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The homotopy type of the space of tight contact structures on the 3-sphere

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Abstract
This work shows that the connected component $\Xi_0(S^3)$ of the standard tight contact structure in the space of (tight) contact structures on S^3 that are fixed at one point has the homotopy type of a point.
The problem is transferred to a family of vector fields on S^2 using Giroux's theory of surfaces in contact manifolds. Their singular points are treated via 3 types of neighbourhoods. A deformation of contact structures is described that deforms the family of vector fields and *eliminates* these neighbourhoods. Building on this construction an algorithm is given that deforms a loop of contact structures in $\Xi_0(S^3)$ until all spheres are convex surfaces with respect to each contact structure. In this situation a homotopy of this loop to the constant one can be constructed.
Via the Serre fibration

$$\mathrm{Diff}(S^3) \to \Xi(S^3)$$

whose fibre over $\xi \in \Xi(S^3)$ is the group of contactomorphisms $\mathrm{Cont}(S^3, \xi)$ the statement implies that every loop of diffeomorphisms of S^3 that fixes a 2-plane in the tangent space of one point is homotopic to a loop of contactomorphisms of the standard contact structure ξ_{st}.

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Contents

Introduction

Contact structures were introduced by Sophus Lie in his work on 'Berührungstransformationen' (contact transformations) in 1896, but he traces the origins back even further.

Their ubiquity has not been noticed until the early 1970s as Lutz and Martinet discovered that there are contact structures on any closed 3-manifold. The importance of contact structures in the theory of classical mechanics became apparent after Gromov's influential work in 1985 and the following rise of their older even-dimensional sibling, symplectic geometry.

For a more detailed account on the history of contact structures than this introduction will and can contain I would like to refer the reader to the book by Geiges [Gei08], without which no historical overview of contact structures could be complete.

A *contact structure* is a maximally non-integrable hyperplane field in the tangent space of a manifold of odd-dimensions. Contact structures occur naturally on those submanifolds of codimension 1 in *symplectic manifolds* that are transverse to a *Liouville vector field*. Such submanifolds arise as energy hypersurfaces in phase spaces of classical mechanical systems such as the much studied 3-body problem.

Symplectic and contact structures are alike in the sense that there are Darboux theorems: As are symplectic structures, contact structures are locally indistinguishable, any two points in contact manifolds have isomorphic neighbourhoods.

One can still ask about their global structure. Bennequin [Ben83] discovered that contact manifolds whose contact structure takes a prescribed form near an embedded 2-disc, an *overtwisted disc*, are of a different kind than those who do not admit an overtwisted disc. The latter ones are called *tight* contact manifolds. In 1989, Eliashberg [Eli89] showed that the overtwisted contact structures have a degree of flexibility and their isotopy classification coincides with their homotopy classification as tangent plane fields.

Tight contact structures are both less flexible and more rare. In fact, one can always construct an overtwisted contact structure from a tight one using a Lutz twist, but the converse direction is in general not possible. For the 3-sphere, Eliashberg [Eli92] found in 1992 that any two tight contact structures are isotopic. In particular, tight contact structures only exist in one homotopy class of plane fields.

We do not have to worry about the correct notion of homotopical equivalence of two contact structures, as by Gray stability any two contact structures ξ_0, ξ_1 that are (smoothly) homotopic through contact structures are already *isotopic*, i.e. there is a path of diffeomorphisms starting at the identity whose time-1 map sends ξ_0 to ξ_1.

This settles the question for overtwisted contact structures on S^3 but leaves the question about the homotopy type of the space of tight contact structures. In his work about tight contact structures Eliashberg states without proof that the space of tight contact structures on S^3 that are fixed at one point is contractible.

This work studies aforementioned space and verifies the claim.

A second reason to study the full homotopy type of this space is the following. A diffeomorphism ψ of a manifold M maps a contact structure ξ to a contact structure $T\psi(\xi)$ via its differential. Thus the group of diffeomorphisms acts on the space $\Xi(M)$ of contact structures on M. Its kernel with respect to ξ, the diffeomorphisms that map the contact structure ξ to itself, form the group of *contactomorphisms* $\mathrm{Cont}(M,\xi)$. In fact, the action of the diffeomorphisms on the group of contact structures

$$\mathrm{Diff}(M) \rightarrow \Xi(M)$$

is a Serre fibration with fibre $\mathrm{Cont}(M,\xi)$ over $\xi \in \Xi(M)$.

Understanding the group of diffeomorphisms is a central task in differential topology. Even for $M = S^3$ the proof of Smale's conjecture that $\mathrm{Diff}\big(S^3\big)$ has the homotopy type of $O(4)$ is a deep result.

Geiges and Zehmisch [GZ10] showed that the group of contactomorphisms of $\big(S^3, \xi_{st}\big)$, the 3-sphere with the standard, tight contact structure, is connected by considering S^3 as the boundary of the 4-ball and filling the latter with holomorphic discs. Similar arguments can only work for tight contact structures as only these can bound compact symplectic manifolds.

The fact that the connected component of ξ_{st} in $\Xi(S^3)$ is contractible once we fix the contact structures in one point implies that every loop of diffeomorphisms that fixes said contact plane is homotopic to a loop of contactomorphisms of ξ_{st}. This may allow further development in understanding the group of diffeomorphisms via the group of contactomorphisms of (S^3, ξ_{st}).

Let me conclude with a few words about the argument and an outline of the present work.

The argument is inspired by Giroux's proof that the space of tight contact structures on S^3 is connected (see [Gir00]) as it is presented in [Gei08]. The argument heavily uses Giroux's theory of characteristic foliations on surfaces (see [Gir91]) and the observation that this foliation recovers the contact structure in a neighbourhood of the surface.

Chapter I begins with a Darboux theorem for families of contact structures and the observation that Gray stability carries over to families of contact structures as well. Using these, in Section I.6, an isotopy is given that makes a given loop of contact structures on S^3 coincide with ξ_{st} outside a compact ball away from the poles of S^3.

The complement of two disc-shaped neighbourhoods of the poles is foliated by 2-spheres and the loop of contact structures is determined by the characteristic foliations they induce on the spheres. These singular foliations can be understood as a *movie* of vector fields on S^2.

It turns out that if all of these vector fields are of a nice form, if they belong to characteristic foliations of *convex surfaces*, then each vector field admits a closed curve that separates positive from negative *singular points*. In this case we can construct a smooth family of such curves (Section I.10), bring them to the equator of S^2 and are then able to construct an isotopy of our given loop to the constant one ξ_{st}, see Section I.11.

Chapter II first reminds of characteristic foliations and of basic notions about dynamical systems. Whereas in the non-parametric applications one usually uses genericity results, most of these fail for sufficiently large dimension of the parameter space. What still remains stable in families is the topology of the phase portraits *near* singular points. For example, a disc that contains a source and whose boundary is transverse to a given vector field X will still be a disc whose boundary is transverse to all vector fields Y close to X, even should they contain uncountably many singular points inside this disc. (In a generic family, each surface only has finitely many singular points, see Section III.1. However, these will be degenerate.) Section II.3.1 reviews the conditions on a vector field to belong to a convex surface and Section II.3.2 phrases these conditions in terms of neighbourhoods of singular points. What parametric neighbourhoods should be and what properties we ask for is explained in Section II.4.

Thus set up we are ready to manipulate our loop of contact structures to deform the movie of vector fields until each vector field belongs to a convex surface. The strategy is to try to remove as many singular points (or neighbourhoods) as possible. Section III.2 contains an elimination deformation that not only works for families of contact structures but also eliminates whole neighbourhoods of singular points. In order to find pairs of neighbourhoods this elimination can be applied to, we will consider in Section III.4 a graph of these neighbourhoods and show that it is a forest. Leaves of this graph can be eliminated. To aid this process we define in Section III.3 a complexity valuation of the vector fields together with said neighbourhoods. Finally Section III.5 describes how to actually perform these deformations, how to deal with overlapping deformations, and that the process terminates and gives the desired result: That each vector field from the movie belongs to a convex surface, that each contact structure is now such that all spheres are convex surfaces with respect to it.

This allows us to construct aforementioned homotopy of the (deformed) loop of contact structures to the constant one.

And now, let me not keep you from reading any longer.

I. Families of contact structures

1. Tight contact structures on S^3

Definition 1.1. *A 2-plane distribution $\xi \subset TM$ on a 3-manifold M given as the kernel of a 1-form $\alpha \in \Omega^1(M)$ such that*

$$\alpha \wedge \mathrm{d}\alpha \quad \textit{is a volume form}$$

is a **contact structure**. *In this case α is called a* **contact form**.

Remark 1.2. We can allow a contact structure ξ to be given as the kernel of only locally defined contact forms. If ξ is the kernel of a single globally defined 1-form α, this form induces an orientation on the 1-dimensional subspaces in TM complementary to ξ. Then we call the contact structure **cooriented**.

As one can obtain a cooriented contact structure from any given one by passing to a double cover, we will only consider cooriented contact structures.

Example 1.3. Consider the unit sphere in $\mathbb{C}^2$ with coordinates x_1, y_1, x_2, y_2. The restriction of the 1-form $x_1\,\mathrm{d}y_1 - y_1\,\mathrm{d}x_1 + x_2\,\mathrm{d}y_2 - y_2\,\mathrm{d}x_2$ is a contact form that we will call α_{st}. Denote its kernel by ξ_{st} and call it the **standard contact structure** on S^3.

The volume form $\alpha_{st} \wedge \mathrm{d}\alpha_{st}$ is positive with respect to the standard orientation on S^3.

Definition 1.4. *Contact structures that are given as the kernel of a contact form α that satisfy*

$$\alpha \wedge \mathrm{d}\alpha > 0 \tag{I.1}$$

are called **positive**.

A diffeomorphism ψ of M allows us to define a 1-form $\left(\psi^{-1}\right)^*\alpha$. It is again a contact form. Its kernel is a contact structure and comprises the image of $\xi = \ker\alpha$ under the differential $T\psi$ of ψ, i.e.

$$\ker\Big(\left(\psi^{-1}\right)^*\alpha\Big) = T\psi(\ker\alpha) = T\psi(\xi)\,.$$

Hence the group of diffeomorphisms acts on the set of contact forms and on the set of contact structures.

Likewise, an isotopy of M induces a smooth path of contact forms and hence a smooth path of contact structure.

Definition 1.5. *We call a contact structure ξ_1* **isotopic** *to a given one ξ_0 if there is an isotopy ψ_t, $t \in [0,1]$, of M with $\psi_0 = \mathrm{id}_M$ and*

$$\xi_1 = T\psi_1(\xi_0)\,.$$

Bennequin [Ben83] observed that there are contact structures on S^3 that are homotopic as plane fields, but not isotopic as contact structures.

Definition 1.6. *An embedded disc Δ in a contact manifold (M,ξ) such that for each point $q \in \partial\Delta$*

- *$T_q(\partial\Delta) \subset \xi_q$ and*
- *$T_q\Delta \neq \xi_q$*

is an **overtwisted disc**. *If (M,ξ) contains an overtwisted disc, then (M,ξ) is called* **overtwisted**, *and* **tight** *otherwise.*

Remark 1.7. This definition is equivalent to the frequently given one requiring there to be single point in the interior Δ with $T_q\Delta = \xi_q$, as is explained in [Gei08, Proposition 4.6.28].

The image of an overtwisted disc under a diffeomorphism is again an overtwisted disc. In particular, a contact structure that is isotopic to an overtwisted contact structure is overtwisted.

Theorem 1.8 (Eliashberg [Eli92]). *Every positive tight contact structure on S^3 is isotopic to ξ_{st}.*

The same publication states without proof that this theorem can be generalised to multiparametric families of tight contact structures. The goal of this work is to give a proof of this generalisation (Theorem 2.1) using different methods than those used in [Eli92].

2. Main results and overview of proof

Denote by $\Xi_0(S^3)$ the space of tight positive contact structures on S^3 with the property that their contact plane at the point $(-i, 0) \in S^3 \subset \mathbb{C}^2$ agrees with the contact plane of ξ_{st}. This is the plane $\ker \alpha_{st,(-i,0)} \cap T_{(-i,0)}S^3 = \ker(\mathrm{d}x_1) \cap T_{(-i,0)}S^3 = \{0\} \times \mathbb{C}^2$.

Theorem 2.1. *Every S^k-parametric family of contact structures in $\Xi_0(S^3)$ is homotopic to the constant one (ξ_{st}).*

We will construct an homotopy of parametric contact structures as follows.

Step 1: There is a parametric Darboux Theorem that yields a first isotopy after which all contact structures agree in a neighbourhood of $(-i, 0)$.

Step 2: There is an isotopy that enlarges the neighbourhood where all contact structures agree with the standard one until it contains a hemisphere.

Step 3: The complement of two discs around the poles is foliated by spheres. The characteristic foliation of any sphere with respect to any contact structure agrees with the characteristic foliation with respect to the standard contact structure outside a disc. Successive elimination of singular points will turn the characteristic foliations on all spheres into nice forms and thereby make the spheres *convex*.

Step 4: An isotopy will bring the *dividing curves* of all spheres to the equator.

Step 5: This will then allow us to find an isotopy between the parametric family of contact structures and the constant family (ξ_{st}).

The space of cooriented 2-planes in $\mathbb{R}^3$ is the double cover of the Grassmannian $G_2(\mathbb{R}^3)$ and can be identified with the unit sphere $S^2 \subset \mathbb{R}^3$. Cooriented contact structures are hence given by specific smooth sections in STS^3, the unit sphere bundle to TS^3.

The space of smooth sections of STS^3 over the compact manifold S^3 is a tame Fréchet manifold, *cf.* [Ham82, Example I.4.1.6]. The contact condition is $\mathcal{C}^1$-open and hence $\mathcal{C}^\infty$-open in STS^3. Consequently the space $\Xi(S^3)$ of (cooriented) contact structures on S^3 is a tame Fréchet manifold.

Lemma 2.2. *The space $\Xi_0(S^3)$ is a tame Fréchet manifold*

Proof. Consider the fibre of STS^3 over the point $(-i, 0)$ and pick an identification with the unit sphere $S^2 \subset \mathbb{R}^3$ such that ξ_{st} takes the value $(1, 0, 0) \in S^2$ in $(-i, 0)$. Denote the subset $\{(x, y, z) \in S^2 \,|\, x > 0\} \cong B^2 \subset \mathbb{R}^2$ by E.

The space $\Xi_\varepsilon(S^3)$ of contact structures on S^3 that lie in the connected component of ξ_{st} and that in the fibre over $(-i, 0)$ take values in E is an open subset in $\Xi(S^3)$ and hence a tame Fréchet manifold.

Consider the projection $\Xi_\varepsilon(S^3) \to E \to \mathbb{R}^2$ that assigns to a contact structure ξ its value in the fibre over $(-i, 0)$. Its differential is surjective and its target is a finite dimensional vector space, so the preimage of $(0, 0)$, the set $\Xi_0(S^3)$, is a Fréchet submanifold, *cf.* [Ham82, Theorem III.2.3.1]. □

Theorem 2.3. *The space $\Xi_0(S^3)$ has the homotopy type of a point.*

Proof. Since the Fréchet manifold $\Xi_0(S^3)$ is metrizable, an infinite-dimensional extension of J. H. C. Whitehead's theorem (see [Pal66] and [Eel66]) implies that $\Xi_0(S^3)$ is contractible. □

3. Families of contact structures

Let $(\xi^s) \subset \Xi_0(S^3)$, $s \in S^k$, be a continuous S^k-family of contact structures that agree with ξ_{st} in $(-i, 0)$. Let them be given as kernels of a family (α^s) of contact forms on S^3. After rescaling these, we may assume that all contact forms agree with α_{st} in the point $(-i, 0)$.

The contact forms are sections in the bundle of differential 1-forms on S^3. Hence, after a Weierstraß-type approximation, bearing in mind that the contact condition is $\mathcal{C}^1$-open, we may assume that the contact forms α^s form a smooth family and that each α^s still agrees with α_{st} in the point $(-i, 0)$.

4. Step 1: Darboux theorem

Let us observe that there is a family of isotopies after which the contact forms agree with α_{st} in a whole neighbourhood of the point $(-i, 0)$. This is a consequence of a parametric version of the Darboux theorem for contact forms.

Proposition 4.1 (Parametric Darboux theorem)**.** *Let M be a 3-dimensional connected manifold, $\mathcal{P}$ a compact manifold and α^ρ a smooth $\mathcal{P}$-family of contact forms on M that is constant at one distinguished point.*

Then around any given point $q \in M$ there are S^ρ-parametric coordinates x^ρ, y^ρ, z^ρ in a neighbourhood U of q such that $q = (0, 0, 0)$ and

$$\alpha^\rho|_U = \mathrm{d}z^\rho + x^\rho\,\mathrm{d}y^\rho - y^\rho\,\mathrm{d}x^\rho \quad \textit{for all } \rho \in \mathcal{P}.$$

Proof. Consider any smooth chart around q. This allows us to assume that $M = \mathbb{R}^3$ and q is the origin. The Reeb vector fields $R_{\alpha^\rho,0}$ in the origin are a smooth $\mathcal{P}$-family of vectors. Denote it by ∂_{z^ρ}.

Assertion 1. *We can solve the problem in the origin, i.e. there are smooth $\mathcal{P}$-families ∂_{x^ρ}, ∂_{y^ρ} of vectors at the origin such that*

$$\alpha^\rho(\mathbf{0}) = (\mathrm{d}z^\rho + x^\rho\,\mathrm{d}y^\rho - y^\rho\,\mathrm{d}x^\rho)(\mathbf{0}),$$

where $\mathrm{d}x^\rho$, $\mathrm{d}y^\rho$ and $\mathrm{d}z^\rho$ are dual to ∂_{x^ρ}, ∂_{y^ρ} and ∂_{z^ρ}, respectively, in $\mathbf{0} \in \mathbb{R}^3$.

Proof. Consider the contact planes $\xi^s(\mathbf{0})$ in the origin. They define a rank-2 vector bundle over $\mathcal{P}$. As any path from q

to the distinguished point p in which α^ρ is constant gives an homotopy of the family $\xi^\rho(q)$ to the constant family $\xi^\rho(p)$. Hence the bundle $\mathbb{R}^2 \to \xi^\bullet(\mathbf{0}) \to \mathcal{P}$ is trivial.

Pick any section ∂_{x^ρ} without zeroes. As $\mathrm{d}\alpha^\rho$ is non-degenerate on the contact planes $\xi^\rho(\mathbf{0})$,

$$\iota_{\partial_{y^\rho}} \mathrm{d}\alpha^\rho = -2\,\mathrm{d}x^\rho$$

uniquely defines a $\mathcal{P}$-parametric vector in the origin with

$$\mathrm{d}\alpha^\rho = 2\,\mathrm{d}x^\rho \wedge \mathrm{d}y^\rho. \qquad \square$$

Choose linear coordinates x^ρ, y^ρ and z^ρ on $\mathbb{R}^3$ such that in the origin the directions of x^ρ, y^ρ and z^ρ are given by ∂_{x^ρ}, ∂_{y^ρ} and ∂_{z^ρ}, respectively. Denote by $\mathrm{d}x^\rho$, $\mathrm{d}y^\rho$, $\mathrm{d}z^\rho$ the 1-forms dual to the coordinates x^ρ, y^ρ and z^ρ. Using these, define

$$\alpha_t^\rho := (1-t)(\mathrm{d}z^\rho + x^\rho\,\mathrm{d}y^\rho - y^\rho\,\mathrm{d}x^\rho) + t\alpha^\rho \quad \text{for } t \in [0,1].$$

This is a smooth family of 1-forms constant in the origin. They are also contact forms in the origin and, as the contact condition is open, contact forms in some neighbourhood U_1 of $\mathbf{0} \in \mathbb{R}^3$ for all $\rho \in \mathcal{P}$.

We will now use Moser's trick to obtain a $\mathcal{P}$-parametric isotopy ψ_t^ρ of $\mathbb{R}^3$ that is the identity away from the origin and that satisfies

$$\left(\psi_t^\rho\right)^* \alpha_t^\rho = \alpha_0^\rho = \mathrm{d}z^\rho + x^\rho\,\mathrm{d}y^\rho - y^\rho\,\mathrm{d}x^\rho \tag{I.2}$$

near the origin. Then the contact forms α^ρ have are of the form stated in the local coordinates

$$x \circ \left(\psi_1^\rho\right),\ y \circ \left(\psi_1^\rho\right),\ z \circ \left(\psi_1^\rho\right).$$

To construct ψ_t^ρ assume that ψ_t^ρ is the flow of a parametric vector field X_t^ρ. A necessary condition for (I.2) is

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\psi_t^\rho\right)^* \alpha_t^\rho\right) = \left(\psi_t^\rho\right)^* \left(\dot\alpha_t^\rho + \mathcal{L}_{X_t^\rho}\,\alpha_t^\rho\right)$$

which equals, using a Cartan's formula for time-dependent vector fields,

$$0 = \left(\psi_t^\rho\right)^* \left(\dot\alpha_t^\rho + \iota_{X_t^\rho} \mathrm{d}\alpha_t^\rho + \mathrm{d}\left(\iota_{X_t^\rho}\alpha_t^\rho\right)\right).$$

A proof of this version of Cartan's formula can be found in [Gei08, Lemma B.1]. As ψ_t^ρ are diffeomorphisms, this is equivalent to

$$0 = \dot\alpha_t^\rho + \iota_{X_t^\rho} \mathrm{d}\alpha_t^\rho + \mathrm{d}\left(\iota_{X_t^\rho}\alpha_t^\rho\right). \tag{I.3}$$

Split X_t^ρ uniquely into its component in Reeb direction (with respect to α_t^ρ) and a vector field Y_t^ρ in the contact structure $\ker \alpha_t^\rho$ and write

$$X_t^\rho =: h_t^\rho R_{\alpha_t^\rho} + Y_t^\rho$$

with $h_t^\rho \in \mathcal{C}^\infty(M)$. The equation (I.3) turns into

$$0 = \dot\alpha_t^\rho + \iota_{Y_t^\rho} \mathrm{d}\alpha_t^\rho + \mathrm{d}h_t^\rho. \tag{I.4}$$

Inserting the Reeb vector field gives

$$0 = \dot\alpha_t^\rho(R_{\alpha_t^\rho}) + R_t^\rho\left(h_t^\rho\right). \tag{I.5}$$

To solve the differential equation, denote the $\mathcal{P}$-parametric flow of the (time-dependent) vector field $R_{\alpha_t^\rho}$ by Ψ_t^ρ.

Assertion 2. *The flow Ψ_t^ρ exists for some time T and all $\rho \in \mathcal{P}$ on a neighbourhood V of $\mathbf{0}$ of $\{z = 0\}$ and there is a neighbourhood U_0 of $\mathbf{0}$ such that all points in U_0 lie in the image of $\Psi_t^\rho(V)$ for all $\rho \in \mathcal{P}$.*

Proof. Remember that in $\mathbf{0}$, the vector $R_{\alpha_t^\rho}$ corresponds to ∂_z^ρ for all $t \in [0, 1]$. In particular, it is transverse to $\{z = 0\}$. Hence for each ρ in the compact parameter space $\mathcal{P}$ and hence also for all $\rho \in \mathcal{P}$ there is an open neighbourhood W of $\mathbf{0}$ in $\{z = 0\}$ in which $R_{\alpha_t^\rho}$ is transverse to $\{z = 0\}$. Shrink W such that $\overline{W}$ still has this property. Regard the flow of the parametric vector field as a flow on $\mathcal{P} \times \mathbb{R}^3$ that has constant $\mathcal{P}$-component. As the smooth vector field satisfies a Lipschitz-inequality on $\mathcal{P} \times \overline{W}$ for all $t \in [0,1]$, every point η in the compact set $\mathcal{P} \times \overline{W}$ has an open neighbourhood U_η in $\mathcal{P} \times \mathbb{R}^3$ and some $0 < t_\eta < 1$ such that the flow Ψ_t^ρ is defined on U_η for time $0 \leq t < t_\eta$.

Hence Ψ_t^ρ is defined on some open neighbourhood $\tilde{V}$ of W in $\mathbb{R}^3$ for all $\rho \in \mathcal{P}$ and $0 \leq t < T$ up to some $T > 0$. Let V be the open set $\tilde{V} \cap \{z = 0\}$ and define

$$\tilde{U} := \left\{ \left(\rho, \Psi_t^\rho(x)\right) \,\middle|\, \rho \in \mathcal{P},\ 0 \leq t < T,\ x \in V \right\}.$$

The set $\tilde{U}$ contains the compact set $\mathcal{P} \times \overline{W}$. Hence there is an open neighbourhood U_0^+ of $\mathbf{0}$ in $\{z \geq 0\}$ such that $\mathcal{P} \times U_0^+ \subset \tilde{U}$. □

We will now solve (I.5) by integration. For all points p in U_0^+ there is a time $s_+^\rho(p)$ as well as a point $x^\rho(p)$ in $\{z = 0\}$ that both smoothly depend on ρ and p such that $p = \Psi_{s^\rho(p)}^\rho(x^\rho(p))$. Using these, define

$$\begin{aligned} h_t^\rho \colon \quad & U_0^+ & \to \quad & \mathbb{R} \\ p = \Psi_{s^\rho(p)}^\rho(x^\rho(p)) \quad & & \mapsto \quad & -\int_0^{s_+^\rho(p)} \dot{\alpha}_t^\rho\Big(R_{\alpha_t^\rho}\big(\Psi_\tau^\rho(x^\rho(p))\big)\Big)\, \mathrm{d}\tau. \end{aligned}$$

As $\dot{\alpha}_t^\rho$ vanishes in $\mathbf{0}$ for all $\rho \in \mathcal{P}$ and $t \in [0,1]$, we may, after shrinking U_0^+, assume that h_t^ρ is well-defined.

Similarly, there is a neighbourhood U_0^- of $\mathbf{0}$ in $\{z \leq 0\}$ such that there is a time $s_-^\rho(p)$ as well as a point $x^\rho(p)$ in $\{z = 0\}$ that both smoothly depend on ρ and p such that $x^\rho(p) = \Psi_{s^\rho(p)}^\rho(p)$. Define again

$$\begin{aligned} h_t^\rho \colon \quad U_0^- \quad & \to \quad \mathbb{R} \\ p \quad & \mapsto \quad -\int_0^{s_-^\rho(p)} \dot{\alpha}_t^\rho\Big(R_{\alpha_t^\rho}\big(\Psi_\tau^\rho(x^\rho(p))\big)\Big)\, \mathrm{d}\tau. \end{aligned}$$

Then h_t^ρ the unique solution of (I.5) in a neighbourhood $U_1 \subset U_0^+ \cup U_0^-$ of $\mathbf{0}$ with initial values $h_t^\rho \equiv 0$ on $\{z = 0\}$. As the functions h_t^ρ also smoothly depend on $\rho \in \mathcal{P}$, this shows that we constructed a $\mathcal{P}$-family of smooth functions.

This then defines X_t^ρ uniquely by equation (I.4) as $\mathrm{d}\alpha_t^\rho$ is non-degenerate on $\ker \alpha_t^\rho$. The contact forms α_t^ρ form a $\mathcal{P}$-family and hence do $\mathrm{d}\alpha_t^\rho$ as well as the functions h_t^ρ. Consequently, we obtain a $\mathcal{P}$-family of vector fields X_t^ρ.

In $\mathbf{0}$, the 1-forms $\dot{\alpha}_t^\rho$ and the functions h_t^ρ vanish and hence so do the vector fields X_t^ρ. If we define ψ_t^ρ as the flow of X_t^ρ, it is defined for all times t in the point $\mathbf{0}$. As a flow of a vector field is always defined on an open domain, it is defined uniquely for $t \in [0,1]$ on a sufficiently small neighbourhood $U \subset U_1$ of q for all $\rho \in \mathcal{P}$. The defining vector fields X_t^ρ form a $\mathcal{P}$-family and thus so do the isotopies ψ_t^ρ. Notice we did not integrate in ρ-direction. Hence we obtained ψ_1^ρ. □

Remark 4.2. Proposition 4.1 also holds for higher dimensional contact manifolds without changes in the proof.

Remark 4.3. In Proposition 4.1 we used the hypothesis that the family of contact forms is fixed at one point only in Assertion 1 to assure that the (symplectic) bundle $\mathbb{R}^2 \to \xi_{\mathbf{0}}^\bullet \to \mathcal{P}$ is trivial.

The theorem holds also true without this hypothesis if for example $\mathcal{P} = S^k$ for $k \neq 2$ as the bundle is trivial if its first Chern class vanishes.

Consider our S^k-family of contact forms α^s that coincide at the point $(-i, 0)$. Applying the Darboux theorem to $\alpha_{st}^{S^3}$ in $(-i, 0)$ gives coordinates x, y, z in a neighbourhood of $(-i, 0)$, in which $\alpha_{st}^{S^3}$ is given as $\mathrm{d}z + x\,\mathrm{d}y - y\,\mathrm{d}x$. The contact forms α^s agree with $\alpha_{st}^{S^3}$ in $(-i, 0)$, so in these coordinates, all α^s are of the form $\mathrm{d}z + x\,\mathrm{d}y - y\,\mathrm{d}x$ in the point $(-i, 0)$. The parametric Darboux theorem then gives a S^k-family of isotopies ψ_t^s, $t \in [0,1]$ of S^3 such that $\psi_0^s = \mathrm{id}_{S^3}$ and

$$\left(\psi_1^s\right)^* \alpha^s = \mathrm{d}z + x\,\mathrm{d}y - y\,\mathrm{d}x$$

in the coordinates just chosen. In particular, the family of contact structures (ξ^s) is isotopic to $\left(T\psi_1^s(\xi^s)\right)$ and the latter one coincides with $\xi_{st}^{S^3}$ on a neighbourhood U of $(-i, 0)$.

We shall denote $T\psi_1^s(\xi^s)$ again by ξ^s.

5. Gray stability

So far we deformed our given contact structures such that they agree near $(-i, 0)$ in S^3. To obtain an isotopy of contact structures on the complement, we will make use of *Gray stability*.

Proposition 5.1 (Gray's stability theorem)**.** *To a family of contact structures (ξ_t), $t \in [0,1]$, on a closed manifold M, there is an isotopy ψ_t, $t \in [0,1]$, of M such that*

$$T\psi_t(\xi_0) \;=\; \xi_t \quad \textit{for each } t \in [0,1].$$

A reference for this statement is [Gei08, Theorem 2.2.2]. To deal with families of contact structures, let us produce a parametric version. The proof using a Moser trick argument is analogous to the non-parametric version.

