FUNDAMENTAL GROUPS OF SPLIT REAL KAC–MOODY GROUPS AND GENERALIZED REAL FLAG MANIFOLDS

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Abstract. We determine the fundamental groups of symmetrizable algebraically simply connected split real Kac–Moody groups endowed with the Kac–Peterson topology. In analogy to the finite-dimensional situation, because of the Iwasawa decomposition $G = KAU^+$, the embedding $K \hookrightarrow G$ is a weak homotopy equivalence, in particular $\pi_1(G) = \pi_1(K)$. It thus suffices to determine $\pi_1(K)$, which we achieve by investigating the fundamental groups of generalized flag varieties. Our results apply in all cases in which the Bruhat decomposition of the generalized flag variety is a CW decomposition — in particular, we cover the complete symmetrizable situation; furthermore, the results concerning only the structure of $\pi_1(K)$ actually also hold in the nonsymmetrizable two-spherical case.

1. Introduction

The structure of maximal compact subgroups in semisimple Lie groups was investigated by Cartan and, later, Mostow. In [30], Mostow gives a new proof of a Cartan's theorem stating that a connected semisimple Lie group G is a topological product of a maximal compact subgroup K and a Euclidean space, implying in particular that G and K have isomorphic fundamental groups. Subsequent case-by-case analysis provided the isomorphism types of these maximal compact subgroups — which in the split real situation turn out to be all classical — and their fundamental groups. Tables of the

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maximal compact subgroups can be found in [16, p. 518], their fundamental groups in [36, 94.33].

Starting in the 1940's, Dynkin diagrams introduced in [6] have been used to describe the structure of simple Lie groups. In this article, we present a uniform result which makes it possible to determine the fundamental group of any algebraically simply connected split real simple Lie group — and, more generally, any algebraically simply-connected semisimple split real topological Kac–Moody group — directly from its Dynkin diagram.

In [39, Thm. 1], Tits for every generalized Cartan matrix $\mathbf{A}$ provides a functor $G_{\mathbf{A}} : \mathfrak{cRings} \to \mathfrak{Groups}$ from commutative rings into groups. Let Π be the Dynkin diagram of $\mathbf{A}$.

Definition 1.1. We set $G(\Pi) := [G_{\mathbf{A}}(\mathbb{R}), G_{\mathbf{A}}(\mathbb{R})]$ and refer to this group as the *algebraically simply-connected semisimple split real Kac–Moody group of type* Π.

Kac–Moody groups endowed with the Kac–Peterson topology have been studied extensively by the second author together with Glöckner and Hartnick in [10] and with Hartnick and Mars in [14]. Our result is applicable to those Kac–Moody groups whose Bruhat decompositions are CW decompositions and for which the embedding $K \hookrightarrow G$ is a weak homotopy equivalence.

In order to fix notations, let $G = G(\Pi)$ be the algebraically simply-connected split real semisimple Kac–Moody group associated to an irreducible diagram $\Pi = (V, E)$ endowed with the Kac–Peterson topology (for definitions, see Section 2). Let $K = K(\Pi)$ be the so-called maximal compact subgroup of the topological group $G(\Pi)$, i.e., the subgroup fixed by the Cartan–Chevalley involution θ of $G(\Pi)$. We stress that in the infinite-dimensional non-Lie case this topological group K is *not* a compact group, only a k-group, in fact a k_ω-group.

Given the Dynkin diagram $\Pi = (V, E)$ with a fixed labelling $\lambda : \{1, \dots, n\} =: I \to V$, we define a modified diagram Π^{adm} with vertex set V and $\{i^\lambda, j^\lambda\} \in V \times V$ edge if and only if $\varepsilon(i,j) = \varepsilon(j,i) = -1$, where $\varepsilon(i,j)$ denotes the parity of the corresponding Cartan matrix entry. To each connected component $\bar{\Pi}^{\mathrm{adm}}$ of Π^{adm} we then assign a colour as follows: Let $\bar{\Pi}^{\mathrm{adm}}$ be coloured red (denoted by r) if it contains a vertex i^λ such that there exists a vertex $j^\lambda \in V$ satisfying $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$; let $\bar{\Pi}^{\mathrm{adm}}$ be coloured green (g) if it is not red and consists only of an isolated vertex; and blue (b) else.

One can then read off the isomorphism type of $\pi_1(G(\Pi))$ from the coloured diagram Π^{adm} as specified in the following theorem.

