alpha \in \mathrm{Aut}(\mathcal{S}_{pq})$ inducing the involution Ω_0 on Δ_0. Furthermore, α fixes at least three pairwise non-collinear points but stabilizes no higher rank singular subspaces of $\mathcal{S}_{pq}$. By 5.4.4, the automorphism α can be extended to an involutory automorphism $\bar{\alpha} \in \mathrm{Aut}(\mathcal{S})$ such that it fixes the point p and such that the fixed points and stabilized lines constitute a thick, non-degenerate polar space of finite rank 2. Let Ω be the involutory automorphism of Δ induced by $\bar{\alpha}$ and set $\Gamma := \langle \Omega \rangle$. The fixed point structure Δ^Γ is isomorphic to the flag complex $\mathrm{Flag}(\mathcal{G}(\mathcal{S}_\varphi))$, which is a thick building of type C_2. Hence, by 2.42, Γ is a descent group of Δ with Tits index $\mathbf{T} = (\mathsf{C}_n, \{\mathrm{id}\}, \{s_3, \ldots, s_n\})$. As Δ satisfies the Moufang condition, 2.41 implies that Δ^Γ is a Moufang quadrangle.

$\square$

5.8   On buildings of type D_n

5.8.1 Proposition

Let Δ be a thick building of type D_n for some $n \geq 5$, let $\Omega \in \mathrm{Aut}(\Delta)$ be an involutory isometry such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with corresponding Tits index

Let $\mathcal{S}$ be a polar space of type D and rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}_{\mathrm{o}}(\mathcal{S}))$. There exists an involution $\alpha \in \mathrm{Aut}(\mathcal{S})$ inducing Ω on Δ and it has the following properties:

(a) If U is a singular subspace of $\mathcal{S}$ which is α-invariant, then U is a line.

(b) There exist at least three lines of $\mathcal{S}$ which are stabilized by α. Moreover, these lines are pairwise opposite.

Proof   Let the Coxeter diagram D_n be labeled as in 2.3. By [MT, 6.2], $\Omega \in \mathrm{Aut}(\mathcal{G}_o(\mathcal{S}))$ is an inclusion-preserving permutation on the set $\mathcal{V}_o(\mathcal{S})$ of all non-trivial singular subspaces of $\mathcal{S}$ which are not submaximal. Moreover, there exist a unique automorphism $\Omega' \in \mathrm{Aut}(\mathcal{G}(\mathcal{S}))$ such that $\Omega'|_{\mathcal{V}_o(\mathcal{S})} = \Omega$.

(a) Let U be a subspace of $\mathcal{S}$ which is stabilized by α and set $k := rk(U)$. The corresponding flag $F := \{U\} \in \mathrm{flag}(\mathcal{G}(\mathcal{S}))$ is stabilized by Ω and hence the corresponding residue of Δ is Γ-invariant and of type $S \backslash \{s_k\}$. We conclude that $s_k = 2$ and hence U is a line.

(b) By 2.40 there are at least three Γ-chambers of Δ and each Γ-chamber is of type $A := S \backslash \{s_2\}$. Let C be a Γ-chamber and let l be the line of $\mathcal{S}$ such that C corresponds to the flag $F = \{l\} \in \mathrm{flag}(\mathcal{G}(\mathcal{S}))$. As C is Γ-invariant, the corresponding line l is stabilized by α.
Let l and g be two lines corresponding to different Γ-chambers. In particular, $l \cap g = \emptyset$ since there are no fixed points. Suppose that $g \cap l^{\perp} \neq \emptyset$ and let $x \in g \cap l^{\perp}$. Since g is α-invariant and α preserves collinearity, $\alpha(x) \in g \cap l^{\perp}$. Then l is a proper subspace of the singular subspace $\langle l, g \cap l^{\perp} \rangle$ which is α-invariant. But this is a contradiction to (a).

$\square$

5.8.2 Theorem

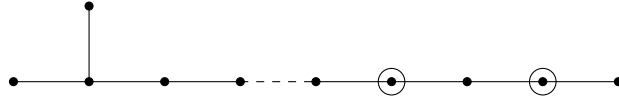Let Δ be a thick building of type D_n for some $n \geq 7$. Let Δ_0 be a residue of Δ of type D_{n-2} and suppose that there is an involutory isometry $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with corresponding Tits index
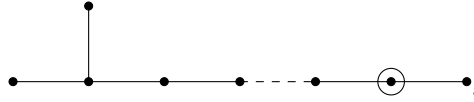
There exists an extension of Ω_0 to an involutory automorphism $\Omega \in \mathrm{Aut}(\Delta)$ such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with Tits index

and such that the fixed point structure Δ^Γ is a Moufang quadrangle.

Proof Let $\mathcal{S} = (P, L)$ be a polar space of type D and finite rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}_o(\mathcal{S}))$ and let $F = \{p, l\} \in \mathrm{flag}(\mathcal{G}_o(\mathcal{S}))$ be the flag consisting of a point p and a line l of $\mathcal{S}$ corresponding to the residue Δ_0. Choose a flag $F' = \{q, g\}$ which is opposite to F. Then

$$\Delta_0 \simeq \mathrm{Flag}(\mathcal{G}_o(\mathcal{R}\mathrm{es}(F))) \simeq \mathrm{Flag}(\mathcal{G}_o(\mathcal{S}_{(l,g)})).$$

According to 5.8.1 there is an involution $\alpha \in \mathrm{Aut}(\mathcal{S}_{(l,g)})$ inducing the involution Ω_0 on Δ_0. Furthermore, there exist at least three pairwise opposite lines of $\mathcal{S}_{(l,g)}$ which are stabilized by α but no α-invariant singular subspaces of rank not equal to two.

By 5.5.8 the involution α can be extended to an involution $\bar{\alpha} \in \mathrm{Aut}(\mathcal{S})$ in such a way that it stabilizes the line l and such that the stabilized lines and singular subspaces of rank 4 constitue a thick, non-degenerate polar space $\mathcal{S}_\alpha$ of finite rank 2. Let Ω be the involutory automorphism of Δ induced by $\bar{\alpha}$ and set $\Gamma := \langle \Omega \rangle$. The fixed point structure Δ^Γ is isomorphisc to the flag complex $\mathrm{Flag}(\mathcal{G}(\mathcal{S}_\alpha))$, which is a thick building of type C_2. Hence, Γ is a descent group of Δ with Tits index $\mathbf{T} = (\mathsf{D}_n, \{\mathrm{id}\}, \{s_1, s_3, s_5, \ldots, s_{n-1}, s_n\})$. As Δ satisfies the Moufang condition, 2.41 implies that Δ^Γ is a Moufang quadrangle. $\qquad\square$

5.8.3 Proposition

Let Δ be a thick building of type D_n for some $n \geq 5$, let $\Omega \in \mathrm{Aut}(\Delta)$ be an involuory isometry such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with corresponding Tits index
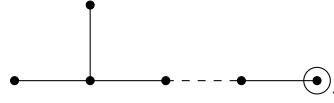
Let $\mathcal{S}$ be a polar space of type D of finite rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}_o(\mathcal{S}))$. Then there exists an involution $\alpha \in \mathrm{Aut}(\mathcal{S})$ inducing Ω on Δ and it has the following properties:

(a) If U is a singular subspace of $\mathcal{S}$ which is α-invariant, then U is a point.

(b) There exist at least three points of $\mathcal{S}$ which are fixed by α. Moreover these points are pairwise non-collinear.

Proof Let Π be the Coxeter diagram D_n labeled as in 2.3.
By [MT, 6.2], $\Omega \in \mathrm{Aut}(\mathcal{G}_o(\mathcal{S}))$ is an inclusion-preserving permutation on the set $\mathcal{V}_o(\mathcal{S})$ of all non-trivial singular subspaces of $\mathcal{S}$ which are not submaximal. Moreover, there exist a unique automorphism $\Omega' \in \mathrm{Aut}(\mathcal{G}(\mathcal{S}))$ such that $\Omega'|_{\mathcal{V}_o(\mathcal{S})} = \Omega$.

(a) Let U be a subspace of $\mathcal{S}$ which is stabilized by α and let $i := \mathrm{rk}(U)$. Then U corresponds to a residue of Δ of type $S\backslash\{s_i\}$ which is stabilized by Ω. We conclude that $s_i = s_1$, i.e. U has rank 1 and thus is a point.

(b) By assumption each Γ-chamber of Δ corresponds to a point of $\mathcal{S}$ which is fixed by α. If two of such points are collinear, α would stabilize the line through these points. But this is a contradiction to (a).

$\square$

5.8.4 Theorem
Let Δ be a thick spherical building of type D_n for some $n \geq 6$. Let Δ_0 be a residue of Δ of type D_{n-1}. Suppose that there is an involutory isometry $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle\Omega_0\rangle$ is a descent group of Δ_0 with corresponding Tits index
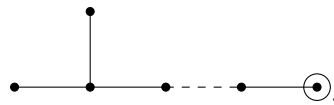
There exists an extension of Ω_0 to an involutory isometry $\Omega \in \mathrm{Aut}(\Delta)$ such that $\Gamma := \langle\Omega\rangle$ is a descent group of Δ with Tits index

and such that the fixed point structure Δ^Γ is a Moufang quadrangle.

Proof   Let $\mathcal{S} = (P, L)$ be a polar space of type D and rank n such that $\Delta \simeq \mathrm{Flag}(\mathcal{G}_\mathrm{o}(\mathcal{S}))$. According to 3.3.15, $\mathcal{S} \simeq \mathbf{P}(\Lambda)$, where Λ is a hyperbolic quadratic space of Witt index $k \geq 6$.

Let p be the point of $\mathcal{S}$ corresponding to the residue Δ_0 and choose a point q of $\mathcal{S}$ which is non-collinear with p. Then $\Delta_0 \simeq \mathrm{Flag}(\mathcal{G}_\mathrm{o}(\mathcal{S}_\mathrm{pq}))$. According to 5.8.3 there is an involutory automorphism $\alpha \in \mathrm{Aut}(\mathcal{S}_{pq})$ fixing at least three pairwise non-collinear points but stabilizing no higher rank singular subspaces. According to 5.4.4, α can be extended to an involution $\varphi \in \mathrm{Aut}(\mathcal{S})$ which fixes the points p and q. Let $\varphi' \in \mathrm{Aut}(\mathcal{G}_o(\mathcal{S}))$ be induced by φ.

Let Ω be the involutory automorphism of Δ induced by φ' and define $\Gamma := \langle \Omega \rangle$. Then the fixed point structure Δ^Γ is isomorphic to the flag complex $\mathrm{Flag}(\mathcal{G}(\mathcal{S}_\varphi))$, which is, by 5.4.4, a thick building of type C_2. Hence, Γ is a descent group of Δ with Tits index $\mathbf{T} = (\mathsf{D}_n, \{\mathrm{id}\}, \{s_3, \dots, s_n\})$. As Δ satisfies the Moufang property, 2.41 gives that Δ^Γ is a Moufang quadrangle. $\qquad \square$

5.8.5 Theorem
Let Δ be a thick building of type $\mathsf{D}_n \times \mathsf{A}_1$ for some $n \geq 6$. Let Δ_0 be a residue of Δ of type $\mathsf{D}_{n-1} \times \mathsf{A}_1$. Suppose that there is an involutory isometry $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with corresponding Tits index

There exists an extension of Ω_0 to an involutory isometry $\Omega \in \mathrm{Aut}(\Delta)$ such that $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with Tits index

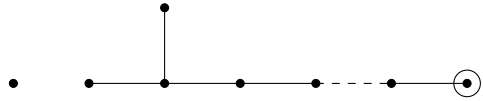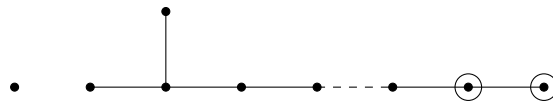and such that the fixed point structure Δ^Γ is a Moufang quadrangle.

Proof   Let Π be the Coxeter diagram $\mathsf{D}_n \times \mathsf{A}_1$ and let (W, S) be the corresponding Coxeter diagram. Let $s \in S$ be such that $st = ts$ for all $t \in S$ and let $J := S \backslash \{s\}$. We label the vertices of Π_J as in 2.3. Let $K := J \backslash \{s_1\} = S \backslash \{s_1, s\}$ and $L := K \backslash \{s_2\}$.

Let Δ_0 be a residue of type $\mathsf{D}_{n-1} \times \mathsf{A}_1$ and let $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ be an involution such that $\Gamma_0 = \langle \Omega_0 \rangle$ is a descent group of Δ_0 with Tits index

Choose a Γ_0-chamber C of Δ_0 and fix a chamber $c \in C$. According to 2.40, $A := \text{Typ}(C) = L \cup \{s\}$ is the type of any Γ_0-chamber. By 4.1.5 and 2.10 we have $\delta(c, \Omega_0(c)) = r_A = r_L s$.

Set $c' := \Omega_0(c)$. The restriction $\Omega_0|_{\mathcal{R}_K(c)}$ is an isometry from $\mathcal{R}_K(c)$ onto $\mathcal{R}_K(c')$. Lemma 4.1.3 yields that the residues $\mathcal{R}_K(c)$ and $\mathcal{R}_K(c')$ are opposite residues of Δ_0, 4.2.5 implies that the projection map $\text{proj}_{\mathcal{R}_K(c)}^{\mathcal{R}_K(c')}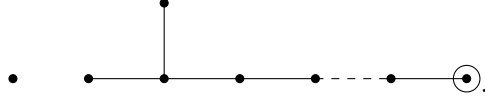$ is a σ-isometry with inverse $\text{proj}_{\mathcal{R}_K(c')}^{\mathcal{R}_K(c)}$, where $\sigma \in \text{Aut}(W_{K \cup \{s\}}, K \cup \{s\})$ is given by

$$\sigma(t) = r_{K \cup \{s\}} \; r_K \; t \; r_K \; r_{K \cup \{s\}} = t$$

for all $t \in K \cup \{s\}$, in particular, the projection maps $\text{proj}_{\mathcal{R}_K(c)}^{\mathcal{R}_K(c')}$ and $\text{proj}_{\mathcal{R}_K(c')}^{\mathcal{R}_K(c)}$ are mutually inverse isometries. We define

$$\alpha := \text{proj}_{\mathcal{R}_K(c)}^{\mathcal{R}_K(c')} \circ \Omega_0|_{\mathcal{R}_K(c)} \in \text{Aut}(\mathcal{R}_K(c)).$$

In view of 4.2.5(c) we have $\alpha = \Omega_0 \circ \text{proj}_{\mathcal{R}_K(c')}^{\mathcal{R}_K(c)}$ and thus $\alpha^2 = \text{id}_{\mathcal{R}_K(c)}$.

There is the following connection between Γ_0-residues of Δ_0 and $\langle \alpha \rangle$-residues of $\mathcal{R}_K(c)$:

> **Claim 1:** If $\mathcal{T} \subseteq \mathcal{R}_K(c)$ is an α-invariant residue, then the unique residue of type $\text{Typ}(\mathcal{T}) \cup \{s\}$ of Δ_0 containing $\mathcal{T}$ is Ω_0-invariant. Conversely, if $\mathcal{T}$ is a Γ_0-residue of Δ_0, then $\text{proj}_{\mathcal{R}_K(c)}(\mathcal{T}) = \mathcal{T} \cap \mathcal{R}_K(c)$ is α-invariant.
>
> **Proof of claim 1:** Let $\mathcal{T} \subseteq \mathcal{R}_K(c)$ be an α-invariant residue and let $\mathcal{T}_s$ be the unique residue of type $\text{Typ}(\mathcal{T}) \cup \{s\}$ such that $\mathcal{T} \subseteq \mathcal{T}_s$. Choose any chamber $d \in \mathcal{T}$. According to 4.1.3(a) $\delta(d, \text{proj}_{\mathcal{R}_K(c')}(d)) = s$ and hence $\text{proj}_{\mathcal{R}_K(c')}(d) \in \mathcal{T}_s$ for any $d \in \mathcal{T}$. Now
>
> $$\mathcal{T} = \alpha(\mathcal{T}) = \Omega_0(\text{proj}_{\mathcal{R}_K(c')}^{\mathcal{R}_K(c)}(\mathcal{T})) \subseteq \Omega_0(\mathcal{T}_s)$$
>
> and since $\text{Typ}(\Omega_0(\mathcal{T}_s)) = \text{Typ}(\mathcal{T}_s)$ we conclude that $\Omega_0(\mathcal{T}_s) = \mathcal{T}_s$. Conversely, let $\mathcal{T}$ be a Γ_0-residue of Δ_0 and let $d \in \mathcal{T}$. By assumption $s \in \text{Typ}(\mathcal{T})$. By 4.1.3(a) we have $\delta(d, \text{proj}_{\mathcal{R}_K(c)}(d)) \in \{1, s\}$ and thus $\mathcal{T} \cap \mathcal{R}_K(c) \neq \emptyset$. In view of 2.28(b) we have $\text{proj}_{\mathcal{R}_K(c)}(\mathcal{T}) = \mathcal{T} \cap \mathcal{R}_K(c)$. Similarly, $\mathcal{T} \cap \mathcal{R}_K(c') \neq \emptyset$ and thus $\text{proj}_{\mathcal{R}_K(c')}(\mathcal{T}) = \mathcal{T} \cap \mathcal{R}_K(c')$. Now
>
> $$\alpha(\text{proj}_{\mathcal{R}_K(c)}(\mathcal{T})) = \Omega_0(\mathcal{T} \cap \mathcal{R}_K(c')) \subseteq \mathcal{T} \cap \mathcal{R}_K(c) = \text{proj}_{\mathcal{R}_K(c)}(\mathcal{T}).$$
>
> $\square$

In particular, if C is a Γ_0-chamber of Δ_0, then $\mathrm{proj}_{\mathcal{R}_K(c)}(C)$ is an $\langle\alpha\rangle$-chamber of $\mathcal{R}_K(c)$. Conversely, if D is an $\langle\alpha\rangle$-chamber of $\mathcal{R}_K(c)$, then the unique residue of Δ_0 of type $L \cup \{s\}$ containing D is a Γ_0-chamber of Δ_0.

> **Claim 2:** The group $\langle\alpha\rangle \leq \mathrm{Aut}(\mathcal{R}_K(c))$ is a descent group of $\mathcal{R}_K(c)$
> with Tits index
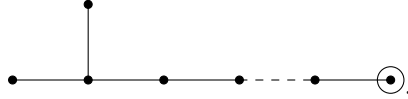
> **Proof of claim 2:** Note that $\mathcal{R}_K(c)$ is the only $\langle\alpha\rangle$-panel. By assumption there exist at least three Γ_0-chambers of Δ_0. Let C and D
> be two of them. Then, by (1), $C' := \mathrm{proj}_{\mathcal{R}_K(c)}(C) = C \cap \mathcal{R}_K(c)$ and
> $D' := \mathrm{proj}_{\mathcal{R}_K(c)}(D) = D \cap \mathcal{R}_K(c)$ are $\langle\alpha\rangle$-chambers of $\mathcal{R}_K(c)$. Note
> that, since $C \neq D$, we have $C' \neq D'$. Thus, there exist at least three
> $\langle\alpha\rangle$-chambers in $\mathcal{R}_K(c)$ and $\langle\alpha\rangle$ is a descent group of $\mathcal{R}_K(c)$ with Tits
> index $\mathbf{T} = (\mathsf{D}_{n-1}, \{\mathrm{id}\}, L)$. $\square$

Using 5.5.8 we obtain that α can be extended to an involutory isometry $\bar{\alpha} \in \mathrm{Aut}(\mathcal{R}_J(c))$ such that $\langle\bar{\alpha}\rangle \leq \mathrm{Aut}(\mathcal{R}_J(c))$ is a descent group of $\mathcal{R}_J(c)$ with Tits index

and such that the fixed point structure $\mathcal{R}_J(c)^{\langle\bar{\alpha}\rangle}$ is a Moufang quadrangle.

Again, by 4.1.3, the residues $\mathcal{R}_J(c)$ and $\mathcal{R}_J(c')$ are opposite in Δ and thus, by 4.2.5 the projection maps $\mathrm{proj}_{\mathcal{R}_J(c')}^{\mathcal{R}_J(c)}$ and $\mathrm{proj}_{\mathcal{R}_J(c)}^{\mathcal{R}_J(c')}$ are mutually inverse isometries. We define an isometry

$$\Omega' := \mathrm{proj}_{\mathcal{R}_J(c')}^{\mathcal{R}_J(c)} \circ \bar{\alpha} \colon \mathcal{R}_J(c) \to \mathcal{R}_J(c').$$

For $x \in \mathcal{C} \backslash \mathcal{R}_J(c)$ we set $c_x := \mathrm{proj}_{\mathcal{R}_J(x)}(c)$. By 4.1.3(a), $\delta(c, c_x) = s$ and thus $c_x \in C$, where C is the unique Γ_0-chamber of Δ_0 containing c. In particular, $\Omega_0(c_x) \in C$ is defined. For any chamber $x \in \mathcal{C}$ we define

$$\Omega(x) := \begin{cases} \Omega'(x), & x \in \mathcal{R}_J(c) \\ (\mathrm{proj}_{\mathcal{R}_J(\Omega_0(c_x))} \circ \Omega' \circ \mathrm{proj}_{\mathcal{R}_J(c)})(x), & \text{otherwise} \end{cases}.$$

Note that the chambers c_x and $\Omega_0(c_x)$ are oppsoite in C, since by 4.1.5 we have $\delta(c_x, \Omega_0(c_x)) = r_A$. Thus, $\mathcal{R}_J(c_x) \neq \mathcal{R}_J(\Omega_0(c_x))$ and moreover, in view of 4.1.3, these residues are opposite.
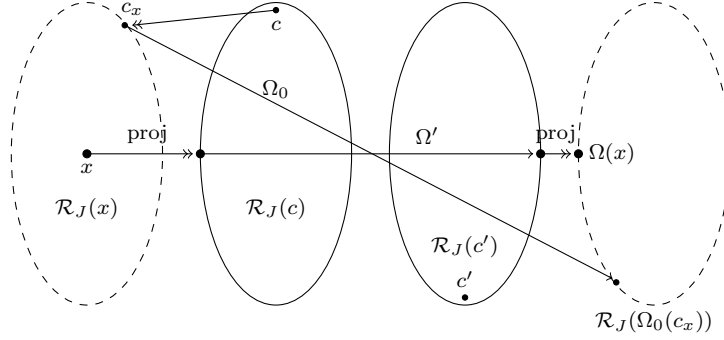
Claim 3: $\Omega \colon \mathcal{C} \to \mathcal{C}$ is an isometry.

Proof of claim 3: Once we showed that Ω is surjective, according to [AB, 5.61], it suffices to show that $\delta(\Omega(x), \Omega(y)) = \delta(x, y)$ for all $x, y \in \mathcal{C}$ such that $\delta(x, y) \in S$.

Let $y \in \mathcal{C}$ be any chamber. If $y \in \mathcal{R}_J(c')$, then, by assumption, there exists $x \in \mathcal{R}_J(c)$ such that $y = \Omega'(x) = \Omega(x)$. So suppose that $y \notin \mathcal{R}_J(c')$. Then $c'_y := \mathrm{proj}_{\mathcal{R}_J(y)}(c')$ is the unique chamber in $\mathcal{R}_J(y)$ satisfying $\delta(c', c'_y) = s$ and thus $c'_y \in C$. Note that

$$\delta(c, \Omega_0(c'_y)) = \delta(\Omega_0(c), c'_y) = \delta(c', c'_y) = s,$$

in particular, $\Omega_0(c'_y) = \mathrm{proj}_{\mathcal{R}_J(\Omega_0(c'_y))}(c)$.

Let $z := \mathrm{proj}_{\mathcal{R}_J(\Omega_0(c'_y))} \circ \Omega'^{-1} \circ \mathrm{proj}_{\mathcal{R}_J(c')}(y)$. By definition, $z \in \mathcal{R}_J(\Omega_0(c'_y))$ and, furthermore, $c_z := \mathrm{proj}_{\mathcal{R}_J(z)}(c) = \Omega_0(c'_y))$. Thus,

$$\begin{aligned}
\Omega(z) &= \mathrm{proj}_{\mathcal{R}_J(y)} \circ \Omega' \circ \mathrm{proj}_{\mathcal{R}_J(c)}(z) \\
&= \mathrm{proj}_{\mathcal{R}_J(y)} \circ \Omega'(\Omega'^{-1}(\mathrm{proj}_{\mathcal{R}_J(c')}(y)) \\
&= \mathrm{proj}_{\mathcal{R}_J(y)} \circ \mathrm{proj}_{\mathcal{R}_J(c')}(y) = y,
\end{aligned}$$

where we used that the residues $\mathcal{R}_J(z)$ and $\mathcal{R}_J(c)$ are opposite as well as the residues $\mathcal{R}_J(y)$ and $\mathcal{R}_J(c')$.

Now choose $x, y \in \mathcal{C}$ such that $\delta(x, y) \in S$.
If $\delta(x, y) \in J$, then the residues $\mathcal{R}_J(x)$ and $\mathcal{R}_J(y)$ coincide and thus $\mathrm{proj}_{\mathcal{R}_J(x)}(c) = c_x = c_y = \mathrm{proj}_{\mathcal{R}_J(y)}(c)$.
Since the composition $\mathrm{proj}^{\mathcal{R}_J(c')}_{\mathcal{R}_J(\Omega_0(c_x))} \circ \Omega' \circ \mathrm{proj}^{\mathcal{R}_J(x)}_{\mathcal{R}_J(c)}$ is an isometry from $\mathcal{R}_J(x)$ onto $\mathcal{R}_J(\Omega_0(c_x))$, we conclude that $\delta(\Omega(x), \Omega(y)) = \delta(x, y)$.
So suppose that $\delta(x, y) = s$ and set $x' := \mathrm{proj}_{\mathcal{R}_J(c)}(x)$ as well as $y' := \mathrm{proj}_{\mathcal{R}_J(c)}(y)$. Since $\delta(x, x'), \delta(y, y') \in \{1_W, s\}$ we conclude that $\delta(x', y') \in \{1_W, s\}$. As $x', y' \in \mathcal{R}_J(c)$ and $s \notin J$ it follows that $\delta(x', y') = 1_W$ and hence $x' = y'$. Now, by definition,

$$\Omega(x) = \mathrm{proj}_{\mathcal{R}_J(\Omega_0(c_x))}(\Omega'(x'))$$

and
$$\Omega(y) = \mathrm{proj}_{\mathcal{R}_J(\Omega_0(c_y))}(\Omega'(x')).$$
Again, $\delta(\Omega(x), \Omega'(x')) \in \{1_W, s\}$ as well as $\delta(\Omega(y), \Omega'(x')) \in \{1_W, s\}$. We conclude that $\delta(\Omega(x), \Omega(y)) \in \{1_W, s\}$.
Our assumption on x and y gives that $\mathcal{R}_J(x) \cap \mathcal{R}_J(y) = \emptyset$. Hence, $\delta(\Omega_0(c_x)), \Omega_0(c_y)) = \delta(c_x, c_y) = s$ and as $s \notin J$ this fact implies that $\mathcal{R}_J(\Omega_0(c_x)) \cap \mathcal{R}_J(\Omega_0(c_y)) = \emptyset$ and thus $\delta(\Omega(x), \Omega(y)) = s$.

$\square$

Claim 4: $\Omega|_{\Delta_0} = \Omega_0$.

Proof of claim 4: Let $x \in \Delta_0 = \mathcal{R}_{K \cup \{s\}}(c)$ be any chamber. Then $\mathrm{proj}_{\mathcal{R}_J(c)}(x) \in \mathcal{R}_K(c)$ and hence $\Omega_0(\mathrm{proj}_{\mathcal{R}_J(c)}(x)) = \mathrm{proj}_{\mathcal{R}_J(c')}(\Omega_0(x))$ by 4.2.5. Moreover, as $\mathrm{proj}_{\mathcal{R}_J(x)}(c) \in \mathcal{R}_K(x)$ we conclude that the residues $\mathcal{R}_J(\Omega_0(c_x))$ and $\mathcal{R}_J(\Omega_0(x))$ coincide. Now

$$\begin{aligned}
\Omega(x) &= \mathrm{proj}_{\mathcal{R}_J(\Omega_0(x))}(\Omega'(\mathrm{proj}_{\mathcal{R}_J(c)}(x))) \\
&= \mathrm{proj}_{\mathcal{R}_J(\Omega_0(x))}(\Omega_0(\mathrm{proj}_{\mathcal{R}_J(c)}(x))) \\
&= \mathrm{proj}_{\mathcal{R}_J(\Omega_0(x))}(\mathrm{proj}_{\mathcal{R}_J(c')}(\Omega_0(x))) = \Omega_0(x),
\end{aligned}$$

since $\Omega'|_{\mathcal{R}_K(c)} = \Omega_0$. $\square$

Claim 5: $\Gamma := \langle \Omega \rangle \leq \mathrm{Aut}(\Delta)$ is a descent of Δ with Tits index
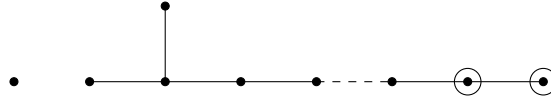
and the fixed point structure Δ^{Γ} is a Moufang quadrangle.

Proof of claim 5: Similar as in the proof of claim 1 one verifies the following connection: If $\mathcal{T} \subseteq \mathcal{R}_J(c)$ is an $\langle \bar{\alpha} \rangle$-residue, then the unique residue of Δ of type $\mathrm{Typ}(\mathcal{T}) \cup \{s\}$ containing $\mathcal{T}$ is Ω-invariant. Conversely, if $\mathcal{T}$ is a Γ-residue of Δ, then $\mathrm{proj}_{\mathcal{R}_J(c)}(\mathcal{T})$ is a $\langle \bar{\alpha} \rangle$-residue.

According to (4), the Γ_0-chamber C is a Γ-chamber and Δ_0 is a Γ-panel containing C. By assumption, Δ_0 contains at least three Γ-chambers. Let $\mathcal{P}$ be a Γ-panel containing C other than Δ_0. Due to the remark above, $\mathrm{proj}_{\mathcal{R}_J(c)}(\mathcal{P})$ is a $\langle \bar{\alpha} \rangle$-panel and as $\langle \bar{\alpha} \rangle$ is a descent group of $\mathcal{R}_J(c)$, $\mathrm{proj}_{\mathcal{R}_J(c)}(\mathcal{P})$ contains at least three $\langle \bar{\alpha} \rangle$-chambers and each is of type L. For any $\langle \bar{\alpha} \rangle$-chamber D of $\mathcal{R}_J(c)$ the unique residue of type $L \cup \{s\}$ containing D is a Γ-chamber. Hence $\mathcal{P}$ contains at least three Γ-chambers. By 2.42, $\mathbf{T} = (\mathsf{D}_n \times \mathsf{A}_1, \{\mathrm{id}\}, A)$ is a Tits index. Note that the relative types of the Tits indices $\mathbf{T}$ and $(\mathsf{D}_n, \{\mathrm{id}\}, L)$ coincide.

As $\langle \bar{\alpha} \rangle$ is a descent group of $\mathcal{R}_J(c)$, there exists a $\langle \bar{\alpha} \rangle$-chamber D' such that $\bar{\delta}(\mathrm{proj}_{\mathcal{R}_J(c)}(C), D')$ equals the longest element in the relative type of the Tits index **T**. Now let $x \in C$ be any chamber and let D be the unique Γ-chamber containing D'. Then, if $x \in C \cap \mathcal{R}_J(c)$,

$$\delta(x, \mathrm{proj}_D(x)) = \min\{r_{\tilde{S}}W_A\} = r_{\tilde{S}},$$

since $A \subseteq J^+(r_{\tilde{S}})$ by [MPW, 20.13(iii)]. Otherwise, if $x \notin \mathcal{R}_J(c)$, we have $\delta(x, \mathrm{proj}_{\mathcal{R}_J(c)}(x)) = s$ and, according to the previous considerations,

$$\delta(\mathrm{proj}_{\mathcal{R}_J(c)}(x), \mathrm{proj}_D(\mathrm{proj}_{\mathcal{R}_J(c)}(x))) = r_{\tilde{S}}.$$

Again, since $s \in A \subseteq J^+(r_{\tilde{S}})$ we have $\ell(sr_{\tilde{S}}) = \ell(r_{\tilde{S}}) + 1$ and thus by (WD2), $\delta(x, \mathrm{proj}_D(\mathrm{proj}_{\mathcal{R}_J(c)}(x))) = r_{\tilde{S}}$. Again, $\delta(x, \mathrm{proj}_D(x)) = \min\{r_{\tilde{S}}W_A\} = r_{\tilde{S}}$. Thus, by 2.42(e), Γ is a descent group of Δ.

According to 2.40, the fixed point structure Δ^Γ is a building of type C_2 (since the relative type of the Tits index $(\mathsf{D}_n, \{\mathrm{id}\}, L)$ is C_2). As the the map $C \mapsto \mathrm{proj}_{\mathcal{R}_J(c)}(C)$ is an isometry from Δ^Γ onto $\mathcal{R}_J(c)^{\langle \bar{\alpha} \rangle}$ we conclude that Δ^Γ is a Moufang quadrangle. $\square$

This finishes the proof.

$\square$

Part III

Descent in twin buildings

Chapter 6

Twin buildings

6.1 Projections in twin buildings

Given a residue $\mathcal{R}$ and a chamber c of a building Δ, there exists a unique chamber $x \in \mathcal{R}$ which is „nearest" c in the sense that it minimizes the distance from c onto $\mathcal{R}$. Although there are no galleries between a chamber $c \in \mathcal{C}_+$ and a chamber $d \in \mathcal{C}_-$, one should think of the numerical codistance $\mathrm{dist}(c, d)$ as a measure of how far away c and d are from each other. Decreasing codistance should be thought of as increasing distance. We will now see that any spherical residue in $\mathcal{C}_\varepsilon$ contains precisely one chamber that is „closest" to a given chamber $d \in \mathcal{C}_{-\varepsilon}$ in the sense that it has maximal codistance from d among all chambers in $\mathcal{R}$.

As is common in the theory of twin buildings, we do not distinguish in notation and terminology between the projections in one building and the projections between the two "halves" of a twin building. Note, however, that the latter exist only for spherical residues. It will always be clear from the context which type of projection we mean.

Throughout this chapter let Π be a Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S). Let $\varepsilon \in \{+, -\}$.

6.1.1 Proposition
Let $\mathcal{R}$ be a residue of Δ_ε of type $J \subseteq S$.

(a) If $\mathcal{T}$ is a residue of $\Delta_{-\varepsilon}$ of type $K \subseteq S$, then

$$\delta_*(\mathcal{R}, \mathcal{T}) := \{\delta_*(c, d) \mid c \in \mathcal{R}, d \in \mathcal{T}\} = W_J \, \delta_*(x, y) \, W_K$$

for any $x \in \mathcal{R}$ and $y \in \mathcal{T}$.

(b) If J is spherical and d is a chamber in $\mathcal{C}_{-\varepsilon}$, then there is a unique chamber $z \in \mathcal{R}$ such that $\delta_*(z, d) = \max\{\delta_*(\mathcal{R}, d)\}$. This chamber z satisfies

$$\delta_*(c, d) = \delta_\varepsilon(c, z)\, \delta_*(z, d)$$

and

$$\mathrm{dist}(c, d) = \mathrm{dist}(z, d) - \mathrm{dist}(c, z)$$

for all $c \in \mathcal{R}$.

Proof   Part (a) is [AB, 5.148] and part (b) is [AB, 5.149].   □

6.1.2 Definition
If $\mathcal{R}, d$ and z are as in 6.1.1(b), then z is called the *projection* of d onto $\mathcal{R}$ and is denoted by $\mathrm{proj}_{\mathcal{R}}(d)$.

6.1.3 Definition
Let $\mathcal{R}$ be a spherical residue of Δ_ε and let $\mathcal{T}$ be an arbitrary residue of $\Delta_{-\varepsilon}$. We define

$$\mathrm{proj}_{\mathcal{R}}(\mathcal{T}) := \{\mathrm{proj}_{\mathcal{R}}(x) \mid x \in \mathcal{T}\}.$$

6.1.4 Lemma
Let $\mathcal{R}$ be a spherical residue of Δ_ε of type $J \subseteq S$ and let $\mathcal{T}$ be a residue of $\Delta_{-\varepsilon}$ of type $K \subseteq S$. Let $w_1 := \min\{\delta_*(\mathcal{R}, \mathcal{T})\}$ and let $r_J := \max\{W_J\}$. The projection $\mathcal{P} := \mathrm{proj}_{\mathcal{R}}(\mathcal{T})$

(a) is given by $\mathcal{P} = \{x \in \mathcal{R} \mid r_J w_1 \in \delta_*(x, \mathcal{T})\}$;

(b) is a residue of type $J' := r_J(J \cap w_1 K w_1^{-1}) r_J^{-1}$.

Proof   Let $x \in \mathcal{R}$ such that $r_J w_1 \in \delta_*(x, \mathcal{T})$ and $y \in \mathcal{T}$ such that $\delta_*(x, y) = r_J w_1$. Note that $\delta_*(y, x) = w_1^{-1} r_J$ and $w_1^{-1} = \min\{\delta_*(\mathcal{T}, \mathcal{R})\}$. Thus, $J \subseteq J^+(w_1^{-1})$ and by 2.8(c) we have $\ell(w_1^{-1} v) = \ell(w_1^{-1}) + \ell(v)$ for all $v \in W_J$. In particular, $w_1^{-1} r_J = \max\{w_1^{-1} W_J\}$ and hence

$$\delta_*(x, y) = r_J w_1 = \max\{W_J w_1\} = \max\{\delta_*(\mathcal{R}, y)\}.$$

We conclude that $x = \mathrm{proj}_{\mathcal{R}}(y) \in \mathrm{proj}_{\mathcal{R}}(\mathcal{T})$.

Conversely, let $y \in \mathcal{T}$, set $x := \mathrm{proj}_{\mathcal{R}}(y)$ and $w := \delta_*(x, y) \in \delta_*(\mathcal{R}, \mathcal{T})$. Consider the element $r_J w \in W_J w_1 W_K$. According to 2.8(b) the element $r_J w$ can be written as $r_J w = w_J w_1 w_K$ with $w_J \in W_J$ and $w_K \in W_K$ and $\ell(r_J w) = \ell(w_J) + \ell(w_1) + \ell(w_K)$. As $w = \max\{\delta_*(\mathcal{R}, y)\} = \max\{W_J w\}$, $w^{-1} = \max\{w^{-1} W_J\}$ and by 2.8(e), $\ell(w^{-1} v) = \ell(w^{-1}) - \ell(v)$ for all $v \in W_J$.

In particular, $w^{-1}r_J = \min\{w^{-1}W_J\}$ and hence $r_Jw = \min\{W_Jw\}$. Suppose that $w_J \neq 1_W$. Then, since $K \subseteq J^+(w_1)$,

$$\ell(w_J^{-1}r_Jw) = \ell(w_1w_K) = \ell(w_1) + \ell(w_K)$$
$$< \ell(w_1) + \ell(w_K) + \ell(w_J) = \ell(r_Jw).$$

But this is a contradiction to the minimality of $\ell(r_Jw)$ in W_Jw. Thus, $w_J = 1_W$ and hence $r_Jw = w_1w_K$. Therefore,

$$r_Jw_1 = (w_1^{-1}r_J)^{-1} = (w_Kw^{-1})^{-1} \in wW_K = \delta_*(x, \mathcal{T})$$

and (a) holds.

Now let $x \in \mathrm{proj}_{\mathcal{R}}(\mathcal{T})$ and $y \in \mathcal{T}$ such that $\delta_*(x, y) = r_Jw_1$. The proof of part (a) implies $x = \mathrm{proj}_{\mathcal{R}}(y)$. For $z \in \mathcal{C}_\varepsilon$ we have

$$\delta_*(z, y) = \delta_\varepsilon(z, x)\ \delta_*(x, y) = \delta_\varepsilon(z, x)\ r_Jw_1.$$

Hence,

$$z \in \mathrm{proj}_{\mathcal{R}}(\mathcal{T}) \Leftrightarrow z \in \mathcal{R},\ r_Jw_1 \in \delta_*(z, \mathcal{T}) = \delta_\varepsilon(z, x)r_Jw_1W_K$$
$$\Leftrightarrow \delta_\varepsilon(z, x) \in W_J,\ (r_Jw_1)^{-1}\delta_\varepsilon(z, x)(r_Jw_1) \in W_K$$
$$\Leftrightarrow \delta_\varepsilon(z, x) \in W_J \cap r_Jw_1W_Kw_1^{-1}r_J^{-1}.$$

Since

$$W_J \cap r_Jw_1W_Kw_1^{-1}r_J^{-1} = r_J(W_J \cap w_1W_Kw_1^{-1})r_J^{-1}$$
$$= r_JW_{J\cap w_1Kw_1^{-1}}r_J^{-1}$$
$$= W_{r_J(J\cap w_1Kw_1^{-1})r_J^{-1}},$$

we conclude that $z \in \mathrm{proj}_{\mathcal{R}}(\mathcal{T})$ if and only if $z \in \mathcal{R}_{r_J(J\cap w_1Kw_1^{-1})r_J^{-1}}(x)$ and (b) holds.

$\square$

6.1.5 Lemma
Let $c \in \mathcal{C}_\varepsilon$ be a chamber and let $\mathcal{T} \subseteq \mathcal{R} \subseteq \mathcal{C}_{-\varepsilon}$ be spherical residues. Then

$$\mathrm{proj}_{\mathcal{T}}(c) = \mathrm{proj}_{\mathcal{T}}(\mathrm{proj}_{\mathcal{R}}(c)).$$

Proof   Since $\mathrm{proj}_{\mathcal{T}}(c) \in \mathcal{T} \subseteq \mathcal{R}$ we use 6.1.1(b) and obtain

$$\mathrm{dist}(\mathrm{proj}_{\mathcal{T}}(c), c) = \mathrm{dist}(\mathrm{proj}_{\mathcal{R}}(c), c) - \mathrm{dist}(\mathrm{proj}_{\mathcal{T}}(c), \mathrm{proj}_{\mathcal{R}}(c)).$$

Using the properties of the projection in the building $\Delta_{-\varepsilon}$ (cf. 2.24) we obtain

$$\text{dist}(\text{proj}_{\mathcal{T}}(c), \text{proj}_{\mathcal{R}}(c)) = \text{dist}(\text{proj}_{\mathcal{T}}(c), \text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c)))$$
$$+ \text{dist}(\text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c)), \text{proj}_{\mathcal{R}}(c)).$$

It follows that

$$\begin{aligned}
\text{dist}(\text{proj}_{\mathcal{T}}(c), c) &= \text{dist}(\text{proj}_{\mathcal{R}}(c), c) - \text{dist}(\text{proj}_{\mathcal{T}}(c), \text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c))) \\
&\quad - \text{dist}(\text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c)), \text{proj}_{\mathcal{R}}(c)) \\
&= \text{dist}(\text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c)), c) \\
&\quad - \text{dist}(\text{proj}_{\mathcal{T}}(c), \text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c))).
\end{aligned}$$

If $\text{proj}_{\mathcal{T}}(c) \neq \text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c))$, then $\text{dist}(\text{proj}_{\mathcal{T}}(c), \text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c))) > 0$ and

$$\text{dist}(\text{proj}_{\mathcal{T}}(c), c) < \text{dist}(\text{proj}_{\mathcal{T}}(\text{proj}_{\mathcal{R}}(c)), c),$$

which is a contradiction since $\text{proj}_{\mathcal{T}}(c)$ is the unique chamber in $\mathcal{T}$ such that $\delta_*(\text{proj}_{\mathcal{T}}(c), c) = \max\{\delta_*(\mathcal{T}, c)\}$. $\qquad\square$

6.2 On parallel and opposite residues

There is a notion of parallel residues in arbitrary buildings. We will extend this notion to spherical residues contained in different halves of a twin building. Since we use projections, the residues need to be spherical, whereas the notion of parallel residues in arbitrary buildings makes no requirements on the type of the residues.

We will then give the definition of opposite residues which exist in every twin building while opposite residues do only exist in spherical buildings. At first sight the convention that opposite residues have the same type seems to be inconsistent with the definition of opposite residues in spherical buildings. We will give an equivalent definition of being opposite which justifies this part of the definition.