Proposition 5.2 (Parametric Gray stability)**.** *To a smooth family (ξ_t^s), $t \in [0,1]$, $s \in S^k$, of S^k-parametric contact structures on a closed manifold M, there is a S^k-parametric isotopy ψ_t^s, $t \in [0,1]$, of M such that*

$$T\psi_t^s(\xi_0^s) \;=\; \xi_t^s \quad \textit{for each } t \in [0,1] \textit{ and } s \in S^k.$$

Proof. Let α^s, $s \in S^k$, $t \in [0,1]$ be a smooth $(S^k \times [0,1])$-family of contact forms to ξ_t^s, i.e. $\xi_t^s = \ker \alpha_t^s$ for all $s \in S^k$ and $t \in [0,1]$. Assume that the isotopies ψ_t^s arise as the flow of a time-dependent S^k-parametric vector field X_t^s on M. Then the condition that $T\psi_t^s\big(\xi_0^s\big) = \xi_t^s$ translates into

$$\lambda_t^s \alpha_0^s = \left(\psi_t^s\right)^* \alpha_t^s \quad \text{for each } s \in S^k,\ t \in [0,1]$$

for some smooth family of functions $\lambda_t^s \colon\ M \;\to\; \mathbb{R}^+$. Differentiation with respect to t yields the necessary condition

$$\dot\lambda_t^s \alpha_0 = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\left(\psi_t^s\right)^* \alpha_t^s\Big) = \left(\psi_t^s\right)^* \Big(\dot\alpha_t^s + \mathcal{L}_{X_t^s}\, \alpha_t^s\Big)\,, \tag{I.6}$$

where the dot denotes the derivative with respect to t. The left hand side can be written as

$$\dot{\lambda}_t^s \alpha_0 = \frac{\dot{\lambda}_t^s}{\alpha_t^s} (\psi_t^s)^* \alpha_t^s = \mu_t^s (\psi_t^s)^* \alpha_t^s$$

with $\mu_t^s := \frac{\mathrm{d}}{\mathrm{d}t}(\log \lambda_t^s) \circ (\psi_t^s)^{-1}$. For the right hand side of equation (I.6), notice that

$$\dot{\alpha}_t^s + \mathcal{L}_{X_t^s} \alpha_t^s = \dot{\alpha}_t^s + \iota_{X_t^s} \mathrm{d}\alpha_t^s + \mathrm{d}\left(\iota_{X_t^s} \alpha_t^s\right) \tag{I.7}$$

by Cartan's Formula. Combining these, equation (I.6) is equivalent to

$$\mu_t^s \alpha_t^s = \dot{\alpha}_t^s + \iota_{X_t^s} \mathrm{d}\alpha_t^s + \mathrm{d}\left(\iota_{X_t^s} \alpha_t^s\right). \tag{I.8}$$

Let us assume that we can choose X_t^s to lie in $\ker \alpha_t^s = \xi_t^s$. Thereby we write equation (I.8) as

$$\mu_t^s \alpha_t^s = \dot{\alpha}_t^s + \iota_{X_t^s} \mathrm{d}\alpha_t^s\,, \tag{I.9}$$

which implies, inserting the Reeb vector field R_t^s,

$$\mu_t^s = \dot{\alpha}_t^s (R_t^s)\,.$$

This uniquely determines μ_t^s. Equation (I.9), that we can write as

$$\iota_{X_t^s} \mathrm{d}\alpha_t^s = \mu_t^s \alpha_t^s - \dot{\alpha}_t^s$$

then uniquely determines X_t^s as the right hand side has no component in Reeb-direction and $\mathrm{d}\alpha_t^s$ is non-degenerate on $\ker \alpha_t^s = \xi_t^s$.

We may integrate these smooth vector fields up to time 1. Its flow ψ_t^s smoothly depends on both $t \in [0,1]$ and $s \in S^k$ and by construction satisfies equation (I.6) and hence

$$T\psi_t^s(\xi_0^s) \;=\; \xi_t^s \quad \text{for each } t \in [0,1] \text{ and } s \in S^k.$$

□

Corollary 5.3 (Parametric Gray stability, relative version)**.** *Let* (ξ_t^s), $t \in [0,1]$, $s \in S^k$ *be a smooth family of* S^k*-parametric contact structures* *on a manifold M that is constant in t on a compact set $K \subset S^k \times M$. Then there is an S^k-parametric isotopy ψ_t^s, $s \in S^k$, of M such that*

$$T\psi_t^s(\xi_0^s) = \xi_t^s \quad \text{for each } t \in [0,1] \text{ and } s \in S^k$$

and ψ_t^s is stationary on K.

Proof. Let α_t^s, $s \in S^k$ be a smooth $(S^k \times [0,1])$-family of contact forms to ξ_t^s. Rescaling them we may assume that they coincide on K. In particular, their derivatives $\dot{\alpha}_t^s$ with respect to t vanish there.

Follow the proof of Proposition 5.2. That $\dot{\alpha}_t^s$ vanishes on K implies that $\mu_t^s = \dot{\alpha}_t^s(R_t^s)$ vanishes there. Consequently, we obtain $\iota_{X_t^s} \, \mathrm{d}\alpha_t^s = \mu_t^s \alpha_t^s - \dot{\alpha}_t^s = 0$ and hence $X_t^s \equiv 0$ on K.

The parametric vector field X_t^s is compactly supported and can be integrated up to time 1. Its flow ψ_t^s is stationary on $M \setminus K$. ☐

6. Step 2: Extending the Darboux neighbourhood

Let us come back to our situation on the 3-sphere. In a first step we used a parametric Darboux theorem to find an isotopy of the contact structures after which they agreed with ξ_{st} on a neighbourhood U of the point $(-i, 0) \in S^3$. We will see that an isotopy of families of contact structures yields contact structures that agree with ξ_{st} on a much larger neighbourhood U of $(-i, 0)$. In particular, we can arrange that all contact structures agree with the standard contact structure on a neighbourhood of a hemisphere. To do this, we will temporarily transform our situation to the euclidean space and give an isotopy there.

On $\mathbb{R}^3$ with coordinates u, v and w, the 1-form $\tilde{\alpha}_{st} := \mathrm{d}w + u\,\mathrm{d}v - v\,\mathrm{d}u$ is a contact form. Its induced contact structure on $\mathbb{R}^3$, $\tilde{\xi}_{st} := \ker \tilde{\alpha}_{st}$, is the **standard contact structure** on $\mathbb{R}^3$.

There is a contactomorphism

$$\psi\colon \; (S^3 \setminus \{(-i, 0)\}, \xi_{st}) \;\to\; (\mathbb{R}^3, \tilde{\xi}_{st}).$$

In [Gei08, Proposition 2.1.8] there is an explicit description of a contactomorphism $(S^3 \setminus \{(0, i)\}, \xi_{st}) \;\to\; (\mathbb{R}^3, \tilde{\xi}_{st})$ as a composition of the

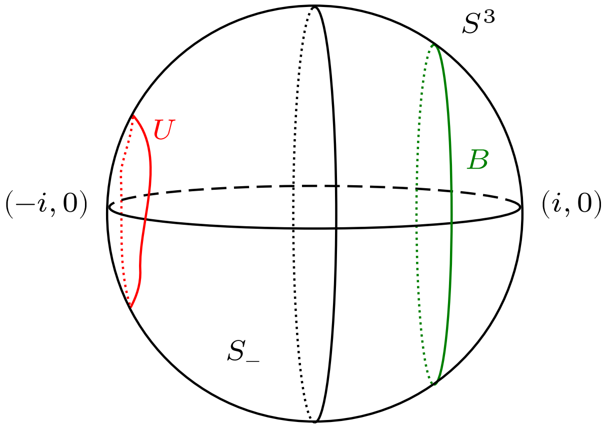


Figure 6.1.: The contact structures ξ^s agree with ξ_{st} on the complement of B

stereographic projection

$$(x_1 + iy_1, x_2 + iy_2) \mapsto \left(\frac{x_1}{1 - y_2}, \frac{y_1}{1 - y_2}, \frac{x_2}{1 - y_2} \right)$$

with the inverse of the map $\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$(r, \varphi, w) \mapsto \left(r, \varphi - w, \tfrac{1}{2} w \left(1 + \tfrac{1}{3} w^2 + r^2 \right) \right)$$

in cylindrical coordinates. Pre-composing it with the contactomorphism $(z, w) \mapsto (w, -z)$ of S^3 that sends $(-i, 0) \mapsto (0, i)$ gives the contactomorphism $(S^3 \setminus \{(-i, 0)\}, \xi_{st}) \to (\mathbb{R}^3, \tilde{\xi}_{st})$.

For any $s \in S^k$ the contact structure ξ^s agrees with ξ_{st} on the neighbourhood U of $(-i, 0)$. Its image $T\psi(\xi^s)$ under ψ therefore agrees with $\tilde{\xi}_{st}$ outside the compact ball $\psi(S^3 \setminus U)$.

We want to find contact isotopies such that the modified contact structures agree with ξ_{st} on the complement of the closed ball $B := \left\{ (z, w) \in S^3 \,\middle|\, \mathfrak{Im}\, z \geq 1/2 \right\}$ that is fully contained in the hemisphere $S_- := \left\{ (z, w) \in S^3 \,\middle|\, \mathfrak{Im}\, z \geq 0 \right\}$ of S^3 as in Figure 6.1.

Observe that ψ sends $(i, 0)$ to $(0, 0, 0)$. In the image of ψ the condition that ξ^s agree with ξ_{st} in the complement of B translates to the condition that $T\psi(\xi^s)$ agree with $\tilde{\xi}_{st}$ outside a smaller compact ball around the origin in $\mathbb{R}^3$ that lies in $\psi(B)$. One quickly checks that the ball of radius $1/4$ lies in the image of B.

By hypothesis $T\psi(\xi^s)$ agree with $\tilde{\xi}_{st}$ outside a ball $B_R(0)$ of radius R around 0 such that $\psi\big(S^3 \setminus U\big) \subset B_R(0)$. Let us assume that $R > 1/4$, otherwise the contact structures ξ^s already agree with ξ_{st} on B.

Consider the maps

$$\begin{array}{cccc} \phi_\lambda\colon & \mathbb{R}^3 & \to & \mathbb{R}^3 \\ & (u,v,w) & \mapsto & \big(\lambda u, \lambda v, \lambda^2 w\big) \end{array}$$

with $\lambda > 0$. It is a contactomorphism of ξ_0. In fact, $\phi_\lambda^*\tilde{\alpha}_{st} = \phi_\lambda^*(\mathrm{d}w + u\,\mathrm{d}v - v\,\mathrm{d}u) = \lambda^2(\mathrm{d}w + u\,\mathrm{d}v - v\,\mathrm{d}u) = \lambda^2\tilde{\alpha}_{st}$.

Choose $\lambda(t) := (1+t) + t/(4R)$. With $\lambda(1) = 1/(4R) < 1$ we also have $\lambda^2(1) < 1/(4R)$ and thus $\phi_{\lambda(1)}$ maps $\psi\big(S^3 \setminus U\big)$ into a ball of radius $1/4$ around the origin and hence into the image of the ball B. The images

$$\tilde{\xi}^s_t := T\phi_{\lambda(t)}\tilde{\xi}^s$$

of $\tilde{\xi}^s$ under $T\phi_{\lambda(t)}$ form a smooth family of contact structures. As for all $s \in S^k$ the contact structure $\tilde{\xi}^s$ agreed with $\tilde{\xi}_0$ outside a ball of radius R and ϕ_λ are contactomorphisms of $\tilde{\xi}_0$, the contact structures $\tilde{\xi}^s_t$ agree with $\tilde{\xi}_0$ outside a ball of radius $1/4$ around the origin, *cf.* Figure 6.2.

The parametric and relative version of Gray stability (*cf.* Corollary 5.3) yields a S^k-family of isotopies Φ^s_t, $s \in S^k$, $t \in (0,1)$, of $\mathbb{R}^3$ that is stationary outside $\psi\big(S^3 \setminus U\big)$ and satisfies

$$T\Phi^s_t\big(\tilde{\xi}^s\big) \;=\; \xi^s_t \quad \text{for each } t \in [0,1] \text{ and } s \in S^k.$$

As Φ^s_t is stationary outside a compact ball, the conjugation $\psi^{-1} \circ \Phi^s_t \circ \psi$ with ψ gives a family of isotopies of S^3 that is stationary near the point $(-i,0)$ and defines isotopies of the contact structures ξ^s with $T\psi^{-1} \circ \Phi^s_1 \circ \psi(\xi^s) = T\psi^{-1} \circ \Phi^s_1\big(\tilde{\xi}^s\big)$. As $T\Phi^s_1\big(\tilde{\xi}^s\big)$ agrees with $\tilde{\xi}_0$ outside the image of the hemisphere S_- and ψ is a contactomorphism, each $T\psi^{-1} \circ \Phi^s_1\big(\tilde{\xi}^s\big)$ agrees with ξ_{st} on the complement of S_-.

Let us denote the contact structures $T\psi^{-1} \circ \Phi^s_1 \circ \psi(\xi^s)$ again by ξ^s. These agree with ξ_{st} on $S^3 \setminus B$, the complement of the ball $B =$

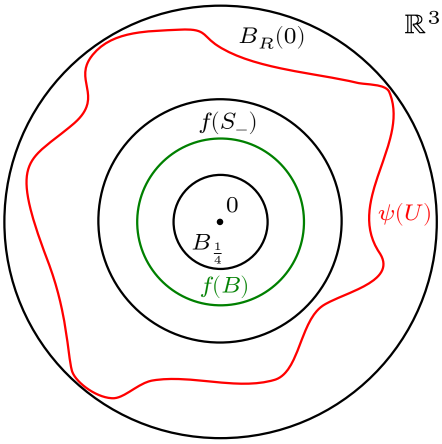


Figure 6.2.: The contact structures $T\psi(\xi^s)$ agree with $\tilde{\xi}_{st}$ outside the ball $B_R(0)$

$\{\mathfrak{Im}\, z \geq 1/2\}$. We also find a smooth family of contact forms α^s for ξ^s such that each α^s coincides with α_{st} on $S^3 \setminus B$.

7. Caps and spheres: Transforming the problem

All contact structures ξ^s agree with the standard contact structure ξ_{st} on the subset $S^3 \setminus B = \{(z,w) \in S^3 \,|\, \mathfrak{Im}\, z < 1/2\}$ of S^3. This set contains the *caps* $C_+ := \{\mathfrak{Re}\, z > 7/8\} \cap S^3$ and $C_- := \{\mathfrak{Re}\, z < -7/8\} \cap S^3$ that are open discs around the north pole $N := (1,0)$ and the south pole $S := (-1,0)$, respectively. Their complement, $S^3 \setminus (C_- \cup C_+) = \{-7/8 \leq \mathfrak{Re}\, z \leq 7/8\}$ is diffeomorphic to $[-1,1] \times S^2$ via

$$\begin{aligned} \Phi\colon \quad [-1,1] \times S^2 \subset [-1,1] \times \mathbb{R}^3 &\to \qquad S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4 \\ (z,u,v,w) &\mapsto \left(\tfrac{7}{8} z, r(z)\, u, r(z)\, v, r(z)\, w\right) \end{aligned}$$

where $r^2(z) + \left(\frac{7}{8} z\right)^2 = 1$. The contact form α_{st} pulls back to

$$\Phi^* \alpha_{st} = \tfrac{7}{8} r \cdot z\, \mathrm{d}u - \tfrac{7}{8} r^{-1} \cdot u\, \mathrm{d}z + r^2 \cdot v\, \mathrm{d}w - r^2 \cdot w\, \mathrm{d}v\,,$$

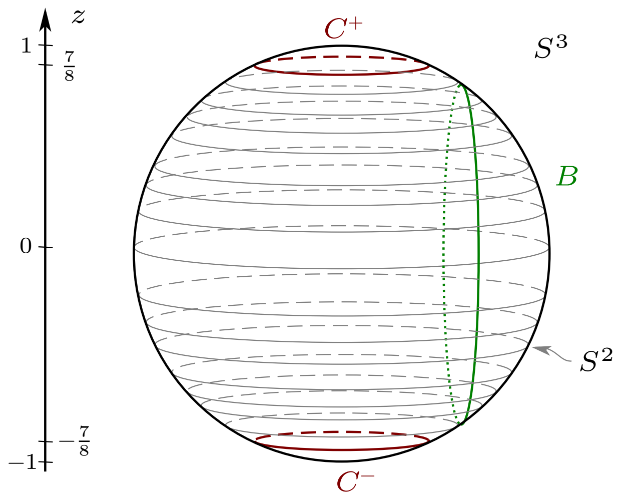


Figure 7.1.: On the caps C^+ and C^- and near them the contact structures ξ^s agree with ξ_{st}. The complement of the caps is foliated by 2-spheres.

which we shall write as

$$\Phi^*\alpha_{st} = \beta_z^{st} + h_z^{st}\,\mathrm{d}z$$

where $\beta_z^{st} := \frac{7}{8}r \cdot z\,\mathrm{d}u + r^2 \cdot v\,\mathrm{d}w - r^2 \cdot w\,\mathrm{d}v$ and $h_z^{st} := -\frac{7}{8}r^{-1}u$. For each fixed z, we regard β_z^{st} as a 1-form on S^2. Likewise, h_z^{st} as a smooth function on S^2: We consider our standard contact structure restricted to $[-1,1] \times S^2$ as a 1-parametric family of 1-forms and functions.

Similarly, we can pull back the contact forms α^s and write them as

$$\Phi^*\alpha^s =: \beta_z^s + h_z^s\,\mathrm{d}z\,.$$

Observation 7.1. For z close to ± 1, the embedding Φ maps the sphere $\{z\} \times S^2$ into $S^3 \setminus B$. Hence for all $s \in S^k$, both β_z^s and h_z^s agree with β_z^{st} and h_z^{st}, respectively, for z sufficiently close to ± 1.

To be able to apply Gray stability, *cf.* Section 5, we need to find a path of parametric contact structures between ξ^s and ξ_{st}. Equivalently, we will construct a parametric path of contact forms between the family α^s and the standard contact form α_{st}.

Let us construct a path that is constant outside $\Phi\big([-1,1]\times S^2\big)$ as the contact forms α^s coincide with α_{st} there already.

Let us assume we are given any such path (α^s_t), $t\in[0,1]$. We may pull it back to $[-1,1]\times S^2$ and write the contact forms α^s_t there as $\beta^s_{t,z}+h^s_{t,z}$ as mentioned above. This turns a path of contact structures into paths of 1-forms and functions on S^2.

Conversely, paths of 1-forms $\beta^s_{t,z}$ and functions $h^s_{t,z}$ on S^2 that are constant in t for z close to ±1 determine paths of 1-forms on S^3. However, without further assumptions, these will not be contact forms.

Observation 7.2. With respect to this splitting $[-1,1]\times S^2$, the contact condition (I.1) for a 1-form $\alpha=\beta_z+h_z\,\mathrm{d}z$ translates into

$$\begin{aligned} 0<\alpha\wedge\mathrm{d}\alpha &= (\beta_z+h_z\,\mathrm{d}z)\wedge\Big(\mathrm{d}\beta_z-\dot\beta_z\wedge\mathrm{d}z+\mathrm{d}h_z\wedge\mathrm{d}z\Big)\\ &=\beta_z\wedge\mathrm{d}\beta_z-\beta_z\wedge\dot\beta_z\wedge\mathrm{d}z+\beta_z\wedge\mathrm{d}h_z\wedge\mathrm{d}z+h_z\,\mathrm{d}\beta_z\wedge\mathrm{d}z \qquad \text{(I.10)}\\ &=\Big(-\beta_z\wedge\dot\beta_z+\beta_z\wedge\mathrm{d}h_z+h_z\,\mathrm{d}\beta_z\Big)\wedge\mathrm{d}z\,, \end{aligned}$$

where the dot denotes derivative with respect to z and all exterior derivatives are with respect to S^2.

Hence the problem to find an S^k-parametric isotopy between the contact structures ξ^s and ξ_{st} translates into finding paths of $S^k\times[-1,1]$-parametric 1-forms and functions on S^2 that satisfy the contact condition (I.10).

A naïve idea would be to choose a convex interpolations between β^s_z and β^{st}_z and between h^s_z and h^{st}_z:

Observation 7.3. For any fixed family of 1-forms β_z on $[-1,1]\times S^2$, the contact condition is convex in h_z: Let h_z and k_z be two families of smooth functions on S^2 such that both $\beta_z+h_z\,\mathrm{d}z$ and $\beta_z+k_z\,\mathrm{d}z$ are contact forms and define

$$\alpha^t:=\beta_z+((1-t)\,h_z+tk_z)\,\mathrm{d}z\,.$$

Then we have

$$\begin{aligned}\alpha^t \wedge \mathrm{d}\alpha^t &= \Big(-\beta_z \wedge \dot{\beta}_z + (1-t)(\beta_z \wedge \mathrm{d}h_z + h_z\,\mathrm{d}\beta_z)\\ &\quad + t\,(\beta_z \wedge \mathrm{d}k_z + k_z\,\mathrm{d}\beta_z)\Big) \wedge \mathrm{d}z\\ &= (1-t)\Big(-\beta_z \wedge \dot{\beta}_z + \beta_z \wedge \mathrm{d}h_z + h_z\,\mathrm{d}\beta_z\Big) \wedge \mathrm{d}z\\ &\quad + t\Big(-\beta_z \wedge \dot{\beta}_z + \beta_z \wedge \mathrm{d}k_z + k_z\,\mathrm{d}\beta_z\Big) \wedge \mathrm{d}z\\ &= (1-t)((\beta_z + h_z\,\mathrm{d}z) \wedge \mathrm{d}(\beta_z + h_z\,\mathrm{d}z))\\ &\quad + t((\beta_z + k_z\,\mathrm{d}z) \wedge \mathrm{d}(\beta_z + k_z\,\mathrm{d}z))\\ &> 0\,.\end{aligned}$$

Consequently, all α^t are contact forms. The contact condition is, however, not convex in β_z for fixed h_z.

We may still construct paths of contact structures provided the 1-forms β_z^s are nice, *cf.* Section 11. These nice 1-forms β_z^s belong to *convex surfaces.*

8. Convex surfaces

Definition 8.1. *If a surface Σ in a contact manifold $(M, \xi = \ker\alpha)$ has a tubular neighbourhood $(-\varepsilon,\varepsilon)\times\Sigma$ such that the contact form α is invariant with respect to the normal direction z, i.e. α can be written as*

$$\alpha|_{(-\varepsilon,\varepsilon)\times\Sigma} = \beta + h\,\mathrm{d}z\,,$$

where neither β nor h depend on z, then the surface Σ is called **convex***.*

Observation 8.2. The contact condition (I.10) for 1-forms given as $\beta_z + h_z\,\mathrm{d}z$ in a neighbourhood of a surface Σ simplifies to

$$0 < (\beta \wedge \mathrm{d}h + h\,\mathrm{d}\beta) \tag{I.11}$$

if the 1-form is invariant with respect to z.

This implies that wherever h vanishes, we have that $0 < \beta \wedge \mathrm{d}h$. In particular, the differential of h does not vanish and hence $\Gamma \coloneqq \{h = 0\}$ is a 1-dimensional submanifold of Σ. This submanifold separates areas where h is positive from those where h is negative.

Definition 8.3. *A* 1*-dimensional submanifold* Γ *of a surface* Σ *is called* **dividing set** *of* Σ *with respect to a contact structure* ξ*, if there is a tubular neighbourhood* $(-\varepsilon, \varepsilon) \times \Sigma$ *of* Σ*, the contact structure* ξ *is given as the kernel of* $\beta + h\,\mathrm{d}z$ *and* $\Sigma = \{h = 0\}$*. The submanifold* Γ *shall be oriented as the boundary of* $\{h < 0\}$*.*

Let us take a short moment to compare the contact condition (I.10) with the contact condition (I.11) for convex surfaces. Notice that the latter is positively linear in h. This property will allow us in Section 11 to find paths of contact forms provided the dividing sets coincide since the linearity allows us to rescale the function h.

In Chapter II we will derive criteria on β and h for the condition that a given surface Σ is convex.

9. Characteristic foliations

Definition 9.1. *Let* Σ *be any surface embedded via* ϕ *into a* 3*-dimensional contact manifold* $(M, \xi = \ker \alpha)$*. The kernel of* $\beta := \phi^*\alpha$ *is the intersection of the contact planes of* ξ *with the tangent space of* Σ*. It is a* 1*-dimensional singular foliation, the* **characteristic foliation** *of* Σ *with respect to* ξ*. We will denote it by* $\xi\Sigma$*.*

The characteristic foliation on a surface determines β up to rescaling. In fact, it also determines the germ of the contact structure near the surface, *cf.* [Gir91, Proposition II.1.2]. A proof in English can be found in [Gei08, Theorem 2.5.22].

Proposition 9.2 (Giroux)**.** *Let* Σ_1 *and* Σ_2 *be two closed surfaces in contact* 3*-manifolds* (M_1, ξ_1) *and* (M_2, ξ_2)*, respectively. Then any diffeomorphism* $\phi\colon\ \Sigma_1 \to \Sigma_2$ *that sends characteristic foliation to characteristic foliation, preserving orientation, extends to a contactomorphism* $\Phi\colon\ (\mathcal{N}(\Sigma_1), \xi_1) \to (\mathcal{N}(\Sigma_2), \xi_2)$*.*

Moreover, any diffeomorphism $\Phi'\colon\ \mathcal{N}(\Sigma_1) \to \mathcal{N}(\Sigma_2)$ *of sufficiently small open neighbourhoods of the surfaces is isotopic to a contactomorphism.*

As the characteristic foliation of a surface Σ determines the contact structure in a neighbourhood of Σ and being convex is a condition on the contact structure in a neighbourhood of Σ, *cf.* Definition 8.1, being convex ultimately is a condition on the characteristic foliation.

Corollary 9.3. *Let Σ be a surface in a 3-dimensional contact manifold $(M, \xi = \ker \alpha)$. If there is a ∂_z-invariant contact structure on $(-1,1) \times \Sigma$ that induces the same characteristic foliation on Σ, then Σ is convex.*

10. Step 4: Bring dividing sets into standard form

Remember our situation from Section 7: The contact structures ξ^s already coincide with ξ_{st} on two caps. On its complement, identified with $[-1,1] \times S^2$, the contact structures ξ^s are given as the kernel of $\beta_z^s + h_z^s \, dz$. We want to construct a path of families of 1-forms β and functions h on S^2 such that these induce a path of parametric contact structures. In Step 3, *cf.* Chapter III, we will see how to find an isotopy of the contact structures ξ^s such that all spheres $\{z\} \times S^2$ that make up the complement of the caps are convex with respect to all ξ^s, $s \in S^k$. In order to find a path of contact structures in Section 11, we will require that we can find on each sphere $\{z\} \times S^2$ a single closed curve that is a dividing set for all contact structures ξ^s and ξ_{st}. To arrange this, we will need to deform the contact structures ξ^s with another isotopy.

Consider S^2 in $\mathbb{R}^3$ with coordinates u, v and w as in Section 7 and for each $z \in [-1,1]$ its embedded copy $\{z\} \times S^2 \subset S^3$.

Lemma 10.1. *For each $z \in [-1,1]$ and $s \in S^k$ there is an embedded curve Γ_z^s in S^2 such that its image in $\{z\} \times S^2$ is a dividing set with respect to ξ^s.*

The curves Γ_z^s smoothly depend on z and s and are contained in the closed hemisphere $\{u \geq 0\} \subset S^2$. For z close to ± 1, they agree with the equator $\{u = 0\}$ for all $s \in S^k$.

Proof. Consider $z_0 \in [-1,1]$ and $s_0 \in S^k$. The sphere $\{z_0\} \times S^2$ is convex with respect to ξ^{s_0}, i.e. there is a tubular neighbourhood of $\{z_0\} \times S^2$ in $[-1,1] \times S^2$ with respect to which the contact structure ξ^{s_0} is given as the kernel of $\beta_{z_0}^{s_0} + h \, dz$, where h does not depend on z. In particular, the function h and the 1-form $\beta_{z_0}^{s_0}$ satisfy the invariant contact condition (I.11). As before, we consider both the 1-form $\beta_{z_0}^{s_0}$ and the function h to be living on S^2. By Section III.6 we find a dividing set on $\{z\} \times S^2$ that is contained in the hemisphere $\{u \geq 0\} \subset S^2$, so we may assume that $h > 0$ on $\{u < 0\}$.

The invariant contact condition is $\mathcal{C}^1$-open in the space of 1-forms on S^2 with respect to β. As the 1-forms β_z^s smoothly depend on both $z \in [-1, 1]$ and $s \in S^k$, there is an open neighbourhood N of (s_0, z_0) in $S^k \times [-1, 1]$ such that for all $(s, z) \in N$ the 1-forms $\beta_z^s + h\,\mathrm{d}z$ satisfy the invariant contact condition (I.11). In general these will not be contact forms, however.

We can find such neighbourhoods N and functions h to all points $(s, z) \in S^k \times [-1, 1]$. Since $S^k \times [-1, 1]$ is compact, there is a finite cover $N_1, \dots, N_m$ of $S^k \times [-1, 1]$ with such neighbourhoods. Let us call the corresponding functions $h_1, \dots, h_m$.

Choose a partition of unity $\phi_1, \dots, \phi_m \colon S^k \times [-1, 1] \to [0, 1]$, $\sum_{i=1}^m \phi_i \equiv 1$, subordinate to the open cover and define

$$H\colon \quad \begin{array}{ccc} S^k \times [-1,1] \times S^2 & \to & \mathbb{R} \\ (s, z, q) & \mapsto & \sum_{i=1}^m \phi_i(s, z) \cdot h_i(q)\,. \end{array}$$

Denote the map $S^2 \to \mathbb{R}$, $q \mapsto H(s, z, q)$ by H_z^s.

Assertion 1. *For each $\{s, z\} \in S^k \times [-1, 1]$, the image of the zero set $\left\{q \in \{z\} \times S^2 \,\middle|\, H_z^s(q) = 0\right\}$ of H_z^s in $\{z\} \times S^2$ is a dividing set with respect to ξ^s.*

Proof. On $\mathbb{R} \times S^2$, where we denote the $\mathbb{R}$-coordinate by ζ, we can define a 1-form as $\beta_z^s + H_z^s\,\mathrm{d}\zeta$. By construction of H this 1-form satisfies the invariant contact condition (I.11). As neither β_z^s nor H_z^s depend on ζ, our 1-form is a thus a contact form on $\mathbb{R} \times S^2$.

There is a tubular neighbourhood $(-\varepsilon, \varepsilon) \times S^2$ of $\{z\} \times S^2$ in $[-1, 1] \times S^2$ such that ξ^s is given as the kernel of $\beta_z^s + H_z^s\,\mathrm{d}\zeta$, where ζ is the $(-\varepsilon, \varepsilon)$-coordinate, *cf.* Proposition 9.2. In particular, the zero set $\left\{q \in \{z\} \times S^2 \,\middle|\, H_z^s(q) = 0\right\}$ of H_z^s is a dividing set of the sphere $\{z\} \times S^2$ with respect to ξ^s, *cf.* Observation 8.2. ☐

As each $\{H_z^s = 0\}$ is a dividing set, the differential of the function H does not vanish along the pre-image of 0, hence $\Gamma := \{H = 0\}$ is a codimension-1 submanifold of $S^k \times [-1,1] \times S^2$. It intersects each sphere $\{z\} \times S^2$ transversely in $\Gamma_z^s := \{H_z^s = 0\}$ for all $s \in S^k$ and thus Γ_z^s depends smoothly on both s and z.

For z close to ± 1, all contact structures ξ^s coincide with ξ_{st} and hence on $\{z\} \times S^2$ the contact structures ξ^s are given as the kernel of $\beta_z^{st} + h_z^{st}\,\mathrm{d}z$. Consequently, the function H_z^s coincides with h_z^{st} for all $s \in S^k$ and $\{H_z^s = 0\}$ is the equator $\{u = 0\}$ of S^2, a single closed curve. Hence all Γ_z^s are single closed curves.

It remains to show that all Γ_z^s are contained in the hemisphere $\{u \geq 0\}$. This follows from the fact that each h^i was strictly positive on $\{u < 0\}$ and hence so is H_z^s for all $s \in S^k$ and $z \in [-1,1]$. $\square$

We saw in the proof of the preceding lemma that the hypersurface $\Gamma = \{H = 0\}$ in $S^k \times [-1,1] \times S^2$ is contained in $S^k \times [-1,1] \times \{u \geq 0\}$, which itself is contained in $S^k \times [-1,1] \times \{u > -1/2\}$. The latter set is diffeomorphic to $S^k \times [-1,1] \times \mathbb{R}^2$ such that $\{u = 0\}$ is mapped to the unit circle $\{r = 1\}$ in $\mathbb{R}^2$. The hypersurface Γ intersects each disc $\{(s,z)\} \times D^2$ in a single closed curve, hence Γ is diffeomorphic to $S^k \times [-1,1] \times S^1$.

The hypersurface Γ bounds a cylinder C diffeomorphic to $S^k \times [-1,1] \times D^2$ inside $S^k \times [-1,1] \times \mathbb{R}^2$. Let $\eta\colon\ S^k \times [-1,1] \times D^2 \to C$ be a diffeomorphism with $\eta(s,z,q) \in \{(s,z)\} \times \mathbb{R}^2$ for all $s \in S^k$ and $z \in [-1,1]$. For z close to ± 1, the curves Γ_z^s are the equator $\{u = 0\}$ in S^2, which we mapped to the unit circle $\{r = 1\}$ in $\mathbb{R}^2$. Hence we may choose η to be $(s,z,q) \mapsto (s,z,q)$ for z close to ± 1. After an isotopy away from $z = \pm 1$ that preserves each level $\{(s,z)\} \times D^2$ we may assume that $\eta(s,z,0) = (s,z,0)$ for all $z \in (-1,1)$.