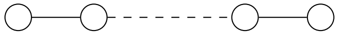
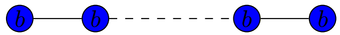
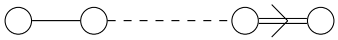
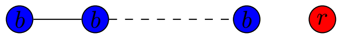
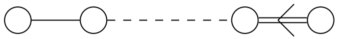
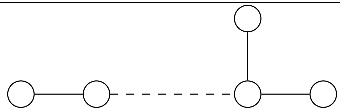
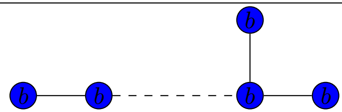
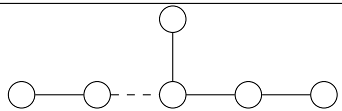
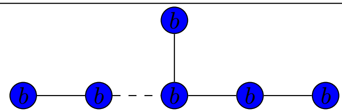
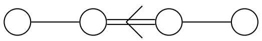
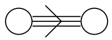
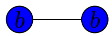
Theorem. *Let* Π *be an irreducible Dynkin diagram such that the Bruhat decomposition of* $G(\Pi)$ *provides a CW decomposition (i.e., such that the conclusion of Proposition* 3.7 *holds) and such that the embedding* $K \hookrightarrow G(\Pi)$ *is a weak homotopy equivalence (i.e., such that the conclusion of Theorem* A.15 *holds). Let* $n(g)$ *and* $n(b)$ *be the number of connected components of* Π^{adm} *of colour* g *and* b*, respectively. Then,*

$$\pi_1(G(\Pi)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b)}.$$

In particular, this statement holds in the symmetrizable case.

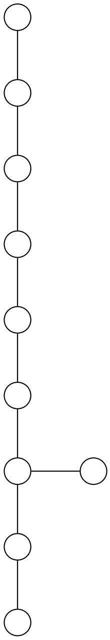
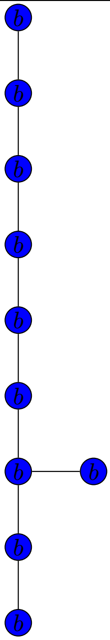
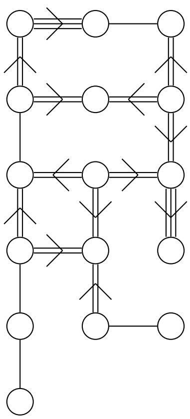
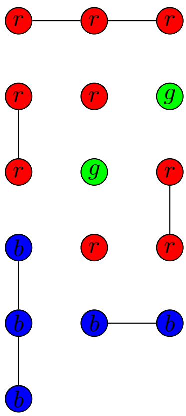
While in the classical finite-dimensional Lie case, one has a topological Iwasawa decomposition $G = K \times A \times U^+$ with A and U^+ contractible, implying $\pi_1(K) \cong \pi_1(G)$, it is currently unknown whether the corresponding Iwasawa decomposition in the general Kac–Moody case is also topological. However, using a fibration result by Palais (see Proposition A.13), in the appendix, Hartnick and the second author prove that the isomorphism between the fundamental groups still exists in the general symmetrizable case, therefore reducing the problem to the computation of $\pi_1(K)$.

Isomorphism types of $\pi_1(G(\Pi))$ for the spherical Dynkin diagrams.[1]

Π	Π^{adm} coloured by γ	$\pi_1(G(\Pi))$
A_1 		$\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$
A_n 		$\pi_1(\mathrm{SL}_{n+1}(\mathbb{R})) \cong C_2 \quad (n \geq 2)$
B_n 		$\pi_1(\mathrm{Spin}(n, n+1)) \cong \begin{cases} \mathbb{Z} & \text{if } n \leq 2, \\ C_2 & \text{if } n > 2. \end{cases}$
C_n 		$\pi_1(\mathrm{Sp}(2n, \mathbb{R})) \cong \mathbb{Z}$
D_n 		$\pi_1(\mathrm{Spin}(n, n)) \cong C_2 \quad (n \geq 3)$
E_n 		$\pi_1(E_n) \cong C_2 \quad (8 \geq n \geq 6)$
F_4 	F_4 	$\pi_1(F_4) \cong C_2$
G_2 		$\pi_1(G_{2,2}) \cong C_2$

[1] Dynkin diagram LaTeX styles kindly provided by Max Horn at [18].

