

Dissertation zur Erlangung
des Doktorgrades der Mathematik

**IDENTIFICATION OF BANDLIMITED
PSEUDO-DIFFERENTIAL OPERATORS**

von

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Declaration

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Abstract

To understand whether a pseudo-differential operator with unit area spreading support is identifiable on a single input signal, we combined and improved existing approaches, and based on that, considered what we call the "pass to limit" method.

We enhanced the rectification method, and by using the adjoint relations, we decomposed the identification map on rectifiable spreading support into the composition of an identification procedure on rectangle spreading support plus the action of a Gabor matrix.

We analyzed the group structure behind discrete time-frequency shifts, and also classified unitary Gabor matrices on prime dimensions, that is, we showed there exist choices of window vectors for a Gabor matrix to be unitary, if and only if its support set is isomorphic to the quotient of proper non-trivial subgroups in $\mathbb{Z}_N \times \mathbb{Z}_N$ (N is a prime number).

We explored properties of periodically weighted delta trains in Wiener-Amalgam spaces, and studied how these properties carry on to the identification map when using these delta trains as identifiers, in particular, we constructed a discretely supported delta train which is a universal identifier that identifies all pseudo-differential operators with rectifiable spreading support (i.e., it identifies all currently known identifiable pseudo-differential operators).

We looked at weak* convergence of periodically weighted delta trains as identifiers, and demonstrated that such convergence itself, even combined with inner approximation of the spreading support, can pass onto the weak* convergence of the identification map. However, it is not conclusive whether the limit remains as an identifier, since bounds do not pass along the weak* convergence.

We also gave geometric insights in aforementioned parts, and briefly discussed identifiability of overspread pseudo-differential operators with linear correlation constraints on the values of their spreading functions.

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1 Introduction

This thesis is written on a level that it does not assume the reader to have prior experiences, the main prerequisite to comprehend related concepts is some understanding of time-frequency analysis and a little knowledge of the sampling theory, [19] and the first few chapters of [50] are good references for these two areas respectively. Alternatively Section 2 also briefly repeats relevant concepts and results. For this particular topic, most important known methods and results can be found in [30], [43], [42], Section 3 also organizes them, and present simplified proofs or improved expositions.

Pseudo-differential operators, when viewed as weighted superposition of time frequency shifts, are natural choices for modelling communication channels. Each communication channel uniquely corresponds to a pseudo-differential operator, which is further determined by its spreading function.

An interesting problem thus arises, that is, whether an unknown channel can be identified and reconstructed by its response from a delicately designed input signal. Previous studies have shown that this is possible when the spreading support area of the operator is strictly smaller than 1, and not possible when the spreading support area of the operator is larger than 1. If the spreading support area is precisely 1, it is only known that identifications are possible when the support area is rectifiable (see Subsection 3.2 for definition), and the answer remains largely unknown if it not.

A core problem of researches on this topic is, thus, to understand the critical case, i.e., the identifiability of pseudo-differential operators with unit spreading support. A seemingly promising idea for attacking this critical case, which is also the main theme of this thesis project, is what we call the "pass to limit" method, that is, given a pseudo-differential operator whose spreading support area is precisely 1, we approximate the spreading support from inside, then each inner approximation gives an identifiable pseudo-differential operator, thus we take an identifier for each of them respectively, and try to produce a limit from this sequence under proper

topology, then seek to show that such a limit is an identifier for the original pseudo-differential operator whose spreading support area is 1.

Considerable amount of work behind this thesis has been invested in consolidating this method from the above primitive idea, namely to develop functional analytic tools in the Wiener-Amalgam space setting for applying this method (see Section 5), and to obtain estimates at each intermediate step by using the Gabor matrices (see Section 4). Unfortunately, the answer to the critical question above remains unclear, we are able to show that it is possible to extract a limit from the identifier sequence, it is also possible to estimate the identification, it is even true that one can construct a limit that identifies all the intermediate inner approximation, i.e., a universal identifier. However, there are obstacles (inspected in detail at the end of this thesis in Section 7) that prevent the limit from being an identifier for the critical case of unit area spreading support.

Nevertheless, like many other cases in mathematical research, by attacking a valuable problem, we obtain meaningful results along the way, these results are presented in Section 6.

An overview of the subsequent sections, especially on their originality and purposes, is listed below:

As mentioned, Section 2 collects preparatory knowledge from different literature, while Section 3 contains improvements of existing results, which include

- Subsection 3.1 uses exponential basis to provide simplified proof for the unit square spreading support case and also provides geometric insight into it.
- Subsection 3.2 streamlines the derivation of the Gabor matrix form by using the weighted Zak transform.
- Subsection 3.3 emphasizes the adjoint relation between short time Fourier transform and pseudo-differential operators under certain conditions, and extended this relation from S_0 to L^2 .

- Subsection 3.4 repeats some results from the literature for this thesis to be self-contained.

Most important intermediate results in Section 3 are the explicit formulas in Corollary 3.2.1 and 3.3.1, which decompose the action of the identification map into the composition of a few much simpler operators that are rather easier to analyze.

Section 4 and Section 5 develop theories for applying the "pass to limit" method, the following original work has been done:

- Subsections 4.1, 4.2, 4.3 analyzes the group structure behind the discrete time frequency shifts, which is fundamental for the major result in the later Subsection 6.1.
- Subsections 4.4 provides a simple formula for the projection onto discrete time-frequency shifts supported on cyclic subgroups, they are not used in other parts of the thesis.
- Subsection 4.5 studies the mathematical model behind correlated MIMO channels, which leads to a minor result in the later Subsection 6.3.
- Subsections 5.1 and 5.2 establishes the possibility to extract a limit for the identification map through inner approximation of the spreading support.
- Subsection 5.3 shows one can view any S'_0 identifier as a weak* limit, which explains why the "pass to limit" method is worth studying, as it reduces an arbitrary identifier to a sequence of identifiers that we already know how to work with.

Most important intermediate results in these parts are the classification of unitary Gabor matrices in Theorem 4.2.2, which characterized the support of such matrices, and the diagonal convergence Lemma 5.2.3, which shows weak* convergence of the identifiers and inner approximation of the spreading support indeed pass onto the convergence of the identification map in the weak* operator topology.

Section 6 elaborates on our results regarding operator identification, which include

- Subsection 6.1 gives a major result that classifies unitarily identifiable pseudo-differential operators, and points out an explicit class of the spreading support of such operators.
- Subsection 6.2 provides a major result that constructs a universal identifier that identifies all currently known identifiable pseudo-differential operators.
- Subsections 6.3 and 6.4 contain minor results concerning identifiability after imposing correlations on the spreading support of otherwise unidentifiable pseudo-differential operators.

Most important results in this part are in Subsections 6.1 and 6.2.

Finally, Section 7 covers supplement contents that are not fully discussed elsewhere in this thesis, for example, related problems, motivation and heuristics behind the "pass to limit" method. More importantly, it inspects current obstacles and formulate a conjecture to propose alternative future strategies.

In the appendix, we include some complementary results that, for readability, we had chosen not to put into the body of this thesis.

2 Preliminaries

In this section we collect fundamental background knowledge that is needed concerning the operator identification problem. We include a list of definitions and notations, as well as some well known results and facts. These materials provide us with proper languages that we can use to precisely pose our main problems and further develop our analysis tools to obtain results.

2.1 Frames and Tight Frames

The concept of frames was first used by Duffin and Sch affer [7] to solve the irregular sampling problem. Given a Hilbert space \mathbb{H} , a set $\{f_n\}_n$ is called a **frame** on \mathbb{H} if for any $g \in \mathbb{H}$ we have

$$K_{\min} \|g\|_{\mathbb{H}}^2 \leq \sum_n |\langle g, f_n \rangle|^2 \leq K_{\max} \|g\|_{\mathbb{H}}^2.$$

for some positive constants K_{\min}, K_{\max} . A frame is said to be **tight** if $K_{\min} = K_{\max}$.

In this thesis the bracket notation $\langle \cdot, \cdot \rangle$ can be used for inner products on Hilbert spaces or sesquilinear dual pairings between dual spaces, in the later case the linear functional can appear in either left or right side, it does not impact our analysis, the reader will see that as the theory evolves both cases can be natural.

Given a frame $\{f_n\}_n$ on \mathbb{H} , its frame operator is defined as

$$S = \sum_n \langle \cdot, f_n \rangle f_n.$$

It is easy to see that S is self-adjoint and positive definite, thus any $g \in \mathbb{H}$ can be reconstructed as

$$g = \sum_n \langle g, f_n \rangle S^{-1} f_n.$$

Moreover, $\{S^{-1} f_n\}_n$ is also a frame, called the **canonical dual** of $\{f_n\}_n$, obviously

one also has

$$g = \sum_n \langle g, S^{-1} f_n \rangle f_n,$$

therefore, $\{f_n\}_n$ spans \mathbb{H} .

In particular, If $\{f_n\}_n$ is a tight frame, then S is a scalar multiple of the identity map, and its canonical dual is (up to a scalar multiple) itself. (see [50] for more on the frame theory)

2.2 Classical Sampling Theory of Band-limited Functions

Denote \mathcal{F} as the Fourier transform for tempered distributions on \mathbb{R}^N . Since there are different ways of scaling in different literature, to avoid confusion, in this thesis we take the following explicit form for L^1 functions:

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot \xi} dx, \quad f \in L^1(\mathbb{R}^N),$$

where the integrals is with respect to the Lebesgue measure. For convenience we also use \hat{f} for $\mathcal{F}f$ and \check{f} for $\mathcal{F}^{-1}f$.

A function f is called **band-limited** to some $U \subset \mathbb{R}^N$ if the support of \hat{f} is in U , i.e.,

$$\text{supp}(\hat{f}) \subseteq U.$$

The **Paley-Wiener space** on U consists of all square integrable functions that are band-limited to U , and is defined as

$$PW(U) = \{\check{f} : \text{supp}(f) \subseteq U, f \in L^2\}.$$

This definition guarantees that if U has finite measure, then all members in $PW(U)$ are continuous since $L^2(U) \subset L^1(U)$ in such cases.

Define sinc x as

$$\operatorname{sinc} x = \begin{cases} 1 & x = 0 \\ \frac{\sin(\pi x)}{\pi x} & x \neq 0 \end{cases}.$$

It is a classical theorem of Shannon (see [2, Chapter 1.1]) that

Theorem 2.2.1 (Shannon Sampling Theorem). *$\{\operatorname{sinc}(x - k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $PW([-1/2, 1/2])$, and every function $f \in PW([-1/2, 1/2])$ can be expanded as*

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) \operatorname{sinc}(x - k).$$

This theorem easily follows from the fact that $\{e^{2\pi i k x}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $L^2([-1/2, 1/2])$ and $\mathcal{F}^{-1}(\operatorname{sinc} x)$ is the characteristic function on $[-1/2, 1/2]$.

From the relation

$$L^2([-1/2, 1/2]) \otimes L^2([-1/2, 1/2]) \cong L^2([-1/2, 1/2]^2),$$

where the isomorphism is given by (see [28, Vol I, p.143, Example 2.6.11])

$$\langle f \otimes \tilde{f}, g \otimes \tilde{g} \rangle \mapsto \langle f, g \rangle \langle \tilde{f}, \tilde{g} \rangle,$$

one derives the sampling theorem for the Paley-Wiener space of the unit square in \mathbb{R}^2 :

Proposition 2.2.1. *$\{\operatorname{sinc}(x - j) \operatorname{sinc}(y - k)\}_{j, k \in \mathbb{Z}}$ is an orthonormal basis for $PW([-1/2, 1/2]^2)$, and every function $f \in PW([-1/2, 1/2]^2)$ can be expanded as*

$$f(x, y) = \sum_{j, k \in \mathbb{Z}} f(j, k) \operatorname{sinc}(x - j) \operatorname{sinc}(y - k).$$

$PW([-1/2, 1/2])$ can also be sampled at irregular nodes other than on \mathbb{Z} . Foundation of the irregular sampling theory was developed in a series of work by Beurling [3], Landau ([33]), Duffin and Schaffer ([7]). Their main results are outlined below, detailed surveys can be found in [50, 1, 11].

Given a sequence of number $\{\lambda_k\}_{k \in \mathbb{Z}}$ in \mathbb{R} , one defines the **lower Beurling density** and **upper Beurling density** respectively as

$$n_B^-(\{\lambda_k\}_{k \in \mathbb{Z}}) := \liminf_{r \rightarrow \infty} \frac{\inf_{\mu(I)=r} n(I)}{r} \quad n_B^+(\{\lambda_k\}_{k \in \mathbb{Z}}) := \limsup_{r \rightarrow \infty} \frac{\sup_{\mu(I)=r} n(I)}{r},$$

where μ is the Lebesgue measure and $n(I)$ is the cardinality of the intersection between $\{\lambda_k\}_{k \in \mathbb{Z}}$ and the interval I .

One further defines its **uniform density** n_U as the proper constant such that

$$\exists L > 0, \quad \left| \lambda_k - \frac{k}{n_U(\{\lambda_k\}_{k \in \mathbb{Z}})} \right| \leq L, \quad \forall k \in \mathbb{Z}.$$

Theorem 2.2.2. For $\{e^{2\pi i \lambda_k \xi}\}_{k \in \mathbb{Z}}$ to be a frame on $L^2([-1/2, 1/2])$:

1. Necessary condition [Landau]: $n_B^- \geq 1$,
2. Sufficient condition [Duffin and Schaffer]: $n_U(\{\lambda_k\}_{k \in \mathbb{Z}}) > 1$,

For convenience of writing, below if $I \subset \mathbb{R}$ (typically we consider intervals), then we use $e_I^{2\pi i \lambda x}$ to denote the exponential $e^{2\pi i \lambda x}$ restricted to I , i.e.,

$$e_I^{2\pi i \lambda x} = \chi_I e^{2\pi i \lambda x}.$$

2.3 Continuous and Discrete Gabor Systems

The **continuous modulation operator** \mathcal{M}_v is defined as

$$(\mathcal{M}_v f)(x) = e^{2\pi i v \cdot x} f(x),$$

and the **continuous translation operator** \mathcal{T}_v is defined as

$$(\mathcal{T}_t f)(x) := f(x - t).$$

It is easy to verify that they commute up to a phase factor

$$\mathcal{T}_t \mathcal{M}_v = e^{-2\pi i v \cdot t} \mathcal{M}_v \mathcal{T}_t,$$

and connected via the Fourier transform

$$\mathcal{F}\mathcal{M}_a = \mathcal{T}_a\mathcal{F}, \quad \mathcal{F}\mathcal{T}_a = \mathcal{M}_{-a}\mathcal{F}.$$

Given $\phi(x)$ with $x \in \mathbb{R}^N$, and fix $a, b \in \mathbb{R}^N$, the following set

$$(\phi, a, b) = \{\mathcal{M}_{ja}\mathcal{T}_{kb}\phi(x)\}_{j,k \in \mathbb{Z}^N},$$

is called a **continuous Gabor system**. The **density of a continuous Gabor system** is defined via the density of its support lattice $a\mathbb{Z} \times b\mathbb{Z}$, i.e., $1/(ab)$.

If (ϕ, a, b) forms a frame for $L^2(\mathbb{R}^N)$, then it is called a **Gabor frame**. The trivial example would be $(\chi_{[0,1]}, 1, 1)$, which, as an immediate consequence from the sampling theorem in the last subsection, is an orthonormal basis for $L^2(\mathbb{R})$.

Gabor initially conjectured that if ϕ_0 is the standard Gaussian, then $(\phi_0, 1, 1)$ would span $L^2(\mathbb{R})$ [17], it turns out that the actual condition for (ϕ_0, a, b) to be a frame is:

Theorem 2.3.1. [45, 36] (ϕ_0, a, b) is a frame for $L^2(\mathbb{R})$ if and only if $ab < 1$.

Let N be a fixed natural number, and denote

$$\omega_N = e^{\frac{2\pi i}{N}},$$

as the first primitive N -th root of unity.

Take $j, k \in \mathbb{Z}$, then on the vector space \mathbb{C}^N , the **discrete modulation operator** is defined as

$$M_N^j = \begin{pmatrix} 1 & & & & \\ & \omega_N & & & \\ & & \ddots & & \\ & & & \omega_N^{N-2} & \\ & & & & \omega_N^{N-1} \end{pmatrix}^j,$$

and the *discrete translation operator* is defined as

$$T_N^k = \begin{pmatrix} 0 & & & & 1 \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}^k.$$

i.e., T represents the cyclic permutation $(1\ 2\ \dots\ N)$ that maps (x_1, x_2, \dots, x_N) to $(x_N, x_1, x_2, \dots, x_{N-1})$.

Lemma 2.3.1 (Commutativity). M_N^j and T_N^k commute up to a phase factor, i.e.,

$$M_N^j T_N^k = \omega_N^{jk} T_N^k M_N^j.$$

Consequently, if

$$kj' \equiv jk' \pmod{N}, \tag{1}$$

then $M_N^j T_N^k$ and $M_N^{j'} T_N^{k'}$ commute.

Proof. It is easy to verify that

$$\begin{aligned}
M_N^1 T_N^1 &= \begin{pmatrix} & & & & & & & & & 1 \\ \omega_N^1 & & & & & & & & & \\ & \omega_N^2 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \omega_N^{N-2} & & & & & & \\ & & & & \omega_N^{N-1} & & & & & \\ & & & & & & & & & \omega^{N-1} \\ & 1 & & & & & & & & \\ & & \omega_N^1 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & \omega_N^{N-3} & & & & & \\ & & & & & & \omega_N^{N-2} & & & \\ & & & & & & & & & \omega_N^{N-2} \end{pmatrix} \\
&= \omega_N \begin{pmatrix} & & & & & & & & & w^{N-1} \\ 1 & & & & & & & & & \\ & \omega_N^1 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \omega_N^{N-3} & & & & & & \\ & & & & \omega_N^{N-2} & & & & & \end{pmatrix} \\
&= \omega_N T_N^1 M_N^1.
\end{aligned}$$

Therefore

$$M_N^j T_N^1 = M_N^{j-1} (M_N^1 T_N^1) = \omega_N M_N^{j-1} T_N^1 M_N^1 = \omega_N M_N^{j-2} (M_N^1 T_N^1) M_N^1 = \dots = \omega_N^j T_N^1 M_N^j,$$

and thus

$$M_N^j T_N^k = (M_N^j T_N^1) T_N^{k-1} = \omega_N^j T_N^1 M_N^j T_N^{k-1} = \omega_N^j T_N^1 (M_N^j T_N^1) T_N^{k-2} = \dots = \omega_N^{jk} T_N^k M_N^j.$$

Using this formula we get

$$M_N^j T_N^k M_N^{j'} T_N^{k'} = M_N^j (T_N^k M_N^{j'}) T_N^{k'} = \omega_N^{-kj'} M_N^{j+j'} T_N^{k+k'},$$

while

$$M_N^{j'} T_N^{k'} M_N^j T_N^k = M_N^{j'} (T_N^{k'} M_N^j) T_N^k = \omega_N^{-j'k} M_N^{j+j'} T_N^{k+k'}.$$

Therefore if (1) is satisfied, then $M_N^j T_N^k$ and $M_N^{j'} T_N^{k'}$ commute. \square

Let

$$W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega_N^1 & \dots & (\omega_N^{N-1})^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\omega_N)^{N-2} & \dots & (\omega_N^{N-1})^{N-2} \\ 1 & (\omega_N)^{N-1} & \dots & (\omega_N^{N-1})^{N-1} \end{pmatrix},$$

be the unitary DFT matrix, i.e.,

$$(W_N)_{mn} = \frac{1}{\sqrt{N}} \omega_N^{(m-1)(n-1)},$$

then these discrete time-frequency shifts are connected via the discrete Fourier transform as

$$T_N^k = W_N M_N^{-k} W_N^* = W_N^* M_N^k W_N, \quad (2)$$

where $*$ denotes the adjoint operation.

If $\vec{y} = W_N \vec{x}$, then we may write \vec{y} as $\hat{\vec{x}}$ and \vec{x} as $\check{\vec{y}}$.

Consider the group $\mathbb{Z}_N \times \mathbb{Z}_N$. If Γ is a subset of $\mathbb{Z}_N \times \mathbb{Z}_N$, and $\vec{d} \in \mathbb{C}^N$ is an arbitrary vector, then we use the notion (\vec{d}, Γ) to denote the set of vectors consists of discrete time frequency shifts supported on Γ , and applied to \vec{d} , i.e.,

$$(\vec{d}, \Gamma) = \{M_N^j T_N^k \vec{d} : (j, k) \in \Gamma\}.$$

The set (\vec{d}, Γ) is referred to as a **discrete Gabor system** with support Γ and window \vec{d} , it is called a **discrete Gabor basis (resp. frame)** when it forms a basis (resp. frame) for the underlying vector space \mathbb{C}^N .

One advantage of using discrete Gabor systems is that for any non-zero window vector $\vec{d} \in \mathbb{C}^N$, $(\vec{d}, \mathbb{Z}_N \times \mathbb{Z}_N)$ is always a tight frame for \mathbb{C}^N :

Lemma 2.3.2 (Frame Properties). *The following holds:*

- 1) $\{\frac{1}{\sqrt{N}} M_N^j T_N^k\}_{j,k=0,1,2,\dots,N-1}$ is an orthonormal basis for the matrix space $\mathbb{C}^{N \times N}$.

2) If $\{A_n\}_{n=1,2,\dots,N^2}$ is an orthonormal basis for $\mathbb{C}^{N \times N}$, then $\{A_n \vec{d}\}_{n=1,2,\dots,N^2}$ is a tight a tight frame for \mathbb{C}^N with frame constant $\|\vec{d}\|^2$ for any $\vec{d} \in \mathbb{C}^N$.

3) $\{M_N^j T_N^k \vec{d}\}_{j,k=0,1,2,\dots,N-1}$ is a tight frame on \mathbb{C}^N with frame constant $N\|\vec{d}\|^2$ for any $\vec{d} \in \mathbb{C}^N$.

Proof. Orthonormality in 1) can be verified trivially through direct computations.

For 2), take any $\vec{x} \in \mathbb{C}^N$, we have

$$\begin{aligned}
\sum_{n=1}^{N^2} |\langle \vec{x}, A_n \vec{d} \rangle|^2 &= \sum_{n=1}^{N^2} |\vec{d}^* A_n^* \vec{x}|^2 \\
&= \sum_{n=1}^{N^2} |\text{tr}(\vec{x} \vec{d}^* A_n^*)|^2 \\
&= \sum_{n=1}^{N^2} |\langle \vec{x} \vec{d}^*, A_n \rangle|^2 \\
&= \|\vec{x} \vec{d}^*\|_{HS}^2 \\
&= \|\vec{d}\|^2 \|\vec{x}\|^2,
\end{aligned}$$

where tr denotes the trace of a matrix and $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm of a matrix.

3) follows from 1) and 2). □

If we write the vectors in (\vec{d}, Γ) into matrix form, then we get the **Gabor matrix**, which we denote as $G_\Gamma(\vec{d})$. The ordering of the columns in $G_\Gamma(\vec{d})$ will not influence our analysis, but to eliminate ambiguity, in this thesis we adopt the convention that columns in $G_\Gamma(d)$ are arranged by the **lexicographical ordering** (i.e., (j, k) precedes (j', k') if either $j < j'$ or $j = j'$ with $k < k'$) on Γ . See [40] for other properties of Gabor frames.

2.4 The Feichtinger Algebra and its Dual

Denote V_ϕ as the *Short Time Fourier Transform* with respect to a window ϕ , i.e.,

$$(V_\phi f)(t, v) = \int_{\mathbb{R}^N} f(x) \overline{\phi(x-t)} e^{-2\pi i x \cdot v} dx = e^{-2\pi i t \cdot v} \int_{\mathbb{R}^N} f(x+t) \overline{\phi(x)} e^{-2\pi i x \cdot v} dx. \quad (3)$$

This form is well defined for f, ϕ being L^2 functions. We may rewrite it as

$$(V_\phi f)(t, v) = \langle f, \mathcal{M}_v \mathcal{T}_t \phi \rangle, \quad (4)$$

where the inner product is with respect to the x variable. More importantly, in this way whenever such a dual pairing is well defined we can further extend it to other spaces on which shifts modulations are also automorphisms. For example, one may take ϕ to be a Schwartz class function and f to be a tempered distribution, or the other way around. (see [19, p.41]).

The *Feichtinger Algebra*, denoted as S_0 , is a core object in time-frequency analysis, and is defined as ([9])

$$S_0(\mathbb{R}^N) = \{f \in L^2(\mathbb{R}) : V_{\phi_0} f \in L^1(\mathbb{R}^{2N})\},$$

where $\phi_0 = e^{-\pi \|x\|^2}$ is the Gaussian function. S_0 is a Banach space equipped with $\|V_{\phi_0} f\|_{L^1}$ norm (see [19, p.246]).

The usefulness of $S_0(\mathbb{R}^N)$ stems from the fact that it is the smallest Banach space allowing a meaningful time-frequency analysis, indeed, the Fourier transform \mathcal{F} , the modulation operator \mathcal{M} , and the translation operator, \mathcal{T} are all isometric automorphisms on $S_0(\mathbb{R}^N)$ (consequently such properties also extend to its dual) and it is continuously embedded in $L^p(\mathbb{R}^N)$ for all $1 \leq p \leq \infty$. In fact, $S_0(\mathbb{R}^N)$ contains the whole Schwartz class $S(\mathbb{R}^N)$, and can be continuously embedded in any Banach space that has these properties and contains at least one, and therefore all, non-trivial Schwartz functions [13]. Moreover, all S_0 functions are continuous.

The dual of $S_0(\mathbb{R}^N)$, denoted as $S'_0(\mathbb{R}^N)$, turns out to be

$$S'_0(\mathbb{R}^N) = \{g \in S'(\mathbb{R}^N) : V_{\phi_0}g \in L^\infty(\mathbb{R}^{2n})\},$$

where $S'(\mathbb{R}^N)$ is the class of tempered distributions on \mathbb{R}^N . S'_0 is also a Banach space equipped with the norm induced from the dual pairing.

For convenience of writing, we introduce the notation \asymp so that $\|\cdot\|_X \asymp \|\cdot\|_Y$ means there exist constants K_{\min} and K_{\max} independent of variables in the context such that

$$K_{\min}\|\cdot\|_X \leq \|\cdot\|_Y \leq K_{\max}\|\cdot\|_X$$

holds for any elements in corresponding spaces with X, Y topologies.

Similarly we use $\|\cdot\|_X \lesssim \|\cdot\|_Y$ to mean there is a constant C , independent of variables in the context such that

$$\|\cdot\|_X \leq C\|\cdot\|_Y,$$

and also $\|\cdot\|_X \gtrsim \|\cdot\|_Y$ means there is a constant C such that

$$\|\cdot\|_X \geq C\|\cdot\|_Y.$$

Gabor expansion provides a useful characterization of $S_0(\mathbb{R}^N)$ and $S'_0(\mathbb{R}^N)$:

Theorem 2.4.1. *[19, Thm 12.1.8] Let $\phi \in S_0(\mathbb{R}^N)$ and a, b be proper constants such that (ϕ, a, b) is a Gabor frame for $L^2(\mathbb{R}^N)$, then for any $f \in S_0(\mathbb{R}^N)$ and $g \in S'_0(\mathbb{R}^N)$*

$$\|f\|_{S_0} \asymp \|\{\langle f, \mathcal{M}_{ja}\mathcal{T}_{kb}\phi \rangle\}_{j,k \in \mathbb{Z}^N}\|_{\ell^1},$$

$$\|g\|_{S'_0} \asymp \|\{\langle g, \mathcal{M}_{ja}\mathcal{T}_{kb}\phi \rangle\}_{j,k \in \mathbb{Z}^N}\|_{\ell^\infty}.$$

Another property we will be using is the Gelfand triple

$$S_0 \subset L^2 \subset S'_0$$

where each embedding is continuous. This embedding also directly follows from

the above theorem.

It follows that for any $f \in L^2$ we have

$$\|f\|_{S'_0} \lesssim \|f\|_{L^2},$$

and for any $g \in S_0$ we have

$$\|g\|_{S'_0} \lesssim \|g\|_{L^2} \lesssim \|g\|_{S_0}.$$

Since S_0 also contains all Schwartz class functions, the embedding $S_0 \subset L^2$ is not just continuous but also dense.

2.5 Wiener-Amalgam Spaces

Let $A(\mathbb{R}^N)$ be the space of functions that are the Fourier transform of $L^1(\mathbb{R}^N)$ functions, and $A'(\mathbb{R}^N)$ be the space of distributions that are the Fourier transform of $L^\infty(\mathbb{R}^N)$ functions. These are Banach spaces with norms

$$\|\hat{f}\|_A = \|f\|_{L^1}; \quad \|\hat{f}\|_{A'} = \|f\|_{L^\infty}.$$

Let $\psi \in C^\infty(\mathbb{R}^N)$ be compactly support in $[-\epsilon_0 - 1/2, \epsilon_0 + 1/2]^N$ for some small fixed ϵ_0 (in this thesis it suffices to assume $\epsilon_0 < 1/9$ for various technical steps) with properties

$$\psi(x) : \begin{cases} = 1 & x \in (-\frac{1}{2} + \epsilon_0, \frac{1}{2} - \epsilon_0)^N \\ \in [0, 1] & x \in [-\frac{1}{2} - \epsilon_0, -\frac{1}{2} + \epsilon_0]^N \cup [\frac{1}{2} - \epsilon_0, \frac{1}{2} + \epsilon_0]^N \end{cases},$$

and

$$\sum_{n \in \mathbb{Z}^N} \mathcal{T}_n \psi \equiv 1.$$

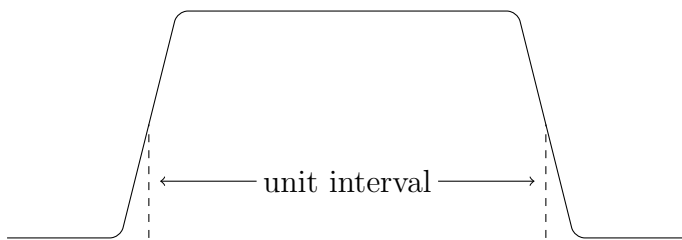


Figure 1: The Window Function ψ on \mathbb{R}

For $N = 1$, $\{\mathcal{M}_{\beta j} \mathcal{T}_\ell \psi\}_{j,k \in \mathbb{Z}}$, with $\beta = 1/(1 + 2\epsilon_0)$ is a Gabor frame, see [22, Thm 4.1.2]. Also, since ψ is compactly supported in $[-\epsilon_0 - 1/2, \epsilon_0 + 1/2]^N$, for each fixed n , there are only finitely many m such that $\text{supp}(\mathcal{T}_n \psi) \cap \text{supp}(\mathcal{T}_m \psi)$ is not empty.

Aside from L^2 , the topologies we will be using are the **Wiener-Amalgam spaces**

$$W^{A,1} = \{f : \|f\|_{W^{A,1}} = \|\{\|\mathcal{T}_n \psi \cdot f\|_A\}\|_{\ell^1} < \infty\},$$

$$W^{A,\infty} = \{f : \|f\|_{W^{A,\infty}} = \|\{\|\mathcal{T}_n \psi \cdot f\|_A\}\|_{\ell^\infty} < \infty\},$$

$$W^{A',\infty} = \{f : \|f\|_{W^{A',\infty}} = \|\{\|\mathcal{T}_n \psi \cdot f\|_{A'}\}\|_{\ell^\infty} < \infty\}.$$

These spaces are among a bigger class of spaces which are sometimes referred to as mixed norm spaces, in the sense that they have different local and global topological properties. For example $W^{A,1}$ consists of functions that are locally the Fourier transform of L^1 functions and, roughly speaking, globally of ℓ^1 decay.

These norms depend on the choice of the window ψ , but different choices of ψ induce equivalent norms, therefore the space does not change. These spaces are also connected to modulation spaces through the Fourier transform (see [6, Prop 2.4]). And in particular, we have (see [9, 10, 14] and references there for more details and history over this equivalence)

Theorem 2.5.1. [42, Prop 2.2] $W^{A,1}$ coincides with the Feichtinger algebra S_0 with equivalent norms, and $W^{A',\infty}$ coincides with the dual of the Feichtinger algebra

bra, S'_0 with equivalent norms, i.e.,

$$\|f\|_{W^{A,1}} \asymp \|f\|_{S_0}; \quad \|f\|_{W^{A',\infty}} \asymp \|f\|_{S'_0}$$

Due to this equivalence, in the rest part of this thesis, although sometimes we write $\|\cdot\|_{S_0}$ and $\|\cdot\|_{S'_0}$, we actually work with $W^{A,1}$ and $W^{A',\infty}$ norms, for ease of writing this knowledge could be silently assumed without particularly referring to the above theorem.