Denote by η_z^s the map $q \mapsto \eta(s,z,q)$ and by $D_q\eta_z^s$ its linearisation as a map $D^2 \to D^2$. For $t \in [0,1]$ define

$$\begin{aligned} \psi_t\colon \quad S^k \times [-1,1] \times D^2 \quad &\to \quad S^k \times [-1,1] \times D^2 \\ (s,z,q) \quad &\mapsto \begin{cases} \left(s, z, \frac{1}{t}\,\eta(s,z,tq)\right), & t > 0 \\ D_0\eta_z^s(q), & t = 0\,. \end{cases} \end{aligned}$$

This is an isotopy of η that preserves the level sets of (s,z) and is stationary for z close to ± 1.

Extend both isotopies to a level-preserving isotopy Ψ_t of $S^k \times [-1,1] \times S^2$ relative to $\{z = \pm 1\}$. Its time-1 map sends $\Gamma \subset C$ to $S^k \times [-1,1] \times \{r = 1\}$.

The isotopy Ψ_t is stationary near $\{z = \pm 1\}$ and hence extends further to an isotopy of $S^k \times S^3$ that is stationary on the caps C_- and C_+. In particular, it induces isotopies of the contact structures ξ^s on S^3 via $T\Psi_t^s(\xi^s)$ where $\Psi_t^s := \Psi_t|_{\{s\}\times S^3}$. Denote the contact structures $T\Psi_1^s(\xi^s)$ again by ξ^s and the induced contact forms by α^s.

As Ψ_t is stationary on the caps C_- and C_+, the contact structures ξ^s and the contact forms α^s still agree with ξ_{st} and α_{st}, respectively, on $C_- \cup C_+$ for all $s \in S^k$.

A contactomorphism sends convex surfaces to convex surfaces and their dividing sets to dividing sets. Consequently, all spheres $\{z\} \times S^2$ are convex with respect to all ξ^s, $s \in S^k$, and for all $z \in [-1,1]$. On any of these we may choose $\Psi_1\big(\Gamma \cap \{(s,z)\} \times S^2\big) = \Psi_1(\Gamma) \cap \{(s,z)\} \times S^2 = \{u = 0\} \subset \{(s,z)\} \times S^2$ as dividing set with respect to ξ^s. In other words, for all spheres the equator $\{u = 0\}$ is a dividing set with respect to all contact structures ξ^s.

11. Step 5: Isotopy to the constant family

We are now ready to construct an isotopy of the contact structures ξ^s to the constant family ξ_{st}. On the caps C_- and C_+, *cf.* Section 7, of S^3 the contact structures ξ^s already coincide with ξ_{st}, so we need to find an isotopy on $[-1,1] \times S^2$.

The tool we will be using is a parametric version of the Gray stability theorem, *cf.* Corollary 5.3: A smooth path of contact forms on $[-1,1] \times S^2$ gives rise to a path of contact structures, which in turn will produce an isotopy of contact structures. This construction is a parametric adaptation of ideas from [Gir00, Lemma 2.6] and their explanation in [Gei08, Lemma 4.9.2].

On $[-1,1] \times S^2$ we wrote the contact structures ξ^s as the kernel of $\beta_z^s + h_z^s \,\mathrm{d}z$, *cf.* Section 7, and ξ^{st} as the kernel of $\beta_z^{st} + h_z^{st} \,\mathrm{d}z$. A convex interpolation between these forms will in general not be through contact forms. Instead, we may use the fact that all spheres $\{x\} \times S^2$ are convex and their equators are common dividing sets with respect to all ξ^s and ξ_{st}: It allows us to find paths to contact forms with large functions h_z^s and h_z^{st}, first. If chosen sufficiently big, their contribution in the inequality of the contact condition dominates the remaining summands and guarantees that convex combinations of the contact forms are indeed through contact forms.

Step I The forms $\beta_z^{st} + h_z^{st}\,\mathrm{d}z$ are contact forms and β_z^{st} and h_z^{st} satisfy the invariant contact condition (I.11). Hence a quick calculation shows that for $\lambda > 1$ the forms $\mu_z := \beta_z^{st} + \lambda h_z^{st}\,\mathrm{d}z$ are also contact forms.

$$\begin{aligned}\mu_z \wedge \mathrm{d}\mu_z &= \big(\beta_z^{st} + \lambda h_z^{st}\,\mathrm{d}z\big) \wedge \Big(\mathrm{d}\beta_z^{st} - \dot\beta_z^{st} \wedge \mathrm{d}z + \lambda\,\mathrm{d}h_z^{st} \wedge \mathrm{d}z\Big)\\ &= \Big(-\beta_z^{st} \wedge \dot\beta_z^{st} + \lambda\beta_z^{st} \wedge \mathrm{d}h_z^{st} + \lambda h_z^{st} \wedge \mathrm{d}\beta_z^{st}\Big) \wedge \mathrm{d}z\\ &= \Big(-\beta_z^{st} \wedge \dot\beta_z^{st} + \beta_z^{st} \wedge \mathrm{d}h_z^{st} + h_z^{st} \wedge \mathrm{d}\beta_z^{st}\Big) \wedge \mathrm{d}z\\ &+ (\lambda-1)\big(\beta_z^{st} \wedge \mathrm{d}h_z^{st} + h_z^{st} \wedge \mathrm{d}\beta_z^{st}\big) \wedge \mathrm{d}z\end{aligned}$$

As before, d denotes the exterior derivative with respect to the S^2 factor. The first summand is a positive area form on S^2 as the forms $\beta_z^{st} + h_z^{st}\,\mathrm{d}z$ are contact forms, the second summand is positive as $\beta_z^{st} + h_z^{st}\,\mathrm{d}z$ satisfy the invariant contact condition. Consequently, the convex combinations

$$\eta_{3,t}^s := \beta_z^{st} + t h_z^{st}\,\mathrm{d}z + (1-t)\lambda h_z^{st}\,\mathrm{d}z$$

are contact forms as well.

The forms $\beta_z^s + h_z^s\,\mathrm{d}z$ are contact forms, but β_z^s and h_z^s do not satisfy the invariant contact condition. All spheres $\{z\} \times S^2$ are convex with respect to all ξ^s, $s \in S^k$, and the equator of S^2 is a dividing set for all ξ^s. Considering $S^2 \subset \mathbb{R}^3$ with coordinates u, v and w, the equator of S^2 is the set $\{u = 0\}$. Since the equator is a dividing set for every $z \in [-1, 1]$ and $s \in S^k$ a construction as in the proof of Lemma 10.1 yields functions H_z^s on S^2 such that the 1-forms $\beta_z^s + H_z^s\,\mathrm{d}z$ do satisfy the invariant contact condition. These functions coincide with h_z^{st} for z close to ± 1, vanish exactly and up to first order on the equator. As also the functions h_z^{st} vanish exactly on $\{u = 0\}$ and up to first order, the functions

$$f_z^s := \frac{h_z^{st}}{H_z^s}$$

defined on $\{u \neq 0\}$ extend uniquely to a smooth family of smooth positive functions $f_z^s \colon S^2 \to \mathbb{R}^+$. Rescaling the 1-forms $\beta_z^s + H_z^s \, \mathrm{d}z$ with the functions f_z^s yields the 1-forms $f_z^s \beta_z^s + h_z^{st} \, \mathrm{d}z$. A quick calculation shows that these still satisfy the invariant contact condition (I.11). First notice that

$$h_z^{st} \, \mathrm{d}f_z^s = h_z^{st} \frac{H_z^s \, \mathrm{d}h_z^{st} - h_z^{st} \, \mathrm{d}H_z^s}{(H_z^s)^2} = f_z^s \, \mathrm{d}h_z^{st} - (f_z^s)^2 \, \mathrm{d}H_z^s \, .$$

Let us omit the indices s and z for simplicity for a moment.

$$\begin{aligned}
&(f\beta) \wedge \mathrm{d}h^{st} + h^{st} \, \mathrm{d}(f\beta) \\
&= f\beta \wedge \mathrm{d}h^{st} + h^{st} \, \mathrm{d}f \wedge \beta + h^{st} f \, \mathrm{d}\beta \\
&= f\beta \wedge \mathrm{d}h^{st} + f \, \mathrm{d}h^{st} \wedge \beta - f^2 \, \mathrm{d}H \wedge \beta + h^{st} f \, \mathrm{d}\beta \\
&= f^2 (H \, \mathrm{d}\beta + \beta \wedge \mathrm{d}H) > 0 \, .
\end{aligned}$$

The second factor is positive as the 1-forms $\beta_z^s + H_z^s \, \mathrm{d}z$ satisfy the invariant contact condition and the functions f_z^s are positive by construction.

As rescaling the contact forms $\beta_z^s + h_z^s \, \mathrm{d}z$ with the positive functions f_z^s results in contact forms that still define the contact structures ξ^s, we will do so, denoting $f_z^s \beta_z^s$ by β_z^s. Notice that we do not change β_z^s for z close to ± 1, where they already coincide with β_z^{st}.

Step II To aid some calculations, observe that the forms

$$A_z^s := -\beta_z^s \wedge \dot{\beta}_z^s + \beta_z^s \wedge \mathrm{d}h_z^s + h_z^s \, \mathrm{d}\beta_z^s$$

are (positive) area forms on S^2 as $\beta_z^s + h_z^s \, \mathrm{d}z$ are contact forms. The invariant contact conditions for $\beta_z^s + h_z^{st} \, \mathrm{d}z$ and $\beta_z^{st} + h_z^{st} \, \mathrm{d}z$ imply that

$$\begin{aligned}
B_z^s &:= \beta_z^s \wedge \mathrm{d}h_z^{st} + h_z^{st} \, \mathrm{d}\beta_z^s \\
B_z^{st} &:= \beta_z^{st} \wedge \mathrm{d}h_z^{st} + h_z^{st} \, \mathrm{d}\beta_z^{st}
\end{aligned}$$

are positive area forms as well.

Consider the convex combinations

$$\eta^s_{1,t} := \beta^s_z + (1-t)h^s_z\,\mathrm{d}z + t\lambda h^{st}_z\,\mathrm{d}z$$

between $\beta^s_z + h^s_z\,\mathrm{d}z$ and $\beta^s_z + \lambda h^{st}_z\,\mathrm{d}z$. Observe that

$$\begin{aligned}\eta^s_{1,t}\wedge \mathrm{d}\eta^s_{1,t} &= \Big[(1-t)\Big(\beta^s_z\wedge \mathrm{d}h^s_z + h^s_z\,\mathrm{d}\beta^s_z - \beta^s_z\wedge\dot\beta^s_z\Big) - t\beta^s_z\wedge\dot\beta^s_z \\ &\quad + \lambda t\big(\beta^s_z\wedge \mathrm{d}h^{st}_z + h^{st}_z\,\mathrm{d}\beta^s_z\big)\Big]\wedge \mathrm{d}z \\ &= \Big[(1-t)A^s_z - t\beta^s_z\wedge\dot\beta^s_z + t\lambda B^s_z\Big]\wedge \mathrm{d}z\,.\end{aligned}$$

Hence, the 1-forms $\eta_{1,t}$ are contact forms, provided λ is chosen large enough.

Finally, consider the convex combinations

$$\eta^s_{2,t} := (1-t)\beta^s_z + t\beta^{st}_z + \lambda h^{st}_z\,\mathrm{d}z\,.$$

A quick calculation shows that

$$\begin{aligned}\eta^s_{2,t}\wedge \mathrm{d}\eta^s_{2,t} &= \big[\lambda(1-t)\big(\beta^s_z\wedge \mathrm{d}h^{st}_z + h^{st}_z\wedge\beta^s_z\big) \\ &\quad + \lambda t\big(\beta^{st}_z\wedge \mathrm{d}h^{st}_z + h^{st}_z\wedge\beta^{st}_z\big) \\ &\quad + t(1-t)\Big(-\beta^s_z\wedge\dot\beta^{st}_z - \beta^{st}_z\wedge\dot\beta^s_z\Big) \\ &\quad - (1-t)^2\beta^s_z\wedge\dot\beta^s_z - t^2\beta^{st}_z\wedge\dot\beta^{st}_z\big]\wedge \mathrm{d}z \\ &= \big[\lambda\big((1-t)B^s_z + tB^{st}_z\big) + C^s_z\big]\wedge \mathrm{d}z\,.\end{aligned}$$

Hence these too are contact forms provided we choose λ sufficiently large.

Having obtained the three paths $\eta_{1,t}$, $\eta_{2,t}$ and $\eta_{3,t}$ of contact forms, we may smoothly concatenate them as follows. Pick $\varepsilon > 0$ and a smooth function $\phi\colon\ [0,1] \to [0,1]$ with $\phi(t) = 0$ for $t < \varepsilon$ and $\phi(t) = 1$ for $t > 1-\varepsilon$ and define

$$\eta^s_t := \begin{cases}\eta^s_{1,\phi(3t)}, & 0\le t<\frac{1}{3}\\ \eta^s_{2,\phi(3t-1)}, & \frac{1}{3}\le t\le\frac{2}{3}\\ \eta^s_{3,\phi(3t-2)}, & \frac{2}{3}<t\le 1.\end{cases}$$

This is a smooth path of contact structures connecting $\beta_z^s + h_z^s\,\mathrm{d}z$ and $\beta_z^{st} + h_z^{st}\,\mathrm{d}z$.

Step III A Moser trick argument will now, similar to the proof of the Gray stability theorem, yield a family of isotopies ψ_t^s, $t \in [0,1]$, of $[-1,1] \times S^2$ relative to $\{-1,1\} \times S^2$ such that $T\psi_t^s(\ker \nu_0^s) = \ker \eta_t^s$.

The latter condition is satisfied if

$$\lambda_t^s \eta_0^s = (\psi_t^s)^* \eta_t^s \quad \text{for each } s \in S^k,\ t \in [0,1] \tag{I.12}$$

for some smooth family of functions $\lambda_t^s\colon\ [-1,1] \times S^2 \ \to\ \mathbb{R}^+$. If we assume that we obtain ψ_t^s as the time-t map of the flow of a parametric vector field X_t^s, as in the proof of Proposition 5.2 we derive the necessary condition

$$\mu_t^s \eta_t^s = \dot{\eta}_t^s + \iota_{X_t^s}\,\mathrm{d}\eta_t^s + \mathrm{d}\Big(\iota_{X_t^s}\eta_t^s\Big)\,. \tag{I.13}$$

for $\mu_t^s := \frac{\mathrm{d}}{\mathrm{d}t}(\log \lambda_t^s) \circ (\psi_t^s)^{-1}$. As before, the dot denotes the derivative with respect to z and the external derivative is taken with respect to the S^2-factor.

Denote by R_t^s the Reeb vector fields to the contact forms η_t^s. We may the vector fields X_t^s uniquely as $Y_t^s + u_t^s R_t^s$ with $Y_t^s \in \ker \eta_t^s$ and $u_t^s\colon\ [-1,1] \times S^2 \ \to\ \mathbb{R}$ smooth functions. Equation (I.13) then amounts to

$$\iota_{Y_t^s}\,\mathrm{d}\eta_t^s = \mu_t^s \eta_t^s - \dot{\eta}_t^s - \mathrm{d}u_t^s\,. \tag{I.14}$$

Insert the Reeb vector field R_t^s yields

$$\mu_t^s = \dot{\eta}_t^s(R_t^s) + R_t^s(u_t^s)\,.$$

Any choice of u_t^s determines μ_t^s. In fact, for μ_t^s chosen that way, the right hand side of equation (I.14) has no component in Reeb direction and non-degeneracy of $\mathrm{d}\theta_t^s$ on the contact structure $\ker \eta_t^s$ uniquely determines Y_t^s. We are left to find u_t^s such that X_t^s vanishes on $\{-1,1\} \times S^2$.

Remember that η_t^s, restricted to $T\big(\{\pm 1\} \times S^2\big)$ is constant in t. Hence, on $\{\pm 1\} \times S^2$ the 1-form $\dot{\eta}_t^s$ is only non-zero on vectors transverse to $\{\pm 1\} \times S^2$, i.e.

$$\dot{\eta}_t^s = k_t^s\,\mathrm{d}z \quad \text{on } \{\pm 1\} \times S^2\,.$$

Consequently, we may choose as u_t^s any family of smooth functions that are zero on $\{-1,1\} \times S^2$ and satisfy $\frac{\partial}{\partial z} u_t^s = k_t^s$ there.

With this choice μ_t^s vanishes on $\{-1,1\} \times S^2$ and by equation (I.14) so do Y_t^s and $X_t^s = Y_t^s + u_t^s R_t^s$.

This determines a smooth family of vector fields X_t^s that vanish on $\{-1,1\} \times S^2$. We can integrate them up to time 1. Its flow ψ_t^s gives a family of isotopies between the contact structures ξ^s and ξ^{st} on $[-1,1] \times S^2$. They are stationary on $\{-1,1\} \times S^2$ and hence extend to isotopies of S^3.

This concludes the proof of Theorem 2.1.

II. Families of characteristic foliations

We still need to modify our family of contact structures ξ^s on $[-1,1]\times S^2$ such that all spheres $\{z\} \times S^2 \subset [-1,1] \times S^2 \subset S^3$ become convex surfaces, *cf.* Definition I.8.1, with respect to all contact structures ξ^s, $s \in S^k$.

We noticed in Section I.9 that the conditions on a surface in a contact manifold on being convex it determined by its characteristic foliation. The goal of this chapter is to understand which characteristic foliations belong to convex surfaces. We will review some theory about characteristic foliations in Section 1, about dynamical systems in Section 2, and recapitulate the situation for an isolated surface in Section 3.1. From there we can understand the conditions on the characteristic foliations in term of neighbourhoods (Section 3.2) and develop properties of these neighbourhoods. This paves the way for the deformations described in the next chapter.

1. Characteristic foliations and vector fields

Let Σ be a sphere $\{z\} \times S^2$ in $[-1,1] \times S^2 \subset S^3$ with contact structure ξ^s. In a neighbourhood of this sphere, the contact structure ξ^s is given as the kernel of

$$\beta_z^s + h_z^s \,\mathrm{d}z\,.$$

As in Section I.7, we consider β_z^s as a 1-form on S^2 and h_z^s as a function on S^2. The characteristic foliation $\xi^s\Sigma$ is given as the kernel of β_z^s. In particular, we may regard it as a singular foliation on S^2.

Definition 1.1. *If X is a vector field on S^2 parallel to the characteristic foliation $\xi^s\Sigma$ (and vanishing in the singular points), we say X* **directs** *the characteristic foliation $\xi^s\Sigma$.*

While we speak of leaves of the characteristic foliation, we shall refer to flow lines of vector fields as trajectories. Each trajectory of X is a leaf of $\xi\Sigma$.

Observation 1.2. Let S^2 oriented as $S^2 \subset \mathbb{R}^3$. If we pick an area form Ω on S^2 that corresponds to this choice, we may uniquely define a vector field X on S^2 using

$$\iota_X \Omega = \beta_z^s \,.$$

In particular, this fixes an orientation on the singular foliation $\xi\Sigma$.

The 1-forms β_z^s on S^2 depend smoothly on both $s \in S^k$ and $z \in [-1,1]$. Hence we obtain a smooth family of vector fields X_z^s that define the characteristic foliations.

Example 1.3. Let us consider the spheres $\{z\} \times S^2$, $z \in [-1,1]$, in (S^3, ξ_{st}). We saw in Section I.7 an embedding of these and pulled the contact form α_{st} back via this embedded to $\beta_z^{st} + u_z^{st}\,\mathrm{d}z$ with $\beta_z^{st} = \frac{7}{8} r \cdot z\,\mathrm{d}u + r^2 \cdot v\,\mathrm{d}w - r^2 \cdot w\,\mathrm{d}v$ and $h_z^{st} = -\frac{7}{8} r^{-1} u$.

To visualise vector fields directing the characteristic foliation of these spheres, consider spherical coordinates given by

$$\begin{array}{rccc} F\colon & [-1,1] \times [0.\pi] \times (-\pi,\pi] & \to & [-1,1] \times S^2 \subset [-1,1] \times \mathbb{R}^3 \\ & (z,\vartheta,\varphi) & \mapsto & \begin{pmatrix} z \\ \cos\vartheta \\ \sin\vartheta\,\cos\varphi \\ \sin\vartheta\,\sin\varphi \end{pmatrix} \end{array}$$

Pulling the 1-form β_z^{st} back via F yields

$$F^*\beta_z^{st} = -\frac{7}{8} r(z) z \sin\vartheta\ \mathrm{d}\vartheta + r^2(z) \sin^2\vartheta\ \mathrm{d}\varphi \,.$$

With respect to the area forms $r^2(z)\sin\vartheta\ \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$ for $\{z\} \times S^2$ we obtain the vector fields

$$X_z^{st} := \frac{7}{8}\,\frac{z}{r(z)}\,\partial_\varphi + \sin\vartheta\,\partial_\vartheta$$

that direct the characteristic foliation of $\{z\} \times S^2$ with respect to ξ_{st}.

These have two zeroes, each in a pole. In the north pole, $\vartheta = 0$, it is a source and the zero in the south pole is a sink. Their complement is foliated with trajectories of X_z^{st} that emanate at the north pole and tend to the south pole.

The vector fields X_z^s agree with X_z^{st} on the set that gets mapped to $\{u \leq \frac{1}{2}\} = \{\cos\vartheta \leq \frac{1}{2}\}$. In particular, this includes the northern hemisphere for each $z \in [-1, 1]$.

2. Characteristic foliations of convex surfaces

Our goal will be to deform the *movie* X_z^s of vector fields by isotopies of the contact structures ξ^s until all spheres $\{z\} \times S^2$ are convex with respect to all contact structures $s \in S^k$. As a first step, we will see which properties each of the final vector fields will have to have, i.e. which characteristic foliations belong to convex surfaces.

Let Σ be a closed sphere in a tight contact manifold (M, ξ). It has a tubular neighbourhood on which the contact structure ξ is given as the kernel of $\beta_z + h_z\,\mathrm{d}z$. Let Ω be an area form on Σ. Then

$$\iota_X \Omega = \beta_0$$

defines a vector field X that directs the characteristic foliation $\xi\Sigma$.

To be able to derive conditions on X that, once satisfied, imply that Σ is convex, let us remember some theory about dynamical systems.

2.1. Regular points and trajectories

Let X be a vector field on a closed manifold M of arbitrary dimension. We will mainly consider planar vector fields, however, we shall make use of the fact that smooth $\mathcal{P}$-parametric vector fields on a surface Σ form a single vector field on $\mathcal{P} \times \Sigma$.

A point $q \in M$ in which X does not vanish is a **regular point** of X. A **trajectory** of the vector field X is a curve $\gamma\colon (-\delta, \delta) \to M$ on M that is defined on an interval $(-\delta, \delta)$ and satisfies $\dot\gamma(t) = X_{\gamma(t)}$ for all $t \in (-\delta, \delta)$. The dot here denotes the derivative with respect to t. We consider smooth vector fields X on closed manifolds M, so we will always be able to extend the domain of γ to $\mathbb{R}$.

Through any point $p \in M$ passes a unique trajectory $\gamma\colon \mathbb{R} \to M$ with $\gamma(0) = p$. We will denote this trajectory by $\gamma(p)$.

Definition 2.1. *To each trajectory* $\gamma\colon \mathbb{R} \to M$ *of a vector field* X *on* M *we associate the* α**-limit set** *to be*

$$\alpha(\gamma) := \left\{q \in M \,\middle|\, \exists (t_n)_{n\in\mathbb{N}} \subset \mathbb{R},\ t_n \to -\infty\colon\ \gamma(t_n) \to q\right\}.$$

Analogously, we define the ω**-limit set** *to be points that comprise the limit of* γ *in positive time, i.e.*

$$\omega(\gamma) := \left\{q \in M \,\middle|\, \exists (t_n)_{n\in\mathbb{N}} \subset \mathbb{R},\ t_n \to +\infty\colon\ \gamma(t_n) \to q\right\}.$$

The unique trajectory through a singular point $q \in M$ is constant and both its α- and ω-limits are the set $\{q\}$.

Definition 2.2. *A codimension-*1 *submanifold* C *of* Σ *is called a* **cross section** *of the vector field* X *if it is transverse to* X.

As transversality is an open condition, we find cross sections through any regular point of X.

Theorem 2.3 (Flow Box Theorem)**.** *Around a regular point* q *of a vector field* X *on a manifold* M *of dimension* m *there are local coordinates* $x_1, x_2, \dots, x_m$ *such that in these coordinates, the vector field* X *is given as* $\partial_{x_1} = \frac{\partial}{\partial x_1}$. *Such coordinates are called* **flow box**.

Proof. Choose a cross section C to X through q together with coordinates $x_2, \dots, x_m$. The flow of the vector field X then defines the coordinate x_1 in an open neighbourhood of q. □

This classical result is also known as *Rectification Theorem*, *Fundamental Theorem for differential equations* or *Tubular Flow Theorem*.

Corollary 2.4 (Long Flow Box Theorem)**.** *Let* $\gamma\colon [0, T] \to M$ *with* $T > 0$ *be an arc of a trajectory of* X *on* M *with* $\gamma(0) \neq \gamma(T)$. *Then there is a neighbourhood* N *of* $\gamma([0, T])$ *with coordinates* $\phi\colon [0, T] \times V \to N$ *for some* $\varepsilon > 0$ *and* $V \subset \mathbb{R}^{m-1}$ *open such that on* N *the vector field* X *is given as* ∂_{x_1} *and* $\gamma = \phi(\bullet, \mathbf{0})$. *These coordinates are called a* **long flow box**.

The arc $\gamma([0, T])$ is compact, so we can cover it with finitely many flow boxes. Similar to the way one extends a trivialisation of a bundle we can construct coordinates along all of $\gamma([0,T])$. A detailed proof is presented in [PM82, Section 3.1].

Remark 2.5. If, in the situation of Corollary 2.4, we are given a parameterised cross section C of X that intersects the arc $\gamma\colon [0,T] \to \Sigma$ in exactly one point, then there is a long flow box that extends these coordinates, i.e. coordinates $\phi\colon U \times V \to \Sigma$ with $U \subset [0,T]$ and $V \subset \mathbb{R}^{m-1}$ open that restrict on $\{c\} \times \mathbb{R}^{m-1}$ to the given coordinates of C for some $c \in U$.

This condition determines ϕ and V uniquely up to the size of V: If there are two flow boxes (V,ϕ) and (V',ϕ') that both extend the coordinates on C then they coincide on the preimage of $\phi(V) \cap \phi'(V')$.

Let me conclude with introducing the *Poincaré map*. A more detailed treatment can be found in [PM82, Section 3.1].

Definition 2.6. *Let C_1 and C_2 two cross sections of a vector field X on a manifold M. Denote the flow of X by ψ_t and by*

$$\begin{aligned} T_{C_1,C_2}\colon \quad C_1 &\to \mathbb{R}^+ \cup \{\infty\} \\ q &\mapsto \min\{\tau \in \mathbb{R}^+ \,|\, \psi_\tau(q) \in C_2\} \end{aligned}$$

the time that is takes the flow of X to map points of C_1 to C_2. This **flow time map** *map allows us to define the* **Poincaré map**

$$\begin{aligned} P_{C_1,C_2}\colon \quad C_1 &\to C_2 \\ q &\mapsto \begin{cases} \psi_{T_{C_1,C_2}(q)}(q), & T_{C_1,C_2}(q) < \infty \\ \textit{undefined} & \textit{otherwise.} \end{cases} \end{aligned}$$

The Poincaré map $P_{C,C} =: P_C$ from a cross section C to itself is called the **Poincaré return map**.

Lemma 2.7. *The flow time map T_{C_1,C_2} to two cross sections C_1 and C_2 is finite on an open domain and is smooth there. The Poincaré map P_{C_1,C_2} is a local diffeomorphism.*

Proof. Let T_{C_1,C_2} be defined in a point $q \in C_1$. If q and $P_{C_1,C_2}(q)$ are disjoint, the claims follow from the Long Flow Box Theorem. Otherwise, if $q = P_{C_1,C_2}(q)$, pick another cross section B trough a point $p \neq q$ in the trajectory $\gamma(q)$ through q, splitting the closed trajectory in two arcs, from q to p and from p to q. Choosing long flow boxes around the two arcs of $\gamma(q)$, observe that $T_{C_1,C_2} = T_{B,C_2} \circ T_{C_1,B}$ and $P_{C_1,C_2} = P_{B,C_2} \circ P_{C_1,B}$ in a neighbourhood of q. Thus the claims follow as in the first case. □

2.2. Singular points

Let X now be a vector field on S^2 and Ω be an area form on S^2. Singular points of the characteristic foliation are zeroes of the directing vector field. We will be referring to both using the same term, **singular points**.

Definition 2.8. *The* **divergence** $\mathrm{div}_\Omega(X)$ *of a vector field* X *with respect to the area form* Ω *is defined as*

$$\mathcal{L}_X \Omega = \mathrm{div}_\Omega(X) \cdot \Omega \,.$$

Observation 2.9. We defined a special vector field X on S^2 that directs the characteristic foliation using $\iota_X \Omega = \beta_0$. For this vector field we obtain

$$\mathrm{d}\beta_0 = \mathrm{d}(\iota_X \Omega)$$

which equals $\mathcal{L}_X \Omega$ by Cartan's Formula. Hence

$$\mathrm{d}\beta_0 = \mathrm{div}_\Omega(X) \cdot \Omega \,.$$

Any positive multiple of X also directs the same characteristic foliation. Let $f\colon S^2 \to \mathbb{R}^+$ be a positive function and consider the vector field fX. A quick calculation yields

$$\mathcal{L}_{fX} \Omega = \mathrm{d}\big(\iota_{fX}\Omega\big) = \mathrm{d}\big(f \iota_X \Omega\big) = \mathrm{d}f \wedge \iota_X \Omega + f \mathrm{d}(\iota_X \Omega) \,.$$

As the 3-form $\mathrm{d}f \wedge \Omega$ vanishes on S^2, we get

$$0 = \iota_X(\mathrm{d}f \wedge \Omega) = X(f)\, \Omega - \mathrm{d}f \wedge \iota_X \Omega$$

and hence

$$\mathcal{L}_{fX} \Omega = \mathrm{d}f \wedge \iota_X \Omega + f \mathrm{d}(\iota_X \Omega) = \big(X(f) + f \mathrm{div}_\Omega(X)\big) \cdot \Omega \,.$$

We obtain the identity

$$\operatorname{div}_\Omega(fX) = X(f) + f \operatorname{div}_\Omega(X)\,. \tag{II.1}$$

Likewise, we can calculate the divergence of the vector field X with respect to the rescaled positive area form $f\Omega$. As Ω is closed, we have $\mathcal{L}_X(f\Omega) = \mathrm{d}\big(\iota_X f\Omega\big) = \mathrm{d}\big(\iota_{fX}\Omega\big) = \mathcal{L}_{fX}\,\Omega$ and hence

$$\operatorname{div}_\Omega(fX) = \operatorname{div}_{f\Omega}(X) = X(f) + f \operatorname{div}_\Omega(X)\,. \tag{II.2}$$

If we restrict again to the vector field X defined by the area form Ω, we can rewrite the contact condition using the divergence of X.

Observation 2.10. The 3-form $\Omega \wedge \mathrm{d}h_0$ vanishes on the surface Σ. Consequently,

$$\begin{aligned} 0 = \iota_X(\Omega \wedge \mathrm{d}h_0) &= \iota_X\Omega \wedge \mathrm{d}h_0 + X(h_0)\,\Omega \\ &= \beta_0 \wedge \mathrm{d}h_0 + X(h_0)\,\Omega\,. \end{aligned}$$

Hence, on Σ the contact condition (I.10),

$$0 < \Big(-\beta_z \wedge \dot{\beta}_z + \beta_z \wedge \mathrm{d}h_z + h_z\,\mathrm{d}\beta_z\Big) \wedge \mathrm{d}z$$

translates into

$$0 < \Big(-\beta_0 \wedge \dot{\beta}_0 + \big(h_0 \operatorname{div}_\Omega(X) - X(h_0)\big)\,\Omega\Big) \wedge \mathrm{d}z\,. \tag{II.3}$$

Observation 2.11. In a singular point q of the characteristic foliation $\xi\Sigma$, the 1-form β_0 vanishes, as does the vector field X. Consequently, the contact condition (II.3) implies

$$0 < h_0(q)\ \operatorname{div}_\Omega(X)(q)\,.$$

In particular, the Jacobian of X has at least one eigenvalue with non-vanishing real part.