Isomorphism types of $\pi_1(G(\Pi))$ for selected indefinite Dynkin diagrams.[2]

Π		Π^{adm} coloured by γ	$\pi_1(G(\Pi))$
E_{10}			$\pi_1(E_{10}) \cong C_2$
X			$\pi_1(G(X)) \cong \mathbb{Z}^2 \times C_2^2$

In [9, Sect. 16], the group $\mathrm{Spin}(\Pi, \kappa)$ — where κ denotes a so-called *admissible colouring* of the vertices of Π — is defined as the canonical universal enveloping group of a $\mathrm{Spin}(2)$-amalgam $\mathcal{A}(\Pi, \mathrm{Spin}(2)) = \{\widetilde{G}_{ij}, \widetilde{\phi}_{ij}^{i} \mid i \neq j \in I\}$ where the isomorphism type of $\widetilde{G}_{ij}$ depends on the (i, j)- and (j, i)-entries of the Cartan matrix of Π, as well as the values of κ on the corresponding vertices.

It is shown in [9, Sect. 17] that there exists a *finite* central extension $\mathrm{Spin}(\Pi, \kappa) \to$

[2]Dynkin diagram LaTeX styles kindly provided by Max Horn at [18].

$K(\Pi)$ which implies that the subspace topology on $K(\Pi)$ inherited from the Kac–Peterson topology on $G(\Pi)$ defines a unique topology on $\mathrm{Spin}(\Pi,\kappa)$ that turns the central extension into a covering map. The resulting group topology on $\mathrm{Spin}(\Pi,\kappa)$ is called the *Kac–Peterson topology* on $\mathrm{Spin}(\Pi,\kappa)$.

In the simply-laced case, there is a unique nontrivial admissible colouring κ and the corresponding group $\mathrm{Spin}(\Pi) := \mathrm{Spin}(\Pi,\kappa)$ double-covers K as shown in [9]. We prove here that in the simply-laced case $\mathrm{Spin}(\Pi)$ is simply connected which then implies that $\pi_1(K) \cong C_2$.

The strategy of proof in the simply-laced case is to study fibre bundles of the form

$$\mathrm{Spin}(3) \to \mathrm{Spin}(\Pi) \to \mathrm{Spin}(\Pi)/\mathrm{Spin}(3)$$

arising from embeddings of $\mathrm{Spin}(3)$ along subdiagrams of type A_2, which yield exact sequences of the form

$$\{1\} = \pi_1(\mathrm{Spin}(3)) \to \pi_1(\mathrm{Spin}(\Pi)) \to \pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(3))$$

and establishes the equivalence of simple-connectedness of $\mathrm{Spin}(\Pi)$ with the simple-connectedness of $\mathrm{Spin}(\Pi)/\mathrm{Spin}(3)$.

A key to the proof both in the simply-laced and in the general case is the computation of the fundamental groups of *generalized flag varieties* — that is, spaces of the form G/P_J for a parabolic subgroup P_J of G corresponding to an index subset $J \subseteq I$. It turns out that the aforementioned space $\mathrm{Spin}(\Pi)/\mathrm{Spin}(3)$ is a universal covering space of an appropriately chosen generalized flag variety. In general, we prove the following theorem.

Theorem. *Let Π be an irreducible Dynkin diagram such that the Bruhat decomposition of $G(\Pi)$ provides a CW decomposition (i.e., such that the conclusion of Proposition 3.7 holds), let I be the index set of the Dynkin diagram, let $J \subseteq I$, and let P_J be a parabolic of type J. Then a presentation of $\pi_1(G/P_J)$ is given by*

$$\left\langle x_i; \quad i \in I \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i, \quad x_k = 1; \quad i,j \in I, k \in J \right\rangle.$$

In particular, this statement holds in the 2-spherical and in the symmetrizable case.

We refer to [40] for the analog result in the finite-dimensional situation.

In order to determine $\pi_1(K)$ in the general case, we compute subgroups of $\pi_1(K)$ corresponding to the index sets of connected components of Π^{adm} using the above theorem and covering maps of the type $K/K_J \to K/(K \cap T)K_J$ where T is a maximal split torus of $G(\Pi)$ and K_J is the subgroup of fixed points of a Levi factor of P_J with both T and P_J invariant under the Cartan–Chevalley involution. We then show that $\pi_1(K)$ is a direct product of appropriately chosen such subgroups.

In a very similar way, the fundamental group of $\mathrm{Spin}(\Pi,\kappa)$ is determined, establishing the following theorem.