We introduce notations $W^{A',\infty}(U)$ to denote the subspace of distributions in $W^{A',\infty}(\mathbb{R}^N)$ that vanish outside U . The norm on $W^{A',\infty}(U)$ inherits from the norm on $W^{A',\infty}(\mathbb{R}^N)$, i.e.,

$$\|\eta\|_{W^{A',\infty}(U)} = \|\eta\|_{W^{A',\infty}(\mathbb{R}^N)}.$$

2.6 Periodically Weighted Delta and Exponential Trains

Denote δ_λ as the Dirac distribution at λ , let $\vec{c}, \vec{d} \in \mathbb{C}^N$ be some vectors, two particular distributions that we will be using frequently in the sequel are the so called ***N -periodically weighted delta trains*** and ***N -periodically weighted exponential trains*** denoted respectively as

$$\mathfrak{g}_{\vec{c}} = \sum_{j \in N\mathbb{Z}} \sum_{k=0}^{N-1} c_k \delta_{\frac{j+k}{\sqrt{N}}},$$

$$\mathfrak{u}_{\vec{d}} = \sum_{j \in N\mathbb{Z}} \sum_{k=0}^{N-1} d_k e^{2\pi i \frac{j+k}{\sqrt{N}} x}.$$

It is clearly that $\mathfrak{g}_{\vec{c}}$ is a well defined tempered distribution and so is $\mathfrak{u}_{\vec{d}}$ since an exponential train is just the Fourier transform of a delta train.

In particular, a periodically weighted delta train is also a periodically weighted exponential train with their weights connected via the discrete Fourier transform:

Lemma 2.6.1.

$$\mathfrak{g}_{\vec{c}} = \mathfrak{u}_{\hat{\vec{c}}}.$$

Proof. Write the vector \vec{c} as

$$\vec{c} = (c_0, c_1, c_2, \dots, c_{N-1}).$$

Apply the distributional definition of the Poisson summation formula (see, e.g., the wikipedia page or [24, Theorem 7.2.1]) we get

$$\begin{aligned} \mathfrak{g}_{\vec{c}} &= \sum_{j \in N\mathbb{Z}} \sum_{k=0}^{N-1} c_k \delta_{\frac{j+k}{\sqrt{N}}} \\ &= \sum_{j \in \mathbb{Z}} \left(\sum_{k=0}^{N-1} c_k \delta_{\frac{k}{\sqrt{N}}} \right) \left(x + \frac{jN}{\sqrt{N}} \right) \\ &= \sum_{k=0}^{N-1} \left(\sum_{j \in \mathbb{Z}} c_k \delta_{\frac{k}{\sqrt{N}}} \right) \left(x + j\sqrt{N} \right) \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\sum_{m \in \mathbb{Z}} c_k e^{2\pi i \frac{k}{\sqrt{N}} \frac{m}{\sqrt{N}}} \right) e^{2\pi i \frac{m}{\sqrt{N}} \xi} \\ &= \sum_{m \in \mathbb{Z}} \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k e^{2\pi i \frac{mk}{N}} \right) e^{2\pi i \frac{m}{\sqrt{N}} \xi}, \end{aligned}$$

If we denote d_m as the coefficient in front of each exponential term in the above equation, i.e.,

$$d_m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k e^{2\pi i \frac{mk}{N}},$$

then clearly $\{d_m\}_{m \in \mathbb{Z}}$ is N -periodic, i.e.

$$d_m = d_{m+N}.$$

Set

$$\vec{d} = (d_0, d_1, d_2, \dots, d_{N-1}),$$

then it is straight forward to verify that

$$\vec{d} = W_N \vec{c}.$$

Combining the above we get

$$\mathfrak{g}_{\vec{e}} = \sum_{m \in N\mathbb{Z}} \sum_{n=0}^{N-1} d_n e^{2\pi i \frac{n+m}{\sqrt{N}} \xi} = \mathfrak{u}_{\vec{d}} = \mathfrak{u}_{\hat{c}}.$$

□

It is easy to see by definition that $\mathfrak{g}_{\vec{e}} \in W^{A', \infty}(\mathbb{R}^N)$ since there are always finitely many deltas in $\text{supp}(\mathcal{T}_n \psi)$ for any $n \in \mathbb{Z}^N$. Thus we also have $\mathfrak{u}_{\vec{d}} \in W^{A', \infty}(\mathbb{R}^N)$, and they are both in $S'_0(\mathbb{R}^N)$.

Take the number $1 \in \mathbb{C}$, viewed as a one dimensional vector $\vec{1}$, we get a very special delta train:

$$\mathfrak{g}_{\vec{1}} = \sum_{k \in \mathbb{Z}} \delta_k,$$

i.e., the unweighted delta train supported on \mathbb{Z} . We will use it frequently in the remaining part of this thesis.

A similarly constructed exponential train is

$$\mathfrak{u}_{\vec{1}} = \sum_{k \in \mathbb{Z}} e^{2\pi i k x},$$

the Poisson summation formula indicates

$$\mathfrak{u}_{\vec{1}} = \mathfrak{g}_{\vec{1}}.$$

2.7 Zak Transform and Weighted Zak Transform

The **Zak transform** Z_a of a function f is formally defined as

$$(Z_a f)(x, w) = \sum_{k \in \mathbb{Z}} f(x + ka) e^{-2\pi i k a w},$$

where $a > 0$ is a parameter. If $a = 1$ then sometimes one simply writes Zf , as is the convention in most literature.

By this formal definition $(Z_a f)(x, w)$ is the Fourier series of the sequence $\{f(x + ka)\}_k$, hence it is well defined in different senses depending on the topology on f (see relevant chapters in [19] for details), most commonly used is $f \in L^2(\mathbb{R})$, and then the right hand side is almost everywhere defined in such cases.

A straightforward way to understand the Zak transform is through its geometry, if we restrict ourselves to a box (depending on a), then as depicted in the following graph, it creates an isomorphism between $L^2(\mathbb{R})$ and L^2 over a box (depending on a) by cutting the function f off each interval $[ka, (k + 1)a)$, and then composing them with the corresponding exponential basis.

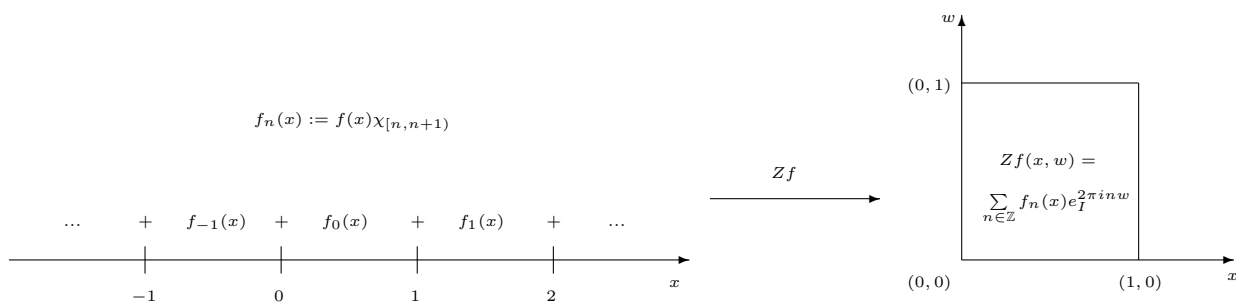


Figure 2: Zak transform with $a = 1$

The isomorphism property of the Zak transform is represented by the following two theorems (see [19, Chapter 8.2]):

Theorem 2.7.1 (Plancherel theorem of the Zak transform). $\sqrt{a}Z_a$ is unitary from $L^2(\mathbb{R})$ onto $L^2([0, a) \times [0, \frac{1}{a}))$.

Theorem 2.7.2 (Quasi-periodicity). Z_a is quasi-periodic with respect to the box $[0, a) \times [0, \frac{1}{a})$, i.e.,

$$(Z_a f)(x + a, w) = e^{-2\pi i a w} (Z_a f)(x, w), \quad (Z_a f)(x, w + \frac{1}{a}) = (Z_a f)(x, w).$$

And underlying variables swap positions if the Zak transform is applied to the Fourier transform of a function ([19, Prop 8.2.2]):

Theorem 2.7.3. For $f \in S_0(\mathbb{R})$, we have

$$(Z_1 \hat{f})(x, w) = e^{2\pi i x \cdot w} (Z_1 f)(-w, x).$$

This property is sometimes used to illustrate the fact that the Fourier transform rotates the time-frequency plane by $\pi/2$. A direct consequence of this theorem is the following

Proposition 2.7.1. For $f \in S_0(\mathbb{R})$, we have

$$(Z_1 \check{f})(x, w) = e^{2\pi i x \cdot w} (Z_1 f)(w, -x).$$

Proof. Apply the above theorem three times we get

$$\begin{aligned} (Z_1 \check{f})(x, w) &= (Z_1 \mathcal{F}^3 f)(x, w) \\ &= e^{2\pi i x \cdot w} (Z_1 \mathcal{F}^2 f)(-w, x) \\ &= e^{2\pi i x \cdot w} e^{-2\pi i w \cdot x} (Z_1 \hat{f})(-x, -w) \\ &= e^{2\pi i x \cdot w} (Z_1 \hat{f})(w, -x). \end{aligned}$$

□

Take $\mathfrak{g}_{\mathbb{T}}$, i.e., the unweighted delta train supported on \mathbb{Z} , then the Zak transform Z_1 can also be viewed as the short time Fourier transform with respect to $\mathfrak{g}_{\mathbb{T}}$. Indeed, direct computation gives

$$\begin{aligned} (V_{\mathfrak{g}_{\mathbb{T}}} f)(t, v) &= e^{-2\pi i t \cdot v} \int_{\mathbb{R}} f(x+t) \left(\sum_{k \in \mathbb{Z}} \delta_k \right) e^{-2\pi i x \cdot v} dx \\ &= e^{-2\pi i t \cdot v} \sum_{k \in \mathbb{Z}} f(t+k) e^{-2\pi i k v} \\ &= e^{-2\pi i t \cdot v} (Z_1 f)(t, v). \end{aligned}$$

Let $\vec{c} = (c_0, c_1, c_2, \dots, c_{N-1}) \in \mathbb{C}^N$, we may extend the definition of the Zak transform to **periodically weighted Zak transform** $Z_{\vec{c}}$ by setting

$$(Z_{\vec{c}} f)(x, w) = \sum_{k \in \mathbb{Z}} \bar{c}_{k \bmod N} f\left(x + \frac{k}{\sqrt{N}}\right) e^{-\frac{2\pi i k w}{\sqrt{N}}}.$$

For convenience of writing, in the rest part of the thesis we will just write c_k instead of $c_{k \bmod N}$.

Similar to the standard Zak transform, the weighted Zak transform can also be viewed as the short time Fourier transform with respect to a weighted delta train:

Lemma 2.7.1.

$$V_{\mathfrak{g}_{\bar{c}}} = e^{-2\pi i t \cdot v} Z_{\bar{c}}.$$

Proof. Straightforward computation shows

$$\begin{aligned} (V_{\mathfrak{g}_{\bar{c}}}f)(t, v) &= e^{-2\pi i t \cdot v} \int_{\mathbb{R}} f(x+t) \left(\sum_{k \in \mathbb{Z}} \bar{c}_k \delta_{\frac{k}{\sqrt{N}}} \right) e^{-2\pi i x \cdot v} dx \\ &= e^{-2\pi i t \cdot v} \sum_{k \in \mathbb{Z}} \bar{c}_k f\left(t + \frac{k}{\sqrt{N}}\right) e^{-\frac{2\pi i k v}{\sqrt{N}}} \\ &= e^{-2\pi i t \cdot v} (Z_{\bar{c}}f)(t, v). \end{aligned}$$

□

Lemma 2.7.2. $Z_{\bar{c}}$ is quasi-periodic with respect to the (bigger) box $[0, \sqrt{N}) \times [0, \sqrt{N})$, i.e.

$$(Z_{\bar{c}}f)(x + \sqrt{N}, w) = e^{-2\pi i \sqrt{N} w} (Z_{\bar{c}}f)(x, w), \quad (Z_{\bar{c}}f)(x, w + \sqrt{N}) = (Z_{\bar{c}}f)(x, w).$$

Proof. Using the periodicity, we may rewrite $Z_{\bar{c}}$ as the sum of N number of unweighted Zak transforms $Z_{\sqrt{N}}$:

$$\begin{aligned} (Z_{\bar{c}}f)(x, w) &= \sum_{j \in \mathbb{Z}} \sum_{k=0}^{N-1} \bar{c}_k f\left(x + \frac{Nj+k}{\sqrt{N}}\right) e^{-2\pi i \frac{Nj+k}{\sqrt{N}} w} \\ &= \sum_{k=0}^{N-1} \sum_{j \in \mathbb{Z}} \bar{c}_k e^{-2\pi i \frac{k}{\sqrt{N}} w} f\left(x + \frac{k}{\sqrt{N}} + j\sqrt{N}\right) e^{-2\pi i j \sqrt{N} w} \\ &= \sum_{k=0}^{N-1} \left((\bar{c}_k e^{-2\pi i \frac{k}{\sqrt{N}} w} \mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f \right)(x, w). \end{aligned}$$

With Theorem 2.7.2 we get

$$\begin{aligned}
(Z_{\bar{c}}f)(x + \sqrt{N}, w) &= \sum_{k=0}^{N-1} ((\bar{c}_k e^{-2\pi i \frac{k}{\sqrt{N}} w} T_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f)(x + \sqrt{N}, w) \\
&= e^{-2\pi i \sqrt{N} w} \sum_{k=0}^{N-1} ((\bar{c}_k e^{-2\pi i \frac{k}{\sqrt{N}} w} \mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f)(x, w) \\
&= e^{-2\pi i \sqrt{N} w} (Z_{\bar{c}}f)(x, w),
\end{aligned}$$

and

$$\begin{aligned}
(Z_{\bar{c}}f)(x, w + \sqrt{N}) &= \sum_{k=0}^{N-1} ((\bar{c}_k e^{-2\pi i \frac{k}{\sqrt{N}} (w + \sqrt{N})} T_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f)(x, w + \sqrt{N}) \\
&= \sum_{k=0}^{N-1} ((\bar{c}_k e^{-2\pi i \frac{k}{\sqrt{N}} w} \mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f)(x, w), \\
&= (Z_{\bar{c}}f)(x, w).
\end{aligned}$$

□

2.8 Pseudo-Differential Operators with Band-limited Symbols

The earliest form of pseudo-differential operators arised in 1930s, and was formulated by Weyl. In quantum mechanics, states in the phase space are taken as unit vectors. Since observables can only be described in a probabilistic way, they are represented by self-adjoint operators through projection valued measures (a consequence of the spectral theorem) on the phase space. The canonical position and momentum operators are the multiplication operator

$$(\mathcal{X}f)(x) = xf(x),$$

and the differential operator

$$(\mathcal{D}f)(x) = \frac{1}{2\pi i} \mathcal{D}_x f.$$

Weyl considered the correspondence between a given function $\sigma(x, w)$ and the operator $\sigma(\mathcal{X}, \mathcal{D})$ via the Fourier transform

$$\sigma(\mathcal{X}, \mathcal{D}) = \int \hat{\sigma}(t, v) e^{2\pi i(t\mathcal{X} + v\mathcal{D})} dt dv, \quad (5)$$

where the exponential map $e^{2\pi i(t\mathcal{X} + v\mathcal{D})}$ is to be quantized by the Schrödinger representation. Detailed derivations of the above can be found in either [15] or [19]. (5) is often referred to as the Weyl correspondence.

The theory of pseudo-differential operators started to flourish in 1960s ([46, Chapter 6]) as an attempt to understand partial differential equations. A partial differential operator has form

$$P(x, \mathcal{D}) = \sum_{|\alpha| \leq N} \sigma_\alpha(x) \mathcal{D}^\alpha, \quad N \in \mathbb{N}.$$

Since the Fourier transform takes differentiation to polynomial multiplication, using the Fourier inversion formula one gets

$$P(x, \mathcal{D})f = \int \sum_{|\alpha| \leq N} \sigma_\alpha(x) (2\pi i w)^\alpha \hat{f}(w) e^{2\pi i x \cdot w} dw,$$

on appropriate spaces, for example, on the Schwartz class.

Setting $\sum_{|\alpha| \leq N} \sigma_\alpha(x) (2\pi i w)^\alpha$ as the function $\sigma(x, w)$, the above rewrites as

$$P(x, \mathcal{D})f = \int \sigma(x, w) \hat{f}(w) e^{2\pi i x \cdot w} dw.$$

Thus today most often in the literature, a pseudo-differential operator takes the form

$$(\mathcal{K}_\sigma f)(x) = \int \sigma(x, w) \hat{f}(w) e^{2\pi i x \cdot w} dw, \quad (6)$$

This definition is also referred to as the Kohn-Nirenberg correspondence and σ is called the **Kohn-Nirenberg symbol**. Which symbol class one can take certainly depends what function class one wants to apply \mathcal{K}_σ , for the purpose of symbolic

calculus (see [49]), in the literature σ is often required to be a function of at most polynomial growth. In this thesis we work with σ from different spaces that will be specified in a few pages, but they will all have compact support.

Pseudo-differential operators can also be rewritten as an integral operators of form $H_h f(t) = \int h(t, t-s) f(s) ds$ which can be used to represent time-varying systems in engineering. We will not be using such forms in this thesis, interested readers can see, for example, [47] to get more details.

We will instead focus on the following form: a pseudo-differential operator can be viewed as weighted superposition of time frequency shifts on appropriate classes (e.g., on S_0):

$$\begin{aligned}
(\mathcal{K}_\sigma f)(x) &= \int \sigma(x, w) \hat{f}(w) e^{2\pi i x \cdot w} dw \\
&= \int \left(\int \int \hat{\sigma}(v, t) e^{2\pi i(x \cdot v + w \cdot t)} dv dt \right) \hat{f}(w) e^{2\pi i x \cdot w} dw \\
&= \int \int \int \hat{\sigma}(v, t) e^{2\pi i(x \cdot v + w \cdot t + x \cdot w)} \hat{f}(w) dv dt dw \\
&= \int \int \hat{\sigma}(v, t) e^{2\pi i x \cdot v} f(x + t) dv dt \\
&= \int \int \hat{\sigma}(v, t) (\mathcal{M}_v \mathcal{T}_{-t} f) dv dt.
\end{aligned}$$

There are various ways of justifying the interchange of dw and dt in the above computation, for example, for each fixed x and $f \in S_0(\mathbb{R})$, $\sigma \in S_0(\mathbb{R}^2)$, we may view $\int \sigma(x, w) \hat{f}(w) e^{2\pi i x \cdot w} dw$ as an L^2 inner product with respect to the w variable, and apply the inverse Fourier transform $\mathcal{F}_{w \rightarrow t}^{-1}$ to both sides in the inner product.

Regarding our topic, it is customary and actually more convenient to replace $\hat{\sigma}$ with $\mathcal{F}_s(\sigma)$ where \mathcal{F}_s is the *symplectic Fourier transform* defined as

$$\mathcal{F}_s(\sigma(x, w))(t, v) := \int \int \sigma(x, w) e^{-2\pi i(v \cdot x - t \cdot w)} dw dx,$$

so that we get the following slight different form

$$(\mathcal{K}_\sigma f)(x) = \int \int \mathcal{F}_s(\sigma)(t, v)(\mathcal{M}_v \mathcal{T}_t f) dv dt.$$

And we extend this integral formula and define a *pseudo-differential operator* \mathcal{H}_η to be

$$(\mathcal{H}_\eta g)(x) = \langle \eta, \overline{\mathcal{M}_v \mathcal{T}_t g} \rangle \quad \left(= \int \int \eta(t, v)(\mathcal{M}_v \mathcal{T}_t g(x)) dv dt \right). \quad (7)$$

whenever the dual pairing is well defined, and the integral formula applies whenever the integration is well defined and consistent with the bracket definition. Possible choices can be $\eta \in S_0(\mathbb{R}^2)$, $g \in S'_0(\mathbb{R})$.

In spite of the above technical subtlety, from time to time we will still use the integration notation in (7) for $g \in S'_0(\mathbb{R})$ instead of the bracket notation in (7), especially considering that most of time in this thesis the distribution g is a delta train, it is customary to write the integral instead of the bracket when a delta distribution is involved. Similarly for short time Fourier transforms we will sometimes also use (3) for distributions despite that it is actually (4) that is in play. The reader should keep this difference in mind.

In this thesis also η is always assumed to be the symplectic Fourier transform of the Kohn-Nirenberg symbol σ . The reason for using \mathcal{F}_s instead of \mathcal{F} is fairly subtle, those results in the subsequent sections do not depend on which form we use, but using \mathcal{F}_s is less misleading in writing. Indeed, we have the spreading function $\eta(t, v)$ and the Kohn-Nirenberg symbol $\sigma(x, w)$. If one inspects the above derivation carefully, one would see that

$$x \mapsto v, \quad w \mapsto t,$$

through the transform. Usually in time-frequency analysis, people would put the time axis as the horizontal axis (i.e., put the t variable in the front). There-

fore, defining the spreading function to be the Euclidean Fourier transform of the Kohn-Nirenberg symbol would create a confusion on how these variables align, while using the symplectic Fourier transform puts the variables in the correct order (i.e., to make the t variable, the one that represents time shifts as the horizontal axis.) since the symplectic structure $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ also swaps variables.

This definition of pseudo-differential operators naturally relates them to communication channels, since a communication channel acts on an input signal by time shifts (due to distance) and frequency shifts (i.e., the Doppler effect due to motion), both superposed with proper weights (due to various factors such as reflection, absorption etc.) represented by η .

Conventionally, η is called the *spreading function*, we will continue to use such a name in this thesis, although in many places η can actually be a distribution. The support of a spreading function is called the *spreading support*. We will call a pseudo-differential operator *overspread* if the Lebesgue measure of its spreading support is larger than 1 and *underspread* if it is smaller than 1.

The terms overspread and underspread stem from communication theory, there they usually mean that the spreading support is inside a rectangle centered at the origin and with area > 1 and < 1 respectively. Underspread essentially means the channel is highly concentrated, and is also the case for most channels. Here we chose to use a slightly different definition since it is more precise, and after our discretization (i.e., what we call rectification) in the later Subsection 3.2, our definition of underspread would mean the degree of freedom of the channel does not exceed the dimension of the input, which coincides with the definition of underspread for discretized channels in communication theory. (See [38] for a comprehensive survey on how communication theory concepts are linked to time frequency analysis tools and the corresponding history on the development of this theory)

For a fixed g , we define a map Φ_g to be

$$\Phi_g : \eta \mapsto \mathcal{H}_\eta g. \quad (8)$$

This definition allow us to extend spreading functions to bigger classes. In this thesis we will come across the following two types of extension:

The first one, we fix $g \in S'_0(\mathbb{R})$ and $U \subset \mathbb{R}^2$, if for all $\eta \in S_0(\mathbb{R}^2)$ with support in U we have

$$\|\Phi_g \eta\|_{L^2(\mathbb{R})} \lesssim \|\eta\|_{L^2(U)}, \quad (9)$$

then Φ_g is a densely defined bounded operator from $L^2(U)$ to $L^2(\mathbb{R})$, thus we can extend its domain to the entire $L^2(U)$, this is the most frequent case in this thesis where g is a periodically weighted delta train and U is a compact set with zero boundary measure, we will discuss it in detail in the next section.

The second one, we fix a compact $U \subset \mathbb{R}^2$ and $g \in S'_0(\mathbb{R})$, and let $f \in S_0(\mathbb{R})$ be arbitrary, then set

$$\Phi_g : \eta \mapsto \langle \eta, V_g f \rangle, \quad (10)$$

this definition is consistent with the previous extension when $\eta \in L^2(U)$. But $V_g f$ is actually in $W^{A,\infty}(\mathbb{R}^2)$ and thus can be tested on S'_0 distributions compactly supported on U , consequently we can further extend the domain of Φ_g (and thus the class of spreading functions) to $S'_0(U)$, this case is discussed in detail in [42] and we will come across it in Subsection 3.3 and Section 5.

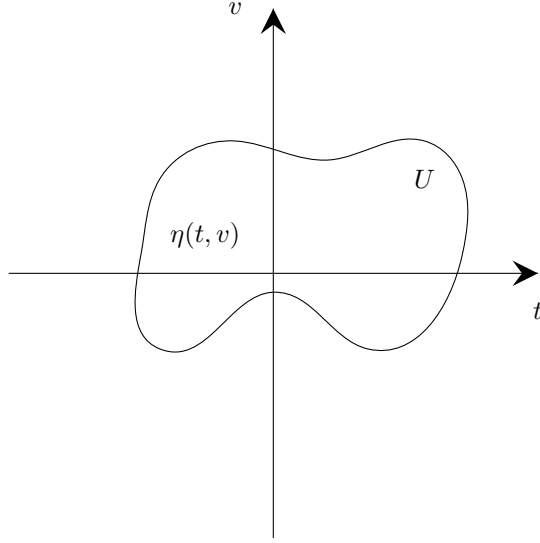


Figure 3: An Example of a Spreading Function Supported on Some Domain U

Let U be a compact or pre-compact (i.e., its closure is compact) set in \mathbb{R}^2 with zero boundary (Lebesgue) measure. In this thesis, we will consider following spaces of pseudo-differential operators, which we call **operator Paley-Wiener spaces**:

$$OPW(U, L^2) = \{\mathcal{H}_\eta : \eta \in L^2(U), \quad \text{supp}(\eta) \subset U\}.$$

$$OPW(U, S'_0) = OPW(U, W^{A', \infty}) = \{\mathcal{H}_\eta : \eta \in W^{A', \infty}(U), \quad \text{supp}(\eta) \subset U\}.$$

Norms on these spaces are defined as

$$\|\mathcal{H}_\eta\|_{OPW(U, X)} = \|\eta\|_X.$$

This makes $OPW(U, L^2)$ a Hilbert space and the rest Banach spaces. The extension in (9) and (10) allows us to work with such spaces, and in particular to test pseudo-differential operators from such spaces on S'_0 distributions.

Our aim is to study when can we find a proper input $g \in S'_0$ such that the map Φ_g defined in (8) and extended by (9) and (10) is injective from $OPW(U, X)$ to $Y(\mathbb{R})$ for proper domain U and norm topologies X, Y , and find a corresponding algorithm to reconstruct η from $\Phi_g \eta$.

3 Methodology in Operator Identification

In the late 1950s, Kailath started to analyze the problem of identifying operators with restricted time and frequency spread (see [29]). Using engineering terms, Kailath proclaimed that a collection of communication channels having common maximum delay a and common maximum Doppler shift b would be identifiable by a single input signal if and only if $ab \leq 1$, i.e., if and only if the spreading function $\hat{\sigma}$ is supported on a rectangle of area at most 1. Using a heuristic argument that the degrees of freedom of identifiable operator must subject to the degree of freedom of the input signal, Kailath asserted without proof that an identifiable operator is necessarily underspread.

Kailath's original conjecture was not accurate but has insight. In this section we will review all known major results on this topic, we also give simplified proof for some of the results. The most important purpose of this section is to introduce those fundamental tools for dealing with this topic, we will develop these existing tools in the next two sections, and then present our main contributions in Section 6.

First let us define some terminology.

Let $U \subseteq \mathbb{R}^2$, and $g \in S'_0(\mathbb{R})$, and X, Y proper topologies such that for every $\eta \in X(U)$ with $\text{supp}(\eta) \subseteq U$ (i.e., $\mathcal{H}_\eta \in OPW(U, X)$), $H_\eta g$ is in $Y(\mathbb{R})$, then we refer to the induced map Φ_g defined in (8) and extended by (9) and (10) as the **identification map**, g is then called the **identifier**, and $\Phi_g \eta$ is called the **response**.

We say $OPW(U, X)$ is **identifiable** if Φ_g is injective on $X(U)$, and we say it is **stably identifiable** if Φ_g is bounded above and below from $X(U)$ into some $Y(\mathbb{R})$. In particular, we say it is **unitarily identifiable** if Φ_g is an isometry from $L^2(U)$ to $L^2(\mathbb{R})$. Moreover, if Φ_g is injective, then its left inverse, denoted as Φ_g^{-1} , is called the **reconstruction map**.

3.1 A Motivating Example: Operator Identification on the Unit Square

In this part we are going to demonstrate the essence of operator identification by working with a simple space, namely $OPW([-1/2, 1/2]^2, L^2)$.

In the literature [30], this was shown using $\mathfrak{g}_{\bar{1}}$, i.e., the unweighted delta train supported on integers. Here we give a different computation using $\mathbf{u}_{\bar{1}}$, and provide a geometric interpretation of the identification process.

By the Poisson summation formula we easily see that

$$\mathfrak{g}_{\bar{1}} = \sum_{n \in \mathbb{Z}} \delta_n = \sum_{k \in \mathbb{Z}} e^{2\pi i k x} = \mathbf{u}_{\bar{1}},$$

thus these two methods are intrinsically same, but working with $\mathbf{u}_{\bar{1}}$ here significantly simplifies computation steps and gives a better geometric perspective.

Theorem 3.1.1. $\Phi_{\mathbf{u}_{\bar{1}}}$ is an isometry from $L^2([-1/2, 1/2]^2)$ onto $L^2(\mathbb{R})$. In particular, it maps the orthonormal exponential basis $\{e_{[-1/2, 1/2]}^{2\pi i m t} e_{[-1/2, 1/2]}^{2\pi i n v}\}_{m, n \in \mathbb{Z}}$ to the orthonormal Gabor basis $(\text{sinc } x, 1, 1)$.

Proof. Using the relation

$$\mathcal{F}(\text{sinc}(x + m)) = \mathcal{F}^{-1}(\text{sinc}(x - m)) = e_{[-1/2, 1/2]}^{2\pi i m \xi},$$

we get that the Kohn-Nirenberg symbol for the spreading function $e_{[-1/2, 1/2]}^{2\pi i m t} e_{[-1/2, 1/2]}^{2\pi i n v}$ is

$$\mathcal{F}_s^{-1}(e_{[-1/2, 1/2]}^{2\pi i m t} e_{[-1/2, 1/2]}^{2\pi i n v}) = \text{sinc}(x + n) \sin(w - m).$$

Apply the correspondence in (6) we obtain

$$(\mathcal{K}_{\sigma \mathbf{u}_{\bar{1}}})(x) = \int \sigma(x, w) \left(\sum_{k \in \mathbb{Z}} \delta_k \right) e^{2\pi i x \cdot w} dw = \sum_{k \in \mathbb{Z}} \sigma(x, k) e^{2\pi i k x}.$$

The integral is well defined for each single δ_k since $\sigma(x, w)$ is continuous, to see that the sum also makes sense, we recall the interpolation property of the sinc

function, i.e., for $k, m \in \mathbb{Z}$, $\text{sinc}(k - m)$ is non-zero if and only $k = m$, we then have

$$\begin{aligned}
(\Phi_{\mathbf{u}_{(1)}}(e_{[-1/2,1/2]}^{2\pi i m t} e_{[-1/2,1/2]}^{2\pi i n v}))(x) &= (\mathcal{H}_{e_{[-1/2,1/2]}^{2\pi i m t} e_{[-1/2,1/2]}^{2\pi i n v}} \mathbf{u}_{\bar{1}})(x) \\
&= \text{sinc}(x + n) \sum_{k \in \mathbb{Z}} \text{sinc}(k - m) e^{2\pi i k x} \\
&= \text{sinc}(x + n) e^{2\pi i m x} \\
&= \mathcal{M}_m \mathcal{T}_{-n} \text{sinc } x,
\end{aligned}$$

which shows the exponential basis is mapped to the Gabor basis. \square

From the computations above, it is clear that a reconstruction formula can be obtained by extracting the basis coefficients and then synthesizing back, i.e.,

$$\eta(t, v) = \sum_{m, n \in \mathbb{Z}} \langle \Phi_{\mathbf{u}_{\bar{1}}} \eta, \mathcal{M}_m \mathcal{T}_{-n} \text{sinc } x \rangle e_{[-1/2,1/2]}^{2\pi i m t} e_{[-1/2,1/2]}^{2\pi i n v}. \quad (11)$$

To geometrically illustrate how $\Phi_{\mathbf{u}_{\bar{1}}}$ acts on $\eta(t, v) \in L^2([-1/2, 1/2])$ we write

$$\eta(t, v) = \sum_{m, n \in \mathbb{Z}} a_{m, n} e_{[-1/2,1/2]}^{2\pi i m t} e_{[-1/2,1/2]}^{2\pi i n v} = \sum_{m \in \mathbb{Z}} \hat{f}_m(v) e_{[-1/2,1/2]}^{2\pi i m t},$$

where $a_{m, n}$ is the basis coefficient and

$$\hat{f}_m(v) = \sum_{n \in \mathbb{Z}} a_{m, n} e_{[-1/2,1/2]}^{2\pi i n v}.$$

The summation is well defined as in L^2 sense.