Throughout this section let Π be a Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S). Let $\varepsilon \in \{+, -\}$.

6.2.1 Definition
Let $\mathcal{R}$ be a spherical residue of Δ_ε and let $\mathcal{T}$ be a spherical residue of $\Delta_{-\varepsilon}$. The residues $\mathcal{R}$ and $\mathcal{T}$ will be called *parallel* if $\mathcal{R} = \mathrm{proj}_\mathcal{R}(\mathcal{T})$ and $\mathcal{T} = \mathrm{proj}_\mathcal{T}(\mathcal{R})$.

6.2.2 Lemma
Let $\mathcal{R} \subseteq \mathcal{C}_\varepsilon$ and $\mathcal{T} \subseteq \mathcal{C}_{-\varepsilon}$ be spherical residues of Δ. Then $\mathcal{R}$ and $\mathcal{T}$ are parallel if and only if the projection maps $\mathrm{proj}_\mathcal{R}$ and $\mathrm{proj}_\mathcal{T}$ induce mutually inverse bijections between $\mathcal{R}$ and $\mathcal{T}$.

Proof The only if part is clear. So suppose that $\mathcal{R}$ and $\mathcal{T}$ are parallel. Choose $x \in \mathcal{R}$, let $y \in \mathcal{T}$ such that $x = \mathrm{proj}_\mathcal{R}(y)$ and set $z := \mathrm{proj}_\mathcal{T}(x)$. By 6.1.1(b), $\mathrm{dist}(x, y) = \mathrm{dist}(x, z) - \mathrm{dist}(z, y)$ and by 2.50 we obtain

$$\mathrm{dist}(\mathrm{proj}_\mathcal{R}(z), y) \geq \mathrm{dist}(\mathrm{proj}_\mathcal{R}(z), z) - \mathrm{dist}(z, y)$$
$$\geq \mathrm{dist}(x, z) - \mathrm{dist}(z, y) = \mathrm{dist}(x, y).$$

But x is the unique chamber in $\mathcal{R}$ such that $\delta_*(x, y) = \max\{\delta_*(\mathcal{R}, y)\}$, so we conclude that $x = \mathrm{proj}_\mathcal{R}(z) = \mathrm{proj}_\mathcal{R}(\mathrm{proj}_\mathcal{T}(x))$. A similar argument shows that $y = \mathrm{proj}_\mathcal{T}(x) = \mathrm{proj}_\mathcal{T}(\mathrm{proj}_\mathcal{R}(y))$. $\square$

There is a notion of opposite chambers and residues for spherical buildings. As we will work with twin buildings whose type is not necessarily spherical, we will not make use of this notion and introduce instead the notion of an opposition relation between the two halves of a twin building.

6.2.3 Definition

Two residues $\mathcal{R} \subseteq \mathcal{C}_\varepsilon$ and $\mathcal{T} \subseteq \mathcal{C}_{-\varepsilon}$ are called *opposite* if $\mathrm{Typ}(\mathcal{R}) = \mathrm{Typ}(\mathcal{T})$ and there exists a pair of chambers $x \in \mathcal{R}, y \in \mathcal{T}$ such that $\delta_*(x, y) = 1_W$.

6.2.4 Lemma

Let $\mathcal{R} \subseteq \mathcal{C}_\varepsilon$ and $\mathcal{T} \subseteq \mathcal{C}_{-\varepsilon}$ be residues of Δ. The following are equivalent:

 (i) The residues $\mathcal{R}$ and $\mathcal{T}$ are opposite.

 (ii) For every chamber $x \in \mathcal{R}$ there exists a chamber in $y \in \mathcal{T}$ which is opposite x and vice versa.

Proof   Let $\mathcal{R}$ and $\mathcal{T}$ be opposite residues as defined in 6.2.3. Let $x \in \mathcal{R}$ and $y \in \mathcal{T}$ such that $\delta_*(x, y) = 1_W$ and set $J := \mathrm{Typ}(\mathcal{R}) = \mathrm{Typ}(\mathcal{T})$. Note that $w_1 := \min\{\delta_*(\mathcal{R}, \mathcal{T})\} = 1_W$. Choose a chamber $z \in \mathcal{R}$. Then $\delta_*(z, \mathcal{T}) = \delta_*(z, y)W_J$ and since $\delta_*(z, y) \in \delta_*(\mathcal{R}, \mathcal{T}) = W_J$ we conclude that $1_W \in \delta_*(z, \mathcal{T})$. Similarly for every $d \in \mathcal{T}$ there exists a $u \in \mathcal{R}$ such that $\delta_*(d, u) = 1_W$.

Set $J := \mathrm{Typ}(\mathcal{R})$ and $K := \mathrm{Typ}(\mathcal{T})$. Since for any chamber $x \in \mathcal{R}$ there exists $y \in \mathcal{T}$ such that $\delta_*(x, y) = 1_W$ we have $\delta_*(x, \mathcal{T}) = \delta_*(x, y)W_K = W_K$ for any chamber $x \in \mathcal{R}$. Similarly $\delta_*(\mathcal{R}, z) = W_J$ for any chamber $z \in \mathcal{T}$. Now let $c \in \mathcal{R}$ be any chamber, $w \in W_K$ and $d \in \mathcal{T}$ such that $w = \delta_*(c, d)$. Then $w \in \delta_*(\mathcal{R}, d) = W_J$ and hence $W_K \subseteq W_J$. Similarly, let $d \in \mathcal{T}$ be any chamber, $w' \in W_J$ and $c \in \mathcal{R}$ such that $w' = \delta_*(c, d)$. Then $w' \in \delta_*(c, \mathcal{T}) = W_K$ and hence $W_J \subseteq W_K$. We conclude that $J = K$. $\square$

Opposite chambers and residues have quite similar properties as opposite chambers and residues in spherical buildings.

6.2.5 Lemma

Let $\mathcal{R}$ and $\mathcal{T}$ be opposite residues of spherical type J in the twin building Δ. Then $\mathcal{R}$ and $\mathcal{T}$ are parallel.

Proof   This follows from proposition [AB, 5.152] together with 6.2.2. $\square$

6.2.6 Lemma

Let $\mathcal{R} \subseteq \mathcal{C}_\varepsilon$ and $\mathcal{T} \subseteq \mathcal{C}_{-\varepsilon}$ be opposite residues of Δ of spherical type $J \subseteq S$. Then $\delta_*(c, \mathrm{proj}_\mathcal{T}(c)) = r_J$ for all $c \in \mathcal{R}$.

Proof   Choose a chamber $c \in \mathcal{R}$. By 6.2.4 there exists $d \in \mathcal{T}$ such that $\delta_*(c, d) = 1_W$ and hence $\delta_*(c, \mathrm{proj}_\mathcal{T}(c)) = \max\{\delta_*(c, \mathcal{T})\} = \max\{W_J\} = r_J$.
   $\square$

6.3   Convex subsets and twin apartments

Throughout this chapter let Π be a Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S). Let $\varepsilon \in \{+, -\}$.

6.3.1 Definition
A pair $(\mathcal{M}_+, \mathcal{M}_-)$ of nonempty subsets $\mathcal{M}_+ \subseteq \mathcal{C}_+$ and $\mathcal{M}_- \subseteq \mathcal{C}_-$ is called *convex* if $\mathrm{proj}_{\mathcal{P}}(c) \in \mathcal{M}_+ \cup \mathcal{M}_-$ for any $c \in \mathcal{M}_+ \cup \mathcal{M}_-$ and any panel $\mathcal{P} \subseteq \mathcal{C}_+ \cup \mathcal{C}_-$ that meets $\mathcal{M}_+ \cup \mathcal{M}_-$.

6.3.2 Lemma
Let $(\mathcal{M}_+, \mathcal{M}_-)$ be a convex pair of Δ. Let $\mathcal{R}$ be a residue of Δ_ε of spherical type $J \subseteq S$ with $\mathcal{R} \cap \mathcal{M}_\varepsilon \neq \emptyset$. Then $\mathrm{proj}_{\mathcal{R}}(x) \in \mathcal{M}_\varepsilon$ for all $x \in \mathcal{M}_+ \cup \mathcal{M}_-$.

Proof   Since $\mathcal{R} \cap \mathcal{M}_\varepsilon \neq \emptyset$ we may choose $c \in \mathcal{R} \cap \mathcal{M}_\varepsilon$. Let $x \in \mathcal{M}_\varepsilon$ and let $\mathcal{P} \subseteq \mathcal{C}_\varepsilon$ be a panel that meets $\mathcal{M}_\varepsilon$. Since $(\mathcal{M}_+, \mathcal{M}_-)$ is a convex pair, it follows by definition that $\mathrm{proj}_{\mathcal{P}}(x) \in \mathcal{M}_\varepsilon$. It follows from [AB, 5.46] that $\mathcal{M}_\varepsilon$ is a convex subset of $\mathcal{C}_\varepsilon$ as defined in [AB, 5.43]. The assertion now follows by [AB, 5.45].
So let $x \in \mathcal{M}_{-\varepsilon}$ and set $w := \delta_*(c, x)$. Then $\delta_*(\mathcal{R}, x) = W_J\, \delta_*(c, x)$ is finite and we may define $d := \max\{\delta_*(\mathcal{R}, x)\}$.

If $w = d$, then $c = \mathrm{proj}_{\mathcal{R}}(x)$ and hence $\mathrm{proj}_{\mathcal{R}}(x) = c \in \mathcal{M}_\varepsilon$. Otherwise there exists $s \in J$ such that $\ell(sw) > \ell(w)$. Let $\mathcal{P} := \mathcal{P}_s(c)$ and note that $\delta_*(\mathcal{P}, x) = \{w, sw\}$. As $s \in J$ is such that $\ell(sw) > \ell(w)$, we conclude that $c \neq \mathrm{proj}_{\mathcal{P}}(x)$. Set $c_s := \mathrm{proj}_{\mathcal{P}}(x)$. Since $(\mathcal{M}_+, \mathcal{M}_-)$ is a convex pair, $c_s \in \mathcal{M}_\varepsilon \cap \mathcal{R}$ and satisfies $\mathrm{dist}(c_s, x) > \mathrm{dist}(c, x)$. Continuing in this way, we obtain a gallery $c, c_s, \ldots$ in $\mathcal{R} \cap \mathcal{M}_\varepsilon$ along which the numerical codistance to x is strictly increasing. Since J is spherical, the process must terminate after finitely many steps with $\mathrm{proj}_{\mathcal{R}}(x) \in \mathcal{R} \cap \mathcal{M}_\varepsilon$.

$\square$

6.3.3 Lemma
Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment of Δ.

(a) Given $c \in \Sigma_\varepsilon$ and $w \in W$ there exists a unique chamber $d \in \Sigma_{-\varepsilon}$ such that $\delta_*(c, d) = w$.

(b) For any three chambers $c, d, e \in \Sigma_+ \cup \Sigma_-$,

$$\delta(c, e) = \delta(c, d) \cdot \delta(d, e),$$

where each δ is to be interpreted as δ_+, δ_- or δ_*, whichever one makes sense.

(c) Σ is convex.

Proof   This is [AB, 5.173]. $\square$

6.3.4 Corollary
Let $\Delta = (\mathcal{C}_+, \mathcal{C}_-, \delta_*)$ be a twin building, let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment of Δ and let $\mathcal{R}$ be a spherical residue of Δ which meets Σ. Then for any $x \in \Sigma$ we have $\mathrm{proj}_{\mathcal{R}}(x) \in \Sigma$.

Proof   By 6.3.3(c), $\Sigma = (\Sigma_+, \Sigma_-)$ is a convex pair of Δ. The assertion now follows by 6.3.2. $\square$

6.3.5 Proposition
Let $c_+ \in \mathcal{C}_+$ and $c_- \in \mathcal{C}_-$ be opposite chambers of Δ. Then there exists a unique twin apartment of Δ containing c_+ and c_-. This twin apartment will be denoted by $\Sigma\{c_+, c_-\}$. Moreover, for any $d \in \Sigma\{c_+, c_-\} \cap \mathcal{C}_\varepsilon$ we have $\delta_\varepsilon(c_\varepsilon, d) = \delta_*(c_{-\varepsilon}, d)$.

Proof   This follows from [AB, 5.179(1)] and [AB, 5.173(2)]. $\square$

6.3.6 Theorem
Suppose that Δ is thick, choose a chamber $c \in \mathcal{C}_+ \cup \mathcal{C}_-$ and suppose that there is a finite subset $X \subseteq 2^S$ with the following properties:

  (i) Each $J \in X$ is spherical.

 (ii) $\forall\, K, J \in X$ such that $K \cap J \neq \emptyset$: $K \cap J \in X$.

(iii) There exists a family of apartments $(\Sigma_J)_{J \in X}$, where Σ_J is an apartment of $\mathcal{R}_J(c)$ containing c and for $J, K \in X$ with $K \subseteq J$ we have $\Sigma_K = \Sigma_J \cap \mathcal{R}_K(c)$.

Then there exists a twin apartment $\Sigma_\Delta = (\Sigma_+, \Sigma_-)$ of Δ such that for all $J \in X$ we have $\Sigma_\Delta \cap \Sigma_J = \Sigma_J$.

Proof   Choose an apartment system $\mathcal{A}$ of Δ, set $\mathcal{A}_c := \{\Sigma \in \mathcal{A} \mid c \in \Sigma\}$ and note that, in view of [AB, 5.179(3)], $\mathcal{A}_c \neq \emptyset$.
Choose a twin apartment $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}_c$. If $\Sigma \cap \Sigma_J = \Sigma_J$ holds for all $J \in X$, we are done. Thus, suppose that there exists a set $J \in X$ such that $\Sigma \cap \Sigma_J \neq \Sigma_J$. Note that $c \in \Sigma \cap \Sigma_J$. Suppose that $c \in \mathcal{C}_\varepsilon$.

Let $c^o \in \Sigma_J$ be the unique chamber such that $\delta_\varepsilon(c, c^o) = r_J$. Then $c^o \notin \Sigma$, since otherwise $\Sigma_J \subseteq \Sigma$ according to the fact that Σ is convex. Thus there

exist chambers $d, e \in \Sigma_J$ such that $\delta_\varepsilon(d, e) = s$ for some $s \in J$ and $d \in \Sigma$ while $e \notin \Sigma$. Set $\mathcal{P} := \mathcal{P}_s(d) \subseteq \mathcal{R}_J(c)$ and note that $\Sigma_J \cap \mathcal{P} = \{d, e\}$ and $\mathrm{proj}_{\mathcal{P}}(c) = d$.

Claim 1: For all $x \in \bigcup_{K \in X} \Sigma_K$: $\mathrm{proj}_{\mathcal{P}}(x) \in \{d, e\}$.

Proof of claim 1: Let $K \in X$ such that $x \in \Sigma_K$. Note that $c \in \Sigma_K \cap \mathcal{R}_J(c)$. We consider two cases:
First, suppose that $L := K \cap J \neq \emptyset$. Then, by (ii), $L \in X$ and according to (iii), $\Sigma_L = \Sigma_K \cap \mathcal{R}_L(c) = \Sigma_K \cap \mathcal{R}_J(c)$. Similarly we obtain $\Sigma_L = \Sigma_J \cap \mathcal{R}_L(c) = \Sigma_K \cap \mathcal{R}_J(c)$. As Σ_K is convex, [AB, 5.45] implies that

$$\mathrm{proj}_{\mathcal{R}_J(c)}(x) \in \Sigma_K \cap \mathcal{R}_J(c) = \Sigma_L = \Sigma_J \cap \mathcal{R}_K(c) \subseteq \Sigma_J.$$

Hence, using 2.28(a),

$$\mathrm{proj}_{\mathcal{P}}(x) = \mathrm{proj}_{\mathcal{P}}(\mathrm{proj}_{R_J(c)}(x)) \in \mathrm{proj}_{\mathcal{P}}(\Sigma_J) = \{d, e\}.$$

If $K \cap J = \emptyset$, then $\mathcal{R}_J(c) \cap \mathcal{R}_K(c) = \{c\}$ and thus $\mathrm{proj}_{\mathcal{R}_J(c)}(x) = c$ by 2.28(b). Hence,

$$\mathrm{proj}_{\mathcal{P}}(x) = \mathrm{proj}_{\mathcal{P}}(\mathrm{proj}_{R_J(c)}(x)) = \mathrm{proj}_{\mathcal{P}}(c) = d.$$

$\square$

Claim 2: There exists a twin apartment Σ' of Δ with the following properties:

(B_1) $\Sigma' \in \mathcal{A}_c$,
(B_2) $\Sigma \cap \Sigma_K \subseteq \Sigma' \cap \Sigma_K$ for all $K \in X$,
(B_3) $\Sigma \cap \Sigma_J \subsetneq \Sigma' \cap \Sigma_J$.

Proof of claim 2: Let $e' \in \Sigma$ be the unique chamber such that $\mathcal{P} \cap \Sigma = \{d, e'\}$. Define

$$\alpha_\varepsilon := \{x \in \Sigma_\varepsilon \mid \mathrm{dist}(d, x) < \mathrm{dist}(e', x)\}$$

and

$$\alpha_{-\varepsilon} := \{y \in \Sigma_{-\varepsilon} \mid \mathrm{dist}(\mathrm{op}_\Sigma(d), y) < \mathrm{dist}(\mathrm{op}_\Sigma(e'), y)\}.$$

According to [AB, 5.190] the pair $\alpha := (\alpha_+, \alpha_-)$ is a twin root of Σ. By definition $\mathcal{P} \cap \alpha_\varepsilon = \{d\}$. Let $\mathcal{A}(\alpha)$ be the set of all twin apartments containing the twin root α. By [AB, 5.198] the convex hull of the set $\{\{e\} \cup \alpha\}$ is a twin apartment Σ' of Δ. This twin apartment Σ' has the desired properties:

(B_2) Let $K \in X$ and $x \in \Sigma \cap \Sigma_K$ be any chamber. Then by claim 1 and [AB, 5.45] $\operatorname{proj}_{\mathcal{P}}(x) \in \{d, e\} \cap \Sigma = \{d\}$ and thus

$$\operatorname{dist}(x, e') = \operatorname{dist}(x, \operatorname{proj}_{\mathcal{P}}(x)) + \operatorname{dist}(\operatorname{proj}_{\mathcal{P}}(x), e') = \operatorname{dist}(x, d) + 1.$$

We conclude that $x \in \alpha_\varepsilon \subseteq \alpha \subseteq \Sigma'$. Hence $\Sigma \cap \Sigma_K \subseteq \Sigma' \cap \Sigma_K$.

(B_1) Since $c \in \Sigma \cap \Sigma_J$ the assertion follows by (B_2).

(B_3) By (B_2) we already know that $\Sigma \cap \Sigma_J \subseteq \Sigma' \cap \Sigma_J$. Moreover, the chamber e lies in $\Sigma' \cap \Sigma_J$ but not in $\Sigma \cap \Sigma_J$.

$\square$

If $\Sigma' \cap \Sigma_J \neq \Sigma_J$, we apply claim 2 to the twin apartment Σ' to obtain a twin apartment $\Sigma'' \in \mathcal{A}_c$ such that $\Sigma' \cap \Sigma_J \subsetneq \Sigma'' \cap \Sigma_J$ and $\Sigma' \cap \Sigma_K \subseteq \Sigma'' \cap \Sigma_K$ for all $K \in X$. Since $|\Sigma_J| < \infty$ we finally end up with a twin apartment $\bar{\Sigma} \in \mathcal{A}_c$ such that $\bar{\Sigma} \cap \Sigma_J = \Sigma_J$ and $\bar{\Sigma} \cap \Sigma_K \subseteq \Sigma \cap \Sigma_K$.

We now apply claim 2 sequentially to the finitely many $K \in X$ (if necessary) to obtain a twin apartment $\Sigma_\Delta \in \mathcal{A}_c$ such that for all $K \in X$ we have $\Sigma_\Delta \cap \Sigma_K = \Sigma_K$.

$\square$

Chapter 7

Isometries on twin buildings

7.1 Basic concepts

In this section we introduce isomorphisms between twin buildings and prove some basic results for those.

Throughout this section let Π be a Coxeter diagram with vertex set I and let (W, S) be the corresponding Coxeter system.

7.1.1 Definition
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta_+, \Delta_-, \delta_*)$ be twin buildings of type Π. Let $\mathcal{X} \subseteq \mathcal{C}_+ \cup \mathcal{C}_-$ and $\mathcal{X}' \subseteq \mathcal{C}'_+ \cup \mathcal{C}'_-$. A mapping $\varphi \colon \mathcal{X} \to \mathcal{X}'$ is called an

(i) *isomorphism*, if there exists $\sigma \in \mathrm{Aut}(\Pi)$ such that for each $\varepsilon \in \{+, -\}$ the restriction $\varphi|_{\mathcal{X} \cap \mathcal{C}_\varepsilon}$ is a σ-isometry from $\mathcal{X} \cap \mathcal{C}_\varepsilon$ onto $\mathcal{X}' \cap \mathcal{C}'_\varepsilon$ and for all $c \in \mathcal{X} \cap \mathcal{C}_\varepsilon$ and $d \in \mathcal{X} \cap \mathcal{C}_{-\varepsilon}$ we have

$$\delta'_*(\varphi(c), \varphi(d)) = \sigma(\delta_*(c, d)).$$

In this case we also call φ a *σ-isometry*.

(ii) *isometry*, if it is an isomorphism with $\sigma = \mathrm{id}_\Pi$.

As usual, an automorphism of a twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ is an isomorphism from $\mathcal{C}_+ \cup \mathcal{C}_-$ onto $\mathcal{C}_+ \cup \mathcal{C}_-$. We denote the corresponding group by $\mathrm{Aut}(\Delta)$.

7.1.2 Lemma
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be twin buildings of type Π, let $\varphi \colon \mathcal{C}_+ \cup \mathcal{C}_- \to \mathcal{C}'_+ \cup \mathcal{C}'_-$ be an isometry and let Σ be a twin apartment of Δ. Then $\varphi(\Sigma)$ is a twin apartment of Δ'.

Proof Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment of Δ. For each $\varepsilon \in \{+, -\}$ the map $\varphi|_{\Sigma_\varepsilon}$ is an isometry from Σ_ε onto $\varphi(\Sigma_\varepsilon)$. As Σ_ε is an apartment of Δ_ε, [AB, 5.67] provides an isometry $\phi \colon W \to \Sigma_\varepsilon$ from the standard thin building (W, δ_W) of type (W, S) onto the apartment Σ_ε. Thus, $\varphi \circ \phi \colon W \to \varphi(\Sigma_\varepsilon)$ is an isometry and hence, by [AB, 5.67], $\varphi(\Sigma_\varepsilon)$ is an apartment of Δ'. Choose a chamber $c \in \varphi(\Sigma_+) \cup \varphi(\Sigma_-)$ and let $x \in \Sigma_+ \cup \Sigma_-$ be the unique chamber such that $\varphi(x) = c$. Let $y := \mathrm{op}_\Sigma(x)$ be the unique chamber in $\Sigma_+ \cup \Sigma_-$ such that $\delta_*(x, y) = 1_W$ and let $d := \varphi(y)$. Then

$$\delta'_*(c, d) = \delta'_*(\varphi(x), \varphi(y)) = \delta_*(x, y) = 1_W$$

and we conclude that $d \in \varphi(\Sigma_+) \cup \varphi(\Sigma_-)$ is a chamber which is opposite to c. Let $e \in \varphi(\Sigma_+) \cup \varphi(\Sigma_-)$ be a chamber opposite to c and let $z \in \Sigma_+ \cup \Sigma_-$ be such that $\varphi(z) = e$. Then x and z are opposite and, by [AB, 5.178], it follows that $z = y$. Hence, $e = d$ and thus the pair $\varphi(\Sigma) = (\varphi(\Sigma_+), \varphi(\Sigma_-))$ is a twin apartment of Δ'. $\square$

The following lemma will be called *standard uniqueness argument for twin buildings*:

7.1.3 Lemma
Let Δ be a twin building of type Π, let Σ be a twin apartment of Δ and let $\varphi \in \mathrm{Aut}(\Delta)$ be an isometry. Then φ fixes Σ pointwise if and only if it stabilizes Σ and fixes some chamber $c \in \Sigma$.

Proof The only-if part is clear.
Let $c \in \Sigma$ be such that $\varphi(c) = c$ and let $\varepsilon \in \{+, -\}$ such that $c \in \Sigma_\varepsilon$. Choose any chamber $d \in \Sigma$ and let $w := \delta_\varepsilon(c, d)$ if $d \in \Sigma_\varepsilon$ and $w := \delta_*(c, d)$ if $d \in \Sigma_{-\varepsilon}$. As φ is an isometry we have $\varphi(\Sigma_+) = \Sigma_+$ as well as $\varphi(\Sigma_-) = \Sigma_-$ and

$$w = \delta_\diamond(c, d) = \delta_\diamond(\varphi(c), \varphi(d)) = \delta_\diamond(c, \varphi(d)),$$

where $\delta_\diamond$ is to be interpreted as δ_ε or δ_*, whichever one makes sense. By [AB, 5.66] and 6.3.3(a), it follows that $\varphi(d) = d$. $\square$

7.1.4 Lemma
Let Δ and Δ' be two twin buildings of type Π. Let $\Sigma = (\Sigma_+, \Sigma_-)$ and $\Sigma' = (\Sigma'_+, \Sigma'_-)$ be twin apartments of Δ and Δ' respectively. For $\varepsilon \in \{+, -\}$ and any pair of chambers $(c, c') \in \Sigma_\varepsilon \times \Sigma'_\varepsilon$ there exists a unique isometry $\phi \colon \Sigma \to \Sigma'$ sending c onto c'.

Proof Let $\varepsilon \in \{+, -\}$ and choose $c \in \Sigma_\varepsilon$ and $c' \in \Sigma'_\varepsilon$. Let (W, δ_W) be the standard thin building of type (W, S) (cf. 2.23). Consider the map $\phi_c \colon \Sigma_\varepsilon \to W, x \mapsto \delta_\varepsilon(x, c)$. Then, for $x, y \in \Sigma_\varepsilon$,

$$\delta_W(\phi_c(x), \phi_c(y)) = \delta_\varepsilon(x, c)\delta_\varepsilon(y, c)^{-1} = \delta_\varepsilon(x, c)\delta_\varepsilon(c, y) = \delta_\varepsilon(x, y),$$

where the last equality follows from 6.3.3(b). By [AB, 5.66], the map ϕ_c is bijective and hence an isometry. Similarly we define an isometry $\phi_{c'}\colon \Sigma'_\varepsilon \to W$ via $\phi_{c'}(x) := \delta'_\varepsilon(x, c')$ and

$$\phi_\varepsilon := \phi_{c'}^{-1} \circ \phi_c \colon \Sigma_\varepsilon \to \Sigma'_\varepsilon$$

which is an isometry sending c onto c'.

Let $d := \mathrm{op}_\Sigma(c) \in \Sigma_{-\varepsilon}$ be the unique chamber opposite c. Similarly, we let $d' := \mathrm{op}_{\Sigma'}(c') \in \Sigma'_{-\varepsilon}$ be the unique chamber opposite c'. Just as the construction above, there exists an isometry $\phi_{-\varepsilon}\colon \Sigma_{-\varepsilon} \to \Sigma'_{-\varepsilon}$ sending d onto d'. We define

$$\phi\colon \Sigma \to \Sigma' \text{ via } \phi(x) := \begin{cases} \phi_\varepsilon(x), & x \in \Sigma_\varepsilon \\ \phi_{-\varepsilon}(x), & x \in \Sigma_{-\varepsilon} \end{cases}.$$

Suppose that $x \in \Sigma_\varepsilon$ and $y \in \Sigma_{-\varepsilon}$. In view of 6.3.5 we have $\Sigma = \Sigma\{c, d\}$ and $\Sigma' = \Sigma\{c', d'\}$ and

$$\delta_{-\varepsilon}(d, y) = \delta_*(c, y) \text{ and } \delta'_{-\varepsilon}(d', \phi(y)) = \delta'_*(c', \phi(y)).$$

Using 6.3.3(b) we obtain

$$\begin{aligned} \delta_*(x, y) &= \delta_\varepsilon(x, c)\delta_*(c, y) = \delta'_\varepsilon(\phi(x), c')\delta_{-\varepsilon}(d, y) \\ &= \delta'_\varepsilon(\phi(x), c')\delta'_{-\varepsilon}(d', \phi(y)) = \delta'_\varepsilon(\phi(x), c')\delta'_*(c', \phi(y)) \\ &= \delta'_*(\phi(x), \phi(y)). \end{aligned}$$

Hence, $\phi\colon \Sigma \to \Sigma'$ is an isometry with the desired properties. Let $\psi\colon \Sigma \to \Sigma'$ be an isometry sending c onto c'. Then $\psi^{-1} \circ \phi\colon \Sigma \to \Sigma$ stabilizes Σ and fixes the chamber c. By 7.1.3, $\psi^{-1} \circ \phi$ is the identity. $\qquad\square$

7.1.5 Lemma
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be twin buildings of type Π. Let $\mathcal{R} \subseteq \mathcal{C}_+$ and $\mathcal{R}' \subseteq \mathcal{C}'_+$ be spherical residues of the same type and let $\varphi\colon \mathcal{R} \to \mathcal{R}'$ be an isometry. Let $(c, d) \in \mathcal{R} \times \mathcal{C}_-$ and $(c', d') \in \mathcal{R}' \times \mathcal{C}'_-$ be pairs of opposite chambers. If $\varphi(\mathrm{proj}_\mathcal{R}(d)) = \mathrm{proj}_{\mathcal{R}'}(d')$, then the map $d \mapsto d'$ extends φ to an isometry from $\mathcal{R} \cup \{d\}$ onto $\mathcal{R}' \cup \{d'\}$.

Proof Let $J \subseteq S$ denote the type of the residues $\mathcal{R}$ and $\mathcal{R}'$. Choose a chamber $x \in \mathcal{R}$. Then

$$\begin{aligned} \delta'_*(\varphi(x), d') &= \delta'_+(\varphi(x), \mathrm{proj}_{\mathcal{R}'}(d'))\, \delta'_*(\mathrm{proj}_{\mathcal{R}'}(d'), d') \\ &= \delta'_+(\varphi(x), \varphi(\mathrm{proj}_\mathcal{R}(d))\, r_J \\ &= \delta_+(x, \mathrm{proj}_\mathcal{R}(d))\, \delta_*(\mathrm{proj}_\mathcal{R}(d), d) = \delta_*(x, d). \end{aligned}$$

$\square$

It follows from the extension theorem of B. Mühlherr and M. Ronan in [MR]
that almost all thick, irreducible, 2-spherical twin buildings of rank at least
3 are completely determined by their local structure. In order to apply this
result we need the following preliminary work:

7.1.6 Proposition

Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be thick 2-spherical twin
buildings of type Π. Let $c \in \mathcal{C}_+$ and $c' \in \mathcal{C}'_+$ be two chambers and suppose
that there is an isometry $\varphi \colon E_2(c) \to E_2(c')$ sending c onto c'. Then there
exist chambers $d \in \mathcal{C}_-$ and $d' \in \mathcal{C}'_-$ which are opposite to c and c' respectively
and such that the map $d \mapsto d'$ extends φ to an isometry from $E_2(c) \cup \{d\}$
onto $E_2(c') \cup \{d'\}$.

Proof The basic idea of the proof is to start with arbitrary chambers d
and d' which are opposite to c and c' respectively. We then go through all
subsets $J \subseteq S$ of cardinality at most 2 one after another (starting with the
subsets of cardinality 1) and "correct" the chamber d', if necessary.

Choose a chamber $d \in \mathcal{C}_-$ which is opposite to c. For $x \in \mathcal{C}'_-$ such that x is
opposite to c' we define $S_x := \{s \in S \mid \varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(d)) = \mathrm{proj}_{\mathcal{P}_s(c')}(x)\}$. If
$x \in \mathcal{C}'_-$ is opposite to c' such that $S_x = S$, then the map $d \mapsto x$ extends the
restriction $\varphi|_{E_1(c)} \colon E_1(c) \to E_1(c')$ to an isometry $E_1(c) \cup \{d\} \to E_1(c') \cup \{x\}$:
Indeed, let $y \in E_1(c)$ and let $s \in S$ such that $y \in \mathcal{P}_s(c)$. As $s \in S = S_x$,
7.1.5 yields that the map $d \mapsto x$ extends the isometry $\mathcal{P}_s(c) \to \mathcal{P}_s(c')$ to an
isometry from $\mathcal{P}_s(c) \cup \{d\}$ onto $\mathcal{P}_s(c') \cup \{x\}$ and whence $\delta'_*(\varphi(y), x) = \delta_*(y, d)$.

Let $d' \in \mathcal{C}'_-$ such that c' and d' are opposite. If $S_{d'} = S$, we are done. So
suppose that there is $s \in S$ such that $\varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(d)) \neq \mathrm{proj}_{\mathcal{P}_s(c')}(d')$.
Since the chambers c and d are opposite chambers of Δ, $\mathrm{proj}_{\mathcal{P}_s(c)}(d) \neq c$
and hence $\varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(d)) \neq c'$. Let $d'' := \mathrm{proj}_{\mathcal{P}_s(d')}(\varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(d)))$. As the
panels $\mathcal{P}_s(c')$ and $\mathcal{P}_s(d')$ are opposite, [AB, 5.153] implies that d'' is opposite
to the chamber c' and, in view of 6.2.5 and 6.2.2,

$$\mathrm{proj}_{\mathcal{P}_s(c')}(d'') = \mathrm{proj}_{\mathcal{P}_s(c')}(\mathrm{proj}_{\mathcal{P}_s(d')}(\varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(d)))) = \varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(d)).$$

In particular, $s \in S_{d''}$. Moreover, we have $S_{d'} \subseteq S_{d''}$:
To see this let $t \in S_{d'}$, i.e. $\varphi(\mathrm{proj}_{\mathcal{P}_t(c)}(d)) = \mathrm{proj}_{\mathcal{P}_t(c')}(d')$. We need to show
that $\varphi(\mathrm{proj}_{\mathcal{P}_t(c)}(d)) = \mathrm{proj}_{\mathcal{P}_t(c')}(d'')$. Note that, by definition, $d'' \in \mathcal{P}_s(d')$.
By 6.1.4(a), the set $\mathrm{proj}_{\mathcal{P}_t(c')}(\mathcal{P}_s(d'))$ is a residue of $\mathcal{P}_t(c')$ whose type is, by
6.1.4(b), $t(\{t\} \cap \{s\})t = \emptyset$, i.e. $\mathrm{proj}_{\mathcal{P}_t(c')}(\mathcal{P}_s(d'))$ is a single chamber. Hence,
$\mathrm{proj}_{\mathcal{P}_t(c')}(d'') = \mathrm{proj}_{\mathcal{P}_t(c')}(d') = \varphi(\mathrm{proj}_{\mathcal{P}_s(c)}(d))$.
Thus, d'' is a chamber which is opposite to c' and $S_{d'} \cup \{s\} \subseteq S_{d''}$. We
replace d' by d''.

Applying the construction described above sequentially to each of the finitely
many $s \in S$ (if necessary), we finally obtain a chamber $d' \in \mathcal{C}'_-$ which is op-

posite to the chamber c' and which satisfies $S_{d'} = S$.

We translate the ideas of the rank 1 case to the rank 2 case without loosing the achievements we obtained so far. Let S' denote the set of all subsets of S of cardinality 2. Just as in the rank 1 case we define for a chamber $x \in \mathcal{C}'_-$ which is opposite to c' a subset

$$J_x := \{J \in S' \mid \varphi(\mathrm{proj}_{\mathcal{R}_J(c)}(d)) = \mathrm{proj}_{\mathcal{R}_J(c')}(x)\}.$$

If $x \in \mathcal{C}'_-$ is opposite to c' such that $J_x = S'$, then the map $d \mapsto x$ extends the isometry $\varphi \colon E_2(c) \to E_2(c')$ to an isometry $E_2(c) \cup \{d\} \to E_2(c') \cup \{x\}$ just as desired: Indeed, let $y \in E_2(c)$ and let $J \in S'$ such that $y \in \mathcal{R}_J(c)$. As $J \in S' = J_x$, 7.1.5 yields that the map $d \mapsto x$ extends the isometry $\mathcal{R}_J(c) \to \mathcal{R}_J(c')$ to an isometry from $\mathcal{R}_J(c) \cup \{d\}$ onto $\mathcal{R}_J(c') \cup \{x\}$ and whence $\delta'_*(\varphi(y), x) = \delta_*(y, d)$.

Let $d' \in \mathcal{C}'_-$ be the a chamber which is opposite to c' and such that $S_{d'} = S$. If $J_{d'} = S'$ we are done.
So suppose that $J \in S'$ is such that $\mathrm{proj}_{\mathcal{R}_J(c')}(d') \neq \varphi(\mathrm{proj}_{\mathcal{R}_J(c)}(d))$.
Let $d'' := \mathrm{proj}_{\mathcal{R}_J(d')}(\varphi(\mathrm{proj}_{\mathcal{R}_J(c)}(d)))$. Since the residues $\mathcal{R}_J(c')$ and $\mathcal{R}_J(d')$ are opposite we have

$$\mathrm{proj}_{\mathcal{R}_J(c')}(d'') = \mathrm{proj}_{\mathcal{R}_J(c')}(\mathrm{proj}_{\mathcal{R}_J(d')}(\varphi(\mathrm{proj}_{\mathcal{R}_J(c)}(d)))) = \varphi(\mathrm{proj}_{\mathcal{R}_J(c)}(d))$$

in view of 6.2.5 and 6.2.2. According to 6.2.6,

$$\delta_*(d, \mathrm{proj}_{\mathcal{R}_J(c)}(d)) = r_J = \delta'_*(d'', \mathrm{proj}_{\mathcal{R}_J(c')}(d''))$$

and hence the calculation

$$\delta'_*(c', d'') = \delta'_+(c', \mathrm{proj}_{\mathcal{R}_J(c')}(d'')) \; r_J = \delta_+(c, \mathrm{proj}_{\mathcal{R}_J(c)}(d)) \; r_J$$
$$= \delta_*(c, d) \; \delta_*(d, \mathrm{proj}_{\mathcal{R}_J(c)}(d)) \; r_J = 1_W$$

shows that the chamber d'' is opposite to the chamber c'. In particular, $J \in J_{d''}$.
We show that the chamber d'' has the following properties:

(i) $S_{d''} = S$

(ii) $J_{d'} \subseteq J_{d''}$

To (i): Let $s \in S$. First suppose that $s \in J$. Then, by 6.1.5,

$$\mathrm{proj}_{\mathcal{P}_s(c)}(d) = \mathrm{proj}_{\mathcal{P}_s(c)}(\mathrm{proj}_{\mathcal{R}_J(c)}(d))$$

as well as

$$\mathrm{proj}_{\mathcal{P}_s(c')}(d'') = \mathrm{proj}_{\mathcal{P}_s(c')}(\mathrm{proj}_{\mathcal{R}_J(c')}(d'')).$$

Now it follows from [MR, 4.2] that

$$\varphi(\text{proj}_{\mathcal{P}_s(c)}(d)) = \varphi(\text{proj}_{\mathcal{P}_s(c)}(\text{proj}_{\mathcal{R}_J(c)}(d)))$$
$$= \text{proj}_{\mathcal{P}_s(c')}(\text{proj}_{\mathcal{R}_J(c')}(d'')) = \text{proj}_{\mathcal{P}_s(c')}(d'').$$

If $s \notin J$, then $\text{proj}_{\mathcal{P}_s(c')}(d'') \in \text{proj}_{\mathcal{P}_s(c')}(\mathcal{R}_J(d'))$ which is, by 6.1.4, a residue of $\mathcal{P}_s(c')$ of type $s(\{s\} \cap J)s = \emptyset$. Hence, $\text{proj}_{\mathcal{P}_s(c')}(\mathcal{R}_J(d'))$ is a single chamber and thus $\text{proj}_{\mathcal{P}_s(c')}(d'') = \text{proj}_{\mathcal{P}_s(c')}(d') = \varphi(\text{proj}_{\mathcal{P}_s(c)}(d))$, as desired.

To (ii): Let $K \in J_{d'}$. Then, by definition, $\text{proj}_{\mathcal{R}_K(c')}(d') = \varphi(\text{proj}_{\mathcal{R}_K(c)}(d))$. We need to show that $\text{proj}_{\mathcal{R}_K(c')}(d'') = \varphi(\text{proj}_{\mathcal{R}_K(c)}(d))$.

Suppose that $J \cap K = \emptyset$. Then, $\text{proj}_{\mathcal{R}_K(c')}(d'') \in \text{proj}_{\mathcal{R}_K(c')}(\mathcal{R}_J(d'))$. By 6.1.4(a), the set $\text{proj}_{\mathcal{R}_K(c')}(\mathcal{R}_J(d'))$ is a residue of $\mathcal{R}_K(c')$ whose type is, by 6.1.4(b), $r_K(K \cap J)r_K = \emptyset$. Hence, it is a single chamber and therefore $\text{proj}_{\mathcal{R}_K(c')}(d'') = \text{proj}_{\mathcal{R}_K(c')}(d') = \varphi(\text{proj}_{\mathcal{R}_K(c)}(d))$.

Suppose that $K \cap J = \{s\}$ for some $s \in S$. Let $x := \text{proj}_{\mathcal{R}_K(c')}(d')$ and $y := \text{proj}_{\mathcal{R}_K(c')}(d'')$ and suppose that $x \neq y$. Since the projection $\text{proj}_{\mathcal{R}_K(c')}(\mathcal{R}_J(d'))$ is a residue of $\mathcal{R}_K(c')$ of type $r_K(K \cap J)r_K = r_K s r_K$ we have $\delta'_+(x, y) = r_K s r_K$.

Note that, by (i),

$$\varphi(\text{proj}_{\mathcal{P}_s(c)}(d)) = \text{proj}_{\mathcal{P}_s(c')}(d'') = \text{proj}_{\mathcal{P}_s(c')}(\text{proj}_{\mathcal{R}_K(c')}(d''))$$

and similarly, as $S_{d'} = S$,

$$\varphi(\text{proj}_{\mathcal{P}_s(c)}(d)) = \text{proj}_{\mathcal{P}_s(c')}(d') = \text{proj}_{\mathcal{P}_s(c')}(\text{proj}_{\mathcal{R}_K(c')}(d')).$$

In particular, $\text{proj}_{\mathcal{P}_s(c')}(x) = \text{proj}_{\mathcal{P}_s(c')}(y)$.

As the chambers d' and c' are opposite, [MR, 3.3] gives that $\delta'_+(c, x) = r_K$ and hence $\delta'_+(x, \text{proj}_{\mathcal{P}_s(c')}(x)) = r_K s$. Now, as $\delta'_+(y, x) = r_K s r_K$ and $\ell(r_K s r_K \ r_K s) = \ell(r_K) = \ell(r_K s) + 1$, (WD2) implies that

$$\delta'_+(y, \text{proj}_{\mathcal{P}_s(c')}(x)) = \delta'_+(y, \text{proj}_{\mathcal{P}_s(c')}(y)) = r_K,$$

a contradiction. Thus, $\text{proj}_{\mathcal{R}_K(c')}(d'') = y = x = \text{proj}_{\mathcal{R}_K(c')}(d')$.

Thus, d'' is a chamber which is opposite to c' and $J_{d'} \cup \{J\} \subseteq J_{d''}$. We replace d' by d''.

Applying the construction described above sequentially to each of the finitely many $J \in S'$ (if necessary), we finally obtain a chamber $d' \in \mathcal{C}'_-$ which is opposite to the chamber c' and which satisfies $J_{d'} = S'$.

$\square$

7.2   Translates

Let Π be a Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let $\sigma \in \mathrm{Aut}(W, S)$. We denote by $\ell \colon W \to \mathbb{N}$ the length function on W with respect to S.

7.2.1 Definition
Let $\Delta = (\mathcal{C}, \delta)$ be a building of type Π. We define a map $\delta^\sigma \colon \mathcal{C} \times \mathcal{C} \to W$ via

$$\delta^\sigma(c, d) := \sigma(\delta(c, d))$$

for all $c, d \in \mathcal{C}$.