Theorem. *Let Π be an irreducible Dynkin diagram such that the Bruhat decomposition of $G(\Pi)$ provides a CW decomposition (i.e., such that the conclusion of Proposition 3.5 holds). Let $n(g)$ be the number of connected components of Π^{adm} of colour g. Let $n(b,\kappa)$ be the number of connected components of Π^{adm} on which κ takes the value 1 and which have colour b. Then*

$$\pi_1(\mathrm{Spin}(\Pi,\kappa)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b,\kappa)}.$$

In particular, this statement holds in the 2-spherical and in the symmetrizable case.

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2. Split-real Kac–Moody groups

In [21, §1.3], with every generalized Cartan matrix $\mathbf{A} = (a_{ij})_{1\leq i,j\leq n} \in \mathbb{Z}^{n\times n}$ Kac associates a quadruple $(\mathfrak{g}_{\mathbb{C}}(\mathbf{A}), \mathfrak{h}_{\mathbb{C}}(\mathbf{A}), \Psi, \check{\Psi})$ of a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}(\mathbf{A})$, an abelian subalgebra $\mathfrak{h}_{\mathbb{C}}(\mathbf{A})$ and linearly independent finite subsets $\Psi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{h}_{\mathbb{C}}(\mathbf{A})^*$ and $\check{\Psi} = \{\check{\alpha}_1, \ldots, \check{\alpha}_n\} \subseteq \mathfrak{h}_{\mathbb{C}}(\mathbf{A})$ called *simple roots* and *simple coroots*, respectively, such that $\alpha_j(\check{\alpha}_i) = a_{ij}$. Associated with such a quadruple is a Lie algebra generating set $\{e_1, \ldots, e_n, f_1, \ldots, f_n\} \cup \mathfrak{h}_{\mathbb{C}}(\mathbf{A})$. The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}(\mathbf{A})$ is called the *complex Kac–Moody algebra* associated with $\mathbf{A}$, and $\mathfrak{h}_{\mathbb{C}}(\mathbf{A})$ its *standard Cartan subalgebra.*

Since $a_{ij} \in \mathbb{R}$, one can analogously define a quadruple $(\mathfrak{g}_{\mathbb{R}}(\mathbf{A}), \mathfrak{h}_{\mathbb{R}}(\mathbf{A}), \Psi, \check{\Psi})$ where $\mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ is a real Lie algebra that embeds naturally into $\mathfrak{g}_{\mathbb{C}}(\mathbf{A})$ as the real form given by the involution induced by complex conjugation. One refers to $\mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ as the *split real Kac–Moody algebra* associated with $\mathbb{R}$ and to $\mathfrak{h}_{\mathbb{R}}(\mathbf{A})$ as its *standard split Cartan subalgebra.*

Let $Q \subseteq \mathfrak{h}_{\mathbb{R}}(\mathbf{A})^*$ be the group generated by Ψ and $Q_\pm$, the subsemigroups generated by $\pm\Psi$, respectively. For $k \in \{\mathbb{C}, \mathbb{R}\}$ and $\alpha \in \mathfrak{h}_k(\mathbf{A})^*$ define the *root space*

$$\mathfrak{g}^k_\alpha := \{X \in \mathfrak{g}_k(\mathbf{A}) \mid \forall H \in \mathfrak{h}_k(\mathbf{A})^* : [H, X] = \alpha(H)X\}.$$

The *set* Δ *of* $\mathfrak{h}_k(\mathbf{A})$ *roots in* $\mathfrak{g}_k(\mathbf{A})$ is defined as $\Delta := \{\alpha \in Q \setminus \{0\} \mid \mathfrak{g}^k_\alpha \neq \{0\}\}$. One has the *root space decomposition*

$$\mathfrak{g}_k(\mathbf{A}) = \mathfrak{h}_k(\mathbf{A}) \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^k_\alpha.$$

The set Δ decomposes as a disjoint union into the subsets $\Delta_\pm := \Delta \cap Q_\pm$ called *positive* (respectively *negative*) *roots.* The restriction of the Lie bracket on $\mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ to

$$\mathfrak{u}^\pm := \bigoplus_{\alpha\in\Delta_\pm} \mathfrak{g}^k_\alpha$$

turns $\mathfrak{u}^+$ and $\mathfrak{u}^-$ into Lie subalgebras of $\mathfrak{g}_{\mathbb{R}}(\mathbf{A})$.