Repeating the computations as in the above theorem we get

$$(\Phi_{\mathbf{u}_{\bar{1}}} \eta)(x) = \sum_{m \in \mathbb{Z}} f_m(x) e^{2\pi i m x}.$$

Applying the Fourier transform to it leads to

$$\mathcal{F}(\Phi_{\mathbf{u}_{\bar{1}}} \eta)(\xi) = \sum_{m \in \mathbb{Z}} \hat{f}_m(\xi - m).$$

Notice that each \hat{f}_m is supported on $[-1/2, 1/2)$, thus

$$\text{supp}(\hat{f}_m(\xi - m)) = [m - \frac{1}{2}, m + \frac{1}{2}),$$

which means, the Fourier transform of the response, i.e., $\mathcal{F}(\Phi_{\mathbf{u}_\Gamma}\eta)(\xi)$, is tiling up the real line by the coefficient functions \hat{f}_m , as depicted by the following figure.

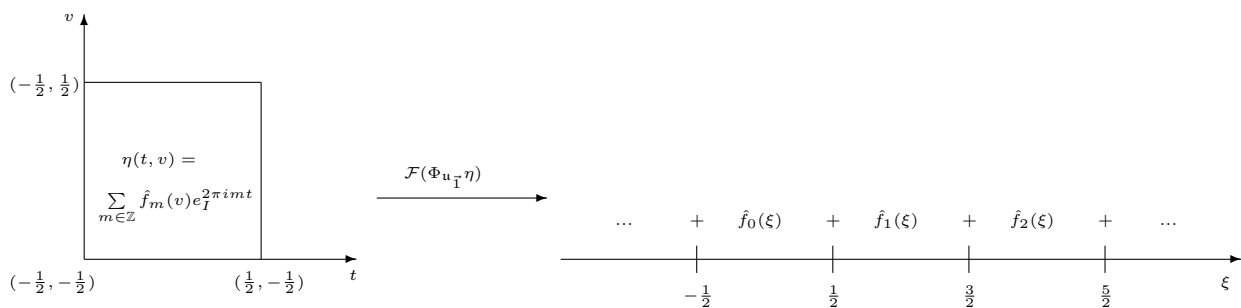


Figure 4: Identification Map on $OPW([-1/2, 1/2]^2, L^2)$

Remark: This geometric interpretation shows that the identification procedure is very sensitive to perturbation. Indeed, if for example, we replace $e^{2\pi i k x}$ in \mathbf{u}_Γ by $e^{2\pi i(k+\epsilon)x}$, then aliasing happens regardless of how small ϵ is. The situation will not improve if \mathbf{u}_Γ is viewed as \mathbf{g}_Γ and we perturbate the deltas, relevant discussion can be found in [23].

3.2 Rectifications

For a general domain U , we would like to carry the idea behind the above proof over to $OPW(U, L^2)$, the basic approach was developed in [43]. Here we provide a different and simpler derivation.

The method consists of two steps, the first step is rectification, i.e., to cover the domain U by squares of side length $1/\sqrt{N}$ with $N \in \mathbb{N}$, as illustrated by the figure below.

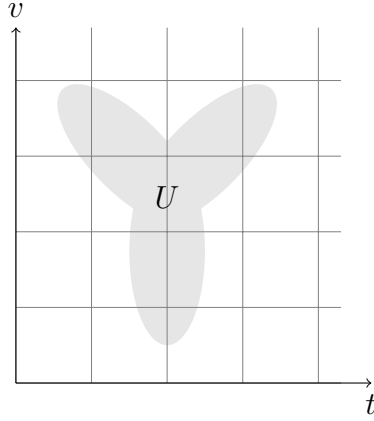


Figure 5: An Example of Rectification

To avoid pathological behavior, we shall always assume that the following holds for the underlying domain U :

1. The closure of U is compact, so that it can be covered by a single $N \times N$ square for some big enough $N \in \mathbb{N}$.
2. ∂U has Lebesgue measure 0.

Let us call such a domain U *admissible*.

If U is admissible, then we say it is *N -th rectifiable* if it can be covered by squares of size $1/\sqrt{N} \times 1/\sqrt{N}$ so that the total area of the squares needed to cover U is less than or equal to 1. We say U is *rectifiable* if U is *N -th rectifiable* for some $N \in \mathbb{N}$. And we refer to the union of those squares that covers U as *the rectification of U* .

It is easy to see that any admissible U with measure strictly less than 1 would be rectifiable.

The second step is to use a N -periodically weighted delta train $\mathbf{g}_{\vec{c}}$ for some properly chosen $\vec{c} \in \mathbb{C}^N$ as the identifier. In the sequel we will refer to the $1/\sqrt{N} \times 1/\sqrt{N}$ shaped squares used in the first step as a box.

To see the mechanism behind it, let us first take a $\vec{c} \in \mathbb{C}^N$, in the proof of Lemma 2.7.2 we have computed for the periodically weighted Zak transform $Z_{\vec{c}}$ that

$$(Z_{\vec{c}}f)(x, w) = \sum_{k=0}^{N-1} ((\bar{c}_k e^{-2\pi i \frac{k}{\sqrt{N}} w} \mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f)(x, w),$$

let us denote

$$h_k(x, w) = e^{-2\pi i \frac{k}{\sqrt{N}} w} (\mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f(x, w),$$

and define a vector

$$\vec{h} = (h_0, h_1, \dots, h_{N-1}),$$

so that

$$(Z_{\vec{c}}f)(x, w) = \sum_{k=0}^{N-1} \bar{c}_k h_k(x, w) = \langle \vec{h}, \vec{c} \rangle,$$

observe that

$$\begin{aligned} h_k(x + \frac{1}{\sqrt{N}}, w) &= (e^{-2\pi i \frac{k}{\sqrt{N}} w} (\mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f)(x + \frac{1}{\sqrt{N}}, w) \\ &= e^{-2\pi i \frac{k}{\sqrt{N}} w} (\mathcal{T}_{(-\frac{k+1}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f(x, w) \\ &= e^{2\pi i \frac{1}{\sqrt{N}} w} h_{k+1}, \end{aligned}$$

and also by Theorem 2.7.2 we have

$$\begin{aligned} h_k(x, w + \frac{1}{\sqrt{N}}) &= e^{-2\pi i \frac{k}{\sqrt{N}} (w + \frac{1}{\sqrt{N}})} (\mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f(x, w + \frac{1}{\sqrt{N}}) \\ &= e^{-2\pi i \frac{k}{\sqrt{N}} w} e^{-2\pi i \frac{k}{N}} (\mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f(x, w) \\ &= e^{-2\pi i \frac{k}{N}} h_k(x, w), \end{aligned}$$

Consequently we get

$$(Z_{\vec{c}}f)(x + \frac{1}{\sqrt{N}}, w) = \sum_{k=0}^{N-1} \bar{c}_k e^{2\pi i \frac{w}{\sqrt{N}}} h_{k+1}(x, w) = e^{2\pi i \frac{w}{\sqrt{N}}} \langle T_N^{-1} \vec{h}, \vec{c} \rangle,$$

and

$$(Z_{\vec{c}}f)(x, w + \frac{1}{\sqrt{N}}) = \sum_{k=0}^{N-1} \bar{c}_k e^{-2\pi i \frac{k}{N}} h_k(x, w) = \langle M_N^{-1} \vec{h}, \vec{c} \rangle,$$

i.e., if we move horizontally by one box (whose side length is $1/\sqrt{N}$), then the weights \bar{c}_k are cyclically shifted in the sum up to an extra scaling factor; and if we move vertically by one box, then each summand is multiplied by a phase factor $e^{-2\pi i k/N}$.

Fix a box in the rectification and mark it as $U_{(0,0)}$, and set

$$U_{(j,k)} = U_{(0,0)} + (j, k).$$

i.e., $U_{(j,k)}$ is the set obtained by shifting every point in $U_{(0,0)}$ by (j, k) .

The figure below is an example for this setting:

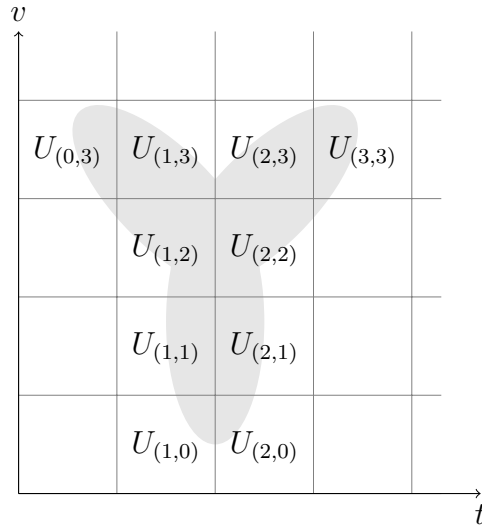


Figure 6: An Example of Rectified and Indexed Domain

Now as in the above, we set

$$h_k(x, w) = e^{-2\pi i \frac{k}{\sqrt{N}} w} (\mathcal{T}_{(-\frac{k}{\sqrt{N}}, 0)} Z_{\sqrt{N}}) f(x, w) \chi_{U_{(0,0)}},$$

and write the functions into vectors, i.e.,

$$\vec{h} = (h_0, h_1, h_2, \dots, h_{N-1}),$$

then the above derivation shows we can write the value of $Z_{\vec{c}}f$ in any $U_{(j,k)}$ as an inner product, i.e.,

$$\chi_{U_{(j,k)}}(Z_{\vec{c}}f)(x, w) = e^{2\pi i \frac{jw}{\sqrt{N}}} \left\langle \vec{h}, M_N^k T_N^j \vec{c} \right\rangle.$$

If we define the set

$$\Gamma = \{(j, k) : U_{(j,k)} \text{ covers } U\},$$

and write $Z_{\vec{c}}f$ also into a vector \vec{z} :

$$\vec{z} = (\chi_{U_{(j_1, k_1)}} Z_{\vec{c}}f, \chi_{U_{(j_2, k_2)}} Z_{\vec{c}}f, \dots),$$

where $(j_1, k_1), (j_2, k_2) \dots \in \Gamma$ and are ordered by the lexicographical ordering, then \vec{z}, \vec{h} are connected via the adjoint of the Gabor matrix $G_{\Gamma}(\vec{c})$, i.e.,

$$\vec{z} = D_{\Gamma} G_{\Gamma}^*(\vec{c}) \vec{h},$$

where

$$D_{\Gamma} = \text{diag}(e^{-2\pi i \frac{j_1}{\sqrt{N}} w}, e^{-2\pi i \frac{j_2}{\sqrt{N}} w}, \dots, e^{-2\pi i \frac{j_N}{\sqrt{N}} w}).$$

Moreover, we notice that $e^{2\pi i k w / \sqrt{N}} h_k$ is simply $Z_{\sqrt{N}}f$ restricted to the box $[k/\sqrt{N}, (k+1)/\sqrt{N}) \times [0, 1/\sqrt{N})$, as depicted by the following figure:

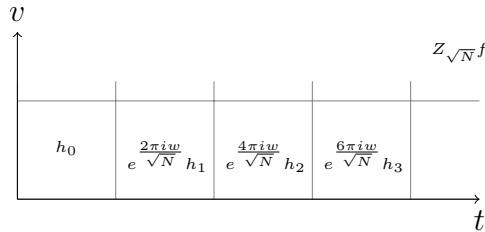


Figure 7: h_k and $Z_{\sqrt{N}}f$

Combing the above derivations we can rewrite $\chi_U Z_{\vec{c}}$ on as a linear transformation

(induced by the Gabor matrix $G_\Gamma^*(\vec{c})$) applied to $Z_{\sqrt{N}}$:

Theorem 3.2.1.

$$\chi_U Z_{\vec{c}} = S_\Gamma D_\Gamma G_\Gamma^*(\vec{c}) D_{\sqrt{N}} A_{\sqrt{N}} Z_{\sqrt{N}}$$

where D_Γ is as defined above,

$$A_{\sqrt{N}} : \eta \mapsto \begin{pmatrix} \chi_{(0, \frac{1}{\sqrt{N}}) \times [0, \frac{1}{\sqrt{N}})} \eta \\ \vdots \\ \chi_{[\frac{k}{\sqrt{N}}, \frac{k+1}{\sqrt{N}}) \times [0, \frac{1}{\sqrt{N}})} \eta \\ \vdots \\ \chi_{[\frac{N-1}{\sqrt{N}}, \sqrt{N}) \times [0, \frac{1}{\sqrt{N}})} \eta \end{pmatrix}$$

is the analysis operator that vectorizes functions or distributions supported on \mathbb{R}^2 , the k -th entry in $A_{\sqrt{N}} \eta$ is just the restriction of η onto the square $[\frac{k}{\sqrt{N}}, \frac{k+1}{\sqrt{N}}) \times [0, \frac{1}{\sqrt{N}})$,

$$D_{\sqrt{N}} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & e^{-2\pi i \frac{k-1}{\sqrt{N}} w} & & \\ & & & \ddots & \\ & & & & e^{-2\pi i \frac{N-1}{\sqrt{N}} w} \end{pmatrix},$$

and

$$S_\Gamma : (f_{(j,k)})_{(j,k) \in \Gamma} \mapsto \eta \text{ such that } \chi_{U_{(j,k)}} \eta = f_{(j,k)}$$

is the synthesis operator that puts the vectorized functions or distributions back to the function or distribution supported on the rectification represented by Γ . And entries in $(f_{(j,k)})_{(j,k) \in \Gamma}$ are arranged by the lexicographic ordering on (j, k) , same as the ordering of columns in the Gabor matrix $G_\Gamma(\vec{c})$.

The following two figures depict the action of $A_{\sqrt{N}}$ and S_Γ (Remember that we always take the lexicographic order in Γ).

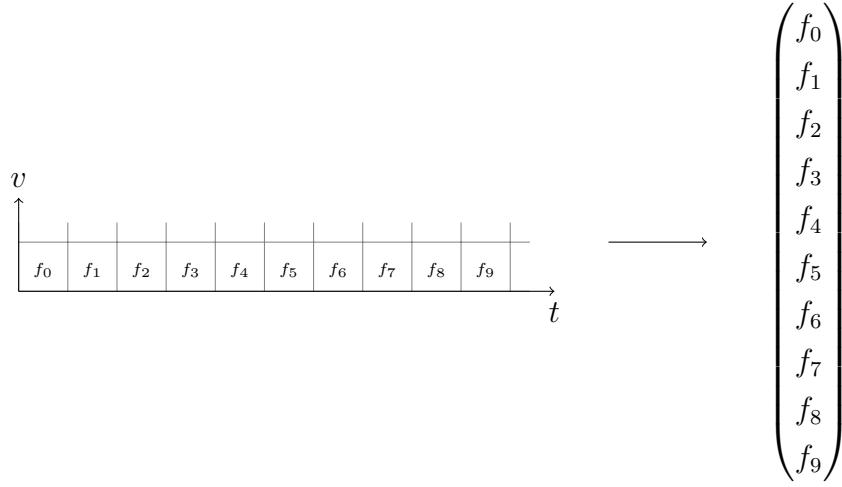


Figure 8: The Analysis Operator $A_{\sqrt{N}}$

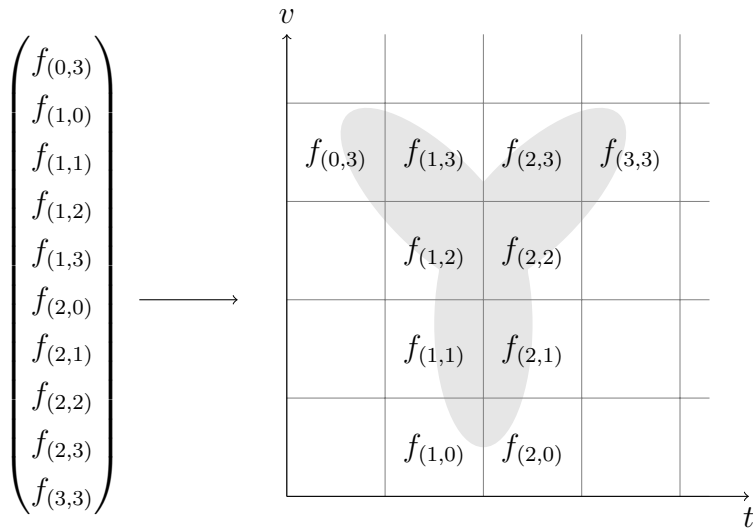


Figure 9: The Synthesis Operator S_{Γ}

This formula plays a core role in our analysis, and it immediately leads to several important consequences

Corollary 3.2.1. *Let $\vec{c} \in \mathbb{C}^N$, $U, \Gamma, D_{\sqrt{N}}, S_{\sqrt{N}}, A_{\sqrt{N}}$ be defined as in the above*

theorem, denote $\vec{1}_N = (1, 0, \dots, 0) \in \mathbb{C}^N$ so that

$$\mathfrak{g}_{\vec{1}_N} = \sum_{k \in \sqrt{N}\mathbb{Z}} \delta_k,$$

is the unweighted delta train supported on $\sqrt{N}\mathbb{Z}$, then we have

$$\chi_U V_{\mathfrak{g}_\varepsilon} = S_\Gamma D_\Gamma G_\Gamma^*(\vec{c}) D_{\sqrt{N}} A_{\sqrt{N}} V_{\mathfrak{g}_{\vec{1}_N}},$$

Proof. Apply Lemma 2.7.1, we get

$$V_{\mathfrak{g}_\varepsilon}(t, v) = e^{-2\pi i t \cdot v} Z_{\frac{1}{\sqrt{N}}}(t, v).$$

$$V_{\mathfrak{g}_{\vec{1}_N}}(t, v) = e^{-2\pi i t \cdot v} Z_{\sqrt{N}}(t, v).$$

Substituting corresponding entries in Theorem 3.2.1 and then dividing both sides by $e^{2\pi i t \cdot v}$ (since this is never 0) leads to the result. \square

From its derivation one can see this proposition holds when $\chi_U V_{\mathfrak{g}_\varepsilon}$ is applied to both $L^2(\mathbb{R})$ and $S_0(\mathbb{R})$ cases.

In the remaining part of this thesis, given a rectification $\Gamma \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$, and a function or distribution η , let $S_{\sqrt{N}}$ as the synthesis operator defined above, we refer to the vector $S_{\sqrt{N}}^{-1}\eta$ as the **vectorization** of η , and denote it as $\vec{\eta}$.

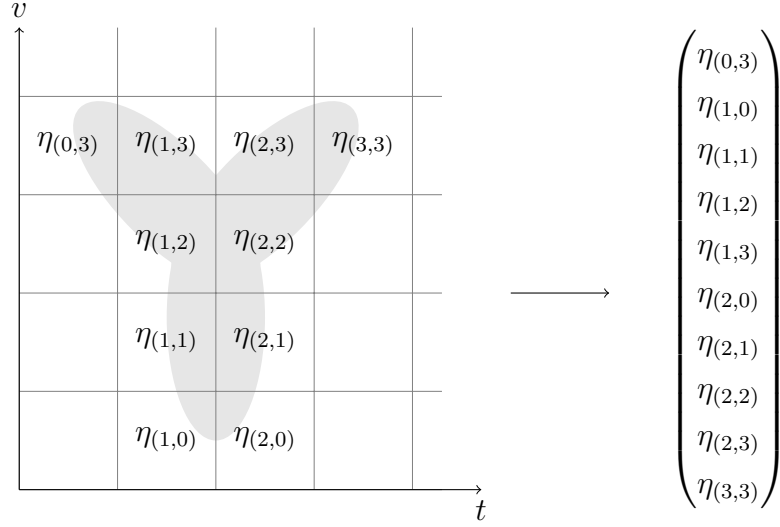


Figure 10: Vectorization of η

3.3 The Adjoint Relation

One easily notices the similarity between Figure 2 and Figure 4. Indeed, from the geometric point of view, the reconstruction formula (11) can also be given via the Zak transform

$$\begin{aligned}
\eta(t, v) &= \chi_{[-1/2, 1/2]^2} \sum_{m \in \mathbb{Z}} \mathcal{F}(\Phi_{u_{\bar{\Gamma}}}\eta)(v + m) e^{2\pi i m t} \\
&= \chi_{[-1/2, 1/2]^2} Z_1(\mathcal{F}(\Phi_{u_{\bar{\Gamma}}}\eta))(v, -t) \\
&= \chi_{[-1/2, 1/2]^2} e^{-2\pi i t v} (Z_1(\Phi_{u_{\bar{\Gamma}}}\eta))(t, v) \\
&= \chi_{[-1/2, 1/2]^2} (V_{g_{\bar{\Gamma}}}(\Phi_{u_{\bar{\Gamma}}}\eta))(t, v),
\end{aligned}$$

where the last step follows from Lemma 2.7.1.

Since $\Phi_{g_{\bar{\Gamma}}}$ (i.e., $\Phi_{u_{\bar{\Gamma}}}$) is unitary from $L^2([-1/2, 1/2]^2)$ to $L^2(\mathbb{R})$, its inverse is also its adjoint, thus the following would holds for any $\eta \in L^2([-1/2, 1/2]^2)$ and $f \in L^2(\mathbb{R})$:

$$\langle \Phi_{g_{\bar{\Gamma}}}\eta, f \rangle = \langle \eta, \chi_{[-1/2, 1/2]^2} V_{g_{\bar{\Gamma}}} f \rangle = \langle \eta, V_{g_{\bar{\Gamma}}} f \rangle.$$

In general, let $a > 0$, and replace the unit square in Subsection 3.1 with the rectangle $[0, a) \times [0, 1/a)$ and replace the corresponding exponential basis accordingly, then repeating the computations there and above will lead us to

Lemma 3.3.1. *Let $\eta \in L^2([0, a) \times [0, 1/a))$ and $\text{supp}(\eta) \subseteq [0, a) \times [0, 1/a)$, and*

$$g_a = \sum_{k \in \mathbb{Z}} \delta_{ka},$$

be the unweighted delta train supported on $a\mathbb{Z}$, then we have

$$\langle \Phi_{g_a} \eta, f \rangle = \langle \eta, \chi_{[0, a) \times [0, 1/a)} V_{g_a} f \rangle = \langle \eta, V_{g_a} f \rangle.$$

Consequently

Corollary 3.3.1. *Let $\vec{c} \in \mathbb{C}^N$, $U, \Gamma, D_{\sqrt{N}}, S_{\sqrt{N}}, A_{\sqrt{N}}, \mathfrak{g}_{\Gamma_N}$ be defined as in Corollary 3.2.1, then we have*

$$\Phi_{\mathfrak{g}_{\vec{c}}} \eta = \Phi_{\mathfrak{g}_{\Gamma_N}} A_{\sqrt{N}}^* D_{\sqrt{N}}^* G_{\Gamma}(\vec{c}) D_{\Gamma}^* S_{\Gamma}^* \eta,$$

holds for any $\eta \in L^2(U)$ with $\text{supp}(\eta) \subseteq U$.

Proof. This follows by using a similar computation as in Theorem 3.2.1, we write $\mathfrak{g}_{\vec{c}}$ as the sum of N unweighted delta trains and then apply Lemma 3.3.1 and Corollary 3.2.1. \square

It follows that

Corollary 3.3.2. *If U is N -th rectifiable and $\vec{c} \in \mathbb{C}^N$, then*

$$\|\Phi_{\mathfrak{g}_{\vec{c}}}\|_{L^2(U) \rightarrow L^2(\mathbb{R})} = \frac{1}{\sqrt[4]{N}} \|G_{\Gamma}(\vec{c})\|_{\ell^2 \rightarrow \ell^2}.$$

Proof. It follows from the adjoint relation that

$$\|\Phi_{\mathfrak{g}_{\vec{c}}}\|_{L^2(U) \rightarrow L^2(\mathbb{R})} = \|S_{\Gamma} D_{\Gamma} G_{\Gamma}^*(\vec{c}) D_{\sqrt{N}} A_{\sqrt{N}} V_{\mathfrak{g}_{\Gamma_N}}\|_{L^2(\mathbb{R}) \rightarrow L^2(U)}.$$

Apparently, $D_{\sqrt{N}}, D_{\Gamma}$ are unitary. If we denote the k -th element in the vector $A_{\sqrt{N}} \eta$ as η_k , i.e., if

$$A_{\sqrt{N}} \eta = (\eta_1, \eta_2, \dots, \eta_N),$$

then

$$\|\eta\|_{L^2}^2 = \|\eta_1\|_{L^2}^2 + \|\eta_2\|_{L^2}^2 + \dots + \|\eta_N\|_{L^2}^2,$$

and similarly if

$$\vec{\eta} = (\eta_1, \eta_2, \dots, \eta_N),$$

then

$$\|\eta_1\|_{L^2}^2 + \|\eta_2\|_{L^2}^2 + \dots + \|\eta_N\|_{L^2}^2 = \|S_\Gamma \vec{\eta}\|_{L^2}^2,$$

And also by Lemma 2.7.1 and Theorem 2.7.1, $\sqrt[4]{N}V_{\mathfrak{g}_{\Gamma_N}}$, which up to a phase factor is equivalent to $\sqrt[4]{N}Z_{\sqrt{N}}$ and thus, is unitary from $L^2(\mathbb{R})$ to $L^2([0, \sqrt{N}) \times [0, 1/\sqrt{N}))$.

Combining all these together leads to the result. \square

The decomposition in Corollary 3.3.1 essentially reduces the action of the identification map to $G_\Gamma(\vec{c})\vec{\eta}$, which also corresponds to communication theory where one can define a discrete channel also as superposition of discrete time-frequency shifts (for example, see [41, 21, 48, 39]), i.e.,

$$H_\eta = \vec{\eta}_1 M^{j_1} T^{k_1} + \vec{\eta}_2 M^{j_2} T^{k_2} + \dots + \vec{\eta}_N M^{j_N} T^{k_N},$$

then such a channel acts on a discrete input \vec{c} as

$$H_\eta \vec{c} = \vec{\eta}_1 M^{j_1} T^{k_1} \vec{c} + \vec{\eta}_2 M^{j_2} T^{k_2} \vec{c} + \dots + \vec{\eta}_N M^{j_N} T^{k_N} \vec{c} = G_\Gamma(\vec{c})\vec{\eta},$$

which coincides with the spirit of our decomposition if we set $\Gamma = \{(j_1, k_1), (j_2, k_2), \dots, (j_N, k_N)\}$.

Corollary 3.2.1 and 3.3.1 show V_g and Φ_g have symmetric forms for periodically weighted delta trains g , which means

Theorem 3.3.1. *If g is an N -periodically weighted delta train, then for any $\eta \in L^2(U)$ with $\text{supp}(\eta) \subseteq U$ and $f \in L^2(\mathbb{R})$ we have*

$$\langle \Phi_g \eta, f \rangle = \langle \eta, V_g f \rangle.$$

In fact, by switching the topology, one can even extend g to be more than delta

trains:

Theorem 3.3.2. [42] *If U is admissible, then for any $\eta \in S'_0(U)$, $f \in S_0(\mathbb{R})$, $g \in S'_0(\mathbb{R})$ we have*

$$\langle \Phi_g \eta, f \rangle = \langle \eta, V_g f \rangle.$$

An immediate consequence of Theorem 3.3.2 is the following

Corollary 3.3.3. *If U is admissible, $g \in S'_0(\mathbb{R})$ induces bounded operators Φ_g and V_g between $L^2(U)$ and $L^2(\mathbb{R})$, then for any fixed $\eta \in L^2(U)$ with $\text{supp}(\eta) \subseteq U$ and $f \in L^2(\mathbb{R})$, then we have*

$$\langle \Phi_g \eta, f \rangle = \langle \eta, V_g f \rangle.$$

Proof. To see they are equal, first we consider $f \in S_0(\mathbb{R})$, and view $\eta \in L^2(U)$ as an $S'_0(U)$ distribution, then by the previous theorem we have

$$\langle \Phi_g \eta, f \rangle = \langle \eta, V_g f \rangle.$$

Now for any $f \in L^2(\mathbb{R})$, since S_0 is dense in L^2 , for any small $\epsilon > 0$, we may choose $\tilde{f} \in S_0$ so that

$$\|\tilde{f} - f\|_{L^2} \leq \epsilon$$

then we apply our assumption that Φ_g and V_g are bounded operators between $L^2(U)$ and $L^2(\mathbb{R})$ to get

$$\begin{aligned} |\langle \Phi_g \eta, f \rangle - \langle \eta, V_g f \rangle| &= |\langle \Phi_g \eta, \tilde{f} \rangle + \langle \Phi_g \eta, f - \tilde{f} \rangle - \langle \eta, V_g \tilde{f} \rangle - \langle \eta, V_g(f - \tilde{f}) \rangle| \\ &= |\langle \Phi_g \eta, f - \tilde{f} \rangle - \langle \eta, V_g(f - \tilde{f}) \rangle| \\ &\leq \|\Phi_g \eta\|_{L^2} \|f - \tilde{f}\|_{L^2} + \|\eta\|_{L^2} \|V_g(f - \tilde{f})\|_{L^2} \\ &\lesssim \|\eta\|_{L^2} \|f - \tilde{f}\|_{L^2} \\ &\lesssim \epsilon \|\eta\|_{L^2}, \end{aligned}$$

which shows the left hand side must be 0 since ϵ can be arbitrarily small. \square

One should be aware that V_g here is the adjoint of the identification map Φ_g , it is not the adjoint of the pseudo differential operator. The adjoint of a pseudo differential operator, takes a different form, see [12, Chapter 1.6] for a formula for

the corresponding spreading function, or [25] for formulas for other properties such as composition, transpose.

Properly applying this relation provides better insights, for example the proof for Theorem 3.1.1 can apparently be significantly simplified to one line using the adjoint relation. Indeed, since the Zak transform Z_1 (which is, up to an exponential factor, equivalent to the adjoint $V_{\mathfrak{g}_\Gamma}$ is an isometry from $L^2(\mathbb{R})$ onto $L^2([-1/2, 1/2]^2)$, we immediately get that $\Phi_{\mathfrak{g}_\Gamma}$ is an isometry from $L^2([-1/2, 1/2]^2)$ onto $L^2(\mathbb{R})$.

In fact, it is not surprising to have this relation if one compares the bracket definition in (4) and (7), it actually has a more profound root: Take η to be a distribution, and proper f, g so that $V_g f$ is a test function, then the Schwartz kernel theorem (see [24]) says

$$\langle \eta, V_g f \rangle = \langle \Phi_g \eta, f \rangle,$$

one can then apply similar arguments as in Corollary 3.3.3 to extend it to other topologies and thus establishing adjoint relations. For example, see [18, Corollary 2.6] for a version on S_0 .

Remark: If U is admissible, then even if $\mu(U) > 1$, one can still embed it into a $N \times N$ square for sufficiently large N , and apply the rectification technique, thus the decomposition results in Corollary 3.2.1, 3.3.1 and 3.3.2 also holds for overspread cases, except that in such cases it is obvious that the identification map induced by periodically weighted delta trains will not be injective, since the Gabor matrix would have more columns than its rows and thus have a kernel.

3.4 Sufficient and Necessary Conditions

Corollary 3.3.1 shows the Gabor matrix $G_\Gamma(\vec{c})$ plays a critical role in the identification of rectifiable domains, for example $\Phi_{\mathfrak{g}_\vec{c}}$ is injective if and only if $G_\Gamma(\vec{c})$ has full rank, now the question becomes whether such matrices exist. It turns out that not only do they exist, but also there are rich choices of them.