7.2.2 Lemma
Let $\Delta = (\mathcal{C}, \delta)$ be a building of type Π and let $\delta^\sigma \colon \mathcal{C} \times \mathcal{C} \to W$ be defined as in 7.2.1. Then the pair $\Delta^\sigma := (\mathcal{C}, \delta^\sigma)$ is a building of type Π and the identity-map on $\mathcal{C}$ is a σ-isometry from Δ onto Δ^σ.

Proof   First, we verify the axioms (WD1)-(WD3):

(WD1) Let $c, d \in \mathcal{C}$ be chambers such that $\delta^\sigma(c, d) = 1_W$. Then

$$\delta(c, d) = \sigma^{-1}(\delta^\sigma(c, d)) = \sigma^{-1}(1_W) = 1_W$$

and thus $c = d$. Conversely, if $c = d$ we have

$$\delta^\sigma(c, d) = \sigma(\delta(c, d)) = \sigma(1_W) = 1_W.$$

(WD2) Let $c, d \in \mathcal{C}$ be chambers and set $w := \delta^\sigma(c, d)$. Let $c' \in \mathcal{C}$ such that $\delta^\sigma(c', c) = s \in S$. By definition this yields $\delta(c, d) = \sigma^{-1}(w)$ and $\delta(c', c) = \sigma^{-1}(s) \in S$. Thus, applying axiom (WD2) of Δ, we obtain

$$\delta(c', d) \in \{\sigma^{-1}(sw), \sigma^{-1}(w)\}.$$

Hence, $\delta^\sigma(c', d) = \sigma(\delta(c', d)) \in \{sw, w\}$.

Suppose that $\ell(sw) = \ell(w) + 1$. Since σ (and thus σ^{-1}) is an automorphism we have $\ell(\sigma^{-1}(sw)) = \ell(sw) = \ell(w) + 1 = \ell(\sigma^{-1}(w)) + 1$. Thus, again by (WD2), $\delta(c', d) = \sigma^{-1}(sw)$ and $\delta^\sigma(c', d) = \sigma(\delta(c', d)) = ws$.

(WD3) Let $c, d \in \mathcal{C}$ and $w := \delta^\sigma(c, d)$. Choose $s' \in S$ and let $s \in S$ be the unique element such that $\sigma(s) = s'$. By (WD3) there exists a chamber $c' \in \mathcal{C}$ such that $\delta(c', c) = s$ and $\delta(c', d) = s\sigma^{-1}(w)$. Thus the chamber $c' \in \mathcal{C}$ satisfies $\delta^\sigma(c', c) = \sigma(s) = s'$ and $\delta^\sigma(c', d) = s'w$.

Hence, $\Delta^\sigma := (\mathcal{C}, \delta^\sigma)$ is a building of type Π.

Consider the map $\mathrm{id}_\mathcal{C} \colon \mathcal{C} \to \mathcal{C}$ given by $\mathrm{id}_\mathcal{C}(c) = c$ for all $c \in \mathcal{C}$. Then

$$\delta^\sigma(\mathrm{id}_\mathcal{C}(c), \mathrm{id}_\mathcal{C}(d)) = \delta^\sigma(c,d) = \sigma(\delta(c,d))$$

shows that $\mathrm{id}_\mathcal{C}$ is a σ-isometry from Δ onto Δ^σ. $\qquad\qquad\square$

The building Δ^σ is called the *translate of Δ with respect to σ*. A similar construction can be done for twin buildings:

7.2.3 Definition
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type Π. We define a map $\delta_*^\sigma \colon (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ via

$$\delta_*^\sigma(c,d) := \sigma(\delta_*(c,d))$$

for all $(c,d) \in (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$.

7.2.4 Lemma
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type Π. For $\varepsilon \in \{+, -\}$ let $\Delta_\varepsilon^\sigma$ be the translate of Δ_ε with respect to σ as described in 7.2.2. Let $\delta_*^\sigma \colon (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ be defined as in 7.2.3. Then the triple $\Delta^\sigma := (\Delta_+^\sigma, \Delta_-^\sigma, \delta_*^\sigma)$ is a twin building of type Π and the identity-map on $\mathcal{C}_+ \cup \mathcal{C}_-$ is a σ-isometry from Δ onto Δ^σ.

Proof Let $\varepsilon \in \{+, -\}$. According to 7.2.2 the pair $\Delta_\varepsilon^\sigma = (\mathcal{C}_\varepsilon, \delta_\varepsilon^\sigma)$ is a building of type Π. We verify the axioms (Tw1)-(Tw3): Let $c \in \mathcal{C}_\varepsilon, d \in \mathcal{C}_{-\varepsilon}$ and set $w := \delta_*^\sigma(c,d) \in W$.

(Tw1) Applying (Tw1) for Δ we obtain

$$\delta_*^\sigma(c,d) = \sigma(\delta_*(c,d)) = \sigma(\delta_*(d,c)^{-1}) = \sigma(\delta_*(d,c))^{-1} = \delta_*^\sigma(d,c)^{-1}.$$

(Tw2) Let $c' \in \mathcal{C}_\varepsilon$ be a chamber such that $\delta_\varepsilon^\sigma(c',c) = s \in S$. Suppose that $\ell(sw) < \ell(w)$. Then $\ell(\sigma^{-1}(w)) = \ell(w) > \ell(sw) = \ell(\sigma^{-1}(sw))$ and $\delta_\varepsilon(c',c) = \sigma^{-1}(s) \in S$. Axiom (Tw2) yields $\delta_*(c',d) = \sigma^{-1}(sw)$ and thus $\delta_*^\sigma(c',d) = sw$.

(Tw3) Let $s \in S$. Since $s' := \sigma^{-1}(s) \in S$, (Tw3) for Δ provides a chamber $c' \in \mathcal{C}_\varepsilon$ such that $\delta_\varepsilon(c',c) = s'$ and $\delta_*(c',d) = s'\sigma^{-1}(w)$. The chamber c' is as desired, since $\delta_\varepsilon^\sigma(c',c) = s$ and $\delta_*^\sigma(c',d) = sw$.

Hence, $\Delta^\sigma := (\Delta_+^\sigma, \Delta_-^\sigma, \delta_*^\sigma)$ is a twin building of type Π.

Let $\mathcal{C} := \mathcal{C}_+ \cup \mathcal{C}_-$ and let $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ be defined by $\mathrm{id}_{\mathcal{C}}(c) = c$ for all $c \in \mathcal{C}$. By 7.2.2, the restriction $\mathrm{id}_{\mathcal{C}}|_{\mathcal{C}_\varepsilon}$ is a σ-isometry from Δ_ε onto $\Delta_\varepsilon^\sigma$ for each $\varepsilon \in \{+, -\}$. Let $c \in \mathcal{C}_\varepsilon$ and $d \in \mathcal{C}_{-\varepsilon}$. Then

$$\delta_*^\sigma(\mathrm{id}_{\mathcal{C}}(c), \mathrm{id}_{\mathcal{C}}(d)) = \sigma(\delta_*(\mathrm{id}_{\mathcal{C}}(c), \mathrm{id}_{\mathcal{C}}(d))) = \sigma(\delta_*(c, d))$$

shows that $\mathrm{id}_{\mathcal{C}}$ is a σ-isometry from Δ onto Δ^σ. $\qquad\qquad\square$

The twin building Δ^σ is called the *translate of Δ with respect to σ*.

7.3   An extension theorem

One of the main results in [Ti74] asserts that a building of spherical type is
determined by its local structure. More precisely, an isomorphism between
thick spherical buildings is uniquely determined by what it does on a small
part of the domain. There is also an existence theorem, which says that an
isomorphism can be arbitrarily prescribed in the neighbourhood of a given
chamber, provided the buildings are irreducible and of rank at least 3.
Even though the rigidity theorem and its proof remain valid for automor-
phisms of thick twin buildings, the situation for the extension theorem is
more complicated. There does exist an extension theorem for a class of twin
buildings, due to work of Mühlherr and Ronan in [MR] that is based on ear-
lier results of Tits [Ti92]. The following statement follows from [MR, 1.1,1.2
and 1.3]:

7.3.1 Theorem
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be thick 2-spherical twin build-
ings of the same type which satisfy condition (co). Let $(c_+, c_-) \in \mathcal{C}_+ \times \mathcal{C}_-$
and $(c'_+, c'_-) \in \mathcal{C}'_+ \times \mathcal{C}'_-$ be two pairs of opposite chambers. Then each isom-
etry from $E_2(c_+) \cup \{c_-\}$ onto $E_2(c'_+) \cup \{c'_-\}$ extends uniquely to an isometry
from Δ onto Δ'.

Throughout this section let Π be a Coxeter diagram with vertex set I and
let (W, S) be the corresponding Coxeter system.

We will be interested in the following consequences of the extension theorem
7.3.1:

7.3.2 Corollary
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be thick 2-spherical twin
buildings of type Π which satisfy condition (co) and let $c \in \mathcal{C}_+$ and $c' \in \mathcal{C}'_+$
be chambers. Let $\Sigma = (\Sigma_+, \Sigma_-)$ and $\Sigma' = (\Sigma'_+, \Sigma'_-)$ be twin apartments of
Δ and Δ' containing the chambers c and c' respectively.
Let $\phi \colon E_2(c) \to E_2(c')$ be an isometry with $\phi(c) = c'$ and let $\psi \colon \Sigma \to \Sigma'$ be
an isometry that coincides with ϕ on the intersection $\Sigma \cap E_2(c)$. Then there
is a unique isometry from Δ onto Δ' extending ϕ and ψ.

Proof   Let Σ, Σ', c and c' be as in the theorem. Let $d := \mathrm{op}_\Sigma(c)$ and
$d' := \mathrm{op}_{\Sigma'}(c')$ and note that $\Sigma' = \Sigma\{c', d'\}$ is the unique twin apartment
of Δ' containing the opposite chambers c' and d'. As $c \in E_2(c) \cap \Sigma$ the
assumption on ϕ and ψ yields $\psi(c) = \phi(c) = c'$.

We proceed in a series of steps:

      Claim 1: $\psi(d) = d'$

Proof of claim 1: Since $d \in \Sigma$ it follows that $\psi(d) \in \Sigma'$. As

$$\delta_*(\psi(d), c') = \delta_*(\psi(d), \psi(c)) = \delta_*(d, c) = 1_W,$$

we conclude that $\psi(d) = \mathrm{op}_{\Sigma'}(c') = d'$.

$\square$

Claim 2: Let $J \subseteq S$ with $|J| \le 2$ and set $\mathcal{R} := \mathcal{R}_J(c)$ as well as $\mathcal{R}' := \mathcal{R}_J(c')$. Then $\phi(\mathrm{proj}_{\mathcal{R}}(d)) = \mathrm{proj}_{\mathcal{R}'}(d')$.

Proof of claim 2: Note that J is spherical. According to 6.3.4 we have $\mathrm{proj}_{\mathcal{R}}(d) \in \Sigma$, since Σ is convex and $\Sigma \cap \mathcal{R} \ne \emptyset$. In particular, $\mathrm{proj}_{\mathcal{R}}(d) \in \Sigma \cap \mathcal{R} \subseteq \Sigma \cap E_2(c)$. The assumption on ϕ and ψ gives $\phi(\mathrm{proj}_{\mathcal{R}}(d)) = \psi(\mathrm{proj}_{\mathcal{R}}(d)) \in \Sigma'$. Similarly, lemma 6.3.4 gives that $\mathrm{proj}_{\mathcal{R}'}(d') \in \Sigma'$.
We use 6.3.5 and obtain

$$
\begin{aligned}
\delta_*'(d', \phi(\mathrm{proj}_{\mathcal{R}}(d))) &= \delta_\varepsilon'(c', \phi(\mathrm{proj}_{\mathcal{R}}(d))) = \delta_\varepsilon'(\phi(c), \phi(\mathrm{proj}_{\mathcal{R}}(d))) \\
&= \delta_\varepsilon(c, \mathrm{proj}_{\mathcal{R}}(d)) = \delta_*(c, d)\, \delta_*(\mathrm{proj}_{\mathcal{R}}(d), d)^{-1} \\
&= 1_W \max\{W_J\}^{-1} = r_J.
\end{aligned}
$$

On the other hand we have

$$\delta_*'(d', \mathrm{proj}_{\mathcal{R}'}(d')) = \max\{\delta_*'(d', c')W_J\} = r_J.$$

In view of 6.3.3(a) we have $\mathrm{proj}_{\mathcal{R}'}(d') = \phi(\mathrm{proj}_{\mathcal{R}}(d))$. $\square$

Claim 3: For all $x \in E_2(c)$ we have $\delta_*(x, d) = \delta_*'(\phi(x), d')$.

Proof of claim 3: Let $x \in E_2(c)$ and let $J \subseteq S$ with $|J| \le 2$ such that $x \in \mathcal{R} := \mathcal{R}_J(c)$. Then $\phi(x) \in \mathcal{R}' := \mathcal{R}_J(c')$ and by using claim 2 we obtain

$$
\begin{aligned}
\delta_*'(\phi(x), d') &= \delta_\varepsilon'(\phi(x), \mathrm{proj}_{\mathcal{R}'}(d'))\, \delta_*'(\mathrm{proj}_{\mathcal{R}'}(d'), d') \\
&= \delta_\varepsilon'(\phi(x), \phi(\mathrm{proj}_{\mathcal{R}}(d)))\, r_J \\
&= \delta_\varepsilon(x, \mathrm{proj}_{\mathcal{R}}(d))\, r_J \\
&= \delta_\varepsilon(x, \mathrm{proj}_{\mathcal{R}}(d))\, \delta_*(\mathrm{proj}_{\mathcal{R}}(d), d) \\
&= \delta_*(x, d).
\end{aligned}
$$

$\square$

Thus, the map $d \mapsto d'$ extends ϕ to an isometry $E_2(c) \cup \{d\} \to E_2(c') \cup \{d'\}$. Applying 7.3.1 we obtain a unique isometry $\varphi \colon \Delta \to \Delta'$ extending ϕ.

We consider the restriction $\varphi|_\Sigma$. According to 7.1.2, $\varphi(\Sigma)$ is a twin apartment of Δ'. Moreover, as $\varphi(\Sigma)$ containins the chambers $\varphi(c) = \phi(c) = c'$ and $\varphi(d) = d'$, we conclude that $\varphi(\Sigma) = \Sigma'$. Hence, $\varphi|_\Sigma$ is an isometry from Σ onto Σ' mapping c onto c'. In view of 7.1.4 we have $\varphi|_\Sigma = \psi$. $\square$

Before we will give another consequence of the extension theorem, which we will apply in the sequel to construct certain automorphisms on twin buildings, we recall the definition of an essential set (cf. 4.2.1): A subset $X \subseteq 2^S$ is called essential, if $S \notin X, \cup_{M \in X} M = S$ and if for each irreducible subset $J \subseteq S$ having cardinality 2 there is a set $M \in X$ such that $J \subseteq M$.

7.3.3 Corollary
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be thick 2-spherical twin buildings of type Π which satisfy condition (co) and let $c \in \mathcal{C}_+$ and $c' \in \mathcal{C}'_+$ be chambers. Let $X \subseteq 2^S$ be an essential set such that each subset $M \in X$ is spherical. Let $\varphi \colon E_X(c) \to E_X(c')$ be a bijective map such that $\varphi(c) = c'$ and such that for all $M \in X$ the restriction $\varphi_M := \varphi|_{\mathcal{R}_M(c)}$ is an isometry from $\mathcal{R}_M(c)$ onto $\mathcal{R}_M(c')$.
Let Σ and Σ' be twin apartments of Δ and Δ' containing the chambers c and c' respectively and suppose that φ maps $\Sigma \cap \mathcal{R}_M(c)$ bijectively onto $\Sigma' \cap \mathcal{R}_M(c')$ for all $M \in X$. Then there exists a unique isometry from Δ onto Δ' extending each φ_M.

Proof Let $x \in E_2(c)$. Then there exists a subset $J \subseteq S$ with $|J| \leq 2$ such that $x \in \mathcal{R}_J(c)$.
If J is irreducible there exists a subset $M \in X$ such that $J \subseteq M$ and thus $x \in E_X(c)$ and $\varphi(x) \in E_X(c')$ is defined. Moreover, the restriction $\varphi_J := \varphi_M|_{\mathcal{R}_J(c)}$ is an isometry from $\mathcal{R}_J(c)$ onto $\mathcal{R}_J(c')$.
If J is reducible, $|J| = 2$ and Π_J is the diagram $\mathsf{A}_1 \times \mathsf{A}_1$. For each $s \in J$ the map $\varphi_s := \varphi|_{\mathcal{P}_s(c)}$ is an isometry from $\mathcal{P}_s(c)$ onto $\mathcal{P}_s(c')$. By 4.2.3, there exists a unique isometry $\varphi_J \colon \mathcal{R}_J(c) \to \mathcal{R}_J(c')$ extending φ_s for each $s \in J$. We define a mapping

$$\bar\varphi \colon E_2(c) \to E_2(c')$$

by $\bar\varphi(x) := \varphi_J(x)$ if $x \in \mathcal{R}_J(c)$. Note that this map is well-defined since for any two subsets $J, K \subseteq S$ of cardinality at most 2 with $J \cap S \neq \emptyset$ the corresponding maps φ_J and φ_K coincide on $\mathcal{R}_{J \cap K}(c)$. By construction, $\bar\varphi$ satisfies the properties of proposition 4.2.4 and whence is an isometry $E_2(c) \to E_2(c')$.

Let $\phi \colon \Sigma \to \Sigma'$ be the unique isometry mapping the chamber c onto the chamber c' (cf. 7.1.4).

Let $x \in E_2(c) \cap \Sigma$. We show that $\bar\varphi(x) = \phi(x)$:

First suppose that $x \in E_2^*(c)$ and let $M \in X$ such that $x \in \mathcal{R}_M(c)$. Then $x \in \Sigma \cap \mathcal{R}_M(c)$ and $\bar{\varphi}(x) = \varphi(x) \in \Sigma' \cap \mathcal{R}_M(c')$. Since $c', \bar{\varphi}(x)$ and $\phi(x)$ are chambers of Σ', we may apply 6.3.3(b) and obtain

$$\delta_+'(\bar{\varphi}(x), \phi(x)) = \delta_+'(\bar{\varphi}(x), c') \, \delta_+'(c', \phi(x))$$
$$= \delta_+(x, c) \, \delta_+(c, x) = 1_W,$$

which implies that $\bar{\varphi}(x) = \phi(x)$.

If $x \notin E_2^*(c)$ there exists a reducible subset $J = \{s, t\} \subseteq S$ such that $x \in \mathcal{R}_J(c)$. Note that, since $x \notin \mathcal{P}_s(c) \cup \mathcal{P}_t(c)$, x is the unique element in Σ such that $\delta_+(c, x) = r_J = st = ts$. We conclude that $\phi(x)$ is the unique element in Σ' such that $\delta_+'(c', \phi(x)) = r_J = st = ts$.
Let c_s' be the unique element in Σ' such that $\delta_+'(c', c_s') = s$ and similarly let c_t' be the unique element in Σ' such that $\delta_+'(c', c_t') = t$. Following the proof of 4.2.3, we obtain

$$\bar{\varphi}(x) = \varphi_J(x) = \mathrm{proj}_{\mathcal{P}_t(\varphi(c_s'))}(\varphi(c_t'))$$

and since $\varphi(c_s') \in \Sigma' \cap \mathcal{P}_t(\varphi(c_s'))$ we conclude that $\bar{\varphi}(x) \in \Sigma'$ by 6.3.2. As $\delta_+'(c', \bar{\varphi}(x)) = \delta_+(c, x) = r_J$ we conclude that $\bar{\varphi}(x) = \phi(x)$.
By 7.3.2, there is a unique isometry $\Phi \colon \Delta \to \Delta'$ extending $\bar{\varphi}$ and ϕ.

Choose $M \in X$ and let $d \in \Sigma$ be the unique chamber such that $\delta_+(c, d) = r_M$. Set $E_1^M(c) := \bigcup_{s \in M} \mathcal{P}_s(c)$ and choose $x \in E_1^M(c)$. Then there exists $s \in M$ such that $x \in \mathcal{P}_s(c)$ and as $\mathcal{P}_s(c) \subseteq E_2^*(c)$ we have

$$\Phi(x) = \bar{\varphi}(x) = \varphi(x) = \varphi_M(x).$$

Clearly, $\Phi(d) = \phi(d)$ is the unique element in Σ' such that $\delta_+'(c', \Phi(d)) = r_M$. By assumption, $\varphi_M(d) \in \Sigma'$ and since φ_M is an isometry from $\mathcal{R}_M(c)$ onto $\mathcal{R}_M(c')$, we have $\delta_+'(c', \varphi_M(d)) = \delta_+(c, d) = r_M$. Whence $\Phi(d) = \varphi_M(d)$.

Now $\Phi^{-1} \circ \varphi_M$ is an isometry from $\mathcal{R}_M(c)$ onto $\mathcal{R}_M(c)$ fixing $E_1^M(c) \cup \{d\}$ pointwise. The rigidity theorem [AB, 5.205] implies that Φ and φ_M coincide on $\mathcal{R}_M(c)$. $\qquad\square$

Chapter 8

Descent

8.1 Γ-residues

The main idea of this section is to generalize the concepts and results of chapter 22 of [MPW] to twin buildings.

Throughout this section let Π be a Coxeter diagram with vertex set I and let (W, S) be the corresponding Coxeter system and denote by $\ell\colon W \to \mathbb{N}$ the length function on W with respect to S. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type Π (not necessarily thick) and let Γ be a subgroup of $\operatorname{Aut}(\Delta)$. We denote by Θ the subgroup of $\operatorname{Aut}(W, S)$ induced by Γ.

8.1.1 Definition
  (i) A Γ-*residue* is a residue of Δ stabilized by Γ.

 (ii) A Γ-*chamber* is a Γ-residue which is minimal with respect to inclusion.

(iii) A Γ-*panel* is a Γ-residue $\mathcal{P}$ such that for some Γ-chamber $C \subseteq \mathcal{P}$, $\mathcal{P}$ is minimal in the set of all Γ-residues containing C. Equivalently, a Γ-panel is a Γ-residue $\mathcal{P}$ that contains a Γ-chamber C such that $\operatorname{Typ}(\mathcal{P}) \backslash \operatorname{Typ}(C)$ is a single Θ-orbit.

8.1.2 Remark
For each $\varepsilon \in \{+, -\}$ the restriction of Γ to the building Δ_ε is a subgroup $\Gamma_\varepsilon \leq \operatorname{Aut}(\Delta_\varepsilon)$ and a Γ-residue (panel/chamber) of Δ contained in Δ_ε is nothing else than a Γ_ε-residue (panel/chamber) of Δ_ε as defined in [MPW, 22.2].

8.1.3 Lemma
Let $\mathcal{R}$ be a Γ-residue and let $\mathcal{T}$ be a residue containing $\mathcal{R}$. Then $\mathcal{T}$ is a Γ-residue if and only if $\operatorname{Typ}(\mathcal{T})$ is Θ-invariant.

Proof Let $\varepsilon \in \{+, -\}$ such that $\mathcal{R} \subseteq \mathcal{T} \subseteq \mathcal{C}_\varepsilon$. For each $\gamma \in \Gamma$ we denote by $\sigma_\gamma \in \Theta$ the accompanying automorphism of γ. Conversely, for each $\sigma \in \Theta$ there exists $\gamma \in \Gamma$ such that $\sigma_\gamma = \sigma$.

Let $\sigma \in \Theta$ and let $\gamma \in \Gamma$ such that $\sigma_\gamma = \sigma$. Suppose that $\mathcal{T}$ is a Γ-residue, let $s \in \mathrm{Typ}(\mathcal{T})$ and choose $x, y \in \mathcal{T}$ such that $\delta_\varepsilon(x, y) = s$. We have $\gamma(x), \gamma(y) \in \mathcal{T}$ and hence

$$\sigma(s) = \sigma_\gamma(s) = \sigma_\gamma(\delta_\varepsilon(x, y)) = \delta_\varepsilon(\gamma(x), \gamma(y)) \in \mathrm{Typ}(\mathcal{T}).$$

Conversely, choose a chamber $c \in \mathcal{R}$ and let $J := \mathrm{Typ}(\mathcal{T})$ be Θ-invariant. Then $\mathcal{T} = \mathcal{R}_J(c)$ and $\gamma(c) \in \mathcal{R}$ for all $\gamma \in \Gamma$. Let $x \in \mathcal{T}$ be any chamber and set $w := \delta_\varepsilon(c, x) \in W_J$. Since $\sigma(w) \in W_J$ for all $\sigma \in \Theta$ we have

$$\delta_\varepsilon(\gamma(c), \gamma(x)) = \sigma_\gamma(\delta_\varepsilon(c, x)) = \sigma_\gamma(w) \in W_J.$$

Thus, $\gamma(x) \in \mathcal{R}_J(\gamma(c)) = \mathcal{R}_J(c) = \mathcal{T}$.

$\square$

8.1.4 Lemma
Let $\varepsilon \in \{+, -\}$, let $\mathcal{R}$ be a spherical Γ-residue in $\mathcal{C}_\varepsilon$ and let $\mathcal{T}$ be a Γ-residue in $\mathcal{C}_{-\varepsilon}$. Then the projection $\mathrm{proj}_{\mathcal{R}}(\mathcal{T})$ is a Γ-residue.

Proof According to 6.1.4, the set $\mathrm{proj}_{\mathcal{R}}(\mathcal{T})$ is a residue.

Let $J := \mathrm{Typ}(\mathcal{R}) \subseteq S$ and let $w_1 := \min\{\delta_*(\mathcal{R}, \mathcal{T})\}$. Let $\gamma \in \Gamma$ and let $\sigma \in \Theta$ be the accompanying automorphism of γ. Since $\mathcal{R}$ is a Γ-residue, its type is Θ-invariant by 8.1.3. Hence, $\sigma(r_J) \in W_J$ and since σ preserves the length, in view of 2.8(d), we conclude that $\sigma(r_J) = r_J$. Let $w \in \delta_*(\mathcal{R}, \mathcal{T})$ and let $c \in \mathcal{R}, d \in \mathcal{T}$ such that $\delta_*(c, d) = w$. Then

$$\sigma(w) = \sigma(\delta_*(c, d)) = \delta_*(\gamma(c), \gamma(d)) \in \delta_*(\mathcal{R}, \mathcal{T}),$$

since $\gamma(c) \in \mathcal{R}$ and $\gamma(d) \in \mathcal{T}$. Since σ preserves the length, in view of 2.8(a), we conclude that $\sigma(w_1) = w_1$. Now, using the characterization in 6.1.4, the fact that $\gamma(y) \in \mathcal{T}$ for all $y \in \mathcal{T}$ and the considerations above,

$$
\begin{aligned}
x \in \mathrm{proj}_{\mathcal{R}}(\mathcal{T}) &\Leftrightarrow \exists\, y \in \mathcal{T} : \delta_*(x, y) = r_J w_1 \\
&\Leftrightarrow \exists\, y \in \mathcal{T} : \sigma(\delta_*(x, y)) = r_J w_1 \\
&\Leftrightarrow \exists\, y \in \mathcal{T} : \delta_*(\gamma(x), \gamma(y)) = r_J w_1 \\
&\Leftrightarrow r_J w_1 \in \delta_*(\gamma(x), \mathcal{T}) \\
&\Leftrightarrow \gamma(x) \in \mathrm{proj}_{\mathcal{R}}(\mathcal{T}).
\end{aligned}
$$

$\square$

8.1.5 Lemma
Let $\mathcal{T}$ be a Γ-residue which is parallel to a spherical Γ-chamber. Then $\mathcal{T}$ is also a Γ-chamber.

Proof Let C be a spherical Γ-chamber parallel to $\mathcal{T}$. According to 2.27(a) and 6.2.2, the projection maps $\mathrm{proj}_C \colon \mathcal{T} \to C$ and $\mathrm{proj}_{\mathcal{T}} \colon C \to \mathcal{T}$ induce mutually inverse bijections. Suppose that there exists a proper Γ-residue $X \subsetneq \mathcal{T}$ and choose a chamber $d \in \mathcal{T} \backslash X$. By 8.1.4, $\mathrm{proj}_C(X) \subseteq C$ is a Γ-residue and hence $\mathrm{proj}_C(X) = C$. Let $c := \mathrm{proj}_C(d) \in C$. As $C = \mathrm{proj}_C(X)$ there exists $x \in X$ such that $\mathrm{proj}_C(x) = c$. Thus,

$$d = \mathrm{proj}_{\mathcal{T}}(\mathrm{proj}_C(d)) = \mathrm{proj}_{\mathcal{T}}(c) = \mathrm{proj}_{\mathcal{T}}(\mathrm{proj}_C(x)) = x \in X,$$

a contradiction. $\qquad\square$

8.1.6 Corollary
Let $\varepsilon \in \{+, -\}$, let C be a spherical Γ-chamber in $\mathcal{C}_\varepsilon$ and let D be a Γ-residue in $\mathcal{C}_{-\varepsilon}$ which is opposite to C. Then D is a Γ-chamber.

Proof By 6.2.5, opposite residues of spherical type are parallel. The assertion now follows from 8.1.5. $\qquad\square$

8.1.7 Corollary
Let $\varepsilon \in \{+, -\}$, let $\mathcal{R}$ be a spherical Γ-residue in $\mathcal{C}_\varepsilon$ and let C be a spherical Γ-chamber in $\mathcal{C}_{-\varepsilon}$. Then the projection $\mathrm{proj}_{\mathcal{R}}(C)$ is a Γ-chamber.

Proof By 8.1.4, $X := \mathrm{proj}_{\mathcal{R}}(C) \subseteq \mathcal{R}$ is a Γ-residue. We show that C and X are parallel and then the assertion follows from 8.1.5.
Let $x \in X$ and $c \in C$ such that $\mathrm{proj}_{\mathcal{R}}(c) = x$. According to 6.1.5 we have

$$\mathrm{proj}_X(c) = \mathrm{proj}_X(\mathrm{proj}_{\mathcal{R}}(c)) = \mathrm{proj}_X(x) = x,$$

and hence $X \subseteq \mathrm{proj}_X(C) \subseteq X$. Applying again 8.1.4, we obtain that $\mathrm{proj}_C(X) \subseteq C$ is a Γ-residue. As C is minimal with respect to inclusion, we conclude that $\mathrm{proj}_C(X) = C$ and X and C are parallel. $\qquad\square$

8.1.8 Lemma
Let $\mathcal{R}_+ \subseteq \mathcal{C}_+$ and $\mathcal{R}_- \subseteq \mathcal{C}_-$ be opposite Γ-residues of spherical type and let $C_+ \subseteq \mathcal{R}_+$ and $C_- \subseteq \mathcal{R}_-$ be Γ-chambers. Then $w := \delta_*(c, \mathrm{proj}_{C_{-\varepsilon}}(c))$ is independent of the choice of the chamber $c \in C_\varepsilon$ for each $\varepsilon \in \{+, -\}$.

Proof Let $\varepsilon \in \{+,-\}$ and $J := \mathrm{Typ}(\mathcal{R}_+) = \mathrm{Typ}(\mathcal{R}_-)$. According to 8.1.7, the set $\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C_\varepsilon)$ is a Γ-chamber contained in $\mathcal{R}_{-\varepsilon}$ and by 6.2.6, $\delta_*(c, \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(c)) = r_J$ for all $c \in C_\varepsilon$. Note that, by [MPW, 22.3(iii)] the Γ-chambers $C_{-\varepsilon}$ and $\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C_\varepsilon)$ are parallel. In view of 2.27(b), the element $v := \delta_{-\varepsilon}(d, \mathrm{proj}_{C_{-\varepsilon}}(d))$ is independent of the choice of the chamber $d \in \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C_\varepsilon)$. Hence, using 2.28(a), for each $c \in C_\varepsilon$,

$$
\begin{aligned}
\delta_*(c, \mathrm{proj}_{C_{-\varepsilon}}(c)) &= \delta_*(\mathrm{proj}_{C_{-\varepsilon}}(c), c)^{-1} \\
&= (\delta_{-\varepsilon}(\mathrm{proj}_{C_{-\varepsilon}}(c), \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(c))\ \delta_*(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(c), c))^{-1} \\
&= r_J\ \delta_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(c), \mathrm{proj}_{C_{-\varepsilon}}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(c))) = r_J\ v
\end{aligned}
$$

is independent of the chamber c. $\square$

8.1.9 Proposition
Let $\mathcal{P}$ be a spherical Γ-residue and suppose that $\mathcal{P}$ contains two Γ-residues C and D of the same type A such that $\mathrm{Typ}(\mathcal{P}) \backslash A$ is a single Θ-orbit. Let Σ_C be an apartment of C. If C and D are opposite in $\mathcal{P}$ the following hold:

(a) There exists a unique apartment $\Sigma_{\mathcal{P}}$ of $\mathcal{P}$ such that $\Sigma_{\mathcal{P}} \cap D \neq \emptyset$ and $\Sigma_{\mathcal{P}} \cap C = \Sigma_C$.

(b) If Σ_C is Γ-invariant, then so is $\Sigma_{\mathcal{P}}$.

Proof Let $\varepsilon \in \{+,-\}$ such that $\mathcal{P} \subseteq \mathcal{C}_\varepsilon$.

(a) Choose a chamber $c \in \Sigma_C$ and let $x \in \Sigma_C$ be the unique chamber such that $\delta_\varepsilon(x, c) = r_A$.
Let $d := \mathrm{proj}_D(x)$. Since C and D are opposite, they are parallel and thus $x = \mathrm{proj}_C(d)$. Now [W03, 9.11(iii)] gives that

$$
\delta_\varepsilon(d, x) = r_{\Theta(s) \cup A}\ r_A
$$

and thus

$$
\delta_\varepsilon(d, c) = \delta_\varepsilon(d, x)\ \delta_\varepsilon(x, c) = r_{\Theta(s) \cup A} r_A r_A = r_{\Theta(s) \cup A}.
$$

In particular, the chambers c and d are opposite in $\mathcal{P}$. Let $\Sigma_{\mathcal{P}}$ be the unique apartment of $\mathcal{P}$ containing the chambers c and d. Applying [AB, 5.45] gives that for all $y \in \Sigma_{\mathcal{P}}$ the chamber $\mathrm{proj}_C(y) \in \Sigma_{\mathcal{P}}$, since $\Sigma_{\mathcal{P}} \cap C \neq \emptyset$. In particular, $x = \mathrm{proj}_C(d) \in \Sigma_{\mathcal{P}}$. As $\Sigma_{\mathcal{P}} \cap C$ is an apartment of C containing the chambers c and x, we conclude that $\Sigma_{\mathcal{P}} \cap C = \Sigma_C$.

Now let $\Sigma'_{\mathcal{P}}$ be another apartment of $\mathcal{P}$ such that $\Sigma'_{\mathcal{P}} \cap C = \Sigma_C$ and $\Sigma'_{\mathcal{P}} \cap D \neq \emptyset$. Again, by [AB, 5.45], $d = \mathrm{proj}_D(x) \in \Sigma'_{\mathcal{P}}$. As d and c are opposite in $\mathcal{P}$, we conclude that $\Sigma_{\mathcal{P}} = \Sigma'_{\mathcal{P}}$.

(b) Let $\gamma \in \Gamma$ and suppose that $\gamma(\Sigma_C) = \Sigma_C$. Let $\Sigma_{\mathcal{P}}$ be the unique apartment of $\mathcal{P}$ containing Σ_C such that $\Sigma_{\mathcal{P}} \cap D \neq \emptyset$. Then $\gamma(\Sigma_{\mathcal{P}})$ is an apartment of $\mathcal{P}$ containing $\gamma(\Sigma_C) = \Sigma_C$. Let $y \in D \cap \Sigma_{\mathcal{P}}$. As D is a Γ-residue, $\gamma(y) \in D \cap \gamma(\Sigma_{\mathcal{P}})$. Thus, $\gamma(\Sigma_{\mathcal{P}}) = \Sigma_{\mathcal{P}}$.

$\square$

8.1.10 Lemma
Let C be a spherical Γ-chamber and suppose that there exists a twin apartment Σ of Δ which is stabilized by Γ such that $\Sigma \cap C \neq \emptyset$. Then there exists a Γ-chamber D which is opposite to C.

Proof Let $\varepsilon \in \{+, -\}$ such that $C \subseteq \mathcal{C}_\varepsilon$. Set $A := \mathrm{Typ}(C)$ and choose $c \in C \cap \Sigma$. Note that, since C is a Γ-chamber, A is Θ-invariant. Since both, C and Σ, are Γ-invariant, we conclude that $\gamma(c) \in C \cap \Sigma$ for all $\gamma \in \Gamma$.
Let $d := \mathrm{op}_\Sigma(c)$, let $\gamma \in \Gamma$ and let $\sigma \in \Theta$ be the accompanying automorphism of γ. Set $w := \delta_\varepsilon(c, \gamma(c)) \in W_A$. Note that $\gamma(d) \in \Sigma$ and thus, by 6.3.5 and 6.3.3(b),

$$\delta_{-\varepsilon}(d, \gamma(d)) = \delta_*(c, \gamma(d)) = \delta_\varepsilon(c, \gamma(c)) \, \delta_*(\gamma(c), \gamma(d))$$
$$= w \, \sigma(\delta_*(c, d)) = w \in W_A.$$

Thus, $\gamma(d) \in \mathcal{R}_A(d)$ for all $\gamma \in \Gamma$. Set $D := \mathcal{R}_A(d)$ and let $x \in D$ be any chamber. Then $w' := \delta_{-\varepsilon}(x, d) \in W_A$ and hence

$$\delta_{-\varepsilon}(\gamma(x), \gamma(d)) = \sigma(\delta_{-\varepsilon}(x, d)) = \sigma(w') \in W_A.$$

We conclude that $\gamma(x) \in \mathcal{R}_A(\gamma(d)) = D$ which implies that D is Γ-invariant. Hence, D is a Γ-residue opposite to C and thus, according to 8.1.6, D is a Γ-chamber. $\square$

8.1.11 Definition
A Γ-chamber C of Δ will be called *thick* if every Γ-panel containing C contains at least three Γ-chambers.

8.1.12 Proposition
Let C be a thick Γ-chamber of Δ. Then every Γ-chamber opposite C is thick.

Proof Let $A := \mathrm{Typ}(C)$ and let D be a Γ-chamber opposite C. Let $\mathcal{P}_D$ be a Γ-panel containing D. Hence $\mathrm{Typ}(\mathcal{P}_D) \backslash A$ is a single Θ-orbit. Let $\mathcal{P}_C$ be the unique residue of the same type as $\mathcal{P}_D$ which contains the Γ-chamber C. By 8.1.3, $\mathcal{P}_C$ is a Γ-panel. By assumption the Γ-panel $\mathcal{P}_C$ contains at least three Γ-chambers.

For every Γ-chamber $C' \subseteq \mathcal{P}_C$, the set $\text{proj}_{\mathcal{P}_D}(C') \subseteq \mathcal{P}_D$ is a Γ-chamber (cf. 8.1.7). As, by construction, the two Γ-panels $\mathcal{P}_C$ and $\mathcal{P}_D$ are opposite residues, they are parallel. In particular, the projection maps $\text{proj}_{\mathcal{P}_C}$ and $\text{proj}_{\mathcal{P}_D}$ are mutually inverse bijections. We conclude that for any two Γ-chambers $X \neq Y \subseteq \mathcal{P}_C$ we have $\text{proj}_{\mathcal{P}_D}(X) \neq \text{proj}_{\mathcal{P}_D}(Y)$. Thus, D is thick.

$\square$

8.1.13 Proposition
Let C be a thick Γ-chamber such that every Γ-panel containing C is of spherical type. Then
$$\mathbf{T} := (\Pi, \Theta, \text{Typ}(C))$$
is a Tits index.

Proof Let $A := \text{Typ}(C)$. The assumption on C yields that A is spherical and the assertion follows from [MPW, 22.13]. $\square$

8.2   Fixed point buildings

Throughout this section let Π be a Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let $\ell \colon W \to \mathbb{N}$ denote the length function on W with respect to S. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S).
Let Γ be a subgroup of $\mathrm{Aut}(\Delta)$ and let $\Theta \leq \mathrm{Aut}(W, S)$ be the subgroup induced by Γ.

We suppose that C_+ is a spherical Γ-chamber in $\mathcal{C}_+$ of type A and that C_- is a Γ-chamber in $\mathcal{C}_-$ which is opposite to C_+ such that $\mathbf{T} := (\Pi, \Theta, A)$ is a Tits index.
We let $(\tilde{W}, \tilde{S})$ be the relative type of $\mathbf{T}$ and denote by $\tilde{\ell} \colon \tilde{W} \to \mathbb{N}$ the length function on $\tilde{W}$ with respect to $\tilde{S}$.

8.2.1 Remark
In view of [MPW, 22.14](i) every Γ-chamber of Δ is of type A. Hence, if $\mathcal{P}$ is a Γ-panel, there exists $s \in S \backslash A$ such that $\mathrm{Typ}(\mathcal{P}) = \Theta(s) \cup A$. Moreover, as $\mathbf{T}$ is a Tits index, each Γ-panel is of spherical type.

8.2.2 Lemma
Let $\mathcal{P}$ and $\mathcal{P}'$ be opposite Γ-panels. The relation of non-opposition induces a bijection between the Γ-chambers of $\mathcal{P}$ and the Γ-chambers of $\mathcal{P}'$.

Proof   We may suppose that $\mathcal{P} \subseteq \mathcal{C}_+$ and $\mathcal{P}' \subseteq \mathcal{C}_-$. Let $s \in S \backslash A$ such that $J := \mathrm{Typ}(\mathcal{P}) = \Theta(s) \cup A$ and let C be a Γ-chamber in $\mathcal{P}$. By 8.1.7 the projection $D := \mathrm{proj}_{\mathcal{P}'}(C)$ is a Γ-chamber. In view of 6.2.5 and 6.2.2 we also have $\mathrm{proj}_{\mathcal{P}}(D) = C$.
Let $c \in C$. Since $\delta_*(c, D) = \delta_*(c, \mathrm{proj}_{\mathcal{P}'}(c)) W_A = r_J W_A$, we conclude that $\delta_*(c, d) \neq 1_W$ for all $d \in D$ because of [MPW, 20.9]. Hence, C and D can not be opposite.
Let X be a Γ-chamber in $\mathcal{P}'$ different from D, let $d := \mathrm{proj}_{\mathcal{P}'}(c) \in D$ and choose a chamber $x \in X$ such that $\delta_-(x, \mathrm{proj}_X(d)) = r_A$. Then

$$\delta_*(x, c) = \delta_-(x, d)\, \delta_*(d, c) = \delta_-(x, \mathrm{proj}_X(d))\, \delta_-(\mathrm{proj}_X(d), d)\, \delta_*(d, c).$$

According to [MPW, 22.14](ii) $\delta_-(d, \mathrm{proj}_X(d)) = \tilde{s} = r_J r_A = r_A r_J$. Thus,

$$\delta_*(x, c) = r_A\, \tilde{s}\, r_J = r_A^2\, r_J^2 = 1_W,$$

and since $\mathrm{Typ}(C) = \mathrm{Typ}(X) = A$ we conclude that the Γ-chambers C and X are opposite. $\square$

8.2.3 Proposition
Every Γ-panel contains at least two Γ-chambers.

Proof Let $\mathcal{P}$ be a Γ-panel. Let $\varepsilon \in \{+,-\}$ be such that $\mathcal{P} \subseteq \mathcal{C}_\varepsilon$ and let C be any Γ-chamber in $\mathcal{C}_\varepsilon$. According to [MPW, 22.3(i) and (ii)], the projection $\mathrm{proj}_{\mathcal{P}}(C)$ is a Γ-chamber parallel to C and by [MPW, 22.15(ii)], $\delta_\varepsilon(C, \mathrm{proj}_{\mathcal{P}}(C)) \in \tilde{W}$.
We will show that for each $n \in \mathbb{N}$ the following statement holds:

> If $\mathcal{P}$ is a Γ-panel in $\mathcal{C}_\varepsilon$ and if there exists a pair of opposite Γ-chambers $(X_+, X_-) \in \tilde{\Delta}_+ \times \tilde{\Delta}_-$ such that $\tilde{\ell}(\delta_\varepsilon(X_\varepsilon, \mathrm{proj}_{\mathcal{P}}(X_\varepsilon))) = n$, then $\mathcal{P}$ has at least two Γ-chambers.