For $i = \{1, \ldots, n\}$ define the *fundamental root reflection* $\sigma_i \in \mathrm{GL}(\mathfrak{h}_{\mathbb{R}}(\mathbf{A})^*)$ by

$$\sigma_i(\lambda) := \lambda - \lambda(\check{\alpha}_i)\alpha_i.$$

Then the *Weyl group* of $\mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ is defined as $W := \langle\sigma_1, \ldots, \sigma_n\rangle \leq \mathrm{GL}(\mathfrak{h}_{\mathbb{R}}(\mathbf{A})^*)$ and forms a Coxeter system together with the set of fundamental root reflections. Finally, define the *set of real roots* $\Phi := W.\Psi \subseteq \Delta$ and $\Phi^\pm := \Delta_\pm \cap \Phi$, the *positive* (respectively *negative*) *real roots.*

The construction in [39] of $G_{\mathbf{A}}(\mathbb{R})$ (see Definition 1.1) provides a representation of $G_{\mathbf{A}}(\mathbb{R})$ on $\mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ by Lie algebra automorphisms, which is denoted by

$$\mathrm{Ad} : G_{\mathbf{A}}(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}(\mathbf{A})),$$

and referred to as the *adjoint representation* of $G_{\mathbf{A}}(\mathbb{R})$. Since the subgroup $\mathrm{Ad}(G(\Pi))$ of $G(\Pi)$ under this representation preserves the commutator subalgebra $\mathfrak{g}'_{\mathbb{R}}(\mathbf{A})$, one obtains an adjoint representation

$$\mathrm{Ad} : G(\Pi) \to \mathrm{Aut}(\mathfrak{g}'_{\mathbb{R}}(\mathbf{A}))$$

for $G(\Pi)$. The kernels of the adjoint representations of $G_{\mathbf{A}}(\mathbb{R})$ and $G(\Pi)$ are given by the respective centres.

An element $X \in \mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ is ad-*locally-finite* if for every element $Y \in \mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ there exists an $\mathrm{ad}(X)$-invariant finite-dimensional subspace W with $Y \in W$. As pointed out in [27, p. 64], this implies that $\mathrm{ad}(X)|_W$ is a (finite) matrix in some basis of W, so the exponential $\exp(\mathrm{ad}(X))$ can be defined in the ususal way. By [39, (KMG5), p. 545] and the uniqueness properties of $G_{\mathbf{A}}(\mathbb{R})$ established in [39, Thm. 1], $\exp(\mathrm{ad}(X)) \in \mathrm{Ad}(G_{\mathbf{A}}(\mathbb{R}))$. Let $F_{\mathfrak{g}_{\mathbb{R}}(\mathbf{A})}$ and $F_{\mathfrak{g}'_{\mathbb{R}}(\mathbf{A})}$ be the subsets of ad-locally-finite elements of the respective algebras. The maps $\exp : F_{\mathfrak{g}_{\mathbb{R}}(\mathbf{A})} \to \mathrm{Ad}(G_{\mathbf{A}}(\mathbb{R}))$ and $\exp : F_{\mathfrak{g}'_{\mathbb{R}}(\mathbf{A})} \to \mathrm{Ad}(G(\Pi))$ given by $X \mapsto \exp(\mathrm{ad}(X))$ can be lifted to exponential functions $\exp : F_{\mathfrak{g}_{\mathbb{R}}(\mathbf{A})} \to G_{\mathbf{A}}(\mathbb{R})$ and $\exp : F_{\mathfrak{g}'_{\mathbb{R}}(\mathbf{A})} \to G(\Pi)$.

For $X \in \mathfrak{h}_{\mathbb{R}}(\mathbf{A}) \subseteq F_{\mathfrak{g}_{\mathbb{R}}(\mathbf{A})}$, one has

$$\begin{aligned} \mathrm{ad}(X)(e_i) &= [X, e_i] = \alpha_i(X)e_i, & \mathrm{Ad}(\exp(X))(e_i) &= e^{\alpha_i(X)} \cdot e_i, \\ \mathrm{ad}(X)(f_i) &= [X, f_i] = -\alpha_i(X)f_i, & \mathrm{Ad}(\exp(X))(e_i) &= e^{-\alpha_i(X)} \cdot f_i, \end{aligned} \tag{1}$$

cf. [25, Sect. 6.1.6], [39, (KMG5), p. 545].