A matrix is said to have **spark** L if any of its $L - 1$ columns are linearly independent. It is said to have **full spark** if it is full rank and its spark equals its rank+1. The following theorem shows choices for good window vectors are rich:

Theorem 3.4.1. [34, 37] *For any $\Gamma \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$, the set of \vec{c} that makes $G_\Gamma(\vec{c})$ full spark is open dense in \mathbb{C}^N .*

It immediately follows that

Theorem 3.4.2 (Sufficient Condition). [43] *If U is N -th rectifiable, then there exists $\vec{c} \in \mathbb{C}^N$ such that $\mathfrak{g}_{\vec{c}}$ stably identifies $OPW(U, L^2)$.*

It is easy to see from this condition that $OPW(U, L^2)$ is identifiable if U is admissible and has area strictly less than 1, since such domains are N -th rectifiable for big enough N . On the other hand, as Kailath original conjectured, identification is not possible if the area of the rectangle spreading support is larger than 1, this was proved in [30].

Theorem 3.4.3. [30] *If U is a rectangle of area larger than 1, then $OPW(U, L^2)$ is not identifiable by any $g \in S'_0(\mathbb{R})$.*

The essence of the proof relies on the following fact, if U is a rectangle of area $s > 1$, then $L^2(U)$ can be expanded by some exponential basis with density s , but $L^2(\mathbb{R})$ can be expanded by a Gaussian windowed Gabor frame whose density is larger than 1 but smaller than s , then with some computation one shows that Φ_g can not map a high density basis to a low density frame without having a kernel.

The assertion in Theorem 3.4.3 extends also to non-rectangle spreading support cases using rectifications, as proved in [43], the principles behind the proof is same as in the proof of the above theorem, one simply replace the exponential basis with a Gabor basis induced from the rectification.

Theorem 3.4.4 (Necessary Condition). [43] *If U is a domain whose interior has Lebesgue measure larger than 1, then $OPW(U, L^2)$ is not identifiable.*

At this point, a typical question that would come to a reader's mind is whether we can replace our identifiers, i.e., delta trains, with something more intuitive. In reality we can only work with finite objects, therefore it would be more favorable to have a better localized identifier so that one can truncate without losing too much. Unfortunately this is not possible, the following is from [31, 3.1], which shows identifiers are not weak* localized, in other words, they can not decay in weak*.

Theorem 3.4.5 (Identifiers Can Not Decay). *If $g \in S'_0(\mathbb{R})$ is an identifier for some OPW space, and $0 \neq f \in S_0(\mathbb{R})$ is arbitrary, then as $x \rightarrow \pm\infty$, we have*

$$\langle T_x f, g \rangle \not\rightarrow 0, \quad \langle T_x f, \hat{g} \rangle \not\rightarrow 0.$$

The proof is not difficult, one simply notices that if η separates, i.e.,

$$\eta(t, v) = f(t)h(v),$$

for some f, h , then it is straightforward from the definition that H_η reduces to a product convolution operator

$$(\mathcal{H}_\eta g)(x) = \int \int f(t)h(v)(\mathcal{M}_v \mathcal{T}_t g(x)) dv dt = \check{h}(x) \langle T_x \tilde{f}, g \rangle,$$

where $\tilde{f}(t) = f(-t)$. If $\langle T_x \tilde{f}, g \rangle \rightarrow 0$ as x approaches infinity, then the right hand side can be made arbitrarily small by properly choosing \check{h} (e.g., take a sequence \check{h}_n , with $\check{h}_n = T\check{h}_{n-1}$, then $\check{h}_n(x) \langle T_x \tilde{f}, g \rangle \rightarrow 0$ as $n, x \rightarrow \infty$), which contradicts the fact that g is an identifier.

Remark:

The following results first discussed in [30] does not improve the sufficient and necessary conditions above, but by including operations like shifting, rotating and flipping the coordinate system, it allows us to, in the identification procedure, reduce complicated situations to simpler ones.

For every dimension $2n$, the **Symplectic Group** over the reals, denoted as

$SP(2n, \mathbb{R})$, is the group of all $2n \times 2n$ real matrices A equipped with matrix multiplication as the group operation, and satisfies

$$A^*JA = J,$$

where (I is the $n \times n$ identity matrix)

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

stands for the symplectic form. In other words, the symplectic group consists of all transforms that keeps the symplectic form invariant. It directly follows from this definition that $SP(2, \mathbb{R})$ consists of all 2×2 matrices with determinant 1.

Since the spreading function η is the symplectic Fourier transform of the the Kohn-Nirenberg symbol, it is not surprising that the following two results holds:

Theorem 3.4.6. [30] *If $A \in Sp(2, \mathbb{R})$, then there exists a unitary operator O_A such that*

$$\Phi_g \eta = \Phi_{O_A(g)}(\eta \circ A).$$

Here (and only here in this thesis) \circ means composition, In other parts of this thesis \circ are frequently used to denote the Hadamard product.

Proposition 3.4.1. [30] *Let $A \in Sp(2, \mathbb{R})$, and X be some topology, if $OPW(U, X)$ is identifiable, then $OPW(A(U), X)$ is also identifiable.*

Hence any coordinate transform that keeps the area invariant also keeps identifiability, in particular, $OPW(U, L^2)$ is identifiable if it is a rotated rectangle of area not larger than 1 or a parallelogram with area not larger than 1 (see [30] for detailed algorithms for identifying parallelograms).

One of the main purpose of this research is to study the critical case where the area of U is precisely 1, we have already seen in Subsections 3.1 and 3.3 that this is possible for admissible domains. The above proposition shows identifiability extends to symplectic transform of admissible domains, which, in particular, includes any

fundamental domain (of some lattice) with determinant 1. However, identifiability is largely unknown if U is not among these cases. Our basic strategy to attack this problem is to approximate the domain from inside where each approximation is N -rectifiable, and then pass to limit by letting $N \rightarrow \infty$.

4 Discrete Gabor Analysis

This section contains technical results that are necessary to present some of our main results in Section 6. As shown in Corollary 3.3.1 and 3.3.2, identifiability on $OPW(U, L^2)$ using a periodically weighted delta as the identifier critically depends on the spectral property of the Gabor matrix, which motivates us to thoroughly inspect it in this section. Results in Subsections 4.1, 4.2, 4.3 are repeated from the author's paper [44], while results in Subsection 4.5 are repeated from the author's paper [35].

4.1 Group Structures of Discrete Time-Frequency Shifts

The finite Heisenberg group of order N^3 consists of tuples (h, j, k) with $h, j, k \in \mathbb{Z}_N$ and the group law

$$(h, j, k) \odot (h', j', k') \mapsto (h + h' - kj', j + j', k + k'),$$

hence it can be viewed as the semiproduct of the additive group \mathbb{Z}_N

$$(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes_{\gamma} \mathbb{Z}_N,$$

where

$$\gamma : \mathbb{Z}_N \mapsto \text{Aut}(\mathbb{Z}_N \times \mathbb{Z}_N),$$

is given as

$$(\gamma(k))((h', j')) = (h' - kj', j').$$

A concise list of properties of the finite Heisenberg group can be found in [32].

It is easy to see using Lemma 2.3.1 that

$$\rho : (h, j, k) \mapsto \omega_N^h M_N^j T_N^k,$$

is a representation of this finite group. Denote \mathbb{H} as the above group represented

by $\{\omega_N^h M_N^j T_N^k\}_{h,j,k=0,1,2,\dots,N-1}$. Then by Lemma 2.3.1, the center of \mathbb{H} is

$$Z(\mathbb{H}) = \{w^h I\}_{h=0,1,2,\dots,N-1} = \{\rho(h, 0, 0)\}_{h=0,1,2,\dots,N-1}.$$

and

$$\mathbb{H}/Z(\mathbb{H}) \cong \mathbb{Z}_N \times \mathbb{Z}_N.$$

From now on we use the notation (MT, Γ) with

$$(MT, \Gamma) = \{M_N^j T_N^k, (j, k) \in \Gamma \subseteq \mathbb{Z}_N \times \mathbb{Z}_N\},$$

to denote the discrete time frequency shifts supported on Γ and similarly

$$(TM, \Gamma) = \{T_N^j M_N^k, (j, k) \in \Gamma \subseteq \mathbb{Z}_N \times \mathbb{Z}_N\}.$$

The following is a straightforward consequence of the Sylow theorems (for example, see exercises in [27]), for the convenience of the reader we include a simple proof here

Lemma 4.1.1. *If N is a prime number, then the only subgroups in $\mathbb{Z}_N \times \mathbb{Z}_N$ are*

$$V_s = \begin{cases} \{(ks, k)\}_{k=0,1,2,\dots,N-1}, & s = 0, 1, 2, \dots, N-1 \\ \{(j, 0)\}_{j=0,1,2,\dots,N-1}, & s = \infty \end{cases}$$

and they pairwise intersect trivially.

Proof. Clearly each V_s is a subgroup of order N . Since N is prime, the only divisors of $|\mathbb{Z}_N \times \mathbb{Z}_N| = N^2$ are $1, N, N^2$, therefore any two proper subgroups must intersect trivially, and thus counting the distinct elements from each V_s shows that there can be no more subgroups other than listed ones. \square

The subscript s has a geometric meaning that will be clear in Subsection 4.3.

One may also verify using Lemma 2.3.1 that for each fixed s , members in (MT, V_s) mutually commute. If V is a subgroup of $\mathbb{Z}_N \times \mathbb{Z}_n$ with order N , and members in (MT, V) commute, then we call V an **isotropic subgroup** (since they jointly fix

their shared eigenvector). They will play a special role in the next subsection. If N is prime, the above listed V_s are only such groups, while if N is not prime, then there might exist other isotropic subgroups. The following lemma shows having mutually commuting members is indeed sufficient for being a subgroup:

Lemma 4.1.2. *If (MT, V) is a maximal abelian subgroup in \mathbb{H} , then V is a subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$.*

Proof. Identity is contained in (MT, V) since it commutes with any other member.

If $(j, k), (j', k'), (m, n) \in V$, by the commuting relation and Lemma 2.3.1 we have

$$km = jn \pmod{N},$$

and

$$k'm = j'n \pmod{N},$$

consequently

$$(k + k')m = (j + j')n,$$

which means $M_N^{j+j'}T_N^{k+k'}$ commutes with $M_N^mT_N^n$. Since $(j, k), (j', k'), (m, n)$ are arbitrary, this implies V is closed under the group operation.

Finally

$$km = jn \quad \Rightarrow \quad -km = -jn,$$

which implies V is closed under taking the inverse in $\mathbb{Z}_N \times \mathbb{Z}_N$.

Combining the above, one may conclude that V is a subgroup. □

Moreover, isotropic subgroups cover the whole group:

Lemma 4.1.3. *1) If $V \subset \mathbb{Z}_N \times \mathbb{Z}_N$ is a cyclic subgroup of order N , then (MT, V) is isotropic.*

2) Any element of $\mathbb{Z}_N \times \mathbb{Z}_N$ is contained in at least one of its isotropic subgroups.

Proof. For 1), let (j, k) be the generator of the subgroup, then Lemma 2.3.1 shows $M_N^j T_N^k$ commutes with $M_N^{nj} T_N^{nk}$.

For 2), first we notice that if a, b coprime, then (a, b) has order N . Indeed, the order must divide N , thus we can write it as N/p for some divisor p of N , then (a, b) having order N/p implies both aN/p and bN/p are multiples of N , i.e., p divides both a, b , but a, b coprime, thus $p = 1$.

Therefore given any (j, k) , the element $(j/\gcd(j, k), k/\gcd(j, k))$ has order N and generates a cyclic subgroup of order N , this subgroup is isotropic by 1), and it contains (j, k) . \square

Here we compute explicitly the diagonalization formulas for (MT, V_s) :

Lemma 4.1.4 (Eigenstructure - Odd Dimension). *Let D be the diagonal matrix in which the ℓ -th entry on the main diagonal is*

$$D_{\ell\ell} = \omega_N^{0+1+2+\dots+(\ell-1)}.$$

Then for N odd, we have:

- 1) *The eigenvectors of $M_N^1, M_N^2, \dots, M_N^{N-1}$ are precisely the Euclidean column basis.*
- 2) *For $s = 0, 1, 2, \dots, N-1$ and $k = 1, 2, \dots, N-1$, each $M_N^{ks} T_N^k \in V_s$ can be diagonalized as*

$$M_N^{ks} T_N^k = \omega_N^{-\frac{k(k-1)s}{2}} D^s W_N M_N^{-k} W_N^* (D^s)^*.$$

Proof. 1) is trivial. For 2), for a vector $x = (x_1, x_2, \dots, x_N)$, if we denote

$$D_x = \text{diag}(x_1, x_2, \dots, x_N),$$

then for any odd N ,

$$\begin{aligned}
M_N^1 D_x T_N^1 &= \begin{pmatrix} & & & & x_1 \\ \omega_N^1 x_2 & & & & \\ & \omega_N^2 x_3 & & & \\ & & \ddots & & \\ & & & \omega_N^{N-2} x_{N-1} & \\ & & & & \omega_N^{N-1} x_N \end{pmatrix} \\
&= \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \frac{h(h-1)}{\omega_N^2} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} x_2 \\ \ddots \\ x_{N-1} \\ x_N \end{pmatrix} \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \frac{h(h-1)}{\omega_N^2} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \\
&= D(D_x T_N^1) D^*.
\end{aligned}$$

Inductively apply this formula, we get that

$$M_N^s T_N^1 = M_N^1 (M_N^{s-1} T_N^1) = D(M_N^{s-1} T_N^1) D^* = \dots = D^s T_N^1 (D^s)^* = D^s W_N M_N^{-1} W_N^* (D^s)^*.$$

Then we may apply Lemma 2.3.1 to get

$$M_N^{ks} T_N^k = \omega_N^{-(s+2s+\dots+(k-1)s)} \underbrace{(M_N^s T_N^1)(M_N^s T_N^1) \dots (M_N^s T_N^1)}_{k \text{ times}} = \omega_N^{-\frac{k(k-1)s}{2}} D^s W_N M_N^{-k} W_N^* (D^s)^*.$$

□

Lemma 4.1.5 (Eigenstructure - Even Dimension). *Let D be the diagonal matrix in which the ℓ -th entry on the main diagonal is*

$$D_{\ell\ell} = \omega_N^{0+1+2+\dots+(\ell-1)},$$

denote

$$\zeta = \omega_N^{\frac{1}{2}} = e^{\frac{\pi i}{N}},$$

and let D' be the diagonal matrix in which the ℓ -th entry on the main diagonal is

$$\tilde{D}_{\ell\ell} = \zeta^{-(\ell-1)}.$$

Then for N even, we have:

1) The eigenvectors of $M_N^1, M_N^2, \dots, M_N^{N-1}$ are precisely the Euclidean column basis.

2) If s is even, then each $M_N^{ks}T_N^k \in V_s$ can be diagonalized as

$$M_N^{ks}T_N^k = \omega_N^{-\frac{k(k-1)s}{2}} D^s W_N M_N^{-k} W_N^* (D^s)^*.$$

3) If s is odd, then each $M_N^{ks}T_N^k \in V_s$ can be diagonalized as

$$M_N^{ks}T_N^k = \zeta^k \omega_N^{-\frac{k(k-1)s}{2}} D^s \tilde{D} W_N M_N^{-k} W_N^* D^{-s} \tilde{D}^*.$$

Proof. 1) is trivial. For 2), we first verify that if N is even and s is also even, then similar to the previous lemma we have

$$\begin{aligned} M_N^s T_N^1 &= \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \omega_N^{\frac{sh(h-1)}{2}} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} & & & & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \omega_N^{-\frac{sh(h-1)}{2}} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \\ &= D^s T_N^1 D^{-s}. \end{aligned}$$

Then we apply Lemma 2.3.1 to get

$$M_N^{ks}T_N^k = \omega_N^{-(s+2s+\dots+(k-1)s)} \underbrace{(M_N^s T_N^1)(M_N^s T_N^1) \dots (M_N^s T_N^1)}_{k \text{ times}} = \omega_N^{-\frac{k(k-1)s}{2}} D^s W_N M_N^{-k} W_N^* (D^s)^*.$$

For 3), denote

$$T' = \begin{pmatrix} & & & -1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

T' is skew circulant, hence

$$\tilde{D}^* T' \tilde{D} = \begin{pmatrix} & & & -\zeta^{-1} \\ \zeta & & & \\ & \ddots & & \\ & & \zeta & \\ & & & \zeta \end{pmatrix} = \begin{pmatrix} & & & \zeta \\ \zeta & & & \\ & \ddots & & \\ & & \zeta & \\ & & & \zeta \end{pmatrix} = \zeta T_N,$$

consequently

$$T' = \zeta \tilde{D} T_N \tilde{D}^* = \zeta \tilde{D} W_N M_N^{-1} W_N^* \tilde{D}^*.$$

Next we verify that if N is even but s is odd, then slightly different from 2) we would have

$$\begin{aligned} M_N^s T_N^1 &= \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \omega_N^{\frac{sh(h-1)}{2}} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} & & & -1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \omega_N^{-\frac{sh(h-1)}{2}} & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \\ &= D^s (T') D^{-s} \\ &= \zeta D^s \tilde{D} W_N M_N^{-1} W_N^* D^{-s} \tilde{D}^*. \end{aligned}$$

Finally we again, apply Lemma 2.3.1 to get

$$\begin{aligned} M_N^{ks} T_N^k &= \omega_N^{-(s+2s+\dots+(k-1)s)} \underbrace{(M_N^s T_N^1)(M_N^s T_N^1) \dots (M_N^s T_N^1)}_{k \text{ times}} \\ &= \zeta^k \omega_N^{-\frac{k(k-1)s}{2}} D^s \tilde{D} W_N M_N^{-k} W_N^* D^{-s} \tilde{D}^*. \end{aligned}$$

□

Lemma 4.1.6 (Orbit). *If x is an eigenvector of $M_N^j T_N^k$, then $M_N^{j'} T_N^{k'} x$ is also an eigenvector of $M_N^j T_N^k$. In particular, if N is odd then for eigenvectors of (MT, V_s) we have*

$$\begin{aligned} M_N^j D^s \vec{u}_\ell &= D^s \vec{u}_{\ell+j}, \\ T_N^k D^s \vec{u}_\ell &= \omega_N^{-\frac{k(k-1)s+2k\ell}{2}} D^s \vec{u}_{\ell-ks}, \end{aligned}$$

where D is as defined in Lemma 4.1.4, and \vec{u}_k is the k -th column of W_N .

Proof. The first statement immediately follows from Lemma 2.3.1. Suppose λ is the eigenvalue, then

$$M_N^j T_N^k (M_N^{j'} T_N^{k'} \vec{x}) = \omega_N^{jk'-kj'} M_N^{j'} T_N^{k'} (M_N^j T_N^k \vec{x}) = \lambda \omega_N^{jk'-kj'} M_N^{j'} T_N^{k'} \vec{x}.$$

For explicit orbits, since diagonal matrices commute, we have

$$M_N^j D^s \vec{u}_h = D^s M_N^j \vec{u}_h = D_s \vec{u}_{h+j},$$

and we apply Lemma 4.1.4 to get

$$\begin{aligned} T_N^k D^s u_h &= D^s (D^{-s} T_N^k D^s) \vec{u}_h \\ &= w^{-\frac{k(k-1)s}{2}} D_s (M_N^{-ks} T_N^k) \vec{u}_h \\ &= w^{-\frac{k(k-1)s}{2} - hk} D^s M_N^{-ks} \vec{u}_h \\ &= w^{-\frac{k(k-1)s+2hk}{2}} D_s \vec{u}_{h-ks}. \end{aligned}$$

□

Remark: The shared eigenvectors of isotropic subgroups are examples of bi-

unimodular sequences (or CAZAC sequences, CAZAC stands for Constant Amplitude Zero Auto Correlation), i.e., sequences that is unimodular before and after applying the discrete Fourier transform. Such sequences are of special interests in engineering and are also connected to the so called cyclic N -roots. See, for example, [20] and [4], for related concepts. Moreover, when the dimension is a prime number, these sets of eigenvectors form the full class of the so called mutually unbiased basis, while if the dimension is a composite number, then they can be used to construct mutually unbiased basis, but not necessarily maximal, see for example [8] for a good survey on this topic.

4.2 Characterizing Unitary Gabor Matrices

In this subsection we will study for which subsets Γ can we have a window vector \vec{c} , such that $G_\Gamma(\vec{c})$ is unitary. We will completely characterize the cases when N is a prime number, give a sufficient condition when N is composite, and make a conjecture on the necessary condition when N is composite.

First we need a few technical results:

Let \vec{x} be a vector of unit length. We introduce the notation

$$P_{\vec{x}} = \vec{x}\vec{x}^*,$$

to denote the orthogonal projection onto the span of the vector $\vec{x} \in \mathbb{C}^N$.

We use \circ for the matrix Hadamard product. Also we write the j -th column in W_N as \vec{u}_j .

Lemma 4.2.1. *The following holds:*

1)

$$\langle M_N^j T_N^k, P_{\vec{x}} \rangle = \langle M_N^j T_N^k \vec{x}, \vec{x} \rangle.$$

2) For any $A \in \mathbb{C}^{N \times N}$, one has

$$M_N^j A (M_N^j)^* = N A \circ P_{\vec{u}_j},$$

Proof. For 1), we have

$$\langle M_N^j T_N^k, P_{\vec{x}} \rangle = \text{tr}(M_N^j T_N^k P_{\vec{x}}) = \text{tr}(M_N^j T_N^k \vec{x} \vec{x}^*) = \text{tr}(\vec{x}^* M_N^j T_N^k \vec{x}) = \langle M_N^j T_N^k \vec{x}, \vec{x} \rangle.$$

For 2), we first write the projection $P_{\vec{u}_j}$ into

$$P_{\vec{u}_j} = \frac{1}{N} M_N^j E (M_N^j)^*,$$

where E is the all ones matrix, i.e., the identity element with respect to Hadamard products. Then we use the fact that Hadamard products commute with diagonal scalings to get

$$N A \circ P_{\vec{u}_j} = A \circ (M_N^j E (M_N^j)^*) = M_N^j (A \circ E) (M_N^j)^* = M_N^j A (M_N^j)^*.$$

□

Lemma 4.2.2. *Let N be prime. If $R \in \mathbb{N}$ and $R < N$, then for*

$$\{j_1, j_2, \dots, j_R\} \subset \{1, 2, \dots, N\},$$

and $a_1, a_2, \dots, a_R \in \mathbb{N}$, we have

1)

$$a_1 \omega_N^{j_1} + a_2 \omega_N^{j_2} + \dots + a_R \omega_N^{j_R} \neq 0.$$

2) Each entry on the main diagonal of $a_1 P_{\vec{u}_{j_1}} + a_2 P_{\vec{u}_{j_2}} + \dots + a_R P_{\vec{u}_{j_R}}$ is $(a_1 + a_2 + \dots + a_R)/N$.

3) Each off diagonal entry of $a_1 P_{\vec{u}_{j_1}} + a_2 P_{\vec{u}_{j_2}} + \dots + a_R P_{\vec{u}_{j_R}}$ is non-zero.

Proof. For 1), N being prime implies that the minimal polynomial of ω_N over \mathbb{Q} is the N -th cyclotomic polynomial $1 + z + z^2 + \dots + z^{N-1}$, which would give us a contradiction if $a_1 \omega_N^{j_1} + a_2 \omega_N^{j_2} + \dots + a_R \omega_N^{j_R} = 0$ since $R < N$.

For 2) and 3), it suffice to notice that each projector $P_{\vec{u}_{j_h}}$ is a circulant matrix

with its first column being

$$\frac{1}{N}(1, \omega_N^{j_h}, \omega_N^{2j_h}, \dots, \omega_N^{(N-1)j_h}),$$

thus 2) can be easily verified through direct computation and 3) follows from 1). \square

Lemma 4.2.3. *If V is an isotropic subgroup in $\mathbb{Z}_N \times \mathbb{Z}_N$, and \vec{x} is a shared eigenvector of members in (MT, V) with unit length, then*

$$P_{\vec{x}} \perp M_N^j T_N^K \quad \text{for any } (j, k) \notin V.$$

Proof. Suppose elements in (MT, V) are diagonalized as $BD_{v_1}B^*, BD_{v_2}B^*, \dots, BD_{v_N}B^*$ where x is the first column of B , and we further denote $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N$ as the vector formed by the main diagonals of $D_{\vec{v}_1}, D_{\vec{v}_2}, \dots, D_{\vec{v}_N}$ respectively.

By Lemma 2.3.2, all members in (MT, V) are linearly independent in $\mathbb{C}^{N \times N}$, consequently $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N$ are linearly independent in \mathbb{C}^N , hence there exists a linear combination such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_N \vec{v}_N = (1, 0, \dots, 0),$$

i.e.,

$$c_1 BD_{\vec{v}_1}B^* + c_2 BD_{\vec{v}_2}B^* + \dots + c_N BD_{\vec{v}_N}B^* = P_{\vec{x}},$$

which means

$$P_{\vec{x}} \in \text{span}((MT, V)),$$

Again by Lemma 2.3.2, if $(j, k) \notin V$ then

$$M_N^j T_N^K \perp (MT, V),$$

which implies the desired result. \square

Define the first order difference set $\Delta\Gamma$ of Γ to be

$$\Delta\Gamma = \{(j - j', k - k') : (j, k), (j', k') \in \Gamma\}.$$

Lemma 4.2.4. *Let \vec{c} be a unit vector, then $G_\Gamma(\vec{c})$ is unitary if and only if*

$$P_{\vec{c}} \perp (MT, \Delta\Gamma).$$

Proof. Each main diagonal entry of $G_\Gamma^*(\vec{c})G_\Gamma(\vec{c})$ is 1 by the assumption that \vec{c} is a unit vector.

Each off diagonal entry of $G_\Gamma^*(\vec{c})G_\Gamma(\vec{c})$ is of form

$$\omega_N^\ell \langle M_N^j T_N^k \vec{c}, \vec{c} \rangle = \omega_N^\ell \langle M_N^j T_N^k, P_{\vec{c}} \rangle,$$

with $(j, k) \in \Delta\Gamma$ and $\ell \in \mathbb{Z}_N$.

Now $G_\Gamma(\vec{c})$ is unitary if and only if $G_\Gamma^*(\vec{c})G_\Gamma(\vec{c})$ is the identity matrix, i.e., all the above mentioned off diagonal entries are 0, which is equivalent to

$$P_{\vec{c}} \perp (MT, \Delta\Gamma).$$

□

Isotropic subgroups are important for Gabor matrices in the sense that taking their quotient group as the support set leads to unitary Gabor matrices.

Theorem 4.2.1. *If Γ consists of precisely one element from each coset of an isotropic subgroup V of $\mathbb{Z}_N \times \mathbb{Z}_N$, i.e.,*

$$\mathbb{Z}_N \times \mathbb{Z}_N = \Gamma \times V = \{(j + j', k + k') : (j, k) \in \Gamma, (j', k') \in V\},$$

and \vec{c} is a shared eigenvector of all members in V with unit length, then (\vec{c}, Γ) forms an orthonormal basis for \mathbb{C}^N .

Proof. We want to show that the Gabor matrix $G_\Gamma(\vec{c})$ is a unitary matrix.

First we notice that the following must hold by our assumption:

$$V \cap \Delta\Gamma = \emptyset.$$

Indeed, if $(j, k), (j', k') \in \Gamma$ and $(j - j', k - k') \in V$, then this would imply that (j, k) and (j', k') belongs to the same coset of V , which contradicts the assumption.

Thus by Lemma 4.2.3, we have

$$P_{\vec{c}} \perp (MT, \Delta\Gamma),$$

which implies $G_\Gamma(\vec{c})$ is unitary by Lemma 4.2.4. \square

If N is a prime number, then the above condition is also necessary:

Theorem 4.2.2. *Let N be a prime number, then there exists a window vector \vec{c} such that (\vec{c}, Γ) forms an orthonormal basis for \mathbb{C}^N if and only if Γ consists of precisely one element from each coset of an isotropic subgroup V of $\mathbb{Z}_N \times \mathbb{Z}_N$, i.e.,*

$$\mathbb{Z}_N \times \mathbb{Z}_N = \Gamma \times V = \{(j + j', k + k') : (j, k) \in \Gamma, (j', k') \in V\}.$$

Proof. Sufficiency is already proved in the last theorem. Let us prove its necessity.

Assume the contrary there exists some \vec{c} such that (\vec{c}, Γ) is an orthonormal basis for \mathbb{C}^N (i.e., $G_\Gamma(\vec{c})$ is unitary) but Γ does not consist of precisely one element from each coset of any isotropic subgroup V .

For convenience, denote the elements listed in lexicographical order in Γ as

$$\Gamma = \{(j_0, k_0), (j_1, k_1), \dots, (j_{N-1}, k_{N-1})\}.$$

Since N is prime, all non-trivial proper subgroups are isotropic, and they are listed in Lemma 4.1.1.

Let us first fix an $s \in \mathbb{Z}_N$, and consider the subgroup V_s as defined in Lemma 4.1.1, i.e.,

$$V_s = \{(0, 0), (s, 1), (2s, 2), \dots, ((N-1)s, s)\},$$

it is easy to verify that

$$\mathbb{Z}_N \times \mathbb{Z}_N = V_\infty \times V_s,$$

where

$$V_\infty = \{(0, 0), (1, 0), (2, 0), \dots, (N - 1, 0)\},$$

is also as defined in Lemma 4.1.1.

This means V_∞ consists of precisely one element from each coset of V_s . Since cosets are unique, there exists ℓ_n such

$$(j_n, k_n) + V_s = (\ell_n, 0) + V_s, \quad (12)$$

for all $n = 0, 1, 2, \dots, N - 1$. Moreover, since by our assumption Γ does not consist of precisely one element from each coset of V_s , the set $\{\ell_n\}_n$ is a proper subset \mathbb{Z}_N .

Now we consider the matrix

$$G_{\Gamma \times V_s}(\vec{c}) = \left(G_{(j_0, k_0) + V_s}(\vec{c}) \mid G_{(j_1, k_1) + V_s}(\vec{c}) \mid \dots \mid G_{(j_{N-1}, k_{N-1}) + V_s}(\vec{c}) \right),$$

Since $G_\Gamma(\vec{c})$ is unitary by our assumption, columns in $G_{\Gamma \times V_s}(\vec{c})$ form a tight frame for \mathbb{C}^N since they consists of N copies of orthonormal basis of form $s(\vec{c}, (ks, k) + \Gamma)$ for $k = 0, 1, 2, \dots, N - 1$.

By (12), we can rewrite $G_{\Gamma \times V_s}(\vec{c})$ as

$$G_{\Gamma \times V_s}(\vec{c}) = \left(G_{(\ell_0, 0) + V_s}(\vec{c}) \mid G_{(\ell_1, 0) + V_s}(\vec{c}) \mid \dots \mid G_{(\ell_{N-1}, 0) + V_s}(\vec{c}) \right).$$

Thus using the tight frame property of $G_{\Gamma \times V_s}(\vec{c})$ and applying Lemma 4.2.1 we get

$$\begin{aligned}
NI &= G_{\Gamma \times V_s}(\vec{c})G_{\Gamma \times V_s}^*(\vec{c}) = \sum_{n=0}^{N-1} G_{(\ell_n, 0) + V_s}(\vec{c})G_{(\ell_n, 0) + V_s}^*(\vec{c}) \\
&= \sum_{n=0}^{N-1} M_N^{\ell_n} G_{V_s}(\vec{c})G_{V_s}^*(\vec{c})M_N^{-\ell_n} \\
&= N \sum_{n=0}^{N-1} (G_{V_s}(\vec{c})G_{V_s}^*(\vec{c})) \circ P_{\vec{u}_{\ell_n}} \\
&= N(G_{V_s}(\vec{c})G_{V_s}^*(\vec{c})) \circ \left(\sum_{n=0}^{N-1} P_{\vec{u}_{\ell_n}} \right).
\end{aligned}$$

Now recall as shown above our construction of $\{\ell_n\}_n$ and our assumption on Γ implies we may apply Lemma 4.2.2 to conclude that the main diagonal of $\sum_{n=0}^{N-1} P_{\vec{u}_{\ell_n}}$ is I while all off diagonal entries in $\sum_{n=0}^{N-1} P_{\vec{u}_{\ell_n}}$ are non-zero.

