Mathematisches Institut
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Dissertation

Martingale Transformations of Jump Processes

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Eine Thesis zur Erlangung des Doktorgrades der Naturwissenschaftlichen Fachbereiche der Justus-Liebig-Universität Giessen

April 29, 2022

Abstract

We provide a large class of functions and their respective parameters to transform a jump-process into a martingale w.r.t. its natural filtration. The proofs are based on a discrete Doob-decomposition and a limiting procedure to continuous time, in turn resulting in a time-continuous Doob-Meyer decomposition. Martingale transformations are then determined by solving the Doob-Meyer decomposition for functions that eliminate the compensator. We discuss several related results and single jump filtrations. The results are provided for single-jump processes and are systematically generalized to the multi-jump case, highlighting the necessity of dependencies between current jumps and the processes paths. Eventually we apply the result to branching random walks as an instructive example.

Acknowledgment:

First and foremost I like to thank Prof. Dr. Ludger Overbeck for his guidance, his patience and the instructive discussions during the last years. His lighthearted, competent nature created an atmosphere for curiosity and creativity.

Additionally I like to thank Prof. Dr. Dr. h. c. Winfried Stute, who supported and inspired me since I was a Bachelor student. His belief in my abilities was what kept me going after my Masters degree and as such this thesis is a continuation of his original impetus.

I like to extend my gratitude to all the dear colleagues and guests at the Mathematical Institute, and of course the many talented students across all the mathematical disciplines. I had the opportunity to work with so many of you and I found appreciation and cooperation, interesting discussions, friendship and shared good food with all of you. What is there else to wish for.

Ganz besonderer Dank gebührt meiner Freundin, Ricarda Gumprich, die nicht nur an mich glaubte, sondern sogar um mich wusste.

Contents

Introduction

Ito's Lemma marks an important and popular result that has experienced several generalizations and augmentations over the past decades. It is formulated in its present form for semimartingales in general (c.f. [Bichteler, 2002]) and used broadly among applicants. The first versions have been formulated for Brownian motions ([It, 1944]) - thus a stochastic process with almost sure continuous paths - was generalized to Wiener processes (e.g. [Kunita and Watanabe, 1967]) and eventually processes with jumps - in particular general processes with cdlg-paths - before it took nowadays form (a passionate and interesting survey of Ito's work can be found in [Kunita, 2010]). But this was not the only direction of evolution for Ito's formula.
In 1970 J.M.C. Clark connected in [Clark, 1970] martingales of the brownian filtration to stochastic integrals w.r.t. to the brownian motion itself. This result reflected that a martingale adapted to the information extracted from the path of a brownian motion was in fact just a functional of the brownian motion. But what if the integrator was a right constant jump process? This question and a pivotal work of P. Bremaud ([Bremaud, 1972]) inspired R.Boel, P. Varaiya and E. Wong to formulate a similar representation theorem for this kind of stochastic process in [Boel et al., 1975]. M.H.A. Davis soon after simplified and generalized this result in [Davis, 1976] and concluded that a process is a local martingale of a jump process $x(t)$ if and only if it is a stochastic integral w.r.t. a fundamental martingale q that is associated to the jump process. This was the counterpart for the representation theorem of [Clark, 1970] for jump processes, i.e. stochastic processes that are fundamentally different to Brownian motion.

Related papers to the representation result for jump processes are [Elliott, 1976], [Elliott, 1977] where the result is adapted to jump times that have accumulation points of any order (and in turn processes that may continue after an accumulation time), [Chou, 1975], [Jacod, 1975], [Jacod, 1976] where the authors yield similar results from the perspective of (marked) point processes and [Gushchin, 2020] where the objects of analysis are the filtrations themselves.

In this thesis we want to investigate a different approach to the representation result. As we've talked about the importance of Ito's lemma in the beginning, we also want to highlight the connection of Ito's formula to another very important result in probability theory: the Doob-Meyer decomposition. This result states that any semimartingale (of reasonable regularity, namely class D) can be decomposed into a martingale part and a previsible compensator. The result in [Clark, 1970] connects the Ito formula to the

Doob-Meyer decomposition as it proposes that the stochastic integral in Ito's formula is a martingale and the classical integral is a previsible compensator (in this case 0). Thus the Ito formula describes a Doob-Meyer decomposition quite naturally ([Kunita and Watanabe, 1967]). In turn one can find martingales of the Brownian motion by determining the harmonic functions in an analytic sense, since they eliminate the compensator in the Doob-Meyer decomposition.

The Doob-Meyer decomposition theorem itself is a time-continuous version of a earlier result by Doob ([Doob and Doob, 1953]) for discrete time adapted processes. There are many proofs of the time-continuous result that apply a limiting procedure to the discrete time Doob-decomposition, e.g. [Rao, 1969], [Bass, 1996], [Jakubowski, 2005] and [Beiglboeck et al., 2010], the latter being the most general and simple proof.

In this thesis we want to combine all of the above strategies to determine functions $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$ that transform a jump process $x = (x(t))_{t\in\mathbb{R}^+}$ with values in a sufficiently nice measurable space $(X, \mathcal{S})$ such that $v = (v(t))_{t\in\mathbb{R}^+}$ with $v(t) = \varphi(t, x(t))$ is a martingale w.r.t. the filtration of x. In analogy to the terminology for Brownian motions we will call functions with this property *harmonic functions of the jump process*. Since the general case follows same as in [Davis, 1976] from the single-jump case, we are going to focus on the single-jump processes first. Here's an outline of the different steps:

(I) Define a discrete time version of the process v.

(II) Determine a discrete time Doob decomposition of the jump process.

(III) Try to survive a limiting procedure in the time parameter.

(IV) Eliminate the compensator by a choice of the function φ.

Each one of these steps has an own potential to force assumptions and restrictions on the function φ, but we made an effort to justify each new condition by the properties of the function as a martingale transformation. For example the limiting procedure has to make use of convergence theorems for Lebesgue-Stieltjes integrals which makes it necessary to bound the supremum of $\varphi\mathbb{1}_{(0,t]}$ for any $t \in \mathbb{R}^+$. This is in turn reasonable for all times $t \in \mathbb{R}^+$ that are strictly less than c - the right endpoint of the distribution of the jump times.

In section 1 we investigate the single-jump case. We provide the necessary notation and results that are needed for this case and follow the strategy outlined above. Main results are the semimartingale representation in theorem 1.14 which is obtained via the discretization method from [Rao, 1969], [Bass, 1996], [Jakubowski, 2005] and [Beiglboeck et al., 2010] and theorem 1.19 which describes the harmonic functions of jump process, i.e. the functions that transform the single-jump process into a martingale w.r.t. its own filtration. The results are cross-verified with the classical results of [Davis, 1976] and the contemporary result of [Gushchin, 2020]

Section 2 lifts the result to the general case (i.e. more than one jump). We again provide further notational tools. The result by [Davis, 1976] can again be reproduced, as we can provide a large family of martingale-transformations in theorem 2.6 and a semimartingale representation in corollary 2.9.

Path-dependent versions and a version of the single-jump case where the process jumps in a countable/discrete measurable space are discussed and provided in the Appendix.

Chapter 1

Single jump

In this section we will work out the first result for the single-jump process. The random process will not only jump at a random time in $\mathbb{R}^+$ but also to a random location in X. Our interest surrounds the ability to be "ready" for this event at any given time. In terms of insurance for example the process might be the first car accident an insured person is involved in. The crash-time T is completely random and the value of the damage is the random location Z of the process after T. Any insurance company needs to be prepared for such a case and it is not unrealistic, that the value of the case of insurance depends in some way on the time of the accident (seasonal effects, driving experience, state of the car, etc.). To prepare for such an event one might be interested in a simple function that accumulates just enough money before the actual event happens, and as such at any given time t. In real insurance this is way more complex than advertised in this little example, but it summarizes intuitively the mathematical problem of this section, of determining a transformation of the single-jump process $x(t)$ into a martingale.

In this simple case (as in the more general case of more than one jump) M.H.A. Davis proves in [Davis, 1976] that every local martingale (w.r.t. the augmented natural filtration $\mathcal{F}_t$ of the process $(x(t))_{t\geq 0}$) can be written as a stochastic integral of a measurable enough function g against a basic martingale q:

$$M_t^g := \int_{(0,t]\times X} g dq$$

Even more he proves, that every such integral is a local martingale. His main result for single-jump processes reads as follows (see [Davis, 1976]):

Theorem 1.1. *(M_t) is a local martingale of $(\mathcal{F}_t)$ if and only if $M_t = M_t^g$ for some $g \in L^1_{loc}(p)$.*

The notion 'for some g' might not be satisfactory and one might be interested in which g exactly.

The goal of this work is to find a way to systematically determine functions of the jump process that result in martingales w.r.t. to the jump process. Not only will we be able

to verify parts of the results of [Davis, 1976], we can add another constructive method to determine martingales of this form with the following strategy:
we search for functions $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$ which transform the process $x = (x(t))_{t\in\mathbb{R}^+}$ into an $(\mathcal{F}_t)$-martingale via $\varphi(t, x(t)) =: v(t)$. We follow the strategy for the proofs of the Doob-Meyer decomposition theorem one can find in [Rao, 1969], [Bass, 1996] and [Beiglboeck et al., 2010]:

- we take a discrete but arbitrary selection of times $t_1 < \cdots < t_K$ to come up with a discrete version of the process $v = (v(t))_{t\in\mathbb{R}^+}$, denoted by $v^(K) = v(t_k)_{k=1,\ldots,K}$.
- This manageable discrete process is still adapted to the natural augmented filtration $\mathcal{F}_k := \mathcal{F}_{t_k}$. Therefore we can determine the Doob-decomposition to end up with a previsible compensator part $A_k^{(K)}$ and a martingale part $M_k^{(K)}$.
- Now we increase the number of discrete times K and get (under certain conditions on φ) a time-continuous Doob-Meyer-decomposition. We will see by then the connection to the already stated result by Davis and the basic family of martingales.
- The next part of this journey will be dedicated to eliminating the compensator part of the Doob-Meyer decomposition via the choice of the function φ. The result by [Gushchin, 2020] can be verified as a related case.

In another approach that is discussed in the Appendix, we want to generalize this method to functionals $\varphi : \mathbb{R}^+ \times D([0,\infty), X) \to \mathbb{R}$ that take into account the whole path of the process. For jump processes the knowledge of the jump times and heights is equivalent to knowing the whole path, but if one is interested in processes that are not constant in between different jumps, this might be a good starting point.

1.1 Definitions

We will use the notation from [Davis, 1976]. Though we differ on the notation for the cumulative distribution function ($\mathbb{P}(T \leq t) = F_t$).

Spaces and random variables: Let $(X, \mathcal{S})$ be a measurable space, more precicely a Blackwell space (see [Dellacherie and Meyer, 1979], III, definition 24). This is going to be the space for the values of our process. Fix $z_0, z_\infty \in X$ as the initial and terminal values of the process and let us define a proper state space for everything random after time 0:

$$(\Omega, \mathcal{F}^0) := \left((\mathbb{R}^+ \times X) \cup \{(\infty, z_\infty)\}, \sigma\{\mathcal{B}(\mathbb{R}^+) \times \mathcal{S}, \{(\infty, z_\infty)\}\}\right).$$

To model the jump time and height we take $T : \Omega \to \mathbb{R}^+$ and $Z : \Omega \to X$ to be the coordinate mapping, picking out the time and space coordinate of the jump of a general state $\omega = (t, z)$, i.e. $\omega = (T(\omega), Z(\omega))$.

The process: The *value of the process at time t* is

$$x(t,\omega) := \begin{cases} z_0 & \text{if } t < T(\omega), \\ Z(\omega) & \text{if } T(\omega) \leq t. \end{cases} \tag{1.1.1}$$

Filtration and probability measure: An increasing sequence of sub-σ-fields $(\mathcal{F}_t)_{t\in I}$ of $\mathcal{F}^0$, (where I might be $\mathbb{N}$, $\mathbb{Q}^+$ or $\mathbb{R}^+$ in our case) is called *filtration*. Given a stochastic process $y = (y(t))_{t\in I}$ the *natural filtration* of the process y is the filtration of σ-fields $\mathcal{F}_t^0 = \sigma\{y(s) : s \leq t, s \in I\}$.
Let $\mathcal{F}_t^0$ be the natural filtration generated by the process $(x(t))_{t\geq 0}$, i.e.

$$\mathcal{F}_t^0 = \sigma(x(s) : s \in [0,t]) = \sigma(\{x^{-1}(s)(B) : s \in [0,t], B \in \mathcal{S}\}). \tag{1.1.2}$$

Let us take the characterisation of a probability measure on $(\Omega, \mathcal{F}^0)$ from [Davis, 1976]: for $\Gamma \in \mathcal{F}^0$ the probability measure $\mathbb{P}$ is defined through

$$\mathbb{P}\left[(T,Z) \in \Gamma\right] = \mu(\Gamma), \tag{1.1.3}$$

where μ is a probability measure on $(Y, \mathcal{Y})$ with

$$\mu((\{0\} \times X) \cup (\mathbb{R}^+ \times \{z_0\})) = 0 \tag{1.1.4}$$

i.e. a jump at time 0, as well as an invisible jump are $\mathbb{P}$-nullsets in the following sense: A $\mathbb{P}$-nullset is a subset $A \subset \Omega$ s.t. there exists a measurable set $B \in \mathcal{F}^0$ with $A \subset B$ and $\mathbb{P}(B) = 0$.

Denote by $\mathcal{N}_0$ the set of all $\mathbb{P}$-nullsets. By $\mathcal{F}, \mathcal{F}_t$ we denote the σ-fields $\mathcal{F}^0, \mathcal{F}_t^0$ augmented with all $\mathbb{P}$-null sets, i.e.

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^0, \mathcal{N}_0) \tag{1.1.5}$$

According to Lemma 0 in [Davis, 1976] the jump time T is then a stopping time of $(\mathcal{F}_t)$ in the following sense:
Given a filtration $(\mathcal{G}_t)_{t\in I}$ a random variable S is called stopping time, if $\{S \leq t\} \in \mathcal{G}_t$ for all $t \in I$.

Let R be a random variable. We write $\mathbb{E}[R] = \int_\Omega R(\omega)d\mathbb{P}(\omega)$ for the expectation of R. Assume $\mathbb{E}[|R|] < \infty$. The conditional expectation of R with respect to a sub-σ-field $\mathcal{M} \subset \mathcal{F}$ is the $\mathbb{P}$-a.s. unique $\mathcal{M}$-measurable random variable $\mathbb{E}[R|\mathcal{M}]$ s.t.

$$\int_A \mathbb{E}[R|\mathcal{M}]d\mathbb{P} = \int_A R d\mathbb{P}, \quad \forall A \in \mathcal{M}.$$

The conditional probability of A w.r.t. to $\mathcal{M}$ will be denoted by $\mathbb{P}(A|\mathcal{M}) := \mathbb{E}[\mathbb{1}_A|\mathcal{M}]$ for all $A \in \mathcal{F}$.

Since we assumed X to be a Blackwell space there exists a *regular version of conditional probability* (c.f. [Shiryaev, 2016], definition 2.7.6) which we will also denote by $\mathbb{P}$, i.e. we can $\mathbb{P}$-a.s. write (see [Shiryaev, 2016], theorem 2.7.3)

$$\mathbb{E}[R|\mathcal{M}](\omega) = \int_\Omega R(\tilde{\omega})d\mathbb{P}(d\tilde{\omega}|\mathcal{M})(\omega).$$

Distributions: The involved distribution functions will be denoted as

$$F_t^A := \mathbb{P}(T \in [0,t], Z \in A) = \mu([0,t] \times A) \tag{1.1.6}$$

$$F_t := F_t^X = \mathbb{P}(T \leq t) = \int_\Omega \mathbb{1}_{T \leq t} d\mathbb{P} \tag{1.1.7}$$

for $t \in \mathbb{R}^+$ and $A \in \mathcal{S}$. The former is the joint distribution function of T and Z, whereas the latter is the marginal distribution function of the jump time T. Since μ is a probability measure on $\mathbb{R}^+ \times X$ and T maps into $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ (i.e. countably generated σ-field containing all singletons), we will also make use of the disintegration property with respect to the distribution of T, i.e. for any nonnegative measurable f on $\mathbb{R}^+ \times X$ we can disintegrate μ w.r.t. $\mu \circ T^{-1} =: dF$ in the following fashion:

$$\int_{\mathbb{R}^+ \times X} f(s,z) d\mu(s,z) = \int_{\mathbb{R}^+} \left(\int_X f(s,z) d\mu^s(z) \right) dF_s \tag{1.1.8}$$

where we set $\mu^s(A) =: \mathbb{P}(Z \in A | T = s) = \mathbb{E}[\mathbb{1}_A \circ Z | T = s]$ for any $A \in \mathcal{S}$. For existence see [Chang and Pollard, 1997] p.293 or the fact, that we assumed X to be a Blackwell space.
The right endpoint of the distribution of T will be denoted by

$$c = \sup\{t \in \mathbb{R}^+ : F_t < 1\}. \tag{1.1.9}$$

There are two different cases for c that are of interest in our discussion and are distinguished in [Davis, 1976], [Gushchin, 2020]:

Case (A) $c = \infty$ or, $c < \infty$ and $F_{c-} = 1$. In this case the marginal distribution function of T is either never exhausted or is continuously exhausted, i.e. the behavior of the process at the right endpoint can be approximated from the left.

Case (B) $c < \infty$ and $F_{c-} < 1$. This means the exhaustion is itself of positive mass. Typically this behavior can be found in discrete distributions but also for random variables like $T \wedge t$, where e.g. T has continuous distribution and $t \in \mathbb{R}^+$ is fixed.

Basic martingales: A process $y = (y(t))_{t \in I}$ is called a $(\mathcal{F}_t)_{t \in I}$-martingale, if the following three properties are satisfied:

(i) $y(t)$ is $\mathcal{F}_t$-measurable for all $t \in I$,

(ii) $y(t) \in L^1(\mathbb{P})$ for all $t \in I$,

(iii) $\mathbb{E}[y(t) | \mathcal{F}_s] = y(s)$ for all $s \in I, s \leq t$.

It is called *sub-martingale* if the last equality is only '$\geq$' and *super-martingale* if it is a '$\leq$'. We say a stochastic process y is a *local (sub/super-)martingale* w.r.t. to a filtration $(\mathcal{F}_t)_{t \in I}$ when there exists a localisation sequence of stopping times - i.e. $(\sigma_n)_{n \in \mathbb{N}}$, σ_n is an $\mathcal{F}_t$ stopping time for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \sigma_n = \infty$ $\mathbb{P}$-a.s. - such that $(y(t \wedge \sigma_n))_{t \in I}$ is an (sub/super-)martingale w.r.t. $\mathcal{F}_t$ for all $n \in \mathbb{N}$.

Let $\{(q(t,A))_{t\in\mathbb{R}^+} : A \in \mathcal{S}\}$ be the basic family of martingales defined through the processes

$$p(t,B) = \mathbb{1}_{Z\in B}\mathbb{1}_{T\le t}$$
$$\tilde{p}(t,B) = \int_{[0,T\wedge t]} \frac{1}{1-F(s-)} dF_s^B$$

by $q(t,B) = p(t,B) - \tilde{p}(t,B)$. Now $(q(t,B))_{t\ge 0}$ is an $(\mathcal{F}_t)$-martingale (see [Davis, 1976], prop. 3) and one can define a Lebesgue-Stieltjes integral for all

$$g \in L^1_{\mathrm{loc}}(p) = \{g \in \mathcal{J} : \int_Y \mathbb{1}_{(0,t]}(s)|g(s,z)|d\mu(s,z) < \infty, \forall t < c\} \tag{1.1.10}$$

by defining:

$$M_t^g := \int_{(0,t]\times X} g(s,z)q(ds,dz) \tag{1.1.11}$$

where in particular

$$\int_{\mathbb{R}^+\times X} g(s,z)p(ds,dz) = g(T,Z) \tag{1.1.12}$$
$$\int_{\mathbb{R}^+\times X} g(s,z)\tilde{p}(ds,dz) = \int_{\mathbb{R}^+\times X} g(s,z)\frac{\mathbb{1}_{s\le T}}{1-F_{s-}}d\mu(s,z) \tag{1.1.13}$$

(see for example [Boel et al., 1975] lemma 3.3). For the latter note that

$$\begin{aligned}\tilde{p}(t,B) &= \int_{(0,T\wedge t]} \frac{1}{1-F(s-)} dF_s^B \\ &= \int_{(0,T\wedge t]} \frac{1}{1-F(s-)} \mu(ds,B).\end{aligned} \tag{1.1.14}$$

Example 1.2.

(a) *As a first example let us look at a very simple process. Take $Z \equiv 1$ and T exponentially distributed, i.e. $F_t = 1-\exp(-\lambda t)$ for some $\lambda > 0$. To that end we set $z_0 := 0$, i.e. $X = \{0,1\}$, $\mathcal{S} = \{\emptyset, X, \{1\}, \{0\}\}$ and the single-jump process is given by*

$$x(t) = \mathbb{1}_{T\le t}.$$

The respective filtration appears as

$$\mathcal{F}_t^0 = \sigma(\{x^{-1}(s)(\{k\}) : s \in [0,t], k \in \{0,1\}\})$$

and the probability measure $\mathbb{P}$ is given by the following characterization:

$$\begin{aligned}
\mathbb{P}((T,Z) \in [0,t] \times \{1\}) &= \mathbb{P}(T \le t, Z = 1) \\
&= \mathbb{P}(T \le t) \\
&= F_t^{\{1\}}, \\
\mathbb{P}((T,Z) \in [0,t] \times \{0\}) &= \mathbb{P}(T \le t, Z = 0) \\
&= F_t^{\{0\}} \\
&= 0 \\
\mathbb{P}((T,Z) \in [0,t] \times X) &= F_t^{\{1\}} \\
\mathbb{P}((T,Z) \in [0,t] \times \emptyset) &= 0
\end{aligned}$$

In this case $c = \infty$ and Case A applies. The basic martingale of this process is given by the process itself, since there is no doubt about the value of Z at any given time $t \in \mathbb{R}^+$. This yields that for $t \in \mathbb{R}^+$:

$$\begin{aligned}
q(t,\{1\}) &= p(t,\{1\}) - \tilde{p}(t,\{1\}) \\
&= \mathbb{1}_{T \le t} - \int_{[0,T\wedge t]} \frac{1}{1-F_{s-}} dF_s^{\{1\}} \\
&= x(t) - \int_{[0,T\wedge t]} \frac{f(t)}{1-F_s} ds \\
&= x(t) - [-\ln(1-F_s)]_0^{T\wedge t} \\
&= x(t) + \ln(\exp(-\lambda(T\wedge t))) \\
&= x(t) - \lambda(T \wedge t) \\
&= q(t,X) \\
q(t,\{0\}) &= p(t,\{0\}) - \tilde{p}(t,\{0\}) \\
&\equiv 0 \\
&\equiv q(t,\emptyset)
\end{aligned}$$

which are $\mathcal{F}_t$-martingales with [Davis, 1976], proposition 3. In this case the basic process coincides with the actual jump process, so the compensator is readily determined as $\tilde{p}$. Note that in this case the choice of the distribution of T is only important for $\tilde{p}$.

(b) *Now the other possible simple process would be $T \equiv 1$, Z standard normal distributed. Now the space $\Omega = \mathbb{R}^+ \times \mathbb{R}$ and $\mathcal{F} = \mathcal{B}^+ \times \mathcal{B}$ but the distribution of T is set to be a single point mass on $\{T = 1\}$, i.e. $F_t = \mathbb{1}_{[1,\infty)}(t)$. The single-jump process is given by*

$$x(t) = Z\mathbb{1}_{[1,\infty)}(t).$$

where the respective filtration is

$$\mathcal{F}_t^0 = \sigma(\{x^{-1}(s)(B) : s \in [0,t], B \in \mathcal{S}\}).$$

The probability measure is characterized for $B = (a, b] \subset \mathbb{R}^+$ as follows:

$$\begin{aligned}\mathbb{P}(\{(T, Z) \in [0, 1) \times B\} \cup \{(T, Z) \in (1, \infty) \times B\}) = 0 \\ \mathbb{P}((T, Z) \in \{1\}, \infty) \times B) &= \mathbb{P}(T = 1, Z \in (a, b]) \\ &= \mathbb{P}(Z \in (a, b]) \\ &= \Phi(b) - \Phi(a)\end{aligned}$$

In this case $c < \infty$ and $F_{c-} = 0$, thus case B applies. The basic martingale of this process is given for any $(a, b] = A \in \mathcal{B}$ by

$$p(t, A) = \begin{cases} 0, & \textit{for } t < 1, \\ \mathbb{1}_{Z \in A}, & \textit{for } t \geq 1 \end{cases}$$

$$\tilde{p}(t, A) = \begin{cases} 0, & \textit{for } t < 1, \\ \Phi(b) - \Phi(a), & \textit{for } t \geq 1. \end{cases}$$

and thus

$$\begin{aligned} q(t, A) &= \begin{cases} 0, & \textit{for } t < 1 \\ \mathbb{1}_{Z \in A} - (\Phi(b) - \Phi(a)), & \textit{for } t \geq 1 \end{cases} \\ &= [\mathbb{1}_{Z \in A} - \mathbb{P}(Z \in A)]\,(1 - \mathbb{1}_{[0,1]}(t)). \end{aligned}$$

In this example we can not directly determine the compensator as the basic process $p(t, A)$ is not given by the single-jump process itself. Again we note that the choice of Z's distribution only impacts the compensator $\tilde{p}$.

(c) *The next step would be to combine the above simple examples. So let Z be standard normal distributed and T exponentially distributed. Note that we assume that Z and T are independent (something we previously did not have to assume, because it held true naturally. Then*

$$\Omega = \mathbb{R}^+ \times \mathbb{R}, \quad \mathcal{F}^0 = \mathcal{B}^+ \star \mathcal{B}.$$

The single-jump process is given by

$$x(t) = Z \mathbb{1}_{T \leq t},$$

the respective filtration

$$\mathcal{F}_t^0 = \sigma(\{x^{-1}(s)(B) : s \in [0, t], B \in \mathcal{B}\}).$$

We can characterize the probability measure for $t \in \mathbb{R}^+$, $(a, b] \in \mathcal{B}$:

$$\begin{aligned}\mathbb{P}((T, Z) \in [0, t] \times (a, b]) &= \mathbb{P}(T \leq t)\mathbb{P}(Z \in (a, b]) \\ &= (1 - \exp(\lambda t))(\Phi(b) - \Phi(a))\end{aligned}$$

In this case $c = \infty$ and case A applies. The basic process and its compensator is given for $t \in \mathbb{R}^+$ and $A = (a, b] \in \mathcal{B}$ by

$$\begin{aligned} p(t, A) &= \mathbb{1}_{Z \in (a,b]} \mathbb{1}_{T \leq t} \\ \tilde{p}(t, A) &= \int_{[0, T \wedge t]} \frac{1}{1 - F_{s-}} dF_s^A \\ &= \int_{[0, T \wedge t]} \frac{f(s)}{1 - F_s} (\Phi(b) - \Phi(a)) ds \\ &= -(\Phi(b) - \Phi(a))\lambda(T \wedge t). \end{aligned}$$

where both processes are unsurprisingly a product of the examples in (a) and (b). Hence the family of basic martingales is given by processes of the form

$$q(t,A) = \mathbb{1}_{Z\in(a,b]}\mathbb{1}_{T\le t} + (\Phi(b) - \Phi(a))\lambda(T\wedge t).$$

This combination has shown that the basic martingales are still easy to determine. But we assumed Z and T to be independent.

(d) *Consequently we now want to omit the independence assumption. As a simple example we take that T is again exponentially distributed, but this time assume that Z given $T = t$ obeys the normal distribution with Variance $\sigma^2 = \sigma^2(t) = \frac{1}{t}$, i.e. a normal distribution that becomes 'sharper' the later the jump happens. In this case we can keep the Ω, $\mathcal{F}^0$ and $x(t)$ as in the previous example. The probability measure becomes more complicated now. Take $t \in \mathbb{R}^+$ and $A = (a,b] \in \mathcal{B}$:*

$$\begin{aligned}\mathbb{P}((T,Z)\in[0,t]\times(a,b]) &= \int_{[0,t]} \mathbb{P}(Z\in(a,b]|T=u)\mathbb{P}(T\in du)\\ &= \int_{[0,t]} \mathbb{P}(Z\in(a,b]|T=u)dF_u\\ &= \int_{[0,t]} \Phi(b\sqrt{u}) - \Phi(a\sqrt{u})dF_u\end{aligned}$$

and thus the basic process and its compensator are given by

$$\begin{aligned}p(t,A) &= \mathbb{1}_{Z\in(a,b]}\mathbb{1}_{T\le t}\\ \tilde{p}(t,A) &= \int_{[0,t]} \frac{1}{1-F_{s-}} dF_s^A\\ &\overset{*}{=} \int_{[0,t]} \frac{f(s)}{1-F_{s-}} \left(\int_{[0,s]} \Phi(b\sqrt{u}) - \Phi(a\sqrt{u})dF_u\right) ds\\ &\quad + \int_{[0,t]} \frac{f(s)F_s}{1-F_{s-}} \left(\Phi(b\sqrt{s}) - \Phi(a\sqrt{s})\right) ds\end{aligned}$$

where in (*) we made use of Fubinis theorem. The family of basic martingales is thus of more complicated nature. Nonetheless we will determine martingale-transformations for these processes.

1.2 Filtration results

Since $\mathcal{F}_t$ is the natural filtration of the process $(x(t))_{t\ge0}$ one might expect to retrieve some of the useful properties of the process. For example: if $T > t$ the process should satisfy $x(u) = z_0$ for all $u \le t$, i.e. no jump occurs until t and the process remains on its initial value z_0. Vice versa the event $\{x(u) = z_0\}$ yields apparently ($\mathbb{P}$-a.s.), that $T > t$. Now T has probability $1 - \mathbb{P}(T \le t) = 1 - F_t$ to be bigger than t, which means that the set $\{T > t\}$ is an atom in the σ-field $\mathcal{F}_t$, as long as

$$t \le c := \sup\{s : F_s < 1\}. \tag{1.2.1}$$

As for the generating set of the σ-field $\mathcal{F}_t$ for any $t \in \mathbb{R}^+$ we note the following:

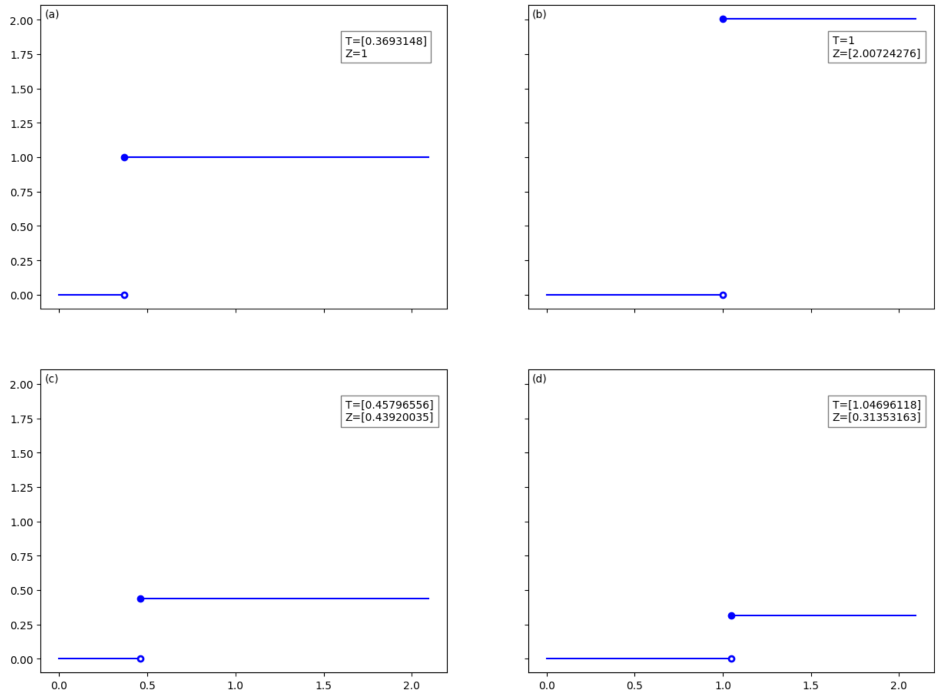

Figure 1.1: Plotted sample paths for example 1.2.

Lemma 1.3. *Let* $\mathcal{B}([0,t]) = [0,t] \cap \sigma(\{[0,s] : s \in [0,t]\}$. *Then:*

$$\mathcal{F}_t = \sigma\left(\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\right)$$

Proof. We first prove $\mathcal{F}_t \subseteq \sigma\left(\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\right)$. For this we take w.l.o.g. $A \in \{x(s)^{-1} : s \in [0,t], B \in \mathcal{S}\}$ which is the generator of $\mathcal{F}^0_t$ by (1.1.2), i.e. $A = x(s)^{-1}(B)$ for some $s \in [0,t], B \in \mathcal{S}$. The case $A \in \mathcal{N}_0$ is trivial. We make out 3 different cases:

- $B = \{z_0\}$.

$$\begin{aligned} A = x(s)^{-1}(\{z_0\}) &= \{\omega \in \Omega : x(s,\omega) = z_0\} \\ &= \{\omega \in \Omega : T(\omega) > s\} \dot\cup \{\omega \in \Omega : T(\omega) \leq s, Z(\omega) = z_0\} \end{aligned}$$

  where we note, that $\{\omega \in \Omega : T(\omega) \leq s, Z(\omega) = z_0\} \subset \mathbb{R}^+ \times \{z_0\}$ and therefore a $\mathbb{P}$-null set (see (1.1.4)). Further since $s \leq t$

$$\begin{aligned} \{T > s\} &= (\{T > s\} \cap \{T > t\}) \,\dot\cup\, (\{T > s\} \cap \{T \leq t\}) \\ &= (\{T > t\}) \,\dot\cup\, (\{\omega \in \Omega : T(\omega) \in (s,t], Z(\omega) \in X\}) \\ &= (\{T > t\}) \,\dot\cup\, ((s,t] \times X)\,. \end{aligned}$$

  Since $(s,t] \times X \in \mathcal{B}([0,t]) \star \mathcal{S}$ we get

$$A \in \sigma\left(\{\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\}\right).$$

- $z_0 \notin B$. This means $x(s)$ maps away from $\{z_0\}$, i.e. $T \leq s$ $\mathbb{P}$-a.s. (see (1.1.4)

$$\begin{aligned} A = x(s)^{-1}(B) &= \{\omega \in \Omega : T(\omega) \in [0,s], Z(\omega) \in B\} \\ &\in \mathcal{B}([0,t]) \star \mathcal{S}, \end{aligned}$$

so that we get again

$$A \in \sigma\left(\{\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\}\right).$$

- For any $B \in \mathcal{S}$ we write
$$B = B \setminus \{z_0\} \dot\cup \{z_0\}$$
and the above respective cases apply:
$$\begin{aligned} A = x(s)^{-1}(B) &= \{\omega \in \Omega : T(\omega) \leq s, Z(\omega) \in B \setminus \{z_0\}\} \dot\cup \{\omega \in \Omega : T(\omega) > s\} \\ &\in \sigma\{\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\}. \end{aligned}$$

Summarizing:

$$\{x^{-1}(s)(B) : s \in [0,t], B \in \mathcal{S}\} \cup \mathcal{N}_0 \subseteq \sigma\left(\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\right)$$

Using the monotonicity of the σ-operation, we yield

$$\mathcal{F}_t = \sigma(\{x^{-1}(s) : s \in [0,t], B \in \mathcal{S}\} \cup \mathcal{N}_0) \subseteq \sigma\left(\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\right).$$

Next we show the converse: $\mathcal{F}_t \supseteq \sigma\left(\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\right)$. For $A = \{T > t\}$ we can easily verify

$$A = \{\omega \in \Omega : T(\omega) \in (t, \infty)\} = x^{-1}(t)(\{z_0\})$$

and therefore $\{T > t\} \in \mathcal{F}_t$. The case $A \in \mathcal{N}_0$ also trivially yields $A \in \mathcal{F}_t$. So let $A \in \{[0,s] : s \in [0,t]\} \times \mathcal{S}$, e.g. $A = [0,s] \times B$ for some $s \in [0,t]$ and $B \in \mathcal{S}$. Then

$$\begin{aligned} A &= \{\omega \in \Omega : T(\omega) \in [0,s], Z(\omega) \in B\} \\ &= x^{-1}(s)(B) \\ &\in \mathcal{F}_t \end{aligned}$$

In total we get:

$$\mathcal{B}([0,t]) \star \mathcal{S} \cup \{\{T > t\}\} \cup \mathcal{N}_0) \subseteq \mathcal{F}_t$$

and by monotonicity we gain the assertion. □

Now we ended up with $\mathcal{F}_t = \sigma(\{x(s) : 0 \leq s \leq t\}, \mathcal{N}_0) = \sigma\left(\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\right)$. We take interest in the fact, that $\forall B \in \mathcal{B}([0,t]) \star \mathcal{S} : B \cap \{T > t\} = \emptyset$. Particularly $\{T > t\}$ is a $\mathbb{P}$-atom of $\mathcal{F}_t$ by the following

Definition 1.4. *Let $(\Omega, \mathcal{A}, \nu)$ be a measure space. A set $A \in \mathcal{A}$ is called ν-atom of $\mathcal{A}$, if*

- $\nu(A) > 0$,
- $\forall B \in \mathcal{A} : \nu(B \cap A) \in \{0, \nu(A)\}$.