The same constructions apply also to $\mathbb{C}$ instead of $\mathbb{R}$. Since $\mathfrak{h}_{\mathbb{C}}(\mathbf{A}) \subseteq F_{\mathfrak{g}_{\mathbb{C}}(\mathbf{A})}$, one can define $T_{\mathbb{C}} := \exp(\mathfrak{h}_{\mathbb{C}}(\mathbf{A}))$. Note that $\exp(\mathfrak{h}_{\mathbb{R}}(\mathbf{A})) =: A_{\mathbb{R}} \subsetneq T_{\mathbb{R}} := T_{\mathbb{C}} \cap G_{\mathbf{A}}(\mathbb{R})$. There is a unique Lie group topology on $T_{\mathbb{R}}$ in which $T_{\mathbb{R}} \cong (\mathbb{R}^{\times})^n$ and $A_{\mathbb{R}} = T_{\mathbb{R}}^{\circ} \cong (\mathbb{R}_{>0})^n$. The centre of $G_{\mathbf{A}}(\mathbb{R})$ is contained in $T_{\mathbb{R}}$.

The intersection $T := G(\Pi) \cap T_{\mathbb{C}}$ is called the *standard split maximal torus* of $G(\Pi)$; again, $A_{\mathbb{R}} \cap T$ is of finite index in T and T contains the centre of $G(\Pi)$.

The Lie algebra $\mathfrak{g}_{\mathbb{R}}(\mathbf{A})$ admits a unique involution θ which maps e_j to f_j for all $j = 1, \ldots, r$ and acts as -1 on $\mathfrak{h}_{\mathbb{R}}(\mathbf{A})$. There exists a unique involutive automorphism $\theta : G_{\mathbf{A}}(\mathbb{R}) \to G_{\mathbf{A}}(\mathbb{R})$ such that $\theta(\exp(X)) = \exp(\theta(X))$ for all $X \in F_{\mathfrak{g}_{\mathbb{R}}(\mathbf{A})}$, and this involutive automorphism is called the *Cartan–Chevalley involution* of $G_{\mathbf{A}}(\mathbb{R})$. We denote by $K_{\mathbf{A}}(\mathbb{R}) := G_{\mathbf{A}}(\mathbb{R})^{\theta} \subset G_{\mathbf{A}}(\mathbb{R})$ the fixed point subgroup of this involution and define $K(\Pi) := K_{\mathbf{A}}(\mathbb{R}) \cap G(\Pi)$.

Let $\alpha \in \Phi$ be a real root. Then $\mathfrak{g}_{\alpha}^{\mathbb{R}}$ is one-dimensional and consists of ad-locally-finite elements. One can therefore define the *root group* $U_{\alpha} := \exp(\mathfrak{g}_{\alpha}^{\mathbb{R}}) \subseteq G_{\mathbf{A}}(\mathbb{R})$. Each root group U_{α} carries a unique Lie group topology such that $U_{\alpha} \cong \mathbb{R}$ as topological groups. Root groups corresponding to positive real roots are called *positive root groups*, root groups corresponding to negative real roots are called *negative root groups*.

Define the *positive* (respectively *negative*) *maximal unipotent subgroup* $U^{\pm}$ of $G_{\mathbf{A}}(\mathbb{R})$ as the group generated respectively by the positive and negative root groups. One has $U^{\pm} \subseteq G(\Pi)$. The groups $U^{\pm}$ are normalized by $T_{\mathbb{R}}$ and intersect $T_{\mathbb{R}}$ trivially. In particular, they intersect the centres of $G_{\mathbf{A}}(\mathbb{R})$ and $G(\Pi)$ trivially and hence embed into both $\mathrm{Ad}(G_{\mathbf{A}}(\mathbb{R})$ and $\mathrm{Ad}(G(\Pi))$.

If $\alpha \in \Phi^+$, then $-\alpha \in \Phi^-$ and the group $G_{\alpha} := \langle U_{\alpha}, U_{-\alpha} \rangle \leq G(\Pi)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$. The groups G_{α} with $\alpha \in \Phi^+$ are called the *rank*-1 *subgroups* and the groups $G_1 := G_{\alpha_1}, \ldots, G_n := G_{\alpha_n}$ are called the *fundamental rank*-1 *subgroups* of $G(\Pi)$.

One can show that the pair $((U_{\alpha})_{\alpha \in \Phi}, T)$ defines an RGD system for $G(\Pi)$. For details concerning RGD systems, we refer the reader to [1, Chap. 8].