Notice that Equation II.1 implies that the divergence in a singular point does not depend on the vector field that directs the characteristic foliation.

This allows us to speak of the sign of singular points of X.

Definition 2.12. *Singular points of $\xi\Sigma$ are* **positive** *or* **negative** *depending on the sign of the divergence of any vector field directing $\xi\Sigma$.*

The linearisation of a vector field X in a singular point q defines a linear map of $\mathbb{R}^2$ via the Jacobian D_qX and an identification of $T_q\Sigma$ with $\mathbb{R}^2$. The eigenvalues of this map do not depend on the chosen identification of $T_q\Sigma$.

Definition 2.13. *If both eigenvalues λ_1 and λ_2 of the linearisation of X in q have non-vanishing real part, the singular point q is* **generic** *or* **non-degenerate**.

If both $\mathfrak{Re}\,\lambda_1$ and $\mathfrak{Re}\,\lambda_2$ have the same sign, the singular point q is an **elliptic** *singular point, if they have different signs, we call q* **hyperbolic**.

An elliptic point with two eigenvalues with negative real part is a **sink**, *one with two positive real parts is a* **source**.

These names for the singular points are motivated by the Grobman–Hartman Theorem.

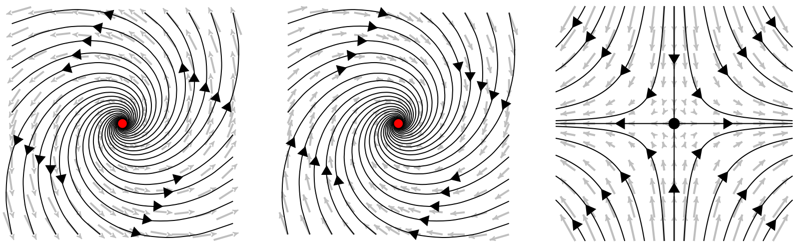
Theorem 2.14 (Grobman–Hartman)**.** *Let X be a vector field on a surface Σ, ψ_t its flow and $q \in \Sigma$ a non-degenerate singular point of X in the interior of Σ. Then there is a neighbourhood $U \subset \Sigma$ of q and a homeomorphism $h\colon U \to U$ that conjugates ψ_t to the linear flow induced by D_qX, i.e.*

$$\psi_t = h \circ e^{tD_qX} \circ h^{-1}\,.$$

A proof can be found in [PM82, Section 2.§4]. In many cases, we can find a conjugation map of higher regularity, see [Jän12, Section III.2.3] for a short overview. The following example and the pictures also appear in [Jän12, Example III.2.20].

Example 2.15. On $\mathbb{R}^2$ with coordinates x and y consider the vector field X_1 given as $(-2x+5y)\,\partial_x + (-5x-3y)\,\partial_y$. The Jacobian of X in the origin $\mathbf{0}$ is

$$D_{\mathbf{0}}X_1 = \begin{pmatrix} -2 & 5 \\ -5 & -3 \end{pmatrix}.$$



(a) source (b) sink (c) hyperbolic point

Figure 2.1.: Types of non-degenerate singular points. The light grey arrows in the background depict the vector field, the solid black, oriented lines are its trajectories.

Both its eigenvalues, $5/2 \pm 3/2\sqrt{11}\, i$, have positive real part, so the vector field X_1 has a source in the origin, *cf.* Figure 2.1a. Changing the sign of the vector field, we obtain another vector field $X_2 := -X_1$. It has a sink in the origin, *cf.* Figure 2.1b.

The linearisation of the vector field $X_3 := 2x\partial_x - 3y\partial_y$ in the origin has the eigenvalues 2 and -3, the vector field has a hyperbolic point in the origin, *cf.* Figure 2.1c. Notice the four trajectories of X_3 that emanate at or tend to the origin. These are called *separatrices*.

We may also distinguish degenerate singular points by the behaviour of X in its vicinity. We will still restrict ourselves to isolated singular points; we will be able to achieve that all singular points are isolated.

Construction 2.16. If the point $q \in \Sigma$ is a degenerate singular point, one eigenvalue of the linearisation of X in q vanishes. The contact condition implies that the other eigenvalue λ is non-zero and real, *cf.* Observation 2.11. Let us assume that the singular point q is isolated, i.e. there is a neighbourhood of q in Σ such that q is the only singular point in this neighbourhood. Choose coordinates x, y around q such that $q = (0, 0)$ and in $T_q\Sigma$ the vector ∂_x is contained in the eigenspace to 0 in q and ∂_y lies in the eigenspace to λ. We may assume that x, y respect the orientation of Σ. In these coordinates, write the vector field X as $a(x, y)\,\partial_x + b(x, y)\,\partial_y$. By the Implicit Function Theorem we may, shrinking our coordinate neighbourhood, assume that $b(x, y)$ vanishes exactly on $\{y = 0\}$.

Definition 2.17. *If for $X(x,y) = a(x,y)\,\partial_x + b(x,y)\,\partial_y$ in local coordinates as in Construction 2.16, the function $x \mapsto a(x,0)$ does not change sign in $x = 0$, the singular point q is* **half-hyperbolic**. *In case a changes sign from negative to positive and $\lambda > 0$, q is a* **degenerate source**. *Analogously we call negative degenerate points* **degenerate sinks** *or* **degenerate hyperbolic points**.

2.3. Neighbourhoods of singular points

Even though there is no Grobman–Hartman theorem for degenerate singular points, we can, even if we do not know the trajectories of X up to conjugation near degenerate points, derive some properties of the topology of the trajectories of X near isolated (degenerate) singular points.

Construction 2.18. Let q be an isolated singular point of the vector field X on Σ and assume that q is a positive singular point. Choose local coordinates x and y around q as in Construction 2.16 and write the vector field X as $a(x,y)\,\partial_x + b(x,y)\,\partial_y$. By the choice of coordinates the function b is strictly positive for $y > 0$ and strictly negative for $y < 0$. Consequently, for any $c \neq 0$ the arc $\{y = c\}$ is a cross section of X.

As q is an isolated singular point, we find a small $\delta_x > 0$ such that $a(-\delta_x, 0) \neq 0$ and $a(\delta_x, 0) \neq 0$. By continuity, there is $\delta_y > 0$ such that $a(-\delta_x, y) \neq 0$ and $a(\delta_x, y) \neq 0$ for all $y \in (-\delta_y, \delta_y)$. Thus the segments $\{\pm\delta_x\} \times [-\delta_y, \delta_y]$ are cross sections of X.

Hence there is a rectangle $[-\delta_x, \delta_x] \times [-\delta_y, \delta_y]$ around q such that all four sides are cross sections of X. Choosing δ_x and δ_y sufficiently small the rectangle will not contain any other singular point.

Each pair of adjacent cross sections of these four intersect each other. If the vector field X passes through two intersecting cross sections with matching orientation, we may smoothly join the two cross section.

Construction 2.19 (Connected sum of cross sections). Let C_1 and C_2 be two cross sections of the vector field X on a surface Σ that intersect transversely in $p \in \Sigma$. Both cross sections are 1-dimensional curves. Assume that both cross sections are given an orientation such that their respective orientation followed by X gives the same orientation on Σ.

Choose a parameterisation x of C_1 respecting its orientation. The flow of the vector field X induces a tubular neighbourhood νC_1 of C_1 whose transverse coordinate we will denote by y. In a small neighbourhood $U \subset \nu C_1$ of the intersection point p the cross section C_2 is a graph $y = c(x)$ over C_1 in νC_1. The neighbourhood U contains $(-3\varepsilon, 3\varepsilon) \times \{0\}$ for some $\varepsilon > 0$. Let $\phi\colon\ (-3\varepsilon, 3\varepsilon) \to [0, 1]$ be a smooth function that is identically zero on $(-3\varepsilon, \varepsilon]$, identically 1 on $[2\varepsilon, 3\varepsilon)$ and is strictly monotone in between. We obtain a new smooth cross section C by taking the part of C_1 that lies before U with respect to the orientation of C_1, the part of C_2 that lies after U and by taking the graph of $\phi \cdot c$ inside U, *cf.* Figure 2.2.

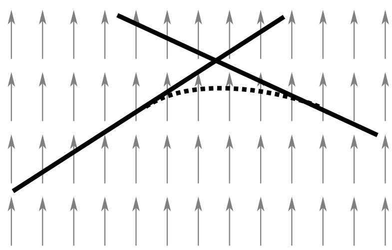


Figure 2.2.: Connected sum of cross sections

2.3.1. Elliptic singular points

Near an elliptic singular point the four cross sections from Construction 2.18 can be joined to a single one bounding an embedded disc.

Corollary 2.20. *Let $q \in \Sigma$ be an isolated, possibly degenerate, source of a vector field X on Σ. Then there is a disc D in Σ that contains q and no other singular point and its boundary ∂D is a cross section of X.*

Proof. Consider coordinates around $q \in \Sigma$ as in Construction 2.16. With respect to those, there is, by Construction 2.18, a rectangle $R = [-\delta_x, \delta_x] \times [-\delta_y, \delta_y]$ around q that does not contain other singular points and its sides are cross sections of X. Orient the top side $C_1 := [-\delta_x, \delta_x] \times \{\delta_y\}$ of R as ∂_x and the bottom side $C_3 := [-\delta_x, \delta_x] \times \{-\delta_y\}$

as $-\partial_x$. By construction of the coordinates, the vector field X crosses the top side from below and the bottom side from above, i.e. from inside R to outside.

The singular point q is a source, so the component function $x \mapsto a(x, 0)$ changes sign from negative to positive in q, i.e. $x = 0$, *cf.* Definition 2.17. Hence, for sufficiently small δ_x, the vector field X crosses the right side $C_2 := \{\delta_x\} \times [-\delta_y, \delta_y]$ and the left side $C_4 := \{-\delta_x\} \times [-\delta_y, \delta_y]$ from inside R.

Orienting all four sides as the boundary of R, we can apply Construction 2.19 to get a closed cross section C of X. It bounds a disc $D \cong D^2$ that is contained in R and hence contains no singular point of X apart from q. □

Analogous statements holds for negative singular points and sinks with obvious sign-changes in the arguments.

2.3.2. Hyperbolic singular points

Lemma 2.21. *Let $q \in \Sigma$ be an isolated, possibly degenerate, hyperbolic singular point of X and assume it is a positive singular point. Then there are up to reparameterisation exactly two trajectories γ_1, γ_2 that have q as their ω-limit. These trajectories are called* **stable separatrices***.*

The union of their images and q, the set $\gamma_1(\mathbb{R}) \cup \{q\} \cup \gamma_2(\mathbb{R})$, is an immersed interval in Σ that is a smooth submanifold of Σ near q.

Proof. Consider coordinates x and y around $q \in \Sigma$ as in Construction 2.16 and a rectangle $R := [-\delta_x, \delta_x] \times [-\delta_y, \delta_y]$ around q as in Construction 2.18. The two trajectories γ_1 and γ_2 through the points $(-\delta_x, 0)$ and $(\delta_x, 0)$, respectively, tend to q: The vector field X is parallel to ∂_x along $\{y = 0\}$ and there are no other singular points of X contained in the rectangle.

Let γ be the trajectory that passes through any point $p = (x, y)$ in the coordinate chart with $y > 0$ and assume that $\gamma(0) = p$. As the y-component b of X is positive on $\{y > 0\}$, the y-component of $\gamma(t)$ increases with increasing time t. In particular, γ will not tend to q without leaving the rectangle R first. Similarly, trajectories through points (x, y) with $y < 0$ cannot tend to q without leaving R.

If the trajectory γ trough $p = (x, y)$ with $y \neq 0$ tends to q, it has to enter the rectangle R at $(-\delta_x, 0)$ or $(\delta_x, 0)$. But then it is a reparameterisation of either γ_1 or γ_2.

The image of each of the two trajectories γ_1 and γ_2 is an immersed interval. Close to the singular point $q = (0, 0)$ their union agrees with $\{(x, 0) \,|\, x < 0\} \cup \{(x, 0) \,|\, x > 0\}$. □

Following the theory of non-degenerate singular points as for example in [PM82, Section 2.§6], we will call the set $\gamma_1(\mathbb{R}) \cup \{q\} \cup \gamma_2(\mathbb{R})$ the **stable submanifold**.

Similarly, to a negative hyperbolic point there are exactly two trajectories that emanate at it. These are called **unstable separatrices** and form the **unstable manifold**.

In case the singular point q is a non-degenerate hyperbolic point, it has both stable and unstable separatrices. This follows from the Implicit Function Theorem that we can apply to both eigenspaces in the case that both eigenvalues are non-zero.

2.3.3. Half-hyperbolic singular points

An argument analogous to the proof of Lemma 2.21 yields a similar statement for half-hyperbolic points.

Lemma 2.22. *Let $q \in \Sigma$ be an isolated half-hyperbolic singular point of a vector field X and assume it is a positive singular point. Then there is, up to reparameterisation, exactly one stable separatrix, i.e. a trajectory γ that has q as its ω-limit.*

Proof. Consider coordinates x and y around $q \in \Sigma$ as in Construction 2.16 and a rectangle $R := [-\delta_x, \delta_x] \times [-\delta_y, \delta_y]$ around q as in Construction 2.18. The component function $x \mapsto a(x, 0)$ of X, restricted to $\{y = 0\}$, does not change sign in $q = (0, 0)$. Hence $a(-\delta_x, 0)$ and $a(\delta_x, 0)$ have the same sign, assume they are both positive. The trajectory γ through the point $(-\delta_x, 0)$ tends to q: The vector field X is parallel to ∂_x along $\{y = 0\}$ and there are no other singular points of X contained in the rectangle.

As in the proof of Lemma 2.21 observe that no trajectory that passes through a point (x, y) with $y \neq 0$ can tend to q without leaving R first.

Since $a(x, 0) > 0$ for all $x > 0$, trajectories through points $(x, 0)$ with $x > 0$ will also leave R before eventually tending to q. Hence γ is the only stable separatrix of q. □

The stable separatrix of a half-hyperbolic point q is, close to q, also contained in an embedded interval that is tangent to X by the same argument as in Lemma 2.21 for hyperbolic singular points.

Construction 2.23. Let R be a rectangle as in Construction 2.18 around a positive half-hyperbolic point $q \in \Sigma$ of a vector field X. Then X points out of R along the top and the bottom as well as along one side. Hence we can apply Construction 2.19 to obtain a half-disc $D \subset R$ whose boundary comprises two smooth cross sections, *cf.* Figure 2.3.

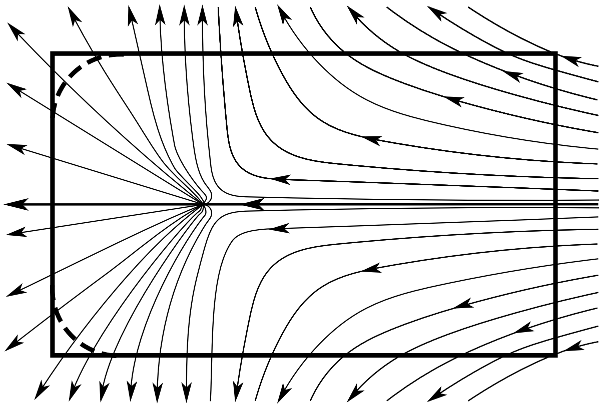


Figure 2.3.: A rectangle and a half-disc around a half-hyperbolic singular point

2.4. Closed trajectories and cycles

In our case all surfaces are spheres and they are contained in tight contact manifolds. Consequently, none of their characteristic foliations contains a closed leaf and hence no vector field directing a characteristic foliation contains a closed trajectory.

Lemma 2.24. *The characteristic foliation $\xi\Sigma$ of a 2-sphere Σ in the tight contact manifold $\left(S^3, \xi_{st}\right)$ does not contain a closed leaf.*

Proof. If there was a closed leaf γ, then it bounds a disc Δ in Σ whose boundary lies in the contact structure ξ and there is no point in $\partial\Delta$ in which the contact planes agree with $T\Sigma$. Hence the contact structure has to be overtwisted, *cf.* Definition I.1.6, but $\left(S^3, \xi_{st}\right)$ is tight. □

Definition 2.25. *A trajectory γ emanates at a singular point $q \in \Sigma$, if $\alpha(\gamma) = \{q\}$ and tends to the singular point $p \in \Sigma$, if $\omega(\gamma) = \{p\}$. In this case γ is a trajectory between q and p.*

A uniformly oriented polygon in Σ consisting of finitely many hyperbolic points and trajectories between them is called a **hyperbolic cycle***.*

The dynamics of any planar dynamical system, that is, one on the plane or the 2-sphere, are quite restricted, as the Poincaré-Bendixson Theorem tells us.

Theorem 2.26 (Poincaré–Bendixson)**.** *The α- and ω-limits of any trajectory of a vector field on $\mathbb{R}^2$ or S^2 are either*

- *a singular point,*
- *a closed trajectory, or*
- *a connected uniformly oriented cycle composed of finitely many hyperbolic points and trajectories connecting them.*

The key ingredient of the proof is the Jordan Curve Theorem. Its important consequence for our argument and the treatment of convex surfaces is that is shows that there is no trajectory that 'spirals against itself', i.e. is contained in its own α- or ω-limit set and is neither constant nor a closed trajectory.

We already saw that the vector field X does not have closed trajectories. A vector field X that directs the characteristic foliation of a sphere in a tight contact manifold does not have hyperbolic cycles, *cf.* [Gei08, Section 4.6.4] or Section III.4. Consequently, all trajectories emanate at and tend to singular points.

3. Conditions on convexity

Consider an oriented closed surface Σ in the contact manifold (M, ξ), $\xi = \ker \alpha$, and choose a tubular neighbourhood $(-1, 1) \times \Sigma$. We observed in Corollary I.9.3 that whether Σ is convex is a condition on its characteristic foliation $\xi\Sigma$ with respect to ξ. We would like to understand these conditions and express them as conditions on a vector field X directing the characteristic foliation $\xi\Sigma$.

3.1. The classic setting

Let us start with a recapitulation of the underlying theory due to Giroux.

By Corollary I.9.3 the surface Σ is convex if and only if there is a ∂_z-invariant 1-form $\beta + h\,\mathrm{d}z$ on $(-1, 1) \times \Sigma$. Let Ω be a positive area form on Σ. Rescaling the 1-form $\beta + h\,\mathrm{d}z$ we can assume that $\beta = \iota_X\Omega$.

Observation 3.1. By inequality Equation II.3 the condition on the ∂_z-invariant 1-form $\beta + h\,\mathrm{d}z$ being a contact form is

$$0 < (\beta \wedge \mathrm{d}h + h\,\mathrm{d}\beta)\,.$$

With

$$\beta \wedge \mathrm{d}h = \iota_X\Omega \wedge \mathrm{d}h = -X(h)\,\Omega$$

and $\mathrm{d}\beta = \mathrm{div}_\Omega(X) \cdot \Omega$ we can write this condition as

$$0 < h\,\mathrm{div}_\Omega(X) - X(h)\,. \tag{II.4}$$

The set $\Gamma = \{h = 0\}$ is a dividing set of Σ, *cf.* Definition I.8.3. In points of Γ, the contact condition implies $0 < -X(h_0)$. In particular, the vector field X points out of $\{h > 0\}$ and into $\{h < 0\}$.

As the divergence $\mathrm{div}_\Omega(X)$ does not depend on the choice of vector field X in singular points of $\xi\Sigma$, *cf.* Observation 2.11, the function h needs to be positive in positive singular points and negative in negative singular points. Consequently, the dividing set $\Gamma = \{h = 0\}$ has to separate negative from positive singular points.

Definition 3.2. *A trajectory that emanates at a negative hyperbolic point and tends to a positive one is called* **retrograde connection***.*

Lemma 3.3. *A surface Σ whose characteristic foliation $\xi\Sigma$ contains a retrograde connection between hyperbolic points is not convex.*

Proof. Let γ be trajectory from a negative hyperbolic point q_- to a positive one q_+. Assume that Σ is convex. Then the contact structure ξ can be written as the kernel of $\beta + h\,\mathrm{d}z$, $\beta \in \Omega^1(\Sigma)$, $h \in \mathcal{C}^\infty(\Sigma)$ in a tubular neighbourhood of Σ. As q_- is a negative singular point, $h(q_-) < 0$ and similarly, $h(q_+) > 0$. Hence, along γ there is a point q with $h(q) = 0$. There, the characteristic foliation points out of $\{h < 0\}$ and into $\{h > 0\}$ in violation of the contact condition, *cf.* Observation 3.1. □

We will observe that in our situation, where all surfaces we consider are spheres, the existence of retrograde connections is the only obstruction to convexity. Central to the argument is the following statement. It can be found in [Theorem 4.8.5 Gei08].

Theorem 3.4. *Let Σ be an oriented surface in a contact manifold (M, ξ) and X a vector field on Σ that directs its characteristic foliation $\xi\Sigma$. Then Σ is a convex surface if and only if there is a collection Γ_C of circles such that*

1. *every curve in Γ_C is transverse to $\xi\Sigma$,*
2. *there is a positive area form Ω_C on Σ such that $\mathrm{div}_{\Omega_C}(X) \neq 0$ on $\Sigma \setminus \Gamma_C$ and along all curves of Γ_C, the vector field points out of $\{\mathrm{div}_{\Omega_C}(X) > 0\}$.*

If there is such a collection of curves together with a volume form then one constructs a smooth function h on Σ that agrees with $\operatorname{sign}\operatorname{div}(X)$ away from Γ and interpolates in between. With this function $\beta + h\,\mathrm{d}z$ will be an $\mathbb{R}$-invariant contact form on $\mathbb{R} \times \Sigma$ implying that Σ is convex.

If Σ is convex, then a dividing set Γ is such a collection of curves. In other words, there is a positive area form Ω_C such that the divergence of X with respect to Ω_C is positive on $\{h > 0\}$, negative on $\{h < 0\}$ and vanishes along $\Gamma = \{h = 0\}$, *cf.* [Gei08, Chapter 4.8].

Let Σ be a sphere in a contact manifold (M, ξ) and its characteristic foliation $\xi\Sigma$ be directed by a vector field X that has no retrograde connection. We will assume for now that all singular points lie isolated and observe later how this assumption can be relaxed.

In a first step we will construct a collection of curves Γ_C that are transverse to X and separate positive from negative singular points and thereafter see how to rescale Ω to obtain an area form Ω_C satisfying the divergence condition of Theorem 3.4.

The following lemma is a construction explained in [Gei08, Chap. 4.8] adapted to our needs.

Lemma 3.5. *Let $\Sigma \cong S^2$ be an oriented sphere in a tight contact 3-manifold (M, ξ). If the singular points of the characteristic foliation $\xi\Sigma$ are isolated and there are no retrograde connections, then there is a subset S_+ of Σ that contains all positive singular points and no negative singular point and its boundary components are transverse to $\xi\Sigma$.*

Proof. Pick disjoint disc neighbourhoods around all sources of X as in Corollary 2.20 and denote their union by S_1. Its boundary is transverse to X and X points out of S_1.

If there is a positive half-hyperbolic point $q \in \Sigma$ whose stable separatrix emanates in S_1, we like to join it to S_1 by a band around its stable separatrix whose boundary components are transverse to X. To do that consider a half-disc D inside a rectangle $R = [-\delta_x, \delta_x] \times [-\delta_y, \delta_y]$ as in Construction 2.23. Without restriction assume that the stable separatrix γ of q contains the point $(\delta_x, 0)$. Denote the right side $\{\delta_x\} \times [-\delta_y, \delta_y]$ of the rectangle by C. That γ emanates at a point in S implies that γ crosses ∂S_1 and consequently the inverse of the Poincaré-map $P_{\partial S_1, C}$ is defined in an open neighbourhood of $(\delta_x, 0)$. Shrinking δ_y (and thus the rectangle R and the half-disc D) we may assume the Poincaré-map to be defined on $C_1 := \{\delta_x\} \times [-2\delta_y, 2\delta_y]$. Inside the long flow box to C_1 we can join C and ∂S_1 by two arcs transverse to X that bound a band B. Finally, rounding the four corners of the union $D \cup B \cup S_1$ of the half-disc, the band and S_1 we obtain a new subset S_2 of Σ that contains only positive singular points and its boundary ∂S_2 is transverse to X. In particular, the vector field X points out of S_2 along ∂S_2.

Iterate this process until after k steps for some $k \in \mathbb{N}$ there are no positive half-hyperbolic points of X left that lie outside S_k and that have stable separatrices that emanate at points in S_k.

Consider now a positive hyperbolic point $q \in \Sigma$ such that both stable separatrices emanate in S_k. As there are no hyperbolic cycles, *cf.*

Section 2.4, there has to be such hyperbolic points unless there are no positive hyperbolic points at all. Similar to the construction for the half-hyperbolic point, we can connect a rectangular neighbourhood of q to S_k using two bands B_1 and B_2 with boundaries transverse to X around the two stable separatrices of q. Round the corners of $R \cup B_1 \cup B_2 \cup S_k$ and denote the new set by S_{k+1}.

Iterate over all positive half-hyperbolic and hyperbolic points: As there is no retrograde connection, all stable separatrices of positive singular points have to emanate at positive points and hence at points in S_n for some $n \in \mathbb{N}$.

There are only finitely many isolated singular points on the compact surface Σ, so after finitely many iterations we arrive at a set that we denote by S_+ that contains all positive singular points, no negative singular points is such that the vector field X points out of S_+ along its boundary ∂S_+. □

Similarly, we construct a subset $S_- \subset \Sigma$ that contains all negative singular points and is such that X points into S_- along its boundary ∂S_-.

Consider a point q in the complement of S_+ and S_-. The trajectory γ through q emanates at a singular point and hence at a point in S_+. It also tends to a singular point and hence to a point in S_-. Consequently, $\Sigma \setminus (S_- \cup S_+)$ is foliated by arcs of trajectories of X and hence is a collection of annuli.

Observation 3.6. The divergence $\mathrm{div}_\Omega(X)$ of X is a smooth function on Σ. Consequently, it is positive in a neighbourhood of a positive singular point of X. Choosing the neighbourhoods in Section 2.3 sufficiently small, we may assume that in those neighbourhoods the divergence is strictly positive on the discs, half-discs or rectangles around positive singular points and strictly negative on the neighbourhoods around negative singular points.

As in [Proposition 4.8.7 Gei08] we want to rescale Ω by a positive function that grows sufficiently fast along the trajectories of X to ensure that the divergence of X with respect to the rescaled area form is positive on the set S_+.

Lemma 3.7. *Under the hypotheses of Lemma 3.5 there is a positive area form Ω_+ that coincides with Ω outside a neighbourhood of S_+ such that on the set S_+ the divergence $\operatorname{div}_{\Omega_+}(X)$ of the vector field X with respect to Ω_+ is positive.*

Proof. We can construct Ω_+ iteratively, following the construction of the sets S_k in the proof of Lemma 3.5. Take $\Omega_1 = \Omega$ and assume that for some $k \in \mathbb{N}$ we already constructed an area form Ω_k that coincides with Ω outside a neighbourhood of S_k and is such that $\operatorname{div}_{\Omega_k}(X) > 0$ on S_k. Consider the construction of the set S_{k+1}. It is contained in the union of S_k, the neighbourhood of a singular point that we will denote by N, and a band B inside a flow box V. Let $f\colon V \to \mathbb{R}^+$ be a positive function that grows sufficiently fast along the flow box to ensure $\operatorname{div}_{f\Omega_k}(X) > 0$ on V, *cf.* Equation II.2. We may choose f to be identically 1 near the end of the flow box V that lies in S_k and constant near the other end, that lies in N as on those sets we already have $\operatorname{div}_{\Omega_k}(X) > 0$. Extend f to a function $F\colon \Sigma \to \mathbb{R}^+$ on all of Σ that is identically 1 outside a small neighbourhood of $V \cup N$ and is constant on N. For $\Omega_{k+1} := F \cdot \Omega_k$ we verify $\operatorname{div}_{\Omega_{k+1}}(X) > 0$ on S_k as $\Omega_{k+1} = \Omega_k$, there. On N, the area form Ω_{k+1} is a constant, positive multiple of Ω_k which in turn coincides with Ω on N. As we assumed $\operatorname{div}_{\Omega}(X) > 0$ on N, this implies that $\operatorname{div}_{\Omega_{k+1}}(X) > 0$. In points of the flow box V, the fact that we chose f to grow fast enough along trajectories of X implies that the divergence of X with respect to Ω_{k+1} is positive there as well. Hence, X has positive divergence with respect to Ω_{k+1} on $S_{k+1} \subset S_k \cup V \cup N$. □

Similarly, we find a positive area form Ω' that is a constant multiple of Ω_+ outside a neighbourhood of S_- such that $\operatorname{div}_{\Omega'}(X) < 0$ on S_-.

As already mentioned, the complement A of $S_- \cup S_+$ is a collection of annuli that are foliated by arcs of trajectories of X. As described in [Chapter 4.8 Gei08] we can modify Ω' near those annuli to another area form Ω_C such that the divergence of X with respect to Ω_C vanishes exactly along the spines of the annuli in A. Then the spines Γ_C of these annuli satisfy the hypotheses of Theorem 3.4 together with Ω_C and Σ is hence a convex surface.

3.2. Conditions on neighbourhoods

So far we considered the classic case assuming that all singular points of $\xi\Sigma$ are isolated. In fact, inspecting the adaptation of the proofs given in the last section we only used this property to obtain standard neighbourhoods of the singular points. In the construction of the sets S_+ and S_-, *cf.* Lemma 3.5, we did not rely on the assumption that each of these neighbourhoods contains only a single singular point.

Definition 3.8. 1. *We call an embedded disc $D \cong D^2$ in Σ whose boundary ∂D is transverse to X and has the property that the divergence of X with respect to an area form Ω is positive on D a* **positive elliptic neighbourhood** *with respect to Ω. Similarly, if the divergence is negative, we call D a negative elliptic neighbourhood.*

2. *Let $R \cong [-1,1] \times [-1,1]$ be a rectangle in Σ such that all four sides are transverse to X. Write X in the induced coordinates as $a(x,y)\,\partial_x + b(x,y)\,\partial_y$. If the divergence of X with respect to Ω is positive on R and $b(x,y) > 0$ for $y > 0$ and $b(x,y) < 0$ for $y < 0$ and $a(-1,0) > 0$ and $a(1,0) < 0$, then we call R a* **positive hyperbolic neighbourhood** *with respect to Ω. The two trajectories through the points $(-\delta_x, 0)$ and $(\delta_x, 0)$ are the* **separatrices** *of the neighbourhood R.*
Similarly, a rectangle R with negative divergence and opposite signs in the conditions on a and b is a negative hyperbolic neighbourhood.

3. *Let $\mathcal{D}$ be a half-disc inside $D^2 \cap \{x \le 0\} \cup [0,1] \times [-1,1] \subset [-1,1] \times [-1,1]$. Let $D \subset S^2$ be a half-disc, i.e. there are neighbourhoods U_D of $D \subset S^2$ and $U_{\mathcal{D}}$ of $\mathcal{D} \subset \mathbb{R}^2$ and a diffeomorphism of these that sends D to $\mathcal{D}$. Assume that the boundary of D consists of the smooth curves c_e and $c_h = \{1\} \times [-1,1]$ that are both transverse to X. If the divergence of X with respect to Ω is positive on D, and we have, for $X = a(x,y)\,\partial_x + b(x,y)\,\partial_y$ as above, that $b(x,y) > 0$ for $y > 0$ and $b(x,y) < 0$ for $y < 0$ and $a(1,0) < 0$, then we call D a* **positive half-hyperbolic neighbourhood** *with respect to Ω. The special half-disc $\mathcal{D} \subset \mathbb{R}^2$ will serve as a model for these neighbourhoods. The trajectory through $(\delta_x, 0)$ is the* **separatrix** *of D.*
Again, we call a half-disc D with negative divergence and reversed

signs in the conditions on a and b a negative half-hyperbolic neighbourhood.

As the divergence of a vector field with respect to any given area form is a smooth function and non-zero in the singular points, each of the neighbourhoods of isolated singular points that we constructed in Section 2.3 is a neighbourhood of one of the types in Definition 3.8 provided the neighbourhood is chosen sufficiently small. Notice that we relaxed the condition that each neighbourhood should only contain a single, isolated singular point.