For any $A \in \mathcal{F}_t$ we see

$$\mathbb{P}(\{T > t\} \cap A) = \begin{cases} \mathbb{P}(T > t), & \text{if } \emptyset \neq A \cap \{T > t\} \notin \mathcal{N}_0 \\ 0, & \text{else,} \end{cases}$$

in the proof of the next

Lemma 1.5. *$\{T > t\}$ is a $\mathbb{P}$-atom of $\mathcal{F}_t$, for all $t < c$.*

Proof. First up we notice, that $\mathbb{P}(T > t) = 1 - F_t > 0$ since $t < c$ (see (1.2.1)). To prove the second part of the definitions properties, we need to dig a little deeper: Let $B \in (\mathcal{B}([0,t]) \star \mathcal{S}) \cup \{(t,\infty) \times X\} \cup \mathcal{N}_0$. Then:

$$B \cap \{T > t\} = \begin{cases} \{T > t\}, & \text{if } B = \{T > t\} \\ N, & \text{if } B \in \mathcal{N}_0, B \cap \{T > t\} \neq \emptyset, \\ \emptyset, & \text{else,} \end{cases}$$

where $N \subset B$ and thus $N \in \mathcal{N}_0$ if $B \in \mathcal{N}_0$ and

$$\{T > t\} = (t,\infty) \times X \notin \mathcal{B}([0,t]) \star \mathcal{S}$$

and $(t,\infty) \cap C = \emptyset$ for all $C \in \mathcal{B}([0,t])$ by definition. Now that we know how any member from the generating set intersects with $\{T > t\}$ we check the intersection of anything that we can construct inside of the σ-field $\sigma\left(\mathcal{B}([0,t]) \star \mathcal{S}, \{T > t\}, \mathcal{N}_0\right)$ from our generating sets:

- $\Omega \cap \{T > t\} = \{T > t\}$ and $\emptyset \cap \{T > t\} = \emptyset$
- Let $B \in (\mathcal{B}([0,t]) \star \mathcal{S}) \cup \{(t,\infty) \times X\} \cup \mathcal{N}_0$. Then

$$\overline{B} \cap \{T > t\} = \begin{cases} \emptyset, & \text{if } B = (t,\infty) \times X \\ \{T > t\} \setminus N, & \text{if } B \in \mathcal{N}_0, B \cap \{T > t\} \neq \emptyset, \\ \{T > t\}, & \text{else.} \end{cases}$$

- Let $(B_n)_{n\in\mathbb{N}} \subseteq (\mathcal{B}([0,t]) \star \mathcal{S}) \cup \{(t,\infty) \times X\} \cup \mathcal{N}_0$. Then $B_n \cap \{T > t\} \in \{\{T > t\}, N, \emptyset\}$ by the same arguments as above. Hence

$$\left(\bigcup_{n\in\mathbb{N}} B_n\right) \cap \{T > t\} = \begin{cases} N', & \text{if } \forall n \in \mathbb{N} B_n \neq \{T > t\} \\ \{T > t\} \cup N'', & \text{else,} \end{cases}$$

  where $N', N'' \in \mathcal{N}_0$ and thus

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} B_n \cap \{T > t\}\right) \in \{0, \mathbb{P}(\{T > t\})\}.$$

Now we note that $\mathbb{P}(\bigcup_{k\in I} N_k) = 0$ end check every possible case of intersection for it's $\mathbb{P}$-measure, just to realize, that only $\mathbb{P}(\{T > t\})$ and 0 occur. Summarizing: $\forall A \in \sigma(\mathcal{B}([0,t]) \star \mathcal{S}) \cup \{\{T > t\}\} \cup \mathcal{N}_0)$:

$$\mathbb{P}(A \cap \{T > t\}) \in \{0, \mathbb{P}(\{T > t\})\}.$$

□

The above discussion was not in vain. For σ-algebras of this form one might use the following lemma - which is a slight modification of Theorem 1 in [Shiryaev, 2016], p.256 - to extract information on a process for it's conditional expectation:

Lemma 1.6. *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and let $\mathcal{F} = \sigma(\mathcal{D}, \mathcal{A})$ be a sub-σ-algebra of $\mathcal{G}$, where $\mathcal{D} = \{D_1, D_2, \dots\}$, $\mathcal{A}$ an arbitrary family of sets s.t.*

- $D \cap A = \emptyset \quad \forall D \in \mathcal{D}, A \in \mathcal{A}$,
- $\forall D \in \mathcal{D}$: *$D$ is a $\mathbb{P}$-atom of $\mathcal{F}$.*

Now take η a random variable on the probabilty space for which $\mathbb{E}[\eta]$ exists. Then

$$\mathbb{1}_D \mathbb{E}[\eta|\mathcal{F}] = \mathbb{1}_D \frac{\int \mathbb{1}_D \eta d\mathbb{P}}{\mathbb{P}(D)}$$

on all atoms $D \in \mathcal{F}$.

Proof. Let $D \in \mathcal{F}$ be an atom. The first step is to show, that for $\xi : \Omega \to \mathbb{R}$ $\mathcal{F}$-measurable

$$\mathbb{P}(D \cap \{\xi \neq const.\}) = 0.$$

Set $K := \sup\{y \in \mathbb{R} : \mathbb{P}(D \cap \{\xi \leq y\}) = 0\}$. Then we have

$$\begin{aligned}
\mathbb{P}(D \cap \{\xi < K\}) &= \mathbb{P}\left(\bigcup_{r<K, r\in\mathbb{Q}} \{\omega \in D : \xi(\omega) < r\} \right) \\
&\leq \sum_{r<K, r\in\mathbb{Q}} \mathbb{P}(\{\omega \in D : \xi(\omega) < r\}) \\
&= 0
\end{aligned}$$

where we used $\{\xi < r, r < K\} \subset \{\xi < K\}$. Now take $y > K$, then we have

$$\mathbb{P}[D \cap \{\xi < y\}] > 0.$$

But we chose D to be an atom, i.e. we get $\mathbb{P}(D \setminus \{\xi < y\}) = \mathbb{P}(D \cap \{\xi \geq y\}) = 0$ and this yields:

$$\begin{aligned}
\mathbb{P}(D \cap \{\xi > K\}) &= \mathbb{P}\left(\bigcup_{r>K, r\in\mathbb{Q}} \{\omega \in D : \xi(\omega) \geq r\} \right) \\
&\leq \sum_{r>K, r\in\mathbb{Q}} \mathbb{P}(\{\omega \in D : \xi(\omega) \geq r\}) \\
&= 0.
\end{aligned}$$

We proved so far, that every $\mathcal{F}$-measurable ξ: $\mathbb{P}(D \cap \{\xi \neq K\}) = 0$, i.e. ξ is constant on atoms $D \in \mathcal{F}$ $\mathbb{P}$-a.s..
Now let us prove the stated equality:

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_D \eta] &= \int \mathbb{1}_D \eta d\mathbb{P} \\
&= \int_D \mathbb{E}[\eta|\mathcal{F}] d\mathbb{P} \\
&= \int_D K d\mathbb{P} \\
&= K\mathbb{P}(D)
\end{aligned}$$

where we used the first part of the proof, the property, that $\mathbb{E}[\eta|\mathcal{F}]$ is constant on atoms. For an arbitrary set $A \in \mathcal{F}$ we start with the defining property of the conditional expectation (see section 1.1):

$$\begin{aligned}\int_A \mathbb{1}_D \mathbb{E}[\eta|\mathcal{F}] d\mathbb{P} &= \int_A \mathbb{1}_D \eta d\mathbb{P} \\ &= \int_{A\cap D} \eta d\mathbb{P} \\ &= \mathbb{E}[\mathbb{1}_{A\cap D}\eta]. \end{aligned} \tag{1.2.2}$$

Now we start with the right side of the stated equality:

$$\begin{aligned}\int_A \mathbb{1}_D \frac{\mathbb{E}[\mathbb{1}_D \eta]}{\mathbb{P}(D)} d\mathbb{P} &= \int \mathbb{1}_{A\cap D} \frac{\mathbb{E}[\eta \mathbb{1}_D]}{\mathbb{P}(D)} d\mathbb{P} \\ &= \mathbb{E}[\eta \mathbb{1}_D] \frac{\mathbb{P}(A\cap D)}{\mathbb{P}(D)} \end{aligned} \tag{1.2.3}$$

We keep in mind, that D is an atom, so either

$$\mathbb{P}(A\cap D) = 0$$

which would yield (1.2.2)$= 0 =$(1.2.3), or

$$\mathbb{P}(D \setminus (A\cap D)) = \mathbb{P}(D\setminus A) = 0$$

which in turn would yield (see (1.2.2), (1.2.3))

$$\begin{aligned}\mathbb{P}(A\cap D) &= \mathbb{P}(D), \\ \mathbb{E}[\mathbb{1}_{A\cap D}\eta] &= K\mathbb{P}(A\cap D) \\ &= K\mathbb{P}(D) \\ &= \mathbb{E}[\mathbb{1}_D \eta]. \end{aligned}$$

□

Another result by A.Gushchin approaches the problem from the perspective of the filtration itself and defines so called *single-jump filtrations* as σ-algebras of certain properties:

Definition 1.7. *(see [Gushchin, 2020], p.139) Let $(\Omega, \mathcal{A})$ be a measurable space, γ a random variable. A* single jump filtration *is σ-field defined for $t \in \mathbb{R}^+$ as*

$$\mathcal{G}_t := \{A \in \mathcal{A} : A \cap \{t < \gamma\} = \emptyset \text{ or } A\cap \{t<\gamma\} = \{t<\gamma\}\}. \tag{1.2.4}$$

In our notation, where T is the jump time of our single-jump process, the single-jump filtration of T is

$$\mathcal{G}_t = \{A \in \mathcal{F} : A\cap\{t<T\} = \emptyset \text{ or } A\cap\{t<T\} = \{t<T\}\}.$$

We note that $\mathcal{F}_t \supset \mathcal{G}_t$ since for any $A \in \mathcal{G}_t$ we have that $A \in \mathcal{F}$ and in case $A \cap \{T > t\} = \emptyset$ we can conclude that $A \in \mathcal{B}([0,t]) \star \mathcal{S}$ and in case $A\cap\{T>t\} = \{T>t\}$ we

can assume $A \supset \{T > t\}$ and thus $A \in \mathcal{F}_t$. The other inclusion is in fact in general not true since Gushchin does not require $\mathcal{F}$ to be complete and for a $\mathbb{P}$-nullset $N \in \mathcal{N}_0 \setminus \mathcal{F}^0$ with $N \cap \{T > t\} = N$ we get $N \in \mathcal{F}_t$ but $N \notin \mathcal{G}_t$.
Although he works in a framework of filtrations that are determined by the jump time T only, the results of Gushchin are nevertheless not far from the result for our natural filtration of the process. He can conclude, that for a càdlàg process to be a local martingale w.r.t. to the single-jump filtration it is enough, if it can be represented as a deterministic function before, and as a function of T and a random variable after the jump time (see [Gushchin, 2020], theorem 2). We will discuss this result further in section 1.4.

1.3 Discrete time results

Let $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$, be Borel-measurable and set for $t \in \mathbb{R}^+$

$$v(t, \omega) := \varphi(t, x(t, \omega)), \quad \forall \omega \in \Omega.$$

Since φ is measurable the new process $v := (v(t))_{t \geq 0}$ is still adapted to the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. In general this process won't be an $(\mathcal{F}_t)$-martingale, but - according to [Davis, 1976], theorem 1, - when it is, it will be an M_t^g for some g, i.e. a stochastic integral.
We define a discrete version of this process v on the set of dyadic numbers $\mathcal{D}$ (see [Beiglboeck et al., 2010]). To that end take for $t \in \mathbb{R}^+$ and $N \in \mathbb{N}$ the set of the N-th dyadic numbers of the interval $[0, t]$:

$$\mathcal{D}_t^N := \left\{ \frac{nt}{2^N} : n \in \{1, \ldots, 2^N\} \right\} \tag{1.3.1}$$

and set

$$t_k := \begin{cases} \frac{kt}{2^N} & \text{for } k \leq 2^N \\ t & \text{for } k > 2^N. \end{cases} \tag{1.3.2}$$

Note that $\mathcal{D}_t^N = \{t_0, \ldots, t_{2^N}\}$.

Now $(v(t_k))_{k \in \mathbb{N}}$ is a discrete version of the process $(v(s \wedge t))_{s \in \mathbb{R}^+}$ (i.e. the process stopped at time t) and it is adapted to its respective σ-field $\mathcal{F}_{t_k}^0 := \sigma\{x(t_l) : l \in \{1, \ldots, k\}\}$ (respectively the augmented version $\mathcal{F}_{t_k}$) and the above lemmata 1.5 and 1.6 apply. As an adapted discrete-time process $(v(t_k))_{k \in \mathbb{N}}$ it qualifies for a Doob-decomposition, a result which we will quote here for convenience with its instructive proof:

Theorem 1.8. *(see for example: [Protter, 2013] p.106) Let $(s_n)_{n \in \mathbb{N}}$ be an $\mathcal{F}_n$ adapted process. Then there is a decomposition $s_n = M_n + A_n$ for all $n \in \mathbb{N}$ where $(M_n)_{n \in \mathbb{N}}$ is an $\mathcal{F}_n$-martingale (called* martingale part*) and $(A_n)_{n \in \mathbb{N}}$ is a previsible process (called* compensator*).*

Proof. We set

$$\begin{aligned} M_0 &:= s_0 \\ M_n &:= M_{n-1} + s_n - \mathbb{E}[s_n|\mathcal{F}_{n-1}] \quad \text{for } n \geq 1 \\ A_0 &:= 0 \\ A_n &:= A_{n-1} + \mathbb{E}[s_n|\mathcal{F}_{n-1}] - s_{n-1} \quad \text{for } n \geq 1. \end{aligned}$$

By piecing together $s_n - s_{n-1}$ we can check the decomposition property. Taking the conditional expectation we verify the martingale property:

$$\begin{aligned} \mathbb{E}[M_n - M_{n-1}|\mathcal{F}_{n-1}] &= \mathbb{E}\left[s_n - \mathbb{E}[s_n|\mathcal{F}_{n-1}]|\mathcal{F}_{n-1}\right] \\ &= \mathbb{E}[s_n|\mathcal{F}_{n-1}] - \mathbb{E}[s_n|\mathcal{F}_{n-1}] \\ &= 0 \end{aligned}$$

for all $n \geq 1$. The compensator is obviously previsible, since it only depends on s_{n-1} which was assumed to be adapted. □

In view of this decomposition for the process $v(t_k)$ we will need the following

Lemma 1.9.

$$\begin{aligned} \mathbb{E}[v(t_k)|\mathcal{F}_{t_{k-1}}] &= \varphi(t_k, x(t_{k-1})) \\ &\quad + \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)]\mu(ds, dz). \end{aligned}$$

Proof. We have

$$\begin{aligned} \mathbb{E}[\varphi(t_k, x(t_k))|\mathcal{F}_{k-1}] &= \mathbb{E}[\varphi(t_k, x(t_k))\mathbb{1}_{T\leq t_{k-1}}|\mathcal{F}_{k-1}] + \mathbb{E}[\varphi(t_k, x(t_k))\mathbb{1}_{T>t_{k-1}}|\mathcal{F}_{k-1}] \\ &= \varphi(t_k, x(t_{k-1}))\mathbb{1}_{T\leq t_{k-1}} \\ &\quad + \mathbb{1}_{T>t_{k-1}} \frac{1}{\mathbb{P}(T > t_{k-1})} \underbrace{\int_\Omega \varphi(t_k, x(t_k))\mathbb{1}_{T>t_{k-1}} d\mathbb{P}}_{=:(I)} \end{aligned}$$

where we used

$$x(t_k, \omega)\mathbb{1}_{T\leq t_{k-1}}(\omega) = Z(\omega)\mathbb{1}_{T\leq t_{k-1}}(\omega) = x(t_{k-1}, \omega)\mathbb{1}_{T\leq t_{k-1}}(\omega)$$

and the equality

$$\begin{aligned} \varphi(t_k, x(t_k)(\omega))\mathbb{1}_{T\leq t_{k-1}}(\omega) &= \varphi(t_k, Z(\omega)\mathbb{1}_{T\leq t_k}(\omega))\mathbb{1}_{T\leq t_{k-1}}(\omega) \\ &= \varphi(t_k, Z(\omega)\mathbb{1}_{T\leq t_{k-1}}(\omega))\mathbb{1}_{T\leq t_{k-1}}(\omega) \\ &= \varphi(t_k, x(t_{k-1})(\omega))\mathbb{1}_{T\leq t_{k-1}}(\omega) \end{aligned}$$

for the first part of the sum and for the second part we use lemma 1.6 and the fact that $\{T > t_{k-1}\}$ is an atom of the σ-field $\mathcal{F}_{k-1}$ (see lemma 1.5).

In particular:

$$
\begin{aligned}
(I) &= \int_\Omega \varphi(t_k, x(t_k))\mathbb{1}_{T>t_{k-1}}\mathbb{1}_{T\le t_k} + \varphi(t_k, x(t_k))\mathbb{1}_{T>t_{k-1}}\mathbb{1}_{T>t_k} d\mathbb{P} \\
&= \int_\Omega \varphi(t_k, Z\mathbb{1}_{T\le t_k})\mathbb{1}_{t_{k-1}<T\le t_k} d\mathbb{P} + \int_\Omega \varphi(t_k, z_0\mathbb{1}_{T>t_k})\mathbb{1}_{T>t_k} d\mathbb{P} \\
&= \int_\Omega \varphi(t_k, Z)\mathbb{1}_{t_{k-1}<T\le t_k} d\mathbb{P} + \int_\Omega \varphi(t_k, z_0)\mathbb{1}_{T>t_k} d\mathbb{P} \\
&= \int_{\mathbb{R}^+\times X} \varphi(t_k, z)\mathbb{1}_{(t_{k-1},t_k]}(s) d\mu(s,z) + \varphi(t_k, z_0)\int_{\mathbb{R}^+\times X} \mathbb{1}_{(t_k,\infty)} d\mu(s,z) \\
&= \int_{\mathbb{R}^+\times X} \varphi(t_k, z)\mathbb{1}_{(t_{k-1},t_k]}(s) d\mu(s,z) + \varphi(t_k, z_0)(1 - F_{t_k}) \\
&= \int_{\mathbb{R}^+\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)]\mathbb{1}_{(t_{k-1},t_k]}(s) d\mu(s,z) + \varphi(t_k, z_0)(1 - F_{t_{k-1}}),
\end{aligned}
$$

where we used

$$
\begin{aligned}
&\varphi(t_k, z_0)(1 - F_{t_k}) \\
=&\varphi(t_k, z_0)(1 - F_{t_{k-1}} - (F_{t_k} - F_{t_{k-1}})) \\
=&\varphi(t_k, z_0)(1 - F_{t_{k-1}}) - \varphi(t_k, z_0)\left(\int_\Omega \mathbb{1}_{T\le t_k} d\mathbb{P} - \int_\Omega \mathbb{1}_{T\le t_{k-1}} d\mathbb{P}\right) \\
=&\varphi(t_k, z_0)(1 - F_{t_{k-1}}) - \int_{\mathbb{R}^+\times X} \varphi(t_k, z_0)\mathbb{1}_{(t_{k-1},t_k]}(s) d\mu(s,z)
\end{aligned}
$$

in the last equality.

If we insert our findings in the original equation above we get:

$$
\begin{aligned}
\mathbb{E}[v(t_k)|\mathcal{F}_{k-1}] = \varphi(t_k, x(t_{k-1}))\mathbb{1}_{T\le t_{k-1}} + \mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}} \\
\times \left[\int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)]\mu(ds, dz) + \varphi(t_k, z_0)(1 - F_{t_{k-1}})\right] \\
= \varphi(t_k, x(t_{k-1})) \\
+ \mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}} \int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)]\mu(ds, dz).
\end{aligned}
$$

In the last equality we've combined $\varphi(t_k, x(t_{k-1}))\mathbb{1}_{T\le t_{k-1}}$ and

$$
\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\varphi(t_k, z_0)(1 - F_{t_{k-1}}) = \mathbb{1}_{T>t_{k-1}}\varphi(t_k, z_0)
$$

into one $\varphi(t_k, x(t_{k-1}))$, since on these different indicator functions the values of $x(t_{k-1})$ are known:

$$
\begin{aligned}
x(t_{k-1}, \omega)\mathbb{1}_{T>t_{k-1}}(\omega) &= z_0\mathbb{1}_{T>t_{k-1}}(\omega), \\
x(t_{k-1}, \omega)\mathbb{1}_{T\le t_{k-1}}(\omega) &= Z(\omega)\mathbb{1}_{T\le t_{k-1}}(\omega)
\end{aligned}
$$

□

Given the form of the conditional expectation from the last lemma we are now fully prepared to state the Doob decomposition of the transformed process $v = (v(t_k))_{k\in\mathbb{N}}$.

Theorem 1.10. *Let $t \in \mathbb{R}^+$, $t < c$ fixed. Let $N \in \mathbb{N}$. For $\mathcal{D}_t^N = (t_k)_{k\in\mathbb{N}}$ (see* (1.3.2) *the Doob decomposition of the process $(v(t_k))_{k\in\mathbb{N}}$ w.r.t. the filtration $(\mathcal{F}_{t_k})_{k\in\mathbb{N}}$ is given by the martingale part $(M_{t_k}^N)$ and a predictable compensator part $(A_{t_k}^N)$, each given respectively by*

$$
\begin{aligned}
M_{t_k}^N - M_{t_{k-1}}^N &= [\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1}))] \\
&\quad - \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)] d\mu(s,z), \\
A_{t_k}^N - A_{t_{k-1}}^N &= [\varphi(t_k, x(t_{k-1})) - \varphi(t_{k-1}, x(t_{k-1}))] \\
&\quad + \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)] d\mu(s,z)
\end{aligned}
$$

for every $k \in \mathbb{N}$ and $M_0^N := \varphi(0, z_0), A_0^K := 0$.

Proof. The two different parts of the Doob-Meyer decomposition of a given process $(v(k))_{k\in\{1,\dots,K\}}$ are given by

$$
M_{t_k}^N = M_{t_{k-1}}^N + v(t_k) - \mathbb{E}[v(t_k)|\mathcal{F}_{t_{k-1}}]
$$

and

$$
A_{t_k}^N = A_{t_{k-1}}^N + \mathbb{E}[v(t_k)|\mathcal{F}_{t_{k-1}}] - v(t_{k-1}).
$$

In our case $v(t_k) = \varphi(t_k, x(t_k))$ and the conditional expectation is taken from Lemma 1.9. $v(t_0) = \varphi(t_0, x(t_0))$ and $t_0 := 0$. $(M_{t_k}^N)_{k\in\mathbb{N}}$ is a $(\mathcal{F}_{t_k})$-martingale by construction and for any $k \in \mathbb{N}$ the random variable $A_{t_k}^N$ is $\mathcal{F}_{t_{k-1}}$-measurable, i.e. the process $(A_{t_k}^N)_{k\in\mathbb{N}}$ predictable. □

Remark 1.11. *Since $d\mu(s,z)$ can be disintegrated with respect to dF_s (see* (1.1.8)*), we can even write*

$$
\begin{aligned}
&\int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)] d\mu(s,z) \\
&= \int_{(t_{k-1},t_k]} \int_X \varphi(t_k, z) d\mu_s(z) dF_s - \int_{(t_{k-1},t_k]\times X} \varphi(t_k, z_0)] d\mu(s,z) \\
&= \int_{(t_{k-1},t_k]} \int_X \varphi(t_k, z) \mathbb{P}[Z \in dz | T = s] dF_s - \varphi(t_k, z_0)[F_{t_k} - F_{t_{k-1}}] \\
&= \int_{(t_{k-1},t_k]} \int_X \varphi(t_k, z) \mathbb{E}[\mathbb{1}_{Z\in dz} | T = s] dF_s - \varphi(t_k, z_0)[F_{t_k} - F_{t_{k-1}}]. \qquad (1.3.3)
\end{aligned}
$$

This property is useful for the cases, when Z only takes finitely many values. A discussion of this special case can be found in the Appendix.

Example 1.12. *This continues the investigation of the examples defined in example 1.2:*

(a) *In the situation of example 1.2 (a) let* $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. *Take* $\varphi(t, y) = \exp(\alpha t(1 - y) + \beta t y)$. *Then the process*

$$v(t) := \varphi(t, x(t)) = \exp\left(\alpha t(1 - \mathbb{1}_{T \le t}) + \beta t \mathbb{1}_{T \le t}\right)$$

has a discrete version $(v(t_k))_{k \in \mathbb{N}}$. *In this case we got*

$$\mu(t, \{1\}) = F_t, \quad \mu(t, \{0\}) \equiv 0$$

and thus

$$\int_{(t_{k-1}, t_k] \times X} d\mu(u, z) = \int_{(t_{k-1}, t_k]} dF_t$$

We note that

$$[\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1}))] = [\exp(\beta t_k) - \exp(\alpha t_k)] \, \mathbb{1}_{T \in (t_{k-1}, t_k]}$$

and end up with the following Doob decomposition:

$$\begin{aligned} M_t^N &= \sum_{k=1}^{2^N} [\varphi(t_k, 1) - \varphi(t_k, 0)] \, \mathbb{1}_{T \in (t_{k-1}, t_k]} \\ &\quad - \sum_{k=1}^{2^N} \frac{\mathbb{1}_{T > t_{k-1}}}{\exp(-\lambda t)} \int_{(t_{k-1}, t_k] \times X} [\varphi(t_k, 1) - \varphi(t_k, 0)] d\mu(u, z) \\ &= \sum_{k=1}^{2^N} \Big\{ [\exp(\beta t_k) - \exp(\alpha t_k)] \, \mathbb{1}_{T \in (t_{k-1}, t_k]} \\ &\quad - \mathbb{1}_{T > t_{k-1}} (1 - \exp(-\lambda(t_k - t_{k-1}))) \, [\exp(\beta t_k) - \exp(\alpha t_k)] \Big\} \\ A_t^N &= \sum_{k=1}^{2^N} \Big\{ \left[\exp(\alpha t_k \mathbb{1}_{T > t_{k-1}} + \beta t_k \mathbb{1}_{T \le t_{k-1}}) - \exp(\alpha t_{k-1} \mathbb{1}_{T > t_{k-1}} + \beta t_{k-1} \mathbb{1}_{T \le t_{k-1}})\right] \\ &\quad + \mathbb{1}_{T > t_{k-1}} (1 - \exp(-\lambda(t_k - t_{k-1}))) \, [\exp(\beta t_k) - \exp(\alpha t_k)] \Big\}. \end{aligned}$$

(b) *In the situation of example 1.2 (b) take*

$$\varphi(t, y) = \sin(ty).$$

The measure $d\mu(u, z)$ *is given in this case as*

$$d\mu(u, z) = \delta_1(u) d\Phi(z)$$

and the increments of φ *due to change of* $x(t)$ *are*

$$[\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1}))] = [\sin(Z t_k)] \, \mathbb{1}_{T \in (t_{k-1}, t_k]}.$$

Then the Doob-decomposition of $(v(t_k))$ *is given by the increments:*

$$\begin{aligned} M_{t_k}^N - M_{t_{k-1}}^N &= \left(\sin(Z t_k) - \int_{(t_{k-1}, t_k]} \sin(z t_k) d\Phi(z) \right) \mathbb{1}_{(t_{k-1}, t_k]}(1) \\ A_{t_k}^N - A_{t_{k-1}}^N &= [\sin(Z t_k) - \sin(Z t_{k-1})] \, \mathbb{1}_{[0, t_{k-1}]}(1) \\ &\quad + \int_{(t_{k-1}, t_k]} \sin(z t_k) d\Phi(z) \mathbb{1}_{(t_{k-1}, t_k]}(1). \end{aligned}$$

(c) *In the situation of example 1.2 (c) we have the measure*

$$d\mu(u,z) = dP_T(u)dP_Z(z)$$

where P_T, P_Z *are the respective distributions of* T *and* Z. *Take* $\varphi(t,y) = y$ *The increments of a function* φ *again reduce to the case where the process has just jumped:*

$$[\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1}))] = [x(t_k) - x(t_{k-1})]\,\mathbb{1}_{T\in(t_{k-1},t_k]}.$$

The Doob decomposition is given by:

$$\begin{aligned} M^N_{t_k} - M^N_{t_{k-1}} &= Z\mathbb{1}_{T\in(t_{k-1},t_k]} \\ A^N_{t_k} - A^N_{t_{k-1}} &= 0. \end{aligned}$$

Thus this simple process already is a martingale, which we also see by computing:

$$\mathbb{E}[Z\mathbb{1}_{T\le t}] = \mathbb{E}[Z]\mathbb{P}(T\le t) = 0.$$

But let us now assume, that Z *is not centered around our chosen* $z_0 = 0$. *Let* $\mathbb{E}[Z] = \xi$. *Then we get from theorem 1.10 that*

$$\begin{aligned} M^N_{t_k} - M^N_{t_{k-1}} &= Z\mathbb{1}_{T\in(t_{k-1},t_k]} - \mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\int_{(t_{k-1},t_k]}\int_X z dP_Z(z)dF_u \\ &= Z\mathbb{1}_{T\in(t_{k-1},t_k]} - \mathbb{1}_{T>t_{k-1}}\frac{F_{t_k} - F_{t_{k-1}}}{1-F_{t_{k-1}}}\xi \\ A^N_{t_k} - A^N_{t_{k-1}} &= \mathbb{1}_{T>t_{k-1}}\frac{F_{t_k} - F_{t_{k-1}}}{1-F_{t_{k-1}}}\xi. \end{aligned}$$

In the case where we assume $T \sim Exp(\lambda)$ *we then get:*

$$\begin{aligned} M^N_{t_k} - M^N_{t_{k-1}} &= Z\mathbb{1}_{T\in(t_{k-1},t_k]} - \mathbb{1}_{T>t_{k-1}}\left[1 - \exp(-\lambda(t_k - t_{k-1})\right]\xi \\ A^N_{t_k} - A^N_{t_{k-1}} &= \mathbb{1}_{T>t_{k-1}}\left[1 - \exp(-\lambda(t_k - t_{k-1})\right]\xi. \end{aligned}$$

(d) *In the situation of example 1.2 (d) we are left with the cryptic measure*

$$d\mu(u,z) = \mathbb{P}(Z\in dz|T=u)\mathbb{P}(T\in du) = \Phi(udz)dF_u$$

Take again $\varphi(t,y) = y$. *Note that*

$$[\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1}))] = Z\mathbb{1}_{t_{k-1}<T\le t_k}$$

and compute the Doob-decomposition:

$$\begin{aligned} M^N_{t_k} - M^N_{t_{k-1}} &= x(t_k) - x(t_{k-1}) \\ A^N_{t_k} - A^N_{t_{k-1}} &= 0 \end{aligned}$$

again. Let us assume, that $Z \sim \mathcal{N}(\xi, \frac{1}{t})$ given $T = t$. Then theorem 1.10 gives us:

$$\begin{aligned} M^N_{t_k} - M^N_{t_{k-1}} &= Z\mathbb{1}_{T\in(t_{k-1},t_k]} - \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int_{(t_{k-1},t_k]} \int_X z d\mu_u(z) dF_u \\ &= Z\mathbb{1}_{T\in(t_{k-1},t_k]} - \mathbb{1}_{T>t_{k-1}} \frac{F_{t_k} - F_{t_{k-1}}}{1-F_{t_{k-1}}}\xi \\ A^N_{t_k} - A^N_{t_{k-1}} &= \mathbb{1}_{T>t_{k-1}} \frac{F_{t_k} - F_{t_{k-1}}}{1-F_{t_{k-1}}}\xi. \end{aligned}$$

This is the same result as in the independent case.

1.4 Limiting procedure

Similar to [Beiglboeck et al., 2010] we now turn to increasing the frequency of our discrete dyadic times $(t_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}^N_t$ (see (1.3.2)) and by that achieving a decomposition of the time-continuous process into an $(\mathcal{F}_t)$-martingale (M_t) and a predictable part (A_t).

Let us state the time continuous version of theorem 1.10. We will realize in its proof, why we will need a few more assumptions on φ now:

Definition 1.13. *We say a function $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$ satisfies the condition (C), if*

(i) $\varphi(\cdot, z) \in \mathcal{C}^1$ *for all* $z \in X$,

(ii) $\sup_{s\in[0,t]} |\varphi(s,\cdot)| \in L^1_{loc}(\mu)$, *for all* $t < c$.

(iii) $\varphi \in L^1_{loc}(\mu)$.

Theorem 1.14. *Under the assumption that φ satisfies the condition (C) the process $(v(t))_{t\in\mathbb{R}^+}$, where $v(t)(\omega) = \varphi(t, x(t)(\omega))$ can be written as*

$$v(t) = M_t + A_t.$$

M_t is a local $(\mathcal{F}_t)$ martingale on $[0, c)$ and A_t is an $(\mathcal{F}_t)$-previsible process. Both processes are given respectively by:

$$\begin{aligned} M_t &= \varphi(t_0, z_0) + \int_{(0,t]} (\varphi(u, z) - \varphi(u, z_0)) dq(u, z), \\ A_t &= \int_{(0,t]} \frac{\partial\varphi}{\partial t}(u, x(u-)) du + \int_{(0,t]} \frac{\mathbb{1}_{T\geq u}}{1 - F_{u-}} (m(u) - \varphi(u, z_0)) F(du), \end{aligned}$$

where $m(u) = \mathbb{E}[\varphi(T, Z)|T = u]$.

Proof. We prove the theorem in 4 different parts:
Construction of M:

We take the result from theorem 1.10 and write ($t_0 = 0, t_{2^N} = t$):

$$\begin{aligned} M_t^N - M_0^N &= \sum_{k=1}^{2^N} \left[M_{t_k}^N - M_{t_{k-1}}^N \right] \\ &= \sum_{k=1}^{2^N} \Big[[\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1}))] \\ &\quad - \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)] d\mu(s, z) \Big] \end{aligned} \tag{1.4.1}$$

Note that

$$\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1})) = [\varphi(t_k, Z) - \varphi(t_k, z_0))] \mathbb{1}_{t_{k-1}<T\le t_k}.$$

and make use of (1.1.12) to write

$$\sum_{k=1}^{2^N} [\varphi(t_k, x(t_k)) - \varphi(t_k, x(t_{k-1}))] = \int_{\mathbb{R}^+\times X} \sum_{k=1}^{2^N} [\varphi(t_k, z) - \varphi(t_k, z_0)] \, \mathbb{1}_{(t_{k-1},t_k]}(u) dp(u, z) \tag{1.4.2}$$

and rearrange

$$\begin{aligned} &\sum_{k=1}^{2^N} \mathbb{1}_{T>t_{k-1}}(\omega) \frac{1}{1-F_{t_{k-1}}} \int_{(t_{k-1},t_k]\times X} [\varphi(t_k, z) - \varphi(t_k, z_0)] \, d\mu(u, z) \\ = &\int_{\mathbb{R}^+\times X} \sum_{k=1}^{2^N} [\varphi(t_k, z) - \varphi(t_k, z_0)] \, \mathbb{1}_{T>t_{k-1}}(\omega) \frac{1}{1-F_{t_{k-1}}} \mathbb{1}_{(t_{k-1},t_k]}(u) d\mu(u, z) \end{aligned} \tag{1.4.3}$$

We explore the limiting behavior of these integrals separately. In (1.4.2) we note that for fixed $u \in \mathbb{R}^+, z \in X$:

$$\sum_{k=1}^{2^N} [\varphi(t_k, z) - \varphi(t_k, z_0)] \, \mathbb{1}_{(t_{k-1},t_k]}(u) \to [\varphi(u, z) - \varphi(u, z_0)] \, \mathbb{1}_{(0,t]}(u)$$

as $N \to \infty$. Thus for a fixed $\omega \in \Omega$ we get that

$$\sum_{k=1}^{2^N} [\varphi(t_k, Z(\omega)) - \varphi(t_k, z_0)] \, \mathbb{1}_{(t_{k-1},t_k]}(T(\omega)) \tag{1.4.4}$$

$$\to [\varphi(T(\omega), Z(\omega)) - \varphi(T(\omega), z_0)] \, \mathbb{1}_{(0,t]}(T(\omega))$$

$$= \int_{(0,t]\times X} [\varphi(u, z) - \varphi(u, z_0)] \, dp(u, z) \tag{1.4.5}$$

where we've used (1.1.12).

For the other limit in (1.4.3) we note that for fixed $u \in \mathbb{R}^+, z \in X, \omega \in \Omega$:

$$\sum_{k=1}^{2^N} [\varphi(t_k, z) - \varphi(t_k, z_0)] \, \mathbb{1}_{T>t_{k-1}}(\omega) \frac{1}{1-F_{t_{k-1}}} \mathbb{1}_{(t_{k-1},t_k]}(u)$$

$$\to [\varphi(u, z) - \varphi(u, z_0)] \, \mathbb{1}_{T\ge u} \frac{1}{1-F_{u-}} \mathbb{1}_{(0,t]}(u)$$

as $N \to \infty$. They are also bounded by an integrable function since

$$\begin{aligned}
&|\sum_{k=1}^{2^N} \varphi(t_k, z) - \varphi(t_k, z_0) \mathbb{1}_{T>t_{k-1}}(\omega) \frac{1}{1 - F_{t_{k-1}}} \mathbb{1}_{(t_{k-1},t_k]}(u)| \\
\leq &\sum_{k=1}^{2^N} |\varphi(t_k, z) - \varphi(t_k, z_0) \mathbb{1}_{T>t_{k-1}}(\omega) \frac{1}{1 - F_{t_{k-1}}} \mathbb{1}_{(t_{k-1},t_k]}(u)| \\
\leq &\sup_{k \in \{1,\dots,2^N\}} \frac{|\varphi(t_k, z) - \varphi(t_k, z_0)|}{1 - F_{t_{k-1}}} \\
\leq &R_t \left(\sup_{k \in \{1,\dots,2^N\}} |\varphi(t_k, z)| + \sup_{k \in \{1,\dots,2^N\}} |\varphi(t_k, z_0)| \right).
\end{aligned}$$

where we used that for any $t < c$ there is a $R_t \in \mathbb{R}^+$, such that

$$\sup_{k \in \{1,\dots,2^N\}} \frac{1}{1 - F_{t_{k-1}}} \leq \sup_{s \in [0,t]} \frac{1}{1 - F_{s-}} = \frac{1}{1 - F_{t-}} = R_t < \infty.$$

We use the dominated convergence theorem to get for any $\omega \in \Omega$:

$$\begin{aligned}
&\sum_{k=1}^{K} \mathbb{1}_{T>t_{k-1}}(\omega) \frac{1}{1 - F_{t_{k-1}}} \int_{(t_{k-1},t_k] \times X} \varphi(t_k, z) - \varphi(t_k, z_0) d\mu(u, z) \\
\to &\int_{(0,t] \times X} [\varphi(u, z) - \varphi(u, z_0)] \mathbb{1}_{T \geq u} \frac{1}{1 - F_{u-}} d\mu(u, z) \qquad (1.4.6) \\
\overset{(1.1.13)}{=} &\int_{\mathbb{R}^+ \times X} [\varphi(u, z) - \varphi(u, z_0)] \mathbb{1}_{(0,t]}(u) d\tilde{p}(u, z, \omega) \qquad (1.4.7)
\end{aligned}$$

for $N \to \infty$.