Thus comparing both sides of the equation we can conclude that

$$G_{V_s}(\vec{c})G_{V_s}^*(\vec{c}) = I.$$

By Lemma 4.2.4, this implies

$$P_{\vec{c}} \perp (MT, \Delta V_s).$$

As V_s is a subgroup, we simply have

$$\Delta V_s = V_s \setminus \{(0, 0)\},$$

thus

$$P_{\vec{c}} \perp (MT, V_s \setminus \{(0, 0)\}).$$

Let s run through $0, 1, 2, \dots, N-1$ we thus get

$$P_{\vec{c}} \perp \bigcup_s (MT, V_s \setminus \{(0, 0)\}) = \{M_N^j T_N^k : k \neq 0, j \in \mathbb{Z}_N\}. \quad (13)$$

Now let us consider the subgroup V_∞ , as stated earlier,

$$\mathbb{Z}_N \times \mathbb{Z}_N = V_0 \times V_\infty,$$

hence similar as in the previous derivation we can find h_n for $n = 0, 1, 2, \dots, N-1$ such that

$$(j_n, k_n) + V_\infty = (0, h_n) + V_s,$$

and the set $\{h_n\}_n$ is a proper subset of \mathbb{Z}_N .

And again we consider

$$G_{\Gamma \times V_\infty}(\vec{c}) = \left(G_{(j_0, k_0) + V_\infty}(\vec{c}) \mid G_{(j_1, k_1) + V_\infty}(\vec{c}) \mid \dots \mid G_{(j_{N-1}, k_{N-1}) + V_\infty}(\vec{c}) \right),$$

whose columns form a tight frame for \mathbb{C}^N for similar reason as above, and also we can rewrite it as

$$G_{\Gamma \times V_\infty}(\vec{c}) = \left(G_{(0, h_0) + V_\infty}(\vec{c}) \mid G_{(0, h_1) + V_\infty}(\vec{c}) \mid \dots \mid G_{(0, h_{N-1}) + V_\infty}(\vec{c}) \right),$$

which by Lemma 2.3.1, differs from the following matrix

$$G_\infty = \left(T_{h_0} G_{V_\infty}(\vec{c}) \mid T_{h_1} G_{V_\infty}(\vec{c}) \mid \dots \mid T_{h_{N-1}} G_{V_\infty}(\vec{c}) \right),$$

by only unitary column scaling.

Indeed, the $(mN + n)$ -th column in $G_{\Gamma \times V_\infty}(\vec{c})$ is $M_N^n T_N^{h_m} \vec{c}$, while the $(mN + n)$ -th column in G_∞ is simply

$$T_N^{h_m} M_N^n \vec{c} = w_N^{-nh_m} M_N^n T_N^{h_m} \vec{c}.$$

Therefore columns in G_∞ also form a tight frame with the same frame constant

as $G_{\Gamma \times V_\infty}(\vec{c})$, and consequently

$$\begin{aligned} NI &= W_N^* N I W_N = W_N^* G_\infty G_\infty^* W_N = \sum_{n=0}^{N-1} W_N^* T_N^{h_n} G_{V_\infty}(\vec{c}) G_{V_\infty}^*(\vec{c}) T_N^{-h_n} W_N \\ &= \sum_{n=0}^{N-1} M_N^{-h_n} G_{V_0}(W_N^* \vec{c}) G_{V_0}^*(W_N^* \vec{c}) M_N^{h_n}, \end{aligned}$$

where the last equality follows upon applying (2).

Now as before we may rewrite the right hand side of the above as a Hadamard product, apply Lemma 4.2.2 and following the same steps to conclude that $G_{V_0}(W_N^* \vec{c}) G_{V_0}^*(W_N^* \vec{c})$ is the identity and thus $G_{V_0}(W_N^* \vec{c})$ is unitary which by Lemma 4.2.4 implies

$$P_{W_N^* \vec{c}} \perp (MT, V_0 \setminus \{(0, 0)\}),$$

i.e. for each $k = 1, 2, \dots, N - 1$ we have

$$0 = \langle T^k, P_{W_N^* \vec{c}} \rangle = \langle T^k W_N^* \vec{c}, W_N^* \vec{c} \rangle = \langle M_N^k \vec{c}, \vec{c} \rangle = \langle M_N^k, \vec{c} \rangle,$$

which implies

$$P_{\vec{c}} \perp (MT, V_\infty \setminus \{(0, 0)\}) = \{M_N, M_N^2, \dots, M_N^{N-1}\}. \quad (14)$$

Combining (13) and (14) leads to

$$P_{\vec{c}} \perp \{M_N^j T_N^k : (j, k) \neq 0\}.$$

By Lemma 2.3.2, this means $P_{\vec{c}}$ must be in the span of $M_N^0 T_N^0 = I$, i.e., it must be a scalar multiple of the identity matrix, which is impossible since $P_{\vec{c}}$ has rank 1 while the identity matrix has rank N , thus we obtained a contradiction. \square

The essence of the above proof can be summarized as having the following 3 steps

- 1) If $G_\Gamma(\vec{c})$ is unitary and Γ does not represent the full coset of some isotropic group V , then $G_V(\vec{c})$ is also unitary.

- 2) If $G_V(\vec{c})$ is unitary, then $\vec{c} \perp (MT, V \setminus \{(0, 0)\})$.
- 3) Isotropic subgroups covers the whole group, hence $G_V(\vec{c})$ can not be unitary for all V .

By Lemma 4.2.4 and Lemma 4.1.3, 2) and 3) always holds regardless of the dimension, hence we conjecture that Theorem 4.2.2 also holds when N is a composite number. 1) can not be repeated in such cases since Lemma 4.2.2 would not hold. The difficulty certainly lies in that we can say very little about either isotropic subgroups in $\mathbb{Z}_N \times \mathbb{Z}_N$ nor sum of ω_N^k over \mathbb{Q} .

In practice, to check whether Γ satisfies the conditions in Theorem 4.2.1 and Theorem 4.2.2, one computes its difference set to see if $\Delta\Gamma$ missed an isotropic subgroup.

Remark: A few months after the author presented Theorem 4.2.2 in a conference in Tallinn, Alihan Kaplan (TU München), a student of Dr. Volker Pohl (TU München) who is a collaborator of the Co-Betreuer, pointed to the Author about Theorem 2 in [26], which covers the sufficiency part in Theorem 4.2.2 for N being prime numbers. The difficult part in Theorem 4.2.2 lies in the necessity, the sufficient part one easily obtains by mere observation. Theorem 2 in [26] even takes a stronger assumption, unnecessary but stems from their settings, that it starts with a unit vector, and assumes there exists an isotropic subgroup that jointly stabilizes it. Nevertheless, we borrowed the term "isotropic" from it as it might better describe the property of those subgroups needed when N is a composite number.

4.3 Visualization of the Support Set for Unitary Gabor Matrices

In this part we give a geometric interpretation of the conditions specified in Theorem 4.2.2. For simplicity, we plot the example for $N = 3$.

As similar to Subsection 3.2, we map $\mathbb{Z}_N \times \mathbb{Z}_N$ to a $N \times N$ grid with (j, k) maps to the box that is the $j + 1$ -th counting from left to right and the $k + 1$ -th counting

from bottom to top. Below is an example for $\mathbb{Z}_3 \times \mathbb{Z}_3$. We should also keep in mind that because of the group structure, the top is identifiable with the bottom edge, and so are the left and right edges, hence we actually have a torus.

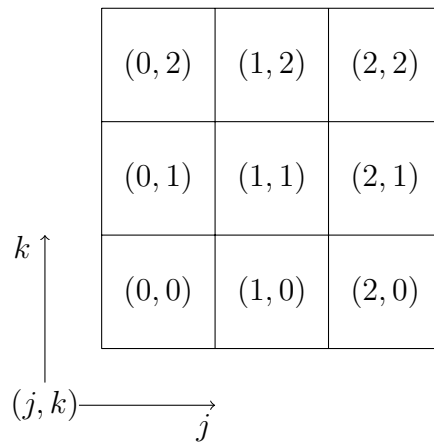


Figure 11: The Support Set $\mathbb{Z}_3 \times \mathbb{Z}_3$

Then each isotropic subgroup V_s as described in Lemma 4.1.1 occupies a straight line passing the origin box $(0, 0)$ with s being its slope. Below is again an example for $\mathbb{Z}_3 \times \mathbb{Z}_3$.

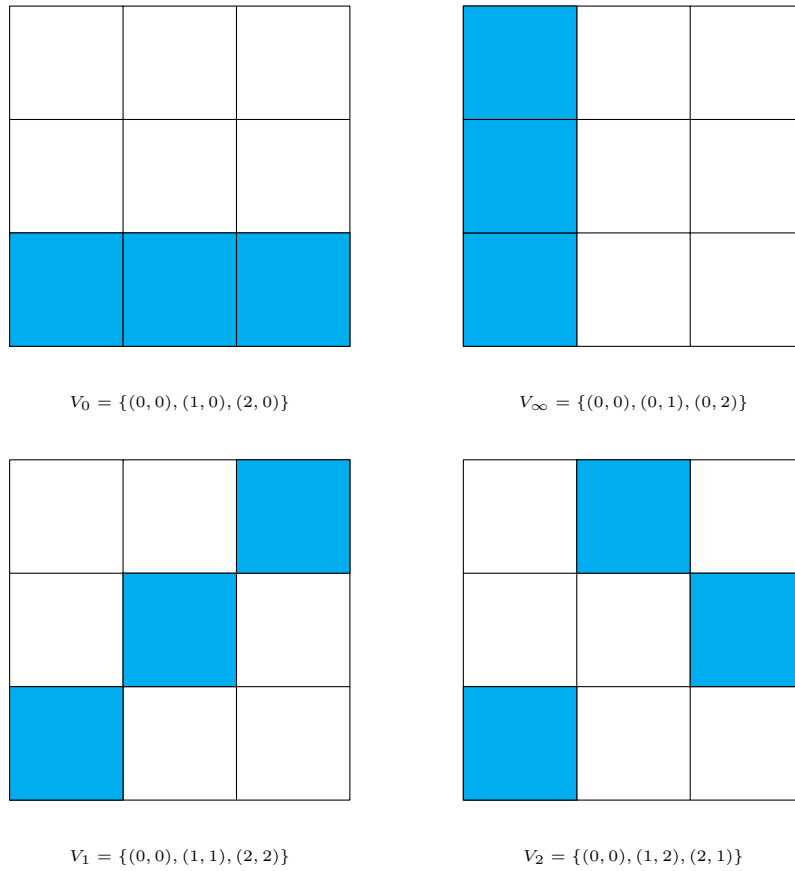


Figure 12: Subgroups in $\mathbb{Z}_3 \times \mathbb{Z}_3$

Some subgroups may not appear to form a straight line in the above figure, but they indeed are since we are on a torus. For example, below is how to view V_2 in $\mathbb{Z}_3 \times \mathbb{Z}_3$, the gray box is moved to the dashed box so that it now forms a straight line of slope 2.

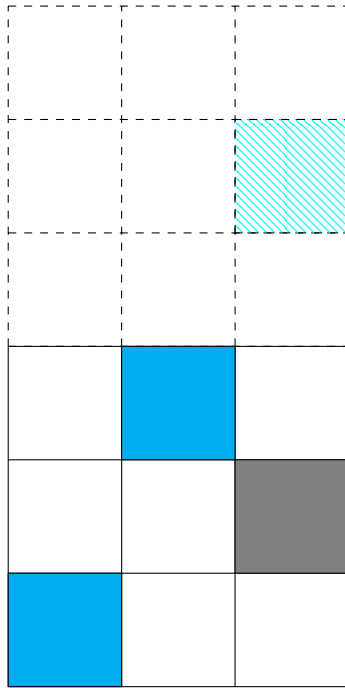


Figure 13: A Closer Inspection of V_2 in $\mathbb{Z}_3 \times \mathbb{Z}_3$

Moreover, cosets of a subgroup are simply shifted copies of the corresponding subgroup. Below is an example for $\mathbb{Z}_3 \times \mathbb{Z}_3$, cosets are distinguished with different colors.

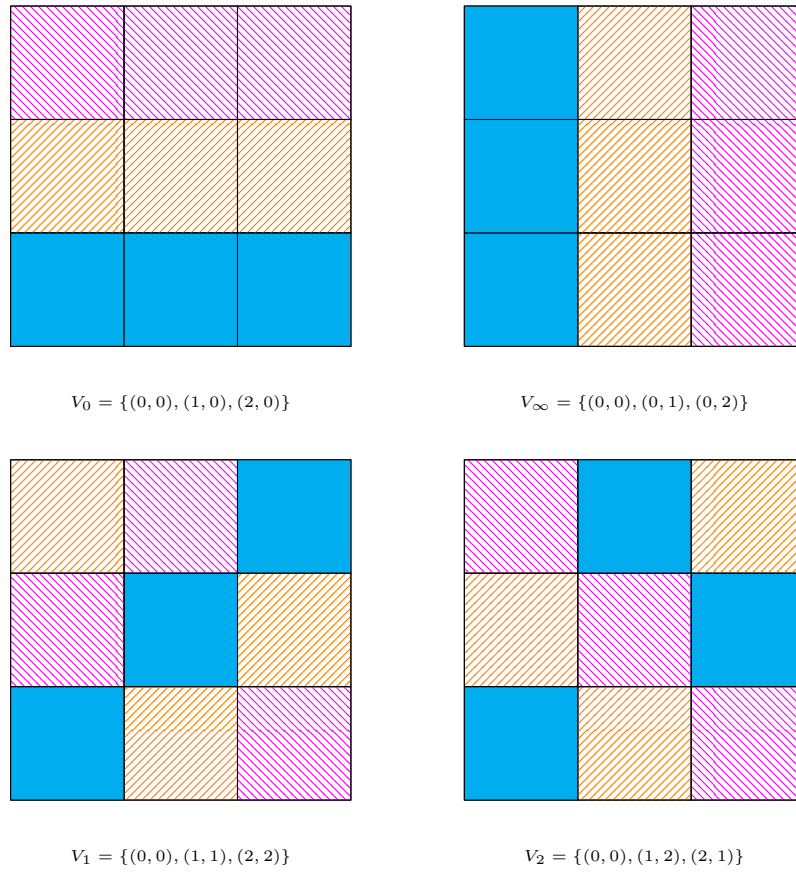


Figure 14: Cosets of Subgroups in $\mathbb{Z}_3 \times \mathbb{Z}_3$

Each Γ is also represented by union of some boxes in the grid, and the formula $\Gamma \times V_s$ can now be visualized as shifting Γ along the line of slope s , and this should, on the torus, tile up the whole grid. Below is an example of a domain $\Gamma \in \mathbb{Z}_3 \times \mathbb{Z}_3$, and it is shifted along the subgroup V_1 (the direction marked by the gray dashed arrow).

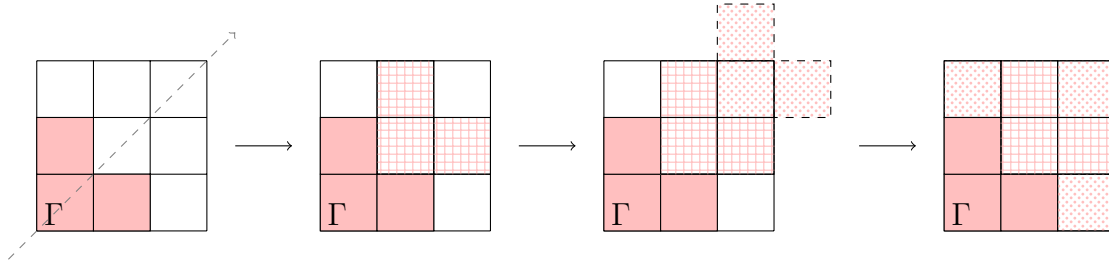


Figure 15: Tiling $\mathbb{Z}_N \times \mathbb{Z}_N$ with Γ along V_1

In this spirit, Theorem 4.2.2 can be restated as:

For N prime, there exists a window vector \vec{c} such that (\vec{c}, Γ) forms an orthonormal basis for \mathbb{C}^N if and only if Γ tiles up $\mathbb{Z}_N \times \mathbb{Z}_N$ along a line represented by some isotropic subgroup V_s (i.e., having slope s).

We remind reader the similarity between the above statement and the Fuglede conjecture (or spectral set conjecture in some literature), which says a set in \mathbb{R}^d admits orthogonal exponential basis if and only if it tiles \mathbb{R}^d . In this sense, Theorem 4.2.2 is version of the Fuglede conjecture under the setting of discrete Gabor analysis.

4.4 Projection onto Subgroups

Lemma 4.4.1 (Projection Formula). *Let N be a prime number, and V_s be as defined in Lemma 4.1.1, then on $\mathbb{C}^{N \times N}$, the orthogonal projection P_s onto the span of (MT, V_s) can be written as*

$$P_s(A) := \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} M_N^{ks} T_N^k A T_N^{-k} M_N^{-ks} & s \in \{0, 1, 2, \dots, N-1\} \\ \frac{1}{N} \sum_{j=0}^{N-1} M_N^j A M_N^{-j} & s = \infty \end{cases},$$

for any $A \in \mathbb{C}^{N \times N}$.

Proof. With Lemma 2.3.1 one may check that for any fixed $s \in \{0, 1, 2, \dots, N-1\}$

we have

$$\begin{aligned}
P_s^2(A) &= \frac{1}{N} \sum_{j=0}^{N-1} M_N^{js} T_N^j \left(\frac{1}{N} \sum_{k=0}^{N-1} M_N^{ks} T_N^k A T_N^{-k} M_N^{-ks} \right) T_N^{-j} M_N^{-js} \\
&= \frac{1}{N^2} \sum_{j,k=0}^{N-1} M_N^{(j+k)s} T_N^{(j+k)} A T_N^{-(j+k)} M_N^{-(j+k)s},
\end{aligned}$$

it is easy to see that for each fixed j and $\ell \in \mathbb{Z}_N$, there is precisely one k such that $j + k = \ell$, hence the above becomes

$$P_s^2(A) = \frac{N}{N^2} \sum_{\ell=0}^{N-1} M_N^{\ell s} T_N^\ell A T_N^{-\ell} M_N^{-\ell s} = \frac{1}{N} \sum_{\ell=0}^{N-1} M_N^{\ell s} T_N^\ell A T_N^{-\ell} M_N^{-\ell s} = P_s(A),$$

which shows P_s is indeed a projection. Similarly for $s = \infty$ we have

$$\begin{aligned}
P_\infty^2(A) &= \frac{1}{N} \sum_{j=0}^{N-1} M_N^j \left(\frac{1}{N} \sum_{k=0}^{N-1} M_N^k A M_N^{-k} \right) M_N^{-j} \\
&= \frac{1}{N^2} \sum_{j,k=0}^{N-1} M_N^{j+k} A M_N^{-(j+k)} \\
&= \frac{N}{N^2} \sum_{\ell=0}^{N-1} M_N^\ell A M_N^{-\ell} \\
&= \frac{1}{N} \sum_{\ell=0}^{N-1} M_N^\ell A M_N^{-\ell} \\
&= P_\infty(A),
\end{aligned}$$

which shows P_∞ is also a projection.

To show orthogonality, it suffice to show $A - P_s(A)$ is orthogonal to each element in (MT, V_s) .

Again we start with a fixed $s \in \{0, 1, 2, \dots, N-1\}$. Since elements in (MT, V_s)

commute, for any $A \in \mathbb{C}^{N \times N}$ and $j \in \{0, 1, 2, \dots, N-1\}$ we have

$$\begin{aligned}
(A - P_s(A))M^{js}T^j &= (A - \frac{1}{N} \sum_{k=0}^{N-1} M_n^{ks}T_N^k AT_N^{-k} M_N^{-ks})M_N^{js}T_N^j \\
&= (AM_N^{js}T_N^j - \frac{1}{N} \sum_{k=0}^{N-1} M_n^{ks}T_N^k (AM_N^{js}T_N^j)T_N^{-k} M_N^{-ks}) \\
&= AM_N^{js}T_N^j - P_s(AM_N^{js}T_N^j),
\end{aligned}$$

where the second line holds since $M_N^{js}T_N^j$ commuting with $M_N^{ks}T_N^k$ implies it also commutes with the inverse, $M_N^{-ks}T_N^{-k}$.

Setting

$$A' = AM_N^{js}T_N^j,$$

we get

$$\text{tr}((A - P_s(A))M_N^{js}T_N^j) = \text{tr}(A' - P_s(A')) = \text{tr}((A' - P_s(A'))M_N^0T_N^0),$$

Therefore it suffice to just show $A - P_s(A) \perp M_N^0T_N^0$ holds for any A .

It follows from the cyclic invariance property of the trace that

$$\text{tr}(P_s(A)) = \frac{1}{N} \sum_{k=0}^{N-1} \text{tr}(M_N^{ks}T_N^k AT_N^{-k} M_N^{-ks}) = \frac{1}{N} \sum_{k=0}^{N-1} \text{tr}(AT_N^{-k} M_N^{-ks} M_N^{ks} T_N^k) = \text{tr}(A),$$

thus

$$\text{tr}((A - P_s(A))M_N^0T_N^0) = \text{tr}(A - P_s(A)) = \text{tr}(A) - \text{tr}(P_s(A)) = 0.$$

Therefore indeed $A - P_s(A) \perp M_0T_0$, and consequently $A - P_s(A) \perp (MT, V_s)$.

The case of V_∞ is similar, one verifies that

$$(A - P_\infty(A))M_N^j = AM_N^j - P_\infty(AM_N^j),$$

and

$$\operatorname{tr}(A - P_\infty(A)) = \operatorname{tr}(A) - \frac{1}{N} \operatorname{tr}\left(\sum_{k=0}^{N-1} M_N^k A M_N^{-k}\right) = \operatorname{tr}(A) - \frac{1}{N} \operatorname{tr}\left(\sum_{k=0}^{N-1} A M_N^{-k} M_N^k\right) = 0,$$

to conclude that $A - P_\infty(A) \perp (MT, V_\infty)$. \square

Lemma 4.4.2. *Let P_s be defined as in Lemma 4.4.1, $G_\Gamma(\vec{c})$ a Gabor matrix as defined before with $|\Gamma| = N$ (i.e., $G_\Gamma(\vec{c})$ is a square matrix) and $\|\vec{c}\|_{\ell^2} = 1$. If N is a prime number, then*

$$\sum_{s \in \{0, 1, 2, \dots, N-1, \infty\}} P_s(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})) = G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}) + NI.$$

Proof. It is easy to see from the definition that

$$\operatorname{tr}(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})) = N.$$

Thus from Lemma 4.4.1, we have

$$\begin{aligned} P_s(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})) &= \frac{1}{N} \sum_{B \in (MT, V_s)} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), B \rangle B \\ &= \frac{1}{N} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), I \rangle I + \frac{1}{N} \sum_{B \in (MT, V_s) \setminus \{I\}} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), B \rangle B. \\ &= \frac{1}{N} \operatorname{tr}(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}))I + \frac{1}{N} \sum_{B \in (MT, V_s) \setminus \{I\}} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), B \rangle B. \\ &= I + \frac{1}{N} \sum_{B \in (MT, V_s) \setminus \{I\}} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), B \rangle B. \end{aligned}$$

Therefore, summing over all s and use Lemma 2.3.2 and Lemma 4.1.1 we get

$$\begin{aligned}
\sum_{s \in \{0,1,2,\dots,N-1,\infty\}} P_s(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})) &= (N+1)I + \frac{1}{N} \sum_{s \in \{0,1,2,\dots,N-1,\infty\}} \sum_{B \in (MT, V_s) \setminus \{I\}} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), B \rangle B \\
&= (N+1)I + \frac{1}{N} \sum_{B \in (MT, \mathbb{Z}_N \times \mathbb{Z}_N) \setminus \{I\}} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), B \rangle B \\
&= NI + \frac{1}{N} \sum_{B \in (MT, \mathbb{Z}_N \times \mathbb{Z}_N)} \langle G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}), B \rangle B \\
&= NI + G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}).
\end{aligned}$$

We remind the reader that the second line in the above equation only holds for prime N . \square

If $\vec{x} \in \mathbb{C}^N$, we denote

$$D_{\vec{x}} = \begin{pmatrix} x_1 & & & & & \\ & x_2 & & & & \\ & & \ddots & & & \\ & & & x_{N-1} & & \\ & & & & x_N & \end{pmatrix},$$

and

$$D_{|\vec{x}|^2} = \begin{pmatrix} |x_1|^2 & & & & & \\ & |x_2|^2 & & & & \\ & & \ddots & & & \\ & & & |x_{N-1}|^2 & & \\ & & & & |x_N|^2 & \end{pmatrix}.$$

It is easy to see that

$$D_{|\vec{x}|^2} = D_{\vec{x}} D_{\vec{x}}^*.$$

The following Lemma shows if a Gabor matrix is supported on one of the subgroups as listed in Lemma 4.1.1, then its spectrum is explicitly computable:

Lemma 4.4.3. *Let D be as defined in Lemma 4.1.4, and N be an odd prime*

number, then

$$G_{V_s}(\vec{c})G_{V_s}^*(\vec{c}) = \begin{cases} ND^s W_N D_{|\vec{c}^{(s)}|^2} W_N^* D^{-s} & s = 0, 1, 2, \dots, N-1 \\ ND_{|\vec{c}|^2} & s = \infty \end{cases},$$

where

$$\vec{c}^{(s)} = W_N^* D^{-s} \vec{c}.$$

Proof. First we consider $s = \infty$, by definition we have

$$G_{V_\infty}(\vec{c}) = \left(\vec{c} \quad M_N^1 \vec{c}, \quad M_N^2 \vec{c}, \quad \dots, \quad M_N^{N-1} \vec{c} \right) = \sqrt{N} D_{\vec{c}} W_N,$$

therefore

$$G_{V_\infty}(\vec{c})G_{V_\infty}^*(\vec{c}) = ND_{\vec{c}} W_N W_N^* D_{\vec{c}}^* = ND_{|\vec{c}|^2}.$$

For $s = 0, 1, 2, \dots, N-1$, we have a similar computation using Lemma 4.1.4:

$$\begin{aligned} G_{V_s}(\vec{c}) &= \left(\vec{c} \quad M_N^s T_N^1 \vec{c}, \quad M_N^{2s} T_N^2 \vec{c}, \quad \dots, \quad M_N^{(N-1)s} T_N^{N-1} \vec{c} \right) \\ &= D^s W_N \left(\vec{c}^{(s)} \quad M_N^{-1} \vec{c}^{(s)}, \quad \omega^{-s} M_N^{-2} \vec{c}^{(s)}, \quad \dots, \quad \omega^{-\frac{(N-1)(N-2)s}{2}} M_N^{-(N-1)} \vec{c}^{(s)} \right) \\ &= D^s W_N D_{\vec{c}^{(s)}} \left(I \quad M_N^{-1} I, \quad \omega^{-s} M_N^{-2} I, \quad \dots, \quad \omega^{-\frac{(N-1)(N-2)s}{2}} M_N^{-(N-1)} I \right), \end{aligned}$$

where the last matrix after dividing by \sqrt{N} differs from Fourier matrices by a unitary scaling, therefore

$$G_{V_s}(\vec{c})G_{V_s}^*(\vec{c}) = ND^s W_N D_{\vec{c}^{(s)}} D_{\vec{c}^{(s)}}^* W_N^* D^{-s} = ND^s W_N D_{|\vec{c}^{(s)}|^2} W_N^* D^{-s}.$$

□

Using the above, we can compute the spectrum of the projection onto the span of (MT, V_s) :

Lemma 4.4.4. *Following all notations and assumptions used in previous lemmas in this subsection, and assume*

$$\Gamma = \{(j_1, k_1), (j_2, k_2), \dots, (j_N, k_N)\},$$

where all elements are arranged in the lexicographic ordering. Then we have

$$P_s(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})) = \begin{cases} \sum_{\ell=1}^N D^s W_N (T_N^{k_\ell-j_\ell} D_{|\vec{c}^{(s)}|^2} T_N^{-(k_\ell-j_\ell)}) W_N^* D^{-s} & s = 0, 1, 2, \dots, N-1 \\ \mathcal{D}(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})) & s = \infty \end{cases},$$

where \mathcal{D} is the diagonal projection, i.e., $\mathcal{D}(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}))$ is the diagonal matrix whose main diagonal entries are same as $G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})$.

Proof. The case of $s = \infty$ is trivial as V_∞ consists of N linearly independent diagonal matrices. Hence, by Lemma 4.4.1, P_∞ is actually the orthogonal projection on to the space of diagonal matrices.

For other values of s , we apply the previous lemma to compute

$$\begin{aligned} P_s(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})) &= \frac{1}{N} \sum_{h=1}^N M_N^{hs} T_N^h G_\Gamma(\vec{c}) G_\Gamma^*(\vec{c}) T_N^{-h} M_N^{-hs} \\ &= \frac{1}{N} \sum_{h=1}^N M_N^{hs} T_N^h \left(\sum_{\ell=1}^N M_N^{j_\ell} T_N^{k_\ell} P_{\vec{c}} T_N^{-k_\ell} M_N^{-j_\ell} \right) T_N^{-h} M_N^{-hs} \\ &= \frac{1}{N} \sum_{\ell=1}^N M_N^{j_\ell} T_N^{k_\ell} \left(\sum_{h=1}^N M_N^{hs} T_N^h P_{\vec{c}} T_N^{-h} M_N^{-hs} \right) T_N^{-k_\ell} M_N^{-j_\ell} \\ &= \frac{1}{N} \sum_{\ell=1}^N M_N^{j_\ell} T_N^{k_\ell} (G_{V_s}(\vec{c}) G_{V_s}^*(\vec{c})) T_N^{-k_\ell} M_N^{-j_\ell} \\ &= \sum_{\ell=1}^N M_N^{j_\ell} T_N^{k_\ell} (D^s W_N D_{|\vec{c}^{(s)}|^2} W_N^* D^{-s}) T_N^{-k_\ell} M_N^{-j_\ell} \\ &= D^s W_N \left(\sum_{\ell=1}^N T_N^{sk_\ell-j_\ell} D_{|\vec{c}^{(s)}|^2} T_N^{-(sk_\ell-j_\ell)} \right) W_N^* D^{-s}, \end{aligned}$$

where the last line follows by applying Lemma 4.1.6 when computing $M_N^{j_\ell} T_N^{k_\ell} D^s W_N$, i.e., we apply Lemma 4.1.6 each time we multiply $M_N^{j_\ell} T_N^{k_\ell}$ with a column in $D^s W_N$.

The above computation can be summarized as following, we first notice that $G_\Gamma(\vec{c})G_\Gamma^*(\vec{c})$ is the sum of projectors $M^j T^k \vec{c}$ with (j, k) in Γ , then Lemma 4.4.1 shows $P_s(G_\Gamma(\vec{c})G_\Gamma^*(\vec{c}))$ can be written as the sum of projectors $M^j T^k \vec{c}$ with (j, k)

in $\Gamma + V_s$, we regroup these summands to write the result as several shifted copies of $G_{V_s}(\vec{c})G_{V_s}^*(\vec{c})$ added together, and apply the previous lemma to diagonalize each shifted copy. \square

4.5 Gabor Matrices Stacked on Top of Arbitrary Matrices

Let $K \in \mathbb{N}$, in this part we consider matrices of form

$$G_\Gamma(A, \vec{c}) = \begin{pmatrix} G_\Gamma(\vec{c}) \\ A \end{pmatrix},$$

where $G_\Gamma(\vec{c})$ is of size $N \times (N+K)$ (hence $|\Gamma| = N+K$), and A is of size $K \times (N+K)$.

The main problem of interest in this subsection is that given Γ and A , how do we choose $\vec{c} \in \mathbb{C}^N$ such that $G_\Gamma(A, \vec{c})$ becomes invertible. This is of particular interest later in Subsection 6.3 where A will encode the linear constraints and Γ represents overspread domains.

Theorem 4.5.1. *If $K = 1$, then for any given Γ and $A \neq 0$, one can find some $\vec{c} \in \mathbb{C}^N$ such that $G_\Gamma(A, \vec{c})$ is invertible.*

Proof. To emphasize that $K = 1$, let us use $G_\Gamma(\vec{a}^*, \vec{c})$ instead of $G_\Gamma(A, \vec{c})$ where

$$G_\Gamma(\vec{a}^*, \vec{c}) = \begin{pmatrix} G_\Gamma(\vec{c}) \\ \vec{a}^* \end{pmatrix},$$

and $\vec{a} \in \mathbb{C}^{N+1}$.