With the same construction as in the proof of Lemma 3.5, iterating over the standard neighbourhoods instead of over the singular points, we obtain a more general statement.

Corollary 3.9. *Let $\Sigma \cong S^2$ be an oriented sphere in a tight contact* 3-*manifold* (M, ξ)*. If there are finitely many elliptic, hyperbolic and half-hyperbolic neighbourhoods with respect to a positive area form* Ω *on* Σ *that contain all singular points of* $\xi\Sigma$ *and if* $\xi\Sigma$ *has no retrograde connections, then there is a subset* S_+ *of* Σ *that contains all positive singular points and no negative singular point and its boundary components are transverse to* $\xi\Sigma$*.*

Then we can construct a positive area form Ω_C and collection of curves Γ_C as above and apply Theorem 3.4 without changes. Summing up, we showed the following sufficient condition on convexity.

Proposition 3.10. *Let $\Sigma \cong S^2$ be an oriented sphere in a tight contact* 3-*manifold* (M, ξ)*. If there are finitely many elliptic, hyperbolic and half-hyperbolic neighbourhoods with respect to a positive area form* Ω *on* Σ *that contain all singular points of* $\xi\Sigma$ *and if* $\xi\Sigma$ *has no retrograde connections, then* Σ *is a convex surface.*

4. Families of vector fields

We will be dealing with smooth $S^k \times [-1, 1]$-families X_z^s, $(s, z) \in S^k \times [-1, 1]$, of vector fields on S^2 that direct the characteristic foliations of the level spheres $\{z\} \times S^2$ inside (S^3, ξ^s) as explained in Section 1.

We will refer to a smooth family of vector fields also as a **parametric vector field**.

Definition 4.1. *Let U be an open neighbourhood of some $(s_0, z_0) \in S^k \times [-1, 1]$ in $S^k \times [-1, 1]$. Let us call a smooth map $\iota\colon U \times M \to S^2$ such that for each $(s, z) \in U$ the restriction ι^s_z of ι to $\{(s, z)\} \times M$ is an embedding of the manifold M (potentially with boundary) into S^2 a* **parametric embedding** *of M.*

Denote by M^s_z the image of M under ι^s_z. This defines a family of embedded submanifolds.

Any family M^s_z of embedded submanifolds that arises as the image of a parametric embedding will be called **smooth family** *of submanifolds.*

We will also want to consider rectangles or half-discs that are manifolds with corners.

Definition 4.2. *Let $R \subset \mathbb{R}^2$ be a compact subset. We will call a parametric embedding of a small neighbourhood V of R in $\mathbb{R}^2$ also a parametric embedding of R. A family R^s_z of compact subsets that is the image of a parametric embedding of R we will call smooth family.*

Let us observe how concepts and constructions from Section 2 can be carried over to the parametric situation.

4.1. Regular points and trajectories

A cross section $C \subset S^2$ of a vector field X is an open submanifold of codimension 1, i.e. an embedded interval, that is transverse to X.

Definition 4.3. *A* **parametric cross section** *of a parametric vector field X^s_z on S^2 is a smooth family C^s_z, $(s, z) \in U$, of embedded intervals $(-1, 1)$ that such for each $(s, z) \in U$ the interval C^s_z is transverse to X^s_z.*

Example 4.4. If $q \in S^2$ is a regular point of $X^{s_0}_{z_0}$ for some $(s_0, z_0) \in S^k \times [-1, 1]$ then there is a cross section C of $X^{s_0}_{z_0}$ through q, i.e. an open embedded interval transverse to the vector field. The cross section C will be transverse the vector fields X^s_z for (s, z) in a small neighbourhood U of (s_0, z_0). Hence the constant family $C^s_z := C$ is a parametric cross section defined on U.

Parametric cross sections allow us to construct parametric flow boxes, analogously to Corollary 2.4.

Construction 4.5. Let $\gamma\colon\ [0,T]\ \to\ S^2$ with $T>0$ and $\gamma(0)\neq\gamma(T)$ be an arc of a trajectory of the vector field $X_{z_0}^{s_0}$ on S^2 for some $(s_0,z_0)\in S^k\times[-1,1]$. Let further C_z^s be a parametric cross section defined on a neighbourhood U of (s_0,z_0) such that $C_{z_0}^{s_0}$ contains the point $\gamma(0)$.

We can regard the parametric vector field X_z^s alternatively as a single smooth vector field $\mathcal{X}$ on the manifold $S^k\times[-1,1]\times S^2$ that is tangent to each of the spheres $\{(s,z)\}\times S^2$.

To the parametric cross section C_z^s there is a parametric embedding $\iota\colon\ U\times(-1,1)\ \to\ S^2$ of $(-1,1)$. It defines an embedding of $U\times(-1,1)$ into $S^k\times[-1,1]\times S^2$ by $(s,z;t)\mapsto(s,z;\iota_z^s(t))$, where ι_z^s again denotes the restriction of ι to $\{(s,z)\}\times(-1,1)$.

In other words, a parametric cross section C_z^s defines a cross section $\mathcal{C}$ of the vector field $\mathcal{X}$ on $S^k\times[-1,1]\times S^2$.

Then $\hat{\gamma}\colon\ [0,T]\ \to\ S^k\times[-1,1]\times S^2$, $t\ \mapsto\ (s_0,z_0,\gamma(t))$ is a trajectory of $\mathcal{X}$ on $S^k\times[-1,1]\times S^2$ and C_z^s defines a cross section $\mathcal{C}$ of $\mathcal{X}$ through $\hat{\gamma}(0)$.

Hence we can construct a long flow box $F\colon\ U\times[0,T]\times(-\varepsilon,\varepsilon)\ \to\ S^k\times[-1,1]\times S^2$ of $\mathcal{X}$ for a potentially smaller neighbourhood U of (s_0,z_0), *cf.* Corollary 2.4 and Remark 2.5. In these coordinates, the vector field $\mathcal{X}$ is given as ∂_t, the $[0,T]$-coordinate. In particular, for each fixed $(s,z)\in U$ the restriction F_z^s of F to $\{(s,z)\}\times[0,T]\times(-\varepsilon,\varepsilon)$ maps into $\{(s,z)\}\times S^2$. As F defines local coordinates on $S^k\times[-1,1]\times S^2$, hence so do its restrictions F_z^s on $\{(s,z)\}\times S^2\cong S^2$. Hence we may regard the composition $\mathrm{pr}_{S^2}\circ F\colon\ U\times[0,T]\times(-\varepsilon,\varepsilon)\ \to\ S^2$ of F with the projection pr_{S^2} of $S^k\times[-1,1]\times S^2$ to S^2 as a flow boxes that depend on a parameter $(s,z)\in U$.

These parametric coordinates will be called **parametric flow box**.

Remark 4.6. Regarding the parametric vector field X_z^s as a single vector field $\mathcal{X}$ on $S^k\times[-1,1]\times S^2$ we observe that both the flow time map as well as the Poincaré map, *cf.* Definition 2.6, are smooth with respect to the parameters s and z.

4.2. Singular points and neighbourhoods

In Section 2.3 we saw that we can find standard neighbourhoods of singular points of a vector field X with respect to a given area form Ω, *cf.* also Definition 3.8. These were closed discs, half-discs or rectangles whose boundary components are transverse to the vector field X and satisfy the condition that the divergence of X with respect to Ω is non-zero.

4.2.1. Elliptic neighbourhoods

Definition 4.7. *A smooth family D_z^s of closed discs such that each disc $D_z^s \cong D^2$ is an elliptic neighbourhood with respect to Ω for X_z^s, cf. Definition 3.8, is called a* **smooth family of elliptic neighbourhoods**.

Lemma 4.8. *To a, possibly degenerate, elliptic singular point there is smooth family of elliptic neighbourhoods.*

Proof. Let $q \in S^2$ be an elliptic singular point of $X_{z_0}^{s_0}$ for some $(s_0, z_0) \in S^k \times [-1, 1]$.

Then there is a elliptic neighbourhood $D \subset S^2$ of q that is a closed disc whose boundary is a cross section to $X_{z_0}^{s_0}$, i.e. transverse to the vector field. Transversality is an open condition, so there is a small neighbourhood $\mathcal{U}$ of (s_0, z_0) in $S^k \times [-1, 1]$ such that ∂D is transverse to all X_z^s, $(s, z) \in \mathcal{U}$. The divergence $\operatorname{div}_\Omega X$ depends smoothly on X, so there is a possibly smaller neighbourhood of (s_0, z_0), still denoted by $\mathcal{U}$, such that the divergence of X_z^s on D is still positive.

Hence the constant family $D_z^s := D$, $(s, z) \in \mathcal{U}$, is a smooth family of elliptic neighbourhoods. □

Remark 4.9. Each neighbourhood D_z^s may contain multiple singular points. However, as D_z^s is a disc with boundary transverse to the vector field X_z^s that does not contain closed trajectories, it has to contain at least one singular point.

We may regard D_z^s as an open submanifold D with boundary inside $S^k \times [-1, 1] \times S^2$. The vector fields X_z^s, regarded as a single vector field $\mathcal{X}$ on $S^k \times [-1, 1] \times S^2$, are transverse to ∂D. Hence these vector fields allow us to define a collar of ∂D inside D that is tangent to the slices $\{(s,z)\} \times S^2$.

Definition 4.10. *An embedding ι of D^2 into S^2 such that the radial directions ∂_r^s coincide with the vector field X_z^s on $\{r > 1/2\}$ will be called* **collared** *with respect to X_z^s.*

A parametric embedding ι_z^s such that each embedding ι_z^s is collared with respect to X_z^s will be called a collared parametric embedding.

4.2.2. Hyperbolic neighbourhoods

Definition 4.11. *A smooth family R_z^s of rectangles such that each rectangle $R_z^s \cong [-1,1] \times [-1,1]$ is a hyperbolic neighbourhood with respect to Ω for X_z^s is called a* **smooth family of hyperbolic neighbourhoods***.*

Lemma 4.12. *To a, possibly degenerate, hyperbolic singular point there is a smooth family of hyperbolic neighbourhoods.*

Proof. Let again X_z^s, $(s,z) \in S^k \times [-1,1]$, be a smooth family of vector fields on S^2 and let $q \in S^2$ be a hyperbolic singular point of $X_{z_0}^{s_0}$ for some $(s_0, z_0) \in S^k \times [-1,1]$.

The contact condition implies that the linearisation of $X_{z_0}^{s_0}$ in q has a non-vanishing eigenvalue. As in Construction 2.16 pick local coordinates x and y of S^2 near q such that in $q = (0,0)$, ∂_x and ∂_y lie in the two eigenspaces of $X_{z_0}^{s_0}$ such that the eigenvalue to ∂_y is non-zero.

Write the vector fields X_z^s near q in these local coordinates as $a_z^s(x,y)\,\partial_x + b_z^s(x,y)\,\partial_y$. That the eigenvalue of $X_{z_0}^{s_0}$ to ∂_y does not vanish implies that $\frac{\partial}{\partial y} b_z^s(x,y) \neq 0$ in (s_0, z_0) and $q = (0,0)$.

Apply the Implicit Function Theorem to obtain neighbourhoods $\mathcal{U}'$ of (s_0, z_0) in $S^k \times [-1,1]$ and V_x and V_y of 0 in $\mathbb{R}$ together with a smooth function $\eta\colon\ \mathcal{U}' \times V_x \to \mathbb{R}$ such that $b_z^s(x,y) = 0$ exactly for $y = \eta(s,z,x)$ for all $(s,z) \in \mathcal{U}'$ and $(x,y) \in V_x \times V_y$.

We obtain a coordinate function $y - \eta(s,z,x)$ that we can complement by its orthogonal complement with respect to Ω. This defines coordinates of S^2 on a neighbourhood V of q that smoothly depend on $(s,z) \in \mathcal{U}'$. Denote them again by just y and x and write the vector fields X_z^s again as $a_z^s(x,y)\,\partial_x + b_z^s(x,y)\,\partial_y$ in these new coordinates. Then $b_z^s(x,y) = 0$ exactly for $y = 0$.

We can now proceed as in Construction 2.18. Along the line $V_x \times \{0\}$, the vector field is parallel to ∂_x, i.e. $X_z^s = a_z^s(x,0)\,\partial_x$. Since q is an isolated singular point of $X_{z_0}^{s_0}$, there are points $(-\delta_x, 0)$ and $(\delta_x, 0)$ such that $a_{z_0}^{s_0}(\pm\delta_x, 0) \neq 0$. By continuity there is some $\delta_y > 0$ as well as some smaller neighbourhood $\mathcal{U} \subset \mathcal{U}'$ of (s_0, z_0) such that the two arcs $\{\pm\delta_x\} \times (-\delta_y, \delta_y)$ are transverse to all vector fields X_z^s for $(s,z) \in \mathcal{U}$. Along the two arcs $[-\delta_x, \delta_x] \times \{\pm\delta_y\}$ the functions b_z^s are non-zero by construction of the local coordinates and the arcs are consequently transverse to vector fields X_z^s.

For each $(s,z) \in \mathcal{U}$ consider the rectangle $R_z^s \coloneqq [-\delta_x, \delta_x] \times [-\delta_y, \delta_y]$ given in the coordinates x and y. Remember that the coordinates depend on (s,z) and hence so do the rectangles R_z^s. The sides of the rectangles R_z^s are, as we just saw, transverse to X_z^s.

The divergence of $X_{z_0}^{s_0}$ does not vanish in q. As the divergence depends smoothly on both the point and the parameter (s,z), we may, by choosing δ_x, δ_y and $\mathcal{U}$ sufficiently small, assume that the divergence of each vector field X_z^s, $(s,z) \in \mathcal{U}$, does not vanish in any point of R_z^s.

Then the rectangles R_z^s, $(s,z) \in \mathcal{U}$, are a smooth family of hyperbolic neighbourhoods. □

Remark 4.13. By construction, the ∂_y-component of any X_z^s vanishes along $\{y = 0\}$ in the coordinates of R_z^s. The sign change of the ∂_x-component of X_z^s along $\{y = 0\}$ mandates the existence of singular points in R_z^s.

Definition 4.14. *We will call an embedding of a hyperbolic neighbourhood* **collared** *if in neighbourhoods of the sides $\{\pm 1\} \times (-3/4, 3/4)$ the vector fields $\partial_{x,z}^s$ in x-direction coincide with either X_z^s or $-X_z^s$.*

Observation 4.15. Let R_z^s, $(s,z) \in U$, be a positive hyperbolic neighbourhood. By construction, for each $(s,z) \in U$ the only trajectories that tend to points in R_z^s are the two separatrices, i.e. the trajectories through the points $(\pm 1, 0)$. Hence all trajectories through points $(\pm 1, y)$, $y \neq 0$, leave R_z^s. They can only do so through the top t and the bottom b, i.e. the cross sections $t \coloneqq (-1,1) \times \{1\}$ and $b \coloneqq (-1,1) \times \{-1\}$. Consequently, the Poincaré maps P^+ between $l^+ \coloneqq \{-1\} \times (0,1)$ and

$r^+ := \{1\} \times (0,1)$ and t and P^- between $l^- := \{-1\} \times (-1,0)$ and $r^- := \{1\} \times (-1,0)$ and b are defined.

The trajectories of X_z^s that pass through the closed interval C that is the complement of $P^+(l^+) \cup P^+(r^+)$ in t emanate at points in R_z^s and trajectories that pass to points in t close to C pass through points in l^+ or r^+ close to the separatrices, *cf.* Figure 4.1.

All the cross sections $l^\pm$, $r^\pm$, t and b are parametric and the Poincaré maps depend smoothly on the parameter. Hence the endpoints of C in t also depend smoothly on (s, z).

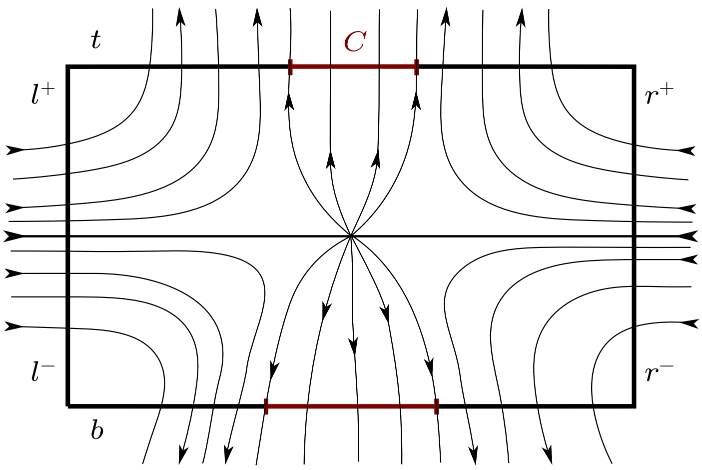


Figure 4.1.: A hyperbolic neighbourhood around a degenerate hyperbolic point

4.2.3. Half-hyperbolic neighbourhoods

Definition 4.16. *A smooth family D_z^s of half-discs $\mathcal{D}$, cf. Definition 3.8, such that each D_z^s is a positive half-hyperbolic neighbourhood with respect to Ω for X_z^s is a* **smooth family of positive half-hyperbolic neighbourhoods** *if every D_z^s satisfies the additional conditions that*

1. *there are no singular points of X_z^s in $\{x \geq 0, y \neq 0\}$ and that*
2. *the only trajectory γ of X_z^s with $\alpha(\gamma) \in D_z^s$ passes through $(1,0) \in D_z^s$.*

We can find such families using analogous constructions as for elliptic and hyperbolic neighbourhoods, applied to a singular point q that is half-hyperbolic.

Notice that the half-hyperbolic neighbourhoods that we construct around an isolated half-hyperbolic point only contain singular points, if at all, along $\{y = 0\}$ and hence satisfy the additionally imposed conditions.

Corollary 4.17. *To a, possibly degenerate, half-hyperbolic singular point there is a smooth family of half-hyperbolic neighbourhoods.*

Definition 4.18. *An embedding of a half-hyperbolic neighbourhood is* **collared** *if it is an embedding of the half-disc $\mathcal{D}$ such that in neighbourhoods of c_e and $\{1\} \times (-3/4, 3/4) \subset c_h$ the vector fields $\partial^s_{z,x}$ coincide with either X^s_z or $-X^s_z$.*

Observation 4.19. Let D^s_z, $(s, z) \in U$, be a positive half-hyperbolic neighbourhood. Denote its two smooth boundary components by e and h, where h is the one that contains the separatrix γ of D^s_z. Denote the complement of $\gamma \cap h$ in h by h^+ and h^-.

Then the Poincaré maps P^+ between h^+ and e and P^- between h^- and e are defined. The trajectories of X^s_z that pass through the closed interval C that is the complement of $P^+(h^+) \cup P^-(h^-)$ in e emanate at points in D^s_z. Again, the end points of this interval depend smoothly on the parameter.

III. Parametric elimination

We observed in Chapter II that whether a sphere $\{z\}\times S^2$ in S^3 is convex with respect to the contact structure ξ^s is a property of its characteristic foliation and phrased the conditions in terms of elliptic and (half-) hyperbolic neighbourhoods and their separatrices. In particular, we saw that retrograde connections between singular points are the obstructions to convexity.

Our goal thus is to deform the contact structures ξ^s such that no vector field X_z^s that directs the characteristic foliation of $\{z\} \times S^2$ with respect to ξ^s has a retrograde connection. We develop a parametric elimination deformation that applies to whole neighbourhoods (Section 2). Applying it to neighbourhoods that are leaves in a graph (Section 4) this deformation can be used achieve our goal (Section 5). This process is aided by a complexity valuation (Section 3) that resembles the order of degeneracy of singular points.

1. Finite number of neighbourhoods of singular points

A single vector field on a surface generically only has finitely many singular points. This is a basic result about transversality of smooth sections to the zero section. Considering parametric families of vector fields, we can still archive transversality to the zero section of the tangent space, however, counting dimensions, this will not imply isolated singular points of the individual vector fields.

Passing to k-th jet extensions for sufficiently large k and applying Thom transversality to these with respect to a suitably crafted submanifold of the k-th jet space, Bruce [Bru86] shows the statement for families.

Theorem 1.1 ([Bru86, Lemma 1.6., Application 2.2.])**.** *Let M and $\mathcal{P}$ be compact manifolds and denote by $\mathfrak{X}_{\mathcal{P}}(M)$ the space of smooth vector* *fields on M, parameterised by $\mathcal{P}$. For a residual set of such sections $\mathcal{X}\colon M \times \mathcal{P} \to TM$, the individual vector fields $X_p := \mathcal{X}|_{M\times\{p\}}$ have isolated singular points.*

As is the case for Thom transversality for sections of the tangent bundle, its proof yields a relative statement.

Corollary 1.2. *Let X_z^s be a parametric vector field on S^2 and $U \subset S^2$ an open subset such that the all vector fields X_z^s have only isolated singular points in U.*

Then there is a parametric vector field Y_z^s on S^2, arbitrarily $\mathcal{C}^\infty$-close to X_z^s that coincides with X_z^s on U and is such that all vector fields Y_z^s have only isolated singular points.

Remember that we wrote contact forms for the contact structures ξ^s on $[-1,1]\times S^2$ as $\beta_z^s + u_z^s\,\mathrm{d}z$ and defined our parametric vector field using $\beta_z^s = \iota_{X_z^s}\Omega$ with an area form Ω on S^2. Conversely, given a parametric vector field Y_z^s, we obtain dual 1-forms γ_z^s by $\iota_{Y_z^s}\Omega$ for which, as the contact condition is $\mathcal{C}^1$-open, the forms $\alpha_t^s := (1-t)\beta_z^s + t\gamma_z^s + u_z^s\,\mathrm{d}z$ are all contact forms on $[-1,1]\times S^2$ for $t \in [0,1]$, provided Y_z^s is sufficiently close to X_z^s.

As for z close to $\{-1,1\}$ the vector fields X_z^s have only two singular points, this perturbation can be done relative to the boundary of $[-1,1]\times S^2$, i.e. the family α_t^s is stationary in t close to $\partial\big([-1,1]\times S^2\big)$. Hence we can apply the parametric and relative version of Gray's Stability Theorem (Corollary I.5.3) to obtain a parametric isotopy of ξ^s such that the perturbed contact structures induce vector fields on S^2 that all have only isolated singular points.

Hence, to each $(s,z) \in S^k \times [-1,1]$ there is a finite number of disjoint smooth families of elliptic, hyperbolic and half-hyperbolic neighbourhoods whose union covers all singular points of X_z^s. As the parameter space $S^k \times [-1,1]$ is compact, we find a finite number of smooth families of neighbourhoods $\mathcal{N}_i$, $i = 1,\dots,k$, that are defined on open subsets U_i of $S^k \times [-1,1]$ such that for each $(s,z) \in S^k \times [-1,1]$ the union of all neighbourhoods $N_{i,z}^s$ for which $(s,z) \in U_i$ contains all singular points of X_z^s.

2. Elimination of (neighbourhoods of) singular points

Suppose we are given a finite collection of neighbourhoods of all singular points as in the previous section. Then Proposition II.3.10 tells us that all those spheres $\{z\} \times S^2$ are convex with respect to ξ^s whose vector fields X_z^s have no retrograde connection between two (half-)hyperbolic neighbourhoods.

In this section we will discuss a homotopy of families of contact structures that eliminates both a hyperbolic and an elliptic neighbourhood. As will be explained in Section 5, this operation will allow us to get rid of all retrograde connections. This elimination homotopy is a parametric version of Giroux's 'Elimination Lemma', *cf.* [Gir91]. Its proof is inspired by the proof of the Elimination Lemma by Fuchs as presented in [Gei08, Lemma 4.6.26].

2.1. Situation

Consider our smooth family of vector fields X_z^s, $(s, z) \in S^k \times [-1, 1]$, on S^2. Let $(s_0, z_0) \in S^k \times [-1, 1]$ and U an open neighbourhood and suppose there are a family of positive elliptic neighbourhoods D_z^s as well as a family of positive hyperbolic neighbourhoods R_z^s, $(s, z) \in U$, disjoint from D_z^s, such that for each $(s, z) \in U$ the trajectory γ_z^s of the vector field X_z^s through the point $(-1, 0) \in R_z^s$, a separatrix of R_z^s, emanates at a point in D_z^s.

Remark 2.1. As we can shrink our neighbourhoods along the flow of the vector fields X_z^s it suffices to require that D_z^s and R_z^s contain disjoint sets of singular points.

To the family D_z^s there is a collared parametric embedding η_z^s of D^2 into S^2 and to R_z^s there is a collared parametric embedding ρ_z^s of a neighbourhood of $[-1, 1] \times [-1, 1]$ in $\mathbb{R}^2$ into S^2.

The embeddings ρ_z^s induce coordinates $[-1, 1]$ on the 'left' side of the rectangles $L_z^s := \{-1\} \times [-1, 1] \subset R_z^s$. The arcs L_z^s form a parametric cross section that intersects each trajectory γ_z^s in the point $0 \in L_z^s$.

To these coordinates of the parametric cross section we find, after shrinking U if necessary, a parametric flow box $F\colon\ U \times [-T, 0] \times$

$(-\varepsilon, \varepsilon) \rightarrow S^2$ around the trajectory $\gamma_{z_0}^{s_0}$ such that for time $t = 0$, the $(-\varepsilon, \varepsilon)$-coordinate corresponds with the coordinates on L_z^s, i.e. $F(s, z; 0, (-\varepsilon, \varepsilon)) = (-\varepsilon, \varepsilon) \subset L_z^s$, and such that the time $t = -T$ side of the flow box is contained in D_z^s, i.e. $F(s, t; T, (-\varepsilon, \varepsilon)) \subset D_z^s$, for all $(s, z) \in U$.

2.2. Adjusting the coordinate neighbourhoods

Rescaling the y-coordinate of the embedding ρ and hence the y-coordinate in the rectangles R_z^s, in L_z^s and in the flow box, we may assume that $\varepsilon = 3/4$. Denote the intersections points of $F(s, z; \bullet, \pm 1/2)$ with ∂D_z^s by p_z^s and $n_z^s \in \partial D_z^s \cong S^1$, respectively. After shrinking U and modifying the embedding of the parametric elliptic neighbourhood as described below we can assume $p_z^s = \frac{\pi}{6}$ and $n_z^s = -\frac{\pi}{6}$.

Construction 2.2. Let us assume that $p_{z_0}^{s_0} \neq -\frac{\pi}{6}$. Note that both p_z^s and n_z^s smoothly depend on (s, z). Shrinking U, we guarantee that $p_z^s \neq -\frac{\pi}{6}$ for all $(s, z) \in U$. Then $a_z^s\colon\ \vartheta \mapsto \vartheta + \left(\frac{\pi}{6} - p_z^s\right)$ is a smooth map that sends p_z^s to $\frac{\pi}{6}$. If $p_z^s = \frac{\pi}{6}$, then n_z^s is in $S^1 \setminus \left\{\frac{\pi}{6}\right\} \cong (0, 2\pi)$. There is a smooth family $b(n)\colon\ [0, 2\pi] \rightarrow [0, 2\pi]$ of diffeomorphisms of the interval $[0, 2\pi]$ that sends $n \in (0, 2\pi)$ to $\frac{5\pi}{3}$. The composition c_z^s of a_z^s with $b(a_z^s(n_z^s))$ sends p_z^s to $\frac{\pi}{6}$ and n_z^s to $-\frac{\pi}{6}$.

Pick a smooth step function $g\colon\ [0, 1] \rightarrow [0, 1]$ that is identically 0 on $\left[0, 1/3\right]$, identically 1 on $\left[2/3, 1\right]$ and strictly monotone in between. Then we may compose each embedding η_z^s of the elliptic neighbourhoods D_z^s with the map $(r, \vartheta) \mapsto g(r)\,\vartheta + (1 - g(r))\,(c_z^s)^{-1}(\vartheta)$. This yields again a collared parametric embedding of an elliptic neighbourhood that is such that $p_z^s = \frac{\pi}{6}$ and $n_z^s = -\frac{\pi}{6}$.

Finally, let us consider the flow box between the rectangles and the elliptic neighbourhood. Denote the flow time map from $y \in \left(-1/2, 1/2\right) \subset L_z^s$ to the cross section ∂D_z^s with respect to X_z^s by $T_z^s(y)$. In the flow box F_z^s, these are the t-coordinates of the intersection of the flow box F_z^s with ∂D_z^s. We may rescale the t-coordinate in all flow boxes such that $T = 5/2$, i.e. that each F_z^s, $(s, z) \in U$, is defined on a neighbourhood of $\left[-5/2, 0\right] \times \left[-1/2, 1/2\right]$, such that ∂D_z^s intersects F_z^s in the points $\left(-3 + \sqrt{1 - y^2}, y\right) \in \left[-5/2, 0\right] \times \left[-1/2, 1/2\right]$ and such that F stays collared near $\{t = 0\} \subset L_z^s$ and near ∂D_z^s.

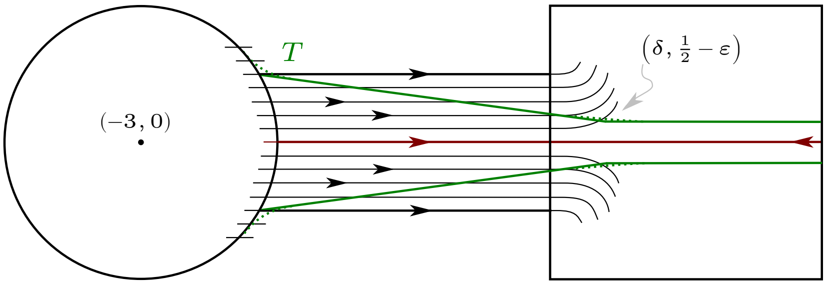


Figure 2.1.: Constructing the half-disc N by merging neighbourhoods of singular points and a long flow box

2.3. Merging neighbourhoods of singular points

As for each $(s, z) \in U$ all three coordinate charts, the elliptic neighbourhood ρ^s_z, the flow box F^s_z and the hyperbolic neighbourhood η^s_z are collared near their common points, we are able to merge their domains and obtain a parametric embedding ι^s_z of a neighbourhood of $N_0 := D^2(-3, 0) \cup [-3 + \sqrt{2}/2, 0] \times [-1/2 \times 1/2] \cup [0, 2] \times [-1, 1] \subset \mathbb{R}^2$ that coincides with η^s_z on $D^2(-3, 0)$, coincides with ρ^s_z on $[0, 2] \times [-1, 1]$ and coincides with F^s_z on their complement. Here, $D^2(-3, 0) \subset \mathbb{R}^2$ denotes the closed 2-disc of radius 1 centred at $(-3, 0)$.

The vector field X^s_z is parallel to the boundary of $\iota^s_z(N_0)$ along the arcs $[-3 + \sqrt{2}/2, 0] \times \{\pm 1/2\}$, points into $\iota^s_z(N_0)$ along $\{2\} \times [-1, 1]$ and points out of $\iota^s_z(N_0)$ along all other parts of its boundary. If we restrict the domain of the flow box F^s_z to the trapezium T between the points $(-3 + \sqrt{2}/2, \mp 1/2)$ and $(\delta, \pm(1/2 - \varepsilon))$ for some small $\varepsilon > 0$ and $\delta > 0$, all four sides of $F^s_z(T)$ will be transverse to the vector fields X^s_z.

In the hyperbolic neighbourhoods R^s_z, the vector fields X^s_z are by construction only tangent to the x-direction along $\{y = 0\}$ and in the collar. Consequently, we may restrict the domain of η^s_z to $R := [\delta, 2] \times [-1/2 + \varepsilon, 1/2 - \varepsilon]$ while preserving the property that each smooth component of $\eta^s_z(\partial R)$ is transverse to X^s_z for all $(s, z) \in U$.

Merging the domains of ρ^s_z, F^s_z and η^s_z we obtain a parametric embedding ι^s_z of $N := D^2(-3, 0) \cup T \cup R \subset \mathbb{R}^2$ such that each smooth component of the piecewise smooth boundary $\iota^s_z(\partial N)$ is transverse to the vector field X^s_z for each $(s, z) \in U$, *cf.* Figure 2.1.