We proceed by induction on n.
First, let $\mathcal{P}$ be a Γ-panel in $\mathcal{C}_\varepsilon$ containing the Γ-chamber C_ε and define $D := \mathrm{proj}_{\mathcal{P}}(C_{-\varepsilon})$. By 8.1.7, D is a Γ-chamber in $\mathcal{P}$ and since C_+ and C_- are opposite, $C_\varepsilon \neq D$ by 8.2.2. Hence our assertion is true for $n = 0$.

Next suppose that $\mathcal{P}$ is a Γ-panel in $\mathcal{C}_\varepsilon$ such that $\tilde{\ell}(\delta_\varepsilon(C_\varepsilon, \mathrm{proj}_{\mathcal{P}}(C_\varepsilon))) = \tilde{s}$ for some $s \in S \backslash A$. Let $\mathcal{P}'$ be the unique residue of type $\Theta(s) \cup A$ containing C_ε. In view of 8.1.3, $\mathcal{P}'$ is a Γ-panel which contains the Γ-chamber $\mathrm{proj}_{\mathcal{P}}(C_\varepsilon)$. Let $\mathcal{Q}$ be the unique residue of type $\Theta(s) \cup A$ containing the Γ-chamber $C_{-\varepsilon}$ and let $X := \mathrm{proj}_{\mathcal{Q}}(C_\varepsilon)$. Again by 8.1.7, X is a Γ-chamber in $\mathcal{Q}$ and in view of 8.2.2, X and $\mathrm{proj}_{\mathcal{P}}(C_\varepsilon)$ are opposite. Now $(\mathrm{proj}_{\mathcal{P}}(C_\varepsilon), X) \in \tilde{\Delta}_\varepsilon \times \tilde{\Delta}_{-\varepsilon}$ is a pair of opposite Γ-chambers such that $\mathrm{proj}_{\mathcal{P}}(C_\varepsilon) \in \mathcal{P}$. The considerations in the first case show that $\mathcal{P}$ has at least two Γ-chambers.

Let $\mathcal{P}$ be a Γ-panel in $\mathcal{C}_\varepsilon$, let $w := \delta_\varepsilon(C_\varepsilon, \mathrm{proj}_{\mathcal{P}}(C_\varepsilon)) \in \tilde{W}$ and suppose that $\tilde{\ell}(w) = n$. Let $s_1, \ldots, s_n \in S \backslash A$ such that $w = \tilde{s}_1 \cdots \tilde{s}_n$. Note that for $c \in C_\varepsilon$

$$w = \delta_\varepsilon(c, \mathrm{proj}_{\mathcal{P}}(c)) = \min\{\delta_\varepsilon(c, \mathcal{P})\}. \tag{8.1}$$

Let $\mathcal{P}_1$ be the unique residue of type $\Theta(s_1) \cup A$ containing C_ε. In view of 8.1.3, this is a Γ-panel. Let $C_1 := \mathrm{proj}_{\mathcal{P}_1}(\mathrm{proj}_{\mathcal{P}}(C_\varepsilon))$. By 8.1.7, C_1 is a Γ-chamber in $\mathcal{P}_1$ and we show that the following hold:

(i) $C_1 \neq C_\varepsilon$ and

(ii) if $\mathrm{proj}_{\mathcal{P}}(C_1) = \mathrm{proj}_{\mathcal{P}}(C_\varepsilon)$ then

$$\tilde{\ell}(\delta_\varepsilon(C_1, \mathrm{proj}_{\mathcal{P}}(C_1))) < \tilde{\ell}(\delta_\varepsilon(C_\varepsilon, \mathrm{proj}_{\mathcal{P}}(C_\varepsilon))).$$

Indeed, let $c \in C_\varepsilon, d := \mathrm{proj}_{\mathcal{P}}(c)$ and $c_1 := \mathrm{proj}_{\mathcal{P}_1}(d)$. Note that, since C_ε and $\mathrm{proj}_{\mathcal{P}}(C_\varepsilon)$ are parallel, due to 2.28(a),

$$c = \mathrm{proj}_{C_\varepsilon}(d) = \mathrm{proj}_{C_\varepsilon}(\mathrm{proj}_{\mathcal{P}_1}(d)) = \mathrm{proj}_{C_\varepsilon}(c_1).$$

Now $\delta_\varepsilon(d, c) = w^{-1}$ and thus $\delta_\varepsilon(d, C_\varepsilon) = w^{-1} W_A$. In particular, if x is a chamber of C_ε there exists $v \in W_A$ such that $\delta_\varepsilon(d, x) = w^{-1}v$. According to [MPW, 20.13(iii)], $A \subseteq J^+(w^{-1})$ and thus, by 2.8(c),

$$\ell(\delta_\varepsilon(d, x)) = \ell(w^{-1}v) = \ell(w^{-1}) + \ell(v) \geq \ell(w^{-1}) = \ell(w). \tag{8.2}$$

Conversely, since $\mathrm{Typ}(\mathcal{P}_1) = \Theta(s_1) \cup A$ and since $\tilde{s}_1 \in W_{\Theta(s_1) \cup A}$ there exists a chamber $x \in \mathcal{P}_1$ such that $\delta_\varepsilon(d, x) = w^{-1}\tilde{s}_1 = \tilde{s}_n \cdots \tilde{s}_2$. By [MPW, 20.31] we have $\ell(w^{-1}\tilde{s}_1) = \ell(\tilde{s}_n \cdots \tilde{s}_2) = \sum_{i=2}^{n} \ell(\tilde{s}_i) = \ell(w) - \ell(\tilde{s}_1) < \ell(w)$. In particular, $\ell(\delta_\varepsilon(d, c_1)) \le \ell(w) - \ell(\tilde{s}_1) < \ell(w)$. In view of 8.2 we conclude that $c_1 \notin C_\varepsilon$. Whence $C_\varepsilon \ne \mathrm{proj}_{\mathcal{P}_1}(\mathrm{proj}_{\mathcal{P}}(C_\varepsilon)) = C_1$ and (i) holds.

This implies $\delta_\varepsilon(c, c_1) = \delta_\varepsilon(C_\varepsilon, C_1) = \tilde{s}_1$. Moreover,

$$\delta_\varepsilon(\mathrm{proj}_{\mathcal{P}}(C_1), C_1) = \delta_\varepsilon(d, c_1) = \delta_\varepsilon(d, c)\delta_\varepsilon(c_1, c)$$
$$= w^{-1}\tilde{s}_1 = \tilde{s}_n \cdots \tilde{s}_2.$$

Hence

$$\tilde{\ell}(\delta_\varepsilon(C_1, \mathrm{proj}_{\mathcal{P}}(C_1))) = \tilde{\ell}(\delta_\varepsilon(c_1, d)) = \tilde{\ell}(\tilde{s}_2 \cdots \tilde{s}_n) = n - 1$$
$$< n = \tilde{\ell}(\delta_\varepsilon(C_\varepsilon, \mathrm{proj}_{\mathcal{P}}(C_\varepsilon)))$$

and (ii) follows.

If the projection $\mathrm{proj}_{\mathcal{P}}(C_1)$ is not equal to the projection $\mathrm{proj}_{\mathcal{P}}(C_\varepsilon)$, then the Γ-panel $\mathcal{P}$ contains at least two Γ-chambers, as desired. In view of (ii), if both projections are equal, it suffices to give a Γ-chamber D opposite C_1 and the assertion follows by induction.

For this, let $\mathcal{T}_1$ be the unique residue of type $\Theta(s_1) \cup A$ containing $C_{-\varepsilon}$. Then $\mathcal{T}_1$ is a Γ-panel and, by construction, $\mathcal{P}_1$ and $\mathcal{T}_1$ are opposite. Since $C_{-\varepsilon}$ is contained in $\mathcal{T}_1$ we have $\delta_{-\varepsilon}(C_{-\varepsilon}, \mathrm{proj}_{\mathcal{T}_1}(C_{-\varepsilon})) = 0$ and it follows by induction that $\mathcal{T}_1$ contains at least two Γ-chambers. By 8.2.2, at least one is opposite to C_1. $\square$

8.2.4 Corollary
For each $\varepsilon \in \{+, -\}$ the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a building of type $(\tilde{W}, \tilde{S})$. This building is thick if and only if every Γ-panel contains at least three Γ-chambers.

Proof   According to 8.2.3, each Γ-panel of Δ_ε contains at least two Γ-chambers. The assertion now follows from [MPW, 22.14(iii)]. $\square$

8.2.5 Proposition
If one of the Γ-chambers C_+ or C_- is thick, all Γ-chambers are thick.

Proof   In accordance with 8.1.12, both Γ-chambers C_+ and C_- are thick. Since the set of Γ-chambers of each half is connected (cf. 8.2.4), it suffices to show that each Γ-chamber contained in a common Γ-panel with C_+ or C_- is thick. For this, let X be a Γ-chamber such that there exists a Γ-panel $\mathcal{P}$ such

that $\mathcal{P}$ contains the Γ-chambers C_ε and X. Let $\mathcal{P}'$ be the unique Γ-panel of type $\mathrm{Typ}(\mathcal{P})$ containing the Γ-chamber $C_{-\varepsilon}$. By construction, $\mathcal{P}$ and $\mathcal{P}'$ are opposite and, due to 6.2.5, parallel. Since the chambers C_+ and C_- are thick, the Γ-panels $\mathcal{P}$ and $\mathcal{P}'$ both contain at least three Γ-chambers. In particular, there exists a Γ-chamber $Y \subseteq \mathcal{P}'$ such that $\mathrm{proj}_{\mathcal{P}'}(C_\varepsilon) \neq Y \neq \mathrm{proj}_{\mathcal{P}'}(X)$. In view of 8.2.2, the Γ-chamber Y is opposite to both, X and C_ε. Now 8.1.12 applied to Y gives that Y is thick. Once again we apply 8.1.12 and obtain that X is thick. $\qquad\square$

As an immediate consequence of 8.2.5 and 8.2.4 we obtain the following corollary:

8.2.6 Corollary
If one of the Γ-chambers C_+ or C_- is thick, then for each $\varepsilon \in \{+,-\}$ the group Γ_ε is a descent group of Δ_ε. In particular, the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a thick building of type $(\tilde{W}, \tilde{S})$.

Proof Proposition 8.2.5 implies that each Γ-panel contains at least three Γ-chambers. The second assertion follows from 2.40. $\qquad\square$

The following observations will be needed for the construction of a codistance function between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_-.

8.2.7 Proposition
Let $\mathcal{R}_+ \subseteq \mathcal{C}_+$ and $\mathcal{R}_- \subseteq \mathcal{C}_-$ be opposite Γ-residues of spherical type $J \subseteq S$. Let C be a Γ-chamber in $\mathcal{R}_+$ and let D be a Γ-chamber in $\mathcal{R}_-$. We define $X := \mathrm{proj}_{\mathcal{R}_-}(C)$ and $Y := \mathrm{proj}_{\mathcal{R}_+}(D)$. Then

(a) For every $c \in C$ we have $c = \mathrm{proj}_C(\mathrm{proj}_{\mathcal{R}_+}(\mathrm{proj}_D(\mathrm{proj}_{\mathcal{R}_-}(c))))$ and for every $d \in D$ we have $d = \mathrm{proj}_D(\mathrm{proj}_{\mathcal{R}_-}(\mathrm{proj}_C(\mathrm{proj}_{\mathcal{R}_+}(d))))$.

(b) $\delta_+(C,Y) = r_{\tilde{J}}\, \delta_-(X,D)\, r_{\tilde{J}}$, where $r_{\tilde{J}} = \max\{\tilde{W}_{\tilde{J}}\}$.

Proof According to 8.1.4, the projection X is a Γ-chamber in $\mathcal{R}_-$ while the projection Y is a Γ-chamber in $\mathcal{R}_+$. Thus, the elements $w := \delta_+(C,Y) \in \tilde{W}$ and $w' := \delta_-(X,D) \in \tilde{W}$ are well-defined by [MPW, 22.3(iii)] and [MPW, 21.8(iii)]. In view of 6.2.6 we have $\delta_*(y, \mathrm{proj}_{\mathcal{R}_-}(y)) = r_J$ for any $y \in \mathcal{R}_+$ and similarly $\delta_*(x, \mathrm{proj}_{\mathcal{R}_+}(x)) = r_J$ for any $x \in \mathcal{R}_-$.

Now let $c \in C$ be any chamber. We set $x := \mathrm{proj}_{\mathcal{R}_-}(c), d := \mathrm{proj}_D(x)$ and $y := \mathrm{proj}_{\mathcal{R}_+}(d)$. In view of 6.1.5 we have

$$d = \mathrm{proj}_D(x) = \mathrm{proj}_D(\mathrm{proj}_{\mathcal{R}_-}(c)) = \mathrm{proj}_D(c).$$

Now

$$\delta_*(d,c) = \delta_-(d,x)\, \delta_*(x,c) = w'^{-1}\, r_J.$$

Hence

$$\begin{aligned}
\delta_+(c,y) &= \delta_+(c,y)\,\delta_*(y,d)\,r_J = \delta_*(c,d)\,r_J \\
&= \delta_*(d,c)^{-1}\,r_J = r_J\,w'\,r_J.
\end{aligned}$$

As $J \subseteq S$ is such that $A \subseteq J$, the triple $\mathbf{T}_J = (\Pi_J, \Theta_J, A)$ is a Tits index (cf. 2.13). Since the absolute type (W_J, J) of $\mathbf{T}_J$ is spherical, also the relative type $(\tilde{W}_{\tilde{J}}, \tilde{J})$ is spherical by 2.16(b). Moreover, part (ii) of [MPW, 20.35] yields that $r_J = r_{\tilde{J}}\,r_A = r_A\,r_{\tilde{J}}$. Due to this

$$\delta_+(c,y) = r_J\,w'\,r_J = r_{\tilde{J}}\,r_A\,w'\,r_A\,r_{\tilde{J}} = r_{\tilde{J}}\,w'\,r_{\tilde{J}},$$

since, by [MPW, 22.14(ii)] $w' \in \tilde{W}_{\tilde{J}}$ and $\tilde{W}_{\tilde{J}} \subseteq C_{W_J}(r_A)$ by [MPW, 20.11(i)].

Since $\mathrm{op}_{\tilde{J}} \colon \tilde{W}_{\tilde{J}} \to \tilde{W}_{\tilde{J}}$ defined by $\mathrm{op}_{\tilde{J}}(\tilde{s}) := r_{\tilde{J}}\,\tilde{s}\,r_{\tilde{J}}$ is an automorphism (cf. 2.9) we have $\tilde{\ell}(r_{\tilde{J}}\,w'\,r_{\tilde{J}}) = \tilde{\ell}(w')$. As $\delta_+(c,Y) = \delta_+(c,y)W_A$ and $A \subseteq J^+(w')$ by [MPW, 20.13(iii)], we conclude that $y \in Y$ is such that $\delta_+(c,y) = \min\{\delta_+(c,Y)\}$. By [AB, 5.34], $y = \mathrm{proj}_Y(c)$ and, as C and Y are parallel, $c = \mathrm{proj}_C(y)$. Thus, (a) follows. Moreover, (b) is satisfied since

$$\delta_+(C,Y) = \delta_+(c,y) = r_{\tilde{J}}\,w'\,r_{\tilde{J}}.$$

$\square$

8.3   2-twinnings

In [M98], B. Mühlherr gives a local criterion for a relation between the chambers of two buildings to be the opposition relation of a codistance function of the buildings in question. More precisely, he requires the existence of an opposition relation on the set of pairs of rank-2-residues, a so-called 2-*twinning*:

8.3.1 Definition
Let Π be a Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let Δ_+ and Δ_- be two buildings of type Π. A set $\mathcal{O} \subseteq (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$ is called a 2-*twinning of the pair* (Δ_+, Δ_-) if the following axioms are satisfied:

(T1) $\mathcal{O} \neq \emptyset$,

(T2) $(x, y) \in \mathcal{O}$ if and only if $(y, x) \in \mathcal{O}$,

(T3) if $J \subseteq S$ is such that $|J| \leq 2$ and if $\mathcal{R}_+$ and $\mathcal{R}_-$ are J-residues of Δ_+ and Δ_- respectively, then either $\mathcal{O} \cap ((\mathcal{R}_+ \times \mathcal{R}_-) \cup (\mathcal{R}_- \times \mathcal{R}_+)) = \emptyset$ or $\mathcal{O} \cap ((\mathcal{R}_+ \times \mathcal{R}_-) \cup (\mathcal{R}_- \times \mathcal{R}_+))$ is the opposition relation of a codistance function between $\mathcal{R}_+$ and $\mathcal{R}_-$.

The precise statement of [M98] reads as follows:

8.3.2 Theorem
Let Π be a Coxeter diagram, let $\Delta_+ = (\mathcal{C}_+, \delta_+)$ and $\Delta_- = (\mathcal{C}_-, \delta_-)$ be two thick buildings of type Π and let $\mathcal{O} \subseteq (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$. Then $\mathcal{O}$ is the opposition relation of a codistance function between Δ_+ and Δ_- if and only if $\mathcal{O}$ is a 2-twinning.

8.3.3 Notation
Throughout this section let Π be a Coxeter diagram with vertex set I, let (W, S) be the corresponding Coxeter system and let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type Π.
Let Γ be a subgroup of $\mathrm{Aut}(\Delta)$, let $\Theta \leq \mathrm{Aut}(W, S)$ be induced by Γ and suppose that for each $\varepsilon \in \{+, -\}$ the group Γ_ε is a spherical descent group of the building Δ_ε.
We suppose that we are given two opposite Γ-chambers $C_+ \subseteq \mathcal{C}_+$ and $C_- \subseteq \mathcal{C}_-$ of type A such that $\mathbf{T} := (\Pi, \Theta, A)$ is a Tits index. Let $(\tilde{W}, \tilde{S})$ be the relative type of $\mathbf{T}$ and let $\tilde{\ell} \colon \tilde{W} \to \mathbb{N}$ be the length function on $\tilde{W}$ with respect to $\tilde{S}$. We assume that each subset $\tilde{J} \subseteq \tilde{S}$ with $|\tilde{J}| \leq 2$ is spherical.

8.3.4 Remark
Let $\varepsilon \in \{+, -\}$.

(a) According to 2.40, all Γ-chambers of Δ are residues of type A and the pair $\Delta_\varepsilon^{\Gamma\varepsilon} = (\mathcal{C}_\varepsilon^{\Gamma\varepsilon}, \bar{\delta}_\varepsilon)$ is a thick building of type $(\tilde{W}, \tilde{S})$.

(b) Let $\mathcal{R}$ be a Γ-residue of Δ_ε and let $J = \mathrm{Typ}(\mathcal{R}) \subseteq S$. Choose elements $s_1, \ldots, s_k \in S \backslash A$ in distinct Θ-orbits such that

$$J = \Theta(s_1) \cup \cdots \cup \Theta(s_k) \cup A.$$

Let $\Gamma_\mathcal{R}$ denote the subgroup of $\mathrm{Aut}(\mathcal{R})$ induced by Γ, let

$$\mathbf{T}_J := (\Pi_J, \Theta_J, A)$$

be as in 2.13 and let

$$\tilde{J} = \{\tilde{s}_1, \ldots, \tilde{s}_k\},$$

where $\tilde{s}_i$ for all $1 \leq i \leq k$ is as in 2.14.
Then $\Gamma_\mathcal{R}$ is a descent group of $\mathcal{R}$ with Tits index $\mathbf{T}_J$ and the fixed point building $\mathcal{R}^{\Gamma_\mathcal{R}}$ is a $\tilde{J}$-residue of the building $\Delta_\varepsilon^{\Gamma\varepsilon}$.

We show that there exists a 2-twinning of the pair $(\Delta_+^{\Gamma_+}, \Delta_-^{\Gamma_-})$ in order to ensure the existence of a codistance function between the Γ-chambers of Δ_+ and Δ_- and hence the existence of a twin building of type $(\tilde{W}, \tilde{S})$ whose chambers are the Γ-chambers of Δ.

8.3.5 Definition
Let $\mathcal{R}_+ \subseteq \mathcal{C}_+$ and $\mathcal{R}_- \subseteq \mathcal{C}_-$ be opposite Γ-residues of spherical type $J \subseteq S$. For $\varepsilon \in \{+, -\}$ and Γ-chambers $C \subseteq \mathcal{R}_\varepsilon$ and $D \subseteq \mathcal{R}_{-\varepsilon}$ we define

$$\tilde{\delta}_*(C, D) := r_A \, \bar{\delta}_*(C, D),$$

where $\bar{\delta}_*(C, D)$ is as in 8.1.8.

8.3.6 Lemma
Let $\mathcal{R}_+, \mathcal{R}_-, C, D$ and $\tilde{\delta}_*$ be as in 8.3.5. Let $s_1, \ldots, s_k \in S \backslash A$ be in distinct Θ-orbits such that $J = \Theta(s_1) \cup \cdots \cup \Theta(s_k) \cup A$ and set $\tilde{J} := \{\tilde{s}_1, \ldots, \tilde{s}_k\}$. Note that, since J is spherical, also $\tilde{J}$ is spherical (cf. 2.16(b)). Then

(a) $\tilde{\delta}_*(C, D) \in \tilde{W}_{\tilde{J}}$

(b) The map

$$\tilde{\delta}_*\colon (\mathcal{R}_+^{\Gamma_{\mathcal{R}_+}} \times \mathcal{R}_-^{\Gamma_{\mathcal{R}_-}}) \cup (\mathcal{R}_-^{\Gamma_{\mathcal{R}_-}} \times \mathcal{R}_+^{\Gamma_{\mathcal{R}_+}}) \to \tilde{W}_{\tilde{J}}$$

is a codistance function.

Proof Let $\varepsilon \in \{+, -\}$. In view of 8.3.4, $\Gamma_{\mathcal{R}_\varepsilon}$ is a descent group of $\mathcal{R}_\varepsilon$ and the set of Γ-chambers contained in $\mathcal{R}_\varepsilon$ constitutes a building of type $(\tilde{W}_{\tilde{J}}, \tilde{J})$. In particular, for any two Γ-chambers $X, Y \subseteq \mathcal{R}_\varepsilon$ we have $\bar{\delta}_\varepsilon(X, Y) \in \tilde{W}_{\tilde{J}}$. According to [MPW, 20.35(ii)] we have $r_A r_J = r_{\tilde{J}}$. Thus, by 8.1.8,

$$\tilde{\delta}_*(C, D) = r_A \, \bar{\delta}_*(C, D) = r_A \, r_J \, \bar{\delta}_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C), D)$$
$$= r_{\tilde{J}} \, \bar{\delta}_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C), D) \in \tilde{W}_{\tilde{J}}.$$

Let $w := \tilde{\delta}_*(C, D) \in \tilde{W}_{\tilde{J}}$. We verify the axioms (Tw1)-(Tw3):

(Tw1) We use 8.2.7 and obtain

$$\tilde{\delta}_*(D, C) = r_{\tilde{J}} \, \bar{\delta}_\varepsilon(\mathrm{proj}_{\mathcal{R}_\varepsilon}(D), C)$$
$$= r_{\tilde{J}} \, r_{\tilde{J}} \, \bar{\delta}_{-\varepsilon}(D, \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C)) \, r_{\tilde{J}}$$
$$= \bar{\delta}_{-\varepsilon}(D, \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C)) \, r_{\tilde{J}}$$
$$= (r_{\tilde{J}} \, \bar{\delta}_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C), D))^{-1} = \tilde{\delta}_*(C, D)^{-1}.$$

(Tw2) Let C' be a Γ-chamber in $\mathcal{R}_\varepsilon$ such that $\bar{\delta}_\varepsilon(C', C) = \tilde{s} \in \tilde{J}$ and suppose that $\tilde{\ell}(\tilde{s}w) < \tilde{\ell}(w)$. We need to show that $\tilde{\delta}_*(C', D) = \tilde{s}w$.

Let $\tilde{w} := \bar{\delta}_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C), D) \in \tilde{W}_{\tilde{J}}$. Since the Γ-residues $\mathcal{R}_+$ and $\mathcal{R}_-$ are opposite, they are parallel and hence $\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C) \neq \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C')$ due to 6.2.2. Note that, by definition, $w = \tilde{\delta}_*(C, D) = r_{\tilde{J}}\tilde{w}$ and by 8.2.7 it follows that

$$\bar{\delta}_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C), \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C')) = r_{\tilde{J}} \, \tilde{s} \, r_{\tilde{J}} \in \tilde{J}.$$

Now, since $r_{\tilde{J}}$ is the unique element in the Coxeter group $\tilde{W}_{\tilde{J}}$,

$$\tilde{\ell}(r_{\tilde{J}} \, \tilde{s} \, r_{\tilde{J}}\tilde{w}) = \tilde{\ell}(r_{\tilde{J}} \, \tilde{s} \, w) = \tilde{\ell}(r_{\tilde{J}}) - \tilde{\ell}(\tilde{s}w)$$
$$= \tilde{\ell}(r_{\tilde{J}}) - \tilde{\ell}(w) + 1 = \tilde{\ell}(\tilde{w}) + 1.$$

We apply axiom (WD2) and obtain

$$\bar{\delta}_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C'), D) = r_{\tilde{J}} \, \tilde{s} \, r_{\tilde{J}} \, \tilde{w}.$$

Thus

$$\tilde{\delta}_*(C', D) = r_{\tilde{J}} \, \bar{\delta}_*(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C'), D) = \tilde{s} \, r_{\tilde{J}} \, \tilde{w} = \tilde{s}w,$$

as desired.

(Tw3) Let $\tilde{s} \in \tilde{J}$. We need to show that there exists a Γ-chamber $C' \subseteq \mathcal{R}_\varepsilon$ such that $\bar{\delta}_\varepsilon(C', C) = \tilde{s}$ and $\tilde{\delta}_*(C', D) = \tilde{s}w$.

Let $\tilde{w} := \bar{\delta}_{-\varepsilon}(\mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C), D) \in \tilde{W}_J$ and note that $w = r_{\tilde{j}}\,\tilde{w}$. We apply (WD3) and obtain that there exists a Γ-chamber $D' \subseteq \mathcal{R}_{-\varepsilon}$ such that

$$\bar{\delta}_{-\varepsilon}(D', \mathrm{proj}_{\mathcal{R}_{-\varepsilon}}(C)) = r_{\tilde{j}}\,\tilde{s}\,r_{\tilde{j}} \in \tilde{J} \text{ and } \bar{\delta}_{-\varepsilon}(D', D) = r_{\tilde{j}}\,\tilde{s}\,r_{\tilde{j}}\,\tilde{w}.$$

Let $C' := \mathrm{proj}_{\mathcal{R}_{\varepsilon}}(D')$. Using 8.2.7 we obtain $\bar{\delta}_{\varepsilon}(C', C) = \tilde{s}$ and thus

$$\tilde{\delta}_*(C', D) = r_{\tilde{j}}\,\bar{\delta}_{-\varepsilon}(D', D) = \tilde{s}\,r_{\tilde{j}}\,\tilde{w} = \tilde{s}w,$$

as desired.

$\square$

8.3.7 Definition
We define

$$\tilde{\mathcal{O}} := \{(C, D) \in (\mathcal{C}_+^{\Gamma+} \times \mathcal{C}_-^{\Gamma-}) \cup (\mathcal{C}_-^{\Gamma-} \times \mathcal{C}_+^{\Gamma+}) \mid C \text{ and } D \text{ are opposite}\}.$$

8.3.8 Proposition
$\tilde{\mathcal{O}}$ is a 2-twinning of the pair $(\Delta_+^{\Gamma+}, \Delta_-^{\Gamma-})$.

Proof We verify the axioms (T1)-(T3):

(T1) By our assumption we have $(C_+, C_-) \in \tilde{\mathcal{O}}$.

(T2) If C is opposite to D, then D is also opposite to C.

(T3) Let $\tilde{J} \subseteq \tilde{S}$ with $|\tilde{J}| \leq 2$, let $\tilde{\mathcal{R}}_+$ and $\tilde{\mathcal{R}}_-$ be residues of $\Delta_+^{\Gamma+}$ and $\Delta_-^{\Gamma-}$ of type $\tilde{J}$ respectively and suppose that $\tilde{\mathcal{O}} \cap ((\tilde{\mathcal{R}}_+ \times \tilde{\mathcal{R}}_-) \cup (\tilde{\mathcal{R}}_- \times \tilde{\mathcal{R}}_+)) \neq \emptyset$. Let $\tilde{\delta}_* : (\tilde{\mathcal{R}}_+ \times \tilde{\mathcal{R}}_-) \cup (\tilde{\mathcal{R}}_- \times \tilde{\mathcal{R}}_+) \to \tilde{W}_{\tilde{J}}$ be the codistance function defined in 8.3.6. Then $\tilde{\mathcal{O}} \cap ((\tilde{\mathcal{R}}_+ \times \tilde{\mathcal{R}}_-) \cup (\tilde{\mathcal{R}}_- \times \tilde{\mathcal{R}}_+))$ is the opposition relation of the codistance function $\tilde{\delta}_*$:

Let $(C, D) \in \tilde{\mathcal{O}} \cap ((\tilde{\mathcal{R}}_+ \times \tilde{\mathcal{R}}_-) \cup (\tilde{\mathcal{R}}_- \times \tilde{\mathcal{R}}_+))$. Since C and D are opposite Γ-chambers of Δ, we have $\delta_*(c, \mathrm{proj}_D(c)) = r_A$ for all $c \in C$. Hence,

$$\tilde{\delta}_*(C, D) = r_A\,\bar{\delta}_*(C, \mathrm{proj}_D(C)) = r_A^2 = 1_W = 1_{\tilde{W}_{\tilde{J}}}.$$

We conclude that C and D are opposite with respect to $\bar{\delta}_*$.
Conversely, let $C \in \tilde{\mathcal{R}}_+$ and $D \in \tilde{\mathcal{R}}_-$ be such that $\tilde{\delta}_*(C, D) = 1_{\tilde{W}_{\tilde{J}}}$. Choose a chamber $c \in C$. Then

$$1_{\tilde{W}_{\tilde{J}}} = 1_W = \tilde{\delta}_*(C, D) = r_A\,\delta_*(c, \mathrm{proj}_D(c)).$$

Let $d \in D$ be a chamber opposite to $\mathrm{proj}_D(c)$. Then

$$\delta_*(d, c) = \delta_-(d, \mathrm{proj}_D(c)) \; \delta_*(\mathrm{proj}_D(c), c) = r_A^2 = 1_W.$$

Since $A = \mathrm{Typ}(C) = \mathrm{Typ}(D)$ and since there exists a pair of opposite chambers $(c, d) \in C \times D$ we conclude that the Γ-chambers C and D are opposite residues of Δ, i.e. $(C, D) \in \tilde{\mathcal{O}}$.

$\square$

8.3.9 Corollary
There exists a codistance function

$$\delta_*^\Gamma \colon (\Delta_+^{\Gamma+} \times \Delta_-^{\Gamma-}) \cup (\Delta_-^{\Gamma-} \times \Delta_+^{\Gamma+}) \to \tilde{W}$$

between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_- . In particular, the triple $(\Delta_+^{\Gamma+}, \Delta_-^{\Gamma-}, \delta_*^\Gamma)$ is a twin building of type $(\tilde{W}, \tilde{S})$.

Proof By 8.3.8 there exists a 2-twinning of the pair $(\Delta_+^{\Gamma+}, \Delta_-^{\Gamma-})$. In view of 8.3.2, this 2-twinning is the opposition relation of a codistance function between $\Delta_+^{\Gamma+}$ and $\Delta_-^{\Gamma-}$. $\square$

Part IV

Classification

Chapter 9

Moufang polygons

9.1 Moufang sets

Moufang sets were introduced by Jacques Tits in [Ti92]. Moufang sets are the rank-one-case of Moufang buildings.

9.1.1 Definition
A *Moufang set* is a pair $\mathbb{M} = (X, \{U_x\}_{x \in X})$, consisting of a set X with $|X| \geq 3$ and a set of *root groups* $\{U_x\}_{x \in X}$ satisfying the following conditions:

(M1) For each $x \in X$, the group $U_x \leq \mathrm{Sym}(X)$ fixes x and acts sharply transitively on $X \backslash \{x\}$.

(M2) For all $x, y \in X$ and each $g \in U_x$ we have $gU_y g^{-1} = U_{g(y)}$.

9.1.2 Remark
As developed in [dMW], every Moufang set is completely determined by the structure of one of the root groups together with one additional permutation of the non-trivial elements of this group.

Conversely, let $(U, +)$ be a group, let $X := U \cup \{\infty\}$ be the disjoint union of U and $\{\infty\}$ and let $\tau \in \mathrm{Sym}(X)$ be a permutation interchanging 0 and ∞. We define groups $\{U_x\}_{x \in X}$ as follows: For each $a \in U$ we let α_a be the permutation of X fixing ∞ and mapping each $x \in U$ to $x + a$. Then $U_\infty := \{\alpha_a \mid a \in U\}$ is a subgroup of $\mathrm{Sym}(X)$ isomorphic to U. Now we define $U_0 := \tau^{-1} U_\infty \tau$ and $U_a := \alpha_a^{-1} U_0 \alpha_a$ for each $0 \neq a \in U$.

One of the main results of [dMW] is a necessary and sufficient condition for the resulting data $\mathbb{M}(U, \tau) := (X, \{U_x\}_{x \in X})$ to be a Moufang set (cf. [dMW, 3.2]).

9.1.3 Remark
An important property of all Moufang sets is the μ-action: Let $\mathbb{M} = \mathbb{M}(U, \tau)$ be a Moufang set and let $a \in U \backslash \{0\}$. By (M1) there exist $g_1, g_2 \in U_0$ such

that $g_1(\infty) = -a$ and $g_2(a) = \infty$ and these maps are uniquely determined by these properties. Thus, the map $\mu_a := g_2 \circ \alpha_a \circ g_1$ is the unique element in the double coset $U_0 \alpha_a U_0$ interchanging 0 and ∞.

We list the Moufang sets which appear as residues in Moufang quadrangles of quadratic, pseudo-quadratic or exceptional type. The formulas for the double μ-maps of Moufang sets of quadratic and pseudo-quadratic form type can be deduced from [TW, 33.11 and 33.13].

Desarguesian Moufang sets   Given a skew field $\mathbb{K}$, the corresponding *Moufang set of linear type* is

$$\mathbb{M}(\mathbb{K}) := \mathbb{M}(\mathbb{K}, \tau), \text{ where } \tau \colon \mathbb{K}^* \to \mathbb{K}^*$$
$$x \mapsto -x^{-1}.$$

Moufang sets of quadratic form type   Given an anisotropic quadratic space $\Lambda = (\mathbb{K}, V, Q)$, the corresponding *Moufang set of quadratic form type* is

$$\mathbb{M}(\Lambda) := \mathbb{M}(V, \tau), \text{ where } \tau \colon V^* \to V^*$$
$$v \mapsto -Q(v)^{-1} \cdot v.$$

For $a, b \in V^*$ the corresponding double μ-map is given by

$$m_{a,b}^\Lambda(v) := (\mu_a \circ \mu_b^{-1})(v) = \frac{Q(a)}{Q(b)} \pi_a(\pi_b(v))$$

for all $v \in V$, where $\pi_a(v) = v - \left(\frac{f_Q(v,a)}{Q(a)} a \right)$.

Moufang sets of pseudo-quadratic form type   Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$ be a pseudo-quadratic space and let $T = T(\Xi)$ as in 1.37. The corresponding *Moufang set of pseudo-quadratic form type* is

$$\mathbb{M}(\Xi) := (T, \tau), \text{ where } \tau \colon T^* \to T^*$$
$$(a, t) \mapsto (at^{-1}, -t^{-1}).$$

For $(a, t) \in T^*$ the corresponding double μ-map is given by

$$m_{(a,t)}^\Xi(b, v) := \left(\mu_{(a,t)} \circ \mu_{(0,1)}^{-1} \right)(b, v) = (bt^\sigma - at^{-1} f(a, b) t^\sigma, tvt^\sigma)$$

for all $(b, v) \in T$.

Moufang sets of type E_n Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8, choose $0_V \neq \varepsilon \in V$ and replace Q by $Q(\varepsilon)^{-1}Q$. Let $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ be a norm splitting of Λ as in 1.24 and let S, π and g be as in 1.27 and [TW, 13.26 and 13.28]. The corresponding *Moufang set of type* E_n is

$$\mathbb{M}(S) := (S, \tau), \text{ where } \tau \colon S^* \to S^*$$

$$(a, t) \mapsto \left(\frac{a f_Q(\varepsilon, \pi(a) + t\varepsilon)\varepsilon - (\pi(a) + t\varepsilon)}{Q(\pi(a) + t\varepsilon)}, \frac{-t + g(a, a)}{Q(\pi(a) + t\varepsilon)} \right).$$

9.1.4 Lemma
Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$ be a proper anisotropic pseudo-quadratic space with associated skew-hermitian form f, let T be the group defined in 1.37 and choose $(0, t) \in Z(T)^*$. For $(b, v) \in T$ we define

$$\Omega^t_{(b,v)} := \{(a, s) \in T^* \mid m^{\Xi}_{(a,s)} \circ m^{\Xi}_{(0,t)}(b, v) \in (b\mathbb{K}, \mathbb{K})\}$$

and set $b^{\perp} = \{x \in V \mid f(b, x) = 0_{\mathbb{K}}\}$. Then $(a, s) \in \Omega^t_{(b,v)} \cup \{(0_V, 0_{\mathbb{K}})\}$ if and only if $a \in b\mathbb{K} \cup b^{\perp}$.

Proof   First, let $a \in b\mathbb{K} \cup b^{\perp}$ and let $s \in \mathbb{K}$ such that $(a, s) \in T$ (which exists since $(a, Q(a)) \in T$). If $s = 0_{\mathbb{K}}$ then, as $(a, s) \in T$, $Q(a) \in \mathbb{K}_0$ implies that $a = 0_V$ since Ξ is anisotropic. Thus we may suppose that $s \neq 0_V$, in particular $(a, s) \in T^*$.
Suppose that there is $r \in \mathbb{K}$ such that $a = br$. Then

$$\begin{aligned}
m^{\Xi}_{(br,s)} \circ m^{\Xi}_{(0,t)}(b, v) &= m^{\Xi}_{(br,s)}(b\sigma(t), tvt) \\
&= (b\sigma(t)\sigma(s) - brs^{-1}f(br, b\sigma(t))\sigma(s), stvt\sigma(s)) \\
&= (b\,(\sigma(st) - rs^{-1}f(br, b\sigma(t))\sigma(s)), stvt\sigma(s))
\end{aligned}$$

is contained in $(b\mathbb{K}, \mathbb{K})$. Suppose that $a \in b^{\perp}$. Then

$$\begin{aligned}
m^{\Xi}_{(a,s)} \circ m^{\Xi}_{(0,t)}(b, v) &= m^{\Xi}_{(a,s)}(b\sigma(t), tvt) \\
&= (b\sigma(t)\sigma(s), stvt\sigma(s)) \in (b\mathbb{K}, \mathbb{K}).
\end{aligned}$$

Conversely, suppose that $(a, s) \in \Omega^t_{(b,v)}$. Then there exist $r, u \in \mathbb{K}$ such that

$$\begin{aligned}
(br, u) &= m^{\Xi}_{(a,s)} \circ m^{\Xi}_{(0,t)}(b, v) \\
&= (b\sigma(t)\sigma(s) - as^{-1}f(a, b\sigma(t))\sigma(s), stvt\sigma(s)) \\
&= (b\sigma(t)\sigma(s) - as^{-1}f(a, b)\sigma(t)\sigma(s), stvt\sigma(s)).
\end{aligned}$$

If $a \notin b^{\perp}$, then $x := -s^{-1}f(a, b)\sigma(t)\sigma(s) \neq 0_{\mathbb{K}}$ and thus

$$a = b\,(r - \sigma(t)\sigma(s))x^{-1}) \in b\mathbb{K}.$$

$\square$

9.1.5 Definition
Let $(X, \{U_x\}_{x \in X})$ and $(Y, \{U_y\}_{y \in Y})$ be two Moufang sets. An isomorphism between $(X, \{U_x\}_{x \in X})$ and $(Y, \{U_y\}_{y \in Y})$ is a bijection $\beta \colon X \to Y$ such that for all $x \in X$ the map $u \mapsto \beta u \beta^{-1}$ defines a group isomorphism from U_x onto $U_{\beta(x)}$.

We also give a definition in terms of groups and permutations:

9.1.6 Definition
Let $\mathbb{M} := \mathbb{M}(U, \tau)$ and $\mathbb{M}' := \mathbb{M}(U', \tau')$ be Moufang sets. An *isomorphism of Moufang sets* $\beta \colon \mathbb{M} \to \mathbb{M}'$ is an isomorphism of groups $\beta \colon U \to U'$ such that $\mathbb{M}(U', \tau') = \mathbb{M}(U', \beta \tau \beta^{-1})$.

In [dMS, 3.1] they give a necessary and sufficient condition on the two permutation maps τ' and $\beta \tau \beta^{-1}$ on U' providing that $\mathbb{M}(U', \tau') = \mathbb{M}(U', \beta \tau \beta^{-1})$.

9.1.7 Lemma
Let $\mathbb{M} = \mathbb{M}(U, \tau)$ and $\mathbb{M}' = \mathbb{M}(U', \tau')$ be Moufang sets, let $\beta \colon \mathbb{M} \to \mathbb{M}'$ be an isomorphism and let $0 \neq a, x \in U$. Then $\beta(\mu_a(x)) = \mu'_{\beta(a)}(\beta(x))$ for the μ-multiplications in the corresponding Moufang sets.

Proof By assumption $\beta \colon U \to U'$ is an isomorphism of groups and hence $\beta(0) = 0'$. We extend β to a bijection $U \cup \{\infty\} \to U' \cup \{\infty'\}$ by defining $\beta(\infty) := \infty'$. Recall that for $0 \neq a \in U$ the map μ_a is the unique map in the double coset $U_0 \alpha_a U_0$ interchanging 0 and ∞ and that μ_a is given by $g_2 \circ \alpha_a \circ g_1$, where $g_1, g_2 \in U_0$ are uniquely determined by $g_1(\infty) = -a$ and $g_2(a) = \infty$. Similarly, $\mu'_{\beta(a)}$ is the unique map in the double coset $U_{0'} \alpha_{\beta(a)} U_{0'}$ interchanging $0'$ and ∞'.
By definition, β induces an isomorphism of root groups $U_0 \to U_{0'}$ via $g \mapsto \beta \circ g \circ \beta^{-1}$. Hence, $g_1' := \beta \circ g_1 \circ \beta^{-1} \in U_{0'}$ and $g_1'(\infty') = -\beta(a)$. Similarly, $g_2' := \beta \circ g_2 \circ \beta^{-1} \in U_{0'}$ and $g_2'(\beta(a)) = \infty'$.
In particular, this implies $g_2' \circ \alpha_{\beta(a)} \circ g_1' \in U_{0'} \alpha_{\beta(a)} U_{0'}$ and

$$(g_2' \circ \alpha_{\beta(a)} \circ g_1')(0') = \infty' \text{ as well as } (g_2' \circ \alpha_{\beta(a)} \circ g_1')(\infty') = 0'.$$

We conclude that $\mu_{\beta(a)} = g_2' \circ \alpha_{\beta(a)} \circ g_1'$. Now, for any $x \in U$,

$$\beta(\mu_a(x)) = \beta \circ g_2 \circ \alpha_a \circ g_1(x) = g_2' \circ \beta \circ \alpha_a \circ \beta^{-1} \circ g_1'(\beta(x))$$
$$= g_2' \circ \alpha_{\beta(a)} \circ g_1'(\beta(x)) = \mu_{\beta(a)}(\beta(x)),$$

since $\beta(\alpha_a(\beta^{-1}(y))) = \beta(\beta^{-1}(y) + a) = y + \beta(a)$ for all $y \in U'$. $\square$

9.2   Moufang polygons and root group sequences

9.2.1 Definition

A *Moufang n-gon* (for $n \geq 3$) is a thick building of type

$$\bullet \!\!\xrightarrow{\quad n \quad}\!\! \bullet$$

which satisfies the Moufang property (cf. 2.33). A *Moufang polygon* is a Moufang n-gon for some n.