First we notice that $G_\Gamma(\vec{a}^*, \vec{c})$ having full rank is same as $G_\Gamma^*(\vec{a}^*, \vec{c})$ having full rank, which is equivalent to

$$\vec{a} \notin \text{range}(G_\Gamma^*(\vec{c})) \text{ and } \text{rank}(G_\Gamma^*(\vec{c})) = N.$$

Denote the set

$$E = \{\vec{c} : \text{rank}(G_\Gamma^*(\vec{c})) = N\},$$

By Theorem 3.4.1, E is open dense in \mathbb{C}^N . Suppose there exists some \vec{a} such that no \vec{c} makes $G_\Gamma(\vec{a}^*, \vec{c})$ full rank, then we must have

$$\vec{a} \in \bigcap_{\vec{c} \in E} \text{range}(G_\Gamma^*(\vec{c})).$$

Therefore, to prove this theorem, it suffices for us to show

$$\bigcap_{\vec{c} \in E} \text{range}(G_\Gamma^*(\vec{c})) = \{\vec{0}\}.$$

For ease of writing let us denote the left hand side of the above as F , i.e.,

$$F = \bigcap_{\vec{c} \in E} \text{range}(G_\Gamma^*(\vec{c})).$$

Recall that

$$\ker(G_\Gamma(\vec{c})) \perp \text{range}(G_\Gamma^*(\vec{c})),$$

thus for any $\vec{c} \in E$

$$\ker(G_\Gamma(\vec{c})) \perp F,$$

since F is a subset of $\text{range}(G_\Gamma^*(\vec{c}))$.

Consequently if

$$\mathbb{C}^{N+1} = \text{span}(\{\ker(G_\Gamma(\vec{c})) : \vec{c} \in E\}),$$

then

$$\mathbb{C}^{N+1} \perp F,$$

which implies F must be $\{\vec{0}\}$.

Since $K = 1$, each $\ker(G_\Gamma(\vec{c}))$ is spanned by a single $(N + 1) \times 1$ vector. Therefore it suffices for us to produce a set of $\{\vec{c}_k\}_k$ so that it forms a basis for \mathbb{C}^{N+1} .

Let $\vec{c} \in E$ such that $G_{\mathbb{Z}_N \times \mathbb{Z}_N}(\vec{c})$ is also full spark, existence of such \vec{c} is also guaranteed by Theorem 3.4.1. Denote the unit kernel vector of $G_\Gamma(\vec{c})$ as \vec{x} , full sparkness of $G_\Gamma(\vec{c})$ also implies that no entries in \vec{x} is 0.

Denote the lexicographically ordered elements in Γ as

$$(j_1, k_1), (j_2, k_2), \dots, (j_{N+1}, k_{N+1}),$$

Using Lemma 2.3.1 we get

$$G_\Gamma(M_N^m T_N^n \vec{c}) = M_N^m T_N^n G_\Gamma(\vec{c}) D_{\vec{y}_{m,n}},$$

where

$$\vec{y}_{m,n} = (\omega_N^{nj_1 - mk_1}, \omega_N^{nj_2 - mk_2}, \dots, \omega_N^{nj_{N+1} - mk_{N+1}}).$$

and accordingly

$$D_{\vec{y}_{m,n}} = \text{diag}(\omega_N^{nj_1 - mk_1}, \omega_N^{nj_2 - mk_2}, \dots, \omega_N^{nj_{N+1} - mk_{N+1}}).$$

The above also shows the unit kernel vector of $G_\Gamma(M_N^m T_N^n \vec{c})$ is $D_{\vec{y}_{m,n}}^* \vec{x}$.

Consider the matrix $Y \in \mathbb{C}^{N^2 \times (N+1)}$ formed by stacking the row vectors $\vec{y}_{m,n}^T$. Denote the columns in Y as Y_1, Y_2, \dots, Y_{N+1} , then

$$\langle Y_h, Y_\ell \rangle = \sum_{m,n \in \mathbb{Z}_N} \omega_N^{nj_h - mk_h - nj_\ell + mk_\ell} = \sum_{m \in \mathbb{Z}_N} \sum_{n \in \mathbb{Z}_N} \omega_N^{n(j_h - j_\ell) + m(k_\ell - k_h)} = 0.$$

Thus columns in Y are mutually orthogonal, thus Y is of full rank $N + 1$ (In fact it is not difficult to see that Y consists of columns from the Kronecker product of the Fourier matrix W_N with itself, which is a matrix for discrete bivariate Fourier transform).

The above shows one can pick a subset $\Lambda \in \mathbb{Z}_N \times \mathbb{Z}_N$ such that $\{\vec{y}_{m,n}\}_{(m,n) \in \Lambda}$ is a basis for \mathbb{C}^{N+1} . Thus any linear combination of $\{\vec{y}_{m,n}\}_{(m,n) \in \Lambda}$ is non-zero, which further implies that any linear combination of $\{D_{\vec{y}_{m,n}}^*\}_{(m,n) \in \Lambda}$ is non-zero.

Then we observe by linearity that any linear combination of the kernel vectors $\{D_{\vec{y}_{m,n}}^* \vec{x}\}_{(m,n) \in \Lambda}$ is the same linear combination of $\{(D_{\vec{y}_{m,n}}^*)_{(m,n) \in \Lambda}$ applied to \vec{x} ,

since the former, which is a diagonal matrix is non zero and no entries in \vec{x} is 0, the result vector can not be $\vec{0}$.

Thus $\{D_{\vec{y}_{m,n}}^* \vec{x}\}_{(m,n) \in \Lambda}$ is a set of basis for \mathbb{C}^{N+1} . □

So far very little is understood when $K > 1$, but the situation deteriorates quickly as K grows. In fact, at $K = 2$ there already exists certain Γ and A such that no \vec{c} leads to invertible $G_\Gamma(A, \vec{c})$, and even to make things even worse, at $K = N$, there exists A such $G_\Gamma(A, \vec{c})$ is always rank deficient regardless of the choice of Γ and \vec{c} . See the author's paper [35] for examples and details.

5 Analysis on Wiener-Amalgam Spaces

In this part we further develop some tools for applying our "passing to limit" arguments, Subsection 5.1 and 5.2 provides basic tools for working with $OPW(U, S_0)$ and $OPW(U, S'_0)$, Subsection 5.3 shows studying arbitrary S'_0 identifiers can be reduced to studying periodically weighted delta trains. Even though as in mentioned in the last section, this method is not as successful as we had expected, we include these tools here, as this method still provides us insights. Obstacles for applying this approach are inspected in the last section.

In this whole section, ψ is the window function for Wiener-Amalgam Space, and ϵ_0 is the fixed small number related to the support of ψ , both are as defined in Subsection 2.5.

5.1 Boundedness and Localization

This part gives boundedness criterion for various scenarios.

Lemma 5.1.1. *[5, Lemma 4.1], [42, Prop 4.2] If $g \in S'_0(\mathbb{R})$, $f \in S_0(\mathbb{R})$ and $\eta \in S'_0(\mathbb{R}^2)$ with $\text{supp}(\eta)$ compact, then*

$$\|V_g f\|_{W^{A,\infty}} \lesssim \|g\|_{W^{A',\infty}} \|f\|_{W^{A,1}},$$

$$\|\Phi_g \eta\|_{W^{A',\infty}} \lesssim \|g\|_{W^{A',\infty}} \|\eta\|_{W^{A',\infty}}.$$

First we need an almost trivial observation:

Lemma 5.1.2. *If both f, g are in A space, then*

$$\|fg\|_A \leq \|f\|_A \|g\|_A,$$

alternatively if f is in A space and g is in A' space, then

$$\|fg\|_{A'} \leq \|f\|_A \|g\|_{A'}.$$

Proof. These inequalities directly follows from the convolution theorem and the

Young inequality. For the first inequality we have

$$\|fg\|_A = \|\mathcal{F}^{-1}(fg)\|_{L^1} = \|\check{f} * \check{g}\|_{L^1} \leq \|\check{f}\|_{L^1} \|\check{g}\|_{L^1} = \|f\|_A \|g\|_A,$$

and for the second inequality we have

$$\|fg\|_{A'} = \|\mathcal{F}^{-1}(fg)\|_{L^\infty} = \|\check{f} * \check{g}\|_{L^\infty} \leq \|\check{f}\|_{L^1} \|\check{g}\|_{L^\infty} = \|f\|_A \|g\|_{A'}.$$

□

It follows that we can boundedly restrict S_0 function and S'_0 distribution to a local area with the help of the window ψ :

Lemma 5.1.3. *Let $f \in S_0(\mathbb{R})$, $g \in S'_0(\mathbb{R})$, then for any fixed $\Lambda \subseteq \mathbb{Z}$ we have,*

$$\left\| \sum_{k \in \Lambda} \mathcal{T}_k \psi \cdot g \right\|_{S'_0} \lesssim \|g\|_{S'_0},$$

and for any given small number $r > 0$, there exists large enough number $M \in \mathbb{N}$ such that

$$\left\| f - \sum_{\substack{|k| < M \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot f \right\|_{S_0} \lesssim r.$$

Proof. For the first inequality, we may apply Lemma 5.1.2 to see that for any $j, k \in \mathbb{Z}$ we have

$$\|\mathcal{T}_j \psi \cdot \mathcal{T}_k \psi \cdot g\|_{A'} \leq \|\mathcal{T}_j \psi\|_A \|\mathcal{T}_k \psi \cdot g\|_{A'} = \|\psi\|_A \|\mathcal{T}_k \psi \cdot g\|_{A'} \lesssim \sup_{k \in \mathbb{Z}} \|\mathcal{T}_k \psi \cdot g\|_{A'} = \|g\|_{W^{A', \infty}} \asymp \|g\|_{S'_0},$$

where we absorbed the constant $\|\psi\|_A$ into \lesssim since ψ is fixed.

It follows that

$$\begin{aligned}
\left\| \sum_{k \in \Lambda} \mathcal{T}_k \psi \cdot g \right\|_{S'_0} &\asymp \left\| \sum_{k \in \Lambda} \mathcal{T}_k \psi \cdot g \right\|_{W^{A', \infty}} \\
&= \sup_{j \in \mathbb{Z}} \left\| \mathcal{T}_j \psi \cdot \sum_{k \in \Lambda} \mathcal{T}_k \psi \cdot g \right\| \\
&= \sup_{j \in \mathbb{Z}} \left\| \mathcal{T}_j \psi \cdot \sum_{k \in \Lambda \cap \{j-1, j, j+1\}} \mathcal{T}_k \psi \cdot g \right\| \\
&\lesssim 3 \|g\|_{S'_0} \\
&\lesssim \|g\|_{S'_0},
\end{aligned}$$

where we absorbed the constant 3 in the last step.

For the second inequality, we apply Lemma 5.1.2 again to see that for any $j, k \in \mathbb{Z}$ we have

$$\|\mathcal{T}_j \psi \cdot \mathcal{T}_k \psi \cdot f\|_A \leq \|\mathcal{T}_j \psi\|_A \|\mathcal{T}_k \psi \cdot f\|_A = \|\psi\|_A \|\mathcal{T}_k \psi \cdot f\|_A \lesssim \|\mathcal{T}_k \psi \cdot f\|_A,$$

where we again absorbed the constant $\|\psi\|_A$ into \lesssim since ψ is fixed.

$$\begin{aligned}
\left\| f - \sum_{\substack{|k| < M \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot f \right\|_{S_0} &\asymp \left\| f - \sum_{\substack{|k| < M \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot f \right\|_{W^{A, 1}} \\
&= \left\| \sum_{\substack{|k| \geq M \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot f \right\|_{W^{A, 1}} \\
&= \sum_{j \in \mathbb{Z}} \left\| \mathcal{T}_j \psi \cdot \sum_{\substack{|k| \geq M \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot f \right\|_A \\
&= \sum_{\substack{|j| \geq M-1 \\ j \in \mathbb{Z}}} \left\| \mathcal{T}_j \psi \cdot \sum_{\substack{|k| \geq M \\ k \in \mathbb{Z} \\ k \in \{j, j-1, j+1\}}} \mathcal{T}_k \psi \cdot f \right\|_A \\
&\lesssim 3 \sum_{\substack{|k| \geq M-1 \\ k \in \mathbb{Z}}} \|\mathcal{T}_k \psi \cdot f\|_A,
\end{aligned}$$

$f \in S_0(\mathbb{R})$ implies the sequence $\{\|\mathcal{T}_k\psi \cdot f\|_A\}_{k \in \mathbb{Z}}$ is an ℓ^1 sequence, hence there exists large enough M for the right hand side above to be less than r . \square

The following lemma is technical for the next subsection:

Lemma 5.1.4. *Let $g \in S'_0(\mathbb{R})$, $f \in S_0(\mathbb{R})$ and $U \subset \mathbb{R}^2$ a compact set, then for $1 \leq p \leq \infty$*

$$\|V_g f\|_{L^p(U)} \lesssim C_u \|g\|_{S'_0} \|f\|_{S_0},$$

where $C_u > 0$ is a constant depending only on U . In particular $C_u = 1$ for $p = \infty$.

Proof. First we look at $p = \infty$. By the definition of ψ we have

$$V_g f = \sum_{k \in \mathbb{Z}^2} \mathcal{T}_k \psi \cdot V_g f. \quad (15)$$

By Lemma 5.1.1, $\mathcal{T}_k \psi \cdot V_g f$ is in A space for all k and thus continuous, in particular

$$\|\mathcal{T}_k \psi \cdot V_g f\|_{L^\infty} \leq \|\mathcal{F}^{-1}(\mathcal{T}_k \psi \cdot V_g f)\|_{L^1} = \|\mathcal{T}_k \psi \cdot V_g f\|_A \leq \|V_g f\|_{W^{A,\infty}} \lesssim \|g\|_{W^{A',\infty}} \|f\|_{W^{A,1}},$$

holds for any k .

It is easy to see by the definition of ψ that for each fixed point $(t, v) \in \mathbb{R}^2$, there are at most 4 different k such that $(t, v) \in \text{supp}(\mathcal{T}_k \psi)$, therefore if we restrict (15) in a small neighborhood of (t, v) , the right hand would have at most 4 summands, and thus also continuous with its uniform norm upper bounded by $4\|V_g f\|_{W^{A,\infty}}$ (follows from the inequality above).

Since this holds for any (t, v) , we can conclude that

$$\|V_g f\|_{L^\infty(U)} \leq 4\|V_g f\|_{W^{A,\infty}} \lesssim \|g\|_{W^{A',\infty}} \|f\|_{W^{A,1}} \asymp \|g\|_{S'_0} \|f\|_{S_0}$$

where we absorbed the constant 4.

Since U is compact, the case of $1 \leq p < \infty$ follows with the constant C_u taken as the Lebesgue measure of U . \square

The following lemma can be summarized as, if a delta train has locally summable weights, then it is bounded in S'_0 .

Lemma 5.1.5. *Let $\Lambda \subset \mathbb{R}$ be a countable set, and g be a weighted delta train supported on Λ with complex weights $\{d_a\}_{a \in \Lambda}$, if there exists some $C > 0$ such that*

$$\sum_{a \in \Lambda \cap [x, x+1]} |d_a| \leq C < \infty,$$

holds for any $x \in \mathbb{R}$, then $g \in S'_0$ with $\|g\|_{S'_0} \lesssim C$.

Proof. Let $k \in \mathbb{Z}$, set

$$\Lambda_k = \Lambda \cap \text{supp}(\mathcal{T}_k \psi),$$

then

$$\|\mathcal{F}^{-1}(\mathcal{T}_k \psi \cdot g)\|_{L^\infty} = \|\mathcal{F}^{-1}\left(\sum_{a \in \Lambda_k} d_a \psi(a-k) \delta_a\right)\|_{L^\infty} = \left\| \sum_{a \in \Lambda_k} d_a \psi(a-k) e^{2\pi i a \xi} \right\|_{L^\infty} \leq \sum_{a \in \Lambda_k} |d_a \psi(a-k)|,$$

since $|\psi|$ is upper bounded 1 everywhere by its definition, we get

$$\sum_{a \in \Lambda_k} |d_a \psi(a-k)| \leq \sum_{a \in \Lambda_k} |d_a| \leq 3C,$$

the last inequality holds since by definition

$$\Lambda_k \subset \left(k - \frac{3}{2}, k + \frac{3}{2}\right),$$

which is an interval of length 3, and by assumption the absolute sum of weights are bounded by C on each unit length interval.

It follows that

$$\|g\|_{S'_0} \asymp \|g\|_{W^{A', \infty}} = \sup_{k \in \mathbb{Z}} \|\mathcal{T}_k \psi \cdot g\|_{A'} = \|\mathcal{F}^{-1}(\mathcal{T}_k \psi \cdot g)\|_{L^\infty} \leq 3C.$$

□

Consequently all periodically weighted delta trains are in S'_0 (even though we have

been assuming and using it so far) since there are only finitely many deltas in any interval $(x, x + 1)$ and periodicity ensures local sum of these weights has a global upper bound. We also point out that this lemma allows a delta train to have infinite support in any unit interval, as long as their weights in these intervals are ℓ^1 sequences, for example we can take a delta train supported on \mathbb{Q} .

5.2 Convergences

The main purpose of this part is to show if a sequence of identifiers g_n weak* converge in S'_0 , and a sequence of spreading function η_n also converges in $L^2(U)$, then the identification map Φ_{g_n} and the response $\Phi_{g_n}\eta_n$ also converges in some sense. Unfortunately the mode of convergence here is fairly weak for us to make useful conclusions.

Lemma 5.2.1. *If $g_n \xrightarrow{w^*} g$ in $S'_0(\mathbb{R})$, then for any $f \in S_0(\mathbb{R})$, we have for the short time Fourier transform*

$$|V_{g_n}f - V_gf| \xrightarrow{\text{pointwise}} 0.$$

Proof. Since translation and modulations are automorphisms on $S_0(\mathbb{R})$ and $S'_0(\mathbb{R})$, for each fixed t and w , $T_{-t}M_{-w}f$ is still in $S_0(\mathbb{R})$, therefore using the weak* convergence we get

$$\begin{aligned} (V_{g_n}f)(t, v) &= \int f(x) \overline{g_n(x-t)} e^{-2\pi i v \cdot x} dx \\ &= \langle f, \mathcal{M}_v \mathcal{T}_t g_n \rangle \\ &= \langle \mathcal{T}_{-t} \mathcal{M}_{-v} f, g_n \rangle \\ &\rightarrow \langle \mathcal{T}_{-t} \mathcal{M}_{-v} f, g \rangle \\ &= \langle f, \mathcal{M}_v \mathcal{T}_t g \rangle \\ &= (V_gf)(t, v), \end{aligned}$$

which shows $V_{g_n}f$ converges pointwise. □

Lemma 5.2.2. *Given a compact U , if $K > 0$ is a constant and*

$$\|g_n\|_{S'_0} < K,$$

for all n and $g_n \xrightarrow{w^} g$ in $S'_0(\mathbb{R})$, and $\eta \in L^2(U) \subset S'_0(U)$ with $\text{supp}(\eta) \subseteq U$, then for any $f \in S_0(\mathbb{R})$ we have*

$$|\langle \Phi_{g_n}\eta - \Phi_g\eta, f \rangle| \rightarrow 0.$$

Here the bracket is not L^2 inner product but the dual pairing between S'_0 and S_0 .

Proof. First, we can verify using Lemma 5.1.1 that both $\Phi_{g_n}\eta, \Phi_g\eta$ are in $S'_0(\mathbb{R})$, thus the bracket is well defined.

By assumption, both $\|g_n\|_{S'_0}$ and $\|g\|_{S'_0}$ are bounded by K , and by Lemma 5.1.4,

$$\|(V_{g_n} - V_g)f\|_{L^\infty} \lesssim \|f\|_{S_0},$$

where the constant K is absorbed since it is fixed.

By the previous lemma, $(V_{g_n} - V_g)f$ pointwise converges to 0, while by our assumption U is compact, thus by the Egorov theorem ([16]), for any small ϵ , there exists a subset $U_\epsilon \subset U$ whose measure is bounded by ϵ and $V_{g_n}f$ converges uniformly on $U \setminus U_\epsilon$, i.e.,

$$\|(V_{g_n} - V_g)f\|_{L^\infty(U \setminus U_\epsilon)} \rightarrow 0,$$

which also implies

$$\|(V_{g_n} - V_g)f\|_{L^2(U \setminus U_\epsilon)} \rightarrow 0,$$

since U is compact.

Now for any given small $r > 0$, we choose ϵ so small (i.e., $\epsilon \lesssim r/\|f\|_{S_0}$) such that

$$\|(V_{g_n} - V_g)f\|_{L^2(U_\epsilon)} \leq \epsilon \|(V_{g_n} - V_g)f\|_{L^\infty} \lesssim \epsilon \|f\|_{S_0} \lesssim r,$$

and choose N so big such that for all $n \geq N$ we have

$$\|(V_{g_n} - V_g)f\|_{L^2(U \setminus U_\epsilon)} \leq r.$$

Then we can compute for all $n \geq N$ that

$$\begin{aligned} |\langle (\Phi_{g_n} - \Phi_g)\eta, f \rangle| &= |\langle \eta, (V_{g_n} - V_g)f \rangle| \\ &\leq |\langle \eta, (V_{g_n} - V_g)f \rangle_{L^2(U \setminus U_\epsilon)}| + |\langle \eta, (V_{g_n} - V_g)f \rangle_{L^2(U_\epsilon)}| \\ &\leq \|\eta\|_{L^2(U \setminus U_\epsilon)} \|(V_{g_n} - V_g)f\|_{L^2(U \setminus U_\epsilon)} + \|\eta\|_{L^2(U_\epsilon)} \|(V_{g_n} - V_g)f\|_{L^2(U_\epsilon)} \\ &\lesssim r \|\eta\|_{L^2(U \setminus U_\epsilon)} + r \|\eta\|_{L^2(U_\epsilon)} \\ &\lesssim r \|\eta\|_{L^2(U)} \\ &\lesssim r \end{aligned}$$

where we absorbed $\|\eta\|_{L^2(U)}$ into the constant since it is fixed. It follows that the right hand side of the above can be arbitrarily small, which implies the left hand side goes to 0. \square

Combining what we have together, we can show that weak* convergence of the identifiers and inner approximation of the spreading support indeed pass onto the convergence of the identification map in the weak* operator topology:

Lemma 5.2.3. *If $U \subset \mathbb{R}^2$ is compact, and $K > 0$ is a constant and*

$$\|g_n\|_{S'_0} < K, \quad \text{supp}(\eta_n) \subseteq U,$$

for all n and $g_n \xrightarrow{w^*} g$ in $S'_0(\mathbb{R})$, and $\eta_n \rightarrow \eta$ in $L^2(U)$, then for any $f \in S_0(\mathbb{R})$ we have

$$|\langle \Phi_{g_n}\eta_n - \Phi_g\eta, f \rangle| \rightarrow 0.$$

Similar as in the previous lemma, here the bracket should be viewed as the dual pairing between S'_0 and S_0 .

Proof. By the previous lemma and the convergence assumption on η_n , for any given small $r > 0$, we can choose N big enough so that for $n \geq N$ we have

$$\|\eta_n - \eta\|_{L^2} \leq r.$$

and

$$|\langle \Phi_{g_n} \eta - \Phi_g \eta, f \rangle| \leq r.$$

Consequently using similar approach as in the previous lemma, we have for all $n \geq N$ that

$$\begin{aligned} |\langle \Phi_{g_n} \eta_n - \Phi_g \eta, f \rangle| &= |\langle \Phi_{g_n} \eta_n - \Phi_{g_n} \eta + \Phi_{g_n} \eta - \Phi_g \eta, f \rangle| \\ &\leq |\langle \Phi_{g_n} \eta_n - \Phi_{g_n} \eta, f \rangle| + |\langle \Phi_{g_n} \eta - \Phi_g \eta, f \rangle| \\ &= |\langle \eta_n - \eta, V_{g_n} f \rangle| + |\langle \Phi_{g_n} \eta - \Phi_g \eta, f \rangle| \\ &\leq \|\eta_n - \eta\|_{L^2(U)} \|V_{g_n} f\|_{L^2(U)} + r \\ &\lesssim \|\eta_n - \eta\|_{L^2(U)} K \|f\|_{S_0} + r \\ &\lesssim r + r \\ &\lesssim r, \end{aligned}$$

where the fifth line follows from Lemma 5.1.4 and we absorbed K and $\|f\|_{S_0} + 1$ into constant since they are fixed. \square

5.3 Density of Periodically Weighted Delta Trains

The purpose of this part is to show any $S'_0(\mathbb{R})$ can be weak* approximated by periodically weighted delta trains, consequently if one wants to take an arbitrary $g \in S'_0(\mathbb{R})$ as an identifier, it suffices to study the periodically weighted delta trains that weak* approximates it, for which we already have methods to analyze.

Let $n \in \mathbb{N}$, and in this part we will denote

$$D_n = \left\{ \sum_{k \in \mathbb{Z}} c_k \delta_{\frac{k}{n}} : c_k \in \mathbb{C} \right\},$$

i.e., D_n consists of all weighted delta trains supported on $\frac{1}{n}\mathbb{Z}$. For any member in D_n , since there are only finitely many deltas on any unit length interval, it is easy to see by Lemma 5.1.5 that $D_n \subseteq S'_0(\mathbb{R})$, thus it is well defined.

First we show non-periodically weighted delta trains are weak* dense.

Lemma 5.3.1. *Set*

$$D = \bigcup_{n \in \mathbb{N}} D_n,$$

then D is weak dense on $S'_0(\mathbb{R})$.*

Proof. By the Hahn-Banach theorem, it suffices to show that if $f \in S_0(\mathbb{R})$ and $\langle f, g \rangle = 0$ for all $g \in D$, then f is 0.

To prove this, we readily notice that for all $k \in \mathbb{Z}$, the single delta $\delta_{k/n}$ is in D_n , and it is easy to see that

$$\left\{ \frac{k}{n} : k \in \mathbb{Z}, n \in \mathbb{N} \right\} = \mathbb{Q}.$$

Now $f \in S_0(\mathbb{R})$ means it is continuous, and if

$$\left\langle f, \delta_{\frac{k}{n}} \right\rangle = 0,$$

for all $k \in \mathbb{Z}, n \in \mathbb{N}$, then f vanishes on \mathbb{Q} , which is a dense subset of \mathbb{R} , thus by continuity of f this implies $f = 0$. \square

Next we show non-periodically weighted delta trains can be weak* approximated by periodically weighted delta trains.

Lemma 5.3.2. *Let $n \in \mathbb{N}$ be fixed, and*

$$g = \sum_{k \in \mathbb{Z}} c_k \delta_{\frac{k}{n}}, \quad c_k \in \mathbb{C},$$

a non-periodically weighted delta train in D_n . For any $m \in \mathbb{N}$, define

$$\tilde{g}_m = \sum_{\substack{|k| \leq m \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot g,$$

and

$$g_m = \sum_{j \in \mathbb{Z}} \mathcal{T}_{2mj} \tilde{g}_m,$$

i.e., g_m is a periodically weighted delta train with period $2m$, then

$$\|g_m\|_{S'_0} \lesssim \|g\|_{S'_0},$$

and for any $f \in S_0(\mathbb{R})$, as $m \rightarrow \infty$ we have

$$|\langle f, g \rangle - \langle f, g_m \rangle| \rightarrow 0.$$

Proof. First, it follows immediately from Lemma 5.1.3 that

$$\|\tilde{g}_m\|_{S'_0} \lesssim \|g\|_{S'_0}.$$

Next, for any fixed $j, k \in \mathbb{Z}$, it is clear that exactly one of the following four cases can hold

$$\begin{cases} \text{supp}(\mathcal{T}_k\psi) \cap \text{supp}(\mathcal{T}_{2mj}\tilde{g}_m) = \emptyset \\ \text{supp}(\mathcal{T}_k\psi) \subset \text{supp}(\mathcal{T}_{2mj}\tilde{g}_m) \\ \text{supp}(\mathcal{T}_k\psi) \subset \text{supp}(\mathcal{T}_{2mj}\tilde{g}_m) \cup \text{supp}(\mathcal{T}_{2m(j+1)}\tilde{g}_m) \\ \text{supp}(\mathcal{T}_k\psi) \subset \text{supp}(\mathcal{T}_{2mj}\tilde{g}_m) \cup \text{supp}(\mathcal{T}_{2m(j-1)}\tilde{g}_m), \end{cases}$$

thus apply Lemma 5.1.2 combined with the above inequality we obtain

$$\begin{aligned} \|g_m\|_{S'_0} &\asymp \|g_m\|_{W^{A',\infty}} \\ &= \sup_{k \in \mathbb{Z}} \|\mathcal{T}_k\psi \cdot (\sum_{j \in \mathbb{Z}} \mathcal{T}_{2mj}\tilde{g}_m)\|_{A'} \\ &\leq 2 \sup_{j, k \in \mathbb{Z}} \|\mathcal{T}_k\psi \cdot \mathcal{T}_{2mj}\tilde{g}_m\|_{A'} \\ &= 2 \sup_{j, k \in \mathbb{Z}} \|\mathcal{T}_{k-2mj}\psi \cdot \tilde{g}_m\|_{A'} \\ &\leq 2\|\tilde{g}_m\|_{W^{A',\infty}} \\ &\asymp \|\tilde{g}_m\|_{S'_0} \\ &\lesssim \|g\|_{S'_0} \end{aligned}$$

where we absorbed the constant 2 into \asymp .

Now we may verify that

$$g - g_m = \sum_{\substack{|k|>m \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot g + \sum_{j \in \mathbb{Z} \setminus \{0\}} \mathcal{T}_{2mj} \tilde{g}_m,$$

in particular, this implies that

$$\text{supp}(g - g_m) \subseteq (-\infty, -m - \frac{1}{2} + \epsilon_0] \cup [m + \frac{1}{2} - \epsilon_0, +\infty).$$

For any $m \in \mathbb{N}$, we set

$$f_m = \sum_{\substack{|k| \leq m-1 \\ k \in \mathbb{Z}}} \mathcal{T}_k \psi \cdot f,$$

then it is easy to see that

$$\text{supp}(f_m) \subseteq [-m + 1 - \frac{1}{2} - \epsilon_0, m - 1 + \frac{1}{2} + \epsilon_0],$$

which is disjoint from the support of $g - g_m$, i.e.,

$$\langle f_m, g - g_m \rangle = 0.$$

By Lemma 5.1.3, for any given $r > 0$, we can also choose m large enough such that

$$\|f - f_m\|_{S_0} \leq r,$$

then combining the above together we get

$$\begin{aligned} |\langle f, g \rangle - \langle f, g_m \rangle| &= |\langle f - f_m, g - g_m \rangle + \langle f_m, g - g_m \rangle| \\ &= |\langle f - f_m, g - g_m \rangle| \\ &\leq \|f - f_m\|_{S_0} \|g - g_m\|_{S'_0} \\ &\lesssim r \|g\|_{S'_0} \\ &\lesssim r, \end{aligned}$$

where we absorbed $\|g\|_{S'_0}$ into the constant in the last line since g is fixed. And this implies convergence since r is arbitrary. \square

Theorem 5.3.1. *The space of locally bounded and periodically weighted delta trains is weak* dense in $S'_0(\mathbb{R})$.*

Proof. Given $f \in S_0(\mathbb{R})$ smooth, and $g \in S'_0(\mathbb{R})$ and a precision $r > 0$, we will construct a periodically weighted delta train $g_N \in S'_0(\mathbb{R})$ with N depending on r such that

$$|\langle f, g_N \rangle - \langle f, g \rangle| \lesssim r.$$

First, we apply Lemma 5.3.1 to find a $g_n \in D_n$ such that

$$|\langle f, g_n \rangle - \langle f, g \rangle| \leq r,$$

Next, we apply Lemma 5.3.2 to find a g_N which has period $2N$ such that

$$|\langle f, g_n \rangle - \langle f, g_N \rangle| \leq r,$$

Combining all these above we get

$$|\langle f, g \rangle - \langle f, g_N \rangle| \leq |\langle f, g \rangle - \langle f, g_n \rangle| + |\langle f, g_n \rangle - \langle f, g_N \rangle| \leq 2r.$$

which completes the proof. □

In the following part we illustrate how to construct the non-periodically weighted delta trains in Lemma 5.3.1 to approximate $\langle f, g \rangle$ for compactly supported and smooth f .