We set $M_0 := \varphi(0, z_0)$ and get in (1.4.1) with the help of (1.4.4) and (1.4.7) for any fixed $\omega \in \Omega$:

$$\begin{aligned}
M_t^N(\omega) - M_0^N(\omega) &= M_t^N(\omega) - \varphi(0, z_0) \\
&\to \int_{\mathbb{R}^+ \times X} [\varphi(u, z) - \varphi(u, z_0)] \mathbb{1}_{(0,t]}(u) dp(u, z, \omega) \\
&\quad - \int_{\mathbb{R}^+ \times X} [\varphi(u, z) - \varphi(u, z_0)] \mathbb{1}_{(0,t]}(u) d\tilde{p}(u, z, \omega) \\
&= \int_{\mathbb{R}^+ \times X} [\varphi(u, z) - \varphi(u, z_0)] \mathbb{1}_{(0,t]}(u) dq(u, z, \omega) \\
&=: M_t(\omega) - \varphi(0, z_0).
\end{aligned}$$

Construction of A:
The second part is the limiting behaviour in the compensator. We use the results of

theorem 1.10 to write ($t_0 = 0, t_{2^N} = t$):

$$\begin{aligned} A_t^N - A_0^N &= \sum_{k=1}^{2^N} \left[A_{t_k}^N - A_{t_{k-1}}^N \right] \\ &= \sum_{k=1}^{2^N} \Big[[\varphi(t_k, x(t_{k-1})) - \varphi(t_{k-1}, x(t_{k-1}))] \\ &\quad + \mathbb{1}_{T > t_{k-1}} \frac{1}{1 - F_{t_{k-1}}} \int_{(t_{k-1}, t_k] \times X} [\varphi(t_k, z) - \varphi(t_k, z_0)] d\mu(s, z) \Big]. \end{aligned} \tag{1.4.8}$$

Note that since $\varphi(\cdot, z) \in \mathcal{C}^1$ for all $z \in X$ we can write

$$\varphi(t_k, x(t_{k-1})) - \varphi(t_{k-1}, x(t_{k-1}) = \int_{(t_{k-1}, t_k]} \frac{d\varphi}{du}(u, x(t_{k-1})) du \tag{1.4.9}$$

and thus:

$$\sum_{k=1}^{2^N} [\varphi(t_k, x(t_{k-1})) - \varphi(t_{k-1}, x(t_{k-1}))] = \int_{\mathbb{R}^+} \sum_{k=1}^{2^N} \frac{d\varphi}{dt}(u, x(t_{k-1})) \mathbb{1}_{(t_{k-1}, t_k]}(u) du. \tag{1.4.10}$$

The second sum of (1.4.8) can again be written as:

$$\begin{aligned} &\sum_{k=1}^{2^N} \mathbb{1}_{T > t_{k-1}}(\omega) \frac{1}{1 - F_{t_{k-1}}} \int_{(t_{k-1}, t_k] \times X} \varphi(t_k, z) - \varphi(t_k, z_0) d\mu(u, z) \\ = &\int_{\mathbb{R}^+ \times X} \sum_{k=1}^{2^N} \varphi(t_k, z) - \varphi(t_k, z_0) \mathbb{1}_{T > t_{k-1}}(\omega) \frac{1}{1 - F_{t_{k-1}}} \mathbb{1}_{(t_{k-1}, t_k]}(u) d\mu(u, z) \end{aligned} \tag{1.4.11}$$

Since the limit of this integral has been discussed in (1.4.7) we focus on the limit of (1.4.10). Take $\omega \in \Omega$ and set $K^N(\omega) := \max\{k : t_{k-1} < T(\omega)\}$. For fixed $u \in [0, t], z \in X$:

$$\begin{aligned} &\sum_{k=1}^{2^N} \frac{d\varphi}{dt}(u, x(t_{k-1}, \omega)) \mathbb{1}_{(t_{k-1}, t_k]}(u) \\ = &\sum_{k=1}^{K^N(\omega)} \frac{d\varphi}{dt}(u, z_0) \mathbb{1}_{(t_{k-1}, t_k]}(u) + \sum_{k=K^N(\omega)}^{2^N} \frac{d\varphi}{dt}(u, Z(\omega)) \mathbb{1}_{(t_{k-1}, t_k]}(u) \\ \to &\frac{d\varphi}{dt}(u, z_0) \mathbb{1}_{(0, T(\omega)]}(u) + \frac{d\varphi}{dt}(u, z_0) \mathbb{1}_{(T(\omega), t]}(u) \\ = &\frac{d\varphi}{dt}(u, x(u-, \omega)) \mathbb{1}_{(0, t]}(u) \end{aligned}$$

where we used that for fixed $\omega \in \Omega : t_{K^N(\omega)} \searrow T(\omega)$ as $N \to \infty$ and thus $\mathbb{1}_{(0, t_{K^N(\omega)}]}(u) \to \mathbb{1}_{(0, T(\omega)]}(u)$ and $\mathbb{1}_{(K^N(\omega), t]}(u) \to \mathbb{1}_{(T(\omega), t]}(u)$ for all $u \in \mathbb{R}^+$. In the last equation we identified

$$z_0 \mathbb{1}_{(0, T(\omega)]}(u) = x(u-, \omega) \mathbb{1}_{(0, T(\omega)]}(u) \text{ and } Z(\omega) \mathbb{1}_{(T(\omega), t]}(u) = x(u-, \omega) \mathbb{1}_{(T(\omega), t]}(u).$$

Further we have

$$
\begin{aligned}
& |\sum_{k=1}^{2^N} \frac{d\varphi}{dt}(u, x(t_{k-1}, \omega))\mathbb{1}_{(t_{k-1},t_k]}(u)| \\
\leq & \sum_{k=1}^{K^N(\omega)} |\frac{d\varphi}{dt}(u, z_0)\mathbb{1}_{(t_{k-1},t_k]}(u)| + \sum_{k=K^N(\omega)+1}^{2^N} |\frac{d\varphi}{dt}(u, Z(\omega))\mathbb{1}_{(t_{k-1},t_k]}(u)| \\
\leq & |\frac{d\varphi}{dt}(u, z_0)|\mathbb{1}_{(0,t]}(u) + |\frac{d\varphi}{dt}(u, Z(\omega))|\mathbb{1}_{(0,t]}(u) \\
\in & L^1_{\text{loc}}(\mathbb{P}).
\end{aligned}
$$

Thus we can use dominated convergence to get

$$
\sum_{k=1}^{2^N} \frac{d\varphi}{dt}(u, x(t_{k-1}, \omega))\mathbb{1}_{(t_{k-1},t_k]}(u) \to \int_{(0,t]} \frac{d\varphi}{dt}(u, x(u-, \omega))du. \tag{1.4.12}
$$

Returning to (1.4.8) we combine (1.4.12) and (1.4.7) to achieve for any $\omega \in \Omega$ and any $t < c$:

$$
\begin{aligned}
A_t^N(\omega) - A_0^N(\omega) \to & \int_{(0,t]} \frac{d\varphi}{dt}(u, x(u-, \omega))du \\
& + \int_{(0,t]\times X} [\varphi(u, z) - \varphi(u, z_0)]\, \mathbb{1}_{T\geq u}(\omega)\frac{1}{1 - F_{u-}}d\mu(u, z)
\end{aligned}
$$

for $N \to \infty$. Setting $A_0^N \equiv 0$ and using the disintegration property of μ (see (1.1.8)) we finally get

$$
\begin{aligned}
A_t := & \int_{(0,t]\times X} [\varphi(u, z) - \varphi(u, z_0)]\, \mathbb{1}_{T\geq u}(\omega)\frac{1}{1 - F_{u-}}d\mu(u, z) \\
= & \int_{(0,t]} \int_X [\varphi(u, z) - \varphi(u, z_0)]\, \mathbb{1}_{T\geq u}\frac{1}{1 - F_{u-}}d\mu_u(dz)dF_u \\
= & \int_{(0,t]} \left[\int_X \varphi(u, z)d\mu_u(dz) - \varphi(u, z_0)\right] \mathbb{1}_{T\geq u}\frac{1}{1 - F_{u-}}dF_u \\
= & \int_{(0,t]} [\mathbb{E}[\varphi(T, Z)|T = u] - \varphi(u, z_0)]\, \mathbb{1}_{T\geq u}\frac{1}{1 - F_{u-}}dF_u.
\end{aligned} \tag{1.4.13}
$$

M **is a martingale:**
Now we have to check that (M_t) is a local $(\mathcal{F}_t)$-martingale and (A_t) is predictable. The former follows from [Davis, 1976], theorem 1 since $\varphi \in L^1_{\text{loc}}(\mu)$ by condition (C), (iii) and

$$
M_t = \varphi(0, z_0) + \int_{(0,t]} \varphi(u, z) - \varphi(u, z_0)dq(u, z) = M_t^g
$$

for $g(u, z) = \varphi(u, z) - \varphi(u, z_0)$.

A **is predictable:**
The predictability of (A_t) can be verified by decomposing the compensator into predictable parts. Clearly the first integral $\int_{(0,t]} \frac{d\varphi}{du}(u, x(u-, \omega))du$ is continuous in t for all

$\omega \in \Omega$ and as such is predictable as a left-continuous process. Now for the remaining part of A_t:

$$\int_{(0,t]} \frac{\mathbb{1}_{T\geq u}}{1-F_{u-}}(m(u)-\varphi(u,z_0))F(du) = \mathbb{1}_{T\geq t}\int_{(0,t]} \frac{1}{1-F_{u-}}(m(u)-\varphi(u,z_0))F(du) + \mathbb{1}_{T<t}\int_{(0,T]} \frac{1}{1-F_{u-}}(m(u)-\varphi(u,z_0))F(du).$$

Now

- $\mathbb{1}_{T<t}(\omega)$ and $\mathbb{1}_{T\geq t}(\omega)$ are left-continuous in t for all $\omega \in \Omega$,
- $\int_{(0,t]} \frac{1}{1-F_{u-}}(m(u)-\varphi(u,z_0))F(du)$ is continuous in t (and deterministic)
- and $\int_{(0,T]} \frac{1}{1-F_{u-}}(m(u)-\varphi(u,z_0))F(du)$ is a random variable which is $\mathcal{F}_T$ measurable.

Thus $\mathbb{1}_{T<t}(\omega)\int_{(0,T(\omega)]} \frac{1}{1-F_{u-}}(m(u)-\varphi(u,z_0))F(du)$ and $\mathbb{1}_{T\geq t}(\omega)(\int_{(0,t]} \frac{1}{1-F_{u-}}(m(u)-\varphi(u,z_0))F(du)$ are left-continuous for all $\omega \in \Omega$.
Hence A_t has left-continuous paths and is predictable in turn.

□

The above theorem has shown, that under relatively strong conditions the process $(v(t))_{t\in\mathbb{R}^+}$ can be decomposed in a Doob-Meyer manner. Since the ultimate goal of φ is to eliminate the compensator A_t we start looking for ways to simplify the actual representation of A_t. If we assume F to be differentiable (especially no jumps) surely F would provide a density $f = F'$ and we would write:

$$A_t = \int_{(0,t]} \frac{\partial\varphi}{\partial t}(u,x(u-)) + \frac{\mathbb{1}_{T\geq u}}{1-F_u}(m(u)-\varphi(u,z_0))\, f(u)du.$$

Determining the right φ is now a matter of solving

$$\frac{\partial\varphi}{\partial t}(u,x(u-)) + \frac{\mathbb{1}_{T\geq u}}{1-F_u}(m(u)-\varphi(u,z_0))\, f(u) = 0. \tag{1.4.14}$$

It is instructive to note that assuming F to be continuous implies that it is of Case A (see section 1.1).

Thus demanding F to be differentiable and even $\varphi(\cdot,z) \in \mathcal{C}^1$ for all $z \in X$ is already quite restrictive. Also we require these regularities w.r.t. the Lebesgue measure, which acts almost like a consultant here and generally would not be involved if T were to have discrete jump times and in turn a noncontinuous distribution function F.
So let us assume something more intrinsic. We take the following notation from [Gushchin, 2020]: for any function $\psi : [0,c) \to \mathbb{R}$ we write $\psi \overset{\text{loc}}{\ll} G$ if there exists a function $f \in L^1_{\text{loc}}(dF)$ s.t.

$$\psi(t) = Z(0) + \int_{(0,t]} f(s)dF_s, \quad \forall t < c \tag{1.4.15}$$

and denote $f(s) =: \frac{d\psi}{dF}(s)$ for $s \in (0,c)$.
This notion leads us to another more general set of requirements:

Definition 1.15. *We say a function $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$ satisfies the condition (C'), if*

(i) $\varphi(\cdot, z) \overset{loc}{\ll} F$ *for all* $z \in X$,

(ii) $\sup_{s\in[0,t]} |\varphi(s,\cdot)| \in L^1_{loc}(\mu)$, *for all* $t < c$,

(iii) $\sup_{s\in[0,t]} |\varphi(s,z_0)| < \infty$ *for all* $t < c$.

(iv) $\varphi \in L^1_{loc}(\mu)$.

A slight modification of the above proof yields then:

Corollary 1.16. *Under the assumption that* φ *satisfies the condition (C') the process* $(v(t))_{t\in\mathbb{R}^+}$, *where* $v(\omega,t) = \varphi(t, x(\omega,t)$ *can be written as*

$$v(t) = M_t + A_t.$$

M_t *is a local* $(\mathcal{F}_t)$ *martingale and* A_t *is an* $(\mathcal{F}_t)$*-previsible process. Both processes are given respectively by:*

$$M_t = \varphi(t_0, z_0) + \int_{(0,t]} (\varphi(u,z) - \varphi(u,z_0))dq(u,z),$$
$$A_t = \int_{(0,t]} \frac{d\varphi}{dF}(u, x(u-)) + \frac{\mathbb{1}_{T\geq u}}{1-F_{u-}}(m(u) - \varphi(u,z_0))dF_u,$$

where $m(u) = \mathbb{E}[\varphi(T,Z)|T = u]$.

Proof. The proof is similar to the proof of theorem 1.14, with the following adjustments: In (1.4.9) we instead write for a fixed ω

$$\varphi(t_k, x(t_{k-1},\omega)) - \varphi(t_{k-1}, x(t_{k-1},\omega)) = \int_{(t_{k-1},t_k]} \frac{d\varphi}{dF}(u, x(t_{k-1},\omega))dF_u \tag{1.4.16}$$

where $\frac{d\varphi}{dF}(\cdot, z)$ is the function that exists since $\varphi(\cdot,z) \overset{\text{loc}}{\ll} F$ for all $z \in X$. Thus we get for fixed $u \in \mathbb{R}^+, u < c$ and $\omega \in \Omega$

$$\sum_{k=1}^{2^N} \varphi(t_k, x(t_{k-1},\omega)) - \varphi(t_{k-1}, x(t_{k-1},\omega)) \to \int_{(0,t]} \frac{d\varphi}{dF}(u, x(u-,\omega))dF_u$$

for $N \to \infty$.
The property (iii) of the condition set (C') ensures that the dominated convergence theorem can still be applied in the following sense:
In the situation of (1.4.2) we are again bounded by

$$|\sum_{k=1}^{N}[\varphi(t_k,z) - \varphi(t_k,z_0)]\mathbb{1}_{(t_{k-1},t_k]}(u)| \leq \sup_{s\in(0,t]} |\varphi(s,z)| + \sup_{s\in(0,t]} |\varphi(s,z_0)|.$$

for all $t < c$, $z \in X$ and any $N \in \mathbb{N}$ (the level of the dyadic partition $\mathcal{D}_t^N$ of the interval $[0,t]$). Now the first supremum is assumed to be in $L^1_{\text{loc}}(\mu)$ by condition (ii) of (C'). The other supremum in turn is in (iii) of (C') assumed to be bounded and thus is of

$L^1_{\text{loc}}(\mu)$ on $[0,c)$. With proposition 4, (ii) in [Davis, 1976] we get $L^1_{\text{loc}}(\mu) = L^1_{\text{loc}}(p)$ and thus use the dominated convergence theorem to yield

$$\int_{\mathbb{R}^+\times X} \sum_{k=1}^{N} [\varphi(t_k,z) - \varphi(t_k,z_0)] \mathbb{1}_{(t_{k-1},t_k]}(u) dp(u,z) = \int_{(0,t]\times X} [\varphi(u,z) - \varphi(u,z_0)] dp(u,z).$$

□

Remark 1.17. *The assumptions made under (C') may seem constructed, but they obey quite realistic circumstances. Heuristically we want to determine a function φ that takes the current value of our single-jump process $x(t)$ and 'bends' the graph of its path s.t. the mean deviation from the initial value is zero. After the process has jumped, the function has no task anymore: the graph of the path should stay constant. But before the jump it bends actively to prepare for the expected jump at any given time.*
Now think of a jump time, that is discrete. Then the distribution function is constant inbetween two atomic values - say $a, b \in \mathbb{R}^+$. The probability to jump in the interval (a,b) is zero and thus (a,b) is a dF-nullset. On this interval the function φ still prepares for the possible incoming jump, but there is dF-a.s. no jump happening inbetween a and b. Thus the function can confidently stay constant inside of the interval, i.e. the signed measure induced by $\varphi(\cdot,z_0)$ gives the interval (a,b) also a value of zero. Hence the function $\varphi(\cdot,z)$ is locally absolutely continuous to dF.
One can see that the other requirements are also well motivated:

(ii) The $\sup_{s\in(0,t]}$ should be locally integrable which is implied, would it be finite (which it doesn't have to be). This ensures, that the integral of φ does not explode on any important (i.e. $d\mu$-massive) set.
Note: due to this property we can conclude, that $\varphi \in L^1_{loc}(\mu)$, since for any $t \in \mathbb{R}^+, t < c$:

$$\int_{(0,t]\times X} |\varphi(s,z)| d\mu(s,z) \le \int_{(0,t]\times X} \sup_{u\in(0,t]} |\varphi(u,z)| d\mu(s,z) < \infty.$$

(iii) Especially for the location z_0 the supremum must be bounded for any $t < c$. This makes sense, if we remember that the function is supposed to compensate the jump of the process. A value of ∞ would be an overreaction.
For $t \nearrow c$ the function can and will in some cases diverge, but for any fixed $t < c$ the value of the supremum should still be finite.

Example 1.18. *This is a sequel to the investigations in examples 1.2 and 1.12. Note that all of these examples satisfy condition (C). Thus we made an effort to adjust the distribution in the setting of (d) to not be continuous anymore.*

(a) The situation in example 1.2 and the choice of φ in example 1.12 can be directly applied to theorem 1.14. Note that

$$\frac{\partial\varphi}{\partial t}(t,y) = (\alpha(1-y) + \beta y)\varphi(t,y)$$

and thus

$$\frac{\partial\varphi}{\partial t}(u,x(u-)) = \alpha\exp(\alpha u)\mathbb{1}_{T\ge u} + \beta\exp(\beta u)\mathbb{1}_{T<u}.$$

The function $m(u)$ is in this case:

$$m(u) = \mathbb{E}[\varphi(T, Z)|T = u] = \exp(\beta u)$$

The Doob-Meyer decomposition is given by

$$\begin{aligned}
M_t &= 1 + \int_{(0,t]\times X} [\exp(\beta u) - \exp(\alpha u)]\, dq(u, z) \\
&= 1 + \int_{(0,t]\times X} [\exp(\beta u) - \exp(\alpha u)]\, dp(u, z) \\
&\quad - \int_{(0,t]\times X} [\exp(\beta u) - \exp(\alpha u)]\, d\tilde{p}(u, z) \\
&= 1 + \left[e^{\beta T} - e^{\alpha T}\right] \mathbb{1}_{T\le t} \\
&\quad - \left[\frac{\lambda}{\beta} e^{\beta(t\wedge T)} - \frac{\lambda}{\beta} - \frac{\lambda}{\alpha} e^{\alpha(t\wedge T)} + \frac{\lambda}{\alpha}\right] \\
&= 1 + \lambda \left(\frac{1}{\beta} - \frac{1}{\alpha)}\right) \\
&\quad - \lambda \left[\frac{1}{\beta} e^{\beta t} - \frac{1}{\alpha} e^{\alpha t}\right] \mathbb{1}_{t<T} \\
&\quad + \left[\left(1 - \frac{\lambda}{\beta}\right) e^{\beta T} - \left(1 - \frac{\lambda}{\alpha}\right) e^{\alpha T}\right] \mathbb{1}_{T\le t} \\
A_t &= \int_{(0,t\wedge T]} \alpha \exp(\alpha u) + \frac{f(u)}{1 - F_u} [\exp(\beta u) - \exp(\alpha u)]\, du \\
&\quad + \int_{(T,t\vee T]} \beta \exp(\beta u) du. \\
&= e^{\alpha(t\wedge T)} - 1 + e^{\beta(t\vee T)} - e^{\beta T} \\
&\quad + \left[\frac{\lambda}{\beta} e^{\beta(t\wedge T)} - \frac{\lambda}{\beta} - \frac{\lambda}{\alpha} e^{\alpha(t\wedge T)} + \frac{\lambda}{\alpha}\right] \\
&= -1 - \lambda \left(\frac{1}{\beta} - \frac{1}{\alpha}\right) + \varphi(t, x(t)) \\
&\quad + \lambda \left[\frac{1}{\beta} e^{\beta t} - \frac{1}{\alpha} e^{\alpha t}\right] \mathbb{1}_{t<T} \\
&\quad - \left[\left(1 - \frac{\lambda}{\beta}\right) e^{\beta T} - \left(1 - \frac{\lambda}{\alpha}\right) e^{\alpha T}\right] \mathbb{1}_{T\le t}.
\end{aligned}$$

(b) *Same procedure for the (b)-example. First*

$$\frac{\partial \varphi}{\partial t}(t, y) = y \cos(ty)$$

thus

$$\frac{\partial \varphi}{\partial t}(u, x(u-)) = Z \cos(uZ) \mathbb{1}_{1<u}$$

and $m(1) = \mathbb{E}[\sin(TZ)|T = 1] = \mathbb{E}[\sin Z|T = 1]$ *and the Doob-Meyer decomposi-*

tion reads as:

$$\begin{aligned} M_t &= \int_{(0,t]\times X} \sin(uz)dq(u,z) \\ &= \sin(Z)\mathbb{1}_{1\le t} - \mathbb{E}[\sin(Z)]\mathbb{1}_{1\le t} \\ A_t &= \int_{(1,t]\times X} Z\cos(uZ)du + \mathbb{E}[\sin Z]\mathbb{1}_{1\le t} \\ &= [\sin(tZ) - \sin(Z)]\,\mathbb{1}_{1\le t} + \mathbb{E}[\sin Z]\mathbb{1}_{1\le t}. \end{aligned}$$

(c) *In the situation of example (c) from before we can determine the Doob-Meyer-decomposition again pretty easily:*

$$\frac{\partial\varphi}{\partial t} = 0$$

and $m(u) = \mathbb{E}[x(T)|T = u] = \mathbb{E}[Z]$. *This yields*

$$\begin{aligned} M_t &= \int_{(0,t]\times X} zdq(u,z) \\ &= Z\mathbb{1}_{T\le t}A_t \qquad\qquad = \int_{(0,t\wedge T]} \mathbb{E}[Z]dF_u \\ &= 0 \end{aligned}$$

Since this process is pretty uninteresting, we now assume that $\mathbb{E}[Z] = \mu > 0$. *Then we have*

$$\begin{aligned} M_t &= Z\mathbb{1}_{T\le t} - \mu\lambda t \\ A_t &= \mu\lambda t \end{aligned}$$

A sample path can be found in figure 1.2 for $\mu = 6$.

(d) *Take* $T \sim Exp(\lambda)$ *and set* $Z := 6\sin(\pi T)$. *Choose* $\varphi(t,y) := y, \forall(t,y) \in \mathbb{R}^+ \times X$. *Then* $\frac{\partial\varphi}{\partial t}(t,y) \equiv 0$ *and the Doob-Meyer decomposition is given by*

$$\begin{aligned} M_t &= \varphi(0,x(0)) + \int_{(0,t]\times X} \varphi(u,z) - \varphi(u,0)dq(u,z) \\ &= \int_{(0,t]\times X} zdp(u,z) - \int_{(0,t]\times X} zd\tilde{p}(u,z) \\ &= Z\mathbb{1}_{T\le t} - \int_{(0,t\wedge T]} 6\sin(\pi u)\frac{f(u)}{1-F_{u-}}\int_X d\mu_u(z)du \\ &= Z\mathbb{1}_{T\le t} - \int_{(0,t\wedge T]} 6\sin(\pi u)\lambda du \\ &= Z\mathbb{1}_{T\le t} - 6\lambda\,(1-\cos(\pi(t\wedge T))) \\ A_t &= \int_{(0,t]} \frac{\partial\varphi}{\partial t}(u,x(u-))du + \int_{(0,t]\times X} zd\tilde{p}(u,z) \\ &= 6\lambda\,(1-\cos(\pi(t\wedge T))) \end{aligned}$$

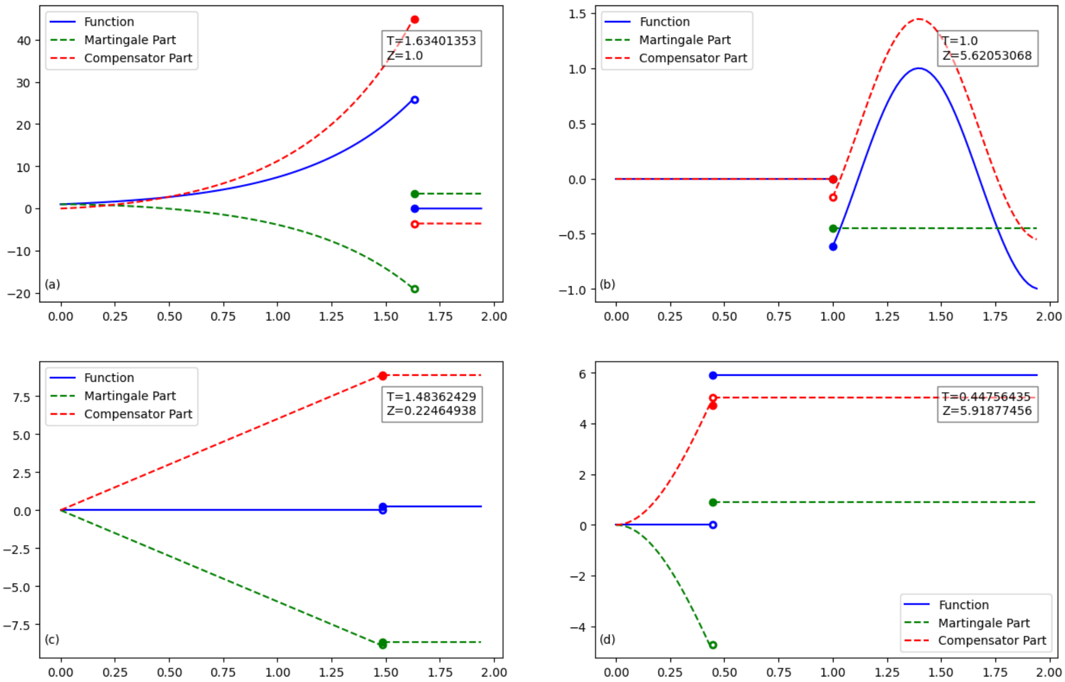

Figure 1.2: Sample paths to the respective cases in example 1.18.

From corollary 1.16 we obtain an equation to eliminate the compensator by the right choice of φ:

$$\frac{d\varphi}{dF}(u, x(u-)) + \frac{\mathbb{1}_{T\geq u}}{1 - F_{u-}}(m(u) - \varphi(u, z_0)) = 0. \tag{1.4.17}$$

This condition on φ could be generalised to only hold true on $\mathbb{R}^+ \setminus N^T$, where $N^T := \{[a, b) \in \mathbb{R}^+ : F([a, b)) = 0\}$. We keep this technicality in mind, but revert to the case, where we want to solve (1.4.17) for all times $t < c$.

1.5 Martingale transformations

Before we state our main result in a few lines, we want to take the special case that F defines a measure absolutely continuous to the Lebesgue-measure λ on $\mathbb{R}^+$ (we write: $F \ll \lambda$) as a guide to a possible solution of (1.4.17), i.e. in the upcoming segment we assume that F yields a density with respect to the Lebesgue measure λ. In addition let φ satisfy the condition (C).

Per assumption $\varphi(\cdot, z) \in \mathcal{C}^1$ for all $z \in X$ and $F \ll \lambda$. Then we have to solve (1.4.14):

$$\frac{\partial\varphi}{\partial t}(u, x(u-)) + \frac{\mathbb{1}_{T\geq u}}{1 - F_u}\,(m(u) - \varphi(u, z_0))\, f(u) = 0.$$

Due to our experience with the process we write

$$\begin{aligned}\varphi(t,x(t)) &= \varphi(t,x(t))\mathbb{1}_{T\le t} + \varphi(t,x(t))\mathbb{1}_{T>t}\\ &= \varphi(t,Z)\mathbb{1}_{T\le t} + \varphi(t,z_0)\left(1-\mathbb{1}_{T\le t}\right)\\ &= \left(\varphi(t,Z)-\varphi(t,z_0)\right)\mathbb{1}_{T\le t} + \varphi(t,z_0)\end{aligned}$$

leading to our first justified assumption:

$$\varphi(t,y) = a_1(t,y)(1-\delta_{z_0}(y)) + a_0(t) \tag{1.5.1}$$

where we've defined

$$a_1(t,y) := \varphi(t,y) - \varphi(t,z_0), \quad a_0(t) := \varphi(t,z_0).$$

The structure of φ yields

- $$\frac{\partial\varphi}{\partial t}(u,y) = \frac{\partial a_1}{\partial t}(u,y)(1-\delta_{z_0}(y)) + \frac{\partial a_0}{\partial t}(u) \tag{1.5.2}$$
- $$m(u) = \mathbb{E}[a_1(T,Z)|T=u] + a_0(u) \tag{1.5.3}$$
- $$\varphi(u,z_0) = a_0(u). \tag{1.5.4}$$

We insert these new findings into (1.4.14)

$$\begin{aligned}0 &= \frac{\partial\varphi}{\partial t}(u,y) + \frac{\delta_{z_0}(y)}{1-F_u}\left(m(u)-\varphi(u,z_0)\right)f(u)\\ &= \Big(\frac{\partial a_1}{\partial t}(u,y)(1-\delta_{z_0}(y)) + \frac{\partial a_0}{\partial t}(u)\Big)\\ &\quad + \frac{\delta_{z_0}(y)}{1-F_u}\left(\mathbb{E}[a_1(T,Z)|T=u] + a_0(u) - a_0(u)\right)f(u)\\ &= \left(\frac{\partial a_1}{\partial t}(u,y) - \frac{\mathbb{E}[a_1(T,Z)|T=u]}{1-F_u}f(u)\right)(1-\delta_{z_0}(y))\\ &\quad + \left(\frac{\partial a_0}{\partial t}(u) + \frac{\mathbb{E}[a_1(T,Z)|T=u]}{1-F_u}f(u)\right)\end{aligned}$$

which has to hold true for any $y \in X$, so especially for $y = z_0$ (first term vanishes as $\delta_{z_0}(z_0) = 1$) and $y \neq z_0$ (second term does not depend on the change of y so it has to be universally 0). This yields two new equations to solve:

- $$\frac{\partial a_1}{\partial t}(u,y) - \frac{\mathbb{E}[a_1(T,Z)|T=u]}{1-F_u}f(u) = 0 \tag{1.5.5}$$
- $$\frac{\partial a_0}{\partial t}(u) + \frac{\mathbb{E}[a_1(T,Z)|T=u]}{1-F_u}f(u) = 0 \tag{1.5.6}$$

Apparently (1.5.5) tells us, that the partial derivative has overcome the dependency on y. This lets us conclude for a_1:

$$a_1(u,y) = b_1(u) + b_2(y), \tag{1.5.7}$$

where b_1 and b_2 are placeholder functions here to illustrate the structural conclusions. One can check this by integrating (1.5.5). For fixed $y \in X$:

$$\begin{aligned}
\frac{\partial a_1}{\partial t}(u,y) &= \frac{f(u)}{1-F_u}\mathbb{E}[a_1(T,Z)|T=u] \\
\Leftrightarrow \int_{[0,t]} \frac{\partial a_1}{\partial t}(u,y)du &= \int_{[0,t]} \frac{f(u)}{1-F_u}\mathbb{E}[a_1(T,Z)|T=u]du \\
\Leftrightarrow a_1(t,y) &= \int_{[0,t]} \frac{f(u)}{1-F_u}\mathbb{E}[a_1(T,Z)|T=u]du + a_1(0,y).
\end{aligned}$$

So $b_1(u) := \int_{[0,t]} \frac{f(u)}{1-F_u}\mathbb{E}[a_1(T,Z)|T=u]du$ and $b_2(y) := a_1(0,y)$.

Inserting this new result (1.5.7) into (1.5.5):

$$\begin{aligned}
0 &= \frac{\partial a_1}{\partial t}(u,y) - \frac{\mathbb{E}[a_1(T,Z)|T=u]}{1-F_u}f(u) \\
&= \left(\frac{\partial b_1}{\partial t}(u) + \frac{\partial b_2}{\partial t}(y)\right) - \frac{\mathbb{E}[b_1(T)|T=u] + \mathbb{E}[b_2(Z)|T=u]}{1-F_u}f(u) \\
&= \frac{\partial b_1}{\partial t}(u) - \frac{f(u)}{1-F_u}b_1(u) + \frac{S(u)f(u)}{1-F_u}
\end{aligned}$$

where we've set $S(u) := \mathbb{E}[b_2(Z)|T=u]$. Now we are left to solve the inhomogenous ODE

$$\frac{\partial b_1}{\partial t}(u) = \frac{f(u)}{1-F_u}b_1(u) + \frac{S(u)f(u)}{1-F_u} \tag{1.5.8}$$

by variation of constants. First we solve the homogenous equation

$$\frac{\partial b_1}{\partial t}(u) = \frac{f(u)}{1-F_u}b_1(u).$$

The solutions are given by the family

$$\{k\frac{1}{1-F_u} : k \in \mathbb{R}\}.$$

The particular solution is given by

$$\begin{aligned}
B_1(u) &= \frac{1}{1-F_u}\int_{[0,u]} \frac{S(v)f(v)}{1-F_v}\left(\frac{1}{1-F_v}\right)^{-1} dv \\
&= \frac{1}{1-F_u}\int_{[0,u]} S(v)f(v)dv \\
&= \frac{1}{1-F_u}\int_{[0,u]} S(v)dF_v.
\end{aligned}$$

Eventually we get a general solution to (1.5.8) with

$$b_1(u) := \frac{1}{1-F_u}\left(\int_{[0,u]} S(v)dF_v + k\right). \tag{1.5.9}$$

We can insert (1.5.9) into (1.5.7):

$$a_1(u,y) = \frac{1}{1-F_u}\left(\int_{[0,u]} S(v)dF_v + k\right) + b_2(y) \tag{1.5.10}$$

where $k \in \mathbb{R}$ and get with (1.5.6)

$$a_0(u) = -\frac{1}{1-F_u}\left(\int_{[0,u]} S(v)dF_v + k\right) + l \tag{1.5.11}$$

where $k, l \in \mathbb{R}$. Set $b(y) := b_2(y) + l$ and $r := k - l$. The final form of φ, obtained by inserting (1.5.10) and (1.5.11) into (1.5.1) then sums up to

$$\begin{aligned}
\varphi(u,y) &= \left(\frac{1}{1-F_u}\left(\int_{[0,u]} S(v)dF_v + k\right) + b_2(y)\right)(1-\delta_{z_0}(y)) \\
&\quad - \frac{1}{1-F_u}\left(\int_{[0,u]} S(v)dF_v + k\right) + l \\
&= l + b_2(y)(1-\delta_{z_0}(y)) - \frac{1}{1-F_u}\left(\int_{[0,u]} S(v)dF_v + k\right)\delta_{z_0}(y) \\
&= (b_2(y)+l)(1-\delta_{z_0}(y)) \\
&\quad - \frac{1}{1-F_u}\left(\int_{[0,u]} \mathbb{E}[b_2(Z)|T=v]dF_v + k - l(1-F_u)\right)\delta_{z_0}(y) \\
&= b(y)(1-\delta_{z_0}(y)) \\
&\quad - \frac{1}{1-F_u}\left(\int_{[0,u]} \mathbb{E}[b_2(Z)|T=v]dF_v + k - l\left(1 - \int_{(0,u]\times X} \mu(dv,dz)\right)\right)\delta_{z_0}(y) \\
&= b(y)(1-\delta_{z_0}(y)) - \delta_{z_0}(y)\frac{1}{1-F_u}\left(\int_{[0,u]\times X} b_2(z) + l\mu(dv,dz) + k - l\right) \\
&= b(y)(1-\delta_{z_0}(y)) - \delta_{z_0}(y)\frac{1}{1-F_u}\left(\int_{[0,u]\times X} b(z)\mu(dv,dz) + r\right).
\end{aligned}$$

The final solution to our initial equation (1.4.14) thus reads as:

$$\varphi(t,y) = b(y)(1-\delta_{z_0}(y)) - \delta_{z_0}(y)\frac{1}{1-F_t}\left(\int_{[0,t]\times X} b(z)\mu(du,dz) + r\right). \tag{1.5.12}$$

For softer requirements on F and φ the martingale-property still holds as the proof of our main result shows:

Theorem 1.19. *For φ defined as in* (1.5.12) *the process $v = (v(t))_{t\in\mathbb{R}^+}$ with $v(t) = \varphi(t, x(t))$ is a local $(\mathcal{F}_t)$-martingale, where $b : X \to \mathbb{R}$, $b \in L^1_{loc}(\mu)$ and $r \in \mathbb{R}$ arbitrary.*

Proof. Set

$$\sigma_k := \begin{cases} k, & c=\infty \\ \infty, & c<\infty, F_{c-}<1 \\ k\mathbb{1}_{T\le t_k} + t_k\mathbb{1}_{T>t_k}, & c<\infty, F_{c-}=0, \end{cases} \tag{1.5.13}$$

where $(t_k)_{n\in\mathbb{N}}\subset\mathbb{R}^+$ increasing with $t_k \nearrow c$. This localization sequence is taken from [Davis, 1976], proof of proposition 4. We note that $\sigma_k\nearrow\infty$ a.s. For every $k\in\mathbb{N}$ the process $v(t\wedge\sigma_k)_{t\in\mathbb{R}^+}$ is naturally adapted and in $L^1(\mu)$ by choice of $b\in L^1_{\text{loc}}(\mu)$, and for any $s\le t\in\mathbb{R}^+$ we validate the martingale property:

$$\begin{aligned}
&\mathbb{E}[\varphi(t\wedge\sigma_k, x(t\wedge\sigma_k))|\mathcal{F}_s] \\
=&\mathbb{E}\left[b(x(t\wedge\sigma_k))\mathbb{1}_{x(t\wedge\sigma_k)\neq z_0} - \mathbb{1}_{x(t\wedge\sigma_k)=z_0}\frac{1}{1-F_{t\wedge\sigma_k}}\left(\int_{(0,t\wedge\sigma_k]\times X} b(z)d\mu(s,z)+r\right)|\mathcal{F}_s\right] \\
\overset{(a)}{=}&\mathbb{E}[b(Z)\mathbb{1}_{T\le t\wedge\sigma_k}|\mathcal{F}_s] - \frac{1}{1-F_{t\wedge\sigma_k}}\left(\int_{(0,t\wedge\sigma_k]\times X} b(z)d\mu(s,z)+r\right)\mathbb{E}[\mathbb{1}_{T>t\wedge\sigma_k}|\mathcal{F}_s] \\
\overset{(b)}{=}&b(Z)\mathbb{1}_{T\le s\wedge\sigma_k} + \mathbb{1}_{T>s\wedge\sigma_k}\frac{1}{1-F_{s\wedge\sigma_k}}\int_{(s,t]\times X} b(z)\mu(du,dz) \\
&-\frac{1}{1-F_{t\wedge\sigma_k}}\left(\int_{(0,t\wedge\sigma_k]\times X} b(z)d\mu(s,z)+r\right)\left[\mathbb{1}_{\sigma_k<T\le s} + \mathbb{1}_{T>s}\frac{1}{1-F_s}\left(1-F_{s\vee(t\wedge\sigma_k)}\right)\right] \\
\overset{(c)}{=}&b(x(s\wedge\sigma_k))\mathbb{1}_{T\le s\wedge\sigma_k} - \mathbb{1}_{T>s\wedge\sigma_k}\frac{1}{1-F_{s\wedge\sigma_k}}\left(\int_{(0,s\wedge\sigma_k]\times X} b(z)\mu(du,dz)+r\right) \\
=&\varphi(s\wedge\sigma_k, x(s\wedge\sigma_k))
\end{aligned}$$

where we used in particular:

(a) We use the equivalence of the sets $\{x(t\wedge\sigma_k)\neq z_0\}$ and $\{T\le t\wedge\sigma_k\}$ as well as $\{x(t)=z_0\}$ and $\{T>t\}$ and use the knowledge on these sets to set the value of $b(x(t))=b(Z)$ on $\{T\le t\}$.