See [TW, 4.2] for an equivalent definition in terms of bipartite graphs. The classification of Moufang polygons was carried out in [TW]. According to [TW, 17.1] Moufang n-gons only exist for $n = 3, 4, 6$ and 8. Moreover, the classification says that each Moufang polygon is uniquely determined by a root group sequence as defined in [TW, 8.7] and these root group sequences in turn are determined by certain algebraic data.

9.2.2 Remark

Let Δ be a Moufang n-gon for some $n \geq 3$, let Σ be an apartment of Δ and let c be a chamber of Σ. Let $\alpha_1, \ldots, \alpha_{2n}$ be the roots of Σ numbered either clockwise or counterclockwise such that $\{c\} = \alpha_1 \cap \cdots \cap \alpha_n$. For each $1 \leq i \leq n$ let U_i denote the root group U_{α_i} and set $U_{[1,n]} := \langle U_1, \ldots, U_n \rangle$.

(a) The sequence
$$\Omega := (U_{[1,n]}, U_1, \ldots, U_n)$$

is a root group sequence as defined in [TW, 8.7]. It is called the *root group sequence of Δ based at (Σ, c)*.
The tuple $\Omega^o := (U_{[1,n]}, U_n, \ldots, U_1)$ is also a root group sequence as defined in [TW, 8.7] and it is called the *opposite root group sequence of Ω*.

(b) Let Ω be as in (a). The root group sequence Ω is uniquely determined (up to opposites) by the pair (Σ, c). By [W03, 11.12], the subgroup $G^\dagger$ of $\mathrm{Aut}(\Delta)$ which is generated by all the root groups of Δ acts transitively on the set of ordered pairs consisting of an apartment of Δ and a chamber contained in this apartment. It follows that the root group sequence Ω is - up to opposites and conjugation in $G^\dagger$ - independent of the choice of the apartment Σ and the numbering of its roots. This fact justifies referring to Ω as *the root group sequence of Δ*.

9.2.3 Definition

Let $\Omega = (U_{[1,n]}, U_1, \ldots, U_n)$ and $\Omega' = (U'_{[1,n]}, U'_1, \ldots, U'_n)$ be two root group sequences as defined in [TW, 8.7] for some $n \geq 3$. An *isomorphism* from Ω

to Ω' is an isomorphism of groups $U_{[1,n]} \to U'_{[1,n]}$ mapping U_i to U'_i for all $1 \le i \le n$.

9.2.4 Remark

Let Δ be a Moufang polygon, let Σ be an apartment of Δ and let Ω be the root group sequence of Δ. Due to [TW, part III] we can assume that the numbering of the roots of the apartment Σ has been chosen so that there is an isomorphism from Ω to one of the root group sequences described in [TW, 16.1-16.9]. We identify Ω with its image under this isomorphism. Thus, Ω is the root group sequence defined by one of the recipes mentioned above in terms of a suitable parameter system Θ and isomorphisms x_i from some part of Θ to the root group U_i, one for each $i \in [1, n]$.

We list those standard root group sequences appearing as root group sequences of Moufang quadrangles of quadratic, pseudo-quadratic or exceptional type:

9.2.5 Notation

($\mathcal{Q}$) *Quadrangles of quadratic form type*

Let $\Lambda = (\mathbb{K}, V, Q)$ be an anisotropic quadratic space with $V \ne \{0\}$ and let f_Q denote the bilinear form associated with Q. For $i = 1, 3$ let x_i be an isomorphism from the additive group of $\mathbb{K}$ to a group U_i and for $i = 2, 4$ let x_i be an isomorphism from the additive group of V to a group U_i. Let $U_{[1,4]}$ be the group generated by the groups U_1, U_2, U_3 and U_4.

The root group sequence

$$\mathcal{Q}_{\mathcal{Q}}(\Lambda) := (U_{[1,4]}, x_1(\mathbb{K}), x_2(V), x_3(\mathbb{K}), x_4(V))$$

with commutator relations $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = [U_1, U_3] = 1$ as well as

$$[x_2(a), x_4(b)^{-1}] = x_3(f_Q(a, b))$$

for all $a, b \in V$ and

$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(tQ(a))$$

for all $t \in \mathbb{K}$ and all $a \in V$ is the *standard root group sequence with respect to* Λ.

($\mathcal{P}$) *Quadrangles of pseudo-quadratic form type*

Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, L_0, Q)$ be an anisotropic pseudo-quadratic space, let f denote the skew-hermitian form associated with Q and let T be the group defined in 1.37. For $i = 1, 3$ let x_i be an isomorphism from T to a group U_i and for $i = 2, 4$ let x_i be an isomorphism from the additive group of $\mathbb{K}$ to a group U_i. Let $U_{[1,4]}$ be the group generated

by the groups U_1, U_2, U_3 and U_4.

The root group sequence

$$\mathcal{Q}_{\mathcal{P}}(\Xi) := (U_{[1,4]}, x_1(T), x_2(\mathbb{K}), x_3(T), x_4(\mathbb{K}))$$

with commutator relations $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$ as well as

$$[x_1(a,t), x_3(b,u)^{-1}] = x_2(f(a,b)),$$

$$[x_2(v), x_4(w)^{-1}] = x_3(0, \sigma(v)w + \sigma(w)v)$$

and

$$[x_1(a,t), x_4(v)^{-1}] = x_2(tv)x_3(av, \sigma(v)tv)$$

for all $(a,t), (b,u) \in T$ and all $v, w \in \mathbb{K}$ is the *standard root group sequence with respect to* Ξ.

($\mathcal{E}$) *Quadrangles of type* $\mathsf{E}_6, \mathsf{E}_7$ *and* E_8

   Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8, choose $0_V \neq \varepsilon \in V$ and replace Q by $Q(\varepsilon)^{-1}Q$. Let $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ be a norm splitting of Λ as in 1.24 and let S be the non-commutative group defined in 1.27. For $i = 1, 3$ let x_i be an isomorphism from S to a group U_i and for $i = 2, 4$ let x_i be an isomorphism from V to a group U_i. Let $U_{[1,4]}$ be the group generated by the groups U_1, U_2, U_3 and U_4.

   The *standard root group sequence with respect to* Λ is defined by

$$\mathcal{Q}_{\mathcal{E}}(\Lambda) := (U_{[1,4]}, x_1(S), x_2(V), x_3(S), x_4(V)),$$

   the defining commutator relations depend on several mappings and can be found in [TW, 16.6].

($\mathcal{F}$) *Quadrangles of type* F_4

   Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type F_4 and let $\hat{\Lambda} = (\mathbb{F}, \hat{V}, \hat{Q})$ be the dual of Λ as defined in 1.31. For $i = 1, 3$ let x_i be an isomorphism from $\hat{V}$ to a group U_i and for $i = 2, 4$ let x_i be an isomorphism from V to a group U_i. Let $U_{[1,4]}$ be the group generated by the groups U_1, U_2, U_3 and U_4.

   The *standard root group sequence with respect to* Λ is defined by

$$\mathcal{Q}_{\mathcal{F}}(\Lambda) := (U_{[1,4]}, x_1(\hat{V}), x_2(V), x_3(\hat{V}), x_4(V)),$$

   the defining commutator relations depend on several mappings and can be found in [TW, 16.7].

9.2.6 Remark

(a) Let $\Omega = (U_{[1,n]}, U_1, \ldots, U_n)$ be one of the root group sequences in [TW, 16.1-16.9]. According to [TW, 8.11 and 7.5] there is a unique Moufang polygon Δ such that Ω is isomorphic to a root group sequence of Δ. The classification of Moufang polygons in [TW] says that, up to isomorphism, there are no other Moufang polygons.

(b) Let Λ be a quadratic space of type E_n. By [TW, 27.20], the root group sequence $\mathcal{Q}_\mathcal{E}(\Lambda)$ is independent (up to isomorphism) of the choice of the element ε and the norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$.

(c) Let Λ be a quadratic space of type F_4. By [TW, 28.43], the root group sequence $\mathcal{Q}_\mathcal{F}(\Lambda)$ is independent (up to isomorphism) of the choice of the complement S_0 and the norm splitting $(\mathbb{E}, \cdot, \{v_1, v_2\})$.

9.2.7 Notation
As in the notion in [W09, 30.15], the Moufang quadrangles corresponding to the cases $(\mathcal{Q}), (\mathcal{P}), (\mathcal{E})$ and $(\mathcal{F})$ described in 9.2.5 are, in order, called: $\mathsf{B}_2^\mathcal{Q}(\Lambda), \mathsf{B}_2^\mathcal{P}(\Xi), \mathsf{B}_2^\mathcal{E}(\Lambda)$ and $\mathsf{B}_2^\mathcal{F}(\Lambda)$.

9.2.8 Remark
In [TW, 35.8 and 35.10-35.12] it is determined to what extend the algebraic structure is an invariant of the corresponding Moufang quadrangle of quadratic, pseudo-quadratic or exceptional type:

 (i) Let Λ and Λ' be proper anisotropic quadratic spaces. Then $\mathcal{Q}_\mathcal{Q}(\Lambda) \simeq \mathcal{Q}_\mathcal{Q}(\Lambda')$ if and only if Λ and Λ' are similar.

 (ii) Let Ξ and Ξ' be proper anisotropic pseudo-quadratic spaces. Then $\mathcal{Q}_\mathcal{P}(\Xi) \simeq \mathcal{Q}_\mathcal{P}(\Xi')$ if and only if Ξ and Ξ' are similar.

(iii) Let Λ and Λ' be quadratic spaces of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8. Then $\mathcal{Q}_\mathcal{E}(\Lambda) \simeq \mathcal{Q}_\mathcal{E}(\Lambda')$ if and only if Λ and Λ' are similar.

(iv) Let Λ and Λ' be quadratic spaces of type F_4. Then $\mathcal{Q}_\mathcal{F}(\Lambda) \simeq \mathcal{Q}_\mathcal{F}(\Lambda')$ if and only if Λ and Λ' are similar.

 (v) Let Λ be a quadratic space of type F_4 and let $\hat\Lambda$ be the dual of Λ as defined in 1.31. Then $\mathcal{Q}_\mathcal{F}^o(\Lambda) \simeq \mathcal{Q}_\mathcal{F}(\hat\Lambda)$.

9.2.9 Remark
Let Λ be a quadratic space of type F_4 which is self-dual as defined in 1.32(d). Then, in view of 9.2.8(v), $\mathcal{Q}_\mathcal{F}(\Lambda) \simeq \mathcal{Q}_\mathcal{F}^o(\hat\Lambda) \simeq \mathcal{Q}_\mathcal{F}^o(\Lambda)$.
A Moufang quadrangle of type F_4 will be called *self-dual* if it is isomorphic to a Moufang quadrangle $\mathsf{B}_2^\mathcal{F}(\Lambda)$ which is defined over a self-dual quadratic space Λ of type F_4.

9.2.10 Remark
Let Δ be a Moufang polygon and let $\mathcal{P}$ be a panel of Δ. For each $x \in \mathcal{P}$ define a subgroup $U_x \leq \mathrm{Sym}(\mathcal{P})$ as in [MPW, 1.19]. Then $\mathbb{M}_{\Delta,\mathcal{P}} := (\mathcal{P}, \{U_x\}_{x\in\mathcal{P}})$ is a Moufang set.

Let $\Sigma, c, \alpha_1, \ldots, \alpha_{2n}, U_1, \ldots, U_n$ and Ω be as in 9.2.2. Let $\mathcal{P}$ be the unique panel of Δ containing c such that U_i acts non-trivially on $\mathcal{P}$. Let $\mathbb{M}_i := \mathbb{M}_{\Delta,\mathcal{P}}$ denote the corresponding Moufang set. If

(i) $\Omega \simeq \mathcal{Q}_{\mathcal{Q}}(\Lambda)$ for some anisotropic quadratic space $\Lambda = (\mathbb{K}, V, Q)$, then $\mathbb{M}_1 \simeq \mathbb{M}(\mathbb{K})$ and $\mathbb{M}_4 \simeq \mathbb{M}(\Lambda)$.

(ii) $\Omega \simeq \mathcal{Q}_{\mathcal{P}}(\Xi)$ for some anisotropic pseudo-quadratic space $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$, then $\mathbb{M}_1 \simeq \mathbb{M}(\Xi)$ and $\mathbb{M}_4 \simeq \mathbb{M}(\mathbb{K})$.

(iii) $\Omega \simeq \mathcal{Q}_{\mathcal{E}}(\Lambda)$ for some quadratic space Λ of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8, then $\mathbb{M}_1 \simeq \mathbb{M}(S)$ and $\mathbb{M}_4 \simeq \mathbb{M}(\Lambda)$.

(iv) $\Omega \simeq \mathcal{Q}_{\mathcal{F}}(\Lambda)$ for some quadratic space Λ of type F_4 with dual $\hat{\Lambda}$, then $\mathbb{M}_1 \simeq \mathbb{M}(\hat{\Lambda})$ and $\mathbb{M}_4 \simeq \mathbb{M}(\Lambda)$.

9.2.11 Notation
Let Δ be a Moufang polygon and let $\mathcal{P}$ be a panel of Δ. We will say that $\mathcal{P}$ is (non-)commutative if the group describing the Moufang set $\mathbb{M}_{\Delta, \mathcal{P}}$ is (non-)commutative. We will say that the panel $\mathcal{P}$ is of quadratic form type, if the corresponding Moufang set $\mathbb{M}_{\Delta, \mathcal{P}}$ is isomorphic to a Moufang set of quadratic form type.

9.3 Property (Ind)

9.3.1 Remark

Let Δ be a Moufang n-gon and let $\mathcal{P}$ and $\mathcal{P}'$ be two panels of Δ of the same type. Choose two chambers $c \in \mathcal{P}$ and $d \in \mathcal{P}'$. Let $G^\dagger$ denote the subgroup of $\mathrm{Aut}(\Delta)$ generated by all the root groups of Δ. By [W03, 11.12], $G^\dagger$ acts transitively on the set of chambers of Δ. Hence there exists an isometry $g \in G^\dagger$ such that $g(c) = d$ and consequently $g(\mathcal{P}) = \mathcal{P}'$. We show that the restriction of g to $\mathcal{P}$ induces an isomorphism of Moufang sets $\mathbb{M}_{\Delta,\mathcal{P}} \to \mathbb{M}_{\Delta,\mathcal{P}'}$:

Clearly, the restriction $\phi := g|_{\mathcal{P}} \colon \mathcal{P} \to \mathcal{P}'$ is a bijection. It remains to show that for any $x \in \mathcal{P}$ we have $\phi U_x \phi^{-1} = U_{\phi(x)}$.

For, choose a chamber $x \in \mathcal{P}$ and an apartment Σ of Δ containing x. Let $y \in \mathcal{P}$ such that $\Sigma \cap \mathcal{P} = \{x,y\}$. Let α be the unique root of Σ containing x but not y, i.e. $\alpha = \{z \in \Sigma \mid \mathrm{dist}(x,z) < \mathrm{dist}(y,z)\}$. The associated root group U_x is given by $\{\varphi|_{\mathcal{P}} \mid \varphi \in U_\alpha\}$. Now $g(x), g(y) \in \mathcal{P}'$, $\Sigma' := g(\Sigma)$ is an apartment of Δ and $\Sigma' \cap \mathcal{P}' = \{g(x), g(y)\}$. Moreover $g(\alpha)$ is the unique root of Σ' containing $g(x)$ but not $g(y)$. Since the permutation group $U_{g(x)}$ is independent of the choice of the apartment, $U_{g(x)} = \{\varphi|_{\mathcal{P}'} \mid \varphi \in U_{g(\alpha)}\}$. According to [AB, 7.25], $g U_\alpha g^{-1} = U_{\varphi(\alpha)}$ and hence $\phi U_x \phi^{-1} = U_{\phi(x)}$.

In particular, each automorphism $g \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{P})$ induces an automorphism of the corresponding Moufang set $\mathbb{M}_{\Delta,\mathcal{P}}$. The converse of this observation is not true in general. There might be more automorphisms of the Moufang set as the stabilizer of any panel provide (for example non-desarguesian planes), cf. [MvM98, Lemma 2] for a proof of this fact.

9.3.2 Definition

Let Δ be a Moufang n-gon and let $\mathcal{P}$ be a panel of Δ. We say that Δ has property (Ind) at $\mathcal{P}$ if the following is satisfied:

(Ind) For every automorphism $\alpha \in \mathrm{Aut}(\mathbb{M}_{\Delta,\mathcal{P}})$ there is an automorphism $\varphi \in \mathrm{Aut}(\Delta)$ inducing α on $\mathcal{P}$.

9.3.3 Proposition

Let $\Lambda = (\mathbb{K}, V, Q)$ be an anisotropic quadratic space such that $\dim_{\mathbb{K}}(V) \geq 3$ and f_Q is not identically zero. Then the Moufang quadrangle $\Delta := \mathsf{B}_2^{\mathcal{Q}}(\Lambda)$ has property (Ind) at each panel $\mathcal{P}$ of Δ satisfying $\mathbb{M}_{\Delta,\mathcal{P}} \simeq \mathbb{M}(\Lambda)$.

Proof   According to 9.2.7, $\mathcal{Q}_{\mathcal{Q}}(\Lambda) = (U_{[1,4]}, x_1(\mathbb{K}), x_2(V), x_3(\mathbb{K}), x_4(V))$ is a root group sequence of Δ. Let $\mathcal{P}$ be a panel of Δ. By 9.2.10, the corresponding Moufang set $\mathbb{M}_{\Delta,\mathcal{P}}$ is either isomorphic to the Desarguesian Moufang set $\mathbb{M}(\mathbb{K})$ or it is isomorphic to the Moufang set $\mathbb{M}(\Lambda)$. Suppose that $\mathbb{M}_{\Delta,\mathcal{P}} \simeq \mathbb{M}(\Lambda)$ and let $\alpha \in \mathrm{Aut}(\mathbb{M}(\Lambda))$. Thus, $\alpha \colon V \to V$ is a group

isomorphism respecting the double μ-maps (cf. 9.1.7). By [MPW, 6.10], therefore, there exist $\phi \in \mathrm{Aut}(\mathbb{K})$ and $t \in \mathbb{K}^*$ such that α is a ϕ-linear t-similitude of V.

For $1 \leq i \leq 4$ we define group automorphisms $\beta_i \colon U_i \to U_i$ as follows:
For $i = 2, 4$ we define $\beta_i \colon x_i(V) \to x_i(V)$ by $\beta_i(x_i(v)) := x_i(\alpha(v))$. Moreover, we let $\beta_1 \colon x_1(\mathbb{K}) \to x_1(\mathbb{K})$ be defined by $\beta_1(x_1(s)) := x_1(\phi(s))$ and $\beta_3 \colon x_3(\mathbb{K}) \to x_3(\mathbb{K})$ by $\beta_3(x_3(s)) := x_3(t\phi(s))$. Since for all $v, w \in V$ and $s \in \mathbb{K}$ we have

$$
\begin{aligned}
[\beta_2(x_2(v)), \beta_4(x_4(w))^{-1}] &= [x_2(\alpha(v)), x_4(\alpha(w))^{-1}] \\
&= x_3(f_Q(\alpha(v), \alpha(w))) \\
&= x_3(t\ \phi(f_Q(v, w))) = \beta_3(x_3(f_Q(v, w)))
\end{aligned}
$$

and

$$
\begin{aligned}
[\beta_1(x_1(s)), \beta_4(x_4(v))^{-1}] &= [x_1(\phi(s)), x_4(\alpha(v))^{-1}] \\
&= x_2(\phi(s)\alpha(v))\ x_3(\phi(s)\ Q(\alpha(v))) \\
&= x_2(\alpha(sv))\ x_3(\phi(s)\ t\ \phi(Q(v))) \\
&= \beta_2(x_2(sv))\ \beta_3(x_3(sQ(v)))
\end{aligned}
$$

the automorphisms β_i induce an automorphism $\beta \colon U_{[1,4]} \to U_{[1,4]}$. By [TW, 7.5], β extends uniquely to an automorphism of Δ.

$\square$

9.3.4 Remark
Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, V, Q)$ be a proper anisotropic pseudo-quadratic space. If $\mathbb{K}/\mathbb{K}_0$ is a separable quadratic extension of fields, σ is the non-trivial element of $\mathrm{Gal}(\mathbb{K}/\mathbb{K}_0)$ and $\dim_{\mathbb{K}}(V) = 4$, then it is shown in [MvM20] that the Moufang quadrangle $\Delta := \mathsf{B}_2^{\mathcal{P}}(\Xi)$ has property (Ind) at each non-commutative panel $\mathcal{P}$. One conjectures that the same is true if $\mathbb{K}$ is a quaternion division algebra over $\mathbb{K}_0$ with standard involution σ and $\dim_{\mathbb{K}}(V) = 4$. However, there isn't any proof, yet.

Chapter 10

Moufang twin buildings and condition (co)

By [AB, 7.83 and 7.116] there is a one-to-one correspondence between the set of spherical Moufang buildings and the set of RGD systems of spherical type as defined in [AB, 7.82]. The concepts and results about spherical Moufang buildings generalize with minor modifications to twin buildings. The algebraic version is a theory of RGD systems of arbitrary type (W, S). This motivates the definition of arbitrary Moufang buildings: An arbitrary building is said to be Moufang if it is part of a Moufang twin building. This definition firstly appeared in [R].

Throughout this section let (W, S) be a Coxeter system which has no isolated nodes in its Coxeter diagram. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a thick twin building of type (W, S) of rank at least 2.

10.0.1 Definition
Let α be a twin root of Δ as defined in [AB, 5.190]. The *root group* U_α is defined to be the set of automorphisms $g \in \mathrm{Aut}(\Delta)$ such that g fixes $\mathcal{P}$ pointwise for every interior panel $\mathcal{P}$ of α.

10.0.2 Definition
Let α be a twin root of Δ. By [AB, 8.17(2)], the root group U_α acts on the set $\mathcal{A}(\alpha)$ consisting of all twin apartments containing the twin root α. We say that Δ is *Moufang* (or, equivalently, a *Moufang twin building*) if the action of the root group U_α on $\mathcal{A}(\alpha)$ is transitive for every twin root α of Δ.

10.0.3 Proposition
If Δ is a Moufang twin building, then every spherical residue of rank at least two of Δ is a Moufang spherical building.

Proof   This is [AB, 8.21]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

According to Ronan [R], general Moufang buildings are defined as follows:

10.0.4 Definition
Let Δ' be a thick building of type (W, S). The building Δ' is called *Moufang* if there exists, for a fixed apartment of Δ', a system $(U_\alpha)_{\alpha\in\Phi}$ of subgroups of $\mathrm{Aut}(\Delta')$, where Φ is the set of all roots of Δ', satisfying the conditions of proposition [AB, 8.56] together with (RGD2).

Suppose that Δ is a Moufang twin building. By [AB, 8.47], the system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$, where Φ denotes the set of all twin roots of a given twin apartment Σ of Δ, $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$ and $T = \mathrm{Fix}_G(\Sigma)$, is a general RGD system as defined in [AB, 8.6.1]. According to [AB, 8.57], a (general) RGD system always gives rise to a (general) Moufang building.
Conversely, if Δ' is a (general) Moufang building then Δ' gives rise to a (general) RGD system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$, where the U_α are as in the definition of a (general) Moufang building. By [AB, 8.81], the twin building associated to the RGD system is Moufang.

10.0.5 Proposition
A thick, irreducible, 2-spherical twin building of rank at least 3 that satisfies (co) is Moufang.

Proof   This is [AB, 8.27]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\Box$

Throughout the rest of this chapter let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a thick twin building of type $\tilde{\mathsf{C}}_2$. Choose a chamber $c \in \mathcal{C}_+ \cup \mathcal{C}_-$ and let $\mathcal{R}$ and $\mathcal{R}'$ denote the two irreducible residues of Δ of rank 2 containing c. We assume that one of these, say $\mathcal{R}$, is a Moufang quadrangle of exceptional type.

Let $\Omega = (U_{[1,4]}, U_1, \ldots, U_4)$ be the root group sequence of $\mathcal{R}$. Since $\mathcal{R}$ is an exceptional Moufang quadrangle we may assume that either

$$\Omega \simeq \mathcal{Q}_\mathcal{E}(\Lambda)$$

for some quadratic space $\Lambda = (\mathbb{K}, V, Q)$ of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8 or

$$\Omega \simeq \mathcal{Q}_\mathcal{F}(\Lambda)$$

for some quadratic space $\Lambda = (\mathbb{K}, V, Q)$ of type F_4.

10.0.6 Proposition
Δ satisfies condition (co).

Proof In view of 2.49(a) it suffices to show that every rank 2 residue of Δ satisfies (co). By 2.49(b), therefore, we show that each panel has at least 4 elements.

Consider the residue $\mathcal{R} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$. By definition, the quadratic space Λ is anisotropic and $\dim_{\mathbb{K}}(V) \geq 4$. In view of [T, 11.2], we conclude that the field $\mathbb{K}$ has infinitely many elements. Let $\mathcal{P} := \mathcal{P}_{s_2}(c)$ and note that $\mathcal{P}$ is a panel of both, $\mathcal{R}$ and $\mathcal{R}'$. Since $\mathcal{R}$ is a Moufang polygon, the panel $\mathcal{P}$ has an induced structure of a Moufang set (cf. [MPW, 1.19]).

Suppose that Λ is of type $\mathsf{E}_6, \mathsf{E}_7$ or E_8. Choose an element $0 \neq \varepsilon \in V$, replace Q by $Q(\varepsilon)^{-1}Q$ and choose a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_4\})$ of Λ. Let S be the group defined in 1.27 with respect to these data. As $|\mathbb{K}| = \infty$, we conclude that the group S as well as the $\mathbb{K}$-vector space V consist of infinitely many elements. By 9.2.10, $\mathbb{M}_{\Delta, \mathcal{P}} \simeq \mathbb{M}(\Lambda)$ or $\mathbb{M}_{\Delta, \mathcal{P}} \simeq \mathbb{M}(S)$. In particular, there is a bijection $\mathcal{P} \to V \cup \{\infty\}$ or $\mathcal{P} \to S \cup \{\infty\}$ and hence $|\mathcal{P}| = \infty$.
Suppose that Λ is of type F_4 and let $\mathbb{F}$ be the subfield of $\mathbb{K}$ as in 1.28(ii). Since $\mathbb{K}^2 \subseteq \mathbb{F}$ (cf. 1.29(b)), we conclude that the field $\mathbb{F}$ has infinitely many elements. Let $\hat{\Lambda} = (\mathbb{F}, \hat{V}, \hat{Q})$ denote the dual of Λ as in 1.31. By 9.2.10, $\mathbb{M}_{\Delta, \mathcal{P}} \simeq \mathbb{M}(\Lambda)$ or $\mathbb{M}_{\Delta, \mathcal{P}} \simeq \mathbb{M}(\hat{\Lambda})$. In particular, there is a bijection $\mathcal{P} \to V \cup \{\infty\}$ or $\mathcal{P} \to \hat{V} \cup \{\infty\}$ and hence $|\mathcal{P}| = \infty$.

Let $\varepsilon \in \{+, -\}$ such that $c \in \mathcal{C}_\varepsilon$ and choose a chamber $d \in \mathcal{C}_{-\varepsilon}$ opposite c. Let $\mathcal{P}' := \mathcal{P}_{s_2}(d)$. By construction, $\mathcal{P}$ and $\mathcal{P}'$ are opposite and hence, by [AB, 5.153], $|\mathcal{P}| = |\mathcal{P}'|$.
Now let $\mathcal{T}$ be any residue of Δ_+ or Δ_- of type B_2 and choose a panel of $\mathcal{T}$ of type $\{s_2\}$. Then, by [AB, 5.157], this panel is isometric to $\mathcal{P}$ (if $\mathcal{T} \subseteq \mathcal{C}_\varepsilon$) or it is isometric to $\mathcal{P}'$ (if $\mathcal{T} \subseteq \mathcal{C}_{-\varepsilon}$). In particular it consists of infinitely many chambers. $\qquad\square$

10.0.7 Proposition
Δ has the Moufang property. In particular, $\mathcal{R}'$ is a Moufang quadrangle.

Proof Since Δ is thick, irreducible, 2-spherical of rank 3 and satisfies (co) by 10.0.6, Δ satisfies the Moufang condition by 10.0.5. The second assertion follows from 10.0.3. $\qquad\square$

Let $\varepsilon \in \{+, -\}$ such that $c \in \mathcal{C}_\varepsilon$. As defined in [W09], a *Bruhat-Tits building* is a thick irreducible affine building whose building at infinity is a spherical Moufang building.

10.0.8 Proposition
The building Δ_ε is a Bruhat-Tits building whose building at infinity is an exceptional Moufang quadrangle.

Proof We denote the building at infinity associated to Δ_ε by $\Delta_\varepsilon^\infty$. Since Δ satisfies the Moufang condition for twin buildings by 10.0.7, the building Δ_ε is a general Moufang building as defined in 10.0.4. Thus, we may apply the main theorem of [vMvS] and obtain that $\Delta_\varepsilon^\infty$ is a Moufang quadrangle.

Since $\Omega \simeq \mathcal{Q}_\mathcal{E}(\Lambda)$ or $\Omega \simeq \mathcal{Q}_\mathcal{F}(\Lambda)$, we have $[U_1, U_3] \neq \{1\} \neq [U_2, U_4]$. By [AB, 11.107], $\mathcal{R}$ can be identified with a subbuilding of $\Delta_\varepsilon^\infty$ and for all $1 \leq i \leq 4$ the root group U_i embeds in the corresponding root group of the quadrangle $\Delta_\varepsilon^\infty$. The classification of Moufang polygons in [TW] claims that every Moufang quadrangle is isomorphic to one of the quadrangles described in [TW, (16.2)-(16.7)]. Let $\Theta = (U'_{[1,4]}, U'_1, \ldots, U'_4)$ be the root group sequence of $\Delta_\varepsilon^\infty$. If $\Delta_\varepsilon^\infty$ is of involutory, quadratic or indifferent type, then $[U'_i, U'_{i+2}] = \{1\}$ for at least one $i \in \{1, 2\}$. Hence, $\Delta_\varepsilon^\infty$ is either of pseudo-quadratic or exceptional type.

By [W09, 24.58], a Bruhat-Tits building whose building at infinity is a Moufang quadrangle of pseudo-quadratic form type has no residues of exceptional type. Hence, Δ_+^∞ needs to be a Moufang quadrangle of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ or F_4. □

Chapter 11

Foundations

11.1 Basic concepts

Let Π be a Coxeter diagram with vertex set I and let (W, S) be the corresponding Coxeter system. For any subset $J \subseteq S$ we let J' denote the set of all irreducible subsets of J which have cardinality 2.

11.1.1 Definition
A *foundation of type* Π is a triple

$$\mathcal{F} := \Big((\Delta_J)_{J \in S'}, (c_J)_{J \in S'}, (\theta_{jik})_{\{s_j, s_i\}, \{s_i, s_k\} \in S'} \Big)$$

such that the following hold:

(F1) $\Delta_J = (\mathcal{C}_J, \delta_J)$ is a building of type Π_J with $c_J \in \mathcal{C}_J$ for each $J \in S'$;

(F2) each *glueing* $\theta_{jik} \colon \mathcal{P}_{s_i}(c_{\{s_j, s_i\}}) \to \mathcal{P}_{s_i}(c_{\{s_i, s_k\}})$ is a bijection sending $c_{\{s_j, s_i\}}$ onto $c_{\{s_i, s_k\}}$;

(F3) $\theta_{kil} \circ \theta_{jik} = \theta_{jil}$ for all $s_i, s_j, s_k, s_l \in S$ such that $s_i \notin \{s_j, s_k, s_l\}$ and $\{s_k, s_i\}, \{s_i, s_l\}, \{s_j, s_i\} \in S'$.

11.1.2 Remark
Let $\mathcal{F}$ be a foundation of type Π. If $s_i, s_j \in S$ are such that $\{s_i, s_j\} \in S'$, axiom (F3) yields $\theta_{jij} \circ \theta_{jij} = \theta_{jij}$. Since θ_{jij} is a bijection, it is the identity on the panel $\mathcal{P}_{s_i}(c_{\{s_i, s_j\}})$. Hence, if $s_i, s_j, s_k \in S$ are such that $\{s_i, s_j\}, \{s_i, s_k\} \in S'$ we have $\theta_{kij} \circ \theta_{jik} = \theta_{jij}$ and thus $\theta_{kij} = \theta_{jik}^{-1}$.

11.1.3 Definition
A foundation $\mathcal{F} = \Big((\Delta_J)_{J \in S'}, (c_J)_{J \in S'}, (\theta_{jik})_{\{s_j, s_i\}, \{s_i, s_k\} \in S'} \Big)$ of 2-spherical type Π will be called a *Moufang foundation* if the following hold:

(MF1) For all $J \in S'$ the building Δ_J is a Moufang polygon.

(MF2) If $s_j, s_i, s_k \in S$ are such that $\{s_j, s_i\}, \{s_i, s_k\} \in S'$, then the glueing θ_{jik} induces an isomorphism between the Moufang sets $\mathbb{M}_{\Delta_{\{s_i,s_j\}}, \mathcal{P}_{s_i}(c_{\{s_i,s_j\}})}$ and $\mathbb{M}_{\Delta_{\{s_i,s_k\}}, \mathcal{P}_{s_i}(c_{\{s_i,s_k\}})}$.

11.1.4 Definition
Let $\mathcal{F}$ be a foundation of type Π and let $K \subseteq S$. The *K-residue of $\mathcal{F}$* is the foundation

$$\mathcal{F}_K := ((\Delta_J)_{J \in K'}, (c_J)_{J \in K'}, (\theta_{jik})_{\{s_j,s_i\},\{s_i,s_k\} \in K'}).$$

11.1.5 Definition
Let $\mathcal{F}$ and $\mathcal{F}'$ be foundations. An *isomorphism of foundations* $\varphi \colon \mathcal{F} \to \mathcal{F}'$ is a system $\varphi := \{\pi, \alpha_J \mid J \in S'\}$ of isomorphisms

$$\pi \colon \Pi \to \Pi', \quad \alpha_J \colon \Delta_J \to \Delta'_{\pi(J)}$$

such that $\alpha_J(c_J) = c'_{\pi(J)}$ and for all $s_i, s_j, s_k \in S$ with $\{s_j, s_i\}, \{s_i, s_k\} \in S'$ and all $x \in \mathcal{P}_{s_i}(c_{\{s_j,s_i\}})$ we have

$$(\theta'_{\pi(j)\pi(i)\pi(k)} \circ \alpha_{\{s_j,s_i\}})(x) = (\alpha_{\{s_i,s_k\}} \circ \theta_{jik})(x).$$

An isomorphism $\varphi \colon \mathcal{F} \to \mathcal{F}'$ is *special* if $\Pi = \Pi'$ and $\pi = \mathrm{id}_\Pi$.

11.1.6 Remark
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type Π, let $\varepsilon \in \{+, -\}$ and let $c \in \mathcal{C}_\varepsilon$ be a chamber. The union of all irreducible rank 2 residues of Δ_ε containing c provides a foundation in a canonical way: For each $J \in S'$ we set

$$\Delta_J := (\mathcal{R}_J(c), \delta_\varepsilon|_{\mathcal{R}_J(c) \times \mathcal{R}_J(c)}) \text{ and } c_J := c$$

and for $s_i, s_j, s_k \in S$ such that $\{s_j, s_i\}, \{s_i, s_k\} \in S'$ we let θ_{jik} be the identity on the panel $\mathcal{P}_{s_i}(c)$. Now the resulting triple

$$\mathcal{F}(\Delta, c) := ((\Delta_J)_{J \in S'}, (c)_{J \in S'}, (\theta_{jik})_{\{i,j\},\{i,k\} \in S'})$$

is a foundation of type Π. It is called *the foundation of Δ based at c.*
It is a (not completely trivial) fact that for any chamber $d \in \mathcal{C}_\varepsilon$ we have $\mathcal{F}(\Delta, d) \simeq \mathcal{F}(\Delta, c)$. Moreover, for $c', d' \in \mathcal{C}_{-\varepsilon}$ we have $\mathcal{F}(\Delta, c') \simeq \mathcal{F}(\Delta, d')$ and the isomorphism class of $\mathcal{F}(\Delta, c')$ is uniquely determined by the isomorphism class of $\mathcal{F}(\Delta, c)$ and vice versa.
If Δ satisfies the Moufang condition, the foundation $\mathcal{F}(\Delta, c)$ is a Moufang foundation for every chamber $c \in \mathcal{C}_+ \cup \mathcal{C}_-$.

11.1.7 Definition
A foundation $\mathcal{F}$ of type Π is called *integrable*, if there exists a twin building Δ of type Π and a chamber c of Δ such that $\mathcal{F} \simeq \mathcal{F}(\Delta, c)$.

11.1.8 Remark
Let $\Delta = (\mathcal{C}, \delta)$ be a Moufang spherical building of type Π and let $c \in \mathcal{C}$ be a chamber. Again, the triple

$$\mathcal{F}(\Delta, c) := ((\mathcal{R}_J(c))_{J \in S'}, (c)_{J \in S'}, (\mathrm{id})_{\{i,j\},\{i,k\} \in S'})$$

is a Moufang foundation of type Π. In view of [AB, 8.81], Δ is part of a Moufang twin building Δ' and $\mathcal{F}(\Delta, c) \simeq \mathcal{F}(\Delta', c)$. Thus, the foundation $\mathcal{F}(\Delta, c)$ is integrable.

11.1.9 Lemma
Let $\mathcal{F}$ be an integrable foundation of type Π and let $J \subseteq S$ such that $|J| \geq 2$. Then the J-residue $\mathcal{F}_J$ is integrable.

Proof This is Theorem 20.1 of [WDis]. $\square$

11.1.10 Definition
A foundation $\mathcal{F} = ((\Delta_J)_{J \in S'}, (c)_{J \in S'}, (\theta_{jik})_{\{i,j\},\{i,k\} \in S'})$ of type Π satisfies

(lco), if for any $J \in S'$ the building Δ_J is spherical and for each chamber $c \in \mathcal{C}_J$ the chamber system defined by the set of chambers opposite c is connected.

(lsco), if Π is 3-spherical and if each panel $\mathcal{P}_s(c_J)$ has at least 17 elements for any $s \in S$ and $J \in S'$ with $s \in J$.

11.1.11 Theorem
Let $\mathcal{F}$ be a Moufang foundation of affine, irreducible type Π which satisfies (lco) and (lsco). Then the following are equivalent:

   (i) $\mathcal{F}$ is integrable.

   (ii) For each irreducible subset $J \subseteq S$ with $|J| = 3$ the J-residue $\mathcal{F}_J$ is integrable.

Proof Let $\mathcal{D}$ be as in [BM], which is an RGD-system since Π is affine. The assertion now follows by Theorem 3.20 of [BM]. The second implication is 11.1.9. $\square$

As has been announced earlier, almost all twin buildings are uniquely determined by the foundation of one of its halves:

11.1.12 Theorem
Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be thick, irreducible, 2-spherical twin buildings of type Π which satisfy (co) and let $c \in \mathcal{C}_+$ and $c' \in \mathcal{C}'_+$ be chambers such that the foundations $\mathcal{F}(\Delta, c)$ and $\mathcal{F}(\Delta', c')$ are isomorphic. Then Δ and Δ' are isomorphic.

Proof Let $\varphi = \{\pi, \alpha_J \mid J \in S'\}\colon \mathcal{F}(\Delta, c) \to \mathcal{F}(\Delta', c')$ be an isomorphism, where S' denotes the set of all irreducible subsets of S of cardinality 2. For all $J \in S'$ the map $\alpha_J\colon \mathcal{R}_J(c) \to \mathcal{R}_{\pi(J)}(c')$ is a $\pi|_J$-isometry which maps c onto c'. By replacing Δ' by $\Delta'^{\pi^{-1}}$ via the π^{-1}-isometry id (cf. 7.2.4), each map $\alpha'_J := \mathrm{id} \circ \alpha_J\colon (\mathcal{R}_J(c), \delta_+) \to (\mathcal{R}_J(c'), \delta_+^{\pi^{-1}})$ becomes an isometry mapping c onto c'. If $s \in S$ and $J, K \in S'$ are such that $s \in K \cap J$, then (IoF2) ensures that $\alpha'_J(x) = \alpha'_K(x)$ holds for all $x \in \mathcal{P}_s(c)$. Hence the mapping $\phi\colon E_2^*(c) \to E_2^*(c')$ defined by $\phi(x) := \alpha_J(x)$ if $x \in \mathcal{R}_J(c)$ is well-defined.

Let $s, t \in S$ be such that $st = ts$. As Π is irreducible we may choose $J, K \in S'$ with $s \in J$ and $t \in K$. Let $x \in \mathcal{P}_s(c)$ and let $y \in \mathcal{P}_t(c)$ such that $x \neq c \neq y$. Then, by 2.28, $c' = \mathrm{proj}_{\mathcal{P}_s(c')}(\phi(y))$ and $c = \mathrm{proj}_{\mathcal{P}_s(c)}(y)$. Hence

$$\delta'_+(\phi(x), \phi(y)) = \delta'_+(\phi(x), c')\delta'_+(c', \phi(y)) = \delta_+(x, c)\delta_+(c, y) = \delta_+(x, y)$$

implies that $\phi|_{\mathcal{P}_s(c) \cup \mathcal{P}_t(c)}$ is an isometry from $\mathcal{P}_s(c) \cup \mathcal{P}_t(c)$ onto $\mathcal{P}_s(c') \cup \mathcal{P}_t(c')$. Using 4.2.3 we extend ϕ to a map $\Phi\colon E_2(c) \to E_2(c')$ which, by construction, satisfies the properties of 4.2.4. Thus, Φ is an isometry $E_2(c) \to E_2(c')$. According to 7.1.6, there exist chambers $d \in \mathcal{C}_-$ and $d' \in \mathcal{C}'_-$ which are opposite to c and c' respectively such that the map $d \mapsto d'$ extends Φ to an isometry $E_2(c) \cup \{d\} \to E_2(c') \cup \{d'\}$.

By 7.3.1, Φ extends uniquely to an isometry from Δ onto $\Delta'^{\pi^{-1}}$. As Δ' is isomorphic to $\Delta'^{\pi^{-1}}$ (cf. 7.2.4), the assertion follows. $\square$

11.2 nm2-foundations

Let $3 \leq n, m \in \mathbb{N}$. Throughout the rest of this section let Π be the 2-spherical Coxeter diagram

We denote the corresponding Coxeter system by (W, S) and let

$$\mathcal{F} = ((\Delta_{\{s_1, s_2\}}, \Delta_{\{s_2, s_3\}}), (c_{\{s_1, s_2\}}, c_{\{s_2, s_3\}}), (\theta_{123}))$$

be a Moufang foundation of type Π.

Note that, in view of 11.1.2, it suffices to consider the glueing θ_{123}. The building $\Delta_{\{s_1, s_2\}}$ is a Moufang n-gon while $\Delta_{\{s_2, s_3\}}$ is a Moufang m-gon. We set $\mathcal{P} := \mathcal{P}_{s_2}(c_{\{s_1, s_2\}})$ and let $\mathbb{M} := \mathbb{M}_{\Delta_{\{s_1, s_2\}}, \mathcal{P}}$ denote the corresponding Moufang set. Similarly, we set $\mathcal{P}' := \mathcal{P}_{s_2}(c_{\{s_2, s_3\}})$ and $\mathbb{M}' := \mathbb{M}_{\Delta_{\{s_2, s_3\}}, \mathcal{P}'}$. Note that $\theta_{123}(c_{\{s_1, s_2\}}) = c_{\{s_2, s_3\}}$.