Recall that the generalized Cartan matrix $\mathbf{A}$ is called 2-*spherical*, if $a_{ij}a_{ji} \leq 3$ for all $i \neq j \in I$; in other words, if the orders of the products $\sigma_i\sigma_j$ are always finite. The generalized Cartan matrix $\mathbf{A}$ is *symmetrizable* if it is the product of a symmetric and a diagonal matrix. These notions are also applied to any and all objects that are derived from $\mathbf{A}$ such as the (extended) Weyl group, the Kac–Moody group, their buildings, etc.

Definition and Remark 2.1. The *Kac–Peterson topology* on $G_{\mathbf{A}}(\mathbb{R})$ equals the finest group topology on $G_{\mathbf{A}}(\mathbb{R})$ such that the natural embeddings $(U_{\alpha} \hookrightarrow G_{\mathbf{A}}(\mathbb{R}))_{\alpha \in \Phi}$ and

$T_{\mathbb{R}} \hookrightarrow G_{\mathbf{A}}(\mathbb{R})$ are continuous when $T_{\mathbb{R}}$ and the root groups U_α are endowed with their Lie group topologies.

The Kac–Peterson topology is k_ω by [14, Prop. 7.10] and, in particular, Hausdorff. Moreover, for every $\alpha \in \Phi^+$, it induces the unique connected Lie group topology on G_α and on $T_{\mathbb{R}}$ by [14, Cor. 7.16]

For more details on the Kac–Peterson topology, see [14, Chap. 7].

Notation 2.2. Throughout this paper, let $G := G(\Pi) := [G_{\mathbf{A}}(\mathbb{R}), G_{\mathbf{A}}(\mathbb{R})]$ be the algebraically simply connected semisimple split real Kac–Moody group associated to an irreducible generalized Dynkin diagram $\Pi = (V, E)$ with (bijective) labelling $\lambda : \{1, \ldots, n\} =: I \to V$. Let $K := K(\Pi)$ be the maximal compact subgroup of G, i.e., the subgroup fixed by the Cartan–Chevalley involution θ.

Denote by $B := B_+$ the positive Borel subgroup of the twin BN-pair of G, by T the standard split maximal torus, and by W the Weyl group of G with generating set $S = \{\sigma_i\}_{i \in I}$. For each $\sigma_i \in S$, take $s_i \in G$ to be a fixed representative of order 4 for σ_i. The group $\widetilde{W} := \langle s_i \mid i \in I \rangle \leq G$ is called the *extended Weyl group*. By [5, Cor. 1.7], one has an Iwasawa decomposition $G = KB$.

The groups G and K are always endowed with the subspace topologies induced by the Kac–Peterson topology on $G_{\mathbf{A}}(\mathbb{R})$ and G/B with the quotient topology.

Unless specified more explicitly, the symbol J will always denote an arbitrary subset of the index set I, the symbol Π_J the subdiagram of Π corresponding to J, the symbol G_J the subgroup $G(\Pi_J)$ of G, and the symbols K_J and B_J the intersections $G_J \cap K$ and $G_J \cap B$, respectively. This is consistent with the notation for the fundamental rank one subgroups: one has $G(\Pi_i) = G_i = G_{\alpha_i}$.

Remark 2.3. Due to the structure theory of RGD systems (cf. [1, Chap. 8], most notably the fact that restricting an RGD system to a subdiagram again yields an RGD system), for each fundamental rank one subgroup G_i there exists an (abstract) isomorphism $\gamma_i : \mathrm{SL}(2, \mathbb{R}) \to G_i$ with the following properties. Let $B_{\mathrm{SL}(2,\mathbb{R})}$ be the group of upper triangular matrices in $\mathrm{SL}(2, \mathbb{R})$ and let $U_{\pm\beta}$ denote the canonical root subgroups of $\mathrm{SL}(2, \mathbb{R})$. Then,

- $\gamma_i(U_{\pm\beta}) = U_{\pm\alpha_i}$.
- $\gamma_i(B_{\mathrm{SL}(2,\mathbb{R})}) = B_i$.
- For each $x \in \mathrm{SL}(2, \mathbb{R})$, $\gamma_i((x^t)^{-1}) = (\gamma_i(x))^\theta$, and hence
- $\gamma_i(\mathrm{SO}(2, \mathbb{R})) = K_i$.

By [14, Cor. 7.16], the restriction of the Kac–Peterson topology to any spherical subgroup H of G coincides with its Lie topology. That is, the groups G_i inherit their Lie group topology from the topological Kac–Moody group G. By the classical theory of Lie groups this yields the existence of a diffeomorphism γ_i with the desired properties; in particular, γ_i is an open map.