(b) We seperate

$$\begin{aligned}
&\mathbb{E}[b(Z)\mathbb{1}_{T\le t\wedge\sigma_k}|\mathcal{F}_s] \\
=&\mathbb{E}[b(Z)\mathbb{1}_{T\le t\wedge\sigma_k}\mathbb{1}_{T\le s}|\mathcal{F}_s] + \mathbb{E}[b(Z)\mathbb{1}_{T\le t\wedge\sigma_k}\mathbb{1}_{T>s}|\mathcal{F}_s]
\end{aligned}$$

Now we note that

$$\{T\le t\wedge\sigma_k\}\cap\{T\le s\} = \begin{cases}\{T\le\sigma_k\}, & \sigma_k\le s \\ \{T\le s\}, & \sigma_k>s\end{cases} \tag{1.5.14}$$

and thus

$$\mathbb{E}[b(Z)\mathbb{1}_{T\le t\wedge\sigma_k}\mathbb{1}_{T\le s}|\mathcal{F}_s] = \mathbb{E}[b(Z)\mathbb{1}_{T\le s\wedge\sigma_k}|\mathcal{F}_s] = b(Z)\mathbb{1}_{T\le s\wedge\sigma_k}.$$

The second term makes use of $\{T>s\}$ being an atom:

$$\begin{aligned}
\mathbb{E}[b(Z)\mathbb{1}_{T\le t\wedge\sigma_k}\mathbb{1}_{T>s}|\mathcal{F}_s] &= \mathbb{1}_{T>s}\frac{1}{1-F_s}\int_{(s,\infty)\times X} b(z)\mathbb{1}_{(0,t\wedge\sigma_k]}(u)d\mu(u,z) \\
&= \mathbb{1}_{T>s}\frac{1}{1-F_s}\int_{(s\wedge\sigma_k,t\wedge\sigma_k)\times X} b(z)d\mu(u,z)
\end{aligned}$$

where we've used that $(s, t \wedge \sigma_k] = (s \wedge \sigma_k, t \wedge \sigma_k$.

Further we evaluate the last term as

$$\mathbb{E}[\mathbb{1}_{T>t\wedge\sigma_k}|\mathcal{F}_s] = \mathbb{E}[\mathbb{1}_{T>t\wedge\sigma_k}\mathbb{1}_{T\leq s}|\mathcal{F}_s] + \mathbb{E}[\mathbb{1}_{T>t\wedge\sigma_k}\mathbb{1}_{T>s}|\mathcal{F}_s].$$

Now $\{T > t \wedge \sigma_k\} \cap \{T \leq s\} = \{\sigma_k < T \leq s\}$ - which is empty for $\sigma_k \geq s$, $\mathcal{F}_s$-measurable either way. For the second term:

$$\{T > t \wedge \sigma_k\} \cap \{T > s\} = \begin{cases} \{T > t\}, & s \leq t < \sigma_k \\ \{T > \sigma_k\}, & s \leq \sigma_k \leq t \\ \{T > s\}, & \sigma_k < s \leq t. \end{cases} \tag{1.5.15}$$

Thus

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{T>t\wedge\sigma_k}|\mathcal{F}_s] &= \mathbb{1}_{\sigma_k<T\leq s} + \mathbb{1}_{T>s}\frac{1}{1-F_s}\int_{(s,\infty)\times X}\mathbb{1}_{(t\wedge\sigma_k,\infty)}(u)d\mu(u,z) \\ &= \mathbb{1}_{\sigma_k<T\leq s} + \mathbb{1}_{T>s}\frac{1}{1-F_s}\left(1 - F_{s\vee(t\wedge\sigma_k)}\right)\end{aligned}$$

(c) On the set $\{T \leq s \wedge \sigma_k\}$ we can substitute Z with $x(s \wedge \sigma_k)$. For the remaining terms we consider 3 different cases:

<u>$s \leq t < \sigma_k$</u> In this case we have $s \wedge \sigma_k = s$ and $t \wedge \sigma_k = t$. Thus the remaining terms are

$$\begin{aligned}&\mathbb{1}_{T>s}\frac{1}{1-F_s}\int_{(s\wedge\sigma_k,t\wedge\sigma_k]\times X} b(z)\mu(du,dz) \\ =&\mathbb{1}_{T>s}\frac{1}{1-F_s}\int_{(s,t]\times X} b(z)\mu(du,dz) \\ =:&I_1\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{1-F_{t\wedge\sigma_k}}\left(\int_{(0,t\wedge\sigma_k]\times X} b(z)d\mu(s,z) + r\right) \\ &\quad\times\left[\mathbb{1}_{\sigma_k<T\leq s} + \mathbb{1}_{T>s}\frac{1}{1-F_s}\left(1 - F_{s\vee(t\wedge\sigma_k)}\right)\right] \\ =&-\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(s,z) + r\right)\left[\mathbb{1}_{T>s}\frac{1}{1-F_s}(1-F_t)\right] \\ =&-\mathbb{1}_{T>s}\frac{1}{-F_s}\left(\int_{(0,t]\times X} b(z)d\mu(s,z) + r\right) \\ =:&I_2.\end{aligned}$$

The sum of these two terms is

$$\begin{aligned} I_1 + I_2 &= \mathbb{1}_{T>s} \frac{1}{1-F_s} \left(\int_{(s,t]\times X} b(z)\mu(du,dz) - \int_{(0,t]\times X} b(z)d\mu(s,z) + r \right) \\ &= \mathbb{1}_{T>s} \frac{1}{1-F_s} \left(- \int_{(0,s]\times X} b(z)d\mu(s,z) + r \right) \\ &= \mathbb{1}_{T>s\wedge\sigma_k} \frac{1}{1-F_{s\wedge\sigma_k}} \left(- \int_{(0,s\wedge\sigma_k]\times X} b(z)d\mu(s,z) + r \right). \end{aligned}$$

$s \le \sigma_k \le t$ Now $s \wedge \sigma_k = s$ and $t \wedge \sigma_k = \sigma_k$ and thus

$$I_1 := \mathbb{1}_{T>s} \frac{1}{1-F_s} \int_{(s,\sigma_k]\times X} b(z)\mu(du,dz)$$

and

$$\begin{aligned} I_2 &:= -\frac{1}{1-F_{t\wedge\sigma_k}} \left(\int_{(0,t\wedge\sigma_k]\times X} b(z)d\mu(s,z) + r \right) \\ &\quad \times \left[\mathbb{1}_{\sigma_k<T\le s} + \mathbb{1}_{T>s} \frac{1}{1-F_s} \left(1 - F_{s\vee(t\wedge\sigma_k)}\right) \right] \\ &= -\frac{1}{1-F_{\sigma_k}} \left(\int_{(0,\sigma_k]\times X} b(z)d\mu(s,z) + r \right) \left[\mathbb{1}_{T>s} \frac{1}{1-F_s} \left(1 - F_{\sigma_k}\right) \right] \\ &= -\mathbb{1}_{T>s} \frac{1}{1-F_s} \left(\int_{(0,\sigma_k]\times X} b(z)d\mu(s,z) + r \right). \end{aligned}$$

Hence the sum is

$$\begin{aligned} I_1 + I_2 &= \mathbb{1}_{T>s} \frac{1}{1-F_s} \left(\int_{(s,\sigma_k]\times X} b(z)\mu(du,dz) - \int_{(0,\sigma_k]\times X} b(z)d\mu(s,z) + r \right) \\ &= \mathbb{1}_{T>s} \frac{1}{1-F_s} \left(- \int_{(0,s]\times X} b(z)d\mu(s,z) + r \right) \\ &= \mathbb{1}_{T>s\wedge\sigma_k} \frac{1}{1-F_s \wedge \sigma_k} \left(- \int_{(0,s\wedge\sigma_k]\times X} b(z)d\mu(s,z) + r \right). \end{aligned}$$

$\sigma_k < s \le t$ Both times are past the localizing stopping time, we note $s\wedge\sigma_k = \sigma_k = t\wedge\sigma_k$ and

$$\begin{aligned} I_1 &:= \mathbb{1}_{T>s} \frac{1}{1-F_s} \int_{(\sigma_k,\sigma_k]\times X} b(z)\mu(du,dz) \\ &= 0 \end{aligned}$$

as well as

$$
\begin{aligned}
I_2 :=& -\frac{1}{1-F_{t\wedge\sigma_k}}\left(\int_{(0,t\wedge\sigma_k]\times X} b(z)d\mu(s,z)+r\right)\\
&\times\left[\mathbb{1}_{\sigma_k<T\le s}+\mathbb{1}_{T>s}\frac{1}{1-F_s}\left(1-F_{s\vee(t\wedge\sigma_k)}\right)\right]\\
=& -\frac{1}{1-F_{\sigma_k}}\left(\int_{(0,\sigma_k]\times X} b(z)d\mu(s,z)+r\right)\\
&\times\left[\mathbb{1}_{\sigma_k<T\le s}+\mathbb{1}_{T>s}\frac{1}{1-F_s}\left(1-F_s\right)\right]\\
=& -\frac{1}{1-F_{\sigma_k}}\left(\int_{(0,\sigma_k]\times X} b(z)d\mu(s,z)+r\right)\left[\mathbb{1}_{\sigma_k<T}\right].
\end{aligned}
$$

Again the sum reduces to

$$
\begin{aligned}
I_1+I_2 &= I_2\\
&= -\mathbb{1}_{\sigma_k<T}\frac{1}{1-F_{\sigma_k}}\left(\int_{(0,\sigma_k]\times X} b(z)d\mu(s,z)+r\right)\\
&= -\mathbb{1}_{s\wedge\sigma_k<T}\frac{1}{1-F_{s\wedge\sigma_k}}\left(\int_{(0,s\wedge\sigma_k]\times X} b(z)d\mu(s,z)+r\right)
\end{aligned}
$$

□

Remark 1.20. *We can also validate the above result by comparing it with [Davis, 1976] prop. 5. We have:*

$$
\begin{aligned}
\varphi(t,x(t)) &= b(x(t))\mathbb{1}_{x(t)\neq z_0}-\mathbb{1}_{y=z_0}\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(s,z)+r\right)\\
&\overset{(a)}{=} b(Z)\mathbb{1}_{T\le t}-\mathbb{1}_{T>t}\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(s,z)+r\int_{(0,c)\times X} d\mu(s,z)\right)\\
&\overset{(b)}{=} -r+\left(b(Z)+r\right)\mathbb{1}_{T\le t}-\mathbb{1}_{T>t}\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)+rd\mu(s,z)\right)
\end{aligned}
$$

where we've used in particular:

(a) We've seen before that μ-a.s. $\{x(t)\neq z_0\}=\{T\le t\}$ and $\{x(t)=z_0\}=\{T>t\}$. We then inserted $b(x(t))\mathbb{1}_{T\le t}=b(Z)\mathbb{1}_{T\le t}$ and multiplied c_1 behind the μ-integral with a fancy $1=\int_{(0,c)\times X} d\mu(s,z)$ (note that $c=\sup\{t : F_t<1\}$ hence the exotic upper bound).

(b) *Separate*

$$\begin{aligned}
&- \mathbb{1}_{T>t}\frac{1}{1-F_t}r\int_{(0,c)\times X} d\mu(u,z)\\
=&- \mathbb{1}_{T>t}\frac{1}{1-F_t}r\left(\int_{(0,t]\times X} d\mu(u,z)+\int_{(t,c)\times X} d\mu(u,z)\right)\\
=&\mathbb{1}_{T>t}\frac{1}{1-F_t}r\left(\int_{(0,t]\times X} d\mu(u,z)+1-F_t\right)\\
=&\mathbb{1}_{T>t}\frac{r}{1-F_t}\int_{(0,t]\times X} d\mu(u,z)+\mathbb{1}_{T>t}r\\
=&\mathbb{1}_{T>t}\frac{r}{1-F_t}\int_{(0,t]\times X} d\mu(u,z)+(1-\mathbb{1}_{T\le t})r
\end{aligned}$$

In this form we can now easily determine the function h prophesized by Davis' result:

$$h(t,z)=b(z)+r.$$

This function also satisfies the required measurability by Davis' result, since we have chosen $b\in L^1_{loc}(\mathbb{P})$. So the function h does not really depend on the jump time T. Only the jump height Z seems to be important and yet this is what we felt during the above discussion all along. The characteristic indicator functions can reduce the dependence on x to the current location (z_0 or not-z_0).

Remark 1.21. *As mentioned earlier another related result is given by Gushchin in:*

Theorem 1.22. *(see [Gushchin, 2020], theorem 2) In order that a right-continuous process $M=(M_t)_{t\in\mathbb{R}^+}$ be a local martingale it is necessary and sufficient that there be a pair (G,H) satisfying conditions M and a random variable L' satisfying*

$$\mathbb{E}\left[|L'|\mathbb{1}_{T\le t}\right]<\infty, t<c, \quad \text{and} \quad \mathbb{E}[L'|T]=0,$$

such that up to $\mathbb{P}$-indistinguishability

$$M_t=(H(T)+L')\mathbb{1}_{T\ge t}+G(t)\mathbb{1}_{t<T}.$$

where the conditions M required in the theorem are the following set of conditions:

(i) $G:[0,c)\to\mathbb{R}, \quad G\overset{loc}{\ll}F,$

(ii) $H:[0,c)\to\mathbb{R}, \quad H\in L^1_{loc}(dF),$

(iii) $G(t)-G(0)=\frac{-1}{1-F(t)}\int_{(0,t]}H(s)dF_s, \quad t<c.$

and additionaly in case B:

(iv) $\lim_{t\nearrow c}G(t)=H(c).$

In our case

$$G(t)=-\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(u,z)+r\right),$$

$$H(T)+L'=b(x(T)) \quad\Leftrightarrow\quad H(T)=b(x(T))-L'$$

For G we have

$$
\begin{aligned}
G(t) - G(0) &= -\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(u,z) + r\right) + r \\
&= \frac{-1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(u,z) + rF_t - F_0\right) \\
&= \frac{-1}{1-F_t}\left(\int_{(0,t]\times X} b(z) + r d\mu(u,z)\right)
\end{aligned}
$$

and $G \overset{loc}{\ll} F$ *quite naturally. In turn for H we have that* $b \in L^1_{loc}(\mu) \subset L^1_{loc}(dF)$ *and thus* $H \in L^1_{loc}(dF)$*.*

With the above blueprint function φ we can now construct many martingales from the process $x(t)$ and can choose a particular starting point by simply choosing an arbitrary $L^1(\mathbb{P})$ function $b : X \to \mathbb{R}$ and an arbitrary constant $r \in X$.

Example 1.23.

(a) *In the situation of example 1.2 (a) we can choose* $b : \mathbb{R} \to \mathbb{R}$ *arbitrarily, since* $b(Z) \equiv b(1)$*. Take* $b \equiv 5$ *and* $r = 0$ *then*

$$5\mathbb{1}_{T\le t} - 5(\exp(\lambda t) - 1)\mathbb{1}_{T>t}$$

is a martingale of the single-jump process $x(t) = \mathbb{1}_{T\le t}$*.*

(b) *In the situation of example 1.2 (b) choose* $b = \sin(y)$ *then*

$$\varphi(t, x(t)) = \sin(Z)\mathbb{1}_{1\le t}$$

is a martingale of $Z\mathbb{1}_{1\le t}$*.*

(c) *In the situation of example 1.2 (c) choose* $b(y) = y$*. Then*

$$\varphi(t, x(t)) = x(t)$$

is a martingale of itself.

(d) *Set* $Z := 6\sin(\pi T)$ *and let* $T \sim Exp(\lambda)$ *for some* $\lambda > 0$*. Take* $b(y) := y$ *for all* $y \in X = \mathbb{R}^+$*. Then we get from theorem 1.19 that the process*

$$
\begin{aligned}
v_t &= Z\mathbb{1}_{T\le t} - \frac{\mathbb{1}_{T>t}}{\exp(-\lambda t)}\left(\int_{(0,t]\times X} z d\mu(u,z) + r\right) \\
&= Z\mathbb{1}_{T\le t} - \mathbb{1}_{T>t}\exp(\lambda t)\left[6\frac{\lambda\exp(-\lambda t)}{\lambda^2 - \pi^2}\left(\frac{\pi}{\lambda}(1 - \cos(\pi t)\exp(-\lambda t)) - \sin(\pi t)\right) + r\right]
\end{aligned}
$$

is a martingale w.r.t. to the single-jump process $x(t)$ *(see also figure 1.3).*

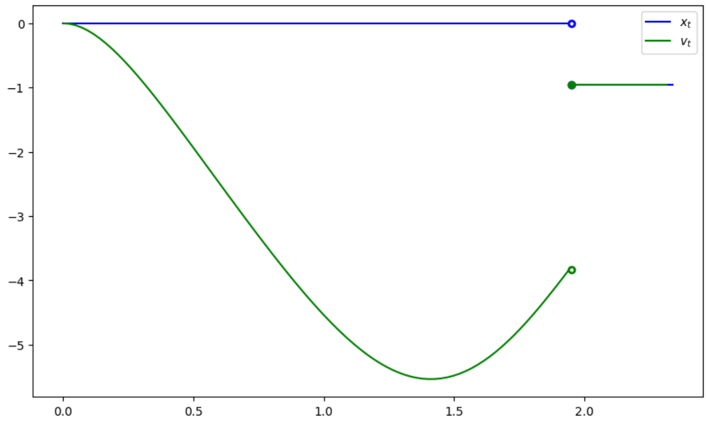

Figure 1.3: A single-jump path and the respective martingales path, constructed as in example 1.23 (d). See also figure 2.1 for a multi-jump version.

While the special first case served as a guide to the more general case, one might be interested in solving the more general equation (1.4.17) analogously to the outline from above. To do that we have to discuss some differences first:

almost all the equations and requirements in the more general case $\varphi \overset{\text{loc}}{\ll} F$ (see notation (1.4.15)) can be deduced by the same arguments as the special case, but *only μ_F-a.s.* We mean by that, that the density in the sense of (1.4.15) of φ w.r.t. F is only μ_F-a.s. a partial derivative like $\frac{\partial \varphi}{\partial t}$ is. The upcoming lemmata will shed some more light on this exotic notion.

Lemma 1.24. *We define $\frac{0}{0} = 1$. Let $\psi : \mathbb{R}^+ \to \mathbb{R}$, $\psi \overset{loc}{\ll} F$ where F is a distribution function. Then for $t < c := \inf\{s : F(s) = 1\}$ and any dyadic sequence of partitions $\mathcal{D}_t^N = \{t_1, \ldots, t_{2^N}\}$ of the interval $[0, t]$ it holds:*

$$\lim_{N\to\infty} \sum_{k=1}^{2^N} \frac{\psi(t_k) - \psi(t_{k-1})}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] = \int_{(0,t]} f dF$$

where f is the density of ψ w.r.t. F in the sense of (1.4.15).
Notation: we will denote $f(t) = \frac{d\psi}{dF}(t)$.

Proof. First we have by the property $\psi \overset{\text{loc}}{\ll} F$ that there exists a function $f \in L^1_{\text{loc}}(dF)$ such that:

$$\begin{aligned}\frac{\psi(t_k) - \psi(t_{k-1})}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] &= \psi(t_k) - \psi(t_{k-1}) \\ &= \int_{(t_{k-1}, t_k]} f dF\end{aligned}$$

Summing over k we get:

$$\sum_{k=1}^{2^N} \frac{\psi(t_k) - \psi(t_{k-1})}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] = \sum_{k=1}^{2^N} \int_{(t_{k-1},t_k]} f dF$$
$$= \int_{(0,t]} f d\nu$$

Under $N \to \infty$ we yield the assertion. □

In our previous discussion the convergence also needed to happen in the time argument of the process $x(\cdot)$ (in particular slightly in the past). In the case that x has not yet jumped, i.e. $T > t$, this is no problem at all, since $x(u) = z_0$ for all $u \in [0,t]$ After the jump the situation will be similar since $x(u) = Z$ for $u \geq T$. We take note of the following

Lemma 1.25. *Let* $\psi : \mathbb{R}^+ \times X \to \mathbb{R}$, $\psi(\cdot, z) \overset{loc}{\ll} F$ *for all* $z \in X$. *Then for* $t < c := \inf\{s : F(s) = 1\}$ *and any dyadic sequence of partitions* $\mathcal{D}_t^N = \{t_1, \ldots, t_{2^N}\}$ *of the interval* $[0,t]$ *it holds:*

$$\lim_{N\to\infty} \sum_{k=1}^{2^N} \frac{\psi(t_k, x(t_{k-1})) - \psi(t_{k-1}, x(t_{k-1}))}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] = \int_{(0,t]} f_{x(u-)}(u) dF_u$$

where f_z *is the density of* $\psi(\cdot, z)$ *w.r.t.* F *in the sense of* (1.4.15).

Proof. For $k \in \{1, \ldots, 2^N\}$ we decompose Ω in two exclusive sets $\{T > t_{k-1}\}$ and $\{T \leq t_{k-1}\}$ and get:

$$\begin{aligned}
[\psi(t_k, x(t_{k-1})) - \psi(t_k, x(t_{k-1}))] &= [\psi(t_k, x(t_{k-1})) - \psi(t_k, x(t_{k-1}))] \mathbb{1}_{T>t_{k-1}} \\
&+ [\psi(t_k, x(t_{k-1})) - \psi(t_k, x(t_{k-1}))] \mathbb{1}_{T\leq t_{k-1}} \\
&= [\psi(t_k, z_0)) - \psi(t_k, z_0))] \mathbb{1}_{T>t_{k-1}} \\
&+ [\psi(t_k, Z)) - \psi(t_k, Z))] \mathbb{1}_{T\leq t_{k-1}}
\end{aligned}$$

For these two differences we use - after summation over k - lemma 1.24 over the intervals $(0, t \wedge T]$ and $(T \wedge t, t]$ respectively and end up with:

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{k=1}^{2^N} \frac{\psi(t_k, x(t_{k-1})) - \psi(t_{k-1}, x(t_{k-1}))}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] \\
&\overset{(i)}{=} \lim_{N\to\infty} \sum_{k=1}^{K^N(\omega)} \frac{\psi(t_k, z_0) - \psi(t_{k-1}, z_0)}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] \\
&+ \lim_{N\to\infty} \sum_{k=K^N(\omega)}^{2^N} \frac{\psi(t_k, Z(\omega)) - \psi(t_{k-1}, Z)}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] \\
&\overset{(ii)}{=} \int_{(0,T(\omega)\wedge t]} f_{z_0} dF + \int_{(T(\omega)\wedge t, t]} f_{Z(\omega)} d\nu \\
&\overset{(iii)}{=} \int_{(0,t]} f_{x(u-,\omega)}(u) d\nu(u)
\end{aligned}$$

where we've used in particular:

(i) For $\omega \in \Omega$ set $K^N(\omega) := \inf\{k \in \{1, \ldots, 2^N\} : t_k \geq T(\omega) \wedge t\}$, then we decompose the sum. Note that for all $k \leq K^N(\omega)$ we can confidently state $x(t_{k-1}, \omega) = z_0$ and for all $k > K^N(\omega)$ we have $x(t_{k-1}, \omega) = Z(\omega)$.

(ii) We write

$$\begin{aligned}
&\sum_{k=1}^{K^N(\omega)} \frac{\psi(t_k, z_0) - \psi(t_{k-1}, z_0)}{F_{t_k} - F_{t_{k-1}}} \left[F_{t_k} - F_{t_{k-1}}\right] \\
=& \sum_{k=1}^{K^N(\omega)} \frac{\psi(t_k, z_0) - \psi(t_{k-1}, z_0)}{F_{t_k} - F_{t_{k-1}}} \int_{(t_{k-1}, t_k]} dF_u \\
=& \int_{\mathbb{R}^+} \sum_{k=1}^{K^N(\omega)} \frac{\psi(t_k, z_0) - \psi(t_{k-1}, z_0)}{F_{t_k} - F_{t_{k-1}}} \mathbb{1}_{(t_{k-1}, t_k]}(u) dF_u.
\end{aligned}$$

The integrand is a pointwise approximation of $\frac{d\psi}{dF}(u, z_0)\mathbb{1}_{(0,T\wedge t]}$ since $K^N(\omega) \searrow T(\omega) \wedge t$. We procede the same way with the second sum.

(iii) In the interval $(0, T \wedge t]$ $x(u-) = z_0$ and in the interval $(T \wedge t, t]$ we have $x(u-) = Z$. We combine this property in the function

$$f_{x(u-)} \begin{cases} f_{z_0}(u), \text{on } (0, T \wedge t], \\ f_Z(u), \text{on } (T \wedge t, t]. \end{cases}$$

□

Remark 1.26. *So the differential quotients converge μ_F-a.e. to the Radon-Nikodym densities. This strengthens our interpretation of the Radon-Nikodym density as a kind of derivative. Still it may not be unique, but it is at least μ_F-a.s. unique.*

Chapter 2

General case

So far we only worked with single-jump processes. But in general one is interested in jump processes with multiple jumps possibly depending on each other. While this situation seems to be more complex, the result from the single-jump case can easily be adapted to the multi-jump case.
Take a single-jump process with random jump time S_1 and random jump height Z_1 which starts in $z_0 \in X$. Intuitively one thinks of a second random jump as another single-jump process that is born at time S_1 at the position Z_1 and that jumps after time S_2 (i.e. after total time $S_1 + S_2$) to the location Z_2, where in turn another single-jump process will be born and so on. The resulting process might have finite or countably infinite random jumps but acts as a single-jump process in between two jump times.
The complexity is not lost under this intuition, it merely hides inside the distributional information of each jump.

2.1 Definitions and assumptions

Spaces and random variables In this section we would like to allow the process to have more than one random jump. To that end we take copies $(Y_n, \mathcal{Y}_n)$ of the template statespace $(Y, \mathcal{Y}) = ((\mathbb{R}^+ \times X) \cup \{(\infty, z_\infty)\}, \sigma\{\mathcal{B}(\mathbb{R}^+) * \mathcal{S}, \{(\infty, z_\infty)\}\})$, and define

$$\Omega := \prod_{n\in\mathbb{N}} Y_n,$$
$$\mathcal{F}^0 := \sigma\{\prod_{n\in\mathbb{N}} \mathcal{Y}_n\}.$$

$(\Omega, \mathcal{F}^0)$ is again a Blackwell space (see [Davis, 1976], p. 624).

Let $(S_n, Z_n) : \Omega \to Y_n$ be the coordinate mappings, picking out the time and space coordinates of the n-th jump of a general state $\omega = (y_1, y_2, \dots)$ with $y_i \in Y_i$ for $i \in \mathbb{N}$. This means S_n will be the random life time of the n-th single-jump process and Z_n marks the birthplace of the next single-jump process.
Let $\omega_k : \Omega \to \Omega_k := \prod_{n=1}^{k} Y_n$ denote the restriction to the first k jumps, i.e.

$$\omega_k(\omega) = (S_1(\omega), Z_1(\omega), \dots, S_k(\omega), Z_k(\omega)).$$

It carries the information of the first k jumps. We will say 'For fixed ω_k...' but won't forget, that we mean for some $v \in \Omega_k$, $\{\omega_k = v\} = \{\omega \in \Omega : \omega_k(\omega) = v\}$ which is in fact a set. But fixing ω_k in this sense is in a way defining the past of any time after the k-th jump and this is well reflected in our notion above.
For some notational convenience we also set $T_0 := 0$. Note that for $\omega \in \{\omega_k = \eta\}$ we get $Z_j(\omega) = \eta^{(2j)}$ for any $j \leq k$, we thus take $Z_j(\eta)$ to deliver the information of the j-th jump location of any given past $\eta \in \Omega_k$, although Z was initially defined on Ω instead of Ω_k for $k \in \mathbb{N}$.

The process: Currently any ω only carries the individual jump times and locations. But to properly connect the single-jump processes at their jump times we will need to run them in a global time frame. We define

$$T_n(\omega) := \sum_{k=1}^{n} S_k(\omega)$$
$$T_\infty(\omega) := \lim_{n\to\infty} T_n(\omega)$$

thus making T_n the (global) random time of the n-th jump and naturally $T_1 \leq T_2 \leq \dots$. The terminal jump time T_∞ marks the exhaustion of the jump processes and will stop the overall process in a graveyard location z_∞. The value of the process at a time $t \in \mathbb{R}^+$ is given by:

$$x(t,\omega) = \begin{cases} z_0, & t < T_1(\omega); \\ Z_i(\omega), & t \in [T_i(\omega), T_{i+1}(\omega)); \\ z_\infty, & t \geq T_\infty(\omega). \end{cases}$$

Additionally set

$$\begin{aligned} &\bar{x}^1(s) := x(t \wedge T_1), \\ &\bar{x}^k(s) := x((T_{k-1} + s) \wedge T_k), \quad \text{for } k \geq 2. \end{aligned} \tag{2.1.1}$$

For $k \in \mathbb{N}$ every $(\bar{x}^k(t))_{t\in\mathbb{R}^+}$ is a single-jump process but for $k \geq 2$ it starts in a random location Z_{k-1}. Each $\bar{x}^k$ has a random life time S_k and the respective jump location Z_k:

$$\bar{x}^1(t,\omega) = \begin{cases} z_0, & \text{for } t < S_1(\omega), \\ Z_1(\omega), & \text{for } t \geq S_1(\omega), \end{cases} \quad \bar{x}^k(t,\omega) = \begin{cases} Z_{k-1}(\omega), & \text{for } t < S_k(\omega), \\ Z_k(\omega), & \text{for } t \geq S_k(\omega). \end{cases}$$

Further we observe that consecutively indexed $\bar{x}$-processes are connected at the jump location of the former and the starting location of the latter process:

$$\bar{x}^k(0) = x(T_{k-1} \wedge T_k) = x(T_{k-1}) = x((T_{k-2} + S_{k-1}) \wedge T_{k-1}) = \bar{x}^{k-1}(S_{k-1}).$$

Thus the different cases for t in the next equality, can be read as $t \in [T_{k-1}, T_k)$ or $t \in (T_{k-1}, T_k]$ arbitrarily. The connection between the multi-jump process $(x(t))_{t\in\mathbb{R}^+}$ and the family of single-jump processes $\{(\bar{x}^k(t))_{t\in\mathbb{R}^+} : k \in \mathbb{R}\}$ is the following:

$$x(t,\omega) = \begin{cases} \bar{x}^1(t,\omega), & \text{for } t \in [0, T_1), \\ \bar{x}^k(t - T_{k-1}(\omega), \omega), & \text{for } t \in [T_{k-1}(\omega), T_k(\omega)), \\ z_\infty, & \text{for } t \geq T_\infty \end{cases} \tag{2.1.2}$$

Inspired by this connection, we will refer to the processes $(\bar{x}^k(t))_{t\in\mathbb{R}^+}$ as *single-jump sections of the process* $(x(t))_{t\in\mathbb{R}^+}$

Filtration and probability measure: As before let $\mathcal{F}_t^0$ denote the natural filtration, generated by the process $(x(t))_{t\geq 0}$.

From [Davis, 1976] we adopt the characterization of the probability measure through the conditional distributions: for $2 \leq i \in \mathbb{N}$ and $\Gamma \in \mathcal{Y}_i$ and $\eta \in \Omega_{i-1}$ the probability measure $\mathbb{P}$ is defined through

$$\mathbb{P}\left[(T_1, Z_1) \in \Gamma\right] = \mu_1(\Gamma), \qquad (2.1.3)$$
$$\mathbb{P}\left[(S_i, Z_i) \in \Gamma | \omega_{i-1} = \eta\right] = \mu_i(\eta; \Gamma),$$

where μ^1 is a probability measure on $(Y_1, \mathcal{Y}_1)$ with

$$\mu^1((\{0\} \times X) \cup (\mathbb{R}^+ \times \{z_0\})) = 0$$

and for $i = 2, 3, \ldots$

$$\mu^i : \Omega_{i-1} \times \mathcal{Y} \to [0, 1]$$

are functions which satisfy

(i) $\mu^i(\cdot; \Gamma) : \Omega_{i-1} \to [0, 1]$ is measurable for each $\Gamma \in \mathcal{Y}$ fixed.

(ii) $\mu^i(\omega_{i-1}(\omega); \cdot) : \mathcal{Y} \to [0, 1]$ is a probability measure for each $\omega \in \Omega$ fixed.

(iii) $\mu^i(\omega_{i-1}(\omega); (\{0\} \times X) \cup (\mathbb{R}^+ \times \{Z_{i-1}(\omega)\})) = 0$ for all $\omega \in \Omega$.

(iv) $\mu^i(\omega_{i-1}(\omega); \{(\infty, z_\infty)\}) = 1$ if $S_{i-1}(\omega) = \infty$.

μ^1 is in fact the same measure as in section 2. The second requirement (ii) makes sure that for $i \geq 2$ and fixed ω_{i-1} the measure $\mu^i(\omega_{i-1}; \cdot)$ acts as a version of μ^1 for the i-th jump process, i.e. is a conditional probability measure. Condition (iii) excludes the possibility of two jumps happening at the same time and consecutive jumps to the same location respectively and the last condition (iv) assigns z_∞ to be a final resting place, once the previous jump-time has been infinite.

More particular, μ^1 is the joint distribution of T_1 and Z_1 whereas for $i \geq 2, \eta \in \Omega_{i-1}$ the probability measure $\mu^i(\eta; \cdot)$ is the joint distribution of T_i and Z_i given η.

In relation to the previously defined single-jump sections of the process $(x(t))_{t\in\mathbb{R}^+}$ they act as

$$\mathbb{P}(\bar{x}^1(t) \in \{z_0\}) = \mathbb{P}(S_1 > t) = \mu^1((t, \infty) \times X), \qquad (2.1.4)$$
$$\mathbb{P}(\bar{x}^i(t) \in \{z\} | \omega_{i-1} = \eta) = \mathbb{P}(S_k > t | \omega_{i-1} = \eta) = \mu^1(\eta; (t, \infty) \times X).$$

Thus they are (given ω_{i-1}) involved in the distribution of single-jump segments.

Denote by $\mathcal{N}_0$ the set of all $\mathbb{P}$-nullsets. By $\mathcal{F}, \mathcal{F}_t$ we denote the σ-fields $\mathcal{F}^0, \mathcal{F}_t^0$ augmented with all $\mathbb{P}$-null sets, i.e.

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^0, \mathcal{N}_0) \qquad (2.1.5)$$

According to Lemma 0 in [Davis, 1976] the jump times T_k are stopping times of $(\mathcal{F}_t)$.

Let $(\mathcal{G}_t)_{t\in I}$ be a filtration and S a stopping time. We define the past up to time S as

$$\mathcal{G}_S := \{A \in \mathcal{H} : A \cap \{S \leq t\} \in \mathcal{G}_t, \forall t \in I\}. \tag{2.1.6}$$

Another important family of stopping times is the following: for $k \in \mathbb{N}$ and $s \in \mathbb{R}^+$

$$U_s^k := (T_{k-1} + s) \wedge T_k. \tag{2.1.7}$$

For fixed $s \in \mathbb{R}^+$ every U_s^k is a stopping time of the filtration $(F_t)_{t\geq 0}$ and yields the following useful property (see [Davis, 1976], Lemma 1):

$$\mathcal{F}_{U_s^k} = \sigma\left(\mathcal{F}_{T_{k-1}}, \{x((T_{k-1}+u) \wedge T_k) : u \in [0,s]\}\right), \tag{2.1.8}$$

i.e. the information of the process $(x(t))_{t\geq 0}$ between two consecutive jumps - say the $k-1$-st and k-th jumps - can be decomposed into information of the first $k-1$ jumps and the path of the process since the $k-1$-st jump.

Note that - given a past $\omega_{k-1} = \eta$ - the natural filtration of the single-jump sections of the process $(x(t))_{t\in\mathbb{R}^+}$ is given by

$$\{\omega_{k-1} = \eta\} \cap \sigma\{x((T_k + s) \wedge T_{k+1}) : s \in [0,t]\} =: \{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}_t^k. \tag{2.1.9}$$

The above equation (2.1.8) can be seen as the collection of natural filtrations of $(\bar{x}^k(t))_{t\in\mathbb{R}^+}$ for any ω_{k-1} (which is $\mathcal{F}_{T_{k-1}}$-measurable).
Let $(\Omega, \mathcal{A})$ be a measurable space. Then we define the trace-σ-field of a set $A \in \mathcal{A}$ with any sub-σ-field $\mathcal{A} \subset \mathcal{G}$ as

$$A \cap \mathcal{A} := \{A \cap B : B \in \mathcal{A}\}. \tag{2.1.10}$$

The next result comes in handy in the proof of our main result:

Lemma 2.1. *For $k \in \mathbb{N}$, $s \in \mathbb{R}^+$.*

(i)

$$\{s \leq T_{k-1}\} \cap \mathcal{F}_s \subseteq \{s \leq T_{k-1}\} \cap \mathcal{F}_{U_0^k}, \tag{2.1.11}$$

(ii)

$$\{T_{k-1} \leq s \leq T_k\} \cap \mathcal{F}_s = \{T_{k-1} \leq s \leq T_k\} \cap \mathcal{F}_{U^k_{s-T_{k-1}}}, \tag{2.1.12}$$

(iii)

$$\{T_k \leq s\} \cap \mathcal{F}_s \supseteq \{T_k \leq s\} \cap \mathcal{F}_{U^k_{S_k}} = \{T_k \leq s\} \cap \mathcal{F}_{U_0^{k+1}}. \tag{2.1.13}$$

(iv) For $\eta \in \Omega_{k-1}$:

$$\{\omega_{k-1} = \eta\} \cap \mathcal{H}_s^k = \{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}_s^k \tag{2.1.14}$$

Proof. (i) Note that with [Davis, 1976], Lemma 1 we have

$$\mathcal{F}_{U_0^k} = \sigma(\mathcal{F}_{T_{k-1}}) = \mathcal{F}_{T_{k-1}}$$

Take $B \in \mathcal{F}_s$. Then $\{s \le T_{k-1}\} \cap B \in \mathcal{F}_\infty$ and

$$\begin{aligned}\{s \le T_{k-1}\} \cap B \cap \{T_{k-1} \le t\} &= \{s \le T_{k-1} \le t\} \cap B \\ &= \begin{cases} \emptyset, & \text{if } t < s, \\ \{s \le T_{k-1} \le t\} \cap B, & \text{if } t \ge s, \end{cases}\end{aligned}$$

and since

- $\{s \le T_{k-1}\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$ for $t \ge s$,
- $B \in \mathcal{F}_s \subseteq \mathcal{F}_t$ for $t \ge s$ and
- $\{T_{k-1} \le t\} \in \mathcal{F}_t$

we get:

$$\{s \le T_{k-1} \le t\} \cap B \in \mathcal{F}_t$$

for all $t \in \mathbb{R}^+$ an thus $\{s \le T_{k_1}\} \cap B \in \mathcal{F}_{T_{k-1}} = \mathcal{F}_{U_0^k}$.