11.2.1 Remark
Let d be any chamber of $\Delta_{\{s_1,s_2\}}$ and set $\mathcal{Q} := \mathcal{P}_{s_2}(d)$. Let $G^\dagger$ denote
the subgroup of $\mathrm{Aut}(\Delta_{\{s_1,s_2\}})$ generated by all the root groups of $\Delta_{\{s_1,s_2\}}$.
According to [W03, 11.12] there exists $g \in G^\dagger$ such that $g(c_{\{s_1,s_2\}}) = d$
and by 9.3.1, g induces an isomorphism of the corresponding Moufang sets
$\mathbb{M} \to \mathbb{M}_{\Delta_{\{s_1,s_2\}},\mathcal{Q}}$. In particular, the map $\gamma_{123} := \theta_{123} \circ (g|_{\mathcal{P}})^{-1} \colon \mathcal{Q} \to \mathcal{P}'$ is
a bijection inducing an isomorphism of Moufang sets $M_{\Delta_{\{s_1,s_2\}},\mathcal{Q}} \to \mathbb{M}'$ and
$\alpha := \{\mathrm{id}_\Pi, g, \mathrm{id}_{\Delta_{\{s_2,s_3\}}}\}$ is an isomorphism of foundations

$$\alpha \colon \mathcal{F} \to ((\Delta_{\{s_1,s_2\}}, \Delta_{\{s_2,s_3\}}), (d, \gamma_{123}(d)), (\gamma_{123})).$$

11.2.2 Lemma
Let $d \in \mathcal{P}$ be a chamber. Then $\mathcal{F} \simeq ((\Delta_{\{s_1,s_2\}}, \Delta_{\}s_2,s_3\}}), (d, \theta_{123}(d)), (\theta_{123}))$.

Proof If $d = c_{\{s_1,s_2\}}$ the assertion is clear. So suppose that $d \neq c_{\{s_1,s_2\}}$.
Since $\Delta_{\{s_1,s_2\}}$ is thick we may choose $z \in \mathcal{P}\backslash\{c_{\{s_1,s_2\}}, d\}$. By property (M1),
there exists a unique $g \in U_z$ such that $g(c_{\{s_1,s_2\}}) = d$. Similarly, there exists
a unique $g' \in U_{\theta_{123}(z)}$ such that $g'((c_{\{s_2,s_3\}})) = \theta_{123}(d)$.
Let $G^\dagger$ denote the subgroup of $\mathrm{Aut}(\Delta_{\{s_1,s_2\}})$ generated by all the root groups
of $\Delta_{\{s_1,s_2\}}$ and similarly let $G^{\dagger'}$ denote the subgroup of $\mathrm{Aut}(\Delta_{\{s_2,s_3\}})$ gen-
erated by all the root groups of $\Delta_{\{s_2,s_3\}}$. There exist $\varphi \in G^\dagger$ and $\phi \in G^{\dagger'}$
inducing the maps g and g' respectively. Then $\alpha := \{\mathrm{id}_\Pi, \varphi, \phi\}$ is an iso-
morphism of foundations

$$\mathcal{F} \to ((\Delta_{\{s_1,s_2\}}, \Delta_{\{s_2,s_3\}}), (d, \theta_{123}(d)), (g' \circ \theta_{123} \circ g^{-1})).$$

Since θ_{123} can be seen to be an isomorphism of Moufang sets $\mathbb{M} \to \mathbb{M}'$ we
have $\theta_{123} U_z \theta_{123}^{-1} = U_{\theta_{123}(z)}$. In particular, $\theta_{123} \circ g \circ \theta_{123}^{-1} \in U_{\theta_{123}(z)}$. Moreover,
as $(\theta_{123} \circ g \circ \theta_{123}^{-1})(\theta_{123}(c)) = \theta_{123}(d)$, we conclude that $\theta_{123} \circ g \circ \theta_{123}^{-1} = g'$.
Hence, $g' \circ \theta_{123} \circ g^{-1} = \theta_{123}$ and the assertion follows. $\square$

11.2.3 Lemma
If $\Delta_{\{s_1,s_2\}}$ satisfies (Ind) at $\mathcal{P}$ or if $\Delta_{\{s_2,s_3\}}$ satisfies (Ind) at $\mathcal{P}'$, the founda-
tion $\mathcal{F}$ is uniquely determined by $\Delta_{\{s_1,s_2\}}$ and $\Delta_{\{s_2,s_3\}}$.

Proof Let $\mathcal{F}' := ((\Delta_{\{s_1,s_2\}}, \Delta_{\{s_2,s_3\}}), (d, \gamma_{123}(d)), (\gamma_{123}))$ be a foundation
of type Π. In view of 11.2.1 we may assume that $d = c_{\{s_1,s_2\}}$.
Let $G^{\dagger'}$ denote the subgroup of $\mathrm{Aut}(\Delta_{\{s_2,s_3\}})$ generated by all the root
groups of $\Delta_{\{s_2,s_3\}}$. Let $g \in G^{\dagger'}$ such that $g(\theta_{123}(c_{\{s_1,s_2\}}) = \gamma_{123}(c_{\{s_1,s_2\}})$.
Following 9.3.1, g induces an isomorphism of the corresponding Moufang
sets $g|_{\mathcal{P}'} \colon \mathbb{M}' \to \mathbb{M}_{\Delta_{\{s_2,s_3\}},\mathcal{P}_{s_2}(\gamma_{123}(c_{\{s_1,s_2\}}))}$.

Suppose that $\Delta_{\{s_1,s_2\}}$ satisfies (Ind) at the panel $\mathcal{P}$. Because of this property there is $\varphi \in \mathrm{Aut}(\Delta_{\{s_1,s_2\}})$ inducing $\phi := \gamma_{123}^{-1} \circ g|_{\mathcal{P}'} \circ \theta_{123} \in \mathrm{Aut}(\mathbb{M})$. Now $\alpha := \{\mathrm{id}_\Pi, \varphi, g\}$ is an isomorphism

$$\alpha \colon \mathcal{F} \to ((\Delta_{\{s_1,s_2\}}, \Delta_{\{s_2,s_3\}}), (\varphi(c_{\{s_1,s_2\}}), \gamma_{123}(\varphi(c_{\{s_1,s_2\}}))), (\gamma_{123})).$$

As $\varphi(c_{\{s_1,s_2\}}) \in \mathcal{P}$, lemma 11.2.2 gives that $\mathcal{F} \simeq \mathcal{F}'$.

If $\Delta_{\{s_2,s_3\}}$ satisfies (Ind) at the panel $\mathcal{P}'$, there exists $\varphi \in \mathrm{Aut}(\Delta_{\{s_2,s_3\}})$ inducing $\phi := g|_{\mathcal{P}'}^{-1} \circ \gamma_{123} \circ \theta_{123}^{-1} \in \mathrm{Aut}(\mathbb{M}')$. Now $\alpha := \{\mathrm{id}_\Pi, \mathrm{id}_{\Delta_{\{s_1,s_2\}}}, g \circ \varphi\}$ is an isomorphism $\mathcal{F} \to \mathcal{F}'$.

$\square$

11.2.4 Remark
Choose an apartment Σ_{12} of $\Delta_{\{s_1,s_2\}}$ containing the chamber $c_{\{s_1,s_2\}}$. By 9.2.2 the Moufang n-gon $\Delta_{\{s_1,s_2\}}$ can be described by the root group sequence $\Omega_{12} = (U_+, U_1, \ldots, U_n)$ based at $(\Sigma_{12}, c_{\{s_1,s_2\}})$. Similarly, $\Delta_{\{s_2,s_3\}}$ can be described by a root group sequence $\Omega_{23} = (U'_+, U'_1, \ldots, U'_m)$. We choose the labeling in such a way that the glueing θ_{123} is an isomorphism $\mathbb{M}_n \to \mathbb{M}'_1$. If at least one of the polygons satisfies (Ind) at the glueing panel, lemma 11.2.3 enables us to briefly describe the foundation by $\mathcal{F} = (\Omega_{12}, \Omega_{23})$.

For the rest of this section we assume that $n = m = 4$, i.e. $\mathcal{F}$ is a foundation of type $\tilde{\mathsf{C}}_2$.

11.2.5 Remark
Let $\pi \in \mathrm{Aut}(\Pi)$ be the non-trivial automorphism of the Coxeter diagram $\tilde{\mathsf{C}}_2$. Then $\alpha := \{\pi, \mathrm{id}_{\Delta_{\{s_1,s_2\}}}, \mathrm{id}_{\Delta_{\{s_2,s_3\}}}\}$ is an isomorphism

$$\alpha \colon \mathcal{F} \to \mathcal{F}^o := ((\Delta_{\{s_2,s_3\}}, \Delta_{\{s_1,s_2\}}), (c_{\{s_2,s_3\}}, c_{\{s_1,s_2\}}), (\theta_{321})).$$

We continue to assume that at least one of the polygons satisfies (Ind) at the glueing panel and let Ω_{12} and Ω_{23} denote the corresponding root group sequences such that $\mathcal{F} = (\Omega_{12}, \Omega_{23})$. The considerations above yield $(\Omega_{12}, \Omega_{23}) = \mathcal{F} \simeq \mathcal{F}^o = (\Omega_{23}^o, \Omega_{12}^o)$.

11.2.6 Lemma
Suppose that $\Delta_{\{s_1,s_2\}} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ for some quadratic space Λ of type E_k for some $k \in \{6, 7, 8\}$ and that $\Delta_{\{s_2,s_3\}} \simeq \mathsf{B}_2^{\mathcal{Q}}(\Lambda')$, where Λ' is an anisotropic quadratic space. Then the glueing is along the panels of quadratic form type.

Proof   Let $\Lambda = (\mathbb{K}, V, Q)$ and $\Lambda' = (\mathbb{K}', V', Q')$. Note that the Moufang sets $\mathbb{M}(V')$ and $\mathbb{M}(\mathbb{K}')$ are commutative. Hence the glueing needs to be an isomorphism $\mathbb{M}(V) \to \mathbb{M}(V')$ or $\mathbb{M}(V) \to \mathbb{M}(\mathbb{K}')$. In view of [KDis, 3.3.5], the latter is impossible, since $\dim_{\mathbb{K}}(V) \geq 6$ and $\mathrm{Def}(\Lambda) \neq V$ by 1.25. $\square$

11.2.7 Lemma
Suppose that $\Delta_{\{s_1,s_2\}} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ for some quadratic space Λ of type E_k for some $k \in \{6,7,8\}$ and that $\Delta_{\{s_2,s_3\}} \simeq \mathsf{B}_2^{\mathcal{P}}(\Xi)$, where Ξ is an anisotropic pseudo-quadratic space. Then the glueing is along the non-commutative panels.

Proof   Let $\Lambda = (\mathbb{K}, V, Q)$ and $\Xi = (\mathbb{K}', \mathbb{K}_0', \sigma', V', Q')$. Suppose that the quadrangles are glued along their commutative panels. Then there is an isomorphims of Moufang sets $\mathbb{M}(V) \to \mathbb{M}(\mathbb{K}')$. In view of [KDis, 3.3.5], this is impossible, since $\dim_{\mathbb{K}}(V) \geq 6$ and $\mathrm{Def}(\Lambda) \neq V$ by 1.25. $\square$

11.2.8 Lemma
Suppose that $\Delta_{\{s_1,s_2\}} \simeq \mathsf{B}_2^{\mathcal{F}}(\Lambda)$ for some quadratic space Λ of type F_4 and that $\Delta_{\{s_2,s_3\}} \simeq \mathsf{B}_2^{\mathcal{Q}}(\Lambda')$, where Λ' is an anisotropic quadratic space. Then the glueing is along panels of quadratic form type.

Proof   Let $\Lambda = (\mathbb{K}, V, Q)$, let $\hat{\Lambda} = (\mathbb{F}, \hat{V}, \hat{Q})$ be the dual of Λ and let $\Lambda' = (\mathbb{K}', V', Q')$. Note that both Moufang sets associated to a Moufang quadrangle of type F_4 are of quadratic form type. Since $\dim_{\mathbb{K}}(V) > 4$ and $V \neq \mathrm{Def}(Q)$ as well as $\dim_{\mathbb{F}}(\hat{V}) > 4$ and $\mathrm{Def}(\hat{Q}) \neq \hat{V}$ by definition, the assertion follows from [KDis, 3.3.5]. $\square$

11.2.9 Theorem
Let $\Lambda_1 = (\mathbb{K}_1, V_1, Q_1)$ and $\Lambda_2 = (\mathbb{K}_2, V_2, Q_2)$ be either quadratic spaces of type E_6 or let Λ_1 and Λ_2 be quadratic spaces of type E_7 with $\mathrm{char}(\mathbb{K}_1) \neq 2$. Let $\Xi_1 = (\bar{\mathbb{K}}_1, \mathbb{K}_0^1, \sigma_1, \bar{V}_1, \bar{Q}_1)$ and $\Xi_2 = (\bar{\mathbb{K}}_2, \mathbb{K}_0^2, \sigma_2, \bar{V}_2, \bar{Q}_2)$ be proper pseudo-quadratic spaces. For $i = 1, 2$ let $\mathcal{F}_i := ((\mathsf{B}_2^{\mathcal{E}}(\Lambda_i), \mathsf{B}_2^{\mathcal{P}}(\Xi_i)), \gamma_i)$ be integrable Moufang foundations of type $\tilde{\mathsf{C}}_2$, where the Moufang quadrangles are glued along their non-commutative panels in each case. Then each isomorphism $\varphi \colon \mathsf{B}_2^{\mathcal{E}}(\Lambda_1) \to \mathsf{B}_2^{\mathcal{E}}(\Lambda_2)$ extends to an isomorphism of foundations $\mathcal{F}_1 \to \mathcal{F}_2$.

Proof   Let $i \in \{1, 2\}$. As $\mathcal{F}_i$ is assumed to be integrable, there exists a twin building $\Delta_i = (\Delta_{i,+}, \Delta_{i,-}, \delta_{i,*})$ of type $\tilde{\mathsf{C}}_2$ and a chamber $c_i \in \mathcal{C}_{i,\varepsilon}$ for some $\varepsilon \in \{+, -\}$ such that $\mathcal{F}_i \simeq \mathcal{F}(\Delta_i, c_i)$. By 10.0.8, the building $\Delta_{i,\varepsilon}$ is a Bruhat-Tits building whose building at infinity is an exceptional Moufang quadrangle. According to the classification of Bruhat-Tits buildings of type $\tilde{\mathsf{C}}_2$ having an exceptional Moufang quadrangle as building at infinity given in [MPW, 14.3 and 17.3], we obtain that Ξ_i is a 4-dimensional proper anisotropic pseudo-quadratic space.
Moreover, if Λ_i is of type E_6, then, by [MPW, 14.3(2)(iii)(a)], $\bar{\mathbb{K}}_i/\mathbb{K}_0^i$ is a separable quadratic extension and σ_i is the non-trivial element of $\mathrm{Gal}(\bar{\mathbb{K}}_i/\mathbb{K}_0^i)$. If Λ_i is of type E_7, then, by [MPW, 14.3(3)(ii)(a)], $\bar{\mathbb{K}}_i$ is a quaternion division algebra, $\mathbb{K}_0^i$ is its center and σ_i is the standard involution on $\bar{\mathbb{K}}_i$.

For $i = 1, 2$ choose $e_i \in V_i$, replace Q_i by $Q_i(e_i)^{-1}Q_i$ (so $Q_i(e_i) = 1_{\mathbb{K}_i}$) and choose a norm splitting $(\mathbb{E}_i, \cdot, \{v_{i_1}, \ldots, v_{i_d}\})$ of Λ_i and set $s_{i_j} := Q_i(v_{i_j})$ for each $1 \leq j \leq d\}$. We may assume that $v_{i_1} = e_i$. Let S_i be the group defined in 1.27 with respect to these data and let T_i be the group defined in 1.37 with respect to the anisotropic quadratic space Ξ_i.

Let $i \in \{1, 2\}$. Set $X_i := S_i/Z(S_i)$ (so if Λ_i is of type E_k, X_i is a vector space over $\mathbb{K}_i$ of dimension 2^{k-3}). If Λ_i is of type E_6 set $\mathbb{D}_i := \mathbb{E}_i$. By [TW, 13.9], X_i is a 4-dimensional vector space over $\mathbb{D}_i$. If Λ_i is of type E_7 let $\mathbb{D}_i := (\mathbb{E}_i/\mathbb{K}_i, s_{i_2}s_{i_3}s_{i_4})$. As $s_{i_2}s_{i_3}s_{i_4} \notin N(\mathbb{E}_i)$ by definition, $\mathbb{D}_i$ is a quaternion division algebra and, by [W06, 3.6], there exists a scalar multiplication $*\colon \mathbb{D}_i \times X_i \to X_i$ extending the scalar multiplication of $\mathbb{K}_i$ on X_i which gives X_i the structure of a vector space over $\mathbb{D}_i$. Note that $\dim_{\mathbb{D}_i}(X_i) = 4$.
By [TW, 13.6], the $\mathbb{K}_i$-vector space V_i acts on X_i and, by [TW, 12.53], this action has a unique extension to a map $X_i \times C(Q_i, e_i) \to X_i$ making X_i into a right $C(Q_i, e_i)$-module, where $C(Q_i, e_i)$ denotes the Clifford algebra of Q_i with basepoint e_i as defined in [TW, 12.47]. As is shown in [W06, 3.8], the centralizer of this action in $\mathrm{End}_{\mathbb{K}_i}(X_i)$ is isomorphic to $\mathbb{D}_i$. Since also $\bar{\mathbb{K}}_i$ centralizes the action of V_i on X_i, we conclude that $\bar{\mathbb{K}}_i$ can be identified with $\mathbb{D}_i$.

Now consider the group T_i. As Ξ_i is a pseudo-quadratic space, $T_i/Z(T_i) \simeq \bar{V}_i$ is a 4-dimensional vector space over $\bar{\mathbb{K}}_i \simeq \mathbb{D}_i$. On the other hand, by assumption, there is an isomorphism of Moufang sets $\gamma_i\colon \mathbb{M}(S_i) \to \mathbb{M}(T_i)$ (and hence of the underlying groups) which carries the structure of a vector space over $\mathbb{D}_i$ from $S_i/Z(S_i) = X_i$ onto $T_i/Z(T_i)$. Thus, γ_i induces an isomorphism of vector spaces.

Suppose that we are given an isomorphism $\varphi\colon \mathsf{B}_2^{\mathcal{E}}(\Lambda_1) \to \mathsf{B}_2^{\mathcal{E}}(\Lambda_2)$. Let $\mathcal{P}$ be a non-commutative panel of $\mathsf{B}_2^{\mathcal{E}}(\Lambda_1)$. Then the restriction $\gamma := \varphi|_{\mathcal{P}}$ induces an isomorphism of Moufang sets: $\gamma\colon \mathbb{M}(S_1) \to \mathbb{M}(S_2)$. Since φ is an isomorphism of Moufang quadrangles, it respects the actions of V_1 on S_1 and V_2 on S_2 and hence it is compatible with the $\mathbb{D}_i$-vector space structure of X_i. Hence there is an isomorphism of skew fields $\phi\colon \mathbb{D}_1 \to \mathbb{D}_2$ such that the map $\varphi_1\colon S_1/Z(S_1) \to S_2/Z(S_2)$ induced by γ is ϕ-semi-linear. Moreover, the map $\gamma' := \gamma_2 \circ \gamma \circ \gamma_1^{-1}$ is an isomorphism $\mathbb{M}(\Xi_1) \to \mathbb{M}(\Xi_2)$ of Moufang sets such that for all $x \in X = S_1/Z(S_1)$ we have $\gamma_2^{-1} \circ \gamma' \circ \gamma_1(x) = \varphi_1(x)$. According to [WDis, 7.15], γ' is well-defined on the first component and thus induces an isomorphism of groups $\gamma_1'\colon T_1/Z(T_1) \to T_2/Z(T_2)$ which can be thought of to be a ϕ-semi-linear map of the underlying vector spaces due to the considerations above.

Let $(0_{X_2}, t) \in Z(T_2)$ be such that $\gamma'(0_{X_1}, 1_{\mathbb{D}_1}) = (0_{X_2}, t)$. Let $b \in X_1$ and $v \in \mathbb{D}_1$ be such that $(b, v) \in T_1$ and let $(a, s) \in T_1^*$. Let $s', v' \in \mathbb{D}_2$ be such

that $\gamma'(a, s) = (\gamma_1'(a), s')$ and $\gamma'(b, v) = (\gamma_1'(b), v')$. In view of 9.1.7 we have

$$
\begin{aligned}
\gamma'(m_{(a,s)}^{\Xi_1} \circ m_{(0_{X_1}, 1_{\mathbb{D}_1})}^{\Xi_1}(b, v)) &= \gamma'(m_{(a,s)}^{\Xi_1}(b, v)) \\
&= \gamma'(\mu_{(a,s)}^{\Xi_1} \circ \mu_{(0_{X_1}, 1_{\mathbb{D}_1})}^{\Xi_1}{}^{-1}(b, v)) \\
&= \mu_{\gamma'(a,s)}^{\Xi_2} \circ \mu_{\gamma'(0_{X_1}, 1_{\mathbb{D}_1})}^{\Xi_2}{}^{-1}(\gamma'(b, v)) \\
&= \mu_{(\gamma_1'(a), s')}^{\Xi_2} \circ \mu_{(0_{X_2}, t)}^{\Xi_2}{}^{-1}(\gamma_1'(b), v') \\
&= m_{(\gamma_1'(a), s')}^{\Xi_2} \circ m_{(0_{X_2}, t^{-1})}^{\Xi_2}(\gamma_1'(b), v').
\end{aligned}
$$

In particular, if $(a, s) \in \Omega_{(b,v)}^{1_{\mathbb{D}_1}}$, where $\Omega_{(b,v)}^{1_{\mathbb{D}_1}}$ is defined as in 9.1.4, then there exist $u, u' \in \mathbb{D}_1$ such that $m_{(a,s)}^{\Xi_1} \circ m_{(0_{X_1}, 1_{\mathbb{D}_1})}^{\Xi_1}(b, v) = (bu, u')$. Hence,

$$
\begin{aligned}
m_{\gamma'(a,s)}^{\Xi_2} \circ m_{(0_{X_2}, t^{-1})}^{\Xi_2}(\gamma'(b, v)) &= \gamma'(m_{(a,s)}^{\Xi_1}(b, v)) \\
&= \gamma'(bu, u') = (\gamma_1'(b)\phi(u), u'') \in (\gamma_1'(b)\mathbb{D}_2, \mathbb{D}_2)
\end{aligned}
$$

implies that $\gamma'(a, s) \in \Omega_{\gamma'(b,v)}^{t^{-1}}$. Note that this is reasonable since $t^{-1} \in \mathbb{K}_0^2$ by [TW, 11.6].

Let $i \in \{1, 2\}$. For a better readability we now identify each element $b \in X_i$ with its image $\gamma_i(b) \in T_i / Z(T_i)$. Let f_i be the skew-hermitian form associated with $\bar{Q}_i$ and let π_i be the polarity of the projective space $\mathbf{P}(X_i)$ induced by f_i, i.e. π is defined by $\pi_i(\langle x \rangle) = x^{\perp}$ for all $x \in X_i$. Note that for all $0_{X_i} \neq b \in X_i$ we have $b \notin b^{\perp}$, since the fact $f_i(b, b) = 0_{\bar{\mathbb{D}}_i}$ implies that $\bar{Q}_i(b) \in \mathbb{K}_0^i$ which in turn implies that $b = 0_{X_i}$ since Ξ_i is anisotropic. In particular, $\langle b \rangle \cap b^{\perp} = \{0_{X_i}\}$ for all $b \in X_i$.

Moreover, for all $b \in X_1$ we have $\gamma_1'(\pi_1(\langle b \rangle)) = \pi_2(\langle \gamma_1'(b) \rangle)$: Indeed, let $a \in b^{\perp}$. Clearly, if $a = 0_{X_1}$ then $\gamma_1'(a) = 0_{X_2} \in \gamma_1'(b)^{\perp}$. Otherwise, by 9.1.4, we have $(a, \bar{Q}_1(a)) \in \Omega_{(b, \bar{Q}_1(b))}^{1_{\mathbb{D}_1}}$ and the considerations above show that $\gamma'(a, \bar{Q}_1(a)) \in \Omega_{\gamma'(b, \bar{Q}_1(b))}^{t^{-1}}$ and thus, by 9.1.4, $\gamma_1'(a) \in \langle \gamma_1'(b) \rangle \cup \gamma_1'(b)^{\perp}$. As $a \notin b\mathbb{D}_1$, $\gamma_1'(a) \in \gamma_1'(b)^{\perp} = \pi_2(\gamma_1'(b))$.

This illustrates that the skew-hermitian forms $\hat{f} := \phi \circ f_1 \circ (\gamma_1'^{-1} \times \gamma_1'^{-1})$ and f_2 induce the same polarity on $\mathbf{P}(X_2)$. Thus, they are "proportional", i.e. there exists $c \in \mathbb{D}_2^*$ such that $\hat{f}(a', b') = c f_2(a', b')$ holds for all $a', b' \in X_2$. By [BC, 7.3.14], $\hat{f} = c f_2$ is a $(\hat{\sigma}, \hat{\varepsilon})$-hermitian form, where $\hat{\sigma}$ is given by $\hat{\sigma}(s) = c\sigma_2(s)c^{-1}$ for all $s \in \mathbb{D}_2$ and $\hat{\varepsilon} = -c\sigma_2(c)^{-1}$. Since $\hat{f}$ is skew-hermitian, we conclude that $c \in \mathrm{Fix}_{\mathbb{D}_2}(\sigma_2) = \mathbb{K}_2 \simeq Z(S_2) \simeq Z(T_2) \simeq \mathbb{K}_0^2$.

Note that $Z(\mathbb{D}_2) = \mathrm{Fix}_{\mathbb{D}_2}(\sigma_2)$ also holds for $\mathbb{D}_2$ being a quaternion division algebra with standard involution σ_2 since we excluded the case that $\mathrm{char}(\mathbb{K}_2) = 2$.

If $\mathrm{char}(\mathbb{K}_2) \neq 2$ the pseudo-quadratic forms $\hat{Q} := \phi \circ \bar{Q}_1 \circ \gamma_1^{-1}$ and $\bar{Q}_2$ are uniquely determined by their associated skew-hermitian forms (cf. [TW,

11.28]) and we conclude that $\hat{Q} = c\bar{Q}_2$. If $\text{char}(\mathbb{K}_2) = 2$, then [TW, 11.19] implies that the skew-hermitian forms are trace-valued and thus again determine the associated pseudo-quadratic forms. In particular, the pair (γ, ϕ) is a similarity from Ξ_1 onto Ξ_2. According to [TW, 35.19 and 7.5], this similarity extends to an isomorphism $\psi \colon \mathsf{B}_2^{\mathcal{P}}(\Xi_1) \simeq \mathsf{B}_2^{\mathcal{P}}(\Xi_2)$.

Furthermore, since $\varphi|_{\mathcal{P}} = \varphi_1$, $\psi|_{\gamma_1(\mathcal{P})} = \gamma_1'$ and $\gamma_1' \circ \gamma_1(x) = \gamma_1 \circ \varphi_1(x)$ holds for all $x \in \mathcal{P}$, the system $\{\mathrm{id}, \varphi, \psi\}$ is an isomorphism of foundations $\mathcal{F}_1 \to \mathcal{F}_2$.

$\square$

Chapter 12

Determining the local structure

12.0.1 Theorem

Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a thick twin building of type $\tilde{\mathsf{C}}_2$ and suppose that for some chamber $c \in \mathcal{C}_+$ the rank 2 residue $\mathcal{R} := \mathcal{R}_{\{s_1,s_2\}}(c)$ is isomorphic to a Moufang quadrangle of exceptional type. Let $\mathcal{R}' := \mathcal{R}_{\{s_2,s_3\}}(c)$ and let $\mathcal{F} := \mathcal{F}(\Delta, c)$ be the foundation of Δ based at c. Then

(E_6) if $\mathcal{R} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ for some quadratic space Λ of type E_6, then the foundation $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$ and either

$$\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda)) \text{ or } \mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}^o(\Lambda), \mathcal{Q}_\mathcal{P}(\Xi)),$$

where Ξ is a proper anisotropic pseudo-quadratic space which is uniquely determined (up to similarity) by $\mathcal{R}$;

(E_7) if $\mathcal{R} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ for some quadratic space Λ of type E_7, then either $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$ and

$$\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda)), \text{ or } \mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}^o(\Lambda), \mathcal{Q}_\mathcal{P}(\Xi)),$$

where Ξ is a proper anisotropic pseudo-quadratic space which is uniquely determined (up to similarity) by $\mathcal{R}$ if $\text{char}(\mathbb{K}) \neq 2$;

(E_8) if $\mathcal{R} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ for some quadratic space Λ of type E_8, then the foundation $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$ and

$$\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda));$$

(F_4) if $\mathcal{R} \simeq \mathsf{B}_2^{\mathcal{F}}(\Lambda)$ for some quadratic space of type F_4, then the foundation $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$ and

$$\mathcal{F} \simeq (\mathcal{Q}_\mathcal{F}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda)) \text{ or } \mathcal{F} \simeq (\mathcal{Q}_\mathcal{F}(\hat{\Lambda}), \mathcal{Q}_\mathcal{Q}^o(\hat{\Lambda})),$$

where $\hat{\Lambda}$ denotes the dual of Λ as defined in 1.31.

Proof   According to 10.0.7, the foundation $\mathcal{F}$ is a Moufang foundation. Moreover, by 10.0.8, the building Δ_+ is a Bruhat-Tits building whose building at infinity is an exceptional Moufang quadrangle. Thus we may use the classification of exceptional Bruhat-Tits buildings of B. Mühlherr, H. Petersson and R. Weiss summarized in [MPW, 14.3 and 17.3] to determine the structure of the residue $\mathcal{R}'$.

(E_6) Suppose that $\mathcal{R}$ is a Moufang quadrangle of type E_6. Then, by [MPW, 14.3 and 17.3] the building Δ_+^∞ is either a Moufang quadrangle of type E_6 or E_7.

  In the first case [MPW, 11.4(i)] implies that $\mathcal{R}'$ is a Moufang quadrangle of quadratic form type. Thus, there exists an anisotropic quadratic space $\Lambda' = (\mathbb{K}', V', Q')$ such that the root group sequence of $\mathcal{R}'$ is isomorphic to the root group sequence $\mathcal{Q}_\mathcal{Q}(\Lambda')$. According to 11.2.6, $\mathcal{R}$ and $\mathcal{R}'$ are glued along their panels of quadratic form type and hence there is an isomorphism of Moufang sets $\gamma \colon \mathbb{M}(\Lambda) \to \mathbb{M}(\Lambda')$, i.e. $\gamma \colon V \to V'$ is an isomorphism of groups which, by 9.1.7, respects the μ-multiplication. Therefore, according to [MPW, 6.10], Λ' is similar to Λ and hence, by [TW, 35.8], $\mathcal{Q}_\mathcal{Q}(\Lambda) \simeq \mathcal{Q}_\mathcal{Q}(\Lambda')$. By 9.3.3, $\mathcal{R}'$ satisfies (Ind) at the glueing panel and hence 11.2.3 implies that the foundation $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$. Thus, $\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda))$. If Δ_+^∞ is of type E_7, [MPW, 12.8(iv)] implies that there exists a proper anisotropic pseudo-quadratic space $\Xi = (\mathbb{E}, \mathbb{K}', \sigma, V', Q')$, where $\mathbb{E}/\mathbb{K}'$ is a separable quadratic extension, σ is the non-trivial element of $\mathrm{Gal}(\mathbb{E}/\mathbb{K}')$ and $\dim_\mathbb{E}(V') = 4$ such that the root group sequence of $\mathcal{R}'$ is isomorphic to the root group sequence $\mathcal{Q}_\mathcal{P}(\Xi)$. According to 11.2.7, $\mathcal{R}$ and $\mathcal{R}'$ are glued along their non-commutative panels. Thus we have $\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}^o(\Lambda), \mathcal{Q}_\mathcal{P}(\Xi))$. Moreover, by 11.2.9, the foundation $\mathcal{F}$ is uniquely determined by the isomorphism class of $\mathcal{R}$.

  Note that, in view of [TW, 38.9], the two foundations given can not be isomorphic.

(E_7) Suppose that $\mathcal{R}$ is a Moufang quadrangle of type E_7. Then, by [MPW, 14.3 and 17.3] the building Δ_+^∞ is either a Moufang quadrangle of type E_7 or E_8.

  In the first case [MPW, 11.4(i)] implies that $\mathcal{R}'$ is a Moufang quadrangle of quadratic form type. Thus, there exists an anisotropic quadratic space $\Lambda' = (\mathbb{K}', V', Q')$ such that the root group sequence of $\mathcal{R}'$ is isomorphic to the root group sequence $\mathcal{Q}_\mathcal{Q}(\Lambda')$. According to 11.2.6, $\mathcal{R}$ and $\mathcal{R}'$ are glued along their panels of quadratic form type and hence there is an isomorphism of Moufang sets $\gamma \colon \mathbb{M}(\Lambda) \to \mathbb{M}(\Lambda')$, i.e. $\gamma \colon V \to V'$ is an isomorphism of groups which, by 9.1.7, respects the μ-multiplication. Therefore, according to [MPW, 6.10], Λ' is similar to

Λ and hence, by [TW, 35.8], $\mathcal{Q}_\mathcal{Q}(\Lambda) \simeq \mathcal{Q}_\mathcal{Q}(\Lambda')$. By 9.3.3, $\mathcal{R}'$ satisfies (Ind) at the glueing panel and hence 11.2.3 implies that the foundation $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$. Thus, $\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda))$.

If Δ_+^∞ is of type E_8, [MPW, 12.9(i)] implies that there exists a proper anisotropic pseudo-quadratic space $\Xi = (\mathbb{D}, \mathbb{K}', \sigma, V', Q')$, where $\mathbb{D}$ is a quaternion division algebra over $\mathbb{K}'$, σ is the standard involution of $\mathbb{D}$ and $\dim_\mathbb{D}(V') = 4$ such that the root group sequence of $\mathcal{R}'$ is isomorphic to the root group sequence $\mathcal{Q}_\mathcal{P}(\Xi)$. According to 11.2.7, $\mathcal{R}$ and $\mathcal{R}'$ are glued along their non-commutative panels. Thus we have $\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}^o(\Lambda), \mathcal{Q}_\mathcal{P}(\Xi))$. Moreover, if $\mathrm{char}(\mathbb{K}) \neq 2$, by 11.2.9, the foundation $\mathcal{F}$ is uniquely determined by the isomorphism class of $\mathcal{R}$.

Note that, in view of [TW, 38.9], the two foundations given can not be isomorphic.

(E_8) Suppose that $\mathcal{R}$ is a Moufang quadrangle of type E_8. Then, by [MPW, 14.3 and 17.3] the building Δ_+^∞ is a Moufang quadrangle of type E_8.

By [MPW, 11.4(i)], $\mathcal{R}'$ is a Moufang quadrangle of quadratic form type. Thus, there exists an anisotropic quadratic space $\Lambda' = (\mathbb{K}', V', Q')$ such that the root group sequence of $\mathcal{R}'$ is isomorphic to the root group sequence $\mathcal{Q}_\mathcal{Q}(\Lambda')$. According to 11.2.6, $\mathcal{R}$ and $\mathcal{R}'$ are glued along their panels of quadratic form type and hence there is an isomorphism of Moufang sets $\gamma \colon \mathbb{M}(\Lambda) \to \mathbb{M}(\Lambda')$, i.e. $\gamma \colon V \to V'$ is an isomorphism of groups which, by 9.1.7, respects the μ-multiplication. Therefore, according to [MPW, 6.10], Λ' is similar to Λ and hence, by [TW, 35.8], $\mathcal{Q}_\mathcal{Q}(\Lambda) \simeq \mathcal{Q}_\mathcal{Q}(\Lambda')$. By 9.3.3, $\mathcal{R}'$ satisfies (Ind) at the glueing panel and hence 11.2.3 implies that the foundation $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$. Thus, $\mathcal{F} \simeq (\mathcal{Q}_\mathcal{E}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda))$.

(F_4) Suppose that $\mathcal{R}$ is a Moufang quadrangle of type F_4. The classification in [MPW, 14.3 and 17.3] shows that, irrespective of the type of Δ_+^∞, $\mathcal{R}'$ is a Moufang quadrangle of quadratic form type in each case. Thus, there exists an anisotropic quadratic space $\Lambda' = (\mathbb{K}', V', Q')$ such that the root group sequence of $\mathcal{R}'$ is isomorphic to the root group sequence $\mathcal{Q}_\mathcal{Q}(\Lambda')$.

Let $\hat{\Lambda}$ denote the dual of Λ. By 11.2.8, the glueing induces an isomorphism between two Moufang sets of quadratic form type. By [MPW, 6.10], therefore, the quadratic space Λ' is similar to Λ or $\hat{\Lambda}$, depending on the glueing panel. Hence, by [TW, 35.8], $\mathcal{Q}_\mathcal{Q}(\Lambda) \simeq \mathcal{Q}_\mathcal{Q}(\Lambda')$ or $\mathcal{Q}_\mathcal{Q}(\hat{\Lambda}) \simeq \mathcal{Q}_\mathcal{Q}(\Lambda')$. As $\mathcal{R}'$ satisfies (Ind) at the glueing panel (cf. 9.3.3), 11.2.3 implies that the foundation $\mathcal{F}$ is uniquely determined by $\mathcal{R}$ and $\mathcal{R}'$. In view of 11.2.5 and 9.2.8(v) we have $\mathcal{F} \simeq (\mathcal{Q}_\mathcal{F}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda))$ or $\mathcal{F} \simeq ((\mathcal{Q}_\mathcal{F}^o(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\hat{\Lambda})) \simeq ((\mathcal{Q}_\mathcal{F}(\hat{\Lambda}), \mathcal{Q}_\mathcal{Q}^o(\hat{\Lambda}))$.

$\square$

Part V

Existence

Chapter 13

Existence of certain twin buildings of higher rank

13.1 Buildings of type $\tilde{\mathsf{E}}_n$

Let Δ be a building of type D_n $(n \geq 4)$ or E_n $(n = 6, 7, 8)$. According to [Ti74, 6.7], each irreducible rank-2-residue of Δ is associated to a Desarguesian projective plane and each of these projective planes is defined over the same skew field $\mathbb{K}$. By [Ti74, 6.12 and 6.13], the defining skew field $\mathbb{K}$ is a field and it determines the building Δ up to isomorphism.

Moreover, for each field $\mathbb{K}$ there exists such a building of type D_n $(n \geq 4)$ or E_n $(n = 6, 7, 8)$ which will be denoted by $\mathsf{D}_n(\mathbb{K})$ or $\mathsf{E}_n(\mathbb{K})$, respectively.

13.1.1 Notation
Throughout this section we fix a quadratic space $\Lambda = (\mathbb{K}, V, Q)$ of type E_n for $n \in \{6, 7, 8\}$. We let $\mathbb{E}/\mathbb{K}$ be a separable quadratic extension such that $\{\mathbb{E}, \{v_1, \ldots, v_d\}\}$ is a norm splitting of Λ.

13.1.2 Proposition
Let Π be the Coxeter diagram $\tilde{\mathsf{E}}_n$ labeled as in 2.3. There exists a twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ of type $\tilde{\mathsf{E}}_n$ such that for some $\varepsilon \in \{+, -\}$ each residue of type $\{s_1, \ldots, s_6\}$ in $\mathcal{C}_\varepsilon$ is isometric to the building $\mathsf{E}_n(\mathbb{E})$.

Proof

$n = 6$ Let Δ_1 be the building $\mathsf{E}_6(\mathbb{E})$, let the corresponding diagram be labeled as in 2.3 and choose a chamber $c \in \Delta_1$. Let Δ_2 be the building $\mathsf{D}_5(\mathbb{E})$. Let $\mathcal{R} := \mathcal{R}_{\{s_2, s_3, s_4, s_4\}}(c)$ be a residue of Δ_1 of type D_4. If $\mathcal{R}'$ is a residue of Δ_2 of type D_4, the classification of spherical buildings yields that there is an isomorphism $\varphi \colon \mathcal{R} \to \mathcal{R}'$. Let $d := \varphi(c)$. We relabel

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the diagram D_5 in such a way that $\varphi(\mathcal{R}_{\{s_i,s_j\}}(c)) = \mathcal{R}_{\{s_i,s_j\}}(d)$ holds for all $\{i,j\} \in \{\{2,4\},\{3,4\},\{4,5\}\}$ and the unique vertex which does not belong to the subdiagram D_4 will be labeled by 0. Then

$$
\begin{aligned}
\mathcal{F} := (&(\mathcal{R}_{\{s_1,s_3\}}(c), \mathcal{R}_{\{s_3,s_4\}}(c), \mathcal{R}_{\{s_2,s_4\}}(c), \mathcal{R}_{\{s_0,s_2\}}(d), \\
&\mathcal{R}_{\{s_4,s_5\}}(c), \mathcal{R}_{\{s_5,s_6\}}(c)), \\
&(c, c, c, d, c, c), \\
&(\theta_{134}, \theta_{342}, \theta_{245}, \theta_{456})),
\end{aligned}
$$

where $\theta_{420} = \varphi|_{\mathcal{P}_{\{s_2\}}(c)}$ and $\theta_{ijk} = \mathrm{id}$ in all the other cases, is a foundation of type $\tilde{\mathsf{E}}_6$. According to [WDis, 25.1], the foundation $\mathcal{F}$ is integrable. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{E}}_6$ and let $x \in \mathcal{C}_+ \cup \mathcal{C}_-$ be such that $\mathcal{F}(\Delta, x) \simeq \mathcal{F}$. Let $J := \{s_1, \ldots, s_6\}$. Then, $\mathcal{F}_J \simeq \mathcal{F}(\Delta_1, c)$ and thus, by [AB, 5.209], $\mathcal{R}_J(x) \simeq \Delta_1 \simeq \mathsf{E}_6(\mathbb{E})$. In view of [AB, 5.157], each spherical residue of type J which is contained in the same half as the chamber x is isometric to the building $\mathsf{E}_6(\mathbb{E})$.

$n = 7$ Let Δ_1 be the building $\mathsf{E}_7(\mathbb{E})$, let the corresponding diagram be labeled as in 2.3 and choose a chamber $c \in \Delta_1$. Let Δ_2 be the building $\mathsf{D}_6(\mathbb{E})$. Let $\mathcal{R} := \mathcal{R}_{\{s_1,\ldots,s_5\}}(c)$ be a residue of Δ_1 of type D_5. If $\mathcal{R}'$ is a residue of Δ_2 of type D_5, the classification of spherical buildings yields that there is an isomorphism $\varphi \colon \mathcal{R} \to \mathcal{R}'$. Let $d := \varphi(c)$. We relabel the diagram D_5 in such a way that $\varphi(\mathcal{R}_{\{s_i,s_j\}}(c)) = \mathcal{R}_{\{s_i,s_j\}}(d)$ holds for all $\{i,j\} \in \{\{1,3\},\{2,4\},\{3,4\}\{4,5\}\}$ and the unique vertex which does not belong to the subdiagram D_5 will be labeled by 0. Then

$$
\begin{aligned}
\mathcal{F} := (&(\mathcal{R}_{\{s_0,s_1\}}(d), \mathcal{R}_{\{s_1,s_3\}}(c), \mathcal{R}_{\{s_3,s_4\}}(c), \mathcal{R}_{\{s_2,s_4\}}(c), \mathcal{R}_{\{s_4,s_5\}}(c), \\
&\mathcal{R}_{\{s_5,s_6\}}(c), \mathcal{R}_{\{s_6,s_7\}}(c)), \\
&(d, c, c, c, c, c, c), \\
&(\theta_{013}, \theta_{134}, \theta_{342}, \theta_{345}, \theta_{245}, \theta_{456}, \theta_{567})),
\end{aligned}
$$

where $\theta_{013} = \varphi^{-1}|_{\mathcal{P}_{\{s_1\}}(d)}$ and $\theta_{ijk} = \mathrm{id}$ in all the other cases, is a foundation of type $\tilde{\mathsf{E}}_7$. According to [WDis, 25.1], the foundation $\mathcal{F}$ is integrable. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{E}}_7$ and let $x \in \mathcal{C}_+ \cup \mathcal{C}_-$ be such that $\mathcal{F}(\Delta, x) \simeq \mathcal{F}$. Let $J := \{s_1, \ldots, s_7\}$. Then, $\mathcal{F}_J \simeq \mathcal{F}(\Delta_1, c)$ and thus, by [AB, 5.209], $\mathcal{R}_J(x) \simeq \Delta_1 \simeq \mathsf{E}_7(\mathbb{E})$. In view of [AB, 5.157], each spherical residue of type J which is contained in the same half as the chamber x is isometric to the building $\mathsf{E}_7(\mathbb{E})$.