Set

$$\psi_n = \psi(nx),$$

i.e.,

$$\psi_n(x) := \begin{cases} 1 & x \in (-\frac{1}{2n} + \frac{\epsilon_0}{n}, \frac{1}{2n} - \frac{\epsilon_0}{n}) \\ 0 & x \in (-\infty, -\frac{1}{2n} - \frac{\epsilon_0}{n}) \cup (\frac{1}{2n} + \frac{\epsilon_0}{n}, \infty) \\ \text{between 0 and 1} & x \in [-\frac{1}{2n} - \frac{\epsilon_0}{n}, -\frac{1}{2n} + \frac{\epsilon_0}{n}] \cup [\frac{1}{2n} - \frac{\epsilon_0}{n}, \frac{1}{2n} + \frac{\epsilon_0}{n}] \end{cases}.$$

and

$$\sum_{k \in \mathbb{Z}} \mathcal{T}_{\frac{k}{n}} \psi_n = 1.$$

We will use linear combinations of their shifts for our approximation, the same way one would follow when using linear combinations of step functions to approximate L^1 functions.

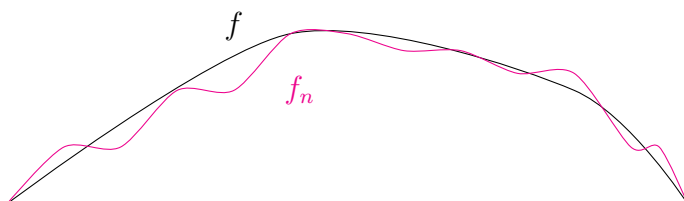


Figure 16: Approximation by Simple Smooth Functions

Lemma 5.3.3. *Let f be a Schwartz class function on \mathbb{R} , and set*

$$f_n(x) = \sum_{k \in \mathbb{Z}} \check{f}\left(\frac{k}{n}\right) (\mathcal{M}_{-\frac{k}{n}} \hat{\psi}_n(x)),$$

then

$$\|f_n - f\|_A \rightarrow 0.$$

Proof. It is clear by definition that both \check{f} and \check{f}_n are continuous, thus by the mean value theorem for integrals, for any fixed k , there exists an $x_k \in [k/n, (k+1)/n]$ such that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\check{f}_n(x) - \check{f}(x)| dx = \frac{1}{n} |\check{f}_n(x_k) - \check{f}(x_k)|.$$

By definition of \check{f}_n and ψ_n , $\check{f}_n(x_k)$ is a convex combination of $\check{f}(\frac{k}{n})$ and $\check{f}(\frac{k+1}{n})$, thus

$$\max(|\check{f}(\frac{k}{n})|, |\check{f}(\frac{k+1}{n})|) \geq |\check{f}_n(x_k)| \geq \min(|\check{f}(\frac{k}{n})|, |\check{f}(\frac{k+1}{n})|).$$

Now, f (and thus \check{f}) is in the Schwartz class means that the derivative of \check{f} exists and is bounded. Consequently we can apply the mean value theorem to further

assert that there exists a point $y_k \in [k/n, (k+1)/n]$ such that

$$|\check{f}_n(x_k) - \check{f}(x_k)| \leq |\max(|\check{f}(\frac{k}{n}) - \check{f}(x_k)|, |\check{f}(\frac{k+1}{n}) - \check{f}(x_k)|)| \leq \frac{1}{n} |\check{f}'(y_k)|.$$

Fix a large $N \in \mathbb{N}$, denote

$$I_N = [-N, N], \quad I_N^c = \mathbb{R} \setminus [-N, N].$$

while on I_N^c , \check{f} is in the Schwartz class implies there exists a constant C such that

$$C \geq \sup_{x \in \mathbb{R}} |x^2 \check{f}'(x)| \quad \Rightarrow \quad |\check{f}'(x)| \leq \frac{C}{x^2}, \quad \forall |x| \geq N$$

therefore

$$\begin{aligned} \|\check{f}_n - \check{f}\|_{L^1(I_N^c)} &= \sum_{|\frac{k}{n}| \geq N} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |\check{f}_n(x) - \check{f}(x)| dx \\ &= \frac{1}{n} \sum_{|\frac{k}{n}| \geq N} |\check{f}_n(x_k) - \check{f}(x_k)| \\ &\leq \frac{1}{n^2} \sum_{|\frac{k}{n}| \geq N} |\check{f}'(y_k)| \\ &\leq \frac{2}{n^2} \sum_{k \geq nN} \frac{C}{y_k^2} \\ &\leq \frac{2C}{n^2} \sum_{k \geq nN} \frac{n^2}{k^2} \\ &= 2C \sum_{k \geq nN} \frac{1}{k^2} \\ &\leq 2C \int_{nN-1}^{+\infty} \frac{1}{x^2} dx \\ &= \frac{2C}{nN-1}, \end{aligned}$$

while on I_N , by the previous argument we simply have

$$\|\check{f}_n - \check{f}\|_{L^\infty} \leq \frac{1}{n} \|\check{f}'\|_{L^\infty},$$

thus

$$\|\check{f}_n - \check{f}\|_{L^1(I_N)} \leq 2N \frac{\|\check{f}'\|_{L^\infty}}{n},$$

consequently, if we choose

$$n \geq \max\left(\frac{2N\|\check{f}'\|_{L^\infty}}{r}, \frac{2C}{Nr} + \frac{1}{N}\right),$$

$$\begin{aligned} \|\check{f}_n - \check{f}\|_{L^1} &= \|\check{f}_n - \check{f}\|_{L^1(I_N)} + \|\check{f}_n - \check{f}\|_{L^1(I_N^c)} \\ &\leq 2N \frac{\|\check{f}'\|_{L^\infty}}{n} + \frac{2C}{nN-1} \\ &\leq 2r. \end{aligned}$$

holds for any given small r , which implies the convergence. \square

Lemma 5.3.4. *Denote*

$$U = \left[-\frac{1}{2} + \epsilon_0, \frac{1}{2} - \epsilon_0\right],$$

i.e., where the window function ψ takes value 1. Let $g \in S'_0(\mathbb{R})$ with

$$\text{supp}(g) \subseteq U,$$

for any $n \in \mathbb{N}$ define

$$g_n = \sum_{k \in \mathbb{Z}} \left\langle \mathcal{M}_{\frac{k}{n}} \psi_n, g \right\rangle e^{2\pi i \frac{k}{n}},$$

then for any smooth $f \in S_0(\mathbb{R})$ with

$$\text{supp}(f) \subseteq U,$$

as $n \rightarrow \infty$, we have

$$|\langle f, g_n \rangle - \langle f, g \rangle| \rightarrow 0.$$

Proof. Since f is compactly supported and smooth, it is in the Schwartz class,

thus let f_n be the smooth approximation defined in Lemma 5.3.3, then we may compute

$$\langle f_n, g \rangle = \sum_{k \in \mathbb{Z}} \check{f}\left(\frac{k}{n}\right) (\mathcal{M}_{\frac{k}{n}} \hat{\psi}_n, g),$$

while

$$\langle f, g_n \rangle = \left\langle \mathcal{M}_{\frac{k}{n}} \check{\psi}_n, g \right\rangle \left\langle f, e^{-2\pi i \frac{k}{n}} \right\rangle = \left\langle \mathcal{M}_{\frac{k}{n}} \check{\psi}_n, g \right\rangle \check{f}\left(\frac{k}{n}\right) = \langle f_n, g \rangle.$$

The support condition on g implies that we can rewrite

$$g = \psi \cdot g.$$

Now for any given small $r > 0$ we choose n large enough such that

$$\|f - f_n\|_A < r.$$

Combining the above together we get

$$\begin{aligned} |\langle f, g - g_n \rangle| &= |\langle f - f_n, g \rangle| \\ &= |\langle f - f_n, \psi \cdot g \rangle| \\ &\leq \|f - f_n\|_A \|\psi \cdot g\|_{A'} \\ &\leq \|f - f_n\|_A \|g\|_{W^{A', \infty}} \\ &\lesssim \|f - f_n\|_A \\ &\lesssim r, \end{aligned}$$

where we absorbed $\|g\|_{W^{A', \infty}}$ as a constant into \lesssim since g is fixed. Since r is arbitrary, this implies convergence. \square

And \hat{g} is the non-periodically weighted delta train as asserted in Lemma 5.3.1, we can further use Lemma 5.3.2 to approximate it with a sequence of periodically weighted delta train \hat{g}_n , and finally Lemma 2.6.1 shows g_n would also be a periodically weighted delta train. For f, g compactly supported in other sets, we may simply apply a scaling on both, then apply the above lemma.

6 Operator Identification Revisited

This part enlists main results of this thesis, Subsection 6.1 and Subsection 6.2 are the two major results of this thesis, while the rest minor results also find their applications. Results in Subsection 6.1 stems from the author's paper [44], while results in Subsection 6.3 are repeated from the author's paper [35].

6.1 Unitarily Identifiable Domains

Theorem 4.2.2 and Theorem 4.2.2 gives sufficient conditions for a domain U to be unitarily identifiable. In this part we give some explicit examples of such domains.

Let us consider $\mathbb{Z}_N \times \mathbb{Z}_N$ and the associated grid mapping as stated in Subsection 4.3. Moreover, we assume each component square box in the grid is of side length $1/\sqrt{N}$, as used in Subsection 3.2. An ***L-shaped domain of order N*** is a simply connected domain which is the union of boxes labeled by $\{(0, 0), (0, 1), \dots, (0, K)\} \cup \{(1, 0), (2, 0), \dots, (N - K - 1, 0)\}$ for some $1 \leq K \leq N - 2$.

Below are some examples of such domains:

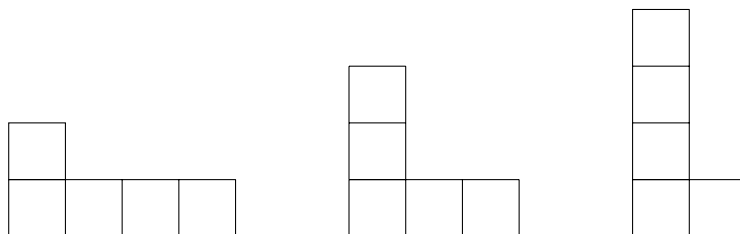


Figure 17: Examples of L -Shaped Domains

Proposition 6.1.1. *An L -shaped domain of order N is unitarily identifiable by the periodically weighted delta train $\mathbf{g}_{\vec{c}}$ where $\vec{c} = (c_1, c_2, \dots, c_N)$ with*

$$\vec{c}_\ell = \begin{cases} \frac{1}{\sqrt[4]{N}} \omega_N^{\frac{\ell(\ell-1)}{2}}, & \ell = 1, 2, \dots, N \\ \frac{1}{\sqrt[4]{N}} \omega_N^{\frac{(\ell-1)^2}{2}}, & \ell = 1, 2, \dots, N \end{cases}.$$

Proof. Given an L -shaped domain of order N , let us call it U , and denote $\Gamma \in \mathbb{Z}_N \times \mathbb{Z}_N$ as its corresponding boxes. We will show

$$\Gamma \times V_1 = \mathbb{Z}_N \times \mathbb{Z}_N,$$

where V_1 is as defined in Lemma 4.1.1.

If this holds, then by Theorem 4.2.1, we may take a shared eigenvector of members in (MT, V_1) , for which we take the first column, denoted as \vec{x} , of the eigenmatrices specified in Lemma 4.1.4 and Lemma 4.1.5, and they are

$$\vec{x}_\ell = \begin{cases} \frac{1}{\sqrt{N}} \omega_N^{\frac{\ell(\ell-1)}{2}}, & \ell = 0, 1, 2, \dots, N \\ \frac{1}{\sqrt{N}} \omega_N^{\frac{(\ell-1)^2}{2}}, & \ell = 0, 1, 2, \dots, N \end{cases}.$$

This makes the Gabor matrix $G_\Gamma(\vec{x})$ unitary, and we can then set

$$\vec{c} = \sqrt[4]{N} \vec{x},$$

and conclude by Corollary 3.3.2 that $\Phi_{\mathfrak{g}_\vec{c}}$ is an isometry from $L^2(U)$ to $L^2(\mathbb{R})$

By Lemma 4.2.4, to show $\Gamma \times V_1$ is the whole group $\mathbb{Z}_N \times \mathbb{Z}_N$ it suffice to establish that

$$\Delta\Gamma \cap V_1 = \emptyset.$$

Suppose

$$\Gamma = \{(0, 0)\} \cup \{(1, 0), (2, 0), \dots, (J, 0)\} \cup \{(0, 1), \dots, (0, K)\},$$

as plotted below:

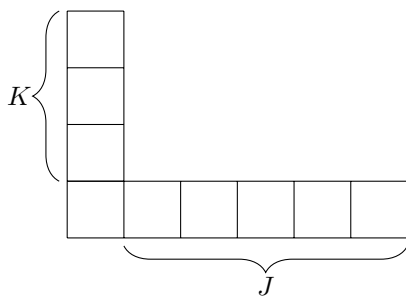


Figure 18: An L -Shaped Domain

Clearly the difference of any elements from the horizontal part $\{(0, 0), (1, 0), (2, 0), \dots, (J, 0)\}$ is always in V_∞ while the difference of any elements from the vertical part $\{(0, 0), (0, 1), \dots, (0, K)\}$ is always in V_0 .

Therefore it suffice it suffice to look at the difference between two element of form $(j, 0)$ and $(0, k)$. If

$$(j, 0) - (0, k) = (j, -k) \in V_1,$$

then

$$j = -k \pmod{N},$$

i.e.,

$$j + k = N,$$

but obviously we have

$$j + k \leq J + K = N - 1,$$

hence their difference can not be in V_1 , which implies

$$\Delta\Gamma \cap V_1 = \emptyset.$$

□

Now consider those $N \in 2\mathbb{N}+1$ with $N = 2J+2K+1$ for some $J, K \in \mathbb{N}$. A ***cross-shaped domain of order N*** is a simply connected domain which is the union of boxes labeled by $\{(0, -K), (0, -K+1), \dots, (0, K)\} \cup \{(1, 0), (2, 0), \dots, (J, 0)\} \cup \{(-1, 0), (-2, 0), \dots, (-J, 0)\}$.

Below are some examples of such domains:

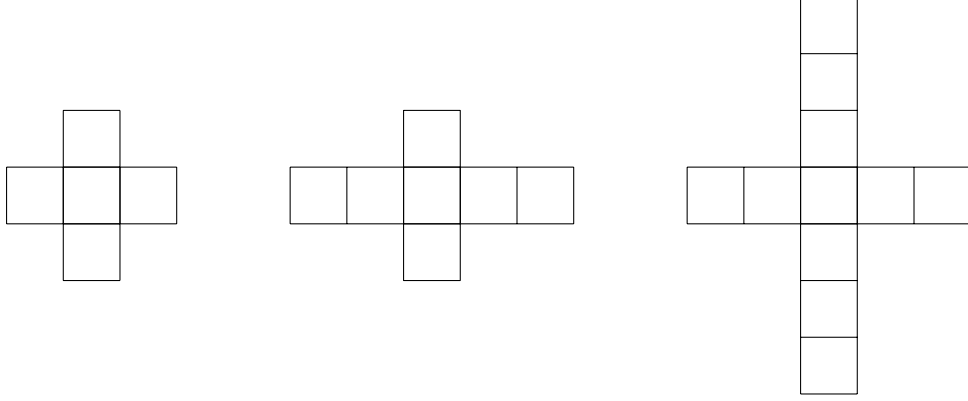


Figure 19: Examples of Cross-Shaped Domains

Proposition 6.1.2. *If $K = 1$, then a cross-shaped domain of order $N = 2J + 2K + 1$ is unitarily identifiable by the periodically weighted delta train $\mathfrak{g}_{\vec{c}}$ where $\vec{c} = (c_1, c_2, \dots, c_N)$ with*

$$\vec{c}_\ell = \frac{1}{\sqrt[4]{N}} \omega_N^{\frac{(J+1)\ell(\ell-1)}{2}}, \quad \ell = 1, 2, \dots, N.$$

Proof. Given an cross-shaped domain of order $N = 2J + 2K + 1$ with $K = 1$, let us call it U , and denote $\Gamma \in \mathbb{Z}_N \times \mathbb{Z}_N$ as its corresponding boxes. We will just show

$$\Gamma \times V_{J+1} = \mathbb{Z}_N \times \mathbb{Z}_N,$$

where V_{J+1} is as defined in Lemma 4.1.1. Then repeating the same argument from the last proposition, we simply take the first column of the shared eigenmatrix of (MT, V_{J+1}) , and scaled it by $\sqrt[4]{N}$ to get \vec{c} .

By Lemma 4.2.4, to show $\Gamma \times V_{J+1}$ is the whole group $\mathbb{Z}_N \times \mathbb{Z}_N$ it suffice to establish that

$$\Delta\Gamma \cap V_{J+1} = \emptyset.$$

By definition

$$\Gamma = \{(0, -1), (0, 0), (0, 1)\} \cup \{(1, 0), (2, 0), \dots, (J, 0)\} \cup \{(-1, 0), (-2, 0), \dots, (-J, 0)\},$$

as plotted below:

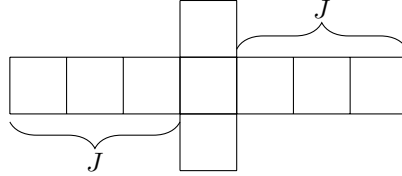


Figure 20: A Cross-Shaped Domain with $K = 1$

Clearly the difference of any elements from the horizontal part $\{(-J, 0), (-J + 1, 0), \dots, (J, 0)\}$ is always in V_∞ while the difference of any elements from the vertical part $\{(0, -1), (0, 0), (0, 1)\}$ is always in V_0 .

Therefore it suffice it suffice to look at the difference between two element of form $(j, 0)$ and $(0, \pm 1)$. If

$$(j, 0) - (0, \pm 1) = (j, \pm 1) \in V_{J+1},$$

then

$$j = \pm(J + 1) \pmod{N}.$$

Now since by definition

$$|j| < J,$$

the only possibility would be

$$j = -(J + 1) \pmod{N},$$

but then

$$|j| = N - J - 1 = 2J + 3 - J - 1 = J + 2 > J,$$

which is a contradiction, therefore $\Delta\Gamma$ can not intersect V_{J+1} non-trivially. \square

In general, any rectification that satisfies Theorem 4.2.1 would be unitarily identifiable:

Proposition 6.1.3. *If U is N -th rectifiable and its rectification Γ satisfies the condition in Theorem 4.2.1, then it is unitarily identifiable by a periodically weighted delta train.*

Proof. If Γ satisfies the condition in Theorem 4.2.1, then we simply take \vec{c} to be a shared eigenvector of the corresponding isotropic subgroup with unit norm, then the Gabor matrix $G_\Gamma(\vec{c})$ is unitary, and thus $\mathfrak{g}_{\vec{c}}$ induces an isometry. \square

If N is prime, then Theorem 4.2.2 shows the condition is also necessary for identifying rectifiable domains using periodically weighted delta trains. In practice, to check whether the condition holds, we apply Lemma 4.2.4 and compute the difference set to see if it misses an isotropic subgroup.

6.2 A Universal Identifier for All Rectifications

In this part we will construct an identifier that identifies all rectifiable domain.

Theorem 6.2.1. *There exists an identifier that identifies all rectifiable domains.*

Proof. The proof is by induction.

First let us consider an arbitrary 4-th rectifiable domains $U^{(1)}$, by Theorem 3.4.1, the set of vectors \vec{c} that makes the Gabor matrix $G_{\mathbb{Z}_4 \times \mathbb{Z}_4}(\vec{c})$ full spark is open dense in \mathbb{C}^4 .

By definition, the rectification of $U^{(1)}$, denoted as Γ , satisfies $|\Gamma| \leq 4$, hence by full sparkness, columns of $G_\Gamma(\vec{c})$ are linearly independent, i.e., $G_\Gamma(\vec{c})$ is injective for all choices of 4-th rectifiable domains. Pick any \vec{c} with unit ℓ^1 norm from the above set, denote it as $\vec{c}^{(1)}$, then $\mathfrak{g}_{\vec{c}^{(1)}}$ identifies all 4-th rectifiable domain.

By Corollary 3.3.2, for any $U \in U^{(1)}$, $\Phi_{\mathfrak{g}_{\vec{c}^{(1)}}}$ is upper and lower bounded from $L^2(U^{(1)})$ to $L^2(\mathbb{R})$, denote the largest upper bound and the smallest lower bound

among all possible $U^{(1)}$ as $\lambda_{\max}^{(1)}$, $\lambda_{\min}^{(1)}$ respectively. i.e.,

$$\lambda_{\max}^{(1)} = \max_{\{\Gamma:|\Gamma|\leq 4\}} \frac{1}{2} \sigma_{\max}(G_{\Gamma}(\vec{c}^{(1)})),$$

$$\lambda_{\min}^{(1)} = \min_{\{\Gamma:|\Gamma|\leq 4\}} \frac{1}{2} \sigma_{\min}(G_{\Gamma}(\vec{c}^{(1)})),$$

where σ_{\max} and σ_{\min} are respectively the largest and smallest singular values of a matrix, and the scaling factor $1/2$ is justified by Corollary 3.3.2. Here max and min are well defined as the rectification Γ is at most 4 boxes from the grid $\mathbb{Z}_4 \times \mathbb{Z}_4$, hence there are only finitely many patterns Γ can take.

By this definition, we have

$$\lambda_{\min}^{(1)} \|\eta\|_{L^2} \leq \|\Phi_{\mathfrak{g}_{\vec{c}^{(1)}}} \eta\|_{L^2} \leq \lambda_{\max}^{(1)} \|\eta\|_{L^2},$$

for all η supported on 4-th rectifiable domains.

Next, suppose we have already chosen a length 2^{n+1} vector $\vec{c}^{(n)}$ such that $\mathfrak{g}_{\vec{c}^{(n)}}$ identifies all 2^{n+1} -th rectifiable domains $U^{(n)}$ with the largest upper bound and the smallest lower bounds of $\Phi_{\mathfrak{g}_{\vec{c}^{(n)}}}$ denoted as $\lambda_{\max}^{(n)}$, $\lambda_{\min}^{(n)}$ respectively, i.e.,

$$\lambda_{\min}^{(n)} \|\eta\|_{L^2} \leq \|\Phi_{\mathfrak{g}_{\vec{c}^{(n)}}} \eta\|_{L^2} \leq \lambda_{\max}^{(n)} \|\eta\|_{L^2},$$

holds for all η supported on 2^{n+1} -th rectifiable domains.

We embed $\vec{c}^{(n)} \in \mathbb{C}^{2^{n+1}}$ into $\mathbb{C}^{2^{n+2}}$ by inserting a 0 after each entry in $\vec{c}^{(n)}$, and denote the resulting vector as \vec{d} , i.e.,

$$\vec{d}^{(n)} = (\vec{c}_1^{(n)}, 0, \vec{c}_2^{(n)}, 0, \dots, \vec{c}_{2^{n+1}}^{(n)}, 0).$$

Set

$$r_n = \min\left(\frac{1}{3^n}, \frac{1}{3} \lambda_{\min}^{(n)}, \frac{1}{3^2} \lambda_{\min}^{(n-1)}, \dots, \frac{1}{3^n} \lambda_{\min}^{(1)}\right),$$

and consider an ℓ^1 ball of radius r_n around $\vec{d}^{(n)}$, we pick an element \vec{z} from this

ball such that $G_{\mathbb{Z}_2(n+2) \times \mathbb{Z}_2(n+2)}(\vec{z})$ has full spark. Existence of such an element is guaranteed by Theorem 3.4.1.

We then set this $\vec{z}^{(n)}$ as $\vec{c}^{(n+1)}$ for the next step.

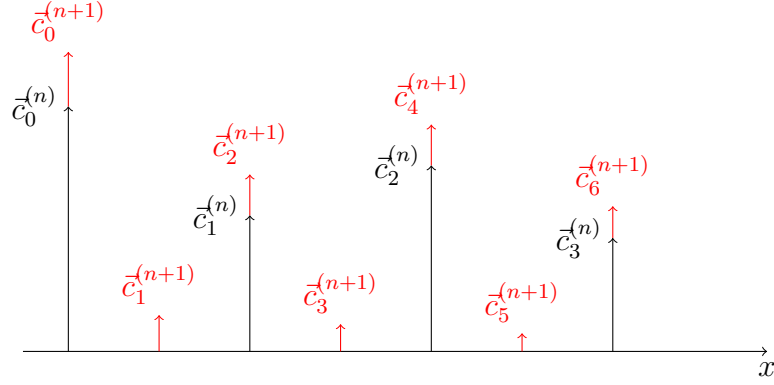


Figure 21: Each $\mathbf{g}_{\vec{c}^{(n)}}$ is a Perturbation of Its Predecessor

Now let us check first that the sequence $\{\mathbf{g}_{\vec{c}^{(n)}}\}_n$ is convergent in S'_0 . Recall Lemma 5.1.5, we have

$$\|\mathbf{g}_{\vec{c}^{(n+1)}} - \mathbf{g}_{\vec{c}^{(n)}}\|_{W^{A', \infty}} = \|\mathbf{g}_{\vec{c}^{(n+1)} - \vec{d}^{(n)}}\|_{W^{A', \infty}} \lesssim r_n \leq \frac{1}{3^n},$$

thus by induction we get

$$\begin{aligned} \|\mathbf{g}_{\vec{c}^{(n+k)}} - \mathbf{g}_{\vec{c}^{(n)}}\|_{W^{A', \infty}} &\lesssim r_{n+k-1} + r_{n+k-2} + \dots + r_n \\ &= \frac{1}{3^{n+k-1}} + \frac{1}{3^{n+k-2}} + \dots + \frac{1}{3^n} \\ &\leq \frac{1}{3^n} \frac{1}{1 - \frac{1}{3}} \\ &= \frac{1}{2 \cdot 3^{n-1}}, \end{aligned}$$

which shows $\{\mathbf{g}_{\vec{c}^{(n)}}\}_n$ is a Cauchy sequence in $W^{A', \infty}$ and thus has a limit. We denote this limit as \mathbf{g} .

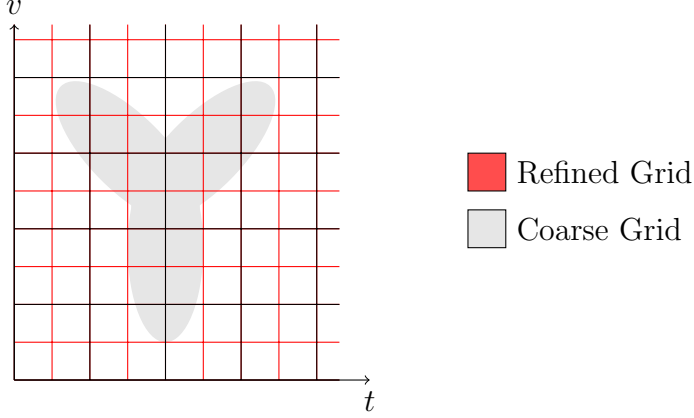


Figure 22: An Example of Refinement

With this construction, on any n -th rectifiable domain $U^{(n)}$ we have for $\eta \neq 0$

$$\begin{aligned}
\|\Phi_{\mathbf{g}}\eta\|_{L^2(\mathbb{R})} &= \|\Phi_{\mathbf{g}_{\mathcal{C}^{(n)}}}\eta + (\Phi_{\mathbf{g}} - \Phi_{\mathbf{g}_{\mathcal{C}^{(n)}}})\eta\|_{L^2(\mathbb{R})} \\
&\geq |\lambda_{\min}^{(n)}\|\eta\|_{L^2(\mathbb{R})} - \|(\Phi_{\mathbf{g}} - \Phi_{\mathbf{g}_{\mathcal{C}^{(n)}}})\eta\|_{L^2(\mathbb{R})}| \\
&\geq |\lambda_{\min}^{(n)} - (r_n + r_{n+1} + \dots)|\|\eta\|_{L^2(\mathbb{R})} \\
&\geq |\lambda_{\min}^{(n)} - (\frac{1}{3}\lambda_{\min}^{(n)} + \frac{1}{3^2}\lambda_{\min}^{(n)} + \dots)|\|\eta\|_{L^2(\mathbb{R})} \\
&= |\lambda_{\min}^{(n)} - \frac{1}{2}\lambda_{\min}^{(n)}|\|\eta\|_{L^2(\mathbb{R})} \\
&= \frac{1}{2}\lambda_{\min}^{(n)}\|\eta\|_{L^2(\mathbb{R})} > 0,
\end{aligned}$$

and similarly

$$\begin{aligned}
\|\Phi_{\mathbf{g}}\|_{L^2(U^{(n)}) \rightarrow L^2(\mathbb{R})} &= \|\Phi_{\mathbf{g}_{\mathcal{C}^{(n)}}} + \Phi_{\mathbf{g}} - \Phi_{\mathbf{g}_{\mathcal{C}^{(n)}}}\|_{L^2(U^{(n)}) \rightarrow L^2(\mathbb{R})} \\
&\leq \lambda_{\max}^{(n)} + \|\Phi_{\mathbf{g}} - \Phi_{\mathbf{g}_{\mathcal{C}^{(n)}}}\|_{L^2(U^{(n)}) \rightarrow L^2(\mathbb{R})} \\
&\leq \lambda_{\max}^{(n)} + (r_n + r_{n+1} + \dots) \\
&\leq \lambda_{\max}^{(n)} + (\frac{1}{3}\lambda_{\min}^{(n)} + \frac{1}{3^2}\lambda_{\min}^{(n)} + \dots) \\
&\leq \lambda_{\max}^{(n)} + \frac{1}{2}\lambda_{\min}^{(n)}.
\end{aligned}$$

Together these mean \mathbf{g} induces a bounded injective identification map on all $U^{(n)}$,

i.e., it identifies all rectifications. □

6.3 Linear Constraints: Identification of Overspread or MIMO Channels

As Theorem 3.4.3 reveals, it is in general not possible to identify overspread domains. However, in practice, values of the spreading function at different locations might be correlated due to, say, mutual interference or similar configurations. If one knows *a priori* these correlation, then one might be able to leverage these extra information. For simplicity, we will assume these correlations are linear.

To start, let us look at a very simple motivating example as depicted in the figure below

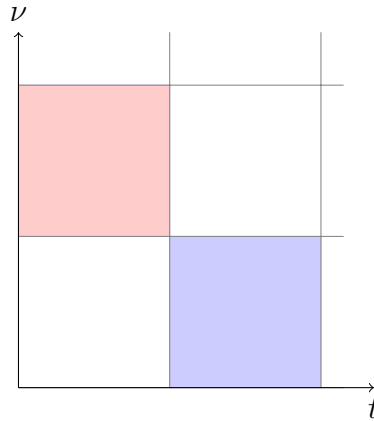


Figure 23: An Example of Linear Correlations

Suppose our spreading function η is supported on the colored boxes (called U) in the above graph, and both the red box (called U_r) and the blue box (called U_b) have area 1. Then Theorem 3.4.3 asserts that $OPW(U, L^2)$ is not identifiable. Now if for some reason, we know that the value of the spreading function on the red box (denoted as η_r) equals the value of the spreading function on the blue box (denoted as η_b), then the whole space would be identifiable. Indeed, if we denote Φ_r and Φ_b as the pseudo differential operator Φ restricted to the red and blue

boxes respectively, and values of their corresponding spreading functions η_r and η_b coincide, then for any test identifier g we have

$$\begin{aligned}
\Phi_b g &= \int \eta_b(t, v) \mathcal{M}_v \mathcal{T}_t g \, dt dv \\
&= \int \eta_r(t - 1, v + 1) \mathcal{M}_v \mathcal{T}_t g \, dt dv \\
&= \int \eta_r(t, v) \mathcal{M}_{v-1} \mathcal{T}_{t+1} g \, dt dv \\
&= \int \eta_r(t, v) \mathcal{M}_v \mathcal{M}_{-1} \mathcal{T}_t \mathcal{T}_1 g \, dt dv \\
&= \int e^{-2\pi i t} \eta_r(t, v) \mathcal{M}_v \mathcal{T}_t (\mathcal{M}_{-1} \mathcal{T}_1 g) \, dt dv,
\end{aligned}$$

thus if we choose g to be $\mathbf{g}_{\bar{1}}$, i.e., the unweighted delta train supported on \mathbb{Z} , and notice that

$$\mathcal{M}_{-1} \mathcal{T}_1 \mathbf{g}_{\bar{1}} = \mathbf{g}_{\bar{1}},$$

then the response $\Phi \mathbf{g}_{\bar{1}}$ can be written as

$$\begin{aligned}
\Phi \mathbf{g}_{\bar{1}} &= (\Phi_r + \Phi_b) \mathbf{g}_{\bar{1}} \\
&= \int \eta_r(t, v) \mathcal{M}_v \mathcal{T}_t \mathbf{g}_{\bar{1}} \, dt dv + \int e^{-2\pi i t} \eta_r(t, v) \mathcal{M}_v \mathcal{T}_t (\mathcal{M}_{-1} \mathcal{T}_1 \mathbf{g}_{\bar{1}}) \, dt dv \\
&= \int (1 + e^{-2\pi i t}) \eta_r(t, v) \mathcal{M}_v \mathcal{T}_t \mathbf{g}_{\bar{1}} \, dt dv,
\end{aligned}$$

Clearly $(1 + e^{-2\pi i t}) \eta_r(t, v) \in L^2(U_r)$ since $\eta_r(t, v) \in L^2(U_r)$, thus by the results in Subsection 3.1, the response $\Phi \mathbf{g}_{\bar{1}}$ can not be 0 if Φ is not 0, i.e., $\mathbf{g}_{\bar{1}}$ identifies the above space.