(ii) This is a refined combination of the other two cases. With (iii) we roughly locate

$$\{T_{k-1} \le s\} \cap \mathcal{F}_s \supseteq \{T_{k-1} \le s\} \cap \mathcal{F}_{U_0^k},$$

The intersection of this set inequality with the corresponding 'other side' of s (i.e. $\{s \le T_k\}$) does not change this relation. Note that for $\omega \in \{T_{k-1} \le s \le T_k\}$ we have $0 \le s - T_{k-1}(\omega)$ and thus $U_0^k(\omega) \subset U^k_{s-T_{k-1}(\omega)}(\omega)$ hence we end up with

$$\begin{aligned}\{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_{U_0^k} &\subseteq \{T_{k-1} \le s < T_k\} \cap \mathcal{F}_{U^k_{s-T_{k-1}}} \\ &\subseteq \{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_s\end{aligned}$$

where we see the last relation by taking any $B \in \{T_{k-1} \le s < T_k\} \cap \mathcal{F}_{U^k_{s-T_{k-1}}}$. For such a set we know $B = \{T_{k-1} \le s < T_k\} \cap A$ (see (2.1.10)), where the set $A \in \mathcal{F}_{U^k_{s-T_{k-1}}}$ and for which we know (by the definition of $\mathcal{F}_{U^k_{s-T_{k-1}}}$, see (2.1.6)) that $A \cap \{U^k_{s-T_{k-1}} \le t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}^+$ - especially for $t = s$, but note that

$$\{U^k_{s-T_{k-1}} \le s\} = \{(T_{k-1} + s - T_{k-1}) \wedge T_k \le s\} = \{s \wedge T_k \le s\} = \Omega$$

and thus

$$\mathcal{F}_s \ni A \cap \{U^k_{s-T_{k-1}} \le s\} = A \cap \Omega = A$$

i.e. $B \in \{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_s$.
For the other inclusion in (2.1.12) we first locate with (i)

$$\{s \le T_k\} \cap \mathcal{F}_s \subseteq \{s \le T_k\} \cap \mathcal{F}_{U_0^{k+1}}.$$

and note that on the intersection with $\{T_{k-1} \le s \le T_k\}$ the time $U^k_{s-T_{k-1}} \le U_0^{k+1} = T_k$, i.e.

$$\{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_{U^k_{s-T_{k-1}}} \subset \{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_{U_0^{k+1}}$$

and we end up with

$$\begin{aligned}\{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_{U_0^{k+1}} &\supseteq \{T_{k-1} \le s < T_k\} \cap \mathcal{F}_{U^k_{s-T_{k-1}}} \\ &\supseteq \{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_s.\end{aligned}$$

To justify the last relation we take $B \in \mathcal{F}_s$. For $\omega \in \{T_{k-1} \le s \le T_k\}$ we conclude $U^k_{s-T_{k-1}(\omega)}(\omega) = (T_{k-1}(\omega) + s - T_{k-1}(\omega)) \wedge T_k(\omega) = s$. Then for any $t \in \mathbb{R}^+$

$$\{T_{k-1} \le s \le T_k\} \cap B \cap \{U^k_{s-T_{k-1}} \le t\} = \{T_{k-1} \le s \le T_k\} \cap B \cap \{s \le t\} \in \mathcal{F}_t$$

where we used that $\{s \le t\} = \emptyset$ for $t < s$ and $\{s \le t\} = \Omega$ for $s \le t$. Hence

$$\{T_{k-1} \le s \le T_k\} \cap B \in \{T_{k-1} \le s \le T_k\} \cap \mathcal{F}_{U^k_{s-T_{k-1}}}$$

by definition.

(iii) Note that

$$U^k_{S_k} = (T_{k-1} + S_k) \wedge T_k = T_k = T_k \wedge T_{k+1} = U_0^{k+1}.$$

Hence

$$\mathcal{F}_{U^k_{S_k}} = \mathcal{F}_{T_k} = \mathcal{F}_{U_0^{k+1}}.$$

Take $B \in \mathcal{F}_{T_k}$. Then by definition of the σ-field $\mathcal{F}_{T_k}$ we know that

$$\{T_k \le t\} \cap B \in \mathcal{F}_t, \forall t \in \mathbb{R}^+$$

especially for $t = s$. Thus $\{T_k \le s\} \cap B \in \mathcal{F}_s$ and by definition of

$$\{T_k \le s\} \cap \mathcal{F}_s = \{\{T_k \le s\} \cap A : A \in \mathcal{F}_s\}$$

we note that $\{T_k \le s\} \cap \{T_k \le s\} \cap B \in \{T_k \le s\} \cap \mathcal{F}_s$.

(iv) Take $B \in \bar{\mathcal{F}}^k_s$ (see (2.1.9)), then $\{\omega_{k-1} = \eta\} \cap B \in \{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}^k_s$ by definition of trace-σ-algebras. But note that $\bar{\mathcal{F}}^k_s \subseteq \mathcal{H}^k_s$ and thus B is a set of $\mathcal{H}^k_s$, hence

$$\{\omega_{k-1} = \eta\} \cap B \in \{\omega_{k-1} = \eta\} \cap \mathcal{H}^k_s.$$

For the other inclusion we take a general set $A \in \mathcal{H}^k_s$. Since $\mathcal{H}^k_s = \sigma\{\mathcal{F}_{T_{k-1}}, \bar{\mathcal{F}}^k_s\}$ we can assume that there exist two sets $B \in \mathcal{F}_{T_{k-1}}$ and $C \in \bar{\mathcal{F}}^k_s$ such that either $A = B \cap C$ or $A = B \cup C$. Either way we have

$$\{\omega_{k-1} = \eta\} \cap B \in \{\emptyset, \{\omega_{k-1} = \eta\}\} \subset \{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}^k_s$$

and thus

$$(\{\omega_{k-1} = \eta\} \cap B) \cup (\{\omega_{k-1} = \eta\} \cap C) \in \{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}^k_s,$$

(similar for the case $A = B \cap C$) which yields

$$\{\omega_{k-1} = \eta\} \cap A \in \{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}^k_s.$$

$\square$

The next lemma is a technical tool we will need in a subsequent proof. We state it here for reference:

Lemma 2.2. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{A}$, $C \in \mathcal{G}$ and a random variable $X \in L^1(\mathbb{P})$ it holds*

$$\mathbb{E}[X|\mathcal{G}]\mathbb{1}_C = \mathbb{E}[X|C \cap \mathcal{G}]\mathbb{1}_C.$$

Proof. Take any $A \in \mathcal{G}$. Then $C \cap A \subset C \in \mathcal{G}$ and

$$\begin{aligned}\int_A \mathbb{E}[X|\mathcal{G}]\mathbb{1}_C d\mathbb{P} &= \int_{A\cap C} X d\mathbb{P} \\ &= \int_{A\cap C} \mathbb{E}[X|C \cap \mathcal{G}]d\mathbb{P}.\end{aligned}$$

□

Basic martingales and applicable integrands We take from [Davis, 1976], page 632-633 the notation for the basic martingale. Let $t \in \mathbb{R}^+, A \in \mathcal{S}$ and $\omega \in \{t \in (T_{j-1}, T_j]\}$. Then the family of basic martingales $q(t, A) := p(t, A) - \tilde{p}(t, A)$ is defined by

$$p(t, A) := \sum_{i=1}^{j} \mathbb{1}_{Z_i \in A}\mathbb{1}_{T_i \le t}$$

and

$$\tilde{p}(t, A) := \Phi_1^A(T_1) + \Phi_2^A(\omega_1; S_2) + \cdots + \Phi_j^A(\omega_{j-1}; t - T_{j-1})$$

where

$$\begin{aligned}\Phi_1^A(s) &:= \int_{(0,s]} \frac{1}{1 - F_{u-}} dF_u^A \\ \Phi_i^A(\omega_{i-1}; s) &:= \int_{(0,s]} \frac{1}{1 - F_{u-}^i} dF_u^{iA},\end{aligned}$$

and

$$\begin{aligned}F_u^{1A} &:= F_u^A, \\ F_u^{iA}(\omega_{i-1}) &:= \mu^i(\omega_{i-1}; [0, u] \times A).\end{aligned}$$

For any $\eta \in \Omega_{k-1}$ and in analogy to the first chapter we denote the right endpoint of the distribution of each jump time S_k as

$$c^k(\eta) := \sup\{t \in \mathbb{R}^+ : F_t^k(\eta) < 1\} \tag{2.1.15}$$

We also follow [Davis, 1976] along the definition of a Lebesgue-Stieltjes integral w.r.t. q. For a function $g : \Omega \times Y \to \mathbb{R}$ and functions $g^1 : Y_1 \to \mathbb{R}$ and $g^k : \Omega^{k-1} \times Y_k \to \mathbb{R}$ for $k = 2, 3, \ldots$ such that

$$g(\omega, t, z) = \begin{cases} g^1(t, z), & t \le T_1(\omega), \\ g^k(\omega_{k-1}(\omega); t, z), & t \in (T_{k-1}(\omega), T_k(\omega)], \\ 0, & t \ge T_\infty(\omega), \end{cases}$$

and

$$g^1(\infty, z) = g^k(\omega_{k-1}; \infty, z) = 0$$

we define the integral as

$$I_t^g(\omega) := \int_{(0,t]\times X} g(\omega, s, z) dq(s, z)(\omega).$$

The nature of q lets us decompose this integral then as follows:

$$I_t^g(\omega) = \int_{(0,t]\times X} g(\omega, s, z) dp(s, z)(\omega) - \int_{(0,t]\times X} g(\omega, s, z) d\tilde{p}(s, z)(\omega)$$

and each of these integrals is given for $t \in [T_{k-1}, T_k)$ by

$$\int_{(0,t]\times X} g(s, z) dp(s, z) = \sum_{n=1}^{k-1} g^n(\omega_{n-1}; T_1, Z_1), \tag{2.1.16}$$

$$\begin{aligned} \int_{(0,t]\times X} g(s, z) d\tilde{p}(s, z) = & \sum_{n=1}^{k-1} \int_{(T_{n-1},T_n]\times X} g^n(\omega_{n-1}; s, z) \frac{1}{1 - F_{s-}^n} d\mu^n(\omega_{n-1}; s, z) \\ & + \int_{(T_{k-1},t]\times X} g^k(\omega_{k-1}; s, z) \frac{1}{1 - F_{s-}^k} d\mu^k(\omega_{k-1}; s, z). \end{aligned} \tag{2.1.17}$$

2.2 Martingale transformations

Again we approach the transformation problem from the perspective of a function of the process. Guided by the results from the previous chapter we set for a measurable function $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$

$$v(t) := \varphi(t, x(t)). \tag{2.2.1}$$

The function φ that we want to determine will be a function as before, but this time the process $x(t)$ has more than two different values in his lifetime (or at least might jump back and forth). We start with $\varphi(t, z_0)$ as long as $t < T_1(\omega)$. Then the first jump happens at $T_1(\omega)$ and we jump to the value $\varphi(T_1(\omega), Z_1(\omega))$. Now the function will already have to prepare for the next jump at $T_2(\omega)$ so we follow $\varphi(t, Z_1(\omega))$ for $t \in [T_1(\omega), T_2(\omega))$ and so on.
This structure leads us to the idea, that φ must be defined piecewise. But this means also, that the function φ will depend on the processes path (at least up until the current time t) and thus might not be determined generally for all paths.

The discussion at the beginning of this chapter lets us approach the problem of determining martingale transformations from the familiar scenario of single-jump processes. Since the process seems to be a 'glued together' version of dependent single-jump processes (see (2.1.2)), we are going to apply the results from the previous chapter to these particular single-jump sections.
The previously defined processes $(\bar{x}^k(t))_{t\geq 0}$ (see (2.1.1)) are themselves single-jump processes in X but for $k \geq 2$ they start in a random location Z_{k-1}, live a random life-time S_k and eventually jump to a random location Z_k in X where they remain for

eternity. This is in fact no problem as long as the respective filtration contains enough information about the processes predecessors: for $t \in \mathbb{R}^+$ set $U_t^k = (T_{k-1} + t) \wedge T_k$ and note the property in (2.1.8). Set

$$\mathcal{H}_t^k := \mathcal{F}_{U_t^k},$$

The next goal is to determine a function φ^k s.t. $\varphi^k(t, \bar{x}^k(t))$ is a martingale w.r.t. the filtration $(\mathcal{H}_t^k)_{t\geq 0}$. To this end we note that $\bar{x}^k$ is of single-jump character *given the information* of the first $k-1$ jumps, consequently we will determine φ^k depending on ω_{k-1}. Further we note that given ω_{k-1} the distribution of the k-th jump-time and location is given by the conditional measure $\mu^k(\omega_{k-1}; \cdot)$, thus we concentrate on determining $\varphi^k(\omega_{k-1}; t, y)$ such that $\bar{v}^k(t) = \varphi^k(\omega_{k-1}; t, \bar{x}^k(t))$ is an $(\mathcal{H}_t^k)$-martingale under the measure $\mu^k(\omega_{k-1}; \cdot)$.

Corollary 2.3. *For $\varphi_1 : \mathbb{R}^+ \times X \to \mathbb{R}$ and $\varphi^k : \Omega_{k-1} \times \mathbb{R}^+ \times X \to \mathbb{R}$ set*

$$\begin{aligned} \bar{v}^1(t) &:= \varphi_1(t, \bar{x}^1(t)), \\ \bar{v}^k(\omega_{k-1}; t) &:= \varphi^k(\omega_{k-1}; t, \bar{x}^k(t)), \quad \text{for } k \geq 2. \end{aligned}$$

Then $\bar{v}^1(t)$ is a local $\mathcal{H}_t^1$-martingale, resp. $\bar{v}^k(\omega_{k-1}; t)$ is a local $\mathcal{H}_t^k$-martingale if for $t \in \mathbb{R}^+, y \in X, \eta \in \Omega_{k-1}$ for $k \geq 2$:

$$\varphi_1(t, y) = b^1(y)(1 - \delta_{z_0}(y)) - \frac{\delta_{z_0}(y)}{1 - F_t^1} \left(\int_{(0,t] \times X} b^1(z) \mu^1(du, dz) + r^1 \right) \tag{2.2.2}$$

$$\begin{aligned} \varphi^k(\eta; t, y) &= b^k(\eta; y)(1 - \delta_{Z_{k-1}(\eta)}(y)) \\ &\quad - \frac{\delta_{Z_{k-1}(\eta)}(y)}{1 - F_t^k(\eta)} \left(\int_{(0,t] \times X} b^k(\eta; z) \mu^k(\eta; du, dz) + r^k(\eta) \right) \end{aligned} \tag{2.2.3}$$

with $b^1 : X \to \mathbb{R}$, $b^1 \in L^1_{loc}(\mu^1)$ and arbitrary constant $r^1 \in \mathbb{R}$, as well as $b^k(\eta; \cdot) : X \to \mathbb{R}, b^k(\eta) \in L^1_{loc}(\mu^k(\eta; \cdot))$ and $r^k(\eta) \in \mathbb{R}$ for all $k \geq 2, \eta \in \Omega_{k-1}$.

Proof. First of all $(\bar{v}^k(t))_{t \in \mathbb{R}^+}$ is adapted to $(\mathcal{H}_t^k)_{t \in \mathbb{R}^+}$ since it is a measurable function of ω_{k-1} and $\bar{x}^k(t)$ which are both $\mathcal{H}_t^k$-measurable. The same holds true for $k = 1$ where $\bar{v}^1$ which is even only a function of $\bar{x}^1(t)$.

For the local martingale property let $s < t \in \mathbb{R}^+$, $k \geq 2$ and $A \in \mathcal{H}_s^k$. Take for $\eta \in \Omega_k$:

$$\sigma_m^k(\eta) := \begin{cases} m, & c^k(\eta) = \infty \\ \infty, & c^k(\eta) < \infty, F^k_{c^k-}(\eta) < 1 \\ k \mathbb{1}_{T_k \leq t_m^{(k)}} + t_m^{(k)} \mathbb{1}_{T > t_m^{(k)}}, & c^k(\eta) < \infty, F^k_{c^k-}(\eta) = 1, \end{cases}$$

where $t_k \nearrow c^k(\eta)$ (see (2.1.15)).

Then we have to show for all $m \in \mathbb{N}$, $s \leq t \in \mathbb{R}^+$

$$\mathbb{E}[\bar{v}^k(\omega_{k-1}; s \wedge \sigma_m^k(\omega_{k-1}))\mathbb{1}_A] = \mathbb{E}[\bar{v}^k(\omega_{k-1}; t \wedge \sigma_m^k(\omega_{k-1})\mathbb{1}_A].$$

We start with the right side:

$$
\begin{aligned}
&\mathbb{E}[\bar{v}^k(\omega_{k-1}; t \wedge \sigma_m^k(\omega_{k-1}))\mathbb{1}_A] \\
&= \int_\Omega \bar{v}(\omega_{k-1}(\omega); t \wedge \sigma_m^k(\omega_{k-1}), \omega)\mathbb{1}_A(\omega) d\mathbb{P}(\omega) \\
&\overset{(a)}{=} \int_{\Omega_{k-1}} \int_{\{\omega_{k-1}=\eta\}} \bar{v}^k(\eta; t \wedge \sigma_m^k(\omega_{k-1}), \omega')\mathbb{1}_A(\omega') d\mathbb{P}(\omega'|\eta) d\nu(\eta) \\
&\overset{(b)}{=} \int_{\Omega_{k-1}} \int_{\{\omega_{k-1}=\eta\}} \mathbb{E}[\bar{v}^k(\eta; t \wedge \sigma_m^k(\omega_{k-1}))\mathbb{1}_A|\mathcal{H}_s^k](\omega') d\mathbb{P}(\omega'|\eta) d\nu(\eta) \\
&\overset{(c)}{=} \int_{\Omega_{k-1}} \int_A \mathbb{E}[\bar{v}^k(\eta; t \wedge \sigma_m^k(\omega_{k-1}))|\mathcal{H}_s^k](\omega')\mathbb{1}_{\{\omega_{k-1}=\eta\}}(\omega') d\mathbb{P}(\omega'|\eta) d\nu(\eta) \\
&\overset{(d)}{=} \int_{\Omega_{k-1}} \int_A \mathbb{E}[\bar{v}^k(\eta; t \wedge \sigma_m^k(\omega_{k-1}))|\{\omega_{k-1}=\eta\} \cap \mathcal{H}_s^k](\omega')\mathbb{1}_{\{\omega_{k-1}=\eta\}}(\omega') d\mathbb{P}(\omega'|\eta) d\nu(\eta) \\
&\overset{(e)}{=} \int_{\Omega_{k-1}} \int_A \mathbb{E}[\bar{v}^k(\eta; t \wedge \sigma_m^k(\omega_{k-1}))|\{\omega_{k-1}=\eta\} \cap \bar{\mathcal{F}}_s^k](\omega') d\mathbb{P}(\omega'|\eta) d\nu(\eta) \\
&\overset{(f)}{=} \int_{\Omega_{k-1}} \int_A \bar{v}^k(\eta; s \wedge \sigma_m^k(\omega_{k-1}), \omega') d\mathbb{P}(\omega'|\eta) d\nu(\eta) \\
&\overset{(a)}{=} \int_\Omega \mathbb{1}_A(\omega)\bar{v}^k(\omega_{k-1}(\omega); s \wedge \sigma_m^k(\omega_{k-1}), \omega) d\mathbb{P}(\omega) \\
&= \mathbb{E}[\bar{v}^k(\omega_{k-1}; s \wedge \sigma_m^k(\omega_{k-1}))\mathbb{1}_A]
\end{aligned}
$$

where we've used in particular:

(a) $(\Omega, \mathcal{F})$ is a Blackwell space and thus enables disintegration. We disintegrate up until the $k-1$-first jump, i.e. we integrate over all pasts $\eta \in \Omega_{k-1}$ in the outer integral and integrate over all possible futures (and presents) from the set $\{\omega' \in \Omega : \omega_{k-1}(\omega') = \eta\}$ in the inner integral. ν denotes the marginal disrtribution of ω_{k-1} and $\mathbb{P}(\cdot|\eta)$ is the conditional measure defined in (2.1.3).

(b) Since $\{\omega_{k-1} = \eta\} \in \mathcal{H}_s^k$ (see (2.1.8)) we can insert the conditional expectation here.

(c) $A \in \mathcal{H}_s^k$ and thus $\mathbb{1}_A$ is $\mathcal{H}_s^k$-measurable and can be pulled outside the conditional expectation. The set of the inner integral is taking its place for the next step.

(d) See lemma 2.2.

(e) See lemma 2.1, (2.1.14).

(f) Use theorem 1.19 on the process $\bar{v}^k(\eta) = (\bar{v}^k(\eta; t))_{t \in \mathbb{R}^+}$ and its natural filtration $\left(\{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}_t^k\right)_{t \in \mathbb{R}^+}$. Then there exists a $\varphi^k(\eta)$ s.t. $\bar{v}(\eta; t) = \varphi^k(\eta; t, \bar{x}^k(t))$ is a local $\{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}_t^k$-martingale. We use the sequence $(\sigma_m^k)_{m \in \mathbb{N}}$ to make use of the martingale property of the stopped process to get rid of the conditional expectation.

For $k = 1$ the process $\bar{v}^1(t)$ is even more directly an application of the single-jump case.

□

Remark 2.4. *The processes $\bar{v}^k$ stay constant after their respective jump time: let $t \in \mathbb{R}^+$:*

$$\begin{aligned}\bar{v}^k(\omega_{k-1};t)\mathbb{1}_{S_k\le t} &= \varphi(t,\bar{x}^k(t))\mathbb{1}_{S_k\le t}\\ &= \varphi^k(\omega_{k-1};t,x((T_{k-1}+t)\wedge T_k))\mathbb{1}_{S_k\le t}\\ &= \varphi^k(\omega_{k-1};t,x(T_k))\mathbb{1}_{S_k\le t}\\ &= \varphi^k(\omega_{k-1};t,Z_k)\mathbb{1}_{S_k\le t}\\ &= b^k(\omega_{k-1};Z_k)\mathbb{1}_{S_k\le t}\end{aligned}$$

(the same is true for $k=1$ but the calculations are without the ω_{k-1} of course). Hence

$$\bar{v}^k(t\wedge T_k) = \bar{v}^k(t) \tag{2.2.4}$$

We started this journey with the goal to determine a function of the multi-jump process, that transforms it into a $\mathcal{F}_s$-martingale. So far we've determined a set of functions, that bend each single-jump section of the multi-jump process into a martingale w.r.t. to the intermediate filtrations $\mathcal{H}^k_s$. In equation (2.1.2) we've seen the connection between the process and its single-jump sections and in lemma 2.1 we've explored the relations of the natural filtration $\mathcal{F}_s$ to its intermediate filtrations $\mathcal{H}^k_{(s-T_{k-1})\vee 0}$. Inspired by these results we now reconstruct a general function $\varphi : \Omega\times\mathbb{R}^+\times X\to\mathbb{R}$ from the martingale transformations of corollary 2.3. Set

$$v(t,\omega) := \begin{cases}\bar{v}^1(t,\omega), & \text{for } t < T_1(\omega)\\ \bar{v}^k(t-T_{k-1}), & \text{for } t\in(T_{k-1}(\omega),T_k(\omega)]\\ \bar{v}^\infty, & \text{for } t\ge T_\infty\end{cases} \tag{2.2.5}$$

where $\bar{v}^\infty\in\mathbb{R}$ is an arbitrary graveyard location.

Lemma 2.5. *For convenience set $v_\infty = 0$. We have*

$$v(t) = v(t\wedge T_1) + \sum_{k=2}^{\infty}[v(t\wedge T_k) - v(T_{k-1})]\mathbb{1}_{T_{k-1}<t} \tag{2.2.6}$$

$$= \bar{v}^1(t) + \sum_{k=2}^{\infty}\left[(\bar{v}^k((t-T_{k-1}) - \bar{v}^{k-1}(S_{k-1})\right]\mathbb{1}_{T_{k-1}<t}. \tag{2.2.7}$$

Proof. We wirst prove the equality of (2.2.6) by evaluating both sides on the disjoint sets $\{T_{n-1}<t\le T_n\}$ for all $n\in\mathbb{N}$. For $n=1$ we see:

$$\begin{aligned}&\left(v(t\wedge T_1) + \sum_{k=2}^{\infty}[v(t\wedge T_k)-v(T_{k-1})]\mathbb{1}_{T_{k-1}<t}\right)\mathbb{1}_{t\in[0,T_1]}\\ =&v(t\wedge T_1)\mathbb{1}_{t\in[0,T_1]} + \sum_{k=2}^{\infty}[v(t\wedge T_k)-v(T_{k-1})]\mathbb{1}_{\{T_{k-1}<t\}\cap\{t\in[0,T_1]\}}\\ =&v(t)\mathbb{1}_{t\in[0,T_1]}\end{aligned}$$

since $t \wedge T_1 = t$ on $\{t \in [0, T_1]\}$ and $\{T_{k-1} < t\} \cap \{t \in [0, T_1]\} = \emptyset$ for all $k \geq 2$. Now for any $n \geq 2$:

$$\begin{aligned}
&\left(v(t \wedge T_1) + \sum_{k=2}^{\infty} [v(t \wedge T_k) - v(T_{k-1})] \, \mathbb{1}_{T_{k-1}<t}\right) \mathbb{1}_{T_{n-1}<t\leq T_n} \\
=& v(t \wedge T_1)\mathbb{1}_{T_{n-1}<t\leq T_n} + \sum_{k=2}^{\infty} [v(t \wedge T_k) - v(T_{k-1})] \, \mathbb{1}_{\{T_{k-1}<t\}\cap\{T_{n-1}<t\leq T_n\}} \\
=& v(T_1)\mathbb{1}_{T_{n-1}<t\leq T_n} + \left(\sum_{k=2}^{n} [v(t \wedge T_k) - v(T_{k-1})]\right) \mathbb{1}_{T_{n-1}<t\leq T_n} \\
=& \left(v(T_1) + \sum_{k=2}^{n-1} [v(T_k) - v(T_{k-1})] + v(t) - v(T_{n-1})\right) \mathbb{1}_{T_{n-1}<t\leq T_n} \\
=& v(t)\mathbb{1}_{T_{n-1}<t\leq T_n}.
\end{aligned}$$

where we used that

$$v(t \wedge T_k)\mathbb{1}_{T_{n-1}<t\leq T_n} = \begin{cases} v(T_k)\mathbb{1}_{T_{n-1}<t\leq T_n}, & \text{for } k \leq n-1 \\ v(t)\mathbb{1}_{T_{n-1}<t\leq T_n}, & \text{else} \end{cases}$$

and

$$\{T_{k-1} < t\} \cap \{T_{n-1} < t \leq T_n\} = \begin{cases} \{T_{n-1} < t \leq T_n\}, & \text{for } k \leq n \\ \emptyset, \text{else.} \end{cases}$$

Thus

$$\begin{aligned}
v(t) &= \sum_{n\in\mathbb{N}} v(t)\mathbb{1}_{T_{n-1}<t\leq T_n} \\
&= \sum_{n\in\mathbb{N}} \left(v(t \wedge T_1) + \sum_{k=2}^{\infty}[v(t \wedge T_k) - v(T_{k-1})]\mathbb{1}_{T_{k-1}<t}\right) \mathbb{1}_{T_{n-1}<t\leq T_n} \\
&= v(t \wedge T_1) + \sum_{k=2}^{\infty}[v(t \wedge T_k) - v(T_{k-1})]\mathbb{1}_{T_{k-1}<t}.
\end{aligned}$$

The second equality (2.2.7) is a direct application of the definition in (2.2.5):

$$\begin{aligned}
v(t \wedge T_1) &= \bar{v}^1(t \wedge T_1) \\
&\overset{(a)}{=} \bar{v}^1(t)
\end{aligned}$$

and

$$\begin{aligned}
v(t \wedge T_k)\mathbb{1}_{T_{k-1}<t} &= \bar{v}^k((t \wedge T_k) - T_{k-1})\mathbb{1}_{T_{k-1}<t} \\
&\overset{(b)}{=} \bar{v}^k(t - T_{k-1} \wedge T_k)\mathbb{1}_{T_{k-1}<t} \\
&\overset{(a)}{=} \bar{v}^k(t - T_{k-1})\mathbb{1}_{T_{k-1}<t}
\end{aligned}$$

where we've used:

(a) See (2.2.4).

(b) We have

$$(t \wedge T_k) - T_{k-1} = \begin{cases} t - T_{k-1}, & \text{if } t \leq T_k \\ T_k - T_{k-1}, & \text{else.} \end{cases}$$

and

$$(t - T_{k-1}) \wedge S_k = \begin{cases} t - T_{k-1}, & \text{if } t - T_{k-1} \leq S_k \\ S_k, & \text{else.} \end{cases}$$

□

Note that the first equation is true for any real-valued process adapted to $(\mathcal{F}_t)$, regardless of the exact definition of v in (2.2.5). Nevertheless we can now construct a general function φ, s.t. $\varphi(\omega, t, x(t))$ is a local $\mathcal{F}_t$ martingale. For a sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ such that $v(t \wedge \sigma_n)$ is an $\mathcal{F}_t$ martingale for all $n \in \mathbb{N}$, the local martingale property only follows from the upcoming proof, if we assume that $T_\infty = \infty$. Take the construction of a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ from [Davis, 1976], proof of theorem 2. With a similar argument we can show that $\varphi(\omega; t, x(t)) \in L^1_{\text{loc}}(p) = L^1_{\text{loc}}(\mu)$ and thus we are now in a position to state our main result:

Theorem 2.6. *For all $n \in \mathbb{N}$ the process defined by*

$$v(t) := \varphi(t, x(t))$$

is a local $(\mathcal{F}_t)$-martingale for any

$$\varphi(\omega, t, y) = \begin{cases} \varphi_1(t, y), & \text{for } t \leq T_1(\omega) \\ \varphi^k(\omega^{k-1}(\omega); t - T_{k-1}(\omega), y), & \text{for } t \in (T_{k-1}(\omega), T_k(\omega)], k \geq 2 \\ v_\infty, & t \geq T_\infty(\omega) \end{cases} \tag{2.2.8}$$

where $v_\infty \in \mathbb{R}$ arbitrary and for any $t \in \mathbb{R}^+, y \in X, \eta \in \Omega_{k-1}$ for $k \geq 2$:

$$\begin{aligned} \varphi^1(t, y) &= b^1(y)(1 - \delta_{z_0}(y)) \\ &\quad - \delta_{z_0}(y) \frac{1}{1 - F^1_t} \left(\int_{(0,t]\times X} b^1(z) \mu^1(ds, dz) + r^1 \right) \\ \varphi^k(\eta; t, y) &= b^k(\eta; y)(1 - \delta_{Z_{k-1}(\eta)}(y)) \\ &\quad - \delta_{Z_{k-1}(\eta)}(y) \frac{1}{1 - F^k_t(\eta)} \left(\int_{(0,t]\times X} b^k(\eta; z) \mu^k(\eta; ds, dz) + r^k(\eta) \right). \end{aligned}$$

with $b^1 : X \to \mathbb{R}$, $b^1 \in L^1_{loc}(\mu^1)$ and arbitrary constant $r^1 \in \mathbb{R}$, as well as $b^k(\eta; \cdot) : X \to \mathbb{R}, b^K(\eta; \cdot) \in L^1_{loc}(\mu^k(\eta; \cdot))$ and $r^k(\eta) \in \mathbb{R}$ for all $k \geq 2, \eta \in \Omega_{k-1}$ such that:

$$r^k(\omega_{k-1}(\omega)) = b^{k-1}(\omega_{k-2}(\omega); Z_{k-1}(\omega)). \tag{2.2.9}$$

Proof. We check the local martingale-property first. To that end set for $k \in \mathbb{N}, \eta \in \Omega_{k-1}$:

$$s^k_n(\eta) := \begin{cases} \inf\{t : F^k_t(\eta) \geq 1 - \frac{1}{n^3}\}, & \text{if } c^k(\eta) = \infty \\ & \text{or } c^k(\eta) < \infty, F^k_{c^k(\eta)-} > 1 - \frac{1}{n^3} \\ c^k(\eta), & \text{else.} \end{cases}$$

Since we possibly deal with jump times that jump at the very last second, we define an emergency break for the term $\frac{1}{1-F_t^k(\eta)}$:

$$\tau_n := T_{j-1} + s_n^j$$

where $j := \min\{k \in \mathbb{N} : T_{k-1} + s_n^k \leq T_k\}$ as the localizing sequence of stopping times. Note that, τ_n is the first time a jump occurs in a tail of its marginal distribution with probability $\frac{1}{n^3}$. Thus the factor $\frac{1}{1-F_t^j}$ is in danger of exploding. τ_n stops the process before this factor gets too large.

Claim 1. *τ_n are stopping times and $\tau_n \nearrow \infty$.*

The former is due to

$$\{\tau_n \leq t\} = \bigcup_{j=1}^{\infty} \{T_{j-1} + s_n^j \leq T_j\} \cap \{T_j \leq t\}$$

which is a countable union of $\mathcal{F}_t$-measurable sets. The sequence of stopping times diverges a.s. since

$$\mathbb{P}(\tau_n \leq T_n) \leq \mathbb{P}\left(\bigcup_{j=1}^{n} \{s_n^j \leq S_j\}\right) \leq \sum_{j=1}^{n} \mathbb{P}(s_n^j \leq S_j)$$

and $\mathbb{P}(s_n^j \leq S_j) = 1 - F^j_{s_n^j} \leq 1 - 1 + \frac{1}{n^3}$. We conclude $\sum_{n=1}^{\infty} \mathbb{P}(\tau_n < T_n) = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$ and with Borel-Cantelli we get

$$\mathbb{P}(\liminf_{n\to\infty}(\tau_n > T_n) = 1$$

and thus $\tau_n \to T_\infty = \infty$ per assumption.

Now let $s, t \in \mathbb{R}^+, s \leq t$. In case that $\tau_n \leq s$ we can easily verify the martingale property, since the process is stopped and hence constant:

$$\mathbb{E}[v(t \wedge \tau_n)|\mathcal{F}_s] = \mathbb{E}[v(\tau_n)|\mathcal{F}_s] = v(\tau_n) = v(s \wedge \tau_n).$$

Therefor we assume $\tau_n > s$ from now on:

$$\mathbb{E}[v(t \wedge \tau_n)|\mathcal{F}_s] = \sum_{k=1}^{\infty} \mathbb{E}[v(t \wedge \tau_n)|\mathcal{F}_s]\mathbb{1}_{T_{k-1} \leq s < T_k}. \tag{2.2.10}$$

For each summand we 'translate' $\mathcal{F}_s$ to the local single jump filtration $\mathcal{H}^k_{s-T_{k-1}}$ belonging to the respective indicator function:

$$\begin{aligned}
\mathbb{E}[v(t \wedge \tau_n)|\mathcal{F}_s]\mathbb{1}_{T_{k-1} \leq s < T_k} &\overset{(a)}{=} \mathbb{E}[v(t \wedge \tau_n)|\{T_{k-1} \leq s < T_k\} \cap \mathcal{F}_s]\mathbb{1}_{T_{k-1} \leq s < T_k} \\
&\overset{(b)}{=} \mathbb{E}[v(t \wedge \tau_n)|\{T_{k-1} \leq s < T_k\} \cap \mathcal{H}^k_{s-T_{k-1}}]\mathbb{1}_{T_{k-1} \leq s < T_k} \\
&\overset{(a)}{=} \mathbb{E}[v(t \wedge \tau_n)|\mathcal{H}^k_{s-T_{k-1}}]\mathbb{1}_{T_{k-1} \leq s < T_k}
\end{aligned}$$

where we used in detail:

(a) See lemma 2.2.

(b) See lemma 2.1.

We decompose $v(t \wedge \tau_m)$ with the help of lemma 2.2.7:

$$\begin{aligned}
&\mathbb{E}[v(t \wedge \tau_m)|\mathcal{H}^k_{s-T_{k-1}}]\mathbb{1}_{T_{k-1}\leq s<T_k}\\
=&\mathbb{E}\Big[\bar{v}^1(t \wedge \tau_m \wedge T_1) &(2.2.11)\\
&+\sum_{n=1}^{\infty}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\Big]\mathbb{1}_{T_{k-1}\leq s<T_k}\\
=&\Big(\mathbb{E}[\bar{v}^1(t \wedge \tau_m \wedge T_1)|\mathcal{H}^k_{s-T_{k-1}}] &(2.2.12)\\
&+\mathbb{E}\left[\sum_{n=1}^{\infty}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\Big)\mathbb{1}_{T_{k-1}\leq s<T_k}. &(2.2.13)
\end{aligned}$$

For $k = 1$ the first summand is a martingale with respect to the filtration $\mathcal{H}^1_s$ according to corollary 2.3, i.e.

$$\mathbb{E}[\bar{v}^1(t \wedge \tau_m)|\mathcal{H}^1_s]\mathbb{1}_{0\leq s<T_1} = \bar{v}^1(s \wedge \tau_m)\mathbb{1}_{0\leq s<T_1}.$$

For $k \geq 2$ the value of $\bar{v}^1(t)$ is known at time s, i.e.:

$$\mathbb{E}[\bar{v}^1(t \wedge \tau_m)|\mathcal{H}^k_{s-T_{k-1}}]\mathbb{1}_{T_{k-1}\leq s<T_k} = \bar{v}^1(T_1)\mathbb{1}_{T_{k-1}\leq s<T_k}.$$

For the second conditional expectation in (2.2.13) we adopt these insights and decompose the sum into already known values (the jumps that already occured until s), the currently happening single-jump process (the jump right after s) and future single-jump processes, i.e. for $k > 1$:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{n=1}^{\infty}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}\\
=&\mathbb{E}\left[\sum_{n=1}^{k-1}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}\\
&+\mathbb{E}\left[[\bar{v}^k(t \wedge \tau_m - T_{k-1}) - \bar{v}^{k-1}(S_{k-1}]\mathbb{1}_{T_{k-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}\\
&+\mathbb{E}\left[\sum_{n=k+1}^{\infty}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}.
\end{aligned}$$

Now the known sections can easily leave the conditional expectation:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{n=1}^{k-1}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}\\
=&\sum_{n=1}^{k-1}[\bar{v}^n(S_n) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{k-1}\leq s<T_k}
\end{aligned}$$

where we made use of the fact, that $t \wedge T_n = T_n$ for all $n \in \{1, \dots, k-1\}$ on the set $\{T_{k-1} \leq s < T_k\}$ and the processes $\bar{v}^n$ stay constant after their respective jump.
For the current single-jump section we use the martingale property of $\bar{v}^k$ from corollary 2.3:

$$\mathbb{E}\left[[\bar{v}^k(t \wedge \tau_m - T_{k-1}) - \bar{v}^{k-1}(S_{k-1})]\mathbb{1}_{T_{k-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}$$
$$=[\bar{v}^k(s \wedge \tau_m - T_{k-1}) - \bar{v}^{k-1}(S_{k-1})]\mathbb{1}_{T_{k-1}\leq s<T_k}.$$

The future jumps can also be determined sectionwise, and as each section is a martingale w.r.t. its current single-jump filtration (see corollary 2.3) we aim to insert the larger σ-field $\mathcal{H}^m_0$ for $m \geq k+1$ via the tower property of conditional expectation. We observe two different cases:

On the set $\{s < \tau_n < \infty\}$ there exists an integer $j \in \mathbb{N}$ such that $\tau_n = T_{j-1} + s^j_n$. We can further assume that on the intersection with the set $\{T_{k-1} \leq s < T_k\}$ the value of j must be larger than $k-1$. In case of $j = k$ we have $\mathbb{1}_{T_k<t\wedge\tau_n} = \mathbb{1}_{T_k<t\wedge(T_{k-1}+s^k_n)} = 0$ by definition of τ_n (i.e. the process had to be stopped prior to the time T_k). Thus

$$\mathbb{E}\left[\sum_{n=k+1}^{\infty}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k} = 0$$

as all terms are 0.
In case of $j > k$ there are still some indicator function $\mathbb{1}_{T_l<t\wedge\tau_n}$ that possibly contribute to the sum (i.e. there might be jumps that could happen in between s and $t \wedge \tau_n$. But:

$$\mathbb{E}\left[\sum_{n=k+1}^{\infty}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}$$
$$=\mathbb{E}\left[\sum_{n=k+1}^{j}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k}$$
$$\mathbb{E}\left[\sum_{n=j+1}^{\infty}[\bar{v}^n(t \wedge \tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\leq s<T_k},$$

where the latter sum is again 0 by the same argument as in the case $j = k$ (no contribution from jumps after the emergency shutdown τ_n). To see that the former sum also vanishes, we use the tower property with the σ-fields $\mathcal{H}^k_{s-T_{k-1}} \subset \mathcal{H}^{k+1}_0 \subset \cdots \subset \mathcal{H}^j_0$:

$$\mathbb{E}\left[\sum_{n=k+1}^{j} [\bar{v}^n(t\wedge\tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\le s<T_k}$$
$$=\sum_{n=k+1}^{j}\mathbb{E}\left[[\bar{v}^n(t\wedge\tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\le s<T_k}$$
$$=\sum_{n=k+1}^{j}\mathbb{E}\left[\mathbb{E}\left[\left(\bar{v}^n(t\wedge\tau_m - T_{n-1}) - \bar{v}^{n-1}(S_{n-1})\right)|\mathcal{H}^n_0\right]\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\le s<T_k}$$
$$=\sum_{n=k+1}^{j}\mathbb{E}\left[\left(\bar{v}^n(0) - \bar{v}^{n-1}(S_{n-1})\right)\mathbb{1}_{T_{n-1}<t\wedge\tau_m}|\mathcal{H}^k_{s-T_{k-1}}\right]\mathbb{1}_{T_{k-1}\le s<T_k}$$
$$=0$$

by assumption of condition (2.2.9).