$n = 8$ Let Δ_1 be the building $\mathsf{E}_8(\mathbb{E})$, let the corresponding diagram be labeled as in 2.3 and choose a chamber $c \in \Delta_1$. Let Δ_2 be the building $\mathsf{D}_8(\mathbb{E})$. Let $\mathcal{R} := \mathcal{R}_{\{s_2,\ldots,s_8\}}(c)$ be the unique residue of Δ_1 of type D_7. If $\mathcal{R}'$

is a residue of Δ_2 of type D_7, the classification of spherical buildings yields that there is an isomorphism $\varphi \colon \mathcal{R} \to \mathcal{R}'$. Let $d := \varphi(c)$. We relabel the diagram D_7 in such a way that $\varphi(\mathcal{R}_{\{s_i,s_j\}}(c)) = \mathcal{R}_{\{s_i,s_j\}}(d)$ holds for all $\{i,j\} \in \{\{2,4\}, \{i, i+1\} \mid 3 \le i \le 7\}$ and the unique vertex which does not belong to the subdiagram D_7 will be labeled by 0. Then

$$\begin{aligned}
\mathcal{F} := (&(\mathcal{R}_{\{s_1,s_3\}}(c), \mathcal{R}_{\{s_3,s_4\}}(c), \mathcal{R}_{\{s_2,s_4\}}(c), \mathcal{R}_{\{s_4,s_5\}}(c), \\
& \mathcal{R}_{\{s_5,s_6\}}(c), \mathcal{R}_{\{s_6,s_7\}}(c), \mathcal{R}_{\{s_7,s_8\}}(c), \mathcal{R}_{\{s_8,s_0\}}(d)), \\
& (c,c,c,c,c,c,c,d), \\
& (\theta_{134}, \theta_{342}, \theta_{345}, \theta_{245}, \theta_{456}, \theta_{567}, , \theta_{678}, \theta_{780})),
\end{aligned}$$

where $\theta_{780} = \varphi|_{\mathcal{P}_{\{s_8\}}(c)}$ and $\theta_{ijk} = \mathrm{id}$ in all the other cases, is a foundation of type $\tilde{\mathsf{E}}_8$. According to [WDis, 25.1], the foundation $\mathcal{F}$ is integrable. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{E}}_8$ and let $x \in \mathcal{C}_+ \cup \mathcal{C}_-$ be such that $\mathcal{F}(\Delta, x) \simeq \mathcal{F}$. Let $J := \{s_1, \ldots, s_8\}$. Then, $\mathcal{F}_J \simeq \mathcal{F}(\Delta_1, c)$ and thus, by [AB, 5.209], $\mathcal{R}_J(x) \simeq \Delta_1 \simeq \mathsf{E}_8(\mathbb{E})$. In view of [AB, 5.157], each spherical residue of type J which is contained in the same half as the chamber x is isometric to the building $\mathsf{E}_8(\mathbb{E})$.

$\square$

13.1.3 Proposition
For $n = 6, 7$ let Π be the Coxeter diagram $\tilde{\mathsf{E}}_{n+1}$ labeled as in 2.3. There exists a twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ of type $\tilde{\mathsf{E}}_{n+1}$ such that for some $\varepsilon \in \{+, -\}$ each residue of type $\{s_1, \ldots, s_n\}$ in $\mathcal{C}_\varepsilon$ is isometric to the building $\mathsf{E}_n(\mathbb{E})$.

Proof Let Δ be the twin building of type $\tilde{\mathsf{E}}_{n+1}$ described in 13.1.2. Then, for some chamber x of Δ, $\mathcal{R}_{J \cup \{s_{n+1}\}}(x) \simeq \mathsf{E}_{n+1}(\mathbb{E})$. In view of the classification, since $\mathcal{R}_J(x)$ is a residue of E_{n+1} of type E_n, $\mathcal{R}_J(c) \simeq \mathsf{E}_n(\mathbb{E})$. By [AB, 5.157], each spherical residue of type J which is contained in the same half as the chamber x is isometric to the building $\mathsf{E}_n(\mathbb{E})$. $\square$

13.2   Buildings of type $\tilde{\mathsf{F}}_4$

Let $\mathbb{L}/\mathbb{K}$ be an inseparable extension of fields with $\mathrm{char}(\mathbb{K}) = 2$ such that $\mathbb{L}^2 \subseteq \mathbb{K} \subseteq \mathbb{L}$. Let $q \colon \mathbb{L} \to \mathbb{K}$ denote the quadratic form which is defined by $q(x) := x^2$ for all $x \in \mathbb{L}$. According to [Ti74, 10.2] there exists a unique building of type F_4 such that its residues of type C_2 are isomorphic to the building $\mathsf{B}_2^{\mathcal{Q}}(\mathbb{K}, \mathbb{L}, q)$. We denote this building by $\mathsf{F}_4(\mathbb{L}/\mathbb{K})$.

13.2.1 Notation
Throughout this section we fix a quadratic space $\Lambda = (\mathbb{K}, V, Q)$ of type F_4. Let $\mathbb{F}$ be as in 1.29, let $(\mathbb{E}, \cdot, \{v_1, v_2\})$ be a norm splitting of some complement of $\mathrm{Def}(\Lambda)$ in V and let $\mathbb{D}$ denote the composite field $\mathbb{E}\mathbb{F}$. Thus $\mathbb{D}/\mathbb{E}$ is an extension such that $\mathbb{D}^2 \subseteq \mathbb{E} \subseteq \mathbb{D}$. Moreover, in view of [T, 11.2], the field $\mathbb{E}$ has infinitely many elements.

13.2.2 Proposition
Let Π be the Coxeter diagram $\tilde{\mathsf{F}}_4$ labeled as in 2.3. There exists a twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ of type $\tilde{\mathsf{F}}_4$ such that for some $\varepsilon \in \{+, -\}$ each residue of type $\{s_1, \ldots, s_4\}$ in $\mathcal{C}_\varepsilon$ is isometric to the building $\mathsf{F}_4(\mathbb{D}/\mathbb{E})$.

Proof Let Δ_1 be the building $\mathsf{F}_4(\mathbb{D}/\mathbb{E})$, let c be a chamber of Δ_1 and let $F_1 := \mathcal{F}(\Delta_1, c)$ be the foundation of Δ_1 based at c. By 11.1.8, the foundation $\mathcal{F}_1$ is integrable and Moufang.

We consider the quadratic space $\Lambda' := (\mathbb{E}, V', Q')$, where $V' := \mathbb{D} \oplus (\mathbb{E})^8$ and $Q' \colon V' \to \mathbb{E}$ is defined by $Q'(x_0, x_1, \ldots, x_8) := x_0^2 + x_1 x_3 + x_2 x_4 + x_5 x_6 + x_7 x_8$ for all $x_0 \in \mathbb{D}, x_1, \ldots, x_8 \in \mathbb{E}$. Let $\Delta_2 := Flag(\mathcal{G}(\mathbf{P}(\Lambda')))$. Thus Δ_2 is a building of type C_4. In view of 11.1.8, the foundation $\mathcal{F}(\Delta_2, d)$ is integrable and Moufang for each chamber d of Δ_2.

Let $\mathcal{R} := \mathcal{R}_{\{s_1, s_2, s_3\}}(c)$ be a residue of Δ_1 of type C_3 and let $\mathcal{R}'$ be a residue of Δ_2 of type C_3. Due to the classification of buildings of type F_4 and the construction of Δ_2, the C_2-residues of $\mathcal{R}$ and $\mathcal{R}'$ of type C_2 are both isomorphic to the building $\mathsf{B}_2^{\mathcal{Q}}(\mathbb{E}, \mathbb{D}, x \mapsto x^2)$. Thus, according to [Ti74, 8.8(ii)] there is an isomorphism $\varphi \colon \mathcal{R} \to \mathcal{R}'$. Let $d := \varphi(c)$. We relabel the diagram C_4 in such a way that $\varphi(\mathcal{R}_{\{s_i, s_{i+1}\}}(c)) = \mathcal{R}_{\{s_i, s_{i+1}\}}(d)$ holds for all $1 \leq i \leq 2$ and the unique vertex which does not belong to the subdiagram C_4 will be labeled by 0. Then

$$\mathcal{F} := ((\mathcal{R}_{\{s_0, s_1\}}(d), (\mathcal{R}_{\{s_i, s_{i+1}\}}(c))_{1 \leq i \leq 3}), (d, (c)_{1 \leq i \leq 3}), (\theta_{012}, \theta_{123}, \theta_{234})),$$

where $\theta_{012} = \varphi^{-1}|_{\mathcal{P}_{s_1}(d)}$ and $\theta_{ijk} = \mathrm{id}$ in all the other cases, is a Moufang foundation of type $\tilde{\mathsf{F}}_4$. Let $s \in \{s_0, s_1, s_2\}$. Then the Moufang set $\mathbb{M}_{\Delta_2, \mathcal{P}_s(d)}$ is isomorphic to the Desarguesian Moufang set $\mathbb{M}(\mathbb{E})$ and since $\mathbb{E}$ has infinitely many elements, we conclude that $|\mathcal{P}_s(d)| = \infty$. Similarly, if $s \in \{s_3, s_4\}$, the Moufang set $\mathbb{M}_{\Delta_1, \mathcal{P}_s(c)}$ is isomorphic to the Desarguesian Moufang set $\mathbb{M}(\mathbb{D})$ and as $\mathbb{D}$ has at least $|\mathbb{E}|$ many elements, we conclude that $|\mathcal{P}_s(c)| = \infty$. In particular, in view of [MR], the foundation $\mathcal{F}$ satisfies (lco) and (lsco) (cf. 11.1.10).

Let $J \subseteq \{s_0, \ldots, s_4\}$ be an irreducible subset of rank 3. Then the residue $\mathcal{F}_J$ is isomorphic to a residue of one of the foundations $\mathcal{F}_1 = \mathcal{F}(\Delta_1, c)$ or $\mathcal{F}_2 := \mathcal{F}(\Delta_2, d)$. As, by 11.1.9, each residue of $\mathcal{F}_1$ or $\mathcal{F}_2$ of rank 3 is integrable, we conclude that $\mathcal{F}_J$ is integrable. According to 11.1.11, therefore,

$\mathcal{F}$ is integrable. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{F}}_4$ and let $x \in \mathcal{C}_+ \cup \mathcal{C}_-$ be such that $\mathcal{F}(\Delta, x) \simeq \mathcal{F}$. Let $J := \{s_1, \ldots, s_4\}$. Then, $\mathcal{F}_J \simeq \mathcal{F}(\Delta_1, c)$ and thus, by [AB, 5.209], $\mathcal{R}_J(x) \simeq \Delta_1 \simeq \mathsf{F}_4(\mathbb{D}/\mathbb{E})$. In view of [AB, 5.157], each spherical residue of type J which is contained in the same half as the chamber x is isometric to the building $\mathsf{F}_4(\mathbb{D}/\mathbb{E})$. $\qquad\square$

Chapter 14

Exceptional fixed point buildings

In [MW] Mühlherr and Weiss apply the theory of descent for buildings to give elementary constructions of the exceptional Moufang quadrangles as the fixed point buildings of involutions of higher rank buildings. As we will extend their constructions, we list the main results here.

14.0.1 Theorem
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_6. Then there exists a separable quadratic extension $\mathbb{E}/\mathbb{K}$ such that $Q_\mathbb{E}$ is hyperbolic and for each such extension $\mathbb{E}/\mathbb{K}$, there exists an involution Ω of the building $\Delta = \mathsf{E}_6(\mathbb{E})$ such that the group $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with Tits index
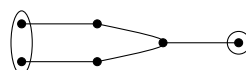
and the fixed point building Δ^Γ isomorphic to $\mathsf{B}_2^\mathcal{E}(\Lambda)$.

Proof   This is [MW, 14.11]. □

14.0.2 Theorem
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_7. Then there exists a separable quadratic extension $\mathbb{E}/\mathbb{K}$ such that $Q_\mathbb{E}$ is hyperbolic and for each such extension $\mathbb{E}/\mathbb{K}$, there exists an involution Ω of the building $\Delta = \mathsf{E}_7(\mathbb{E})$ such that the group $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with Tits index

and the fixed point building Δ^Γ is isomorphic to $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$.

Proof   This is [MW, 13.12].   □

14.0.3 Theorem
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_8. Then there exists a separable quadratic extension $\mathbb{E}/\mathbb{K}$ such that $Q_{\mathbb{E}}$ is hyperbolic and for each such extension $\mathbb{E}/\mathbb{K}$, there exists an involution Ω of the building $\Delta = \mathsf{E}_8(\mathbb{E})$ such that the group $\Gamma := \langle \Omega \rangle$ is is a descent group of Δ with Tits index
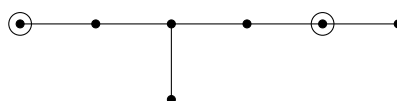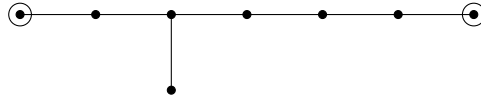
$$\odot\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\odot$$

and the fixed point building Δ^Γ is isomorphic to $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$.

Proof   This is [MW, 11.21].   □

14.0.4 Theorem
Let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type F_4 and let $\mathbb{F}$ be as in 1.29. Then there exists a separable quadratic extension $\mathbb{E}/\mathbb{K}$ such that $Q_{\mathbb{E}}$ is pseudo-split and for each such extension $\mathbb{E}/\mathbb{K}$, there exists an involution Ω of the building $\Delta = \mathsf{F}_4(\mathbb{E}\mathbb{F}, \mathbb{E})$ such that the group $\Gamma := \langle \Omega \rangle$ is a descent group of Δ with Tits index

$$\odot\!\!-\!\!-\!\!-\!\!\bullet\!\!=\!\!=\!\!\bullet\!\!-\!\!-\!\!-\!\!\odot$$

and the fixed point building Δ^Γ is isomorphic to $\mathsf{B}_2^{\mathcal{F}}(\Lambda)$.

Proof   This is [MW, 17.14].   □

Chapter 15

Existence of exceptional twin buildings of type $\tilde{\mathsf{C}}_2$

15.1 Case E_6

Throughout this section let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_6 and fix a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_3\})$ of Λ.

15.1.1 Theorem
There exists a twin building Δ of type $\tilde{\mathsf{C}}_2$ such that for some chamber c of Δ we have

$$\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda)).$$

Proof Let $\tilde{\Delta} = (\Delta_+, \Delta_-, \delta_*)$ be the twin building of type $\tilde{\mathsf{E}}_6$ constructed in 13.1.2. Let Π be the Coxeter diagram $\tilde{\mathsf{E}}_6$ labeled as in 2.3 and let (W, S) be the corresponding Coxeter system. Then each residue of Δ_+ of type $\{s_1, \ldots, s_6\}$ is isomorphic to the building $\mathsf{E}_6(\mathbb{E})$. As each rank 2 residue of $\tilde{\Delta}$ is either of type A_2 or of type $\mathsf{A}_1 \times \mathsf{A}_1$, $\tilde{\Delta}$ satisfies condition (co) by [MR, 1.5].

Choose a chamber $x \in \mathcal{C}_+$ and let $\Delta_0 := \mathcal{R}_{\{s_1, \ldots, s_6\}}(x) \simeq \mathsf{E}_6(\mathbb{E})$ be the unique residue of Δ_+ of type $\{s_1, \ldots, s_6\}$ containing the chamber x.

According to 1.26, the quadratic form $Q_\mathbb{E}$ is pseudo-split and hence we may apply theorem 14.0.1 to obtain an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with Tits index

$$\mathbf{T}_0 := \quad \text{}$$
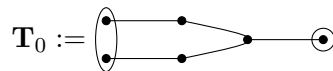
and such that the fixed point building $\Delta_0^{\Gamma_0}$ is isomorphic to the Moufang quadrangle $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$.
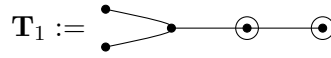
We fix a Γ_0-chamber $C \subseteq \Delta_0$ and let Δ_1 denote the unique residue of type $\{s_0, s_2, \ldots, s_5\}$ of Δ_+ containing C. Then $\Delta_{01} := \Delta_0 \cap \Delta_1$ is the unique residue of type D_4 containing C. By assumption, Δ_{01} is a Γ_0-panel of Δ_0 and hence contains at least three Γ_0-chambers. According to 2.42, the restriction $\Omega_{01} := \Omega_0|_{\Delta_{01}} \in \mathrm{Aut}(\Delta_{01})$ is an involution and the group $\Gamma_{01} := \langle \Omega_{01} \rangle$ is a descent group of Δ_{01} with Tits index

$$\underset{\displaystyle .}{\begin{array}{c}\bullet \\ \diagdown \\ \bullet\!\!-\!\!\!\!-\!\!\bullet\!\!-\!\!\!\!-\!\!\odot\end{array}}$$
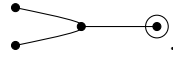
By 2.41, the fixed point building $\Delta_{01}^{\Gamma_{01}}$ can be equipped with a Moufang structure $\mathbb{M}$ such that the pair $(\Delta_{01}^{\Gamma_{01}}, \mathbb{M})$ is a Moufang set. It follows from [MW, 10.4] that

$$(\Delta_{01}^{\Gamma_{01}}, \mathbb{M}) \simeq \mathbb{M}(\Lambda). \tag{15.1}$$

By 5.8.5, the involution Ω_{01} can be extended to an involution $\Omega_1 \in \mathrm{Aut}(\Delta_1)$ such that the group $\Gamma_1 := \langle \Omega_1 \rangle$ is a descent group of Δ_1 with Tits index

$$\mathbf{T}_1 := \begin{array}{c}\bullet \\ \diagdown \\ \bullet\!\!-\!\!\!\!-\!\!\bullet\!\!-\!\!\!\!-\!\!\odot\!\!-\!\!\!\!-\!\!\odot\end{array}$$

and such that the fixed point building $\Delta_1^{\Gamma_1}$ is a Moufang quadrangle.

Choose a chamber $c \in C$ and let $c' := \Omega_0(c) = \Omega_1(c) \in C$. We define subsets $M_1 := \{s_2, \ldots, s_5, s_0\}, M_0 := \{s_1, \ldots, s_6\}, M_{01} := M_0 \cap M_1$ and $A = \{s_3, s_4, s_5\}$ of S and set $X := \{M_1, M_0, M_{01}, A\}$. The set X is a finite essential set as defined in 4.2.1.

Let $\sigma \in \mathrm{Aut}(\Pi)$ be the automorphism fixing the vertices $4, 2$ and 0 and interchanging the vertices 1 and 6 as well as 3 and 5 respectively. Let $\tilde{\Delta}^\sigma = (\Delta_+^\sigma, \Delta_-^\sigma, \delta_*^\sigma)$ be the translate of $\tilde{\Delta}$ with respect to σ (cf. 7.2.4). For $i = 0, 1$ let $\sigma_i := \sigma|_{\Pi_{M_i}}$ and note that $\sigma_i \in \mathrm{Aut}(\Pi_{M_i})$. With this setup Ω_i is a σ_i-isometry of the building Δ_i.

We define a mapping φ from $E_X(c)$ onto $E_X(c')$ (where the last is considered as a subset of the building Δ_+^σ) via

$$\varphi(x) := \begin{cases} \mathrm{id} \circ \Omega_0(x), & x \in \mathcal{R}_{M_0}(c) \\ \mathrm{id} \circ \Omega_1(x), & x \in \mathcal{R}_{M_1}(c) \end{cases},$$

which is well-defined since for each $M \in X$ we have $M \subseteq M_0$ or $M \subseteq M_1$ and since Ω_0 and Ω_1 coincide on the intersection of their domains $\mathcal{R}_{M_{01}}(c)$. The map φ is bijective, the inverse on $\mathcal{R}_{M_i}(c')$ is given by $\Omega_i \circ \mathrm{id}^{-1}$ for each $i = 0, 1$.

For each $M \in X$ let $\varphi_M := \varphi|_{\mathcal{R}_M(c)}$ be the restriction of φ to the residue $\mathcal{R}_M(c)$. For $x, y \in \mathcal{R}_M(c)$ we have

$$
\begin{aligned}
\delta_+^\sigma(\varphi_M(x), \varphi_M(y)) &= \delta_+^\sigma(\mathrm{id}(\Omega_i(x)), \mathrm{id}(\Omega_i(y))) \\
&= \sigma(\delta_+(\Omega_i(x), \Omega_i(y))) \\
&= \sigma^2(\delta_+(x,y)) = \delta_+(x,y),
\end{aligned}
$$

where $i = 0$ or $i = 1$ depending on wether $M \subseteq M_0$ or $M \subseteq M_1$. In particular, for each $M \in X$ the map φ_M is an isometry from $(\mathcal{R}_M(c), \delta_+)$ onto $(\mathcal{R}_M(c'), \delta_+^\sigma)$.

By 4.1.5, the chambers c and c' are opposite in C. Let Σ_A be the unique apartment of C containing c and c'. Since $\Omega_0(\Sigma_A)$ is an apartment of C containing the chambers $\Omega_0(c) = c'$ and $\Omega_0(c') = c$, we conclude that Σ_A is Γ_0-invariant. Similarly, as Ω_0 and Ω_1 coincide on C, Σ_A is also Γ_1-invariant. Choose a Γ_0-chamber D of Δ_{01} different from C. Then the Γ_0-chambers C and D are opposite in Δ_{01}. By 8.1.9(a), there exists a unique apartment of Δ_{01} containing the apartment Σ_A and intersecting D non-trivially. We denote this apartment by $\Sigma_{M_{01}}$. Moreover, by 8.1.9(b), the apartment $\Sigma_{M_{01}}$ is Γ_0- and Γ_1-invariant.
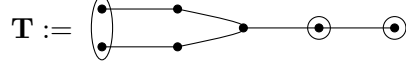
Since $(\Delta_i^{\Gamma_i}, \bar{\delta}_i)$ is a spherical building for each $i \in \{0, 1\}$, we may choose a Γ_i-chamber C_i of $\Delta_i^{\Gamma_i}$ such that $\bar{\delta}_i(C_i, C) = r_{\tilde{S}_i}$, where $r_{\tilde{S}_i}$ is the longest element of the relative type of the Tits index $\mathbf{T}_i$. Thus, by [MPW, 20.35], $\delta_+(c, \mathrm{proj}_{C_i}(c)) = r_{\tilde{S}_i} = r_{M_i} r_A$ and in particular there exists a chamber $z \in C_i$ such that $\delta(c, z) = r_{M_i}$. As the opposition map stabilizes A we conclude that C and C_i are opposite residues of Δ_i. Let $\mathcal{P}_i$ be the unique residue of Δ_i of type D_4 containing the Γ_i-chamber C_i. Then $\mathcal{P}_i$ and Δ_{01} are opposite spherical Γ_i-residues of Δ_i. Again, by 8.1.9(a), there exists a unique apartment of Δ_i containing the apartment $\Sigma_{M_{01}}$ and intersecting $\mathcal{P}_i$ non-trivially. We denote this apartment by Σ_{M_i}. In view of 8.1.9(b), the apartment Σ_{M_i} is Γ_i-invariant.

With the notations as above, the set X satisfies the conditions of theorem 6.3.6. Thus there exists a twin apartment $\Sigma_{\tilde{\Delta}^\sigma}$ of $\tilde{\Delta}^\sigma$ such that for all $M \in X$ we have $\Sigma_M \subseteq \Sigma_{\tilde{\Delta}^\sigma}$. Now, for $M \in X$ and φ as above,

$$
\varphi(\Sigma_{\tilde{\Delta}^\sigma} \cap \mathcal{R}_M(c)) = \varphi(\Sigma_M) = \Sigma_M = \Sigma_{\tilde{\Delta}^\sigma} \cap \mathcal{R}_M(c').
$$

According to the extension theorem 7.3.3, there is a unique isometry $\tilde{\Omega}$ from $\tilde{\Delta}$ onto $\tilde{\Delta}^\sigma$ extending $\mathrm{id} \circ \Omega_0$ and $\mathrm{id} \circ \Omega_1$. Set $\Omega := \mathrm{id}^{-1} \circ \tilde{\Omega}$. Then Ω is an isometry from $\tilde{\Delta}$ onto $\tilde{\Delta}$ extending Ω_0 and Ω_1. Let $\Gamma := \langle \Omega \rangle \le \mathrm{Aut}(\tilde{\Delta})$. Let $\mathcal{P}$ be a Γ-panel containing C. Then $\mathcal{P}$ is spherical and $\mathcal{P}$ is a Γ_i-panel of Δ_i for $i = 0$ or $i = 1$. Hence $\mathcal{P}$ contains at least three Γ_i-chambers, i.e. at least three Γ-chambers.

By 8.1.10, there exists a Γ-chamber which is opposite to C in $\tilde{\Delta}$ and by 8.1.13

$$\mathbf{T} := \text{(diagram)}$$

is a Tits index.   Using 8.2.6 we obtain that for $\varepsilon \in \{+, -\}$ the group $\Gamma_\varepsilon := \Gamma|_{\Delta_\varepsilon}$ is a descent group of Δ_ε.

Let $\tilde{\Pi}$ be the relative type of the Tits index $\mathbf{T}$. In view of 2.16, the diagram $\tilde{\Pi}$ is connected, of rank 3, its corresponding Coxeter system is affine and, by construction, has two subdiagrams of type B_2. We conclude that the relative type of $\mathbf{T}$ is $\tilde{\mathsf{C}}_2$. In particular, for each $\varepsilon \in \{+, -\}$ the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a building of type $\tilde{\mathsf{C}}_2$.

According to 8.3.9 there exists a codistance function δ_*^Γ between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_-. Hence, $\Delta := (\Delta_+^{\Gamma_+}, \Delta_-^{\Gamma_-}, \delta_*^\Gamma)$ is a twin building of type $\tilde{\mathsf{C}}_2$.

Let $\mathcal{F}(\Delta, C)$ be the foundation of Δ based at C. The irreducible rank 2 residues of Δ containing C are precisely $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$. As $\Delta_0^{\Gamma_0} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ and $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$ are glued along a panel of quadratic form type (cf. 15.1), theorem 12.0.1 implies that $\mathcal{F}(\Delta, C) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$. $\square$

15.1.2 Theorem
There exists a twin building Δ of type $\tilde{\mathsf{C}}_2$ such that for some chamber c of Δ we have

$$\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_{\mathcal{E}}^o(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$$

for some anisotropic pseudo-quadratic space Ξ.

Proof Let $\tilde{\Delta} = (\Delta_+, \Delta_-, \delta_*)$ be the twin building of type $\tilde{\mathsf{E}}_7$ constructed in 13.1.3. Let Π be the Coxeter diagram $\tilde{\mathsf{E}}_7$ labeled as in 2.3 and let (W, S) be the corresponding Coxeter system. Then each residue of Δ_+ of type $\{s_1, \dots, s_6\}$ is isomorphic to the building E_6. As each rank 2 residue of $\tilde{\Delta}$ is either of type A_2 or of type $\mathsf{A}_1 \times \mathsf{A}_1$, $\tilde{\Delta}$ satisfies condition (co) by [MR, 1.5].

Choose a chamber $x \in \mathcal{C}_+$ and let $\Delta_0 := \mathcal{R}_{\{s_1,\dots,s_6\}}(x) \simeq \mathsf{E}_6(\mathbb{E})$ be the unique residue of Δ_+ of type $\{s_1, \dots, s_6\}$ containing the chamber x.

According to 1.26, the quadratic form $Q_{\mathbb{E}}$ is pseudo-split and hence we may apply theorem 14.0.1 to obtain an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with Tits index

$$\mathbf{T}_0 := \text{(diagram)}$$
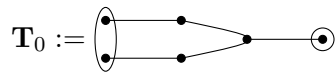
and such that the fixed point building $\Delta_0^{\Gamma_0}$ is isomorphic to the Moufang quadrangle $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$.

We fix a Γ_0-chamber $C \subseteq \Delta_0$ and let Δ_1 denote the unique residue of type $\{s_0, s_1, s_3, \ldots, s_7\}$ of Δ_+ containing C. Then $\Delta_{01} := \Delta_0 \cap \Delta_1$ is the unique residue of type A_5 containing C. By assumption, Δ_{01} is a Γ_0-panel of Δ_0 and hence contains at least three Γ_0-chambers. According to 2.42, the restriction $\Omega_{01} := \Omega_0|_{\Delta_{01}} \in \mathrm{Aut}(\Delta_{01})$ is an involution and the group $\Gamma_{01} := \langle \Omega_{01} \rangle$ is a descent group of Δ_{01} with Tits index

$$\text{[diagram]}$$
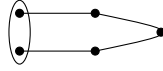
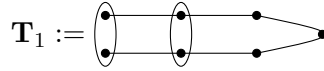By 2.41, the fixed poi9nt building $\Delta_{01}^{\Gamma_{01}}$ can be equipped with a Moufang structure $\mathbb{M}$ such that the pair $(\Delta_{01}^{\Gamma_{01}})$ is a Moufang set. It follows from [MW, 14.10] that

$$(\Delta_{01}^{\Gamma_{01}}) \simeq \mathbb{M}(S). \tag{15.2}$$

By 5.8.5, the involution Ω_{01} can be extended to an involution $\Omega_1 \in \mathrm{Aut}(\Delta_1)$ such that the group $\Gamma_1 := \langle \Omega_1 \rangle$ is a descent group of Δ_1 with Tits index

$$\mathbf{T}_1 := \text{[diagram]}$$

and such that the fixed point building $\Delta_1^{\Gamma_1}$ is a Moufang quadrangle.

Choose a chamber $c \in C$ and let $c' := \Omega_0(c) = \Omega_1(c) \in C$. We define subsets $M_1 := \{s_1, s_3, \ldots, s_7, s_0\}, M_0 := \{s_1, \ldots, s_6\}, M_{01} := M_0 \cap M_1$ and $A = \{s_3, s_4, s_5\}$ of S and set $X := \{M_1, M_0, M_{01}, A\}$. The set X is a finite essential set as defined in 4.2.1.

Let $\sigma \in \mathrm{Aut}(\Pi)$ be the automorphism fixing the vertices 4 and 2 and interchanging the vertices 0 and 7, 1 and 6 as well as 3 and 5 respectively. Let $\tilde{\Delta} = (\Delta_+^{\sigma}, \Delta_-^{\sigma}, \delta_*^{\sigma})$ be the translate of $\tilde{\Delta}$ with respect to σ (cf. 7.2.4). For $i = 0, 1$ let $\sigma_i := \sigma|_{\Pi_{M_i}}$ and note that $\sigma_i \in \mathrm{Aut}(\Pi_{M_i})$. With this setup, Ω_i is a σ_i-isometry of the building Δ_i.

We define a mapping φ from $E_X(c)$ onto $E_X(c')$ (where the last is considered as a subset of the building Δ_+^{σ}) via

$$\varphi(x) := \begin{cases} \mathrm{id} \circ \Omega_0(x), & x \in \mathcal{R}_{M_0}(c) \\ \mathrm{id} \circ \Omega_1(x), & x \in \mathcal{R}_{M_1}(c) \end{cases},$$

which is well-defined since for each $M \in X$ we have $M \subseteq M_0$ or $M \subseteq M_1$ ans since Ω_0 and Ω_1 coincide on the intersection of their domains $\mathcal{R}_{M_{01}}(c)$. The map φ is bijective, the inverse on $\mathcal{R}_{M_i}(c')$ is given by $\Omega_i \circ \mathrm{id}^{-1}$ for each $i = 0, 1$.

For each $M \in X$ let $\varphi_M := \varphi|_{\mathcal{R}_M(c)}$ be the restriction of φ to the residue $\mathcal{R}_M(c)$. For $x, y \in \mathcal{R}_M(c)$ we have

$$\delta_+^\sigma(\varphi_M(x), \varphi_M(y)) = \delta_+^\sigma(\mathrm{id}(\Omega_i(x)), \mathrm{id}(\Omega_i(y)))$$
$$= \sigma(\delta_+(\Omega_i(x), \Omega_i(y))) = \sigma^2(\delta_+(x, y)) = \delta_+(x, y),$$

where $i = 0$ or $i = 1$ depending on wether $M \subseteq M_0$ or $M \subseteq M_1$. In particular, for each $M \in X$ the map φ_M is an isometry from $(\mathcal{R}_M(c), \delta_+)$ onto $(\mathcal{R}_M(c'), \delta_+^\sigma)$.

By 4.1.5, the chambers c and c' are opposite in C. Let Σ_A be the unique apartment of C containing c and c'. Since $\Omega_0(\Sigma_A)$ is an apartment of C containing the chambers $\Omega_0(c) = c'$ and $\Omega_0(c') = c$, we conclude that Σ_A is Γ_0-invariant. Similarly, as Ω_0 and Ω_1 coincide on C, Σ_A is also Γ_1-invariant. Choose a Γ_0-chamber D of Δ_{01} different from C. Then the Γ_0-chambers C and D are opposite in Δ_{01}. By 8.1.9(a) there exists a unique apartment of Δ_{01} containing Σ_A and intersecting D non-trivially. We denote this apartment by $\Sigma_{M_{01}}$. Moreover, by 8.1.9(b) this apartment is Γ_0- and Γ_1-invariant.

Since $(\Delta_i^{\Gamma_i}, \bar{\delta}_i)$ is a spherical building for each $i \in \{0, 1\}$, we may choose a Γ_i-chamber C_i of $\Delta_i^{\Gamma_i}$ such that $\bar{\delta}_i(C_i, C) = r_{\tilde{S}_i}$, where $r_{\tilde{S}_i}$ is the longest element of the relative type of the Tits index $\mathbf{T}_i$. Thus, by [MPW, 20.35], $\delta_+(c, \mathrm{proj}_{C_i}(c)) = r_{\tilde{S}_i} = r_{M_i} r_A$ and in particular there exists a chamber $z \in C_i$ such that $\delta(c, z) = r_{M_i}$. As the opposition map stabilizes A we conclude that C and C_i are opposite residues of Δ_i. Let $\mathcal{P}_i$ be the unique residue of Δ_i of type A_5 containing the Γ_i-chamber C_i. Then $\mathcal{P}_i$ and Δ_{01} are opposite spherical Γ_i-residues of Δ_i. Again by 8.1.9(a) there exists a unique apartment of Δ_i containing the apartment $\Sigma_{M_{01}}$ and intersecting $\mathcal{P}_i$ non-trivially. We denote this apartment by Σ_{M_i}. In view of 8.1.9(b), the apartment Σ_{M_i} is Γ_i-invariant.

With the notations as above, the set X satisfies the conditions of theorem 6.3.6. Thus there exists a twin apartment $\Sigma_{\tilde{\Delta}^\sigma}$ of $\tilde{\Delta}^\sigma$ such that for all $M \in X$ we have $\Sigma_M \subseteq \Sigma_{\tilde{\Delta}^\sigma}$. Now, for $M \in X$ and φ as above,

$$\varphi(\Sigma_{\tilde{\Delta}^\sigma} \cap \mathcal{R}_M(c)) = \varphi(\Sigma_M) = \Sigma_M = \Sigma_{\tilde{\Delta}^\sigma} \cap \mathcal{R}_M(c').$$

According to the extension theorem 7.3.3, there is a unique isometry $\tilde{\Omega}$ from $\tilde{\Delta}$ onto $\tilde{\Delta}^\sigma$ extending $\mathrm{id} \circ \Omega_0$ and $\mathrm{id} \circ \Omega_1$. Set $\Omega := \mathrm{id}^{-1} \circ \tilde{\Omega}$. Then Ω is an isometry from $\tilde{\Delta}$ onto $\tilde{\Delta}$ extending Ω_0 and Ω_1. Let $\Gamma := \langle \Omega \rangle \leq \mathrm{Aut}(\tilde{\Delta})$.

Let $\mathcal{P}$ be a Γ-panel containing C. Then $\mathcal{P}$ is spherical and $\mathcal{P}$ is a Γ_i-panel of Δ_i for $i = 0$ or $i = 1$. Hence $\mathcal{P}$ contains at least three Γ_i-chambers, i.e. at least three Γ-chambers.

By 8.1.10, there exists a Γ-chamber which is opposite to C in $\tilde{\Delta}$ and by 8.1.13

is a Tits index.   Using 8.2.6 we obtain that for $\varepsilon \in \{+, -\}$ the group $\Gamma_\varepsilon := \Gamma|_{\Delta_\varepsilon}$ is a descent group of Δ_ε.

Let $\tilde{\Pi}$ be the relative type of the Tits index $\mathbf{T}$. In view of 2.16, the diagram $\tilde{\Pi}$ is connected, of rank 3, its corresponding Coxeter system is affine and, by construction, has two subdiagrams of type B_2. We conclude that the relative type of $\mathbf{T}$ is $\tilde{\mathsf{C}}_2$. In particular, for each $\varepsilon \in \{+, -\}$ the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a building of type $\tilde{\mathsf{C}}_2$.

According to 8.3.9 there exists a codistance function δ_*^Γ between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_-. Hence, $\Delta := (\Delta_+^{\Gamma_+}, \Delta_-^{\Gamma_-}, \delta_*^\Gamma)$ is a twin building of type $\tilde{\mathsf{C}}_2$.

Let $\mathcal{F}(\Delta, C)$ be the foundation of Δ based at C. The irreducible rank 2 residues of Δ containing C are precisely $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$. As $\Delta_0^{\Gamma_0} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ and $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$ are glued along a non-commutative panel (cf. 15.2), theorem 12.0.1 implies that there exists an anisotropic pseudo-quadratic space Ξ such that $\mathcal{F}(\Delta, C) \simeq (\mathcal{Q}_{\mathcal{E}}^o(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$. $\qquad\square$

15.2   Case E_7

Throughout this section let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_7 and fix a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_4\})$ of Λ.

15.2.1 Theorem

There exists a twin building Δ of type $\tilde{\mathsf{C}}_2$ such that for some chamber c of Δ we have

$$\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_\mathcal{E}(\Lambda), \mathcal{Q}_\mathcal{Q}^o(\Lambda)).$$

Proof   Let $\tilde{\Delta} = (\Delta_+, \Delta_-, \delta_*)$ be the twin building of type $\tilde{\mathsf{E}}_7$ constructed in 13.1.2. Let Π be the Coxeter diagram $\tilde{\mathsf{E}}_7$ labeled as in 2.3 and let (W, S) be the corresponding Coxeter system. Then each residue of Δ_+ of type $\{s_1, \ldots, s_7\}$ is isomorphic to the building $\mathsf{E}_7(\mathbb{E})$. As each rank 2 residue of $\tilde{\Delta}$ is either of type A_2 or of type $\mathsf{A}_1 \times \mathsf{A}_1$, $\tilde{\Delta}$ satisfies condition (co) by [MR, 1.5].

Choose a chamber $x \in \mathcal{C}_+$ and let $\Delta_0 := \mathcal{R}_{\{s_1, \ldots, s_7\}}(x) \simeq \mathsf{E}_7(\mathbb{E})$ be the residue of Δ_+ of type $\{s_1, \ldots, s_7\}$ containing the chamber x.

According to 1.26, the quadratic form $Q_\mathbb{E}$ is pseudo-split and hence we may apply theorem 14.0.2 to obtain an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with Tits index
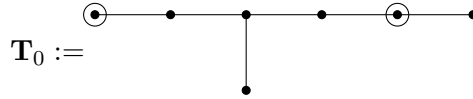
and such that the fixed point building $\Delta_0^{\Gamma_0}$ is isomorphic to the Moufang quadrangle $\mathsf{B}_2^\mathcal{E}(\Lambda)$.

We fix a Γ_0-chamber $C \subseteq \Delta_0$ and let Δ_1 denote the unique residue of type $\{s_0, \ldots, s_5, s_7\}$ of Δ_+ containing C. Then $\Delta_{01} := \Delta_0 \cap \Delta_1$ is the unique residue of type $\mathsf{D}_5 \times A_1$ containing C. By assumption, Δ_{01} is a Γ_0-panel of Δ_0 and hence contains at least three Γ_0-chambers. According to 2.42, the restriction $\Omega_{01} := \Omega_0|_{\Delta_{01}}$ is an involutory automorphism of Δ_{01} and the group $\Gamma_{01} := \langle \Omega_{01} \rangle$ is a descent group of Δ_{01} with Tits index
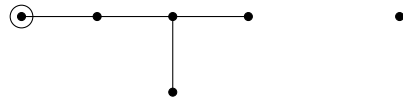
By 2.41, the fixed point building $\Delta_{01}^{\Gamma_{01}}$ can be equipped with a Moufang structure $\mathbb{M}$ such that the pair $(\Delta_{01}^{\Gamma_{01}}, \mathbb{M})$ is a Moufang set. It follows from [MW, 10.4] that

$$(\Delta_{01}^{\Gamma_{01}}, \mathbb{M}) \simeq \mathbb{M}(\Lambda). \tag{15.3}$$

By 5.8.5, the involution Ω_{01} can be extended to an involution $\Omega_1 \in \mathrm{Aut}(\Delta_1)$ such that the group $\Gamma_1 := \langle \Omega_1 \rangle$ is a descent group of Δ_1 with Tits index

$$\mathbf{T}_1 :=$$
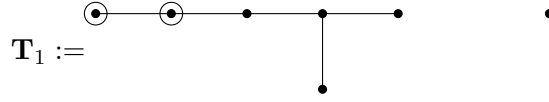
and such that the fixed point building $\Delta_1^{\Gamma_1}$ is a Moufang quadrangle.

Choose a chamber $c \in C$ and let $c' := \Omega_0(c) = \Omega_1(c) \in C$. We define subsets $M_1 := \{s_0, s_1, \ldots, s_5, s_7\}, M_0 := \{s_1, \ldots, s_7\}, M_{01} := M_0 \cap M_1$ and $A = \{s_2, \ldots, s_5, s_7\}$ of S and set $X := \{M_1, M_0, M_{01}, A\}$. The set X is a finite essential set as defined in 4.2.1. Note that for each $M \in X$ we have $M \subseteq M_0$ or $M \subseteq M_1$. Hence we may define a mapping

$$\varphi \colon E_X(c) \to E_X(c') \text{ via } \varphi(x) := \begin{cases} \Omega_0(x), & x \in \mathcal{R}_{M_0}(c) \\ \Omega_1(x), & x \in \mathcal{R}_{M_1}(c) \end{cases},$$

which is well-defined since Ω_0 and Ω_1 coincide on the intersection of their domains $\mathcal{R}_{M_{01}}(c)$. As both Ω_0 and Ω_1 are involutions, the map φ is bijective. For each $M \in X$ let $\varphi_M := \varphi|_{\mathcal{R}_M(c)}$ be the restriction of φ to the residue $\mathcal{R}_M(c)$ and note that all these maps are isometries from $\mathcal{R}_M(c)$ onto $\mathcal{R}_M(c')$.

By 4.1.5, the chambers c and c' are opposite in C. Let Σ_A be the unique apartment of C containing c and c'. Since $\Omega_0(\Sigma_A)$ is an apartment of C containing the chambers $\Omega_0(c) = c'$ and $\Omega_0(c') = c$, we conclude that Σ_A is Γ_0-invariant. Similarly, as Ω_0 and Ω_1 coincide on C, Σ_A is also Γ_1-invariant. Choose a Γ_0-chamber D of Δ_{01} different from C. Then the Γ_0-chamber C and D are opposite. By 8.1.9(a) there exists a unique apartment of Δ_{01} containing Σ_A and intersecting D non-trivially. We denote this apartment by $\Sigma_{M_{01}}$. Moreover, by 8.1.9(b), this apartment is Γ_0- and Γ_1-invariant.