Consider $\iota^{s_0}_{z_0}(N)$. The vector field $X^{s_0}_{z_0}$ points out of $\iota^{s_0}_{z_0}(N)$ along the curves $\iota^{s_0}_{z_0}\big(\partial D^2(-3,0)\cap\partial N\big)$ and $\iota^{s_0}_{z_0}(\partial T\cap\partial N)$, as well as along $\iota^{s_0}_{z_0}\big([0,2]\times\{-1/2+\varepsilon, 1/2-\varepsilon\}\big)$. Consequently, we can round the corners $\big(-3+\sqrt{2}/2, \mp 1/2\big)$ and $\big(0, \pm(1/2-\varepsilon)\big)$ of N, *cf.* Construction II.2.19, i.e. there is a closed neighbourhood N' of N, coinciding with N away from those corners such that the boundary of N' comprises only two smooth components, one being $\{2\}\times[-1/2+\varepsilon, 1/2-\varepsilon]$, such that both smooth components of $\iota^{s_0}_{z_0}(\partial N')$ are transverse to $X^{s_0}_{z_0}$.

To ease our constructions in the future, consider N'', the intersection $N'\cap[-4+\varepsilon',3]\times[-1,1]$ for some small $\varepsilon'>0$. For ε' sufficiently small, the set R is a tetragon such that all four smooth components of $\iota^{s_0}_{z_0}(\partial R)$ are transverse to $X^{s_0}_{z_0}$. Shrinking U, we can assume that $\iota^s_z(\partial N'')$ is likewise transverse to X^s_z for all $(s,z)\in U$.

Pick an (orientation preserving) diffeomorphism r of open neighbourhoods of $[-4,3]\times[-2,2]$ and N'' that satisfies $r\big([-4,3]\times[-2,2]\big)=N''$ and that

- preserves the x-axis, i.e. $r\big([-4,3]\times\{0\}\big)=[-4+\varepsilon',3]\times\{0\}$,
- maps the rectangles $E:=[-4,-1]\times[-2,2]$ into $D^2(3,0)\cap N''$ and $[-3,-2]\times[-1,1]$ into the disc around $(-2,0)\in N''$ with radius $1/2$, i.e. the region that contains the singular points of the vector fields X^s_z in the elliptic neighbourhoods,
- maps $F:=[-1,1]\times[-2,2]$ into the trapezium T,
- and satisfies $r(H)=R$ for $H:=[1,3]\times[-2,2]$.

The composition of ι^s_z with r is a parametric embedding of the rectangle $[-4,3]\times[-2,2]\subset\mathbb{R}^2$.

On the trapezium $T\subset N''\subset\mathbb{R}^2$, all vector fields X^s_z, $(s,z)\in U$, were parallel to the ∂_x-direction of $\mathbb{R}^2$. Hence we may assume that, writing the vector fields X^s_z in the coordinates ∂_x and ∂_y of $r\circ\iota^s_z$ as $X^s_z=X^s_{1,z}\partial_x+X^s_{2,z}\partial_y$, the ∂_y component $X^s_{2,z}$ is strictly positive on $F\cap\{y>0\}$ and strictly negative on $F\cap\{y<0\}$. Remember that $X^s_{2,z}<0$ on $H\cap\{y<0\}$, and $X^s_{2,z}>0$ on $H\cap\{y>0\}$ by construction.

Since $\iota^s_z(\partial R)$ is transverse to X^s_z, we can choose r such that also $X^s_{2,z}<0$ on $[-4,-1]\times[-2,-1]$ and that $X^s_{2,z}>0$ on $[-4,-1]\times[1,2]$ and, further, that $X^s_{1,z}<0$ on both $[-4,-3]\times[-1,1]$ and $[5/2,3]\times[-1,1]$.

For the elimination process we restrict U to a product $D_0\times I_0$ of the closure of an open interval around $z_0\in[-1,1]$ and the closure of an open neighbourhood around s_0 in S^k such that $D_0\times I_0\subset U$.

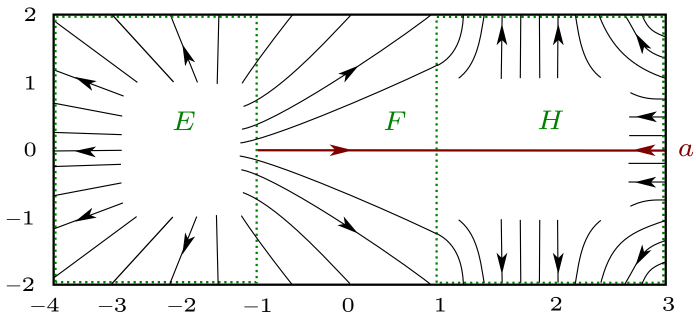


Figure 2.2.: The rectangle R_z^s and some trajectories of a vector field X_z^s

2.4. Neighbourhood for elimination

Denote the parametric embedding that we constructed in the last section by ι_z^s and denote the image of $[-4,3]\times[-2,2]$ under ι_z^s by R_z^s. Write Ω_z^s for the pull-back of the area form Ω to $[-4,3]\times[-2,2]$. The vector fields X_z^s, $(s,z)\in D_0\times I_0$, have the following properties.

N1: On the segment $a := [-1,3]\times\{0\}$ the vector field X_z^s is parallel to ∂_x.

N2: The ∂_y-component $X_{2,z}^s$ of X_z^s is positive on $[-4,-1]\times[1,2]$ as well as on $[-1,3]\times(0,2]$ and negative on $[-4,-1]\times[-2,-1]$ as well as on $[-1,3]\times[-2,0)$.

N3: It is $X_{1,z}^s < 0$ on $[-4,-3]\times[-1,1]$ and on $[5/2,3]\times[-1,1]$.

N4: There are no singular points of X_z^s in $[-4,3]\times[-2,2]$ outside of $[-3,-1]\times[-1,1]\cup a$ and all singular points are positive.

N5: The divergence of X_z^s with respect to Ω_z^s is positive on both E and H.

Additionally we ensured that

A1: all trajectories of X_z^s through points in F intersect E in negative time.

2.5. Contact forms of normal form

In the elimination process we will deform the contact structures in ∂_z direction. It is therefore beneficial to obtain control over the contact forms in ∂_z direction. Suitable contact forms $\beta^s_z + u^s_z\,\mathrm{d}z$ have the following property.

A2: The function u^s_z is positive and is constant on E and H.

Lemma 2.3. *If ι^s_z, $(s,z) \in D_0 \times I_0$, is a parametric embedding of $[-4,3]\times[-2,2]$ satisfying properties (N5) and (A1) and $U \subseteq D_0 \times I_0$ any open subset, then there is an isotopy of the parametric contact structures ξ^s, supported in any neighbourhood of $U \subset S^k \times [-1,1]$ and any neighbourhood of $R^s_z \subset S^2$ that*

1. *does not change the vector fields X^s_z that the contact structures induce on S^2,*
2. *is such that after the isotopy the contact structures satisfy property (A2) on R^s_z for $(s,z) \in U$ and*
3. *for (s,z) in the complement of U where $\beta^s_z + u^s_z\,\mathrm{d}z$ satisfied property (A2) before the deformation, the contact forms will also do so after the deformation.*

Proof. As first step, let us observe that we find a smooth family of functions f^s_z on S^2, that are identically 1 outside a neighbourhood of the image of $[-4,3]\times[-2,2]$ under ι^s_z, such that the vector fields $f^s_z X^s_z$ have positive divergence on all of $[-4,3]\times[-2,2]$.

In Section II.2.2 we saw that for a smooth function f we have $\mathrm{div}_\Omega(fX) = X(f) + f\,\mathrm{div}_\Omega(X)$, *cf.* Equation (II.1) on page 35. Hence, pick a smooth family of positive functions f^s_z, $(s,z) \in I_0 \times D_0$, on $[-4,3]\times[-2,2]$ that take the value 1 on E, grow sufficiently fast along the trajectories of X^s_z in F such that $X^s_z(f^s_z) > -f^s_z\,\mathrm{div}_{\Omega^s_z}(X^s_z)$ and that take a constant value $c \in \mathbb{R}$ on H. (To see that we can construct a family of function f^s_z that is smooth in (s,z), consider the vector fields X^s_z again as a single vector field $\mathcal{X}$ on $S^k \times [-1,1] \times S^2$.)

Property (N5) together with the fact that f^s_z is constant on both E and H guarantees that $\mathrm{div}_{\Omega^s_z}(f^s_z X^s_z) > 0$ on $H \cup E$.

Extend $f^s_z \circ (\iota^s_z)^{-1}$ to a smooth family of functions on S^2, again denoted by f^s_z, such that each f^s_z is constantly 1 outside a small neighbourhood of $\iota^s_z([-4,3]\times[-2,2])$. Consider the rescaled contact forms $f_z^s(\beta_z^s + u_z^s\,\mathrm{d}z)$. They induce the vector fields $f_z^s X_z^s$ on S^2 that have positive divergence on the image of ι_z^s.

That the divergence is positive implies that the forms $\mathrm{d}(f_z^s\beta_z^s)$ are positive area forms on R_z^s. Consequently, there is a sufficiently large $\lambda \in \mathbb{R}$ such that in the compact sets R_z^s, $(s,z) \in D_0 \times I_0$, the 1-forms $f_z^s\beta_z^s + \lambda\,\mathrm{d}z$ are contact forms:

$$\begin{aligned}(f_z^s\beta_z^s + \lambda\,\mathrm{d}z) \wedge \mathrm{d}(f_z^s\beta_z^s + \lambda\,\mathrm{d}z)&\\ &= (f_z^s\beta_z^s + \lambda\,\mathrm{d}z) \wedge \left(\mathrm{d}(f_z^s\beta_z^s) - \frac{\mathrm{d}}{\mathrm{d}z}(f_z^s\beta_z^s) \wedge \mathrm{d}z\right)\\ &= \left(-f_z^s\beta_z^s \wedge \frac{\mathrm{d}}{\mathrm{d}z}(f_z^s\beta_z^s) + \lambda\,\mathrm{d}(f_z^s\beta_z^s)\right) \wedge \mathrm{d}z\,.\end{aligned}$$

As the contact condition is open, the forms $f_z^s\beta_z^s + \lambda\,\mathrm{d}z$ are contact forms in sufficiently small open neighbourhoods N_z^s of R_z^s. Pick a smooth family of bump functions $\phi_{1,z}^s\colon\ S^2 \to \mathbb{R}$ that take the value 1 on R_z^s and vanish outside N_z^s. Additionally, let $\phi_2\colon\ S^k \times [-1,1] \to \mathbb{R}$ be a smooth bump function that takes the value 1 on U and vanishes outside a small neighbourhood of U. Denote by ϕ_z^s the product $(s,z;q) \mapsto \phi_{1,z}^s(q)\,\phi_2(s,z)$. Then for $t \in [0,1]$ the forms

$$\alpha_t^s \coloneqq f_z^s\beta_z^s + ((1 - t\phi_z^s)u_z^s + t\phi_z^s\lambda)\,\mathrm{d}z$$

are contact forms on $[-1,1] \times S^2$ as convex combinations of contact forms with coinciding 1-forms $f_z^s\beta_z^s$, *cf.* Observation I.7.3. Outside of N_z^s, the contact forms α_t^s are stationary in t.

By the relative and parametric version of the Gray stability theorem (Corollary I.5.3) to the family of contact structures $\xi_t^s \coloneqq \ker\alpha_t^s$ there is a parametric isotopy ψ_t^s of $[-1,1] \times S^2$ such that

$$T\psi_t^s(\xi^s) = \xi_t^s\,,$$

or equivalently, if we denote by $\left(\psi_t^s\right)_*$ the pullback with its inverse,

$$\left(\psi_t^s\right)_*\alpha^s = \mu_t^s\alpha_t^s$$

for some smooth family of positive functions μ_t^s. This isotopy is stationary outside the sets N_z^s.

Denote by f^s the map $(z;q) \mapsto f^s_z(q)$. As the functions f^s are positive, the forms α^s_t/f^s are also contact forms for the contact structures ξ^s_z and

$$\frac{1}{f^s}\alpha^s_t = \beta^s_z + \left((1-t\phi^s_z)\frac{u^s_z}{f^s_z} + t\phi^s_z\frac{\lambda}{f^s_z}\right)\mathrm{d}z\,. \tag{III.1}$$

These contact forms all induce the vector fields X^s_z on S^2. For $t=1$, $(s,z) \in U$ and on R^s_z the contact form α^s_t/f^s is given as $\beta^s_z + \phi^s_z\lambda/f^s_z\,\mathrm{d}z$ and the function $\phi^s_z\lambda/f^s_z = \lambda/f^s_z$ is positive on R^s_z and constant on both E and H.

If u^s_z is positive on R^s_z and constant on E and H for some $(s,z) \notin U$, the convex combination $(1-\phi^s_z)\,u^s_z/f^s_z + \phi^s_z\lambda/f^s_z$ is again positive and, as f^s_z is constant on $E \cup H$, also constant on both E and H. □

If we are just looking at a region where the divergence is positive, we can, with the same argument, but without rescaling the contact forms, obtain a similar statement.

Corollary 2.4. *Let D^s_z, $(s,z) \in D_0 \times I_0$, be an parametric embedding of a disc in S^2 such that the contact forms in D^s_z are given as $\beta^s_z + u^s_z\,\mathrm{d}z$ with $u^s_z > 0$ and $\mathrm{d}\beta^s_z > 0$, i.e. the divergence of the vector fields X^s_z induced by the contact forms is positive. Let further $K \subset D_0 \times I_0$ be the closure of an open subset.*

Then there is a parametric contact isotopy Ψ^s_t that is stationary outside small neighbourhoods of K and D^s_z that

1. *does not change the vector fields X^s_z that the contact structures induce on S^2,*
2. *is such that after the isotopy the contact forms in a neighbourhood of D^s_z are given by $\beta^s_z + h^s_z\,\mathrm{d}z$ with $h^s_z > 0$ and h^s_z is constant on D^s_z for $(s,z) \in K$.*
3. *For all $(s,z) \notin K$ where u^s_z was constant on D^s_z, h^s_z is also constant on D^s_z.*

Additionally, h^s_z may be chosen arbitrarily large on D^s_z.

2.6. Parametric Elimination

Theorem 2.5 (Parametric Elimination)**.** *Let $D_0 \subset S^k$ be a closed disc, $I_0 \subset [-1,1]$ a closed interval and R^s_z, $(s,z) \in D_0 \times I_0$, be a* *parametric embedding of* $[-4,3] \times [-2,2]$ *via* ι_z^s *such that there the contact forms are of the form* $\beta_z^s + u_z^s \,\mathrm{d}z$ *that satisfies property (A2) and that the vector fields* X_z^s *that are induced on* S^2 *by the contact forms satisfy the conditions (N1) to (N5).*

Then there is, for any $I_1 \subsetneq I_0$ *that is the closure of an open interval and any* $D_1 \subsetneq D_0$ *that is the closure of an open disc, a parametric contact isotopy* Ψ_t^s *that is stationary outside the* R_z^s *such that after the deformation the contact forms*

1. *satisfy condition (A2) on* R_z^s *for* $(s,z) \in D_0 \times I_0$

and induce vector fields Y_z^s *that*

1. *satisfy the conditions (N1) to (N5) in* R_z^s *for all* $(s,z) \in D_0 \times I_0$ *and*
2. *have no singular points in* R_z^s *for* $(s,z) \in D_1 \times I_1$*.*

Proof. We seek a parametric isotopy Ψ_t^s of contact structures on $[-1,1] \times S^2$ that is stationary away from the neighbourhoods R_z^s and changes the induced family of vector fields X_z^s.

Remember that these vector fields arose as follows. We wrote the contact forms α^s as $\beta_z^s + u_z^s \,\mathrm{d}z$ on $[-1,1] \times S^2$, where β_z^s were 1-forms on S^2. In particular, the β_z^s arise as the pull-back of α^s under the embedding $S^2 \cong \{z\} \times S^2 \subset [-1,1] \times S^2$. The contact forms $\beta_z^s + u_z^s \,\mathrm{d}z$ induce the vector fields X_z^s on S^2 by $\iota_{X_z^s}\Omega = \beta_z^s$.

Let us assume for a moment that we found a parametric isotopy Ψ_t^s of contact structures as required. It is, in particular, a parametric diffeotopy of $[-1,1] \times S^2$. Consider, for some $(s,z) \in U$, the vector field Y_z^s that the contact form $\big(\Psi_1^{s,-1}\big)^*\alpha^s$ of the deformed contact structure $T\Psi_1^s(\xi^s) = \ker\big(\Psi_1^{s,-1}\big)^*\alpha^s$ induces on S^2 embedded as $\{z\} \times S^2$. It is given by

$$\begin{aligned} \iota_{Y_z^s}\Omega &= (\iota_z)^*\Big(\Psi_1^{s,-1}\Big)^*\alpha^s \\ &= \Big(\Psi_1^{s,-1} \circ \iota_z\Big)^*\alpha^s \,, \end{aligned}$$

where $\iota_z \colon\ S^2 \ \to\ [-1,1] \times S^2$ is the embedding $q \mapsto (z,q)$. Consequently, the vector field Y_z^s coincides with the vector field that the original contact form α^s induces on S^2 embedded as $\Psi_1^{s,-1}\big(\{z\} \times S^2\big)$.

Hence, to find Ψ_t^s, we will deform the spheres $\{z\} \times S^2$ inside R as graphs over their original embedding such that the contact forms α^s induce vector fields Y_z^s on these that satisfy the required conditions. We will construct a suitable parametric function g_z^s, $(s,z) \in S^k \times [-1,1]$, on S^2 that is identically 0 outside the R_z^s, $(s,z) \in D_0 \times I_0$, and write

$$\begin{aligned} j_z^s\colon \quad S^2 &\to \quad [-1,1] \times S^2 \\ q &\mapsto \quad (z + g_z^s(q);\, q)\,. \end{aligned}$$

Then we define Ψ_t^s as the inverse of

$$\begin{aligned} \Psi_t^{s,-1}\colon \quad [-1,1] \times S^2 &\to \quad [-1,1] \times S^2 \\ (z;q) &\mapsto \quad (z + t\, g_z^s(q);\, q)\,. \end{aligned}$$

A quick calculation yields that

$$(j_z^s)^*(\beta_z^s + u_z^s\, \mathrm{d}z) = \beta_{z+g_z^s}^s + u_{z+g_z^s}^s\, \mathrm{d}g_z^s\,.$$

Hence, the vector fields Y_z^s that the contact forms $\beta_z^s + \mathrm{d}z$ induce on the deformed spheres are given by

$$\iota_{Y_z^s}\Omega = (j_z^s)^*(\beta_z^s + \mathrm{d}z) = \beta_{z+g_z^s}^s + u_{z+g_z^s}^s\, \mathrm{d}g_z^s = \iota_{X_{z+g_z^s}^s}\Omega + u_{z+g_z^s}^s\, \mathrm{d}g_z^s$$

and we can write Y_z^s as

$$Y_z^s = X_{z+g_z^s}^s + Z_z^s,$$

where the vector fields Z_z^s are defined by $\iota_{Z_z^s}\Omega = u_{z+g_z^s}^s\, \mathrm{d}g_z^s$. Write the area form Ω in the coordinates x and y on R_z^s as $a_z^s\, \mathrm{d}x \wedge \mathrm{d}y$. Then Z_z^s takes the form

$$Z_z^s(x,y) = \frac{u_{z+g_z^s(x,y)}^s(x,y)}{a_z^s(x,y)} \left(\frac{\partial g_z^s}{\partial y}(x,y)\, \partial_x - \frac{\partial g_z^s}{\partial x}(x,y)\, \partial_y \right).$$

Let $\varphi\colon\ S^k \times [-1,1] \ \to\ [0,1]$ be a smooth bump function that is constant 0 outside $D_0 \times I_0$ and takes the value 1 on $D_1 \times I_1$. Let us make the ansatz

$$g_z^s(x,y) := \varphi(s,z) \cdot g_1(x) \cdot g_2(y)\,.$$

Construction of g_1 and g_2
Pick $g_1 \colon \mathbb{R} \to \mathbb{R}$ as a smooth non-negative bump function that vanishes outside $[-4, 3]$ and is constantly 1 on $[-7/2, 5/2]$.

Define a smooth function $g_2 \colon \mathbb{R} \to \mathbb{R}$ that vanishes outside $[-2, 2]$, satisfies $g_2(0) = 0$ and $g_2'(y) < 0$ for $y \in [-1, 1]$ and $g_2'(0) = -2 \max\bigl((a^s_z/u^s_z)\, X^s_{1,z}\bigr)$ as in Figure 2.3. The maximum is taken over all $(s, z) \in D_0 \times I_0$ and all $(x, 0)$ with $x \in [-1, 3]$.

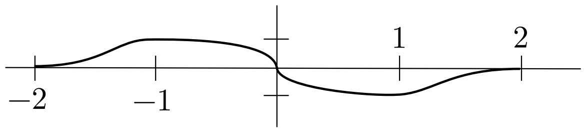


Figure 2.3.: The function g_2

This function might be chosen arbitrarily $\mathcal{C}^0$-close to zero to ensure that the product $\varphi\, g_1\, g_2$ is sufficiently small. In particular, to guarantee that $\Psi_t^{s,-1}$ is an embedding, it suffices to ensure

$$\begin{aligned} 0 &< \frac{\mathrm{d}}{\mathrm{d}z}\Bigl(z + \varphi(s, z)\, g_1(x)\, g_2(y)\Bigr) \\ &= 1 + \frac{\partial \varphi}{\partial z}(s, z)\, g_1(x)\, g_2(y)\,. \end{aligned}$$

This is satisfied, provided that g_2 is sufficiently small.

Verification We ensured that the maps $\Psi_t^{s,-1}$ are embeddings and the deformation hence can be applied as described. Let us observe that we can use this deformation to achieve the required conditions.

(N1) The vector fields X^s_z are parallel to ∂_x along the segment $a = [-1, 3] \times \{0\}$. The ∂_y-component of Z^s_z amounts to

$$\begin{aligned} Z^s_{2,z}(x, y) &= -\frac{u^s_{z+g^s_z(x,y)}}{a^s_z(x, y)}\, \frac{\partial g^s_z}{\partial x}(x, y) \\ &= -\frac{u^s_{z+g^s_z(x,y)}(x, y)}{a^s_z(x, y)}\, \varphi(s, z)\, g_2(y)\, g_1'(x)\,. \end{aligned}$$

For $y = 0$, this vanishes as $g_2(0) = 0$. Consequently, the ∂_y-component of each vector field Y^s_z vanishes along the segment a.

(N2) For $x \in [-4, 5/2]$ and $y > 0$, the ∂_y-component of Z^s_z is non-negative, as u^s_z and a^s_z are positive, $g_1'(x) \geq 0$ and $\varphi(s,z)\, g_2(y) \geq 0$. By property (N2), the ∂_y-component of X^s_z is positive on $[-4,-1] \times [1,2]$ and on $[-1, 5/2] \times (0,2]$, hence the same holds true for the sum $Y^s_z = X^s_{z+g^s_z} + Z^s_z$.

Analogously, the ∂_y-component of Y^s_z is negative on $[-4,-1] \times [-2,-1]$ and on $[-1, 5/2] \times [-2, 0)$.

(N3) The ∂_x-component of Z^s_z amounts to

$$\begin{aligned} Z^s_{1,z}(x,y) &= \frac{u^s_{z+g^s_z(x,y)}}{a^s_z(x,y)} \frac{\partial g^s_z}{\partial y}(x,y) \\ &= \frac{u^s_{z+g^s_z(x,y)}}{a^s_z(x,y)}\, \varphi(s,z)\, g_1(x)\, g_2'(y)\,. \end{aligned}$$

Both u^s_z and a^s_z are positive and $\varphi(s,z)\, g_1(x) \geq 0$. For $y \in [-1,1]$, $g_2'(y) < 0$, and hence $Z^s_{1,z} \leq 0$. As by property (N3) the ∂_x-component $X^s_{1,z}$ is negative on $[-4,-3] \times [-1,1]$ and on $[5/2, 3] \times [-1,1]$, this is also true for the ∂_x-component $Y^s_{1,z}$ of the sum $Y^s_z = X^s_{z+g^s_z} + Z^s_z$.

(N5) In a point $(x,y) \in E$ we have

$$\begin{aligned} \operatorname{div}_{\Omega^s_z}(Y^s_z)\, \Omega^s_z &= \mathrm{d}\Big(\beta^s_{z+g^s_z} + u^s_{z+g^s_z}\, \mathrm{d}g^s_z\Big) \\ &= \mathrm{d}\beta^s_{z+g^s_z} + \mathrm{d}u^s_{z+g^s_z} \wedge \mathrm{d}g^s_z \end{aligned}$$

and since u^s_z are constant on E, $\mathrm{d}u^s_z = 0$, and

$$= \mathrm{d}\beta^s_{z+g^s_z} = \frac{1}{a^s_{z+g^s_z}}\, \operatorname{div}_{\Omega^s_{z+g^s_z}}\Big(X^s_{z+g^s_z}\Big)\, \mathrm{d}x \wedge \mathrm{d}y\,.$$

The divergence of $X^s_{z+g^s_z}$ is positive in $(x,y) \in E$ by property (N5) and the functions a^s_z are positive. An analogous argument shows that the divergence of Y^s_z is positive on H.

(N4) Property (N2) implies that there are no singular points for $x \in [-1, 5/2]$ and $y \neq 0$ as well as on $[-4,-1]\times[1,2]$ and $[-4,-1]\times[-2,-1]$. Property (N3) excludes singular points on $[5/2, 3] \times [-1, 1]$ and on $[-4, -3] \times [-1, 1]$.

Consider the rectangle $[5/2, 3] \times [1, 2]$. There, $X^s_{2,z} > 0$ and $Z^s_{2,z} < 0$ as $g_2 < 0$ and $g_1' < 0$. However, choosing g_2 sufficiently small, we can ensure that $Y^s_{2,z} = X^s_{2,z} + Z^s_{2,z}$ is still positive. Likewise, we can choose g_2 such that $Y^s_{2,z} < 0$ on $[5/2, 3] \times [-2, -1]$.

We verified that the divergence of Y^s_z is positive on E. Hence any singular point in E is positive.

Consider

$$\begin{aligned}\left(\Psi_1^{s,-1}\right)^*&(\beta^s_z + u^s_z\,\mathrm{d}z)\\ &= \left(\beta^s_{z+g^s_z} + u^s_{z+g^s_z}\,\mathrm{d}g^s_z\right) + u^s_{z+g^s_z}\left(1 + \frac{\partial\varphi}{\partial z}(s,z)\,g_1\,g_2\right)\mathrm{d}z\end{aligned}$$

and write these contact forms as $\mu^s_z + h^s_z\,\mathrm{d}z$. The condition that $\Phi_t^{s,-1}$ are embeddings implies $1 + \partial_z\varphi(s,z)\,g_1 g_2 > 0$ and hence $h^s_z > 0$. We showed that Y^s_z is parallel to ∂_x along $a = [-1,3] \times \{0\}$, hence μ^s_z is a multiple of $\mathrm{d}y$ there, and so is its derivative $\dot\mu^s_z$ in ∂_z-direction. Consequently, in singular points $(x,0)$ with $x > -1$ the form $\mu^s_z \wedge \dot\mu^s_z$ vanishes and the contact condition for $\mu^s_z + h^s_z\,\mathrm{d}z$ implies

$$\begin{aligned}0 &< -\mu^s_z \wedge \dot\mu^s_z + \mu^s_z \wedge \mathrm{d}h^s_z + h^s_z\,\mathrm{d}\mu^s_z\\ &= h^s_z\,\mathrm{d}\mu^s_z\end{aligned}$$

as $\mu^s_z = 0$ in singular points of Y^s_z. Then $h^s_z > 0$ implies $\mathrm{d}\mu^s_z > 0$ and the singular point $(x,0)$ is positive.

No singular points in R^s_z for $(s,z) \in D_1 \times I_1$: There are no singular points outside E and $[-1, 5/2] \times \{0\}$. Let us first consider the set $[-1, 5/2] \times \{0\}$. There, the ∂_x-components $Y^s_{1,z}$ of the deformed vector fields Y^s_z satisfy

$$Y^s_{1,z} = X^s_{1,z+g^s_z(x,0)} + \frac{u^s_{z+g^s_z(x,0)}}{a^s_z(x,0)}\,\frac{\partial g^s_z}{\partial y}(x,0)\,.$$

For $y = 0$, the function g_2 vanishes and hence $z + g_z^s = z$. The function g_1 is identically 1 for $x \in [-1, 5/2]$. Consequently, $Y_{1,z}^s$ amounts to

$$Y_{1,z}^s = X_{1,z}^s + \frac{u_z^s(x,0)}{a_z^s(x,0)} \varphi(s,z)\, g_2'(0)\,.$$

For $(s,z) \in D_1 \times I_1$ we have $\varphi(s,z) = 1$ and as

$$g_2'(0) = -2 \max\Big((a_z^s/u_z^s) X_{1,z}^s\Big)\,,$$

the ∂_x-component of Y_z^s is given by

$$Y_{1,z}^s = X_{1,z}^s - 2\frac{u_z^s(x,0)}{a_z^s(x,0)} \max\Big((a_z^s/u_z^s) X_{1,z}^s\Big) < 0\,.$$

On the subsets $[-4,-1] \times [-2,-1]$ and $[-4,-1] \times [1,2]$ of E there are no singular points. To make sure that there will be no singular points in their complement, i.e. on $[-4,-1] \times [-1,1]$, even though $g_2'(y)$ might be quite small we can make the effect of the deformation on the vector fields stronger by enlarging u_z^s on E: On E, the divergence of the vector fields X_z^s is positive, as are the functions u_z^s. We can apply Corollary 2.4 to a small neighbourhood of E to make u_z^s arbitrarily large on E. Denote the changed functions by U_z^s. This deformation does not change the vector fields X_z^s and U_z^s are constant on E and $U_z^s = u_z^s$ on H, so it does not change properties (N1) to (N5) of the vector fields Y_z^s. It does also not change the property that there are no singular points for $(s,z) \in D_1 \times I_1$, $x \in [-1,3]$ and $y = 0$ as

$$-2\frac{U_z^s(x,0)}{a_z^s(x,0)} \max\Big((a_z^s/u_z^s) X_{1,z}^s\Big) \leq -2\frac{u_z^s(x,0)}{a_z^s(x,0)} \max\Big((a_z^s/u_z^s) X_{1,z}^s\Big)\,.$$

The value of $g_2'(y) < 0$ on $[-1,1]$ is bounded from above by a negative constant c. Hence, if we choose U_z^s sufficiently large on E then for $(x,y) \in [-7/2,-1] \times [-1,1]$ the ∂_y-component of Y_z^s is negative as the following calculation shows.

$$\begin{aligned} Y_{1,z}^s(x,y) &= X_{1,z+g_z^s(x,y)}^s + Z_z^s \\ &= X_{1,z+g_z^s(x,y)}^s + \frac{U_{z+g_z^s(x,y)}^s}{a_z^s(x,y)} \frac{\partial g_z^s}{\partial y}(x,y) \end{aligned}$$

$$
\begin{aligned}
&= X^s_{1,z+g^s_z(x,y)} + \frac{U^s_{z+g^s_z(x,y)}}{a^s_z(x,y)}\, g_2'(y) \\
&< X^s_{1,z+g^s_z(x,y)} + U^s_{z+g^s_z(x,y)}\, \frac{c}{a^s_z(x,y)} \\
&< 0
\end{aligned}
$$

As there are no singular points of Y^s_z on $[-4, -7/2]$ by property (N3), this implies that Y^s_z does not have singular points in $[-4,3] \times [-2,2]$ for $(s,z) \in D_1 \times I_1$.

(A2) Apply the deformation that we described so far. Verifying property (N4) we saw that after the deformation, the contact forms are given by $\mu^s_z + h^s_z\, \mathrm{d}z$, where h^s_z is given by

$$
h^s_z = u^s_{z+g^s_z}(1 + \partial_z \varphi(s,z)\, g_1\, g_2) > 0\,.
$$

This function is no longer constant on E and H. As we verified that the divergence of Y^s_z is positive on both E and H, we can apply a further deformation as in Corollary 2.4 to guarantee that the contact forms are given by $\mu^s_z + H^s_z\, \mathrm{d}z$ such that H^s_z are constant on E and H. As this deformation does not change the vector fields Y^s_z, it preserves the properties (N1) to (N5) and the fact that the vector fields Y^s_z have no singular points in $[-4,3] \times [-2,2]$ for $(s,z) \in D_1 \times I_1$. □

Let us have a look at the effects of the deformation of Theorem 2.5.