To summarize we get for $k>1$ (under $\infty>\tau_m>s$):

$$\mathbb{E}[v(t\wedge\tau_m)|\mathcal{F}_s]\mathbb{1}_{T_{k-1}\le s<T_k}$$
$$=\left(\bar{v}^1(T_1)+\sum_{n=1}^{k-1}[\bar{v}^n(S_n)-\bar{v}^{n-1}(S_{n-1})]+[\bar{v}^k(s\wedge\tau_m-T_{k-1})-\bar{v}^{k-1}(S_{k-1})]\right)\mathbb{1}_{T_{k-1}\le s<T_k}$$
$$=\bar{v}^k(s\wedge\tau_m-T_{k-1})\mathbb{1}_{T_{k-1}\le s<T_k}$$

and for $k=1$:

$$\mathbb{E}[v(t\wedge\tau_m)|\mathcal{F}_s]\mathbb{1}_{0\le s<T_1}=\bar{v}^1(s\wedge\tau_m)\mathbb{1}_{0\le s<T_1}$$

and thus after summing over k and using the definition of v we get:

$$\mathbb{E}[v(t\wedge\tau_m)|\mathcal{F}_s]=v(s\wedge\tau_m).$$

Measurability follows directly from the definition of φ. Every section φ^k is adapted to the filtration $(\mathcal{H}^k_t)$. Given the result of lemma 2.1 the σ-field $\mathcal{F}_s$ coincides with $\mathcal{H}^k_{s-T_{k-1}}$ as well as the function $\varphi(\omega,s,x(s,\omega)))$ does coincide with the section $\varphi^k(\omega_{k-1}(\omega);s-T_{k-1}(\omega),x(s\wedge T_k(\omega))))$ as long as $s\in(T_{k-1}(\omega,T_k(\omega)]$. □

Remark 2.7. *The glue-condition in* (2.2.9) *makes sure, that every function* φ^k *starts at the jump-location of the previous function* φ^{k-1}*. We've seen in the proof of the local martingale-property of* $v(t)$*, that this is in fact a sufficient condition to guarantee local martingales. The process might otherwise be set off by the value of the series*

$$\mathbb{E}\left[\sum_{k=2}^{\infty}\left(r^k(\omega_{k-1})-b^{k-1}(\omega_{k-2};Z_{k-1})\right)\mathbb{1}_{T_{k-1}<t}|\mathcal{F}_s\right]\mathbb{1}_{T_{k-1}>s}.$$

But this offset might also be eliminated by assuming that the expected offset may vanish for each jump.

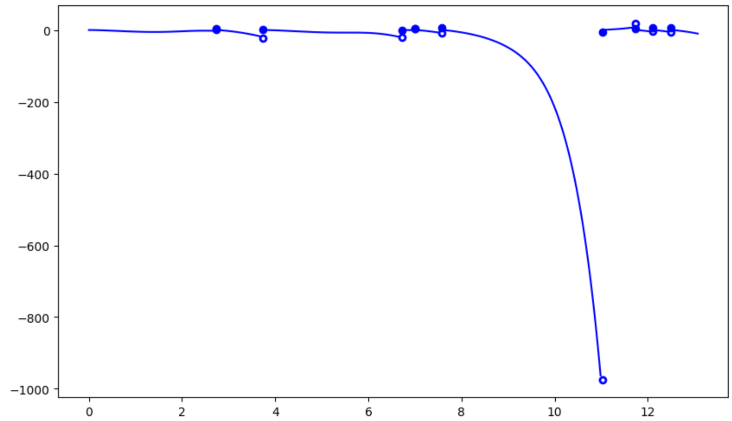

Figure 2.1: Sample path of martingale w.r.t. to the multi-jump process, where each jump is constructed as in example 1.23 (d).

The sequence of stopping times τ_n stops at the first time that the process jumped near or at a right endpoint of a distribution of the jump times. Thus it avoids explosions in the term $\frac{1}{1-F_t^k(\eta)}$ and acts as an emergency break.

2.3 Semi-martingale representation and verification

The result from the last section was a blueprint for functions φ that transform the multi-jump process x into a local martingale w.r.t. to its own natural (augmented) filtration. During the proof we used the result for single-jump processes to determine φ on each section $[T_{k-1}, T_k)$ respectively. But this is not the only result we can adopt for our multi-jump process.
In the first chapter we came up with a semi-martingale representation for the single-jump process. The multi-jump process is a combination of single-jump processes and so is the process $v(t)$ a combination of $\bar{v}^k$ processes (see lemma 2.5 and (2.2.5)). These processes can be decomposed into a martingale part and a previsible compensator:

Corollary 2.8. *Let $1 < k \in \mathbb{N}$, let $\varphi^k : \Omega_{k-1} \times \mathbb{R}^+ \times X \to \mathbb{R}$ s.t. $\varphi^k(\eta; \cdot, \cdot)$ satisfies condition (C') w.r.t. to the measure $\mu^k(\eta; \cdot)$ for all $\eta \in \Omega_{k-1}$. Let $t < c_k(\eta)$. For all $\eta \in \Omega_{k-1}$ each $\bar{v}^k(\eta; t) = \varphi^k(\eta; t, \bar{x}^k(\eta; t))$ can be decomposed into a $\mathcal{H}_t^k$-martingale $M^k(\eta)$ and a previsible compensator $A^k(\eta)$ w.r.t. $\mu^k(\eta; \cdot)$ which are given by*

$$M_t^k(\eta) = \varphi^k(\eta; 0, Z_{k-1}(\eta)) + \int_{(0,t]\times X} \left[\varphi^k(\eta; s, z) - \varphi^k(\eta; s, Z_{k-1}(\eta))\right] dq^k(s, z)$$

$$A_t^k(\eta) = \int_{(0,t]} \frac{d\varphi^k}{dF^k}(\eta; s, Z_{k-1}(\eta)) + \frac{\mathbb{1}_{T_k \geq s}}{1 - F_{s-}^k} \left[m^k(\eta; s) - \varphi^k(\eta; s, Z_{k-1}(\eta))\right] dF_s^k.$$

Proof. We use corollary 1.16 on the process $\bar{v}^k(\omega_{k-1}; t)$ and the natural filtration of $\bar{x}^k(\omega_{k-1}; t)$ which is $\{\omega_{k-1} = \eta\} \cap \bar{\mathcal{F}}_t^k$. □

This yields the following interesting result for the process v:

Corollary 2.9. *The process $v(t) = \varphi(t, x(t))$ can be decomposed as follows:*

$$v(t) = M^1_{t\wedge T_1} + \sum_{k=2}^{\infty} \left[M^k_{(t-T_{k-1})\wedge S_k}(\omega_{k-1}) - M^{k-1}_{S_{k-1}}(\omega_{k-1}) \right] \mathbb{1}_{T_{k-1}\le t}$$
$$+ A^1_{t\wedge T_1} + \sum_{k=2}^{\infty} \left[A^k_{(t-T_{k-1})\wedge S_k}(\omega_{k-1}) - A^{k-1}_{S_{k-1}}(\omega_{k-1}) \right] \mathbb{1}_{T_{k-1}\le t}.$$

Proof. We use the representation from lemma 2.5 and the decomposition from corollary 2.8. □

We take this oppurtunity to verify our result with [Davis, 1976], theorem 2. Davis notes here, that any local martingale w.r.t. $\mathcal{F}_t$ is a stochastic integral of a piecewise defined function $g \in L^1_{\text{loc}}(p)$ against the basic martingale process q. In combination with the choice of φ from theorem 2.6 our semi-martingale representation in corollary 2.9 states that

$$\varphi(t, x(t)) = M^1_{t\wedge T_1} + \sum_{k=2}^{\infty} \left[M^k_{(t-T_{k-1})\wedge T_k}(\omega_{k-1}) - M^{k-1}_{S_{k-1}}(\omega_{k-2}) \right] \mathbb{1}_{T_{k-1}\le t}.$$

Now let $g \in L^1_{\text{loc}}(p)$ with

$$g(\omega, t, z) := \begin{cases} g^1(t, z), & \text{for } t \le T_1(\omega), \\ g^k(\omega_{k-1}; t, z), & \text{for } t \in (T_{k-1}(\omega), T_k(\omega)], \\ 0, & \text{for } t \ge T_\infty(\omega). \end{cases} \tag{2.3.1}$$

From Davis' result we deduce the existence of g for the process $(v(t))_{t\in\mathbb{N}}$. The following choice connects both results:

Corollary 2.10. *For φ defined as in theorem 2.6 we set for $t \in \mathbb{R}^+, z \in X$ and $\omega \in \Omega$:*

$$g^1(t, z) := \varphi^1(t, z) - \varphi^1(t, z_0)$$
$$g^k(\omega_{k-1}; t, z) := \varphi^k(\omega_{k-1}(\omega); t - T_{k-1}(\omega), z) - \varphi^k(\omega_{k-1}(\omega); t - T_{k-1}(\omega); Z_{k-1}(\omega)),$$

and define $g : \Omega \times \mathbb{R}^+ \times X \to \mathbb{R}$ as in (2.3.1). Then it holds that

$$v(t) = \int_{(0,t]\times X} g(s, z) dq(s, z).$$

Proof. We prove the equation by induction over the single-jump sections of the process. Let $t \in \mathbb{R}^+$, $z \in X$. For $\omega \in \{t \in (0, T_1]\}$ we have

$$\begin{aligned} g(\omega, t, z) &= g^1(t, z) \\ &= \varphi^1(t, z) - \varphi^1(t, z_0) \\ &= b^1(z) + \frac{1}{1 - F^1_t} \left(\int_{(0,t]\times X} b^1(y) d\mu^1(s, y) + r^1 \right). \end{aligned}$$

By definition of $dq(s,z)$-integration we have

$$
\begin{aligned}
&\int_{(0,t]\times X} g(s,z)dq(s,z)(\omega) \\
=&\int_{(0,t]\times X} g(s,z)dp(s,z)(\omega) - \int_{(0,t]\times X} g(s,z)d\tilde{p}(s,z)(\omega) \\
=&g^1(T_1(\omega), Z_1(\omega))\mathbb{1}_{T_1\leq t}(\omega) - \int_{(0,t]\times X} g^1(s,z)\frac{1}{1-F^1_{s-}}d\mu^1(s,z).
\end{aligned} \tag{2.3.2}
$$

The definition of g^1 thus yields

$$
g^1(T_1(\omega), Z_1(\omega)) = b^1(Z_1(\omega)) + \frac{1}{1-F^1_{T_1(\omega)}}\left(\int_{(0,T_1(\omega)]\times X} b^1(z)d\mu^1(s,z) + r^1\right) \tag{2.3.3}
$$

and

$$
\begin{aligned}
&\int_{(0,t]\times X} g^1(s,z)\frac{1}{1-F^1_{s-}}d\mu^1(s,z) \\
=&\int_{(0,t]\times X}\left[b^1(z) + \frac{1}{1-F^1_s}\left(\int_{(0,s]\times X} b^1(y)d\mu^1(u,y) + r^1\right)\right]\frac{1}{1-F^1_{s-}}d\mu^1(s,z) \\
=&\int_{(0,t]\times X} b^1(z)\frac{1}{1-F^1_{s-}}d\mu^1(u,y) \\
&\quad + \int_{(0,t]\times X}\frac{1}{1-F^1_s}\frac{1}{1-F^1_{s-}}\int_{(0,s]\times X} b^1(y)d\mu^1(u,y)\mu^1(s,z) \\
&\quad + \int_{(0,t]\times X} r^1\frac{1}{1-F^1_s}\frac{1}{1-F^1_{s-}}d\mu^1(s,z) \\
\overset{(a)}{=}&\frac{1}{1-F^1_t}\int_{(0,t]\times X} b^1(z)d\mu^1(s,z) + r^1\left(\frac{1}{1-F^1_t} - 1\right)
\end{aligned} \tag{2.3.4}
$$

where we've used

(a) We can compute the second integral with the help of Fubini's theorem:

$$
\begin{aligned}
&\int_{(0,t]\times X}\frac{1}{1-F^1_s}\frac{1}{1-F^1_{s-}}\int_{(0,s]\times X} b^1(y)d\mu^1(u,y)d\mu^1(s,z) \\
=&\int_{(0,t]}\frac{1}{1-F^1_s}\frac{1}{1-F^1_{s-}}\int_{(0,s]\times X} b^1(y)d\mu^1(u,y)dF^1_s \\
=&\int_{(0,t]\times X} b^1(y)\left(\int_{[u,t]}\frac{1}{1-F^1_s}\frac{1}{1-F^1_{s-}}dF^1_s\right)d\mu^1(u,y) \\
=&\int_{(0,t]\times X} b^1(y)\left(\frac{1}{1-F^1_t} - \frac{1}{1-F^1_{u-}}\right)d\mu^1(u,y) \\
=&\frac{1}{1-F^1_t}\int_{(0,t]\times X} b^1(z)d\mu^1(s,z) - \int_{(0,t]\times X} b^1(z)\frac{1}{1-F^1_{u-}}d\mu^1(s,z)
\end{aligned}
$$

and the third integral with

$$\begin{aligned}\int_{(0,t]\times X} r^1 \frac{1}{1-F^1_s}\frac{1}{1-F^1_{s-}} d\mu^1(s,z) &= r^1 \int_{(0,t]\times X} \frac{1}{1-F^1_s}\frac{1}{1-F^1_{s-}} dF^1_s \\ &= r^1\left(\frac{1}{1-F^1_t} - \frac{1}{1-F^1_0}\right) \\ &= r^1\left(\frac{1}{1-F^1_t} - 1\right).\end{aligned}$$

Inserting (2.3.3) und (2.3.4) into (2.3.2) yields

$$\begin{aligned}&\int_{(0,t]\times X} g(s,z)dq(s,z) \\ &= \left[b^1(Z_1(\omega)) + \frac{1}{1-F^1_{T_1(\omega)}}\left(\int_{(0,T_1(\omega)]\times X} b^1(z)d\mu^1(s,z) + r^1\right)\right]\mathbb{1}_{T_1\le t} \\ &\quad - \frac{1}{1-F^1_t}\int_{(0,t]\times X} b^1(z)d\mu^1(s,z) + r^1\left(\frac{1}{1-F^1_t} - 1\right)\end{aligned}$$

more precisely for $t = T_1$:

$$\int_{(0,T_1]\times X} g(s,z)dq(s,z) = b^1(Z_1) + r^1 \tag{2.3.5}$$

and for $t < T_1$:

$$\int_{(0,t]\times X} g(\omega,s,z)dq(s,z)(\omega) = -\frac{1}{1-F^1_t}\int_{(0,t]\times X} b^1(z)d\mu^1(s,z) + r^1\left(\frac{1}{1-F^1_t} - 1\right). \tag{2.3.6}$$

We also note that for $t = T_1$

$$\begin{aligned}v(t) - v(0) = v(T_1) - v(0) &= \varphi(T_1, x(T_1)) - \varphi(0, x(0)) \\ &= \varphi^1(T_1, Z_1) - \varphi^1(0, z_0) \\ &= b^1(Z_1) + r^1\end{aligned}$$

and for $t < T_1$ we have that $x(t) = z_0$, hence

$$v(t) - v(0) = \varphi^1(t, z_0) - \varphi^1(0, z_0) = -\frac{1}{1-F^1_t}\left(\int_{(0,t]\times X} b^1(z)d\mu^1(s,z) + r^1\right) + r^1$$

which proves the assertion for $n = 1$.
Now assume that the assertion holds for some $n \in \mathbb{N}$. Take $\omega \in \{t \in (T_n, T_{n+1}]\}$. Then by definition of g:

$$\begin{aligned}&g(t,z) \\ =&g^{n+1}(\omega_n; t, z) \\ =&\varphi^{n+1}(\omega_n; t, z) - \varphi^{n+1}(\omega_n; t, Z_n) \\ =&b^{n+1}(\omega_n; z) + \frac{1}{1-F^{n+1}_t(\omega_n)}\left(\int_{0,t-T_n]\times X} b^{n+1}(\omega_n; y)d\mu^{n+1}(\omega_n; u, y) + r^{n+1}(\omega_n)\right).\end{aligned}$$

Now by definition of $dq(s,z)$-integration we have

$$\begin{aligned}&\int_{(0,t]\times X} g(s,z)dq(s,z)\\ =&\sum_{k=1}\int_{(T_{k-1},T_k]\times X} g^k(\omega_{k-1};s,z)dq(s,z)+\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dq(s,z).\end{aligned}$$

For each $k\in\{1,\dots,n\}$ we now by the induction assumption that:

$$\int_{(T_{k-1},T_k]\times X} g^k(\omega_{k-1};s,z)dq(s,z)=b^k(\omega_{k-1};Z_k)+r^k(\omega_{k-1})$$

thus the first n integrals reduce under the 'glue-condition' (2.2.9) to

$$\begin{aligned}\sum_{k=1}^{n}\int_{(T_{k-1},T_k]\times X} g^k(\omega_{k-1};s,z)dq(s,z)&=\sum_{k=1}^{n}\left[b^k(\omega_{k-1};Z_k)+r^k(\omega_{k-1})\right]\\ &=\sum_{k=1}^{n}\left[-r^{k+1}(\omega_k)+r^k(\omega_{k-1})\right]\\ &=-r^{n+1}(\omega_n)+r^1. \end{aligned}\tag{2.3.7}$$

For the last integral we note:

$$\begin{aligned}&\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dq(s,z)\\ =&\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dp(s,z)-\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)d\tilde{p}(s,z)\end{aligned}$$

where

$$\begin{aligned}&\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dp(s,z)\\ =&g^{n+1}(\omega_n;T_{n+1},Z_{n+1})\mathbb{1}_{T_{n+1}\le t}\\ =&\left[\varphi^{n+1}(\omega_n;S_{n+1},Z_{n+1})-\varphi^{n+1}(\omega_n;S_{n+1},Z_n)\right]\mathbb{1}_{T_{n+1}\le t}\\ =&\Bigg[b^{n+1}(\omega_n;Z_{n+1})\\ &\quad+\frac{1}{1-F^{n+1}_{T_{n+1}}(\omega_n)}\left(\int_{(0,t-T_n]\times X} b^{n+1}(\omega_n;y)d\mu^{n+1}(\omega_n;u,y)+r^{n+1}(\omega_n)\right)\Bigg]\mathbb{1}_{T_{n+1}\le t}\end{aligned}$$

and (under omission of ω_n-dependencies for better readability)

$$\begin{aligned}
&\int_{(T_n,t]\times X} g^{n+1}(s,z)d\tilde{p}(s,z) \\
=&\int_{(0,t-T_n]\times X} \left[\varphi^{n+1}(s,z)-\varphi^{n+1}(s,Z_n)\right]\frac{1}{1-F^{n+1}_{s-}}d\mu^{n+1}(s,z) \\
=&\int_{(0,t-T_n]\times X} \Bigg[b^{n+1}(z)+\frac{1}{1-F^{n+1}_{s}} \\
&\quad\times\left(\int_{(0,s]\times X} b^{n+1}(y)d\mu^{n+1}(u,y)+r^{n+1}\right)\Bigg]\frac{1}{1-F^{n+1}_{s-}}d\mu^{n+1}(s,z) \\
=&\int_{(0,t-T_n]\times X} b^{n+1}(z)\frac{1}{1-F^{n+1}_{s-}}d\mu^{n+1}(s,z) \\
&\quad+\int_{(0,t-T_n]\times X}\int_{(0,s]\times X} b^{n+1}(y)d\mu^{n+1}(u,y)\frac{1}{1-F^{n+1}_{s}}\frac{1}{1-F^{n+1}_{s-}}d\mu^{n+1}(s,z) \\
&\quad+\int_{(0,t-T_n]\times X} r^{n+1}\frac{1}{1-F^{n+1}_{s}}\frac{1}{1-F^{n+1}_{s-}}d\mu^{n+1}(s,z) \\
\overset{(b)}{=}&\frac{1}{1-F^{n+1}_{t}}\int_{(0,t-T_n]\times X} b^{n+1}(z)d\mu^{n+1}(s,z)+r^{n+1}\left(\frac{1}{1-F^{n+1}_{t}}-1\right)
\end{aligned}$$

where we've used in particular:

(b) Both - the second and the third - integral can again be computed as before in (a) with the help of Fubini's theorem.

We combine the above p- and $\tilde{p}$-integrals again to yield

$$\begin{aligned}
&\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dq(s,z) \\
=&\Bigg[b^{n+1}(\omega_n;Z_{n+1}) \\
&\quad+\frac{1}{1-F^{n+1}_{T_{n+1}}(\omega_n)}\left(\int_{(0,T_{n+1}]\times X} b^{n+1}(\omega_n;y)d\mu^{n+1}(\omega_n;u,y)+r^{n+1}(\omega_n)\right)\Bigg]\mathbb{1}_{T_{n+1}\le t} \\
&\quad-\frac{1}{1-F^{n+1}_{t}(\omega_n)}\left(\int_{(0,t-T_n]\times X} b^{n+1}(z)d\mu^{n+1}(\omega_n;s,z)+r^{n+1}(\omega_n)\right)+r^{n+1}(\omega_n).
\end{aligned}$$

In particular for $t=T_{n+1}$:

$$\begin{aligned}
\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dq(s,z) &= b^{n+1}(\omega_n,Z_{n+1})+r^{n+1} \\
&=\varphi^{n+1}(\omega_n;t-T_n,Z_{n+1})+r^{n+1} \qquad (2.3.8)
\end{aligned}$$

and for $t\in(T_n,T_{n+1})$:

$$\int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dq(s,z) \qquad (2.3.9)$$

$$=-\frac{1}{1-F^{n+1}_{t}(\omega_n)}\left(\int_{(0,t-T_n]\times X} b^{n+1}(z)d\mu^{n+1}(\omega_n;s,z)+r^{n+1}(\omega_n)\right)+r^{n+1}(\omega_n)$$

$$=\varphi^{n+1}(\omega_n;t-T_n,Z_n)+r^{n+1}(\omega_n). \qquad (2.3.10)$$

Together with (2.3.7) and the results in (2.3.8) and (2.3.9) we conclude:

$$\begin{aligned}\int_{(0,t]\times X} g(s,z)dq(s,z) &= -r^{n+1}(\omega_n) + r^1 + \int_{(T_n,t]\times X} g^{n+1}(\omega_n;s,z)dq(s,z)\\ &= -r^{n+1}(\omega_n)\varphi^{n+1}(\omega_n;t-T_n,x(t)) + r^{n+1}(\omega_n) - \varphi(0,z_0)\\ &= v(t) - v(0).\end{aligned}$$

□

Chapter 3

Application example

As a complement to the first two chapters we now discuss an application that has not yet been covered by the examples during the first 2 chapters. We want to investigate the branching random walk, i.e. a process that describes an ensemble of particles that at random times $T_1, T_2, \ldots$ either branch into 2 particles or die. The 2 possibly born particles may choose randomly and independent of each other a location in $\mathbb{Z}$ and are able to branch themselves in the future. The notational framework of the previous chapters combined with the construction of a multi-jump process from single-jump sections allow us to describe a branching process with very flexible assumptions on branching times, -mechanisms and birth distributions. Some examples will be provided where the particles are independent and non interacting, as well as an example of a generalization to path dependent particle systems.

3.1 Single-branch random walk

3.1.1 Definitions

The individuals of our branching random walk may take values in $\mathbb{Z}$, i.e. a particle may reside at 0 and then either randomly branches (or reproduces) into two new particles, each at a random location 'around' zero (i.e. in our case $\{-1, 1\}$) or the particle dies and leaves no descendents behind. Take $\tilde{\Omega}_1 := \bigcup_{n\in\mathbb{N}} \mathbb{Z}^n$ the set of all vectors of integer valued coordinates, and let any vector $y = (y_1, \ldots, y_n)$ describe a population where n individuals are alive and at the locations $y_1, \ldots, y_n$. We want to describe the branching process as a measure valued process, i.e. let us describe the associated discrete measure w.r.t. a particle distribution $(y_1, \ldots, y_n)$ as:

$$\hat{y} = \sum_{i=1}^{n} \delta_{y_i}.$$

The set of measures of this form may be denoted by $X := \{\sum_{i=1}^{n} \delta_{y_i} : n \in \mathbb{N}, y_1, \ldots, y_n \in \mathbb{Z}\}$. We choose the Prochorow metric d_P to define a topology on X (Note that $(\mathbb{Z}, |\cdot|)$ is trivially separable and thus (X, d_P) is separable. Hence the topology is equivalent to the weak topology). Take $\mathcal{S} := \mathcal{B}(X)$ to be the Borel-σ-algebra w.r.t. this topology. $(X, \mathcal{S})$ is thus a Blackwell space, since $(X, \mathcal{S})$ is separable.

Next we combine the set X with a time-frame. Set

$$\Omega := \mathbb{R}^+ \times X, \mathcal{A} := \mathcal{B}(\mathbb{R}^+) \star \mathcal{S}.$$

Our branching random walk will take its values in X whereas the time of value-changes (i.e. the jump time or *branch time*) will take its values in $\mathbb{R}^+$. A member $\omega \in \Omega$ thus looks like:

$$\omega = (t, z)$$

where z is a linear combination of Dirac-measures on $\mathbb{Z}$ and t is a nonnegative time. Let $Z : \Omega \to X$ and $T : \Omega \to \mathbb{R}^+$ be the coordinate mappings s.t. $\omega = (T(\omega), Z(\omega))$. Fix an initial value $z_0 \in X$ (where we explore $z_0 = \delta_0$ as a simple example and $z_0 = \sum_{n=1}^{K} a_n \delta_{y_n}$ as a more general case below). We define the single-branch random walk at time t as

$$x(t, \omega) = \begin{cases} z_0, & t < T(\omega) \\ Z(\omega), & T(\omega) \leq t. \end{cases}$$

Thus the process remains on its initial measure z_0 until the random time T happens and the process changes its value to the random measure Z.
The natural filtration of the above process will be denoted by $\mathcal{F}_t^0 := \sigma\{x(s) : s \in [0, t]\}$. Note that for any $t \in \mathbb{R}^+$ this σ-algebra is not yet complete. We want to exclude cases of invisible branches and jumps at time zero and thus augment the filtration with the respective events of measure zero in a few lines. We define a probability measure on $(\Omega, \mathcal{A})$ by the definition

$$\mathbb{P}((Z, T) \in \gamma) = \mu(\gamma)$$

where μ is a probability measure (the joint distribution of T and Z) and $\gamma \in \mathcal{A}$. Additionally μ may have the property $\mu(\{z_0\} \times \mathbb{R}^+) = \mu(X \times \{0\}) = 0$. Let $F_t := \mathbb{P}((Z, T) \in X \times [0, t])$ denote the marginal distribution function of T.
Denote by $\mathcal{N}^0$ the set of all $\mathbb{P}$-nullsets and augment the natural filtration with these sets:

$$\mathcal{F}_t := \sigma(\mathcal{F}_t, \mathcal{N}^0).$$

Now let us take a function $\psi : X \to \mathbb{R}$. We denote the space of locally integrable functions by

$$L^1_{\text{loc}}(\mu) := \{\psi : X \to \mathbb{R} : \int_{(0,t] \times X} |\psi(z)| d\mu(u, z) < \infty, \forall t < c\}.$$

3.1.2 Martingales of the Single-branch random walk

While the single-branch case might seem uninteresting it yields the key to a general multi-branch random walk. The initial measure z_0 can be chosen arbitrary in X, in particular it can be chosen to be the measure one ends up with after another single-branch process has branched, which will be the approach for our generalization to multi-branch processes. But before we start constructing a multi-branch random walk, we first assert the results from the previous chapters to this particular single-jump process.

Let $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$ be a measurable function. For $t \in \mathbb{R}^+$ define $v(t) = \varphi(t, x(t))$. The process $v = (v(t))_{t \in \mathbb{R}^+}$ is adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ as long as φ is measurable, but can we pose additional conditions on φ s.t. v is even a $\mathcal{F}_t$-martingale? The answer is given in theorem 1.14, but the application to the branching process is in itself an interesting topic.

We can choose an arbitrary function $b \in L^1_{\text{loc}}(\mu)$ and construct a martingale with respect to the augmented natural filtration of the branching random walk by setting

$$\varphi(t,y) := b(y)(1-\delta_{z_0}) - \delta_{z_0}\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(u,z) + r\right).$$

The Dirac-measure $\delta_{z_0} : X \to \{0,1\}$ for some distribition $z_0 = \sum_{n=1}^K \delta_{y_n} \in X$ is to be read as

$$\delta_{z_0}(\nu) = \prod_{i=1}^n \int_{\{y_i\}} d\nu = \begin{cases} 1, & y = z_0 \\ 0, & y \neq z_0. \end{cases}$$

Let μ be a measure on a measurable space $(\Omega, \mathcal{A})$. For any μ-integrable function $f : \Omega \to \mathbb{R}$ we write

$$\langle f, \mu\rangle := \int_\Omega f(\omega)d\mu(\omega).$$

Note that all assumptions on branching time and distribution of descendents are located in μ (and F). We will explore some combinations of these assumptions now.

One initial particle Let $z_0 = \delta_0$, i.e. we assume that the initial distribution is just one particle. The situation after the first branching-time is given by the possible populations (and their measure-valued description) $\{0, z_1, \dots\}$, where $z_i \in X$. So far we can assert that the state space is discrete. Given any time $t \in \mathbb{R}^+$ the situation the single-branch random walk is currently in can be described by the following sets:

$$\{T > t\}, \{T \le t, Z = 0\}, \{T \le t, Z = z_1\}, \dots$$

Noting that these sets are pairwise disjoint we can decompose the integral w.r.t. μ in the following series:

$$\int_{(0,t]\times X} b(z)d\mu(u,z) = \int_{(0,t]\times\{0\}} b(0)d\mu(u,z) + \sum_{i=1}^\infty \int_{(0,t]\times\{z_1\}} b(z_1)d\mu(u,z).$$

We now assume further, that the particle either dies or splits in two. Then the set of possible populations is given by

$$\{0\} \cup \{z \in X : z = \delta_{y_1} + \delta_{y_2}, y_1, y_2 \in \mathbb{Z}\} =: \{0\} \cup X^{[2]}$$

The above integral becomes

$$\int_{(0,t]\times X} b(z)d\mu(u,z) = \int_{(0,t]\times\{0\}} b(0)d\mu(u,z) + \sum_{\nu\in X^{[2]}} \int_{(0,t]\times\nu} b(\nu)d\mu(u,z).$$

Another assumption on the possible locations of birth might be, that $y_1, y_2 \in \{-1, 1\}$, i.e. that the particle branches either up or down of its current location. Under these circumstances the set of possible populations after the reproductions time is given explicitly by:

$$\{0, 2\delta_{-1}, \delta_{-1} + \delta_1, 2\delta_1\}$$

and thus the situation at time t can be described by the following partition:

$$\{T > t\}, \{T \leq t, Z = 0\}, \{T \leq t, Z = 2\delta_{-1}\}, \{T \leq t, Z = \delta_{-1} + \delta_1\}, \{T \leq t, T = 2\delta_1\}.$$

The probability measure μ can be uniquely described by the values on each of these generating sets at any time and we can now decompose the integral in our martingale paradigm as follows:

$$\begin{aligned} \int_{(0,t]\times X} b(z)d\mu(u,z) = &\int_{(0,t]\times\{0\}} b(z)d\mu(u,z) \\ &+ \int_{(0,t]\times\{2\delta_{-1}\}} b(z)d\mu(u,z) \\ &+ \int_{(0,t]\times\{\delta_{-1}+\delta_1\}} b(z)d\mu(u,z) \\ &+ \int_{(0,t]\times\{2\delta_1\}} b(z)d\mu(u,z) \end{aligned} \tag{3.1.1}$$

Each of these integrals can be computed, given the explicit description of the measure μ.
We assume now that the branching mechanism is independent of the branching time and thus are able to describe them even more precisely because

$$\mu((0,t] \times \{z\}) = \mu((0,t] \times X)\mu(\mathbb{R}^+ \times \{z\}).$$

For example one of the above integrals is then computed as

$$\int_{(0,t]\times\{0\}} b(z)d\mu(u,z) = \int_{(0,t]} b(0)dF_u \cdot \mathbb{P}(Z = 0).$$

The marginal distributions of reproduction and branching time now play a major role in the construction of martingales. More assumptions on the branching probability and/or the locations of the descendants target the marginal distribution of Z. For example we might assume that the particle has a certain probability of branching denoted by $r \in (0,1)$. Then the probability of dying is given by $1 - r$. Further we may assume that in the event of branching the two descendents choose their place of birth independently of each other from $\{-1, 1\}$ and let us denote the probability of choosing 1 as birthplace with $u \in (0,1)$. Then the marginal distribution of Z is given by the following characterizing values:

$$\begin{aligned} \mathbb{P}(Z = 0) &= 1 - r \\ \mathbb{P}(Z = 2\delta_{-1}) &= r(1-u)^2 \\ \mathbb{P}(Z = 2\delta_1) &= ru^2 \\ \mathbb{P}(Z = \delta_{-1} + \delta_1) &= ru(1-u) + r(1-u)u = 2ru(1-u). \end{aligned}$$

The integral in (3.1.1) then becomes:

$$\int_{(0,t]\times X} b(z)d\mu(u,z) = (1-e^{-\lambda t})\Big[(1-r)b(0) \\ + r(1-u)^2 b(2\delta_{-1}) \\ + ru^2 b(2\delta_1) \\ + 2ru(1-u)b(\delta_{-1}+\delta_1)Big].$$

So far we've tried to illuminate the impact of assumptions on the result from chapter 1. Next we turn to further explore the applications by explicit models:

Example 3.1. *Take a single-branch random walk that starts with one individual at location 0. Assume that T and Z are independent. Let $T \sim Exp(\lambda)$ and $\mathbb{P}(Z=0) = \frac{1}{2} = \mathbb{P}(Z = \delta_1 + \delta_{-1}|Z \neq 0)$ and $\mathbb{P}(Z = 2\delta_1|Z \neq 0) = \frac{1}{4} = \mathbb{P}(Z = \delta_{-1}|Z \neq 0)$. Now choose $b(\nu) = \langle 1, \nu\rangle = \sum_{n\in\mathbb{Z}} \nu(n)$ and $r = 0$. Then*

$$v(t) = \sum_{n\in\mathbb{Z}} Z(\omega)(n)\mathbb{1}_{T\leq t} - \mathbb{1}_{T>t}\exp(\lambda t)\left[\int_{(0,t]\times X}\left(\sum_{n\in\mathbb{Z}}\nu(n)\right)d\mu(u,\nu) + r\right].$$

is a martingale w.r.t $\mathcal{F}_t$. The sum $\sum_{n\in\mathbb{Z}}\nu(n)$ can be evaluated at the finite values of Z that have a positive probability, i.e. $\{Z=0\},\{Z=\delta_1+\delta_{-1}\},\{Z=2\delta_1\},\{Z=2\delta_{-1}\}$. Thus we end up with

$$\sum_{n\in\mathbb{Z}}(\delta_1+\delta_{-1})(n) = \sum_{n\in\mathbb{Z}}(2\delta_1)(n) = \sum_{n\in\mathbb{Z}}(2\delta_{-1})(n) = 2$$

and the trivial $\sum_{n\in\mathbb{Z}} 0(n) = 0$.
Further we can represent the first term of the process $v(t)$ as

$$b(x(t,\omega))\mathbb{1}_{T\leq t}(\omega) = \sum_{n\in\mathbb{Z}} Z(\omega)(n)\mathbb{1}_{T\leq t}(\omega) = 2\mathbb{1}_{\{Z\neq 0\}\cap\{T\leq t\}}.$$

The second term of $v(t)$ is given by

$$\exp(\lambda t)\left[\int_{(0,t]\times X} b(z)d\mu(u,z)\right] = \exp(\lambda t)\left[2\int_{(0,t]} 2dF_u\mathbb{P}(Z\neq 0)\right] \\ = \frac{2}{2}\exp(\lambda t)\left[1-\exp(-\lambda t)\right] \\ = \exp(\lambda t) - 1.$$

For the above martingale this yields:

$$v(t) = 2\mathbb{1}_{T\leq t, Z\neq 0} - \mathbb{1}_{T>1}(\exp(\lambda t)-1). \tag{3.1.2}$$

The first term is a process that counts the members of the population at time t, given that the process branched already. The second term states the compensator for this event.

Example 3.2. *Under the assumption that the behaviour of the particles descendents depend in some way on the time of reproduction, the model is less explicit:*

$$v(t) = b(x(t))\mathbb{1}_{T\leq t} - \mathbb{1}_{T>t}\frac{1}{1-F_t}\left(\int_{(0,t]\times X} b(z)d\mu(u,z) + r\right) \tag{3.1.3}$$

is a local martingale w.r.t. $\mathcal{F}_t$. *Take for example* $b \in L^1_{loc}(\mu)$ *defined by*

$$\begin{aligned} b(0) &= 0 \\ b(2\delta_{-1}) &= -1 \\ b(\delta_{-1}+\delta_1) &= 0 \\ b(2\delta_1) &= 1 \end{aligned}$$

The measurable space $\mathbb{R}^+ \times X$ *decomposes into 5 disjoint subsets:*

$$\mathbb{R}^+ \times X = \mathbb{R}^+ \times \{0\} \cup \mathbb{R}^+ \times \{2\delta_{-1}\} \cup \mathbb{R}^+ \times \{\delta_{-1}+\delta_1\} \cup \mathbb{R}^+ \times \{2\delta_1\} \cup \mathbb{R}^+ \times \mathcal{N}$$

where $\mathcal{N} = X\setminus\{0, 2\delta_{-1}, \delta_{-1}+\delta_1, 2\delta_1\}$ *is a* μ*-nullset. Denote by* $\mu_s(z) = \mathbb{P}(Z = z|T = s)$ *the conditional distribution of* Z *given the value of* T. *The above martingale then becomes:*

$$v(t) = b(Z)\mathbb{1}_{T\leq t} - \mathbb{1}_{T>t}\frac{1}{1-F_t}\left(\int_{(0,t]} [\mu_s(2\delta_1) - \mu_s(2\delta_{-1})]dF_s + r\right).$$

As the single-branch random walk is only of very limited use and most of the interesting features of such a process appear in the multi-branch case, we take the opportunity to analyze the features and information involved in a single-branch. If we would know the location of our first ancestor, we are able to deduce a martingale w.r.t. to this single ancestors lifetime. The time of branching (the life-time) and the behavior at it's death are directly depending on this information (and not on any other individuals life). Thus we can very locally compensate for each of these ancestor-dependent information.