Since $(\Delta_i^{\Gamma_i}, \bar{\delta}_i)$ is a spherical building for each $i \in \{0, 1\}$, we may choose a Γ_i-chamber C_i of $\Delta_i^{\Gamma_i}$ such that $\bar{\delta}_i(C_i, C) = r_{\tilde{S}_i}$, where $r_{\tilde{S}_i}$ is the longest element of the relative type of the Tits index $\mathbf{T}_i$. Thus, by [MPW, 20.35], $\delta_+(c, \mathrm{proj}_{C_i}(c)) = r_{\tilde{S}_i} = r_{M_i} r_A$ and in particular there exists a chamber $z \in C_i$ such that $\delta(c, z) = r_{M_i}$. As the opposition map acts trivially on the diagram Π_{M_i} we conclude that C and C_i are opposite residues of Δ_i. Let $\mathcal{P}_i$ be the unique residue of Δ_i of type $\mathsf{D}_5 \times A_1$ containing the Γ_i-chamber C_i. Then $\mathcal{P}_i$ and Δ_{01} are opposite spherical Γ_i-residues of Δ_i. Again by 8.1.9(a), there exists a unique apartment of Δ_i containing the apartment $\Sigma_{M_{01}}$ and intersecting $\mathcal{P}_i$ non-trivially. We denote this apartment by Σ_{M_i}. In view of 8.1.9(b) the apartment Σ_{M_i} is Γ_i-invariant.

With the notations as above, the set X satisfies the conditions of theorem 6.3.6. Thus there exists a twin apartment $\Sigma_{\tilde{\Delta}}$ of $\tilde{\Delta}$ such that for all $M \in X$
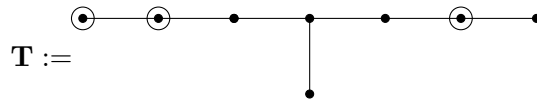
we have $\Sigma_M \subseteq \Sigma_{\tilde{\Delta}}$. Now, for $M \in X$ and φ as above,

$$\varphi(\Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c)) = \varphi(\Sigma_M) = \Sigma_M = \Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c').$$

According to the extension theorem 7.3.3, there exists a unique isometry $\Omega \in \mathrm{Aut}(\Delta)$ extending Ω_0 and Ω_1. Set $\Gamma := \langle \Omega \rangle \leq \mathrm{Aut}(\Delta)$.

Let $\mathcal{P}$ be a Γ-panel containing C. Then $\mathcal{P}$ is spherical and $\mathcal{P}$ is a Γ_i-panel of Δ_i for $i = 0$ or $i = 1$. Hence $\mathcal{P}$ contains at least three Γ_i-chambers, i.e. at least three Γ-chambers.

By 8.1.10, there exists a Γ-chamber which is opposite to C in $\tilde{\Delta}$ and by 8.1.13

$$\mathbf{T} := \qquad \begin{array}{c}\text{(diagram)}\end{array}$$

is a Tits index.   Using 8.2.6 we obtain that for $\varepsilon \in \{+, -\}$ the group $\Gamma_\varepsilon := \Gamma|_{\Delta_\varepsilon}$ is a descent group of Δ_ε.

Let $\tilde{\Pi}$ be the relative type of the Tits index $\mathbf{T}$. In view of 2.16, the diagram $\tilde{\Pi}$ is connected, of rank 3, its corresponding Coxeter system is affine and, by construction, has two subdiagrams of type B_2. We conclude that the relative type of $\mathbf{T}$ is $\tilde{\mathsf{C}}_2$. In particular, for each $\varepsilon \in \{+, -\}$ the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a building of type $\tilde{\mathsf{C}}_2$.

According to 8.3.9 to obtain a codistnace function δ_*^Γ between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_-. Hence, $\Delta := (\Delta_+^{\Gamma_+}, \Delta_-^{\Gamma_-}, \delta_*^\Gamma)$ is a twin building of type $\tilde{\mathsf{C}}_2$.

Let $\mathcal{F}(\Delta, C)$ be the foundation of Δ based at the chamber C. The irreducible rank 2 residues of Δ containing C are precisely $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$. As $\Delta_0^{\Gamma_0} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ and $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$ are glued along a panel of quadratic form type (cf. 15.3), theorem 12.0.1 implies that $\mathcal{F}(\Delta, C) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$. $\square$

15.2.2 Theorem
There exists a twin building Δ of type $\tilde{\mathsf{C}}_2$ such that for some chamber c of Δ we have

$$\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_{\mathcal{E}}^o(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$$

for some anisotropic pseudo-quadratic space Ξ.

Proof Let $\tilde{\Delta} = (\Delta_+, \Delta_-, \delta_*)$ be the twin building of type $\tilde{\mathsf{E}}_8$ constructed in 13.1.3. Let Π be the Coxeter diagram $\tilde{\mathsf{E}}_8$ labeled as in 2.3 and let (W, S) be the corresponding Coxeter system. Then each residue of Δ_+ of type $\{s_1, \ldots, s_7\}$ is isomorphic to the building $\mathsf{E}_7(\mathbb{E})$. As each rank 2 residue of $\tilde{\Delta}$ is either of type A_2 or of type $\mathsf{A}_1 \times \mathsf{A}_1$, $\tilde{\Delta}$ satisfies condition (co) by [MR, 1.5].

Choose a chamber $x \in \mathcal{C}_+$ and let $\Delta_0 := \mathcal{R}_{\{s_1, \ldots, s_7\}}(x) \simeq \mathsf{E}_7(\mathbb{E})$ be the unique residue of Δ_+ of type $\{s_1, \ldots, s_7\}$ containing the chamber x.

According to 1.26, the quadratic form $Q_{\mathbb{E}}$ is pseudo-split and hence we may apply theorem 14.0.2 to obtain an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with Tits index

$$\mathbf{T}_0 :=$$
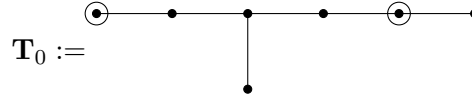
and such that the fixed point building $\Delta_0^{\Gamma_0}$ is isomorphic to the a Moufang quadrangle $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$.

We fix a Γ_0-chamber $C \subseteq \Delta_0$ and let Δ_1 denote the unique residue of type $\{s_2, \ldots, s_8, s_0\}$ of Δ_+ containing C. Then $\Delta_{01} := \Delta_0 \cap \Delta_1$ is the unique residue of type D_6 containing C. By assumption, Δ_{01} is a Γ_0-panel of Δ_0 and hence contains at least three Γ_0-chambers. According to 2.42, the restriction $\Omega_{01} := \Omega_0|_{\Delta_{01}}$ is an involutory automorphism of Δ_{01} and the group $\Gamma_{01} := \langle \Omega_{01} \rangle$ is a descent group of Δ_{01} with Tits index

$$.$$

By 2.41, the fixed point building $\Delta_{01}^{\Gamma_{01}}$ can be equipped with a Moufang structure $\mathbb{M}$ such that the pair $(\Delta_{01}^{\Gamma_{01}}, \mathbb{M})$ is a Moufang set. It follows from [MW, 13.11] that

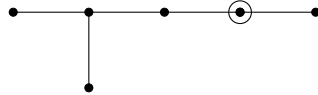$$(\Delta_{01}^{\Gamma_{01}}, \mathbb{M}) \simeq \mathbb{M}(S). \tag{15.4}$$

By 5.8.2, the involution Ω_{01} can be extended to an involution $\Omega_1 \in \mathrm{Aut}(\Delta_1)$ such that the group $\Gamma_1 := \langle \Omega_1 \rangle$ is a descent group of Δ_1 with Tits index

$$\mathbf{T}_1 :=$$
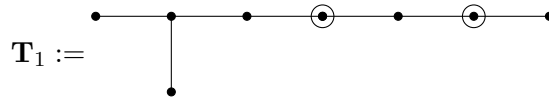
and such that the fixed point building $\Delta_1^{\Gamma_1}$ is a Moufang quadrangle.

Choose a chamber $c \in C$ and let $c' := \Omega_0(c) = \Omega_1(c) \in C$. We define subsets $M_1 := \{s_2, \ldots, s_8, s_0\}, M_0 := \{s_1, \ldots, s_7\}, M_{01} := M_0 \cap M_1$ and $A = \{s_2, \ldots, s_5, s_7, s_0\}$ of S and set $X := \{M_1, M_0, M_{01}, A\}$. The set X is a finite essential set as defined in 4.2.1. Note that for each $M \in X$ we have $M \subseteq M_0$ or $M \subseteq M_1$. Hence we may define a mapping

$$\varphi \colon E_X(c) \to E_X(c') \text{ via } \varphi(x) := \begin{cases} \Omega_0(x), & x \in \mathcal{R}_{M_0}(c) \\ \Omega_1(x), & x \in \mathcal{R}_{M_1}(c) \end{cases},$$

which is well-defined since Ω_0 and Ω_1 coincide on the intersection of their domains $\mathcal{R}_{M_{01}}(c)$.

For each $M \in X$ let $\varphi_M := \varphi|_{\mathcal{R}_M(c)}$ be the restriction of φ to the residue $\mathcal{R}_M(c)$ and note that all these maps are isometries from $\mathcal{R}_M(c)$ onto $\mathcal{R}_M(c')$.

By 4.1.5, the chambers c and c' are opposite in C. Let Σ_A be the unique apartment of C containing c and c'. Since $\Omega_0(\Sigma_A)$ is an apartment of C containing the chambers $\Omega_0(c) = c'$ and $\Omega_0(c') = c$, we conclude that Σ_A is Γ_0-invariant. Similarly, as Ω_0 and Ω_1 coincide on C, Σ_A is also Γ_1-invariant. Choose a Γ_0-chamber D of Δ_{01} different from C. Then the Γ_0-chamber C and D are opposite. By 8.1.9(a) there exists a unique apartment of Δ_{01} containing Σ_A and intersecting D non-trivially. We denote this apartment by $\Sigma_{M_{01}}$. Moreover, by 8.1.9(b), this apartment is Γ_0- and Γ_1-invariant.

Since $(\Delta_i^{\Gamma_i}, \bar{\delta}_i)$ is a spherical building for each $i \in \{0,1\}$, we may choose a Γ_i-chamber C_i of $\Delta_i^{\Gamma_i}$ such that $\bar{\delta}_i(C_i, C) = r_{\tilde{S}_i}$, where $r_{\tilde{S}_i}$ is the longest element of the relative type of the Tits index $\mathbf{T}_i$. Thus, by [MPW, 20.35], $\delta_+(c, \mathrm{proj}_{C_i}(c)) = r_{\tilde{S}_i} = r_{M_i} r_A$ and in particular there exists a chamber $z \in C_i$ such that $\delta(c, z) = r_{M_i}$. As the opposition map acts trivially on the diagram Π_{M_i} we conclude that C and C_i are opposite residues of Δ_i. Let $\mathcal{P}_i$ be the unique residue of Δ_i of type D_6 containing the Γ_i-chamber C_i. Then $\mathcal{P}_i$ and Δ_{01} are opposite spherical Γ_i-residues of Δ_i. Again by 8.1.9(a), there exists a unique apartment of Δ_i containing the apartment $\Sigma_{M_{01}}$ and intersecting $\mathcal{P}_i$ non-trivially. We denote this apartment by Σ_{M_i}. In view of 8.1.9(b) ,the apartment Σ_{M_i} is Γ_i-invariant.

With the notations as above, the set X satisfies the conditions of theorem 6.3.6. Thus there exists a twin apartment $\Sigma_{\tilde{\Delta}}$ of $\tilde{\Delta}$ such that for all $M \in X$ we have $\Sigma_M \subseteq \Sigma_{\tilde{\Delta}}$. Now, for $M \in X$ and φ as above,

$$\varphi(\Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c)) = \varphi(\Sigma_M) = \Sigma_M = \Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c').$$

According to the extension theorem 7.3.3, there exists a unique isometry $\Omega \in \mathrm{Aut}(\Delta)$ extending Ω_0 and Ω_1. Set $\Gamma := \langle \Omega \rangle \leq \mathrm{Aut}(\Delta)$.

Let $\mathcal{P}$ be a Γ-panel containing C. Then $\mathcal{P}$ is spherical and $\mathcal{P}$ is a Γ_i-panel of Δ_i for $i = 0$ or $i = 1$. Hence $\mathcal{P}$ contains at least three Γ_i-chambers, i.e. at least three Γ-chambers. By 8.1.10, there exists a Γ-chamber which is opposite to C in $\tilde{\Delta}$ and by 8.1.13

$$\mathbf{T} := $$

is a Tits index. Using 8.2.6 we obtain that for $\varepsilon \in \{+, -\}$ the group $\Gamma_\varepsilon := \Gamma|_{\Delta_\varepsilon}$ is a descent group of Δ_ε.

Let $\tilde{\Pi}$ be the relative type of the Tits index $\mathbf{T}$. In view of 2.16, the diagram $\tilde{\Pi}$ is connected, of rank 3, its corresponding Coxeter system is affine and,

by construction, has two subdiagrams of type B_2. We conclude that the relative type of $\mathbf{T}$ is $\tilde{\mathsf{C}}_2$. In particular, for each $\varepsilon \in \{+, -\}$ the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a building of type $\tilde{\mathsf{C}}_2$.

According to 8.3.9 there exists a codistance function δ_*^Γ between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_-. Hence, $\Delta := (\Delta_+^{\Gamma_+}, \Delta_-^{\Gamma_-}, \delta_*^\Gamma)$ is a twin building of type $\tilde{\mathsf{C}}_2$.

Let $\mathcal{F}(\Delta, C)$ be the foundation of Δ based at C. The irreducible rank 2 residues of Δ containing C are precisely $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$. As $\Delta_0^{\Gamma_0} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ and $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$ are glued along a non-commutative panel (cf. 15.4), theorem 12.0.1 implies that there exists an anisotropic pseudo-quadratic space Ξ such that $\mathcal{F}(\Delta, C) \simeq (\mathcal{Q}_{\mathcal{E}}^o(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$. $\qquad\square$

15.3   Case E_8

Throughout this section let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type E_8 and fix a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_6\})$ of Λ.

15.3.1 Theorem
There exists a twin building Δ of type $\tilde{\mathsf{C}}_2$ such that for some chamber c of Δ we have

$$\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda)).$$

Proof   Let $\tilde{\Delta} = (\Delta_+, \Delta_-, \delta_*)$ be the twin building of type $\tilde{\mathsf{E}}_8$ constructed in 13.1.2. Let Π be the Coxeter diagram $\tilde{\mathsf{E}}_8$ labeled as in 2.3 and let (W, S) be the corresponding Coxeter system. Then each residue of Δ_+ of type $\{s_1, \ldots, s_8\}$ is isomorphic to the building $\mathsf{E}_8(\mathbb{E})$. As each rank 2 residue of $\tilde{\Delta}$ is either of type A_2 or of type $\mathsf{A}_1 \times \mathsf{A}_1$, $\tilde{\Delta}$ satisfies condition (co) by [MR, 1.5].

Choose a chamber $x \in \mathcal{C}_+$ and let $\Delta_0 := \mathcal{R}_{\{s_1, \ldots, s_8\}}(x) \simeq \mathsf{E}_8(\mathbb{E})$ be the unique residue of Δ_+ of type $\{s_1, \ldots, s_8\}$ containing the chamber x.

According to 1.26, the quadratic form $Q_{\mathbb{E}}$ is pseudo-split and hence we may apply theorem 14.0.3 to obtain an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with Tits index
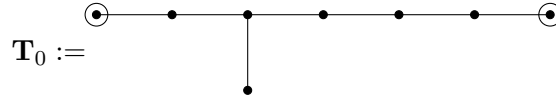
$$\mathbf{T}_0 :=$$

and such that the fixed point building $\Delta_0^{\Gamma_0}$ is isomorphic to the Moufang quadrangle $\mathsf{B}_2^{\mathcal{E}}(\Lambda)$.

We fix a Γ_0-chamber $C \subseteq \Delta_0$ and let Δ_1 denote the unique residue of type $\{s_2, \ldots, s_8, s_0\}$ of Δ_+ containing C. Then $\Delta_{01} := \Delta_0 \cap \Delta_1$ is the unique residue of type D_7 containing C. By assumption, Δ_{01} is a Γ_0-panel of Δ_0 and hence contains at least three Γ_0-chambers. According to 2.42, the restriction $\Omega_{01} := \Omega_0|_{\Delta_{01}}$ is an involutory automorphism of Δ_{01} and the group $\Gamma_{01} := \langle \Omega_{01} \rangle$ is a descent group of Δ_{01} with Tits index

.
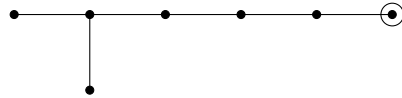
By 2.41, the fixed point building $\Delta_{01}^{\Gamma_{01}}$ can be equipped with a Moufang structure $\mathbb{M}$ such that the pair $(\Delta_{01}^{\Gamma_{01}}, \mathbb{M})$ is a Moufang set. It follows from [MW, 10.4] that

$$(\Delta_{01}^{\Gamma_{01}}, \mathbb{M}) \simeq \mathbb{M}(\Lambda). \tag{15.5}$$

By 5.8.4, the involution Ω_{01} can be extended to an involution $\Omega_1 \in \mathrm{Aut}(\Delta_1)$ such that the group $\Gamma_1 := \langle \Omega_1 \rangle$ is a descent group of Δ_1 with Tits index

$$\mathbf{T}_1 :=$$
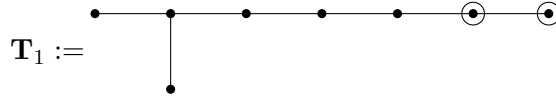
and such that the fixed point building $\Delta_1^{\Gamma_1}$ is a Moufang quadrangle.

Choose a chamber $c \in C$ and let $c' := \Omega_0(c) = \Omega_1(c) \in C$. We define subsets $M_1 := \{s_2, \ldots, s_8, s_0\}, M_0 := \{s_1, \ldots, s_8\}, M_{01} := M_0 \cap M_1$ and $A = \{s_2, \ldots, s_7\}$ of S and set $X := \{M_1, M_0, M_{01}, A\}$. The set X is a finite essential set as defined in 4.2.1. Note that for each $M \in X$ we have $M \subseteq M_0$ or $M \subseteq M_1$. Hence we may define a mapping

$$\varphi \colon E_X(c) \to E_X(c') \text{ via } \varphi(x) := \begin{cases} \Omega_0(x), & x \in \mathcal{R}_{M_0}(c) \\ \Omega_1(x), & x \in \mathcal{R}_{M_1}(c) \end{cases},$$

which is well-defined since Ω_0 and Ω_1 coincide on the intersection of their domains $\mathcal{R}_{M_{01}}(c)$.
For each $M \in X$ let $\varphi_M := \varphi|_{\mathcal{R}_M(c)}$ be the restriction of φ to the residue $\mathcal{R}_M(c)$ and note that all these maps are isometries from $\mathcal{R}_M(c)$ onto $\mathcal{R}_M(c')$.

By 4.1.5, the chambers c and c' are opposite in C. Let Σ_A be the unique apartment of C containing c and c'. Since $\Omega_0(\Sigma_A)$ is an apartment of C containing the chambers $\Omega_0(c) = c'$ and $\Omega_0(c') = c$, we conclude that Σ_A is Γ_0-invariant. Similarly, as Ω_0 and Ω_1 coincide on C, Σ_A is also Γ_1-invariant. Choose a Γ_0-chamber D of Δ_{01} different from C. Then the Γ_0-chamber C and D are opposite. By 8.1.9(a), there exists a unique apartment of Δ_{01} containing Σ_A and intersecting D non-trivially. We denote this apartment by $\Sigma_{M_{01}}$. Moreover, by 8.1.9(b), this apartment is Γ_0- and Γ_1-invariant.

Since $(\Delta_i^{\Gamma_i}, \bar{\delta}_i)$ is a spherical building for each $i \in \{0, 1\}$, we may choose a Γ_i-chamber C_i of $\Delta_i^{\Gamma_i}$ such that $\bar{\delta}_i(C_i, C) = r_{\tilde{S}_i}$, where $r_{\tilde{S}_i}$ is the longest element of the relative type of the Tits index $\mathbf{T}_i$. Thus, by [MPW, 20.35], $\delta_+(c, \mathrm{proj}_{C_i}(c)) = r_{\tilde{S}_i} = r_{M_i} r_A$ and in particular there exists a chamber $z \in C_i$ such that $\delta(c, z) = r_{M_i}$. As the opposition map acts trivially on the diagram Π_{M_i} we conclude that C and C_i are opposite residues of Δ_i. Let $\mathcal{P}_i$ be the unique residue of Δ_i of type D_7 containing the Γ_i-chamber C_i. Then $\mathcal{P}_i$ and Δ_{01} are opposite spherical Γ_i-residues of Δ_i. Again by 8.1.9(a), there exists a unique apartment of Δ_i containing the apartment $\Sigma_{M_{01}}$ and intersecting $\mathcal{P}_i$ non-trivially. We denote this apartment by Σ_{M_i}. In view of 8.1.9(b), the apartment Σ_{M_i} is Γ_i-invariant.

With the notations as above, the set X satisfies the conditions of theorem 6.3.6. Thus there exists a twin apartment $\Sigma_{\tilde{\Delta}}$ of $\tilde{\Delta}$ such that for all $M \in X$

we have $\Sigma_M \subseteq \Sigma_{\tilde{\Delta}}$. Now, for $M \in X$ and φ as above,

$$\varphi(\Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c)) = \varphi(\Sigma_M) = \Sigma_M = \Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c').$$

According to the extension theorem 7.3.3, there exists a unique isometry $\Omega \in \mathrm{Aut}(\Delta)$ extending Ω_0 and Ω_1. Set $\Gamma := \langle \Omega \rangle \le \mathrm{Aut}(\Delta)$.
Let $\mathcal{P}$ be a Γ-panel containing C. Then $\mathcal{P}$ is spherical and $\mathcal{P}$ is a Γ_i-panel of Δ_i for $i = 0$ or $i = 1$. Hence $\mathcal{P}$ contains at least three Γ_i-chambers, i.e. at least three Γ-chambers. By 8.1.10 there exists a Γ-chamber which is opposite to C in $\tilde{\Delta}$ and by 8.1.13

$$\mathbf{T} :=$$

is a Tits index. Using 8.2.6 we obtain that for $\varepsilon \in \{+, -\}$ the group $\Gamma_\varepsilon := \Gamma|_{\Delta_\varepsilon}$ is a descent group of Δ_ε.
Let $\tilde{\Pi}$ be the relative type of the Tits index $\mathbf{T}$. In view of 2.16 the diagram $\tilde{\Pi}$ is connected, of rank 3, its corresponding Coxeter system is affine and, by construction, has two subdiagrams of type B_2. We conclude that the relative type of $\mathbf{T}$ is $\tilde{\mathsf{C}}_2$. In particular, for each $\varepsilon \in \{+, -\}$ the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a building of type $\tilde{\mathsf{C}}_2$.

According to 8.3.9 there exists a codistance function δ_*^{Γ} between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_-. Hence, $\Delta := (\Delta_+^{\Gamma_+}, \Delta_-^{\Gamma_-}, \delta_*^{\Gamma})$ is a twin building of type $\tilde{\mathsf{C}}_2$.

Let $\mathcal{F}(\Delta, C)$ be the foundation of Δ based at C. The irreducible rank 2 residues of Δ containing C are precisely $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$. As $\Delta_0^{\Gamma_0} \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$, theorem 12.0.1 implies that $\mathcal{F}(\Delta, C) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$. $\square$

15.4　Case F_4

Throughout this section let $\Lambda = (\mathbb{K}, V, Q)$ be a quadratic space of type F_4. Fix a complement S_0 of $\mathrm{Def}(\Lambda)$ in V and a norm splitting $(\mathbb{E}, \cdot, \{v_1, v_2\})$ of $(\mathbb{K}, S_0, Q|_{S_0})$. Let $\mathbb{F}$ be as in 1.29 and let $\mathbb{D}$ denote the composite field $\mathbb{E}\mathbb{F}$. Thus, $\mathbb{D}/\mathbb{E}$ is an extension such that $\mathbb{D}^2 \subseteq \mathbb{E} \subseteq \mathbb{D}$.

15.4.1 Theorem
There exists a twin building Δ of type $\tilde{\mathsf{C}}_2$ such that for some chamber c of Δ we have
$$\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_{\mathcal{F}}(\Lambda), \mathcal{Q}_Q^o(\Lambda)).$$

Proof　Let $\tilde{\Delta} = (\Delta_+, \Delta_-, \delta_*)$ be the twin building of type $\tilde{\mathsf{F}}_4$ constructed in 13.2.2. Let Π be the Coxeter diagram $\tilde{\mathsf{F}}_4$ labeled as in 2.3 and let (W, S) be the corresponding Coxeter system. Then each residue of Δ_+ of type $\{s_1, \ldots, s_4\}$ is isomorphic to the building $\mathsf{F}_4(\mathbb{D}/\mathbb{E})$. By construction, $\tilde{\Delta}$ satisfies condition (co).

Choose a chamber $x \in \mathcal{C}_+$ and let $\Delta_0 := \mathcal{R}_{\{s_1,\ldots,s_4\}}(x) \simeq \mathsf{F}_4(\mathbb{D}/\mathbb{E})$ be the unique residue of Δ_+ of type F_4 containing the chamber x.

According to 1.30, the quadratic form $Q_{\mathbb{E}}$ is pseudo-split and hence we may apply theorem 14.0.4 to obtain an involution $\Omega_0 \in \mathrm{Aut}(\Delta_0)$ such that $\Gamma_0 := \langle \Omega_0 \rangle$ is a descent group of Δ_0 with Tits index

$$\mathbf{T}_0 := \odot\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\bullet\!\!=\!\!\bullet\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\odot.$$

and such that the fixed point building $\Delta_0^{\Gamma_0}$ is isomorphic to the Moufang quadrangle $\mathsf{B}_2^{\mathcal{F}}(\Lambda)$.

We fix a Γ_0-chamber $C \subseteq \Delta_0$ and let Δ_1 denote the unique residue of type C_4 of Δ_+ containing C. Then $\Delta_{01} := \Delta_0 \cap \Delta_1$ is the unique residue of type C_3 containing C. By assumption Δ_{01} is a Γ_0-panel of Δ_0 and hence contains at least three Γ_0-chambers. According to 2.42, the restriction $\Omega_{01} := \Omega_0|_{\Delta_0 \cap \Delta_1} \in \mathrm{Aut}(\Delta_{01})$ is an involution and the group $\Gamma_{01} := \langle \Omega_{01} \rangle$ is a descent group of Δ_{01} with Tits index

$$\odot\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\!\!-\!\!\bullet\!\!=\!\!\!=\!\!\blacksquare.$$

By 2.41, the fixed point building $\Delta_{01}^{\Gamma_{01}}$ can be equipped with a Moufang structure $\mathbb{M}$ such that the pair $(\Delta_{01}^{\Gamma_{01}}, \mathbb{M})$ is a Moufang set. It follows from [MW, 17.12] that

$$(\Delta_{01}^{\Gamma_{01}}, \mathbb{M}) \simeq \mathbb{M}(\Lambda). \tag{15.6}$$
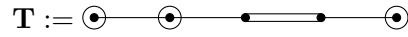
Due to the construction of $\tilde{\Delta}$, the polar space corresponding to Δ_{01} is the polar space associated with a regular but not hyperbolic quadratic space Λ. Thus, by 5.7.2, the involution Ω_{01} can be extended to an involution $\Omega_1 \in \mathrm{Aut}(\Delta_1)$ such that the group $\Gamma_1 := \langle \Omega_1 \rangle$ is a descent group of Δ_1 with Tits index

$$\mathbf{T}_1 := \odot\!\!-\!\!-\!\!\odot\!\!-\!\!-\!\!-\!\!\bullet\!\!=\!\!\bullet$$

and such that the fixed point building $\Delta_1^{\Gamma_1}$ is a Moufang quadrangle.

Choose a chamber $c \in C$ and let $c' := \Omega_0(c) = \Omega_1(c) \in C$. We define subsets $M_1 := \{s_0, s_1, s_2, s_3\}, M_0 := \{s_1, s_2, s_3, s_4\}, M_{01} := M_0 \cap M_1$ and $A = \{s_2, s_3\}$ of S and set $X := \{M_1, M_0, M_{01}, A\}$. The set X is a finite essential set as defined in 4.2.1. Note that for each $M \in X$ we have $M \subseteq M_0$ or $M \subseteq M_1$. Hence we may define a mapping

$$\varphi\colon E_X(c) \to E_X(c') \text{ via } \varphi(x) := \begin{cases} \Omega_0(x), & x \in \mathcal{R}_{M_0}(c) \\ \Omega_1(x), & x \in \mathcal{R}_{M_1}(c) \end{cases},$$

which is well-defined since Ω_0 and Ω_1 coincide on the intersection of their domains $\mathcal{R}_{M_{01}}(c)$. For each $M \in X$ the restriction $\varphi_M := \varphi|_{\mathcal{R}_M(c)}$ is an isometry from $\mathcal{R}_M(c)$ onto $\mathcal{R}_J(c')$.

By 4.1.5, the chambers c and c' are opposite in C. Let Σ_A be the unique apartment of C containing c and c'. Since $\Omega_0(\Sigma_A)$ is an apartment of C containing the chambers $\Omega_0(c) = c'$ and $\Omega_0(c') = c$, we conclude that Σ_A is Γ_0-invariant. Similarly, as Ω_0 and Ω_1 coincide on C, Σ_A is also Γ_1-invariant. Choose a Γ_0-chamber D of Δ_{01} different from C. Then the Γ_0-chambers C and D are opposite in Δ_{01}. By 8.1.9(a), there exists a unique apartment of Δ_{01} containing Σ_A and intersecting D. We denote this apartment by $\Sigma_{M_{01}}$. Moreover, by 8.1.9(b), this apartment is Γ_0- and Γ_1-invariant.

Since $(\Delta_i^{\Gamma_i}, \bar{\delta}_i)$ is a spherical building for each $i \in \{0, 1\}$, we may choose a Γ_i-chamber C_i of $\Delta_i^{\Gamma_i}$ such that $\bar{\delta}_i(C_i, C) = r_{\tilde{S}_i}$, where $r_{\tilde{S}_i}$ is the longest element of the relative type of the Tits index $\mathbf{T}_i$. Thus, by [MPW, 20.35], $\delta_+(c, \mathrm{proj}_{C_i}(c)) = r_{\tilde{S}_i} = r_{M_i} r_A$ and in particular there exists a chamber $z \in C_i$ such that $\delta(c, z) = r_{M_i}$. As the opposition map acts trivially on the diagram Π_{M_i} we conclude that C and C_i are opposite residues of Δ_i. Let $\mathcal{P}_i$ be the unique residue of Δ_i of type C_3 containing the Γ_i-chamber C_i. Then $\mathcal{P}_i$ and Δ_{01} are opposite spherical Γ_i-residues of Δ_i. Again, by 8.1.9(a), there exists a unique apartment of Δ_i containing the apartment $\Sigma_{M_{01}}$ and intersecting $\mathcal{P}_i$. We denote this apartment by Σ_{M_i}. In view of 8.1.9(b), the apartment Σ_{M_i} is Γ_i-invariant.

With the notations as above, the set X satisfies the conditions of theorem 6.3.6. Thus there exists a twin apartment $\Sigma_{\tilde{\Delta}}$ of $\tilde{\Delta}$ such that for all $M \in X$

we have $\Sigma_M \subseteq \Sigma_{\tilde{\Delta}}$. Now, for $M \in X$ and φ as above,

$$\varphi(\Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c)) = \varphi(\Sigma_M) = \Sigma_M = \Sigma_{\tilde{\Delta}} \cap \mathcal{R}_M(c').$$

According to the extension theorem 7.3.3, there exists a unique isometrie $\Omega \in \mathrm{Aut}(\tilde{\Delta})$ extending Ω_0 and Ω_1. Set $\Gamma := \langle \Omega \rangle \leq \mathrm{Aut}(\tilde{\Delta})$.

Let $\mathcal{P}$ be a Γ-panel containing C. Then $\mathcal{P}$ is spherical and $\mathcal{P}$ is a Γ_i-panel of Δ_i for $i = 0$ or $i = 1$. Hence $\mathcal{P}$ contains at least three Γ_i-chambers, i.e. at least three Γ-chambers.

By 8.1.10, there exists a Γ-chamber which is opposite to C in $\tilde{\Delta}$ and by 8.1.13

$$\mathbf{T} := \begin{array}{c} \circledcirc\!\!-\!\!-\!\!\circledcirc\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\circledcirc \end{array}$$

is a Tits index.   Using 8.2.6 we obtain that for $\varepsilon \in \{+, -\}$ the group $\Gamma_\varepsilon := \Gamma|_{\Delta_\varepsilon}$ is a descent group of Δ_ε.

Let $\tilde{\Pi}$ be the relative type of the Tits index $\mathbf{T}$. In view of 2.16, the diagram $\tilde{\Pi}$ is connected, of rank 3, its corresponding Coxeter system is affine and, by construction, has two subdiagrams of type B_2. We conclude that the relative type of $\mathbf{T}$ is $\tilde{\mathsf{C}}_2$. In particular, for each $\varepsilon \in \{+, -\}$ the fixed point structure $\Delta_\varepsilon^{\Gamma_\varepsilon}$ is a building of type $\tilde{\mathsf{C}}_2$.

According to 8.3.9 there exists a codistance function δ_*^Γ between the Γ-chambers of Δ_+ and the Γ-chambers of Δ_-. Hence, $\Delta := (\Delta_+^{\Gamma_+}, \Delta_-^{\Gamma_-}, \delta_*^\Gamma)$ is a twin building of type $\tilde{\mathsf{C}}_2$.

Let $\mathcal{F}(\Delta, C)$ be the foundation of Δ based at C. The irreducible rank 2 residues of Δ containing C are precisely $\Delta_0^{\Gamma_0}$ and $\Delta_1^{\Gamma_1}$. As $\Delta_0^{\Gamma_0} \simeq \mathsf{B}_2^{\mathcal{F}}(\Lambda)$ and in view of (15.6), theorem 12.0.1 implies that $\mathcal{F}(\Delta, C) \simeq (\mathcal{Q}_{\mathcal{F}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$.
$\square$

Part VI

Conclusion

Combining the results of the previous chapters we finally obtain the following classification:

15.4.2 Theorem
Let Δ be a Moufang quadrangle of type E_6. Then there exist, up to isomorphism, exactly two twin buildings of type $\tilde{\mathsf{C}}_2$ which have a residue isomorphic to Δ.

Proof Fix a quadratic space Λ of type E_6 such that $\Delta \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ and note that Λ is uniquely determined up to similarity by [TW, 35.11]. Let Δ_1 be the twin building constructed in 15.1.1 and let c_1 be a chamber of Δ_1. Then $\mathcal{F}(\Delta_1, c_1) \simeq ((\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$. Let Δ_2 be the twin building constructed in 15.1.2 and let c_2 be a chamber of Δ_2. Then $\mathcal{F}(\Delta_2, c_2) \simeq ((\mathcal{Q}_{\mathcal{E}}^o(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$, where Ξ is a proper anisotropic pseudo-quadratic space which is, up to similarity, uniquely determined by Δ. In view of [TW, 38.9], the twin buildings Δ_1 and Δ_2 are non-isomorphic. Thus, there exist at least two twin buildings of type $\tilde{\mathsf{C}}_2$ which have a residue isomorphic to Δ.

Let $\Delta' = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{C}}_2$ having a residue isomorphic to Δ. Choose such a residue $\mathcal{R} \subseteq \mathcal{C}_+$ and a chamber $c \in \mathcal{R}$. In view of 12.0.1, either $\mathcal{F}(\Delta', c) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$ or $\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_{\mathcal{E}}^o(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$, where Ξ is a proper anisotropic pseudo-quadratic space which is, up to similarity, uniquely determined by Δ. According to 11.1.12, therefore, the twin building Δ' is either isomorphic to Δ_1 or it is isomorphic to Δ_2. $\qquad\square$

15.4.3 Theorem
Let Δ be a Moufang quadrangle of type E_7. Then there exist at least two twin buildings of type $\tilde{\mathsf{C}}_2$ which have a residue isomorphic to Δ. If the characteristic of the defining field is not equal to 2, there exist exactly two twin buildings of type $\tilde{\mathsf{C}}_2$ which have a residue isomorphic to Δ.

Proof Fix a quadratic space Λ of type E_7 such that $\Delta \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ and note that Λ is uniquely determined up to similarity by [TW, 35.11]. Let Δ_1 be the twin building constructed in 15.2.1 and let c_1 be a chamber of Δ_1. Then $\mathcal{F}(\Delta_1, c_1) \simeq ((\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$. Let Δ_2 be the twin building constructed in 15.2.2 and let c_2 be a chamber of Δ_2. Then $\mathcal{F}(\Delta_2, c_2) \simeq ((\mathcal{Q}_{\mathcal{E}}^o(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$, where Ξ is a proper anisotropic pseudo-quadratic space. In view of [TW, 38.9], the twin buildings Δ_1 and Δ_2 are non-isomorphic. Thus, there exist at least two twin buildings of type $\tilde{\mathsf{C}}_2$ which have a residue isomorphic to Δ.

Suppose that $\text{char}(\mathbb{K}) \neq 2$ and let $\Delta' = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{C}}_2$ having a residue isomorphic to Δ. Choose such a residue $\mathcal{R} \subseteq \mathcal{C}_+$ and a chamber $c \in \mathcal{R}$. In view of 12.0.1, either $\mathcal{F}(\Delta', c) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{Q}}^o(\Lambda))$

or $\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}^o_{\mathcal{E}}(\Lambda), \mathcal{Q}_{\mathcal{P}}(\Xi))$, where Ξ is a proper anisotropic pseudo-quadratic space which is, up to similarity, uniquely determined by Δ. According to 11.1.12, therefore, the twin building Δ' is either isomorphic to Δ_1 or it is isomorphic to Δ_2. □

15.4.4 Theorem
Let Δ be a Moufang quadrangle of type E_8. Then there exists a twin building of type $\tilde{\mathsf{C}}_2$ having a residue isomorphic to Δ and this twin building is uniquely determined up to isomorphism.

Proof Fix a quadratic space Λ of type E_8 such that $\Delta \simeq \mathsf{B}_2^{\mathcal{E}}(\Lambda)$ and note that Λ is uniquely determined up to similarity by [TW, 35.11]. Let Δ' be the twin building constructed in 15.3.1 and let c' be a chamber of Δ'. Then $\mathcal{F}(\Delta', c') \simeq ((\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}^o_{\mathcal{Q}}(\Lambda))$. Thus, there exists a twin building of type $\tilde{\mathsf{C}}_2$ which has a residue isomorphic to Δ.

Let $\Delta'' = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{C}}_2$ having a residue isomorphic to Δ. Choose such a residue $\mathcal{R} \subseteq \mathcal{C}_+$ and a chamber $c \in \mathcal{R}$. In view of 12.0.1, $\mathcal{F}(\Delta'', c) \simeq (\mathcal{Q}_{\mathcal{E}}(\Lambda), \mathcal{Q}^o_{\mathcal{Q}}(\Lambda))$. According to 11.1.12, therefore, the twin building Δ'' is isomorphic to Δ'. □

15.4.5 Theorem
Let Δ be a Moufang quadrangle of type F_4. If Δ is self-dual as defined in 9.2.9 there exists a unique twin building of type $\tilde{\mathsf{C}}_2$ having a residue isomorphic to Δ. Otherwise there exist, up to isomorphism, exactly two twin buildings of type $\tilde{\mathsf{C}}_2$ having a residue isomorphic to Δ.

Proof Fix a quadratic space Λ of type F_4 such that $\Delta \simeq \mathsf{B}_2^{\mathcal{F}}(\Lambda)$ and let $\hat{\Lambda}$ denote the dual of Λ as defined in 1.31. Note that, by 1.32(a), $\hat{\Lambda}$ is a quadratic space of type F_4 and that Λ and $\hat{\Lambda}$ are uniquely determined up to similarity by [TW, 35.12]. Let Δ_1 be the twin building constructed in 15.4.1 with respect to the quadratic space Λ and let c_1 be a chamber of Δ_1. Then $\mathcal{F}(\Delta_1, c_1) \simeq ((\mathcal{Q}_{\mathcal{F}}(\Lambda), \mathcal{Q}^o_{\mathcal{Q}}(\Lambda))$. Let Δ_2 be the twin building constructed in 15.4.1 with respect to the quadratic space $\hat{\Lambda}$ and let c_2 be a chamber of Δ_2. Then $\mathcal{F}(\Delta_2, c_2) \simeq ((\mathcal{Q}_{\mathcal{F}}(\hat{\Lambda}), \mathcal{Q}^o_{\mathcal{Q}}(\hat{\Lambda}))$. In view of [TW, 38.9], the twin buildings Δ_1 and Δ_2 are isomorphic if and only if the quadratic space Λ is self-dual as defined in 1.32(d).

Let $\Delta' = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type $\tilde{\mathsf{C}}_2$ having a residues isomorphic to Δ. Choose such a residue $\mathcal{R} \subseteq \mathcal{C}_+$ and a chamber $c \in \mathcal{R}$. In view of 12.0.1, $\mathcal{F}(\Delta', c) \simeq (\mathcal{Q}_{\mathcal{F}}(\Lambda), \mathcal{Q}^o_{\mathcal{Q}}(\Lambda))$ or $\mathcal{F}(\Delta, c) \simeq (\mathcal{Q}_{\mathcal{F}}(\hat{\Lambda}), \mathcal{Q}_{\mathcal{Q}}(\hat{\Lambda}))$. According to 11.1.12, therefore, the twin building Δ' is isomorphic to Δ_1 or it is isomorphic to Δ_2. □

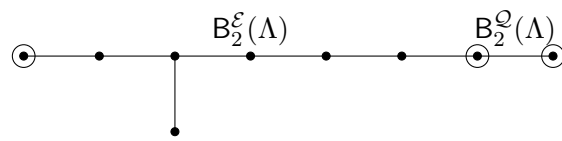# Indices for the exceptional $\tilde{\mathsf{C}}_2$-twin buildings
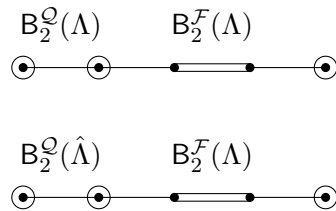
Λ quadratic space of type E_6 (15.4.2)

$$\mathsf{B}_2^{\mathcal{E}}(\Lambda) \qquad \mathsf{B}_2^{\mathcal{Q}}(\Lambda)$$

$$\mathsf{B}_2^{\mathcal{P}}(\Xi) \qquad \mathsf{B}_2^{\mathcal{E}}(\Lambda)$$
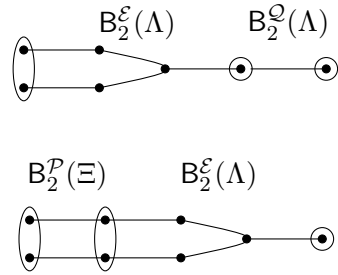
Λ quadratic space of type E_7 (15.4.3)

$$\mathsf{B}_2^{\mathcal{Q}}(\Lambda) \qquad \mathsf{B}_2^{\mathcal{E}}(\Lambda)$$

$$\mathsf{B}_2^{\mathcal{E}}(\Lambda) \qquad \mathsf{B}_2^{\mathcal{P}}(\Xi)$$
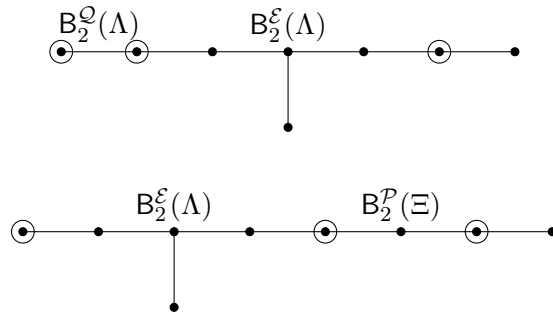
Λ quadratic space of type E_8 (15.4.4)

$$\mathsf{B}_2^{\mathcal{E}}(\Lambda) \qquad \mathsf{B}_2^{\mathcal{Q}}(\Lambda)$$

Λ quadratic space of type F_4 (15.4.5)

$$\mathsf{B}_2^{\mathcal{Q}}(\Lambda) \qquad \mathsf{B}_2^{\mathcal{F}}(\Lambda)$$

$$\mathsf{B}_2^{\mathcal{Q}}(\hat{\Lambda}) \qquad \mathsf{B}_2^{\mathcal{F}}(\Lambda)$$

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