Remark 2.6. Even if we started with elliptic and hyperbolic neighbourhoods that contained isolated hyperbolic points, unless they are non-degenerate we cannot guarantee that we do not create many more singular points inside E or along the x-axis in H. We could apply a second deformation that ensures all singular points are isolated, possibly destroying the property that there are only singular points outside E along the x-axis, there would still many, possible degenerate, singular points. We deal with these intricacies by considering neighbourhoods of singular points instead.

Observation 2.7. There is some small $\varepsilon > 0$ such that for $\varphi(s,z) < \varepsilon$ there are no singular points in $[-4,3] \times [-2,2]$ outside E and $\{y = 0\} \cap H$.

In particular, the elliptic and hyperbolic neighbourhoods that we used are still elliptic and hyperbolic neighbourhoods.

For any value of z, the rectangle $[-4,3]\times[-2,2]$ is, after rounding two corners and arranging $\operatorname{div}>0$, *cf.* Lemma 2.3, a half-hyperbolic neighbourhood. Enlarging it along the flow lines near E makes sure that it contains the rectangle $[-4,3]\times[-2,2]$.

Remark 2.8. Conversely, a half-hyperbolic neighbourhood can be used for an elimination deformation.

Let D_z^s, $(s,z)\in U$, be a positive half-hyperbolic neighbourhood. Restricting to a rectangle inside the half-disc that we may parametrise with $[-4,4]\times[-2,2]$, we obtain a neighbourhood R_z^s that satisfies the conditions (N1) to (N4). In addition, its the divergence of the vector fields X_z^s is positive on R_z^s. Hence, there is a deformation of the contact forms $\beta_z^s+u_z^s\,\mathrm{d}z$ in a neighbourhood of R_z^s as in Corollary 2.4, such that u_z^s is positive and constant on each R_z^s. In particular, this neighbourhood satisfies condition (A2) and we can apply Theorem 2.5.

We will need to perform the elimination on overlapping sets and use a partial elimination as follows.

Observation 2.9. If for some $(s,z)\in D_0\times I_0$ the vector fields X_z^s on R_z^s satisfy the conditions (N2), (N3) and (N4) only on $\{-1,3\}\times[-2,2]$, then we can still perform the deformation, but the vector field X_z^s after the deformation will satisfy (N2), (N3) and (N4) again only on $\{-1,3\}\times[-2,2]$ and there will be critical points on $[-1,3]\times[-2,2]$ even if $(s,z)\in D_1\times I_1$.

3. Complexity

Deformations as in Section 2 may reduce the number of retrograde connections of a given vector field X_z^s by *removing* one of the hyperbolic points. A priori, however, we cannot ensure that this deformation does not create new retrograde connections for either X_z^s or a vector field nearby.

Let us introduce a complexity valuation on vector fields and the chosen collection of neighbourhoods that measure how far a given sphere $\{z\}\times S^2$ is from being convex with respect to ξ^s and how far our collection of neighbourhoods is from recognising it. We will present an iterative way to deform the contact structures ξ^s and to modify the collections of neighbourhoods that decreases the complexity until all spheres are convex with respect to all contact structures.

The complexity will depend on the choice of parametric neighbourhoods $N^s_{z,i}$, $i = 1, \dots, K$, defined on open sets $U_i \subset S^k \times [-1, 1]$, that cover the singular points of all X^s_z, $(s, z) \in S^k \times [-1, 1]$.

Definition 3.1. *A finite number of parametric neighbourhoods N_i, $i = 1, \dots, K$, is* **simple** *if whenever a singular point q of a vector field X^s_z is contained in both N_i and N_j, all singular points of N_i are contained in N_j or all singular points in N_j are contained in N_i.*

Let us assume for now that our collection of neighbourhoods N_i is simple.

To an isolated singular point q of a vector field X we might assign the maximal number of singular points that vector fields close to X have that are close to q. For non-degenerate singular points, this will be 1 as these vary smoothly with the parameter, for birth-death singularities this might be 2, for example. We want to associate a similar count to the neighbourhoods that contain the singular points.

Consider a neighbourhood $N^s_{z,i}$, $(s, z) \in U_i$ and associate to it the length of the longest string of neighbourhoods $N_{i_1}, \dots, N_{i_m}$ such that

- $N^s_{z,i_j} \cap N^s_{z,i}$ contains singular points for all $j = 1, \dots, m$ and
- for every $j \neq j'$, $N^s_{z,i_j} \cap N^s_{z,i_{j'}}$ does not contain singular points.

Denote this count by $d^s_{z,i}$. It is at most equal to the number K of all neighbourhoods and at least 1 for neighbourhoods that do contain singular points. For (half-hyperbolic) neighbourhoods that do not contain singular points, this count is 0.

To any given parameter $(s, z) \in S^k \times [-1, 1]$ let us assign the sum of all indices of its neighbourhoods of singular points. A singular point might be contained in multiple neighbourhoods $N^s_{z,i}$, so let us take

$$\begin{array}{rccc} \check{d}\colon & S^k \times [-1, 1] & \to & \mathbb{N} \\ & (s, z) & \mapsto & \min\limits_J \sum\limits_{j \in J} d^s_{z,j}\,, \end{array}$$

where $J \subset \{1, 2, \dots, K\}$ are the indices of neighbourhoods such that all singular points of X^s_z are contained in $\bigcup_{j \in J} N^s_{z,j}$.

As the neighbourhoods N_i are defined on open sets U_i in the parameter space $S^k \times [-1,1]$, the maps $d_j\colon (s,z) \mapsto \check{d}^s_{z,j}$ are not upper semi-continuous and neither is the map $\check{d}$. Instead, let us consider $d \coloneqq \limsup \check{d}$, which is.

Definition 3.2. *The map* $d\colon\ S^k \times [-1,1] \ \to\ \mathbb{N}_0$ *will be called* **complexity** *of the vector fields* X^s_z *together with a chosen set of simple neighbourhoods* $N^s_{z,i}$.

Example 3.3. Consider a family X_t, $t \in (-1,1)$, of vector fields on an open disc B such that for $t < 0$ there are two non-degenerate singular points, an elliptic e_t and one hyperbolic h_t, for $t = 0$ there is a single half-hyperbolic singular point q and for $t > 0$ there are no singular points. This phenomenon occurs generically in 1-parametric families of vector fields, *cf.* [Sot74].

Let there be, for $t \in (-\varepsilon, \varepsilon)$, a half-hyperbolic neighbourhood N around q that contains all singular points of X_t. Assume that for $t < -\varepsilon/2$ we find an elliptic neighbourhood E around e_t and for t in a slightly larger open set we find a hyperbolic neighbourhood H around h_t, *cf.* Figure 3.1. These three together form a collection of neighbourhoods that contain all singular points of the vector fields X_t.

To the neighbourhood E we associate for any $t < -\varepsilon/2$ the order 1 as there is exactly one neighbourhood, E, that contains singular points of E. The same is true for the hyperbolic neighbourhood H. For $t \in \left[-\varepsilon/2, 0\right)$ the only neighbourhood that contains all singular points of N is N itself, so again its order is 1. For $t \in \left(-\varepsilon, -\varepsilon/2\right)$, the longest string of neighbourhoods that contain the two singular points e_t and h_t of N consists of both E and H. Consequently, the order of N is 2. For $t > 0$ the longest string of neighbourhoods that contain the singular points of N is empty as there are no singular points, and N has order 0.

In total the sum of the orders $\check{d}$ is 2 on $t \in \left(-1, -\varepsilon/2\right)$, 1 on $\left[-\varepsilon/2, 0\right]$ and 0 otherwise. The complexity d hence evaluates to

$$t \mapsto \begin{cases} 2, & t \in \left(-1, -\varepsilon/2\right] \\ 1, & t \in \left(-\varepsilon/2, 0\right] \\ 0, & t \in (0,1)\,. \end{cases}$$

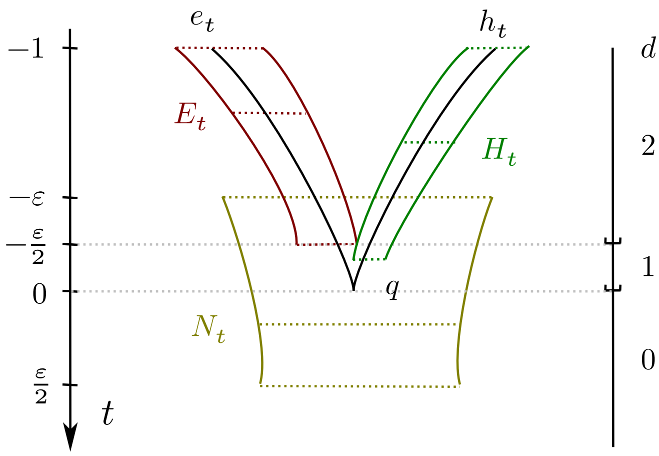


Figure 3.1.: Neighbourhoods of singular points and their order d for a 1-parametric family of vector fields. The horizontal direction depicts the surface the vector fields are defined on.

Notice that unlike the order of degeneracy, that we could also define in this case, the surface of maximal order is not X_0 but only close to it. The reason is that, just looking at the neighbourhoods instead of singular points, we do not distinguish the situations on X_t for $t \in (-\varepsilon/2, 0)$.

4. Graph of singular points

We saw that we can apply an elimination deformation to two neighbourhoods of singular points that are connected by a separatrix of the hyperbolic neighbourhood. We will see that we can indeed find such suitable pairs.

Definition 4.1. *Let X_z^s be a vector field and N_i a collection of neighbourhoods that are defined in (s, z) and that are such that each singular point of X_z^s is contained in exactly one N_i. To the pair define a* **graph** G_z^s *whose vertices are all neighbourhoods N_i that contain singular points and whose edges are the (finitely many) unstable separatrices of positive (half-)hyperbolic neighbourhoods and the stable separatrices of negative (half-)hyperbolic neighbourhoods.*

We will show that this graph does not contain cycles by contradiction: If there is a graph $G_{z_0}^{s_0}$ that contains a cycle, then the contact structure ξ^{s_0} is isotopic to an overtwisted contact structure.

4.1. Closed Legendrian curves

Assume that there is a cycle $\Delta \subset G_{z_0}^{s_0}$. As a first step, let us observe that we can find an isotopy of ξ^{s_0} such that there is a closed Legendrian curve on $\{z_0\} \times S^2$ that agrees with Δ away from some singular points. This argument is based on arguments in [Gei08, Section 4.6.4] that shows that starting from to a hyperbolic cycle, *cf.* Section II.2.4, we find a closed trajectory after perturbation of the contact structure. As Δ is a more general cycle we will construct a closed Legendrian curve that will meet singular points of the characteristic foliation of $\{z_0\} \times S^2$.

Definition 4.2. *A smooth curve in a contact manifold that is in every point tangent to the contact structure is a* **Legendrian curve***.*

Observation 4.3. Let $\Sigma \subset (M, \xi)$ be a surface in a contact manifold and let X be a vector field that directs its characteristic foliation. A smooth curve $c\colon [0,1] \to \Sigma \subset M$ is a Legendrian curve precisely if it is **parallel** to a vector field X, i.e. in all points $c(t)$ that are not singular points of X, $\dot{c}(t)$ is a multiple of $X_{c(t)}$.

The following statement appears in the proof of [Theorem 4.6.33 Gei08] with a slightly different proof.

Lemma 4.4. *Let q be a non-degenerate hyperbolic point of the vector field X and let γ_0, γ_1 be two trajectories of X that lie in two different separatrices of q. Let U be any open neighbourhood U of q and $g_0 \in \gamma_0 \setminus U$ and $g_1 \in \gamma_1 \setminus U$ two points on the two trajectories. Then there is a vector field Y that vanishes outside U and is $\mathcal{C}^\infty$-close to the zero section as well as a smooth curve c defined on $[0,1]$ that is parallel to $X + Y$ and satisfies $c(0) = g_0$ and $c(1) = g_1$.*

Proof. If the trajectories γ_0 and γ_1 lie in both stable or both unstable separatrices, take Y to be the zero vector field. To construct c, follow the trajectory γ_0 from g_0, use a chart of the stable or unstable manifold around q and then follow γ_1 until g_1. Finally reparameterise to $[0,1]$.

Let us hence assume that γ_0 lies in a stable separatrix and γ_1 in an unstable one. Pick two disjoint cross section C_0 and C_1 through points $p_0 \in \gamma_0$ and $p_1 \in \gamma_1$ such that both are contained in U and are parameterised by $(-\delta, \delta)$ such that γ_i intersects C_i in 0, $i = 0, 1$. By the Grobman–Hartman theorem, *cf.* Theorem II.2.14, we can choose C_0 so small that the Poincaré-map P from C_0 to C_1 is defined on one connected component of $(-\delta, 0) \cup (0, \delta)$. Let us assume that P is defined on $(0, \delta)$ and that $P((0, \delta)) \subset (0, \delta) \subset C_1$.

Pick a small flow box $V_0 \cong (-\delta, \delta) \times [0, \delta_y]$ with coordinates x and y around γ_0 starting from C_0 such that $C_0 \equiv (-\delta, \delta) \times \{0\}$ and $\{0\} \times [0, \delta_y] \subset \gamma_0$. Let ϕ_0 be a bump function, supported inside V_0 such that ϕ_0 is constantly 1 in a neighbourhood of $(0, \delta_y/2)$. Define the vector field X_0 to be $\phi_0 \, \partial_x$ in the coordinates of V_0 and identically 0 outside V_0.

Likewise, pick another flow box $V_1 \cong (-\delta, \delta) \times [-\delta_y, 0]$ around γ_1 such that $(-\delta, \delta) \times \{0\} \cong C_1$ and $\{0\} \times [-\delta_y, 0] \subset \gamma_1$. This is possible provided δ_y was chosen sufficiently small. Choose again a bump function ϕ_1 that is supported inside V_1 and is constantly 1 on a neighbourhood of $(0, -\delta_y/2)$ and define $X_1 := \phi_1 \, \partial_x$.

Denote $(-\delta, \delta) \times \{\delta_y\} \subset V_0$ by C_0' and $(-\delta, \delta) \times \{-\delta_y\}$ by C_1'. By construction the Poincaré-map with respect to X between C_0 and C_0' is the identity. With respect to the vector field $X + \varepsilon X_0$ for some small $\varepsilon > 0$ the Poincaré-map between C_0 and C_0' sends 0 to some $\tau_0 > 0$. The Poincaré-map between C_0' and C_1' sends τ_0 to some $\tau_1 > 0$. If ε was chosen sufficiently small, there is some $\lambda > 0$ such that the Poincaré-map with respect to $X + \varepsilon X_0 - \lambda X_1$ between C_1' and C_1 sends τ_1 back to 0, *cf.* Figure 4.1.

The vector field $Y := \varepsilon X_0 - \lambda X_1$ vanishes outside U and the trajectory through $0 \in C_0$ passes through $0 \in C_1$. Thus, the points g_0 and g_1 lie on the same trajectory with respect to $X + Y$ and we may take the segment of this trajectory between g_0 and g_1 to be the curve c after a reparameterisation. Choosing ε small yields a small λ and we can find Y arbitrarily $\mathcal{C}^\infty$-close to the zero section. □

Observation 4.5. Pick a bump function φ on $[-1, 1]$ that takes the value 1 on $\{z_0\}$ and vanishes outside a small neighbourhood. The 1-forms $\beta_z^{s_0} + t \cdot \iota_Y \Omega + u_z^{s_0} \, du$ are contact forms for all $t \in [0, 1]$ and

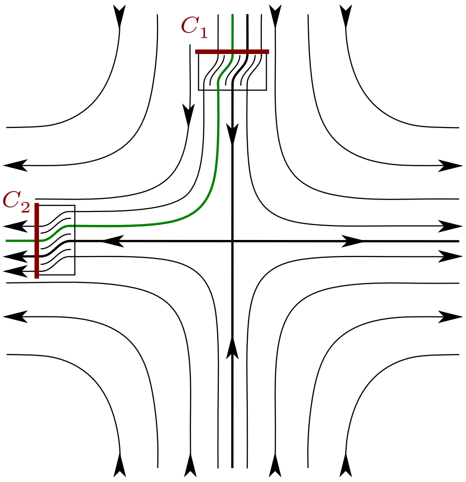


Figure 4.1.: Constructing a smooth Legendrian arc near a hyperbolic singular point

$z \in [-1, 1]$, provided that Y was chosen sufficiently close to zero. Hence the Gray stability theorem, *cf.* Proposition I.5.1, yields a contact isotopy of ξ^{s_0} such that the deformed contact structure induces the vector field $X + Y$ on $\{z_0\} \times S^2$. Consequently, the curve c is a Legendrian for the deformed contact structure.

Lemma 4.6. *Let q be a positive non-degenerate elliptic singular point of X and γ_0 and γ_1 two trajectories of X that emanate at X. Let U be an open neighbourhood of q and $g_0 \in \gamma_0 \setminus U$ and $g_1 \in \gamma_1 \setminus U$ be two points on the two trajectories. Then there is a contact isotopy of ξ^{s_0} such that the deformed contact structure admits a Legendrian curve c with $c(0) = g_0$ and $c(1) = g_1$.*

This is [Lemma 4.6.23 Gei08]. It is proved by finding a closed disc around q with $\operatorname{div}(X) > 0$, constructing a characteristic foliation on it that also has positive divergence and has trajectories that join γ_0 and γ_1, and then applying Gray stability.

Lemma 4.7. *Let N be a positive hyperbolic neighbourhood for X and γ_0 and γ_1 two trajectories of X with α- or ω-limit in N. Let* *further $g_0 \in \gamma_0 \setminus N$ and $g_1 \in \gamma_1 \setminus N$ be two points on the two trajectories. Then there is a contact isotopy of ξ^{s_0} such that the deformed contact structure admits a Legendrian curve c with $c(0) = g_0$ and $c(1) = g_1$.*

Proof. Let us first consider the case that the α- or ω-limits $q_0 = (x_0, 0)$ and $q_1 = (x_1, 0)$ in N of the trajectories γ_0 and γ_1, respectively, are different points.

Then for $\bar{x} := (x_0 + x_1)/2$ the point $(\bar{x}, 0) \in [-1, 1] \times [-1, 1] \cong N$ lies between q_0 and q_1. Pick a bump function $\varphi(x, y)$ that is supported inside N and takes the value 1 on the segment a between q_0 and q_1. Define $f\colon [-1, 1] \times [-1, 1] \to \mathbb{R}$ by $(x, y) \mapsto \varepsilon\varphi(x, y)(\bar{x} - x)$ for some $\varepsilon > 0$ By Sard's Theorem there is an arbitrary small $\varepsilon > 0$ such that for $X + f\partial_x$ all singular points on the segment between q_0 and q_1 are non-degenerate. The vector field $X + f\partial_x$ also has no singular points outside $\{y = 0\}$. For $\varepsilon > 0$ sufficiently small we find a contact isotopy of ξ^s such that the perturbed contact structure induces the vector field $X + f\partial_x$ on $\{z_0\} \times S^2$.

The trajectories through g_0 and g_1 will tend to non-degenerate singular points q_0' and q_1' in the segment a. The set $\{y = 0\}$ is parallel to $X + f\partial_x$ and hence a Legendrian curve. Applying Lemma 4.4 and Lemma 4.6 to q_0' and q_1' and the segment of $\{y = 0\}$ between them we find further perturbations of ξ^s such that there is a single Legendrian curve that runs from g_0 close to q_0', along $\{y = 0\}$, close to q_1', and to g_1.

In case $q_0 = (\bar{x}, 0) = q_1$ apply the same deformations, taking care to pick $\varepsilon > 0$ sufficiently small such that q_0' and q_1', which will not be the same point, are still contained in N. □

Lemma 4.8. *Let N be a positive elliptic neighbourhood for X and γ_0 and γ_1 two trajectories of X that emanate at points in X. Let further $g_0 \in \gamma_0 \setminus N$ and $g_1 \in \gamma_1 \setminus N$ be two points on the two trajectories. Then there is a contact isotopy of ξ^{s_0} such that the deformed contact structure admits a Legendrian curve c with $c(0) = g_0$ and $c(1) = g_1$.*

Proof. There are perturbations of X that are arbitrarily $\mathcal{C}^\infty$-small and supported inside N such that all singular points in N are non-degenerate. They still are all positive.

The graph G of singular points inside N, i.e. the set comprising the singular points and the unstable separatrices of the hyperbolic singular points, is connected: We find a closed set S^+ that contains G such that ∂S^+ is transverse to X, *cf.* Lemma II.3.5. All trajectories through points in ∂S^+ pass through ∂N in positive time and all trajectories through points in ∂N emanate at singular points in N and hence pass through ∂S^+. The Poincaré map from ∂S^+ to ∂N is hence defined everywhere and surjective, hence ∂N being connected implies that ∂S^+ is connected. Consequently, the graph G was connected.

Pick a path in G between the points $\alpha(\gamma_0) =: q_0$ and $\alpha(\gamma_1)$. Iteratively applying Lemma 4.4 and Lemma 4.6 gives us a Legendrian curve c between g_0 and g_1. □

Lemma 4.9. *Let N be a positive half-hyperbolic neighbourhood for X and γ_0 and γ_1 two trajectories of X with α- or ω-limit in N. Let further $g_0 \in \gamma_0 \setminus N$ and $g_1 \in \gamma_1 \setminus N$ be two points on the two trajectories. Then there is a contact isotopy of ξ^{s_0} such that the deformed contact structure admits a Legendrian curve c with $c(0) = g_0$ and $c(1) = g_1$.*

Proof. Depending on whether $\alpha(\gamma_0)$ and $\alpha(\gamma_1)$ lie in the *hyperbolic* part $\{0,1\} \times [-1,1]$ or in its *elliptic* complement, applying either perturbations of Lemma 4.7, or Lemma 4.8, or both again yields a contact structure ξ^s for which there is a Legendrian c that only contains singular points of N as required. □

The five lemmata imply that if the graph $G^{s_0}_{z_0}$ has a cycle Δ, then there is an isotopy of ξ^{s_0} to a contact structure that admits a closed Legendrian curve. It will in general contain singular points of the characteristic foliation.

4.2. Thurston–Bennequin invariant and singular points

A closed Legendrian curve c in an oriented contact 3-manifold (S^3, ξ) has a tubular neighbourhood that is (orientation preserving) contactomorphic to $\big(S^1 \times \mathbb{R}^2, \ker(\cos\vartheta\,\mathrm{d}x - \sin\vartheta\,\mathrm{d}y)\big)$, *cf.* [Gei08, Example 2.5.10], and is called *standard neighbourhood.* Let us use the coordinate ϑ to parameterise the Legendrian curve c. Let $\Sigma \cong \{z_0\} \times S^2 \subset S^3$ be a surface that contains c.

As described in [Jän12, Section IV.3] consider a tubular neighbourhood of the curve c in Σ that is embedded into the standard neighbourhood of c. For each point of the Legendrian c we obtain an angle $\breve{\phi}\colon [0, 2\pi] \to S^1$ that the contact planes form with the tangent space of Σ with respect to the coordinates of the standard neighbourhood and define a lift $\phi\colon [0, 2\pi] \to \widetilde{S}^1 \cong \mathbb{R}$.[1] Along the curve c a vector field X on Σ that defines the characteristic foliation is a positive multiple of the vector field $-\sin(\phi(\vartheta))\,\partial_\vartheta$ where $\partial_\vartheta = \dot{c}$.

In singular points $c(\vartheta)$ of X, the tangent space of Σ and the contact plane coincide, so they enclose an angle of 0 or π. In these points, the lift ϕ is an integral multiple of π, i.e. $\phi(\vartheta) = k\pi$ for some $k \in \mathbb{Z}$. The parity of k indicates the sign of the singular point, it is even in positive singular points and odd in negative ones. Combining this and the sign of $-\sin(\phi(\vartheta))\,\partial_\vartheta$ we can read off the level sets of $k\pi$ of ϕ from the vector field X along c.

The difference $\phi(2\pi) - \phi(0)$ is an integral multiple of 2π and the multiplicity counts the twisting of the contact framing relative to the surface framing along c. It does not depend on the orientation of c. In a simply connected contact manifold this multiplicity does not depend on the Seifert-surface of c and it is called the **Thurston–Bennequin** invariant $\operatorname{tb}(c)$ of c, *cf.* [Gei08, Section 3.5.1].

For homologically trivial closed Legendrian curves in tight contact manifolds, the Bennequin inequality, *cf.* [Gei08, Section 4.6.5], implies

$$\operatorname{tb}(c) \leq -\chi(\Sigma_c),$$

where $\chi(\Sigma_c)$ is the Euler characteristic of a Seifert surface Σ_c of c.

In our case the contact structure ξ is isotopic to a tight contact structure, hence itself tight, Σ is a sphere and the embedded Legendrian c consequently an unknot and we have a disc as Seifert surface. Consequently, the inequality implies

$$\operatorname{tb}(c) \leq -1\,.$$

If c contains no singular points, then $\operatorname{tb}(c) = 0$, contradicting the inequality. If all singular points of c are of the same sign, then ϕ

[1]This is the negative of the angle considered in [Jän12, Section IV.3].

only ever intersects the same level set $k\pi$ and hence again $2\pi\,\mathrm{tb}(c) = \phi(2\pi) - \phi(0) = 0$. Consequently, c must contain both positive and negative singular points.

As c arose out of a cycle Δ of a graph G^s_z, all its trajectories between positive and negative singular points emerge at negative points and tend to positive ones. Assume that $c(\vartheta_0)$ and $c(\vartheta_1)$ are adjacent singular points on c, $\vartheta_0 < \vartheta_1$, and $c(\vartheta_0)$ is a negative singular point, i.e. $\phi(\vartheta) = (2k+1)\pi$. Then for any $\vartheta \in (\vartheta_0, \vartheta_1)$ the vector field X in $c(\vartheta)$ points in the direction of $\dot{c}(\vartheta)$ and hence $-\sin(\phi(\vartheta))$ should be positive, i.e. $\phi(\vartheta) > (2k+1)\pi$. Thus $\phi(\vartheta_1) = (2k+2)\pi$. In an analogous fashion, we see that if $c(\vartheta_0)$ was positive and $c(\vartheta_1)$ negative, then again $\phi(\vartheta_1) = \phi(\vartheta_0) + \pi$.

This implies that any closed Legendrian curve c that arose from a cycle Δ has $\mathrm{tb}(c) > 0$, which is impossible.

Summing up, this shows the following statement.

Theorem 4.10. *Every graph G^s_z does not contain cycles, it is hence a forest.*

We will see that leaves of this graphs comprise neighbourhoods that can be used in an elimination deformation.

5. Strategy for elimination

Let us now observe how to find a sequence of elimination deformations that in the end leaves us with a family of contact structures with respect to which all spheres $\{z\} \times S^2$ are convex.

We start with a family of vector fields X^s_z on S^2 for which we can assume, *cf.* Section 1, that each has only finitely many singular points. Choose a finite number of parametric elliptic and (half-)hyperbolic neighbourhoods N_i, $i = 1, \dots, K$, on S^2, each defined on an open set $U_i \subset S^k \times [-1, 1]$ that each contain at most one singular point of each X^s_z and together cover all singular points of the vector fields X^s_z, *cf.* Section II.4.2.

This collection is simple, *cf.* Definition 3.1, so let $d\colon\ S^k \times [-1, 1] \to \mathbb{N}_0$ be the complexity of the vector fields X^s_z together with this collection of neighbourhoods that we defined in Section 3. It is upper semi-continuous and hence attains its maximum d on the compact parameter space and its pre-image $\mathcal{K}$ is closed in $S^k \times [-1, 1]$ and thus compact.

Lemma 5.1. *If $d = 2$, then every vector field X_z^s belongs to a convex surface.*

Proof. Every vector field X_z^s on S^2 has at least two singular points. As there are no closed trajectories, there must be a positive and a negative singular point and these are contained in different neighbourhoods. Consequently, there are exactly two neighbourhoods, one positive and one negative. If one of these was (half-)hyperbolic, one of its separatrices has to connect back to the same neighbourhood, creating a cycle, but a cycle cannot exist.

Hence for each vector field there are exactly two elliptic neighbourhoods that contain all singular points. In particular, there are no retrograde connections and $\{z\} \times S^2$ is convex with respect to ξ^s. □

Let us assume $d > 2$.

5.1. Step 1: Find possible deformations

Consider a parameter $(s_0, z_0) \in \mathcal{K}$. There is a collection of neighbourhoods N_j, $j \in J \subset \{1, \dots, K\}$, such that each singular point of $X_{z_0}^{s_0}$ is contained in exactly one N_j. To the collection associate the graph $G_{z_0}^{s_0}$.

Choose a collection $J \subset \{1, \dots, K\}$ with the fewest neighbourhoods possible, i.e. with $|J|$ minimal. As the collection is simple, this is equivalent to choosing the largest neighbourhoods, i.e. those that contain the most singular points of $X_{z_0}^{s_0}$.

Let us first consider the case that the graph $G_{z_0}^{s_0}$ has edges. As it is a forest, there are leaves, i.e. vertices of order 1.

A leaf cannot be a hyperbolic neighbourhood, as each of these has two edges. It can be a half-hyperbolic neighbourhood. It can also be an elliptic neighbourhood that is connected to a (half-)hyperbolic neighbourhood via a separatrix.

Let us consider these cases separately. As before, we will restrict ourselves to the case of positive neighbourhoods, the negative ones can be treated analogously with signs reversed.

Elliptic connected to a hyperbolic Let E be a positive elliptic neighbourhood that is a leaf and is connected to a hyperbolic neighbourhood H whose other separatrix is not connected to another elliptic point that is a leaf. We only consider stable separatrices of positive neighbourhoods, so H is a positive neighbourhood.

Denote the open set on which E is defined by U_E and the domain of H by U_H. There is a neighbourhood $U \subset U_E \cap U_H$ of (s_0, z_0) such that both E and H are defined on U and are connected by a separatrix γ of H. By Section 2 there is a smaller open neighbourhood, let us call it U again, such that we can apply an elimination deformation inside any product $D_0 \times I_0 \subset U$ where D_0 is the closure of an open neighbourhood of s_0 and I_0 the closure of an open neighbourhood of z_0. It eliminates all singular points of X_z^s inside a neighbourhood R of the singular points of $E \cup H$ for all (s, z) that are contained in the closures $D_1 \times I_1$ of a smaller open disc around s_0 and a smaller open interval around z_0.

After the deformation, the neighbourhoods E and H will not be elliptic and hyperbolic neighbourhoods, respectively, anymore. Hence we construct a modified collection of neighbourhoods for the deformed vector fields.

Lemma 5.2. *We can modify our collection of neighbourhoods to obtain a simple collection of neighbourhoods for the vector fields X_z^s after the elimination.*

Proof. Let us first consider those neighbourhoods N_i that are no longer elliptic or (half-)hyperbolic neighbourhoods for the vector fields X_z^s after the elimination. These are the neighbourhoods E and H, but also all other neighbourhoods that contain singular points of $E \cup H$, but not all of them.

Case 1: In the most basic case there are, for all (s, z) in the interior $U_I := \operatorname{int}(D_0 \times I_0)$, no other neighbourhoods that contain singular points of E_z^s and H_z^s. We can find a half-hyperbolic neighbourhood N that is defined for each (s, z) in the interior $\operatorname{int}(D_0 \times I_0) =: U_I$ and that contains the rectangle R in which the elimination takes place, *cf.* Observation 2.7. As there are no singular points in N for $(s, z) \in D_1 \times I_1$, we may restrict the domain U_I of N to $U_N := U_I \setminus (D_1 \times I_1)$.

There is a neighbourhood U_∂ of the boundary of $D_0 \times I_0$ inside $D_0 \times I_0$ such that the sets E_z^s and H_z^s are still elliptic and hyperbolic neighbourhoods, respectively, for all $(s, z) \in U_\partial$ after the elimination, again Observation 2.7. Restrict the open sets U_E and U_H on which the neighbourhoods E and H, respectively, are defined by removing from these sets the closure of an open ball B_I that contains $D_1 \times I_1$ and is such that $\partial \overline{B}_I \subset U_\partial$. Finally add the half-hyperbolic neighbourhood N to the collection N_i of neighbourhoods.