A set of initial particles: Were we to analyze a single-branch process for a set of ancestors, one problem would be to choose, which ancestor is randomly the first to branch. As an illustrating example we think of two particles at two arbitrary positions $y_1, y_2 \in \mathbb{Z}$. Each of these particles may branch independent of the other particle, and let us assume that each particle has its own lifetime $T^{(1)}, T^{(2)}$ with marginal distributions $F_t^{(1)}$ and $F_t^{(2)}$ respectively. The time of the first branch is given by $T := \min(T^{(1)}, T^{(2)})$ and its distribution is given by

$$\mathbb{P}(T \leq t) = 1 - (1 - F_t^{(1)})(1 - F_t^{(2)}).$$

Now there is one difficulty that could arise in this simple example. Let us assume that the two particles start in the same location, say 0. After the first branching the population is described by one of the following measures:

$$\{\delta_0, 2\delta_{-1} + \delta_0, \delta_{-1} + \delta_0 + \delta_1, \delta_0 + 2\delta_1\}$$

and the probability of each of these populations is composed of two different situations: either the particle y_1 branches first, or the other particle y_2 branches first. As a result the population after the branching consists of descendents of either particle y_1 or of particle y_2, but the respective other not-yet-branched particle lives on. In the single-branch random walk case this is no problem at all. We can compute the transition probabilities for each possible population post-branch and hence describe the probability measure μ same as before on each of the different populations. But keep in mind that we want to generalize the model to multi-branch random walks. And our strategy to fuze single-branch random walks together relies on information about the initial populations at each intermediate branching-time. In the situation described above we could not tell from the population after the first branching, which of the particles y_1, y_2 reproduced and thus cannot describe the distribution of the particles lifetimes. There are two ways to overcome this obstacle, one of which involves using more information to describe the processes history (for example the age of each particle), the other is to impose that each particle has the same exponential lifetime (thus a constant branching rate).

Let us assume that for $2 \leq K \in \mathbb{N}$ the initial measure $z_0 = \sum_{n=1}^{K} a_n \delta_{z_n}$ where $a_n \in \mathbb{N}$ and $z_n \in \mathbb{Z}$ for all $n \in \{1, \dots, K\}$. The total initial population size is given by $N := \sum_{n=1}^{K} a_n$. Now the probability involved in branching is also a question of who is branching. To get the idea for the distributional assumptions we think of the process right before and right after the branching:

$$x(T) - x(T-) = \begin{cases} -\delta_l, & \text{if a particle at } l \text{ died} \\ \delta_{l+r_1} + \delta_{z_l+r_2} - \delta_{z_l}, & \text{if a particle at } l \text{ branched, } r_1, r_2 \in \{-1, 1\}. \end{cases}$$

Let us denote $\mathcal{X}$ as the set of all possible populations after the first branch. Choose $b \in L^1_{\text{loc}}(\mu)$, then we get that for $T \sim \text{Exp}(\lambda)$

$$v(t) = b(Z)\mathbb{1}_{T \leq t} - \mathbb{1}_{T>t} \frac{1}{\exp(-\lambda N t)} \left[\int_{(0,t] \times X} b(z) d\mu(u, z) + r \right]$$

is a local $\mathcal{F}_t$-martingale. We further decompose the integral:

$$\begin{aligned} \int_{(0,t] \times X} b(z) d\mu(u, z) &= \int_{(0,t]} \int_X b(z) d\mu_u(z) dF_u \\ &= \int_{(0,t]} \sum_{z \in \mathcal{X}} b(z) \mu_u(z) dF_u \end{aligned} \tag{3.1.4}$$

We have N particles, each with an exponential branching rate. Thus the branching time occurs at $T = \min\{S^{(1)}, \dots, S^{(N)}\}$ where $S^{(i)}$ are i.i.d. $\text{Exp}(\lambda)$ random variables. Then a random particle location is chosen from the initial particle distribution $\{z_1, \dots, z_K\}$ where each particle is equally likely to branch (under the i.i.d. assumptions of their lifetimes). The probability to choose a particle in the location $l \in \mathbb{Z}$ is given by $\frac{a_l}{N}$. Eventually the type of branching is decided randomly, so either the particle dies (probability $1 - r$) or branches (probability r), and if so how the descendents choose their locations of birth, where u denotes the probability that a born particle is located at $l + 1$.

Each of these random events is assumed to be independent from the others and then the probability measure μ is characterized as follows:

$$\begin{aligned}
\mathbb{P}(x(t) = z_0 - \delta_{z_l} + 2\delta_{z_l+1}) &= (1 - \exp(-\lambda N t))\frac{a_l}{N} r u^2 \\
\mathbb{P}(x(t) = z_0 - \delta_{z_l} + 2\delta_{z_l-1}) &= (1 - \exp(-\lambda N t))\frac{a_l}{N} r (1-u)^2 \\
\mathbb{P}(x(t) = z_0 - \delta_{z_l} + \delta_{z_l+1} + \delta_{z_l-1}) &= (1 - \exp(-\lambda N t))\frac{a_l}{N} 2ru(1-u) \\
\mathbb{P}(x(t) = z_0 - \delta_{z_l}) &= (1 - \exp(-\lambda N t))\frac{a_l}{N}(1-r) \\
\mathbb{P}(x(t) = z_0) &= \exp(-\lambda N t).
\end{aligned}$$

Hence the sum in (3.1.4) becomes

$$\begin{aligned}
\sum_{z\in\mathcal{X}} b(z)\mu_u(z) = \sum_{l\in\mathbb{Z}} \Big[& b(z_0 - \delta_l)\frac{a_l}{N}(1-r) \\
& + b(z_0 - \delta_l + 2\delta_{l-1})\frac{a_l}{N} r (1-u)^2 \\
& + b(z_0 - \delta_l + 2\delta_{l+1})\frac{a_l}{N} r u^2 \\
& + b(z_0 - \delta_l + \delta_{l-1} + \delta_{l+1})\frac{2a_l}{N} r u (1-u)\Big].
\end{aligned} \tag{3.1.5}$$

Remark 3.3. *(i) In the above discussion the values N and a_l are defined beforehand, but they can also be computed from the initial population by*

$$N = \langle 1, z_0 \rangle, \quad a_l = \langle \delta_{\{z_l\}}, z_0 \rangle.$$

(ii) While the i.i.d. assumptions on the particles lifetimes is explored here for Exp(λ) distributions, the result also holds true for other lifetime distributions with non-constant rates. But as mentioned before the constant branching rate is mandatory for the description of multi-branch random walks in this simple setting.

Example 3.4. *Let $r = \frac{1}{2} = u = 1 - u$ and let $z_0 = \delta_{-1} + \delta_1$. Take $b(\nu) := \langle 1, \nu \rangle$. Then*

$$v(t) = \langle 1, Z \rangle \mathbb{1}_{T\le t} - \mathbb{1}_{T>t}\frac{1}{1 - F_t}\int_{(0,t]\times X} b(z)d\mu(u,z)$$

is a local $\mathcal{F}_t$ martingale, where

$$\begin{aligned}
\int_{(0,t]\times X} b(z)d\mu(u,z) &= (1 - e^{-\lambda 2t})2\left[\frac{1}{4} + \frac{3}{16} + \frac{3}{16} + \frac{6}{16}\right] \\
&= 2(1 - e^{-\lambda 2t})
\end{aligned}$$

and

$$\langle 1, Z \rangle \mathbb{1}_{T\le t} = \mathbb{1}_{\{T\le t, Z=\delta_{-1}\}\cup\{T\le t, Z=\delta_1\}} + 3\mathbb{1}_{\{T\le t, Z\ne\delta_{-1}\}\cup\{T\le t, Z\ne\delta_1\}}$$

3.2 Multi-branch

Now we consider the general multi-branch case. We follow the same reasoning as in chapter 2, i.e. we glue together instances of single-branch processes to construct a general branching process. The correlation of the different branching processes is preserved and described by the conditional distributions. The necessity of a constant branching rate is due to the fact that we use the single-branch processes from section 3.1 with initial distributions $z_0 = \sum_{n=1}^{K} a_n \delta_{z_n}$ (see section 3.1). We collect some notation first and define the conditional probability measures accordingly.
Throughout this section we assume that the particles branch independent of each other, i.e. the branching times of the particles are independent of each other and that the branching mechanisms of the particles are independent of each other. Also we assume that the branching mechanism of any particle is independent of the branching time. As in chapter 2 we take copies $(Y_n, \mathcal{Y}_n)$ of the template space $(Y, \mathcal{Y})$ from section 3.1 and define

$$\Omega := \prod_{n\in\mathbb{N}} Y_n, \quad \mathcal{F}^0 := \sigma\left(\prod_{n\in\mathbb{N}} \mathcal{Y}_n\right).$$

The process is given by

$$x(t)(\omega) := \begin{cases} z_0, & t < T_1(\omega) \\ Z_k(\omega), & t \in [T_k(\omega), T_{k_1}(\omega)) \\ z_\infty, t \leq T_\infty(\omega). \end{cases}$$

And the probability measure is characterized by the set of conditional measures s.t. $\forall k = 2, 3, \dots$ and $\Gamma \in \Omega_k$

$$\mathbb{P}((T_1, Z_1) \in \Gamma) = \mu^1(\Gamma)$$
$$\mathbb{P}((T_k, Z_k) \in \Gamma | \eta) = \mu^k(\eta; \Gamma),$$

where the functions μ^1 and μ^k satisfy the same conditions as in section 2.1.

We construct the multi-branch random walk by the same strategy as in chapter 2. To that end set

$$\bar{x}^1(t)(\omega) := \begin{cases} z_0, & t < T_1(\omega) \\ Z_1(\omega), & T_1(\omega) \leq t, \end{cases} \bar{x}^k(t)(\omega) := \begin{cases} Z_{k-1}(\omega), & t < T_{k-1} \\ Z_k(\omega), & T_k(\omega) \leq t. \end{cases}$$

We call these processes *single-branch sections of the multi-branch process.* They are adapted to the filtrations $\mathcal{H}_s^k$ that are defined as

$$\mathcal{H}_s^k := \mathcal{F}_{U_s^k}$$

where $U_s^k := (T_{k-1} + s) \wedge T_k$.
The multi-branch random walk is then constructed by the following composition:

$$x(t)(\omega) := \begin{cases} \tilde{x}^1(t)(\omega), & t \leq T_1(\omega) \\ \tilde{x}^k(t)(\omega), & t \in [T_{k-1}(\omega), T_k(\omega)), \\ z_\infty, & t \geq T_\infty(\omega). \end{cases} \tag{3.2.1}$$

Note that for the right choice of $\varphi^k : \Omega_{k-1} \times \mathbb{R}^+ \times X \to \mathbb{R}$ the process $\tilde{v}^k := \left(\varphi^k(\omega_{k-1}; t, \tilde{x}^k(t))\right)_{t \in \mathbb{R}^+}$ is a local $\mathcal{H}^k$-martingale, due to corollary 2.3. Thus we can compose another process

$$v(t)(\omega) := \varphi(t, x(t))(\omega) = \begin{cases} \varphi^1(t, x(t))(\omega), & t \le T_1(\omega) \\ \varphi^k(\omega_{k-1}; t - T_{k-1}, x(t)), & t \in [T_{k-1}(\omega), T_k(\omega)) \\ z_\infty, & t \ge T_\infty(\omega). \end{cases} \tag{3.2.2}$$

By theorem 2.6 this process is a local $\mathcal{F}_t$-martingale for the right choice of $\varphi^1, \varphi^2, \dots$.

Let r denote the probability of a particle to reproduce, u denote the probability with which a particle is born one unit above the parent particle and $p_l^k := p_l^k(\omega_{k-1})$ denotes the probability that a particle at positions $l \in \mathbb{Z}$ is the particle which performs the k-th branching action. Under the above assumptions and for $z_0 = \delta_0$ and for $\nu \in \Omega_{k-1}, n \in \mathbb{Z}$ the transition probabilities for the process are as follows:

$$\begin{aligned}
\mathbb{P}(T_1 \le t, Z_1 = 2\delta_1) &= (1 - \exp(-\lambda t)) r u^2 \\
\mathbb{P}(x(t) = 2\delta_{-1}) &= (1 - \exp(-\lambda t)) r (1-u)^2 \\
\mathbb{P}(T_1 \le t, Z_1 = \delta_{-1} + \delta_1) &= (1 - \exp(-\lambda t)) r u (1-u) \\
\mathbb{P}(T_1 \le t, Z_1 = 0) &= (1 - \exp(-\lambda t))(1-r) \\
\mathbb{P}(T_k \le t, Z_k = Z_{k-1}(\nu) - \delta_n + 2\delta_{n+1} | \nu) &= (1 - \exp(-\lambda t)) p_n^k(\nu) r u^2 \\
\mathbb{P}(T_k \le t, Z_k = Z_{k-1}(\nu) - \delta_n + 2\delta_{n-1} | \nu) &= (1 - \exp(-\lambda t)) p_n^k(\nu) r (1-u)^2 \\
\mathbb{P}(T_k \le t, Z_k = Z_{k-1}(\nu) - \delta_n + \delta_{n-1} + \delta_{n+1} | \nu) &= (1 - \exp(-\lambda t)) p_n^k(\nu) r u (1-u) \\
\mathbb{P}(T_k \le t, Z_k = Z_{k-1}(\nu) - \delta_n | \nu) &= (1 - \exp(-\lambda t)) p_n^k(\nu)(1-r).
\end{aligned}$$

Remark 3.5. *These are the transition probabilities under the assumption that each branching is independent of the previous branching. But in the slightly more general case where each branching might depend on the previous branches one can adjust the probabilites for branching to $r = r(\omega_{k-1})$ and/or $u = u(\omega_{k-1})$. For example the reproduction probability might depend on the current population size, the previous branching method, or even the positions of the current population.*

Now let $t \in [T_{k-1}, T_k)$. Then the process x is about to branch the k-th time and thus the current path of v will be described by:

$$\begin{aligned}
\varphi^k(\omega_{k-1}; t - T_{k-1}, x(t)) = b^k(\omega_{k-1}; x(t)) \mathbb{1}_{T \le t} - \mathbb{1}_{T > t} \frac{1}{1 - F^k_{t - T_{k-1}}(\omega_{k-1})} \\
\times \left(\int_{(0,t] \times X} b^k(\omega_{k-1}; z) d\mu^k(\omega_{k-1}; u, z) + r^k(\omega_{k-1}) \right)
\end{aligned}$$

for some $b^k(\omega_{k-1}; \cdot) \in L^1_{\text{loc}}(\mu)$ and $r^k(\omega_{k-1}) = \varphi(T_{k-1}, Z_{k-1})$. The interesting part here is again the integral. Since we assumed that each particle in our population obeys the same behavior as the original ancestor, we can decompose the integral same as in

(3.1.4) and (3.1.5) for every $\omega \in \Omega$:

$$\begin{aligned}
&\int_{(0,t]\times X} b^k(\omega_{k-1}(\omega); z) d\mu^k(\omega_{k-1}(\omega; u, z) \\
&= \int_{(0,t]} \int_X b^k(\omega_{k-1}(\omega); z) d\mu_u^k(\omega_{k-1}(\omega); z) dF_u^k(\omega_{k-1}(\omega)) \\
&= \int_{(0,t]} \sum_{z\in\mathcal{X}^k} b^k(\omega_{k-1}(\omega); z) \mu_u^k(\omega_{k-1}(\omega); z) dF_u^k(\omega_{k-1}(\omega)
\end{aligned}$$

where $\mathcal{X}^k$ is the set of all possible populations after the first k jumps and $\mu_u^k(\omega_{k-1}(\omega); \cdot)$ denotes the conditional distribution of Z_k given $T_k = u$ and ω_{k-1}. The sum can be decomposed further as:

$$\begin{aligned}
&\sum_{z\in\mathcal{X}^k} b^k(\omega_{k-1}(\omega); z) \mu_u^k(\omega_{k-1}(\omega); z) \\
&= \sum_{l\in\mathbb{Z}} \Big[b^k(\omega_{k-1}(\omega); Z_{k-1}(\omega) - \delta_l) p_l^k (1-r) \\
&\quad + b^k(\omega_{k-1}(\omega); Z_{k-1}(\omega) - \delta_l + 2\delta_{l-1}) p_l^k r(1-u)^2 \\
&\quad\quad + b^k(\omega_{k-1}(\omega); Z_{k-1}(\omega) - \delta_l + 2\delta_{l+1}) p_l^k r u^2 \\
&\quad\quad\quad + b^k(\omega_{k-1}(\omega); Z_{k-1}(\omega) - \delta_l + \delta_{l-1} + \delta_{l+1}) p_l^k r u(1-u) \Big].
\end{aligned}$$

The outer integral describes the probability that any particle branches before $t - T_{k-1}$ after the last branching event and thus is given by

$$F_{t-T_{k-1}}^k(\omega_{k-1}(\omega)) = \left[1 - \exp\left(-\lambda\langle 1, Z_{k-1}(\omega)\rangle (t - T_{k-1})\right)\right].$$

3.3 Semimartingale representation

In this section we will determine the semimartingale representation of certain functions of the branching random walk. This is a direct application of corollary 2.9. In our framework the process takes its values in the space of weighted Dirac-measures, i.e. the random variable Z_k is a random distribution of the population and T_k is the random branching time that leads to the population described by Z_k. Throughout this section we assume that $T_\infty = \infty$ $\mathbb{P}$-a.s.; under the assumptions from section 3.1 - namely binary branching or dying at each branching time and only two birthplaces for descendents of a branching particle to choose from - the values of Z_k are finite for all $k \in \mathbb{N}$ and thus the possible values of any function $\varphi : \mathbb{R}^+ \times X \to \mathbb{R}$ of the process with $\varphi(t, x(t))$ are finite for each fixed $t \in \mathbb{R}^+$ if $T_\infty = \infty$. Further we assumed before that each particle has an exponential lifetime and thus a continuously differentiable distribution function. Now let φ satisfy condition (C) (see definition 1.13). The results from chapter 2, corollary 2.9 imply that the semimartingale representation is then given by:

$$\begin{aligned}
\varphi(t, x(t)) &= M_{t\wedge T_1}^1 + \sum_{k=2}^{\infty} \Big[M_{(t-T_{k-1})\wedge S_k}^k(\omega_{k-1}) - M_{S_{k-1}}^{k-1}(\omega_{k-1}) \Big] \mathbb{1}_{T_{k-1}\le t} \\
&\quad + A_{t\wedge T_1}^1 + \sum_{k=2}^{\infty} \Big[A_{(t-T_{k-1})\wedge S_k}^k(\omega_{k-1}) - A_{S_{k-1}}^{k-1}(\omega_{k-1}) \Big] \mathbb{1}_{T_{k-1}\le t}
\end{aligned}$$

where each M^k is a local $\mathcal{H}^k$ martingale and A^k are each previsible processes w.r.t. $\mathcal{H}^k$. In the last section we constructed a function φ by combining functions φ^k that apply to the random intervals $[T_{k-1}, T_k]$. But the discussion about $\varphi(t, x(t))$ has also shown, that any function φ of the process x can always be represented as a combination of such functions φ^k:

$$\varphi(t, x(t)) := \sum_{k=1}^{\infty} \varphi^k(t, x(t)) \mathbb{1}_{T_{k-1}<t\leq T_k}.$$

These functions φ^k are given by the respective semimartingale $M^k_{t-T_{k-1}} + A^k_{t-T_{k-1}}$ and thus depend on the knowledge of the previous branching time T_{k-1}. From corollary 2.8 we take the form of each of these terms for $\eta \in \Omega_{k-1}$:

$$M^k_t(\eta) = \varphi^k(\eta; 0, Z_{k-1}(\eta)) + \int_{(0,t]\times X} \left[\varphi^k(\eta; s, z) - \varphi^k(\eta; s, Z_{k-1}(\eta))\right] dq^k(s, z)$$

$$A^k_t(\eta) = \int_{(0,t]} \frac{d\varphi^k}{dF^k}(\eta; s, Z_{k-1}(\eta)) + \frac{\mathbb{1}_{S_k\geq s}}{1 - F^k_{s-}} \left[\varphi^k(\eta; s, z) - \varphi^k(\eta; s, Z_{k-1}(\eta))\right] d\mu^k(\eta; s, z)$$

where $m^k(\eta; s) := \mathbb{E}_{\mu^k(\eta;\cdot)} \left[\varphi^k(\eta; T_k, Z_k)|T_k = s\right]$. Under the assumption of exponential branching times one might also write

$$\begin{aligned} A^k_t(\eta) = &\int_{(0,t]} \frac{d\varphi^k}{dt}(\eta; s, Z_{k-1}(\eta))dt \\ &+ \int_{(0,t]\times X} \frac{\mathbb{1}_{S_k\geq s}}{1 - F^k_{s-}} \left[\varphi^k(\eta; s, z) - \varphi^k(\eta; s, Z_{k-1}(\eta))\right] d\mu^k(\eta; s, z). \end{aligned}$$

Example 3.6. *1. As an example we explore the process*

$$v(t) := \exp\left(\langle f, x(t)\rangle\right),$$

where for a measure ν and a measurable function $f : X \to \mathbb{R}$ we denote $\langle f, \nu\rangle = \int_X f(z)d\nu(z)$. Let $x(t)$ be the symmetric branching random walk from example 3.4. Then $v(t) = \varphi(t, x(t))$ where $\varphi(t, z) = \exp(\langle f, z\rangle) =: \varphi(z)$, hence $\frac{d}{dt}\varphi \equiv 0$. Take $\omega \in \{T_{k-1} \leq t < T_k\}$. Then

$$\begin{aligned} v(t, \omega) &= \exp\left(\langle f, x(t, \omega)\rangle\right) \\ &= \exp\left(\langle f, Z_{k-1}(\omega)\rangle\right). \end{aligned}$$

The semimartingale representation is given by

$$\begin{aligned} M^k_{t-T_{k-1}}(\omega) = &\exp\left(\langle f, Z_{k-1}(\omega)\rangle\right) \qquad (3.3.1) \\ &+ \int_{(0,t]\times X} \left[\exp\left(\langle f, z\rangle\right) - \exp\left(\langle f, Z_{k-1}(\omega)]\rangle\right) dq^k(s, z)\right. \\ A^k_{t-T_{k-1}}(\omega) = &\int_{(0,t\wedge S_k]\times X} \frac{1}{1 - F^k_{s-}} \left[\varphi(z) - \varphi(Z_{k-1}(\omega))\right] d\mu^k(\omega_{k-1}(\omega); s, z) \\ &\qquad (3.3.2) \end{aligned}$$

We further simplify (and omit ω in favor of readability):

$$\begin{aligned}
&\int_{(0,(t-T_{k-1})\wedge S_k]\times X} \frac{1}{1-F^k_{s-}} \left[\varphi(z)-\varphi(Z_{k-1})\right] d\mu^k(\omega_{k-1};s,z)\\
&= \int_{(0,(t-T_{k-1})\wedge S_k]} \frac{1}{1-F^k_{s-}} \int_X \left[\varphi(z)-\varphi(Z_{k-1})\right] d\mu^k_s(\omega_{k-1};z) dF^k(\omega_{k-1};s)\\
&= \int_{(0,(t-T_{k-1})\wedge S_k]} \exp(\lambda s\langle 1, Z_{k-1}\rangle) \left\{ \sum_{z\in\mathcal{Z}^k} \left[\varphi(z)-\varphi(Z_{k-1})\right] r^k_z \right\} dF^k(\omega_{k-1};s)\\
&= \left\{ \sum_{z\in\mathcal{Z}^k} \left[\varphi(z)-\varphi(Z_{k-1})\right] r^k_z \right\} \int_{(0,(t-T_{k-1})\wedge S_k]} (\lambda\langle 1, Z_{k-1}\rangle) ds\\
&= \left\{ \sum_{z\in\mathcal{Z}^k} \left[\varphi(z)-\varphi(Z_{k-1})\right] r^k_z \right\} \left[(t-T_{k-1})\wedge S_k\right] (\lambda\langle 1, Z_{k-1}\rangle)
\end{aligned}$$

where for $k\in\mathbb{N}, \eta\in\Omega_{k-1}$:

$$\begin{aligned}
\mathcal{Z}^{k-1}(\eta) :=&\{z\in X \mid \exists m\in\mathcal{X}^{k-1} :\\
&\qquad z - Z_{k-1}(\eta)+\delta_m \in \{0, 2\delta_{m-1}, \delta_{m-1}+\delta_{m+1}, 2\delta_{m+1}\}\}
\end{aligned}$$

is the set of possibly obtainable measures with the k-th branching, given the measure Z_{k-1}[1] *and*

$$z\in X, k\in\mathbb{N}, \eta\in\Omega_{k-1} : r^k_z(\eta) := \mu^k(\eta;\mathbb{R}^+\times\{z\})$$

is the respective transition probability to any measure z given measure Z_{k-1}. In our particular case (symmetric branching random walk) the possible outcomes of the k-th branching event are finite. Each particle of the current ensemble can either die or split in 3 different patterns, thus $|\mathcal{Z}^k| \le 4\langle 1, Z_{k-1}\rangle$. The transition probabilities r^k_z are thus composed of the probability for choosing the specific particle that can lead to z, the probability for the right branching method (death or split) and the respective choices of birthplace:

$$r^k_z(\eta) := \begin{cases} \frac{\langle\delta_m, Z_{k-1}(\eta)\rangle}{\langle 1, Z_{k-1}(\eta)\rangle}\frac{1}{2}, & \text{if } \exists m\in\mathcal{X}^{k-1}(\eta) : z-Z_{k-1}(\eta)+\delta_m = 0,\\ \frac{\langle\delta_m, Z_{k-1}(\eta)\rangle}{\langle 1, Z_{k-1}(\eta)\rangle}\frac{1}{8}, & \text{if } \exists m\in\mathcal{X}^{k-1}(\eta) : z-Z_{k-1}(\eta)+\delta_m = 2\delta_{m\pm 1},\\ \frac{\langle\delta_m, Z_{k-1}(\eta)\rangle}{\langle 1, Z_{k-1}(\eta)\rangle}\frac{1}{4}, & \text{if } \exists m\in\mathcal{X}^{k-1}(\eta) : z-Z_{k-1}(\eta)+\delta_m = \delta_{m-1}+\delta_{m+1}. \end{cases}$$

Note further that $\langle\cdot,\cdot\rangle$ is bilinear and thus

$$\exp(\langle f, Z_{k-1}-\delta_m+\nu\rangle) = \exp(\langle f, Z_{k-1}\rangle)\exp(-\langle f,\delta_m\rangle)\exp(\langle f,\nu\rangle).$$

[1] Note that $\mathcal{X}^{k-1}(\eta) := \{m\in\mathbb{Z} : \langle\delta_m, Z_{k-1}(\eta)\rangle \neq 0\}$

For the next term this helps a bit:

$$\begin{aligned}
&\sum_{z\in\mathcal{Z}^k} [\varphi(z) - \varphi(Z_{k-1})]\, r_z^k \\
=&\sum_{z\in\mathcal{Z}^k} [\exp(\langle f, z\rangle) - \exp(\langle f, Z_{k-1}\rangle)]\, r_z^k \\
=&\sum_{m\in\mathcal{X}^{k-1}} \Big[(\exp(\langle f, Z_{k-1} - \delta_m\rangle) - \exp(\langle f, Z_{k-1}\rangle))\frac{\langle \delta_m, Z_{k-1}(\eta)\rangle}{2\langle 1, Z_{k-1}(\eta)\rangle} \\
&\quad (\exp(\langle f, Z_{k-1} - \delta_m + 2\delta_{m-1}\rangle) - \exp(\langle f, Z_{k-1}\rangle))\frac{\langle \delta_m, Z_{k-1}(\eta)\rangle}{8\langle 1, Z_{k-1}(\eta)\rangle} \\
&\quad\; (\exp(\langle f, Z_{k-1} - \delta_m + 2\delta_{m+1}\rangle) - \exp(\langle f, Z_{k-1}\rangle))\frac{\langle \delta_m, Z_{k-1}(\eta)\rangle}{8\langle 1, Z_{k-1}(\eta)\rangle} \\
&\quad\;\; (\exp(\langle f, Z_{k-1} - \delta_m + \delta_{m-1} + \delta_{m+1}\rangle) - \exp(\langle f, Z_{k-1}\rangle))\frac{\langle \delta_m, Z_{k-1}(\eta)\rangle}{4\langle 1, Z_{k-1}(\eta)\rangle}\Big] \\
=&\frac{1}{2}\exp(\langle f, Z_{k-1}\rangle) \sum_{m\in\mathcal{X}^{k-1}} \frac{\langle \delta_m, Z_{k-1}(\eta)\rangle}{\langle 1, Z_{k-1}(\eta)\rangle}\Big[(\exp(-f(m)) - 1) \\
&\quad + \frac{\exp(2f(m-1) - f(m)) - 1}{4} \\
&\quad\; + \frac{\exp(2f(m+1) - f(m)) - 1)}{4} \\
&\quad\;\; + \frac{\exp(f(m-1) - f(m) + f(m+1)) - 1}{2}\Big].
\end{aligned}$$

The final form of (3.3.2) *is hence given by:*

$$\begin{aligned}
A^k_{t-T_{k-1}} =& [(t - T_{k-1}) \wedge S_k]\,(\lambda\langle 1, Z_{k-1}\rangle)\frac{\exp(\langle f, Z_{k-1}\rangle)}{2} \\
&\times \sum_{m\in\mathcal{X}^{k-1}} \frac{\langle \delta_m, Z_{k-1}(\eta)\rangle}{\langle 1, Z_{k-1}(\eta)\rangle}\Big[(\exp(-f(m)) - 1) \\
&\quad + \frac{\exp(2f(m-1) - f(m)) - 1}{4} \\
&\quad\; + \frac{\exp(2f(m+1) - f(m)) - 1)}{4} \\
&\quad\;\; + \frac{\exp(f(m-1) - f(m) + f(m+1)) - 1}{2}\Big]
\end{aligned}$$

2. *As another and last example we now look at the slightly adjusted process:*

$$v(t) := \exp(t\langle f, x(t)\rangle)$$

so that $\varphi(t,\nu) := \exp(t\langle f, \nu\rangle)$ *for some* $t \in \mathbb{R}^+$ *and* $\nu \in X$. *In this case the derivative* $\frac{d}{dt}\varphi(t,\nu) = \langle f, \nu\rangle\varphi(t,\nu)$. *For* $\omega \in \{T_{k-1} \le t < T_k\}$ *we can now* *determine the compensator again:*

$$\begin{aligned}
A^k_{t-T_{k-1}} &= \int_{(T_{k-1},t]} \frac{d\varphi}{dt}(s, Z_{k-1})ds \\
&\quad + \int_{(T_{k-1},t]\times X} \frac{\mathbb{1}_{T_k \geq s}}{1-F^k_{s-}} \left[\varphi(s,z) - \varphi(s, Z_{k-1}(\eta))\right] d\mu^k(\eta; s, z) \\
&= \int_{(T_{k-1},t]} \langle f, Z_{k-1}\rangle \varphi(s, Z_{k-1})ds + \int_{(T_{k-1},t]} \exp(\lambda s\langle 1, Z_{k-1}\rangle)\mathbb{1}_{T_k\geq s} \\
&\quad \times \left(\int_X \left[\varphi(s,z) - \varphi(s, Z_{k-1}(\eta))\right] d\mu^k_s(\omega_{k-1}; z)\right) dF^k(\omega_{k-1}; s).
\end{aligned}$$

Using the fact that $dF^k(\omega_{k-1}; s) = \lambda\langle 1, Z_{k-1}\rangle \exp(-\lambda s\langle 1, Z_{k-1}\rangle)$ *and that* $\varphi(s,z) = \varphi(s, Z_{k-1})\varphi(s,\nu)$ *for some* $\nu \in \{2\delta_{m-1}, \delta_{m-1}+\delta_{m+1}, 2\delta_{m+1}\}$ *and some* $m \in \mathcal{X}^{k-1}$*, we can further simplify the second integral to:*

$$\begin{aligned}
&\int_{(T_{k-1},t\wedge T_k]} \exp(\lambda s\langle 1, Z_{k-1}\rangle) \left(\int_X \left[\varphi(s,z) - \varphi(s, Z_{k-1})\right] d\mu^k_s(\omega_{k-1}; z)\right) dF^k(\omega_{k-1}; s) \\
=&\lambda\langle 1, Z_{k-1}\rangle \int_{(T_{k-1},t\wedge T_k]} \varphi(s, Z_{k-1}) \left[\sum_\nu (\varphi(s,\nu) - 1)\right] r^k_\nu ds
\end{aligned}$$

where in this case the transition probabilities $r^k_\nu(\eta) := \mathbb{P}(Z_k = Z_{k-1} + \nu|\eta)$ *and are given by the same values as in the previous example. We revert back to the whole term for*

$$\begin{aligned}
&A^k_{t-T_{k-1}} \\
&= \int_{(T_{k-1},t]} \langle f, Z_{k-1}\rangle \varphi(s, Z_{k-1}) \\
&\qquad + \lambda\langle 1, Z_{k-1}\rangle \varphi(s, Z_{k-1})\mathbb{1}_{T_k\geq s} \left[\sum_\nu (\varphi(s,\nu)-1) r^k_\nu\right] ds \\
&= \int_{(T_{k-1},t]} \varphi(s, Z_{k-1}) \left\{\langle f, Z_{k-1}\rangle + \langle \lambda, Z_{k-1}\rangle \mathbb{1}_{T_k\geq s} \left[\sum_\nu (\varphi(s,\nu)-1) r^k_\nu\right]\right\} ds.
\end{aligned}$$

From this form one can easily deduce that only $f \equiv 0$ *can provide us with a martingale, as the compensator does not vanish otherwise.*

Chapter 4

Discussion

As a closing section we want to discuss the results of this work, their implications, some possible generalizations, developments and open problems.

We started this journey with the idea to combine a rather intuitive strategy with a classical result: a discrete time approximation of the time continuous Doob-Meyer decomposition of a process adapted to the single-jump processes filtration to help with the determination of martingales of said jump process. While the discrete version (theorem 1.10) of the process has been a straightforward application of Doob's decomposition theorem and some taylormade results and properties of conditional expectation (lemma 1.9), the limiting procedure (theorem 1.14 and corollary 1.16) lead us to first technical assumptions (conditions C and C'). After the limiting procedure has been dealt with, we were left with an instructive result (theorem 1.14 and corollary 1.16) that shows how to compute the time-continuous Doob-Meyer decomposition of any acceptable function φ of time and the value of the single-jump process at that time. This enabled us to determine structural conditions a function would have to show to be able to transform the process into a martingale w.r.t. to its natural filtration. Finally we highlighted the connection to the result by [Davis, 1976] and moved on to the generalization of our result to multi-jump processes by glueing together so called single-jump sections of the process (theorem 2.6). The ominous 'glue-condition' as well as another instructive result shows that processes of this certain form are in fact martingales of the multi-jump process. Here assumed that the accumulation time $T_\infty = \infty$. In the final section of chapter 2 we then went on to obtain a version of the Doob-Meyer decomposition for multi-jump processes under the above assumptions (theorem 2.10). Chapter 3 finally focuses on applying the results to a Branching random walk. We find ourselves confronted with a paradigm for martingales, delivering a practical amount of parameters to control different aspects of the process.

Now let us turn to the interesting avenues for further research. Although they were not unrealistic for our purpose, one might be interested in more general/popular conditions for the functions φ to still admit a consistent time continuous Doob-Meyer decomposition. As we already stated the most general and simple proof of the time-continuous Doob-Meyer decomposition can be found in [Beiglboeck et al., 2010]. The strategy there makes use of the uniform integrability that comes with the assumption that the submartingale is of class D. While our result is convergent $\mathbb{P}$-a.s., we can yield an L^1-convergent result under the assumption of uniform integrability of the discrete approximations of M and A by Vitali's theorem. Furthermore one can explore the connection between conditions C' and class D to locate our result in the classical literature.

The application in chapter 3 showed, that the results of the first two chapters could be adapted to hold for branching processes. As a matter of fact we applied them to branching random walks, which opened up some options for similar result on branching Brownian motion. In analogy to the approximation of Brownian motion through random walks one might try to approximate branching Brownian motion by rescaling the time and space of branching random walks to yield similar results in the limit. Another possibility is to combine the path-valued approach from Appendix A and the measure-valued example from chapter 3. For this let T be a random jump time in $\mathbb{R}^+$, Z be a random number of descendents (integer-valued) and B_t the path of Brownian motion stopped at time $t \in \mathbb{R}^+$ (see [Levental et al., 2013] for notation). The single-branch Brownian motion then would be defined as a process

$$x_t(\omega) := \begin{cases} \delta_{B_t(\omega)}, & T(\omega) < t \\ Z(\omega)\delta_{B_{T(\omega)}(\omega)}, & T(\omega) \leq t. \end{cases}$$

This simple process is measure valued, but with measures on the path space $\mathbb{D} := D(\mathbb{R}^+, \mathbb{R})$. As long as the branching time T has not been overcome, the process is the measure that follows the 'excavation' of the Brownian motions path B_t. Once the branching time is hit, it is the constant measure $Z(\omega)\delta_{B_{T(\omega)}(\omega)}$ and the symbolically phrased 'excavation' of the Brownian motions path has stopped. But now the integer-valued random variable Z determines how many new particles may start at the branching location $B(T(\omega), \omega))$. The program for this application can follow the strategy of the first chapter to determine discrete Doob-decompositions of the single-branch Brownian motion. Assuming the Brownian motion to be independent of the jump time (T) and height (Z) and vice-versa should even be a straightforward application of the results, as the products of independent processes can be viewed separately most of the times. Generalizing the single-branch Brownian motion to the multi-branch case might be achieved by a similar technique as we explored in chapter 2, by fuzing single-branch instances.

Another interesting perspective is the representation of a martingale of the jump process, as an integral w.r.t. the fundamental family of martingales. In light of the factorization lemma or the related result for Brownian motion, this observation turns our attention to the underlying dynamics of randomness. While fundamentally different to Brownian motion, the jump processes still obey a general basic process that can be described seperately and explains the behaviour of processes that are structurally related to jump processes (e.g. branching random walks). An interesting topic might be, how the combination of jump processes and Brownian motion can be factorized into their respective fundamental processes. Applying the above strategy of discrete approximations to branching Brownian motions may yield constructive results for determining martingales of said processes.

The application to branching random walks could also be generalized or adjusted: the branching mechanism may allow an arbitrary number of descendents, the progeny may distribute over $\mathbb{R}^d$, etc. But eventually incorporating multitype processes seems to be the most interesting part to us, as it enables us to assume more dependencies between for example reproduction and age.

Another generalization concerning the accumulation time T_∞ might be achieved in a similar way as [Elliott, 1976] and [Elliott, 1977] generalize [Davis, 1976]. The connection of [Gushchin, 2020] to the results of chapter 2 is suspected to hold in the multi-jump case. An augmentation of [Gushchin, 2020] to multi-jump processes could draw intuition from our results.