In general, linear constraints on the values of spreading functions supported on of a rectified domains can be characterized by a set of linear equations

$$A \vec{\eta} = 0,$$

where as in Subsection 3.2, $\vec{\eta}$ is the vectorization of the spreading function with respect to the rectification. For instance, the condition in the above example that

values of the spreading in the red and the blue boxes coincides can be written as

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \eta_r \\ \eta_b \end{pmatrix} = 0.$$

As derived in Subsection 3.2, the identification procedure on a rectifiable domain using properly designed periodically weighted delta train essentially reduces to solving the simple linear system

$$G_\Gamma(\vec{c})\vec{\eta} = \vec{y}.$$

Here, with extra linear constraints we attempt to solve the above two equation simultaneously, and thus can write them into a compacter form as follows

$$\begin{pmatrix} G_\Gamma(\vec{c}) \\ A \end{pmatrix} \vec{\eta} = \begin{pmatrix} \vec{y} \\ 0 \end{pmatrix}.$$

In the usual case, if U is an overspread domain, then $G_\Gamma(\vec{c})$ will have more columns than rows, thus force it to have a non-trivial kernel, but with sufficiently many linear constraints, the concatenated matrix $\begin{pmatrix} G_\Gamma(\vec{c}) \\ A \end{pmatrix}$ could be square, thus making it possible to be inverted. In particular, if there is only 1 row in A , then Theorem 4.5.1 shows it is always possible to pick some good window vector for this to happen. See counter examples in [35] for the case of A having more than 1 row.

6.4 Identifying Overspread Channels with Real Valued Symbols

For illustration purpose, we consider a simpler case that $\eta \in L^2(\mathbb{R}^2)$ is supported on the square $(-1, 1) \times (-1, 1)$ and $\hat{\eta}$ is a real valued function. We write the restriction of η to the k -th quadrant as η_{k-1} , as depicted in the figure below.

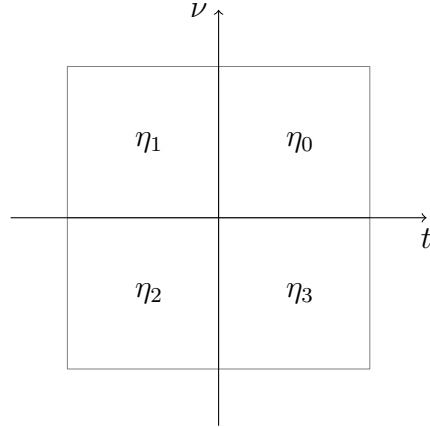


Figure 24: Real Valued Symbol Bandlimited to 2×2 Square

We may expand η_0 by exponential basis:

$$\eta_0(t, v) = \sum_{j \in \mathbb{Z}} f_j(t) e_{(0,1)}^{2\pi i j v}.$$

Now since $\hat{\eta}$ is real valued, we have

$$\eta_1(t, v) = \sum_{j \in \mathbb{Z}} \bar{f}_j(-t) e_{(0,1)}^{2\pi i j v},$$

$$\eta_2(t, v) = \sum_{j \in \mathbb{Z}} \bar{f}_j(-t) e_{(-1,0)}^{2\pi i j v},$$

$$\eta_3(t, v) = \sum_{j \in \mathbb{Z}} f_j(t) e_{(-1,0)}^{2\pi i j v},$$

Thus we can combine them and write η as

$$\eta(t, v) = \begin{cases} \sum_{j \in \mathbb{Z}} \tilde{f}_j(t) e_{(-1,1)}^{2\pi i j v} & t \in (-1, 0] \\ \sum_{j \in \mathbb{Z}} f_j(t) e_{(-1,1)}^{2\pi i j v} & t \in (0, 1) \end{cases},$$

where

$$\tilde{f}_j(t) = \bar{f}_j(-t),$$

i.e., \tilde{f}_j is f_j reflected (with respect to the origin) and then complex conjugated.

With similar computation as in Subsection 3.1, we see that $\mathcal{F}(\frac{1}{2}\Phi_{\mathbf{u}_\mp}\eta)(\xi)$, i.e., the frequency side of the response of such a pseudo-differential operator tested on the unweighted exponential train supported on \mathbb{Z} , is as depicted in the following figure

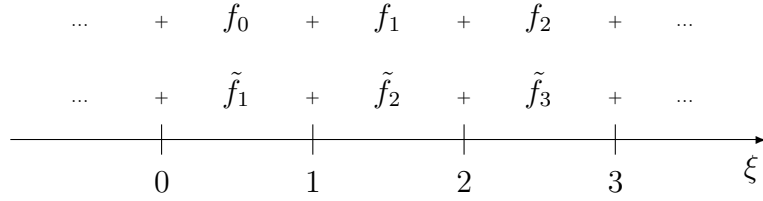


Figure 25: Fourier Transform of the Response

It is easy to see from above that if the response is 0, then

$$f_0 = -\tilde{f}_1, \quad f_1 = -\tilde{f}_2, \quad f_2 = -\tilde{f}_3, \quad \dots,$$

which means

$$\dots = f_0 = f_2 = f_4 = \dots = f_{2k} = \dots, \quad k \in \mathbb{Z},$$

and

$$\dots = f_1 = f_3 = f_5 = \dots = f_{2k+1} = \dots, \quad k \in \mathbb{Z},$$

but since η is in L^2 , we necessarily have

$$\|f_j\|_{L^2} \rightarrow 0, \quad \text{as } |j| \rightarrow \infty,$$

hence the response is 0 only if $f_j = 0$ for all j , i.e., η is 0, which implies injectivity.

We wrap the conclusion of the above derivation into the following theorem:

Theorem 6.4.1. *The subspace of all real values symbols from $OPW(U, L^2)$, where U is the box $(-1, 1)^2$, is identifiable by \mathbf{u}_\mp .*

If U has area 4 and is symmetric with respect to a point on the time-frequency plane, then one could expect to adopt a similar approach as above to show its identifiability.

7 Open Problems and Future Directions

To study whether $OPW(U, L^2)$ is identifiable for U of area 1, the core method we intended to adopt is the so called "pass to limit" method, that is, we approximate U from inside by rectifiable domains, we analyze identifiers on these sub-domains using Gabor matrices, and try to show the weak* limit of these identifiers is an identifier for U with some tools from Wiener-Amalgam spaces.

This strategy is inspired by the following fact: If we denote X as the time axis on the time-frequency plane, and X_K as its restriction to the interval $(-K/2, K/2)$, then the space of convolution operators $OPW(X, S'_0)$ can be identified by δ_0 , which is the weak* limit of $\sum_{n \in K\mathbb{N}} \delta_n$ as $K \rightarrow \infty$, and each $\sum_{n \in K\mathbb{N}} \delta_n$ identifies $OPW(X_K, S'_0)$.

Technically this method reduces to the following subquestions that have assumptions from strong to weak:

Let $\{g_n\}_n$ be a sequence of S'_0 distributions that is weak* convergent to some g on S'_0 , and let $\{U_n\}_n$ be a sequence of sets so that $U_n \rightarrow U$ in the topology induced by the Lebesgue measure, then

- If each g_n stably identifies $OPW(U, X)$ with the same lower bound for all n , then whether g identifies $OPW(U, X)$,
- If each g_n identifies $OPW(U, X)$, then whether g identifies $OPW(U, X)$,
- If each g_n stably identifies $OPW(U_n, X)$ with the same lower bound for all n , then whether g identifies $OPW(U, X)$,
- If each g_n identifies $OPW(U_n, X)$, then whether g identifies $OPW(U, X)$.

Aside from solving the critical case of U being an area 1 non-rectifiable domain, affirmative answers to the questions will also have consequences on another interesting problem of whether the sum of a convolution operator and a multiplication operator can be identified by a single input. We record the motivation and setting

of this problem below:

Denote X, Y respectively as the time axis and the frequency axis on the time-frequency plane, it is easy to see that if the spreading function has the form $h(t) \otimes \delta_Y$, i.e., it is concentrated on the time axis, then the pseudo-differential operator reduces to a convolution operator, indeed, we have

$$(H_{h(t) \otimes \delta_Y} f)(x) = \int (h(t) \otimes \delta_Y) e^{2\pi i x \cdot v} f(x - t) dt dv = \int h(t) f(x - t) dt,$$

and similarly if the spreading function has the form $\delta_X \otimes m(v)$, i.e., it is concentrated on the frequency axis, then the pseudo-differential operator reduces to a multiplication operator, since

$$\begin{aligned} (H_{\delta_X \otimes m(v)} f)(x) &= \int (\delta_X \otimes m(v)) e^{2\pi i x \cdot v} f(x - t) dt dv \\ &= f(x) \int m(v) e^{2\pi i x \cdot v} dv \\ &= \check{m}(x) f(x). \end{aligned}$$

A convolution operator can easily be identified by testing it on a single δ , while a multiplication operator can also be easily identified by testing it on the constant function 1. So the next question is what happens if we have the sum of a multiplication operator and a convolution operator.

As shown above, such an operator can be written as a pseudo-differential operator whose spreading function is concentrated on the axis, i.e., it has from

$$(H_{h(t) \otimes \delta_Y + \delta_X \otimes m(v)} f)(x) = (h * f)(x) + \check{m}(x) f(x).$$

This identification problem is related to what we have shown in Subsection 6.1 in the following way: For simplicity let us consider spreading functions concentrated on the positive part of the axis (denoted as X^+ and Y^+ respectively) only. Since the two positive axis can be viewed as the "limit" of L shaped domains (see the figure below), we may consider the spreading function concentrated on the axis

as the limit (under some suitable topology) of spreading functions (distributions) supported on these L shaped domains.

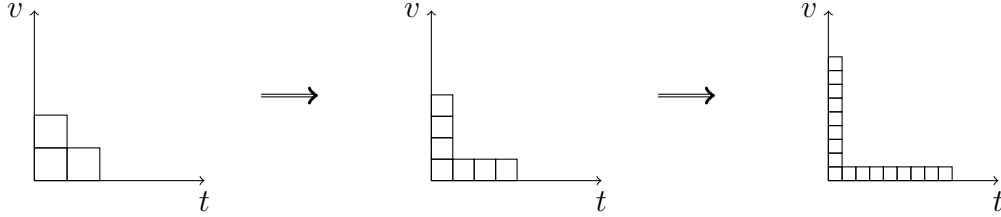


Figure 26: Axis as Limit of L -Shaped Domains

i.e., let L_n be the L shaped domain induced from the group $\mathbb{Z}_n \times \mathbb{Z}_n$ and as defined in Subsection 6.1, and denote L as the union of the positive parts of two axis, then we reduce the identifiability of $OPW(L, S'_0)$ to two sub-problems

1. To identify $OPW(L_n, S'_0)$ for each n .
2. If $g_n \in S'_0$ identifies $OPW(L_n, S'_0)$, and $g_n \rightarrow g$ weak*, then whether g identifies $OPW(L, S'_0)$.

By the results in Subsection 6.1, we know at least for $OPW(L_n, L^2)$, we can find explicit unitary identifiers, and as discussed in Appendix II, these identifiers are also injective on $OPW(L_n, S'_0)$, thus leaving us only the second sub-problem, which is same as one of the questions listed at the beginning of this section.

We now inspect core difficulties of this "pass to limit" method:

- As Corollary 3.3.2 shows, for g_n being periodically weighted delta trains supported $1/n$ apart, the scaling factor $1/\sqrt{n}$ actually makes the lower bound of Φ_{g_n} , as an operator from $OPW(U_n, L^2)$ to $L^2(\mathbb{R})$, go to 0. i.e., the stronger assumption that g_n stably identifies $OPW(U_n, X)$ with the same lower bound for all n can actually not be fulfilled using delta trains of higher and higher densities.
- Recall Lemma 5.2.3, we see that the weak* convergence of the identifiers g_n together with the measure convergence of the support set U_n (which would

imply the norm convergence of the spreading function η_n by setting η_n as η restricted to U_n) only imply the weak* convergence of the response $\Phi_{g_n}\eta_n$. For this reason, even if we can show $\Phi_{g_n}\eta$ are lower bounded, it will still not prevent the weak* limit $\Phi_g\eta$ from vanishing. In fact, due to the same reason, if Φ_{g_n} has vanishing lower bounds as $n \rightarrow \infty$, we can not use it to assert the limit Φ_{g_n} is not injective either. Therefore, regardless of whether we are trying to obtain or disproving a lower bound, such efforts actually does not help us in either directions.

- It is therefore more feasible to inspect whether the pre-adjoint V_g has dense range in $S_0[U]$. However, since g is non-constructively obtained via the Banach Alaoglu theorem, we can not compute V_g explicitly. For the sequence V_{g_n} , although they are surjective onto $S_0(U_n)$, their behaviors are much less predictable toward the boundary area $U \setminus U_n$, so we can not get density by the passing to limit argument.

In short, the weak* limit comes to us too easy and cheap that it also has little helpful properties we could use, indeed, it comes from the boundedness of g_n in S'_0 and other than that, we have no idea how this limit looks like and do not know how to describe it either, by the density result in Theorem 5.3.1 we know it can literally be any S'_0 distribution, and thus very wild.

Nevertheless, again in light of Theorem 5.3.1, the author still considers this as a good approach, since whatever S'_0 distribution one wants to take as a candidate identifier, it can always be written as the weak* limit as some bounded periodically weighted delta trains, therefore it makes sense to study the limiting behavior.

Finally, recall that in the case of U being a rectifiable domains with area precisely 1, any periodically weighted delta train g that is an identifier will induce an isomorphism between $L^2(\mathbb{R})$ and $L^2(U)$, physically this can be interpreted as a single response carries at most information from channels of unit spreading support. We thus conclude this thesis by formulating an analog conjecture for non-rectifiable domains:

Conjecture: Let U be a domain of measure 1 and boundary measure 0, and $g \in S'_0(\mathbb{R})$, then Φ_g is injective from $L^2(U)$ to $L^2(\mathbb{R})$ if and only if it also has dense range, and similarly the adjoint operator V_g has dense range in $L^2(U)$ if and only if it is also injective from $L^2(\mathbb{R})$ to $L^2(U)$.

Appendix I: Rectification by Rectangles

In [43], the rectification is carried out not using small squares but small rectangles of the same area (i.e., area $1/N$). To better demonstrate the rectification approach, we chose to hide such complexity and proceeded with rectangles in Section 3, however, a domain U with $\mu(U)$ rectifiable by rectangles of area $1/N$ may not be rectifiable by squares of the same area (e.g., when the length of one side of the rectangle is not in the field $\mathbb{Q}(\sqrt{N})$), which shows it makes sense to also discuss it a bit.

The purpose of this subsection is to show that this approach using rectangles can also be reproduced by the aforementioned method with slight adjustment. It is safe to skip this subsection for readers who are only interested in our main problem since, essences behind rectangle rectifications and square rectification are same, one eventually decomposes the identification map, and obtain a Gabor matrix. We chose to go with the square rectification as it is easier to present, we include the rectangle rectification here for completeness, as it covers more cases.

For $\vec{c} = (c_0, c_1, \dots, c_{N-1}) \in \mathbb{C}^N$, we simply introduce a parameter $M \geq \sqrt{N}$ to define a new weighted Zak transform as linear combinations of Z_M

$$(Z_{M,\vec{c}}f)(x, w) = \sum_{k=0}^{N-1} ((\bar{c}_k e^{-2\pi i \frac{k}{M} w} T_{(-\frac{k}{M}, 0)} Z_M) f)(x, w),$$

and take

$$h_k(x, w) = e^{-2\pi i \frac{k}{M} w} (T_{(-\frac{k}{M}, 0)} Z_M) f(x, w),$$

then one easily verifies that, similar as the computations in Subsection 3.2, we have

$$h_k(x + \frac{1}{M}, w) = h_{k+1}(x, w), \quad h_k(x, w + \frac{M}{N}) = e^{-\frac{2\pi i}{N}},$$

i.e., horizontal shifts induces circulant permutations of weights and vertical shifts induce modulations of weights.

We then also define a periodically weighted delta train as

$$\mathfrak{g}_{M,\vec{c}} = \sum_{j \in M\mathbb{Z}} \sum_{k=0}^{N-1} c_k \delta_{\frac{j+k}{M}},$$

then similar as before we have

$$V_{\mathfrak{g}_{M,\vec{c}}} = e^{-2\pi i t \cdot v} Z_{M,\vec{c}},$$

which can be easily verified by viewing $\mathfrak{g}_{M,\vec{c}}$ as linear combination of N unweighted delta trains supported on $M\mathbb{Z} + k/M$.

Now given a domain U , we consider a grid induced by $(1/M, 0)$ and $(0, M/N)$ and cover U with this grid. If U is admissible $\mu(U) < 1$, then we can choose M, N large enough so that the total rectangles needed to cover U are not more than N . Same as before We mark one rectangle as the reference rectangle $(0, 0)$ and get the index of other rectangles by their relative position to the reference rectangle see the figure below.

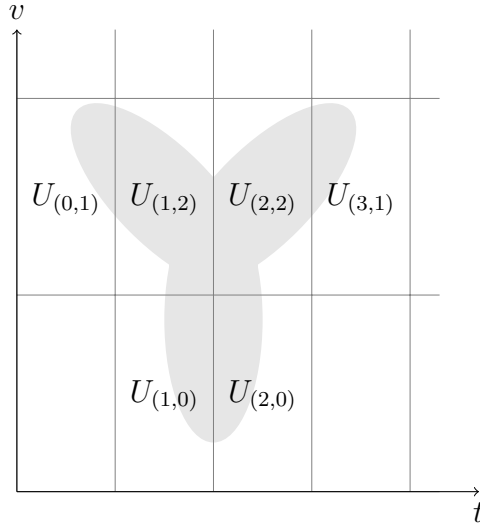


Figure 27: Rectification by Rectangles

The corresponding rectification set, i.e., indices of those rectangles used to cover

U , from $\mathbb{Z}_N \times \mathbb{Z}_N$, we again denote as Γ .

$$\chi_U V_{\mathfrak{g}_{M,\varepsilon}} = S_\Gamma D_\Gamma G_\Gamma^*(\vec{c}) D_{\sqrt{N}} A_{M,N} V_{\mathfrak{g}_{\Gamma_N}},$$

where D_Γ , $D_{\sqrt{N}}$ and $V_{\mathfrak{g}_{\Gamma_N}}$ have the same meaning as in Corollary 3.2.1, S_Γ now synthesizes a vectorized function to the new rectification Γ defined above which consists of small rectangles instead of small squares, and $A_{M,N}$ vectorizes a function the same way as in Corollary 3.2.1, except that here one uses rectangles of length $(1/M) \times (M/N)$ instead of squares of side length $1/\sqrt{N}$.

It then follows from the adjoint relation in Corollary 3.3.3 that

$$\Phi_{\mathfrak{g}_{M,\varepsilon}} \chi_U = \Phi_{\mathfrak{g}_{\Gamma_N}} A_{M,N}^* D_{\sqrt{N}}^* G_\Gamma(\vec{c}) D_\Gamma^* S_\Gamma^*,$$

which is analogous to the decomposition in Corollary 3.3.1.

As mentioned, if $\mu(U) = 1$, then there exist domains that are only rectifiable by rectangles (e.g., unions of precisely N rectangles of size $(1/M) \times (M/N)$ with properly selected N), and also for $\mu(U) < 1$ using rectangles for rectification might lower the dimension N in certain cases, but for our problem of main interest, it makes no difference which approach take, as they share the same form of decomposition. We will, in the rest part of this thesis, continue to use squares for rectifications as it is slightly simpler.

Appendix II: An Identification Result on S'_0

In practice, the class $OPW(U, L^2)$ is rather limited since it does not even contain the identity operator, i.e., with η being δ_0 . Therefore identifiability on S'_0 was also studied for rectangles:

Theorem. [42, Theorem 5.2] *Let $a, b > 0$, $ab = 1$, and $U = [0, a] \times [0, b]$ be a rectangle with area precisely 1, let $c > 0$ and $U_c \subset U$ be a rectangle in U with area $1 - c$, then $OPW(U_c, S'_0)$ is identifiable by the same unweighted delta train g_a as defined in Lemma 3.3.1:*

$$g_a = \sum_{k \in \mathbb{Z}} \delta_{ka}.$$

It actually follows that $OPW(U, S'_0)$ is identifiable by a periodically weighted delta train if U is compact and rectifiable with $\mu(U) < 1$. Indeed, This is a direct consequence of Corollary 3.2.1, Theorem 7 and Theorem 3.3.2. Indeed, the decomposition in Corollary 3.2.1 is just a consequence of the quasi-periodicity in Lemma 2.7.2, although we only explicitly proved this decomposition for L^2 case, it actually holds whenever the Zak transform is well defined, and in this cases, it was already proved in [42] that the Zak transform is well defined from $W^{A,1}(\mathbb{R})$ to $W^{A,1}(U)$ when U is compact.

Therefore together with the adjoint relation in Theorem 3.3.2, the decomposition passes over to the identification map Φ_g . i.e., the formula in Corollary 3.3.1

$$\Phi_{g_c} \eta = \Phi_{g_{\Gamma_N}} A_{\sqrt{N}}^* D_{\sqrt{N}}^* G_{\Gamma}(\vec{c}) D_{\Gamma}^* S_{\Gamma}^* \eta,$$

also holds for $\eta \in S'_0(U)$.

It is then easy to see that the analysis operator is injective from $S'_0(U)$ to the space of vectorized spreading functions, the synthesis operator is also injective from the space vectorized spreading functions to $S'_0(U)$.

Now since we have assumed U is compact, rectifiable and with area less than 1, we can choose N so big, such that in the rectification it takes not more than $N - 1$

squares of side length $1/\sqrt{N}$ to cover U , consequently $A_{\sqrt{N}}^* D_{\sqrt{N}}^* G_{\Gamma}(\vec{c}) D_{\Gamma}^* S_{\Gamma}^* \eta$ is supported in a rectangle inside $[0, \sqrt{N}] \times [0, 1/\sqrt{N}]$, and of area $1 - 1/N$.

Then Theorem 7 shows the identification map $\Phi_{g_{\Gamma, N}}$ is injective on such a rectangle, thus it suffices for us to choose the weight vector \vec{c} so that the Gabor matrix $G_{\Gamma}(\vec{c})$ is injective, such choices are many as of Theorem 3.4.1.

To this moment, one might be tempted to formulate a statement that is similar to what we had in section 3, i.e., to connect the identifiability of $OPW(U, S'_0)$ to the area of the domain U .

For distributional spaces, such idea is clearly not good, the following example provides some explanation:

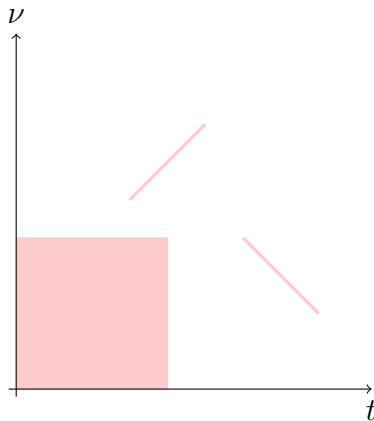


Figure 28: Expanding the Distributional Space without Changing Supporting Area

Suppose the area of the square (let us call it U) on the bottom left corner of the picture is less than 1, then $OPW(U, S'_0)$ is identifiable. Next we can add arbitrary null sets to U without changing the measure of the domain while the space was enlarged. For example, in this picture, we added 2 line segments (as outlined in red color), all distributions supported on these two line segments (e.g., deltas on these line segments) are now in the space. One could add infinitely many line segments, straight lines or other null sets, and should not expect the resulting space to be still identifiable. In this sense, area is not the correct criterion for identifiability.

List of Frequently Used Symbols and Notations

Symbol	Meaning
i, e, π	the corresponding mathematical constants
supp	the support of a function
ψ	the window function for Wiener-Amalgam spaces, see Subsection 2.5
κ	the spectral condition number of a matrix, i.e., the largest singular value divided by the smallest
$\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$	the corresponding mathematical symbols for complex numbers, real numbers, rational numbers, integers and natural numbers respectively
\mathbb{Z}_N	the cyclic group of order N
N	a fixed natural number
$\ \cdot\ _X$	the norm induced by the topology X , the subscript X can be omitted if it is clear from the context
χ_I	the characteristic function on a set I
L^p	space of functions with finite p -norm with respect to the Lebesgue measure
ℓ^p	space of sequences with finite p -norm

Symbol	Meaning
\mathcal{F}	the unitary Fourier transform on \mathbb{R}^N i.e., $(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^N} f(x)e^{2\pi i x \cdot \xi} dx$
\hat{f}, \check{f}	short notations for $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$
$\langle f, g \rangle$	inner product or sesquilinear dual pairing, in the later case the linear functional can appear in either left or right side, it does not impact our analysis, to better capture the natural of the problem it is actually better not to insist on a consist form.
A, A'	spaces of Fourier transform of L^1 and L^∞ functions respectively, see Subsection 2.5
$W^{X,p}$	Wiener-Amalgam spaces that are locally in X and globally with ℓ^p decay, see Subsection 2.5
$W^{X,p}(U)$	$W^{X,p}(\mathbb{R}^2)$ restricted to U , see Subsection 2.5
$W^{X,p}[U]$	Equivalence classes in $W^{X,p}(\mathbb{R}^2)$ that coincide on U , see Subsection 2.5
$\sigma(x, w)$	the Kohn-Nirenberg symbol of a pseudo-differential operator
$\eta(t, v)$	the spreading function of a pseudo-differential operator
$\vec{\eta}$	vectorization of a spreading function with respect to a rectification, see Subsection 3.2

Symbol	Meaning
\mathcal{M}_v	the continuous modulation operator, i.e., $(\mathcal{M}_v f)(x) = e^{2\pi i v \cdot x} f(x)$
\mathcal{T}_t	the continuous translation operator, i.e., $(\mathcal{T}_t f)(v) = f(x - t)$.
$\mathcal{K}_\sigma, \mathcal{H}_\eta$	a pseudo-differential operator with symbol σ or spreading function η
$OPW(U, X)$	Space of pseudo-differential operators with their spreading functions supported on U and equipped with norm induced by $X(U)$ or $X[U]$, see Subsection 2.8
ω_N	the first primitive N -th root of unity $e^{2\pi i/N}$
M_N^j, T_N^k	discrete time-frequency shifts, see Subsection 2.3
W_N	the $N \times N$ unitary DFT matrix, see Subsection 2.3
$\hat{\vec{x}}, \check{\vec{x}}$	short notations for $W_N \vec{x}$ and $W_N^* \vec{x}$
\vec{u}_j	the j -th column in W_N
\circ	the matrix Hadamard product
\hat{f}, \check{f}	$\mathcal{F}f, \mathcal{F}^{-1}f$ in the continuous case or $Wf, W^{-1}f$ in the discrete case
$*$	denotes the adjoint operation. i.e., L^* is the adjoint of a linear operator L

Symbol	Meaning
(\vec{d}, Γ)	a discrete Gabor system with window \vec{d} and support Γ , see Subsection 2.3
$G_\Gamma(\vec{d})$	the matrix form of (\vec{d}, Γ) , columns are arranged by the lexicographical ordering on Γ
(ϕ, a, b)	a continuous system $\{\mathcal{M}_{ma}\mathcal{T}_{nb}g(x)\}_{m,n \in \mathbb{Z}}$
χ_U	the characteristic function on U
tr, det	trace, determinant of a matrix
$V_g f$	The short time Fourier transform of f with window g , i.e., $(V_g f)(t, v) = \int_{\mathbb{R}^N} f(x) \overline{g(x-t)} e^{-2\pi i x \cdot v} dx.$
S_0, S'_0	The Feichtinger algebra and its dual, see Subsection 2.4
S, S'	The Schwarz class and tempered distributions
\asymp	$\ \cdot\ _X \asymp \ \cdot\ _Y$ means there exist constants K_{\min} and K_{\max} such that $K_{\min} \ \cdot\ _X \leq \ \cdot\ _Y \leq K_{\max} \ \cdot\ _X$ holds for any elements in corresponding spaces
\lesssim	$\ \cdot\ _X \lesssim \ \cdot\ _Y$ means there exist constants C such that $\ \cdot\ _X \leq C \ \cdot\ _Y$
\gtrsim	$\ \cdot\ _X \gtrsim \ \cdot\ _Y$ means there exist constants C such that $\ \cdot\ _X \geq C \ \cdot\ _Y$

Symbol	Meaning
δ_λ	The Dirac distribution at λ
$\mathfrak{g}_{\vec{c}}$	N -periodically weighted delta trains induced by $\vec{c} \in \mathbb{C}^N$, see Subsection 2.4
$\mathfrak{u}_{\vec{d}}$	N -periodically weighted exponential trains induced by $\vec{d} \in \mathbb{C}^N$, see Subsection 2.4
C_{emb}	embedding constant for the Gelfand triple $S_0(W^{A,1}) \subset L^2 \subset S'_0(W^{A,\infty})$
Φ_g	the identification map induced by g , i.e., $\Phi_g \eta = \mathcal{H}_\eta g$
Z_a	the Zak transform with parameter a , i.e., $(Z_a f)(x, w) = \sum_{k \in \mathbb{Z}} f(x + ka) e^{-2\pi i k a w}$
$Z_{\vec{c}}$	the periodically weighted Zak transform induced by $\vec{c} \in \mathbb{C}^N$, see Subsection 2.7
$e_I^{2\pi i \lambda x}$	the exponential function restricted to the set I , i.e., $= \chi_I e^{2\pi i \lambda x}$
A_N, S_Γ	The analysis and the synthesis operator related to the weighted Zak transform, see Subsection 3.2
(MT, Γ)	$= \{M_N^j T_N^k, (j, k) \in \Gamma \subseteq \mathbb{Z}_N \times \mathbb{Z}_N\}$
(TM, Γ)	$= \{T_N^j M_N^k, (j, k) \in \Gamma \subseteq \mathbb{Z}_N \times \mathbb{Z}_N\}$
$\Delta\Gamma$	first order difference set of Γ , i.e., $= \{(j - j', k - k') : (j, k), (j', k') \in \Gamma\}$

Symbol	Meaning
$G_{\Gamma}(A, \vec{c})$	the concatenated matrix with the Gabor matrix $G_{\Gamma}(\vec{c})$ on top of A , see Subsection 4.5
V_s	the cyclic subgroup in $\mathbb{Z}_N \times \mathbb{Z}_N$ consists of all $\{(j, k)\}$ with $j/k = s$
$D_{\vec{x}}$	$= \text{diag}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$
$D_{ \vec{x} ^2}$	$= \text{diag}(\vec{x}_1 ^2, \vec{x}_2 ^2, \dots, \vec{x}_N ^2)$
$P_{\vec{x}}$	the orthogonal projection onto the span of the vector \vec{x}
P_s	the orthogonal projection onto the span of the (MT, V_s)

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