This is a collection of neighbourhoods for the vector fields X_z^s after the elimination and it is simple if the collection was simple before.

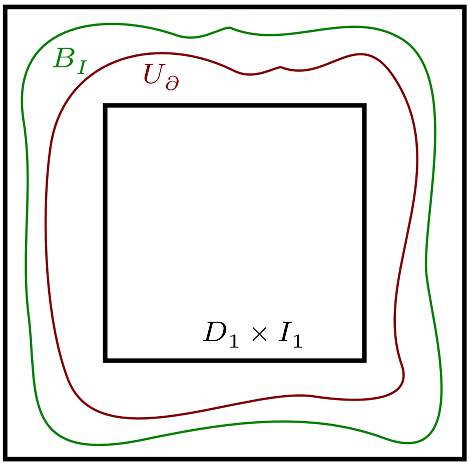


Figure 5.1.: The relevant sets in the parameter space inside $D_0 \times I_0$ to an elimination deformation

Case 2: Consider the case that there is an additional neighbourhood N_i, defined on U_i, that contains a strict subset of singular points of either E_z^s or H_z^s for all $(s, z) \in U_I \cap U_i$. It suffices to restrict its domain. We do not have to add an additional neighbourhood as all its singular points after the elimination are already contained in E, H or N.

Case 3: The next case to consider is that there is a neighbourhood N_i, defined on U_i that contains all singular points of E_z^s and H_z^s for all $(s, z) \in U_I \cap U_i$. Notice that N_i also contains the separatrix γ for these (s, z). Otherwise γ would be a separatrix of N_i and form a cycle by connecting back to N_i, which cannot happen, *cf.* Section 4. Hence, after eventually enlarging it along the flow lines of X_z^s we may assume that N_i contains the rectangle R. Consequently, N_i is also an elliptic or (half-)hyperbolic neighbourhood after the elimination without further changes.

Notice that there can be no neighbourhoods as before that contain only some singular points of E_z^s and some singular points of H_z^s as the collection was assumed to be simple.

Case 4a: Assume that there is a neighbourhood N_i, defined on U_i, that contains for every $(s, z) \in U_I \cap U_i$ all singular points of E_z^s, not those of H_z^s, and also other singular points.

We will see later, in Lemma 5.7, the we ensure that there is no (half-)hyperbolic neighbourhood N_i that contains all singular points E but not those of H.

Hence N_i is an elliptic neighbourhood. Restricting its domain may result in some singular points not being contained in any neighbourhood. If we add the half-hyperbolic neighbourhood N, then our collection is no longer simple. Instead, we may construct a new neighbourhood by joining N_i and H_z^s along a neighbourhood of the separatrix to γ as follows.

Consider, within U_H, an open set $U_{N\gamma}$ for which the trajectory γ emanates at singular points that are contained in N_i. (This is the case for all U_I and we may assume $U_I \subset U_{N\gamma}$.) As we find collared parametric embeddings to the neighbourhoods, we may construct a new neighbourhood N_i' by adding to N_i the neighbourhood H together with a neighbourhood of the separatrix γ by merging along the collars. It is a half-hyperbolic neighbourhood defined on $U_i \cap U_{N\gamma}$. Add this neighbourhood to our collection and restrict the domain of N_i by removing $D_0 \times I_0$, the closure of U_I. We found sufficient neighbourhoods for all $(s, z) \in U_I \cap U_i$. As its complement, $U_I \setminus U_i$, may be non-empty, also add the neighbourhood N. This results in a collection of neighbourhoods for the vector fields X_z^s after the elimination that is again simple.

Case 4b: Assume that there is a neighbourhood N_j, defined on U_j, that contains for every $(s, z) \in U_I \cap U_j$ all singular points of H_z^s, not those of E_z^s, and also other singular points. The neighbourhood N_j is hyperbolic or half-hyperbolic. Then, analogue to the previous case, we construct a new neighbourhood N'_j. It will be a half-hyperbolic neighbourhood if N_j was hyperbolic and an elliptic neighbourhood if N_j was half-hyperbolic.

Case 5: Assume there are an elliptic neighbourhood N_i and a (half-)hyperbolic neighbourhood N_j, defined on U_i and U_j, such that N_i contains the singular points of E, N_j contains the singular points of H, and both of them also contain other singular points.

There is an open set $U_{N\gamma}$ on which γ tends to singular points that are contained in N_j. Likewise, there is an open set $U_{\gamma N}$ on which γ is a separatrix of the (half-)hyperbolic neighbourhood N_j. As before, construct new neighbourhoods N'_i and N'_j.

The make sure our collection stays simple, we construct a third new neighbourhood where γ connects N_i and N_j. There is an open set $U_{N\gamma N}$ for which γ connects N_i and N_j. Similar to the previous case, we construct a new neighbourhood N'_{ij} by merging N_i and N_j along a neighbourhood of γ, add it to the collection and restrict the domains of both N_i and N_j. We can define the neighbourhood N'_{ij} on the open set $U_{N\gamma N} \cap U_I$.

Restrict the neighbourhoods N_i and N_j by removing $D_0 \times I_0$ from their domains of definition as before and add the neighbourhoods N'_i, N'_j, and N'_{ij} to the collection. Then this is a simple collection for the vector fields X^s_z after the elimination.

Case 6: Assume that there are two elliptic neighbourhoods N_{i_1} and N_{i_2} that contain the singular points of E and two (half-)hyperbolic neighbourhoods $N_{j,1}$ and $N_{j,2}$ that contain the singular points of H. Follow the construction from the previous case and construct new neighbourhoods $N'_{i,1}$, $N'_{i,2}$, $N'_{j,1}$, and $N'_{j,2}$ as well as all possible ways to merge these, N'_{i_1,j_1}, N'_{i_1,j_2}, ..., and add them to the collection, restricting the domains of the old neighbourhoods as before.

Case 7: Let there be a neighbourhood N_i, defined on U_i, that, for different $(s, z) \in U_I \cap U_i$, belongs to different cases. As the boundary of N_i is a parametric cross section and both E^s_z and H^s_z necessarily contain singular points, the subsets of $U_i \cap U_i$ belonging to the different cases are open. Hence we may treat N_i as a finite number of sets, each defined on a subset of U_i, and apply the constructions of the appropriate case.

In the general case, there will be multiple sets U_i that each belong to one of the cases mentioned above and we may iterate over them, applying the constructions described above. □

Lemma 5.3. *The elimination homotopy does not increase the complexity d anywhere. Additionally, for any $(s,z) \in B_I \cap \mathcal{K}$ such that in the collection of neighbourhoods before the elimination there is no neighbourhood N_i, defined in (s,z), that contains singular points of both E_z^s and H_z^s, the elimination homotopy decreases the complexity d.*

Proof. Denote the complexity valuation d and the map $\check{d}$ before the deformation by d^{old} and $\check{d}^{old}$, respectively.

For (s,z) outside $D_0 \times I_0$ or in the boundary of $D_0 \times I_0$ we not change the vector fields X_z^s nor the collection of neighbourhoods, so the complexity remains.

Let us first consider the case that (s,z) is contained in the interior of $(D_0 \times I_0) \setminus B_I$. The neighbourhoods E and H are defined in (s,z) and we did not change the collection of neighbourhoods except adding N. Consequently, the minimal sum of the orders d_z^s over all neighbourhoods that cover all singular points did not increase and $\check{d}(s,z) \leq \check{d}^{old}(s,z)$. In fact, the order d_z^s of N after the elimination is at least the sum of those of E and H, so $\check{d}(s,z) = \check{d}^{old}(s,z)$.

Consider the case $(s,z) \in B_I$. After the elimination there are no neighbourhoods N_i that contain a strict subset of the singular points in the half-hyperbolic neighbourhood N. Consequently, the order d_z^s of N after the elimination is 1 where N is defined, i.e. outside $D_1 \times I_1$. Let J^0 be a collection of neighbourhoods before the elimination such that $\sum_{j \in J^0} d_{z,j}^s$ is minimal and such that $\bigcup_{j \in J^0} N_j$ contain all singular points of X_z^s.

If the singular points of E_z^s and H_z^s are contained in two different N_{j_0} and N_{j_1}, then the collection J^1 that consists of J^0 with N but without N_{j_0} and N_{j_1} is a collection of the neighbourhoods after the elimination, contains all singular points of X_z^s and has $\sum_{j \in J^1} d_{z,j}^s < \sum_{j \in J^0} d_{z,j}^s$. Hence $\check{d}(s,z) < \check{d}^{old}(s,z)$.

If J^0 contains a single neighbourhood N_{j_0} that contains the points of both E_z^s and H_z^s then this neighbourhood is still defined in (s,z) after the elimination and hence J^0 is also a collection of neighbourhoods after the elimination and $\check{d}(s,z) \leq \sum_{j \in J^0} d_{z,j}^s = \check{d}^{old}(s,z)$.

We have seen that $\check{d} \leq \check{d}^{old}$ everywhere. This implies $d \leq d^{old}$. On the complement of $\mathcal{K}$, the map $\check{d}$ is strictly smaller than on $\mathcal{K}$. Together with the fact that $\check{d} < \check{d}^{old}$ on an open set V intersected with $\mathcal{K}$ this implies $d < d^{old}$ on $V \cap \mathcal{K}$. □

In fact, what we showed is the following.

Corollary 5.4. *The elimination homotopy does not increase the complexity d anywhere. Additionally, the elimination homotopy decreases the complexity for any $(s, z) \in B_I \cap \mathcal{K}$ such that*

- *in the collection of neighbourhoods before the elimination there is no neighbourhood N_i, defined in (s, z), that contains singular points of the half-hyperbolic neighbourhood N_z^s*

if either

1. *there are neighbourhoods N_j, defined in (s, z), that each contains a strict subset of the singular points of N_z^s and that together contain all singular points of N_z^s, or*
2. *if $(s, z) \in D_1 \times I_1$.*

The first condition was satisfied by the neighbourhoods E_z^s and H_z^s.

Lemma 5.5. *There is an open neighbourhood $V \subset S^k \times [-1, 1]$ of (s_0, z_0) such that the elimination homotopy decreases the complexity d for every $(s, z) \in V \cap \mathcal{K}$.*

Proof. Denote the complexity valuation d and the map $\check{d}$ before the deformation by d^{old} and $\check{d}^{old}$, respectively.

We saw in the proof of Lemma 5.3 that in particular for $(s, z) \in D_1 \times I_1$ the value of $\check{d}$ decreases if there is no neighbourhood N_i that is defined in (s, z) and covers all singular points of both E_z^s and H_z^s.

If there are such neighbourhoods $N_{i_1}, \dots, N_{i_m}$ such that $(s_0, z_0) \notin \overline{U}_{i_j}$ for all $j = 1, \dots, m$ then $V' := B_I \setminus \cup_{j=1}^m \overline{U}_{i_j}$ is open and contains (s_0, z_0).

Let us assume $(s_0, z_0) \in \overline{U}_{i_j}$ for some j. By construction of the d_z^s there is an open neighbourhood W of (s_0, z_0) such that the order d_z^s of N_{i_j} is at least 2 since both E_z^s and H_z^s exist on W and they contain strict subsets of singular points. After the elimination, this count decreases in $U_I \cap W$.

Take V to be an open ball in the open set V' from above intersected with all finitely many such neighbourhoods W. □

The restrictions given in terms of neighbourhoods that cover all singular points may seem worse than it actually is. Those parameter sets that are excluded here are those, where we already collected singular points into one neighbourhood with reduced order. Those bigger neighbourhoods will be dealt with *as whole neighbourhoods* in a further step.

We will want to cover all of $\mathcal{K}$ with these open sets. Notice that the fact that E is a leaf for $X_{z_0}^{s_0}$ does not mean it is a leaf for vector fields X_z^s nearby. Problems could arise if for any parameter (s, z) we tried to eliminate the singular points of E along two different separatrices that connect to two different hyperbolic neighbourhoods.

We can prevent this if we chose the neighbourhoods $D_0 \times I_0$ in the parameter space sufficient small. Endow the parameter space $S^k \times [-1, 1]$ with an arbitrary metric. As we constructed the neighbourhoods at the beginning of this section we can choose the domains U_i of the (half-)hyperbolic neighbourhoods N_i sufficiently small to ensure that for each parameter (s_0, z_0) that lies in the closure of U_i no elliptic neighbourhood E that intersects N_i is a leaf with respect to any other separatrix, i.e. one that is not contained in N_i for (s, z) close to (s_0, z_0). We will preserve this property if we choose $D_0 \times I_0$ sufficiently small as follows.

Construction 5.6. The sets U_E and U_H on which the sets E and H are defined is open, as is the set U_γ of parameters such that the trajectory γ of H with respect to X_z^s, $(s, z) \in U_\gamma$, emanates at a point in E. Consequently, there is some $\delta > 0$ such that the open ball $B_\delta(s_0, z_0)$ of radius δ around (s_0, z_0) in $S^k \times [-1, 1]$ is contained in $U_E \cap U_\gamma \cap U_H$.

Denote by N_j all (finitely many) (half-)hyperbolic neighbourhoods that intersect E for some parameters except those that also intersect H. These are defined on open sets U_j and $(s_0, z_0) \notin \overline{U}_j$ for all j. Hence $U := S^k \times [-1, 1] \setminus \bigcup_j \overline{U}_j$ is open and contains (s_0, z_0).

Choose $D_0 \times I_0$ inside $B_{\delta/3}(s_0, z_0) \cap U$.

Half-hyperbolic Let N be a half-hyperbolic neighbourhood that is a leaf for $G_{z_0}^{s_0}$. There is a neighbourhood $U \subset S^k \times [-1, 1]$ of (s_0, z_0) such that N is a half-hyperbolic neighbourhood for all $(s, z) \in U$.

To the neighbourhood N we may, just as in the previous case, apply the elimination deformation in a neighbourhood of (s_0, z_0) with the following changes: We do not need to introduce new neighbourhoods and there is no need to find a radius-$\delta/3$ disc.

Corollary 5.4 implies again that the complexity does not increase. On a subset of $D_1 \times I_1$ we find again, *cf.* Lemma 5.5, an open set V such that the complexity decreases on $V \cap \mathcal{K}$ as in the previous case.

Elliptic connected to a hyperbolic connected to an elliptic Let E_1 be a positive elliptic neighbourhood that is a leaf and is connected to a (positive) hyperbolic neighbourhood H whose other separatrix is connected to an elliptic neighbourhood E_2 that is another leaf.

There is an open ball $B_\delta(s_0, z_0)$ of radius δ around (s_0, z_0) such that this situation persists for all (s, z) in this ball. Let U be an open set such that there are no half-hyperbolic neighbourhoods that contain the singular points of E_1 or E_2 and not those of H, *cf.* Construction 5.6, such that U is also contained in the open ball of radius $\delta/3$ around (s_0, z_0).

Then we find, similar to the construction in Section 2, a disc shaped neighbourhood N that contains all singular points of E_1, H and E_2 that is defined on a neighbourhood U.

We want to *replace* the neighbourhoods E_1, H, and E_2 by N on U. Using a construction analogous to Lemma 5.2 produces a collection of neighbourhoods that is simple and contains N.

Analogous to Lemma 5.3 and Lemma 5.5 we see that this decreases the complexity for those points in U for which there is no neighbourhood N_i that contains all singular points of E_1, H and E_2.

Elliptic connected to half-hyperbolic Let E be a positive elliptic neighbourhood that is a leaf and is connected to a half-hyperbolic neighbourhood H via its trajectory γ.

A neighbourhood of E, γ and H is an elliptic neighbourhood and we can treat this case in the same way as the previous one.

There are no edges If the graph $G_{z_0}^{s_0}$ has no edges, then all neighbourhoods are elliptic. Consider a positive elliptic neighbourhood E^+

that is a disc and a trajectory through a point $q \in \partial E^+$. It tends to a singular point that lies in a negative elliptic neighbourhood E_q^-.

The same is true for the trajectory through any other point $p \in \partial E^+$. As the Poincaré map between ∂E^+ and ∂E^- is defined on open sets and is a local diffeomorphism, all trajectories tend to the same negative elliptic neighbourhood E^-.

As these sets are subsets of S^2, there consequently cannot be more neighbourhoods that are not contained in either E^+ or E^-. Hence $\check{d}(s_0, z_0) = 2$.

There is an open neighbourhood of (s_0, z_0) such that both E^+ and E^- are defined. For a potentially smaller open neighbourhood U there are no singular points of any X_z^s, $(s, z) \in U$, outside E^+ and E^-. Hence $\check{d}(s, z) = 2$ on U and consequently $d(s_0, z_0) = 2$.

5.2. Step 2: Perform deformations

To every point (s, z) of the set $K \subset S^k \times [-1, 1]$ on which d attains its maximum we find, as described in the first step, operations that are modifications of our collection of neighbourhoods and isotopies of the contact structures ξ^s that decrease the complexity on $V_{(s,z)} \cap \mathcal{K}$ for some open set $V_{(s,z)}$ in $S^k \times [-1, 1]$.

The sets $V_{(s,z)}$ cover the compact set $\mathcal{K}$. Pick a finite cover of K, i.e. a finite number of points (s_i, z_i), $i = 1, \dots, n$, such that the $V_{(s_i,z_i)}$ cover K.

To each point (s_i, z_i) we associated a modification. We will perform these one by one in the order given by i.

Let us observe first that Construction 5.6 indeed suffices to guarantee that two overlapping modifications operate along the same separatrices.

Lemma 5.7. *Let E be a leaf of $G_{z_0}^{s_0}$ that is an elliptic neighbourhood and connected to a (half-)hyperbolic neighbourhood H, and $D_0 \times I_0$ the closure of an open neighbourhood of (s_0, z_0) as in Step 1. If for any $(s, z) \in D_0 \times I_0$ there is a (half-)hyperbolic neighbourhood N that contains the singular points of E, then it will also contain the singular points of H.*

Proof. Assume that there exists a (half-)hyperbolic neighbourhood N, defined in $(s, z) \in D_0 \times I_0$, that contains the singular points of E but not those of H.

By Construction 5.6 we chose $D_0 \times I_0$ disjoint from any (half-) hyperbolic neighbourhoods that already existed before performing any modifications of this iteration by Construction 5.6.

We do not create hyperbolic neighbourhoods. Hence there is a modification, around the point $(s_1, z_1) \in \mathcal{K}$, say, that produced the half-hyperbolic neighbourhood N. Consider the vector field X_z^s before any modifications of this step. Denote, with respect to the vector field X_z^s, the separatrix of H that connects to E by γ_0 and the separatrix that is used in the modification to (s_1, z_1) by γ_1.

The elliptic neighbourhood E is a leaf for both (s_0, z_0) and (s_1, z_1). There are radii $\delta_0 > 0$ and $\delta_1 > 0$ such that γ_0 connects to E for all parameters in $B_{\delta_0}(s_0, z_0)$ and such that γ_1 connects to E for all parameters in $B_{\delta_1}(s_1, z_1)$. In particular, the distance of (s_0, z_0) and (s_1, z_1) is larger than $\min\{\delta_0, \delta_1\}$. On the other hand, $(s, z) \in B_{\delta_0/3}(s_0, z_0)$ and that H is defined in (s, z) implies $(s, z) \in B_{\delta_1/3}(s_1, z_1)$ which is a contradiction by the triangle inequality. □

This implies that whenever two modifications overlap, they are modifications along the same separatrix or one modification completely contains the other.

Consider two modifications that overlap both in the parameter space and on S^2. If one modification does not deform the contact structures ξ^s, then we could apply them both without restrictions. So assume that both modifications deform the vector fields X_z^s, i.e. are both elimination deformations on the rectangles R^1 and R^2, say, defined on the closed sets $D_0^1 \times I_0^1$ and $D_0^2 \times I_0^2$, respectively, in the parameter space. Assume that we want to apply the deformation on R^1 before the one on R^2.

Same trajectory Consider the case that these are eliminations to the elliptic and hyperbolic neighbourhoods E^1 and H^1, and E^2 and H^2, respectively, and that neither does E^1 nor H^1 contain all singular points of $E^2 \cap H^2$ nor does E^2 or H^2 contain all singular points of $E^1 \cup H^1$.

Case 1: Assume that on all $(D_0^1 \times I_0^1) \cap (D_0^2 \times I_0^2)$ the neighbourhoods E^1 and E^2 contain the same singular points. Using the flow of the vector fields near the sides of the rectangle R^1 that are cross sections we archive that both $\{-4\} \times [-2, 2]$ and $[-4, 1] \times \{\pm 2\} \subset R^1$ are contained in R^2. Applying the elimination to R^1 will then preserve the properties (N2), (N3), and (N4), *cf.* Section 2, on $[-4, -1] \times [-2, 2] \subset R^2$ needed for the elimination there.

The hyperbolic neighbourhoods are connected to the elliptic ones by the same separatrix. Wherever the two hyperbolic neighbourhoods overlap, their ∂_x-coordinates on $\{y = 0\}$ are multiples of each other and the ∂_y-coordinate is orthogonal with respect to Ω, *cf.* Section II.4.2. Hence we can guarantee in any case, even if the singular points of H^2 are a strict subset of the singular points of H^1, that the elimination on R^1 preserves the conditions (N1), (N2), (N3), and (N4) on $[-1, 3] \times [-2, 2] \subset R^2$.

Case 2: If the singular points of E^1 are a strict subset of those of E^2, then $[-4, -1] \times [-2, 2] \subset R^1$ is contained in R^2 and we can proceed as in the previous case.

Case 3: Consider the case that the singular points of E^2 are a strict subset of those of E^1. Arrange as before that $[-4, -1] \times [-2, 2] \subset R^2$ is contained in R^1. If the singular points in H^2 are not a subset of those in H^1, *cf.* Figure 5.2, then then performing the elimination on R^1 will destroy the properties (N2), (N3), and (N4) of R^2 on $[-4, -1] \times [-2, 2]$. We can still perform the elimination on R^2, *cf.* Observation 2.9. On $(D_1^1 \times I_1^1) \cap (D_0^2 \times I_0^2)$ the part $[-4, -1] \times [-2, 2]$ of R^2 does not contain any singular points after the elimination on R^1 and R^2 is a half-hyperbolic rectangle, *cf.* Figure 5.3. The partial elimination will hence still eliminate all singular points of R^2 on $(D_1^1 \times I_1^1) \cap (D_1^2 \times I_1^2)$.

If the singular points in H^2 are a subset of those in H^1, then we can arrange $R^2 \subset R^1$. The singular points of R^2 are already eliminated by the deformation on R^1. Hence we may restrict the deformation on R^2. On the set U_∂^1 the elimination on R^1 preserves the property that E^1 and H^1 are elliptic and hyperbolic neighbourhoods. On a potentially smaller neighbourhood $U \subset D_0^1 \times I_0^1$ of the boundary of $D_0^1 \times I_0^1$, the sets E^2 and H^2 will still be elliptic and hyperbolic neighbourhoods, respectively, and the trajectory of H^2 will still connect to E^2. Hence restrict the deformation on R^2 to $(D_0^2 \times I_0^2) \setminus U$ by choosing the bump

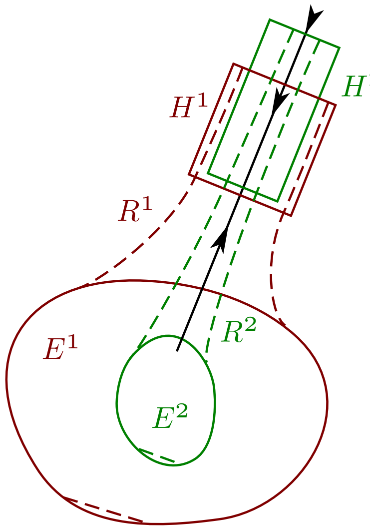


Figure 5.2.: Overlapping deformations on different neighbourhoods along the same trajectory

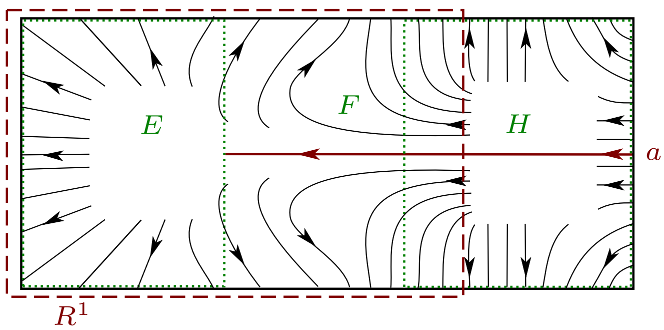


Figure 5.3.: The rectangle R^2 after the first elimination in the situation of Figure 5.2

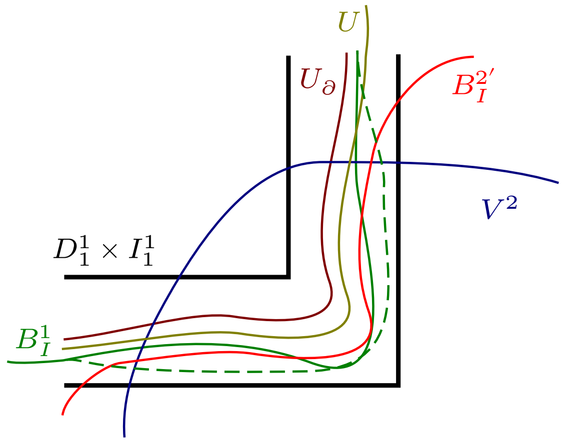


Figure 5.4.: Choosing B_I^1 sufficiently large ensures $V^2 \subset B_I^1 \cup B_I^{2'}$

function $\phi(s, z)$ accordingly. Let us denote all sets with respect to this restricted elimination with the index $2'$. To the restricted domain of elimination $D_0^{2'} \times I_0^{2'}$, we get again a neighbourhood $U_\partial^{2'}$ of its boundary as well as an open ball $B_I^{2'}$ as mentioned in Case 1 of Step 1.

Let us observe that we may perform this restriction such that we still reduce the complexity on $(V^1 \cap V^2) \cap \mathcal{K}$, where V^i should denote the open sets on which the deformation on R^i shall decrease d. Denote the parameter around which we constructed the deformation to R^2 by (s_2, z_2) and call the open sets on which H^1 and E^1 are defined by U_H^1 and U_E^1, respectively. If (s_2, z_2) does not lie in $\partial U_H^1 \cap \partial U_E^2$, then V^2 is disjoint from both U_H^1 and U_E^2 and the restricted elimination on R^2 still decreases the complexity on $V^2 \cap \mathcal{K}$.

So assume that $(s_2, z_2) \in \partial U_H^1 \cap \partial U_E^2$. The restricted elimination on R^2 decreases the complexity on a ball $B_I^{2'}$ that in general does not contain V^2. The elimination on R^1 decreases the complexity on the set B_I^1. The construction of the latter was independent of the deformation of the vector fields, it determines the collection of neighbourhoods after the elimination. We could haven chosen B_I^1 as large as we wanted inside $D_0^1 \times I_0^1$. Hence, enlarging B_I^1 and choosing $B_I^{2'}$ sufficiently large ensures that $B_I^1 \cup B_I^{2'}$ contains the open set V^2, *cf.* Figure 5.4.

Performing both eliminations will hence still decrease complexity on $\left(V^1 \cup V^2\right) \cap \mathcal{K}$.

Larger neighbourhoods Let again the deformation on R^1 and R^2 be constructed from elliptic and hyperbolic neighbourhoods E^1 and H^1, and E^2 and H^2.

Case 4: Assume that E^2 contains singular points of both E^1 and H^1. Then it has to contain the separatrix of H^1 that connects to E^1 as otherwise it would be a cycle in G_z^s. As our collection is simple, the neighbourhood E^2 additionally contains all singular points of both E^1 and H^1. Then R^1 is contained in R^2 and we apply both deformations without changes.

If H^2 contains some singular points of both E^1 and H^1 then it contains them all and likewise contains the separatrix connecting H^1 and E^1. Arrange $R^1 \subset R^2$ and we are able to apply both deformations.

Case 5: Let E^1 or H^1 contain the singular points of both E^2 and H^2. In these cases, restrict the domain of the deformation on R^2 as in Case 3. Notice that the neighbourhood E^1 or H^1 is defined on a neighbourhood of the set that we remove from the domain of the deformation. The elimination to R^2 did not assume to decrease the complexity on this set, so we do not have to do any changes to ensure the complexity decreases as wanted.

Deformations to half-hyperbolic neighbourhoods Consider the case that the deformation to R^1 is an elimination to a half-hyperbolic neighbourhood H^1 and the one to R^2 is again an elimination to the elliptic and hyperbolic neighbourhoods E^2 and H^2, respectively.

By construction, *cf.* Construction 5.6 and Lemma 5.7, the half-hyperbolic neighbourhood H^1 contains singular points of both E^2 and H^2. Consequently, it contains all singular points of E^2 and H^2 and we proceed again as in Case 5.

That this still reduces the complexity on the open set V^2 follows as in Case 3. We used the fact that we decrease d on B_I^1. Let us remark that a priori an elimination to a half-hyperbolic point only decreases complexity inside $D_1 \times I_1$. This does not pose a problem here, as on all of $D_0^2 \times I_0^2$, where we planned to deform on R^2, both neighbourhoods

E and H exist. This satisfies condition 1 of Corollary 5.4 and implies that the deformation to R^1 does decrease d on $B^1_I \cap (D^2_0 \times I^2_0) \cap \mathcal{K}$.

The situation that the deformation to R^2 is constructed from a half-hyperbolic neighbourhood and that to R^1 is to a pair of elliptic and hyperbolic neighbourhoods and the situation that both are constructed from half-hyperbolic points are treated in analogue ways.

General case In general, there may be points in $(D^1_0 \times I^1_0) \cap (D^2_0 \times I^2_0)$ that belong to different cases. Because the boundaries of all neighbourhoods involved are cross sections and hence disjoint from all singular points, all parameters (s, z) that belong to Cases 2–5 form connected components in the complement of those (s, z) that belong to Case 1.

Consequently, the adjustments can be made by iterating over the cases we described. As we do only finitely many deformations in this iteration, only finitely many neighbourhoods of deformations intersect and we can choose appropriate adjustments.

Apply deformations After we made all deformations compatible, we can now construct a new collection of neighbourhoods for the deformed vector fields and then apply all deformations. The complexity d decreases on all of $\mathcal{K}$.

Remark 5.8. Every tree in the forest G^s_z has, if it has edges, at least two leaves. Hence we can exclude one specific leaf from being used in an elimination, or, more generally, from any modification that involves only one leaf.

Remember from Example II.1.3 that the characteristic foliations of all spheres agrees with those with respect to (S^3, ξ_{st}) on the northern hemisphere. These hemispheres contain a single singular point that is a source. Every time this singular point is contained in a leaf, exclude this leaf from the modifications.

Consequently, all deformations of the vector fields X^s_z happen relative to of the contact all happen relative to the hemisphere. The isotopies of the contact structures ξ^s happen outside the hemisphere S_- of S^3 on which they already agree with ξ_{st}.

Remark 5.9. For z in a neighbourhood of $\{-1, 1\}$, all vector fields X^s_z also agreed with the vector fields X^{st}_z that are induced by ξ_{st}. Each of these has exactly one source, one sink and all trajectories emanate at the source and tend to the sink, *cf.* Example II.1.3. In particular, we every collection neighbourhoods of these has a complexity of $d(s,z) = 2$ and these spheres will not be part of any modification. This ensures that the isotopy of contact structures is relative to $S^k \times \{-1, 1\}$.

5.3. Step 3: Iterate

The construction decreases the maximal value of d. If the maximum value of d is still larger than 2 repeat it to further decrease d. As d takes finite values, this terminates after finitely many steps.

If d takes the value 2 on all $(s,z) \in S^k \times [-1, 1]$, all spheres $\{z\} \times S^2$ are convex with respect to all contact structures ξ^s.

6. The dividing curves of the spheres

For any vector field X_z^s that directs the characteristic foliations of the, now convex, sphere $\{z\} \times S^2$ with respect to ξ^s we can find a dividing curve, *cf.* Definition I.8.3 and Section II.3.1. The vector fields X_z^s agree with X_z^{st} on the northern hemisphere and there is a only a single singular point that is positive. Consequently, for each vector field X_z^s we find a dividing curve that is contained in the southern hemisphere.

This allows us to apply the deformations starting from Section I.10 and conclude the proof of Theorem I.2.1.

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Gießen, den 30. Oktober 2018

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