Appendix A

Single Jump and path dependent

In chapter 1 we approached the single-jump process with a function $\varphi(t, x(t))$ to form a martingale. We ended up with a Markov process, since only the present state of the process is inserted into the second argument of the function φ. While this is a first successful step, we would like to generalize the methods with a different approach:

Let φ be a function of time and the whole path of the process until time t.

This enables us to use the notation for path-spaces and possibly incorporate non-markovian features more easily in future research projects.

A.1 Definitions

The state space and random variables: We will keep the state space Ω and the random variables S, Z, T as before, since our only change will affect the process x. Before $x(t)$ was just the evaluation of the path at the time t. Now we want x_t to hold the values of x until time t and constant afterwards. This is the notation from [Levental et al., 2013]. Keep in mind, that this is still the single-jump case.

The path process: For any $\omega \in \Omega$ the *path of the process up to time* t will be denoted by

$$x_t(\omega, \cdot) = x(\omega, \cdot)\mathbb{1}_{[0,t]}(\cdot) + x(\omega, t)\mathbb{1}_{(t,\infty)}(\cdot)$$

and is a member of the space of cdlg-paths $D(\mathbb{R}^+, X)$. Since the above description is rather long, we denote the relevant paths for $z \in X$, $u \in \mathbb{R}^+$:

$$z_u := \{z_0\mathbb{1}_{[0,u)}(s) + z\mathbb{1}_{[u,\infty)}(s) : s \in \mathbb{R}^+\} \tag{A.1.1}$$

for the path that jumps to the value $z \in X$ at time $u \in \mathbb{R}^+$. The path that never jumps will be denoted by $z^{(0)}$. I.e. for $\omega \in \Omega$ we have

$$x_t(\omega) = \begin{cases} z^{(0)}, & \text{if } t < T(\omega), \\ Z(\omega)_{T(\omega)}, & \text{if } T(\omega) \leq t. \end{cases}$$

This notation is reminiscent of the first approach. In fact the evaluation of the path for fixed $t \in \mathbb{R}^+$ at any time $s \in \mathbb{R}^+$ would be:

$$x_t(\omega, s) = \begin{cases} z_0, & \text{if } s \wedge t < T(\omega) \\ Z(\omega), & \text{if } T(\omega) \leq s \wedge t. \end{cases} \tag{A.1.2}$$

The message from this is, that x_t is - for a fixed t - a stopped process $x(\cdot)$. So by increasing the time t we push back the time at which we stop the process. If we let $t \to \infty$ the resulting path would be just the whole path of $x(\cdot)$ from the first approach.

On $\mathbb{D} := D(\mathbb{R}^+, X) \times \mathbb{R}^+$ we take the distances $\tilde{d}$ from [Levental et al., 2013] defined by

$$\tilde{d}((x,t),(y,s)) := \|x - y\|_\infty + |t - s|$$

where $\|x - y\|_\infty := \sup_{0 \leq t < \infty} \|x(t) - y(t)\|$.

Filtration and probability measure: Take $\mathcal{E}_t^0 = \sigma(x_u : u \in [0,t])$, the natural σ-field of the path process $(x_t)_{t\geq 0}$. We note that $\mathcal{E}_t^0 = \mathcal{F}_t^0$ since they both include the history of $x(u)$ up until the time t. This enables us to use the exact same σ-field as in the first approach: $\mathcal{F}_t$

Our path approach would start with functions - or, better: functionals

$$\varphi : \mathbb{R}^+ \times D(\mathbb{R}^+, X) \to \mathbb{R},$$

and we define a new stochastic process by

$$v_t := \varphi(t, x_t),$$

Following our strategy from before, we need a discrete version of the above process. We define $v := (v_{t_k})_{k\in\{1,\dots,2^N\}}$ and $\mathcal{F}_{t_k} := \mathcal{F}_k$ for an arbitrary set of times $t_1 < \cdots < t_2^N \in \mathbb{R}^+$.

We also keep the probability measure $\mathbb{P}$ as defined in (2.1.3).

The discrete Version of the process is adapted to the filtration $\mathcal{F}_{t_k}$ from before, i.e. a Doob-Meyer-decomposition is imminent. Let us prepare:

Lemma A.1.

$$\mathbb{E}[v_{t_k}|\mathcal{F}_{t_{k-1}}] = \varphi(t_k, x_{t_{k-1}}) + \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_k}} \int_{\{T>t_{k-1}\}} [\varphi(t_k, x_{t_k}) - \varphi(t_k, z^{(0)})] d\mathbb{P}$$

Proof. We seperate Ω into the two exclusive sets $\{T \leq t_{k-1}\}$ and $\{T > t_{k-1}\}$ and get:

$$\begin{aligned}
\mathbb{E}[v_{t_k}|\mathcal{F}_{t_{k-1}}] &= \mathbb{E}[v_{t_k}\mathbb{1}_{T\leq t_{k-1}}|\mathcal{F}_{t_{k-1}}] + \mathbb{E}[v_{t_k}\mathbb{1}_{T>t_{k-1}}|\mathcal{F}_{t_{k-1}}] \\
&\overset{(1)}{=} v_{t_k}\mathbb{1}_{T\leq t_{k-1}} + \mathbb{1}_{T>t_{k-1}} \frac{1}{\mathbb{P}(T > t_{k-1})} \int_{\{T>t_{k-1}\}} v_{t_k} d\mathbb{P} \\
&\overset{(2)}{=} v_{t_k}\mathbb{1}_{T\leq t_{k-1}} + \frac{\mathbb{1}_{T>t_{k-1}}}{\mathbb{P}(T > t_{k-1})} \left\{ \int_{\{t_{k-1}<T\leq t_k\}} v_{t_k} d\mathbb{P} + \varphi(t_k, z_k^{(0)}) \int \mathbb{1}_{T>t_k} d\mathbb{P} \right\} \\
&\overset{(3)}{=} v_{t_k}\mathbb{1}_{T\leq t_{k-1}} + \mathbb{1}_{T>t_{k-1}} \left\{ \frac{1}{1-F_{t_{k-1}}} \int_{\{t_{k-1}<T\leq t_k\}} [v_{t_k} - \varphi(t_k, z_k^{(0)})] d\mathbb{P} + \varphi(t_k, z_k^{(0)}) \right\}
\end{aligned}$$

where we used in particular:

(1) On the set $\{T \leq t_{k-1}\}$ v_{t_k} is $\mathcal{F}_{t_{k-1}}$-measurable, since we only take paths that jumped prior to t_{k-1} (i.e. they remain constant afterwards). In fact we have

$$v_{t_k}\mathbb{1}_{T\leq t_{k-1}} = \varphi(t_k, x_{t_k})\mathbb{1}_{T\leq t_{k-1}} = \varphi(t_k, x_{t_{k-1}})\mathbb{1}_{T\leq t_{k-1}}.$$

Therefore we can move $v_{t_k}\mathbb{1}_{T\leq t_{k-1}}$ out of the conditional expectation. For the second part of the sum we used, that $\{T > t_{k-1}\}$ is an atom of

$$\mathcal{F}_{t_{k-1}} = \sigma\left(\{x(s)\mathbb{1}_{T\leq t_{k-1}} : s \leq t_{k-1}\}, \{T > t_{k-1}\}, \mathcal{N}_0\right).$$

The conditional expectation is then calculated with Lemma 1.6.

(2) Further seperate into $\{T \leq t_k\}$ and $\{T > t_k\}$ and use $v_{t_k}\mathbb{1}_{T>t_k} = \varphi(t_k, z_k^{(0)})\mathbb{1}_{T>t_k}$ as well as $\{T > t_k\} \cap \{T > t_{k-1}\} = \{T > t_k\}$ to get

$$\int v_{t_k}\mathbb{1}_{T>t_{k-1}}d\mathbb{P} = \int v_{t_k}\mathbb{1}_{t_{k-1}<T\leq t_k}d\mathbb{P} + \int \varphi(t_k, z_k^{(0)})\mathbb{1}_{T>t_k}d\mathbb{P}$$

.

(3) Write

$$\varphi(t_k, z_k^{(0)})\int \mathbb{1}_{T>t_k}d\mathbb{P} = \varphi(t_k, z_k^{(0)})(1 - F_{t_{k-1}} + F_{t_{k-1}} - F_{t_k})$$

and sort with respect to the fraction $\frac{1}{1-F_{t_{k-1}}}$.

□

A.2 Doob-Meyer-decomposition

Due to our last lemma, we only need to piece together the different parts of the Doob-Meyer-decomposition.

Corollary A.2. *The Doob-Meyer-decomposition of the discrete time process* (v_{t_k}) *consists of* $(M_{t_k}^N)_{k\in\mathbb{N}}$ *and* $(A_{t_k}^N)_{k\in\mathbb{N}}$ *which are given by*

$$\begin{aligned}
M_{t_k}^N - M_{t_{k-1}}^N &= [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{t_{k-1}<T\leq t_k} \\
&\quad - \mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\int(\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)}))\mathbb{1}_{t_{k-1}<T\leq t_k}d\mathbb{P}
\end{aligned}$$

$$\begin{aligned}
A_{t_k}^N - A_{t_{k-1}}^N &= [\varphi(t_k, x_{t_{k-1}}) - \varphi(t_{k-1}, x_{t_{k-1}})] \\
&\quad + \mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\int[\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{t_{k-1}<T\leq t_k}d\mathbb{P}.
\end{aligned}$$

Proof. Use $M_{t_k}^N - M_{t_{k-1}}^N = v_{t_k} - \mathbb{E}[v_{t_k}|\mathcal{F}_{t_{k-1}}]$ and $A_{t_k}^N - A_{t_{k-1}}^N = \mathbb{E}[v_{t_k}|\mathcal{F}_{t_{k-1}}] - v_{t_{k-1}}$ and insert Lemma A.1:

$$\begin{aligned}
M_{t_k}^N - M_{t_{k-1}}^N &= v_{t_k} - \varphi(t_k, x_{t_{k-1}}) \\
&\quad - \mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_k}}\int_{\{T>t_{k-1}\}}[\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]d\mathbb{P} \\
&\overset{(i)}{=} [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{t_{k-1}<T\leq t_k} \\
&\quad - \mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_k}}\int_{t_{k-1}<T\leq t_k}[\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]d\mathbb{P}
\end{aligned}$$

and

$$\begin{aligned}
A^N_{t_k} - A^N_{t_{k-1}} &= \varphi(t_k, x_{t_{k-1}}) - v_{t_{k-1}} \\
&\quad + \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_k}} \int_{\{T>t_{k-1}\}} [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})] d\mathbb{P} \\
&\overset{(i)}{=} [\varphi(t_k, x_{t_{k-1}}) - \varphi(t_{k-1}, x_{t_{k-1}})] \\
&\quad + \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int_{t_{k-1}<T\leq t_k} [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})] d\mathbb{P}.
\end{aligned}$$

Where we used in particular:

(i) In the integral we decompose $[\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})] \, (\mathbb{1}_{T\leq t_k} + \mathbb{1}_{T>t_k})$ and note that:

$$\begin{aligned}
[\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{\{T>t_k\}} &= [\varphi(t_k, z_k^{(0)}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{T>t_k} \\
&= 0
\end{aligned}$$

Further we use that

$$\begin{aligned}
[\varphi(t_k, x_{t_k}) - \varphi(t_k, x_{t_{k-1}})] &= [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_{k-1}^{(0)})]\mathbb{1}_{t_{k-1}<T\leq t_k} \\
&= [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{t_{k-1}<T\leq t_k}
\end{aligned}$$

□

Before we advance to the limiting procedure, we take a closer look at the various differences in the above result:

(I)

$$[\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{t_{k-1}<T\leq t_k}.$$

Other than in our first approach we are confronted with a difference of a functional on two paths. In this case the difference will vanish everywhere except right at the jump time T (i.e. where $t_{k-1} < T \leq t_k$) where it will describe the change in φ under the jump height. This particular increment can be found twice in the representation of $M^N_{t_k} - M^N_{t_{k-1}}$. Given the same basic martingale process we find this difference to be a disintegration against the basic process p:

$$\begin{aligned}
&[\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})]\mathbb{1}_{t_{k-1}<T\leq t_k} \\
&= \int_Y [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})] p(dz, du) \mathbb{1}_{t_{k-1}<T\leq t_k}
\end{aligned}$$

where dz inserts the jump height and du the jump time. Note that $x_u = x_T$ for $T \leq u$, hence we wrote x_{t_k} for the path that actually describes a random jump at a random time.

The second time this term appears as a part of the representation of $M^N_{t_k} - M^N_{t_{k-1}}$ it is actually a disintegration against the compensator of p, namely $\tilde{p}$:

$$\mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int (\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)}))\mathbb{1}_{t_{k-1}<T\leq t_k} d\mathbb{P}$$
$$= \int (\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)}))\tilde{p}(z,u)\mathbb{1}_{t_{k-1}<T\leq t_k}$$

(II)

$$[\varphi(t_k, x_{t_{k-1}}) - \varphi(t_{k-1}, x_{t_{k-1}})].$$

This will approximate the slope of the function $\varphi(\cdot, y)$ for $x_{t_{k-1}} = y$. Think of $x_{t_{k-1}}$ as a random parameter choosing a function φ from a family of functions. For $T > t_{k-1}$ the above difference will capture the slope of such a function prior to the jump time T. After such a jump the function would have to stay constant for it to have any chance of transforming the process into a martingale (but that's something for a later discussion below).

A.3 Limiting procedure for the path dependent approach

Like previously we now increase the resolution of our discrete time-scale. If we would try this without any experience we would end up with the same complication we ran into before. Take $(t_u^N)_{N\in\mathbb{N}}$, $(t_d^N)_{N\in\mathbb{N}}$ s.t. $t_u^N \searrow u \in \mathbb{R}+$, $t_d^N \nearrow u$ for $N \to \infty$. For the different parts of $M^N_{t_u^N} - M^N_{t_d^N}$ we get:

$$\mathbb{1}_{T>t_u^N} \to \mathbb{1}_{T\geq t}$$
$$\varphi(t_u^N, x_{t_u^N}) - \varphi(t_u^N, x_{t_d^N}) \to \varphi(u, x_u) - \varphi(u, x_{u-})$$
$$\frac{1}{1-F_{t_d^N}} \to \frac{1}{1-F_{t-}}.$$

Since x_t is in turn just working with the random variables T, Z, the path-process has the same "'skeleton-process"' as $(x(t))_{t\in\mathbb{R}^+}$. Not surprisingly the time continuous Doob-Meyer decomposition reflects this dependency again:

Proposition 1. *For $\varphi \in \mathbb{C}^{1,0}$ the time continuous version of the martingale part is given by*

$$M_t = \int_{[0,t]\times\mathcal{S}} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] q(du, dz)$$
$$= \int_{[0,t]\times\mathcal{S}} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] p(du, dz) + \int_{(0,t]\times\mathcal{S}} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] d\tilde{p}(u, z)$$

Proof. Take for $t \in \mathbb{R}$ and $A \in \mathcal{S}$ the fundamental process

$$q(t, A) = p(t, A) - \tilde{p}(t, A)$$

from section 1.1. Then we write:

$$\varphi(t_k, x_{t_k}) - \varphi(t_k, z_{t_k}^{(0)}) = \int_{(t_{k-1}, t_k]\times X} \varphi(t_k, z_u) - \varphi(t_k, z^{(0)}) p(du, dz)$$

and check that

$$\sum_{k=1}^{2^N}\left(\varphi(t_k,x_{t_k})-\varphi(t_k,z_{t_k}^{(0)})\right)=\int_{\mathbb{R}^+\times X}\sum_{k=2}^{2^N}\left(\varphi(t_k,z_u)-\varphi(t_k,z^{(0)})\right)\mathbb{1}_{(t_{k-1},t_k]}(u)dp(u,z)$$
$$\to\int_{[0,t]\times X}\varphi(u,z_u)-\varphi(u,z^{(0)})p(du,dz)$$

for $N\to\infty$ since the integrand is approximated pointwise. The other part is a little bit more complicated, but we saw earlier, that this should end up as an integral against the process $\tilde{p}$, which looks like (we omit the trivial integrand $\varphi(u,z_u^{(0)})$ here to save some space):

$$\int_{(0,t]\times X}\varphi(u,z_u)d\tilde{p}(u,z)=\int_{[0,t]\times X}\varphi(u,z_u)\frac{1}{1-F_{u-}}\mathbb{1}_{T\geq u}d\mu(u,z)$$
$$=\int_{(0,t]}\int_X\varphi(u,z_u)\frac{1}{1-F_{u-}}\mathbb{1}_{T\geq u}\mu_u(dz)dF_u.$$

In fact the discrete version looks like:

$$\sum_{k=2}^{2^N}\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\int\varphi(t_k,x_{t_k})\mathbb{1}_{t_{k-1}<T\leq t_k}d\mathbb{P}$$
$$=\sum_{k=2}^{2^N}\int_{\mathbb{R}^+}\int_X\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\varphi(t_k,z_u)\mathbb{1}_{(t_{k-1},t_k]}(u)\mu_u(dz)dF_u$$

For $N\to\infty$ the mesh size converges to zero and in turn the above sum over k will converge in the following way:

$$\sum_{k=2}^{2^N}\int\int_X\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\varphi(t_k,z_s)\mathbb{1}_{(t_{k-1},t_k]}(s)\mathbb{P}(Z\in dz|T=s)\mathbb{P}(T\in ds)$$
$$=\int_{\mathbb{R}^+}\int_X\sum_{k=2}^{2^N}\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\varphi(t_k,z_s)\mathbb{1}_{(t_{k-1},t_k]}(s)\mathbb{P}(Z\in dz|T=s)\mathbb{P}(T\in ds)$$
$$\to\int_{(0,t]}\int_{\mathcal{S}}\frac{1}{1-F_{u-}}\mathbb{1}_{T\geq u}\varphi(u,z_u)\mathbb{P}(Z\in dz|T=u)\mathbb{P}(T\in du)$$

where we used again that the integrand converges pointwise:

$$\sum_{k=2}^{2^N}\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\varphi(t_k,x_{t_k})\mathbb{1}_{(t_{k-1},t_k]}(u)$$
$$\to\frac{1}{1-F_{u-}}\mathbb{1}_{T\geq u}\varphi(u,x_u)\mathbb{1}_{(0,t]}(u)$$

since $\mathbb{1}$ is right-continuous with left limits, F is right-continuous with left limits and φ is continuous in its first and second component (note that continuity in the second component is for paths that 'pretty much look alike' in terms of jump-time *and* -height, since we have to use the metric $\tilde{d}$ here). □

Now we want to do the same for the compensator part.

Proposition 2. *For $\varphi \in \mathbb{C}^{(1,0)}$ the time continuous version of the compensator part is given by*

$$A_t = \int_{(0,t]} \frac{\partial \varphi}{\partial t}(u, z_{u-})du + \int_{(0,t]} \int_X \frac{1}{1 - F_{u-}} \mathbb{1}_{T \geq u} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] d\mu(u, z)$$

Proof. We sum our result from corollary A.2 over k

$$\sum_{k=2}^{2^N} A^N_{t_k} - A^N_{t_{k-1}} = \sum_{k=2}^{2^N} [\varphi(t_k, x_{t_{k-1}}) - \varphi(t_{k-1}, x_{t_{k-1}})] \frac{t_k - t_{k-1}}{t_k - t_{k-1}}$$
$$+ \sum_{k=2}^{2^N} \mathbb{1}_{T > t_{k-1}} \frac{1}{1 - F_{t_{k-1}}} \int [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})] \mathbb{1}_{t_{k-1} < T \leq t_k} d\mathbb{P}$$

For $N \to \infty$ we get due to φ being $\mathbb{C}^{1,0}$:

$$\sum_{k=2}^{2^N} [\varphi(t_k, x_{t_{k-1}}) - \varphi(t_{k-1}, x_{t_{k-1}})] \frac{t_k - t_{k-1}}{t_k - t_{k-1}} \to \int_{(0,t]} \frac{\partial \varphi}{\partial t}(u, z_{u-})du.$$

The other term converges similar as in corollary 1. □

Remark A.3. *The path process is now openly represented as a stochastic transformation of the fundamental family of martingales $q(t, z)$ for the martingale part. The compensator takes again a form similar to the one we obtained in the first approach and the result in theorem 1.14 and again we will have to assume some intrinsic regularities to yield results about the characteristics of φ. Though we can easily generalize these results for functions φ that only satisfy the condition (C'), we will stop right here, as the similarities between the path dependent approach and the original value-at-time-t-oriented approach are recognizable already. In fact one might construct the path of the process up until time $t \in \mathbb{R}^+$ from the information about $\{T > t\}$ or $\{T \leq t\} \times \{Z\}$ respectively. Even more the jump time and height can be reconstructed from a particular path. This equivalence of notation seems to be unique for the single-jump processes that stay constant in between two jumps. For branching Brownian motion for example, the path-dependent notation seems to be preferable.*

A.4 Compensating the path dependent compensator

We see ourselves again confronted with a solution of an ODE. But keep in mind, that φ is now a functional in its second argument. Regarding the solution of the compensator-ODE, we've treated the second argument only as a parameter so far. The differential operator (or a.s. differential operator) was only aware of the first argument. In our path dependent approach this is not different. The path is just a parameter to us, for times t prior to the jump time T we will have the same argument $z^{(0)}$, which is just a constant path. After T we will have jumped to a random height Z, but the path will not change after that - our parameter here will then be Z_T. Due to these similarities to the point-dependent approach, we can easily adopt the result from section 1.5 for our path dependent approach:

Corollary A.4. *$v(t) = \varphi(t, x_t)$ is an $(\mathcal{F}_t)$-martingale, where*

$$\varphi(t, y) = b(y)(1 - \delta_{z^{(0)}}(y)) - \delta_{z^{(0)}}(y)\frac{1}{1 - F_t}\left(\int_{(0,t]\times X} b(z_s)d\mu(s, z) + r\right),$$

and $b : D([0, \infty), X) \to \mathbb{R}$ with $b \in L^1_{loc}(\mathbb{P})$ and $r \in \mathbb{R}$ arbitrary.

Appendix B

Discrete jump heights

This section discusses an alternative way for more constrained processes. We use the notation of chapter 1. The sections below can be read right after theorem 1.10, where the discrete Version of the jump process is provided with a general Doob-decomposition. An interesting special case arises under the following assumption:

Let $Z(\omega) \in \{z_1, \ldots, z_n\} \subset X$.

B.1 Markovian approach

The interesting part is the now available ability to 'pull' the difference $[\varphi(t_k, z) - \varphi(t_k, z_0)]$ out of the integral in the compensator part of the discrete Doob-Meyer compensator (see theorem refDiscResultDM)

$$
\begin{aligned}
A_{t_k}^{(N)} - A(N)_{t_{k-1}} &= [\varphi(t_k, x(t_{k-1})) - \varphi(t_{k-1}, x(t_{k-1}))] \\
&\quad + \mathbb{1}_{T>t_{k-1}} \frac{1}{1 - F_{t_{k-1}}} \int_{(t_{k-1}, t_k] \times X} [\varphi(t_k, z) - \varphi(t_k, z_0)] d\mu(s, z).
\end{aligned}
$$

For this we write

$$
[\varphi(t_k, Z) - \varphi(t_k, z_0)] = \sum_{i=1}^{n} [\varphi(t_k, z_i) - \varphi(t_k, z_0)] \mathbb{1}_{Z=z_i}.
$$

The measures $dF_s^{\{z_i\}}$ on $\mathbb{R}^+$ concentrate now on the atoms of the Z-distribution. We can write

$$
F_t^{\{z_i\}} = \mu([0,t] \times \{z_i\}) = \mathbb{P}(T \in [0,t], Z = z_i) = \mathbb{P}(T \in [0,t] | Z = z_i) \mathbb{P}(Z = z_i)
$$

and would get

$$
\begin{aligned}
\int_{\mathbb{R}^+ \times X} g(s, z) d\mu(s, z) &= \int_{\mathbb{R}^+} \sum_{i=1}^{n} g(s, z_i) \mathbb{P}(T \in dt | Z = z_i) \mathbb{P}(Z = z_i) \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^+} g(s, z_i) dF_u^{\{z_i\}} \qquad \text{(B.1.1)}
\end{aligned}
$$

with $B = \{z_i\}$ and F_k^B as defined in section 1.1 before.

We immediately get

$$\begin{aligned}
&\int_{(t_{k-1},t_k]\times X}[\varphi(t_k,z)-\varphi(t_k,z_0)]\mu(ds,dz)\\
=&\int_Y[\varphi(t_k,Z(\omega))-\varphi(t_k,z_0)]\mathbb{1}_{t_{k-1}<T\le t_k}(\omega)d\mathbb{P}(\omega)\\
=&\sum_{i=1}^n\int_Y[\varphi(t_k,z_i)-\varphi(t_k,z_0)]\mathbb{1}_{Z=z_i}(\omega)\mathbb{1}_{t_{k-1}<T\le t_k}(\omega)d\mathbb{P}(\omega)\\
=&\sum_{i=1}^n[\varphi(t_k,z_i)-\varphi(t_k,z_0)]\int_Y\mathbb{1}_{Z=z_i}(\omega)\mathbb{1}_{t_{k-1}<T\le t_k}(\omega)d\mathbb{P}(\omega). \qquad \text{(B.1.2)}
\end{aligned}$$

The remaining integral might also be written as

$$\begin{aligned}
&\int_{\mathbb{R}^+\times X}\mathbb{1}_{Z=z_i}(\omega)\mathbb{1}_{t_{k-1}<T\le t_k}(\omega)d\mathbb{P}(\omega)\\
=&\int_{\mathbb{R}^+\times X}\mathbb{1}_{Z=z_i}(\omega)\mathbb{1}_{T\le t_k}(\omega)d\mathbb{P}-\int_Y\mathbb{1}_{Z=z_i}(\omega)\mathbb{1}_{T\le t_{k-1}}(\omega)d\mathbb{P}\\
=&\,\mathbb{P}(T\le t_k, Z\in\{z_i\})-\mathbb{P}(T\le t_{k-1},Z\in\{z_i\})\\
=&\,F_{t_k}^{\{z_i\}}-F_{t_{k-1}}^{\{z_i\}}.
\end{aligned}$$

For $K\to\infty$ we will eventually end up with

$$\begin{aligned}
&\sum_{i=1}^n\sum_{k=1}^{2^N}[\varphi(t_k,z_i)-\varphi(t_k,z_0)]\left[F_k^{\{z_i\}}-F_{k-1}^{\{z_i\}}\right]\\
\overset{N\to\infty}{\to}&\sum_{i=1}^n\int_{[0,t]}[\varphi(u,z_i)-\varphi(u,0)]dF_u^{\{z_i\}}\\
\overset{(B.1.1)}{=}&\int_{[0,t]\times X}[\varphi(u,z)-\varphi(u,0)]d\mu(u,z)\\
\overset{(1.1.14)}{=}&\sum_{i=1}^n\int_{[0,t]}[\varphi(u,z_i)-\varphi(u,0)]\mathbb{P}(Z=\{z_i\}|T=u)dF_u\\
=&\int_{[0,t]}\sum_{i=1}^n[\varphi(u,z_i)-\varphi(u,0)]\mathbb{E}\left[\mathbb{1}_{Z=\{z_i\}}|T=u\right]dF_u\\
=&\int_{[0,t]}\mathbb{E}\left[\sum_{i=1}^n\varphi(u,z_i)\mathbb{1}_{Z=\{z_i\}}|T=s\right]\mathbb{P}(T\in ds)-\varphi(u,0)[F_t-F_0]\\
=&\int_{[0,t]}\mathbb{E}\left[\varphi(u,Z)|T=s\right]\mathbb{P}(T\in ds)-\varphi(u,0)[F_t-F_0]
\end{aligned}$$

which is just a side note for now. We want to concentrate on the fact that we now have:

$$\int_{(t_{k-1},t_k]\times X}[\varphi(t_k,z)-\varphi(t_k,z_0)]\mu(ds,dz)=\sum_{i=1}^{n}\sum_{k=1}^{2^N}[\varphi(t_k,z_i)-\varphi(t_k,z_0)]\left[F_{t_k}^{\{z_i\}}-F_{t_{k-1}}^{\{z_i\}}\right]. \tag{B.1.3}$$

Upon closer inspection $\varphi(t_k,x(t_k))-\varphi(t_k,x(t_{k-1}))$ is not zero if and only if the jump time $T\in(t_{k-1},t_k]$. With the above assumption of discrete-valued jump heights we get:

$$[\varphi(t_k,x(t_k))-\varphi(t_k,x(t_{k-1}))]=\sum_{i=1}^{n}(\varphi(t_k,z_i)-\varphi(t_k,z_0))[p(t_k,\{z_i\})-p(t_{k-1},\{z_i\})] \tag{B.1.4}$$

Let us sum up the above discussion in a

Corollary B.1. *The Doob-Meyer decomposition of the discrete process $(x(t_k))$ with discrete-valued jump heights is given by the martingale part $(M_{t_k}^N)$ and the previsible compensator part $(A_{t_k}^N)$, each given respectively by*

$$\begin{aligned}
M_k^N-M_{k-1}^N&=\sum_{i=1}^{n}(\varphi(t_k,z_i)-\varphi(t_k,z_0))\left[M_{t_k}^{(i),N}-M_{t_{k-1}}^{(i),N}\right],\\
A_k^N-A_{k-1}^N&=[\varphi(t_k,x(t_{k-1}))-\varphi(t_{k-1},x(t_{k-1}))]\\
&\quad+\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\sum_{i=1}^{n}(\varphi(t_k,z_i)-\varphi(t_k,z_0))\left[F_{t_k}^{\{z_i\}}-F_{t_{k-1}}^{\{z_i\}}\right].
\end{aligned}$$

were $M_{t_k}^{(i),N}=p(t_k,\{z_i\})-\sum_{j=1}^{k-1}\mathbb{1}_{T>t_j}\frac{1}{1-F_j}\left[F_{j+1}^{\{z_i\}}-F_j^{\{z_i\}}\right]$ is an $(\mathcal{F}_{t_k})$-martingale.
In particular: $M_{t_k}^{(i),N}$ is the martingale part of the Doob-Meyer decomposition of the process $(p(t_k,\{z_i\}))_k$

Proof. Use (B.1.3) and (B.1.4) to achieve the stated representation. The only thing left to show is that the $M_k^{(i)}$ are $(\mathcal{F}_{t_k})$-martingales and part of the particular Doob-Meyer decomposition. The conditional expectation of the process $(p(t_k,\{z_i\}))_k$ is given by

$$\begin{aligned}
\mathbb{E}[p(t_k,\{z_i\})|\mathcal{F}_{t_{k-1}}]&=\mathbb{E}[\mathbb{1}_{Z=z_i}\mathbb{1}_{T\le t_k}|\mathcal{F}_{t_{k-1}}]\\
&=\mathbb{E}[\mathbb{1}_{Z=z_i}\mathbb{1}_{T\le t_{k-1}}|\mathcal{F}_{t_{k-1}}]+\mathbb{E}[\mathbb{1}_{Z=z_i}\mathbb{1}_{t_{k-1}<T\le t_k}|\mathcal{F}_{t_{k-1}}]\\
&=\mathbb{1}_{Z=z_i}\mathbb{1}_{T\le t_{k-1}}+\mathbb{1}_{T>t_{k-1}}\frac{\int\mathbb{1}_{Z=z_i}\mathbb{1}_{t_{k-1}<T\le t_k}d\mathbb{P}}{1-F_{t_{k-1}}}\\
&=\mathbb{1}_{Z=z_i}\mathbb{1}_{T\le t_{k-1}}+\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\left[F_{t_k}^{\{z_i\}}-F_{t_{k-1}}^{\{z_i\}}\right]
\end{aligned}$$

By the instructions from the proof of the Doob-Meyer decomposition we get as usual

$$M_{t_k}^{(i),N}-M_{t_{k-1}}^{(i),N}=[p(t_k,\{z_i\})-p(t_{k-1},\{z_i\})]-\mathbb{1}_{T>t_{k-1}}\frac{1}{1-F_{t_{k-1}}}\left[F_{t_k}^{\{z_i\}}-F_{t_{k-1}}^{\{z_i\}}\right]$$

which is an $(\mathcal{F}_{t_k})$-martingale by construction. □

The above special case with discrete Z is also quite instructive on how to deal with the general case. For that let us set $x^n(t)$ to be constant on the sets of a partition:

$$x^n(t,\omega) := \sum_{i=0}^{n} z_i \mathbb{1}_{B_i}(\omega), \tag{B.1.5}$$

with $B_0 = \{T \geq t\}$ and $B_i = \{T \leq t, Z \in C_i\}$ for $i \in \{1,\dots,n\}$ where $C^{(n)} := (C_i)_{i\in\{1,\dots,n\}}$ is a partition of $X \setminus \{z_0\}$ (to account for all the values of Z when the jump already happened) and $z_i = \sup_{\omega \in C_i} Z(\omega)$ for $i = 1, 2, \dots$. Inside of the transforming function φ this would look like

$$v^n(k) := \varphi(t_k, x^n(t_k)) = \sum_{i=0}^{n} \varphi(t_k, z_i) \mathbb{1}_{B_i}$$

By this approximation we restrict our process to being discrete-valued and the above corollary applies. Choose a monotonously increasing sequence of such partitions $C^{(n)}$ (i.e. $C^{(n)} < C^{(n+1)}$ or $C^{(n+1)}$ is finer than $C^{(n)}$) and note that

- $v^n(k) \geq v(k)$ for all $k \in \{1,\dots,K\}$ and $n \in \mathbb{N}$ and
- $v^{n+1}(k) \leq v^n(k)$ (i.e. pointwise monotone decreasing).

Hence $v^n \to v$ pointwise and monotone. Take corollary B.1 for the process $(v^n(t_k))_{k\in\{1,\dots,2^N\}}$ (denote martingale and compensator as $M_k^{n,N}$ and $A_k^{n,N}$ respectively) and we end up with:

$$M_k^{n,N} - M_{k-1}^{n,N} = \sum_{i=1}^{n} \left(\varphi(t_k, z_i) - \varphi(t_k, z_0)\right) \left[M_k^{(i),n,N} - M_{t_{k-1}}^{(i),n,K} \right],$$

where

$$M_k^{(i),n,N} = p(t_k, B_i) - \sum_{j=1}^{k-1} \mathbb{1}_{T > t_j} \frac{1}{1 - F_{t_j}} \left[F_{t_{j+1}}^{B_i} - F_{t_j}^{B_i} \right]$$

and

$$F_{t_j}^{B_i} = \int_\Omega p(t_j, B_i) d\mathbb{P} = \mu([0, t_j], B_i).$$

B.2 Pathdependent aproach

Assume again just finite different jump heights, i.e. values for the random variable Z:

$$Z(\omega) \in \{z^{(1)}, \dots, z^{(n)}\} \in \mathcal{S}. \tag{B.2.1}$$

Under these assumptions we denote the path which jumps at time u to the level $z^{(i)}$ by $z_u^{(i)}$, so we will get:

$$x_t \mathbb{1}_{T \leq t} = \sum_{i=1}^{n} z_T^{(i)} \mathbb{1}_{Z = z^{(i)}} \mathbb{1}_{T \leq t}.$$

We take the the notation of section 2 and set:

$$F_t^{(i)} := \mathbb{P}(T \leq t, Z = z^{(i)}),$$

$$M_t^{(i)} := p(t, \{z^{(i)}\}) - \int_{(0,t]\times X} \mathbb{1}_{T\geq u} \frac{1}{1-F_{u-}} d\mu(u, \{z^{(i)}\}).$$

Note that the assumption on Z yields, that the different jump heights are now atoms of the conditional distribution $\mathbb{P}(Z \in \cdot | T = u) = \mu_u(\cdot)$.

We summarize the limiting results under discrete jump heights for the martingale part seperately:

Corollary B.2. *Under the assumption of discrete jump heights Z the limiting martingale part of the process is given by*

$$M_t = \varphi(0, z_0^{(0)}) + \sum_{i=1}^{n} \int_{(0,t]\times X} \left(\varphi(u, z_u^{(i)}) - \varphi(u, z_u^{(0)})\right) dM_u^{(i)}.$$

Proof. Take the result from corollary1 and use the partition-property and the "‘conditional-atom-property"’:

$$M_t = \int_{[0,t]\times\mathcal{S}} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] dp(u, z) + \int_{(0,t]\times X} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] d\tilde{p}(u, z)$$

then for one we've got

$$\int_{(0,t]\times X} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] dp(u, z)$$
$$= \sum_{i=1}^{n} \int_{(0,t]\times X} \left[\varphi(u, z_u^{(i)}) - \varphi(u, z^{(0)})\right] dp(u, z^{(i)})$$

and on the other hand we know

$$\int_{(0,t]\times\mathcal{S}} \left[\varphi(u, z_u) - \varphi(u, z^{(0)})\right] d\tilde{p}(u, z)$$
$$= \sum_{i=1}^{n} \int_{(0,t]} \left[\varphi(u, z_u^{(i)}) - \varphi(u, z^{(0)})\right] d\tilde{p}(u, z^{(i)}).$$

Since $d\tilde{p}(u, z^{(i)}) = \frac{1}{1-F_{u-}} \mathbb{1}_{T\geq u} d\mu(u, z^{(i)})$ we end up with the claimed equality. □

Now we do the same for the compensator part:

Corollary B.3. *Under the above assumption the limiting compensator part of the process is given by*

$$A_t = \int_{(0,t]} \frac{\partial\varphi}{\partial t}(u, x_{u-}) du + \sum_{i=1}^{n} \int_{(0,t]\times X} \frac{\mathbb{1}_{T\geq u}}{1-F_{u-}} [\varphi(u, x_u) - \varphi(u, z_u^{(0)})] d\mu(u, z^{(i)})$$

Proof. In the form of Lemma A.2 we get

$$\begin{aligned}
A_{t_K} &= \sum_{k=1}^{K} [\varphi(t_k, x_{t_{k-1}}) - \varphi(t_{k-1}, x_{t_{k-1}})] \\
&\quad + \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int [\varphi(t_k, x_{t_k}) - \varphi(t_k, z_k^{(0)})] \mathbb{1}_{t_{k-1}<T\leq t_k} d\mathbb{P} \\
&= \sum_{k=2}^{K} \frac{\varphi(t_k, x_{t_{k-1}}) - \varphi(t_{k-1}, x_{t_{k-1}})}{t_k - t_{k-1}} [t_k - t_{k-1}] \\
&\quad + \sum_{k=2}^{K} \sum_{i=1}^{n} \mathbb{1}_{T>t_{k-1}} \frac{1}{1-F_{t_{k-1}}} \int [\varphi(t_k, z_k^{(i)}) - \varphi(t_k, z_k^{(0)})] \mathbb{1}_{Z=z^{(i)}} \mathbb{1}_{t_{k-1}<T\leq t_k} d\mathbb{P} \\
&\to \int_{(0,t]} \frac{\partial}{\partial t} \varphi(u, x_{u-}) du \\
&\quad + \sum_{i=1}^{n} \int_{[0,t]} \mathbb{1}_{T\geq u} \frac{1}{1-F_{u-}} \int [\varphi(u, z_u^{(i)}) - \varphi(u, z_u^{(0)})] d\mu(u, z^{(i)})
\end{aligned}$$

where we used the same pointwise approximation arguments as previously.

□

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Wiesbaden, April 29, 2022