Definition 2.4. Using the Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$ ([1, Thm. 6.56, Rem. (1)]), let

$$\begin{aligned} \delta : G/B \times G/B \quad &\to \quad W \\ \delta(gB, hB) = w \quad &\Longleftrightarrow \quad g^{-1}h \in BwB \end{aligned}$$

be the Weyl distance function on G/B, and let l_S be the length function that associates to each element the (unique) length of a corresponding reduced expression in S. Let $\leq$

be the strong Bruhat order on W. Recall that for $w_1, w_2 \in W$ one has $w_1 \le w_2$ if there exist reduced expressions $s_{i_1} \cdots s_{i_{l_S(w_1)}}$ of w_1 and $s_{j_1} \cdots s_{j_{l_S(w_2)}}$ such that the former is a (not necessarily consecutive) substring of the latter.

For $w \in W$ and a chamber $gB \in G/B$, define

$$C_w(gB) := \{hB \in G/B \mid \delta(gB, hB) = w\},$$
$$C_{\le w}(gB) := \bigcup_{v \le w} C_v(gB)$$

and

$$C_{<w}(gB) := C_{\le w}(gB) \setminus C_w(gB).$$

In particular, one has $C_w(B) = BwB/B$ and $C_{\le\sigma}(B) = B\langle s\rangle B/B$ for $\sigma \in S$ with representative $s \in \widetilde{W} \subseteq G$ in the extended Weyl group $\widetilde{W}$. A set $C_{\le\sigma}(gB)$ is called a *σ-panel.*

Moreover, for a subset $\{\sigma_i\}_{i\in J} \subseteq S$ with representatives $\{s_i\}_{i\in J} \subseteq \widetilde{W}$ define P_J to be the standard parabolic subgroup corresponding to the index set J: that is, $P_J := B\langle\{s_i\}_{i\in J}\rangle B$.

Throughout this paper, $C_w(gB)$ and $C_{\le w}(gB)$ will always be endowed with the subspace topologies induced by G/B.

Lemma 2.5. *Let $\sigma_i \neq \sigma_j \in S$. Then the following hold:*

(a) *$P_i = G_iB = K_iB$. In particular, $C_{\le\sigma_i}(B) = K_iB/B$.*

(b) *$Bs_is_jB = Bs_iBBs_jB$. In particular, $C_{\le\sigma_i\sigma_j}(B) = K_iBK_jB/B$.*

Proof. Assertions (a) and (b) follow from [1, Rem. 8.51] and [1, Rem. (2) after Thm. 6.56], respectively, and the Iwasawa decomposition $G_i = K_iB_i$. □

3. The fundamental group of the generalized flag variety G/P_J

For a moment, let Π be an irreducible simply-laced diagram distinct from A_1, and let $G = G(\Pi)$ and $K = K(\Pi)$ be as in the preceding section. Moreover, let $\mathrm{Spin}(\Pi)$ be the double cover of $K(\Pi)$ constructed in [9, Lem. 16.18] (see Definition 4.6 below). By construction, any A_2-subdiagram of Π yields an embedding $\mathrm{Spin}(3) \hookrightarrow \mathrm{Spin}(\Pi)$ and, since $\mathrm{Spin}(3)$ inherits the Lie topology from the Kac–Peterson topology on $\mathrm{Spin}(\Pi)$ by [14, Cor. 7.16], one obtains a locally trivial fibre bundle

$$\mathrm{Spin}(3) \to \mathrm{Spin}(\Pi) \to \mathrm{Spin}(\Pi)/\mathrm{Spin}(3)$$

by [32] (see Proposition A.13). It will turn out in Section 4 below that $\mathrm{Spin}(\Pi)/\mathrm{Spin}(3)$ is a universal covering space of the generalized flag variety G/P_J where $J \subset I$ equals the set consisting of the two types involved in the chosen A_2-subdiagram. The fundamental group of $\mathrm{Spin}(\Pi)$ then follows from the homotopy exact sequence

$$\{1\} = \pi_1(\mathrm{Spin}(3)) \to \pi_1(\mathrm{Spin}(\Pi)) \to \pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(3)) = 1.$$

This motivates our interest in the fundamental group and covering theory of *generalized flag varieties G/P_J.*

Throughout this section, let $J \subseteq I$, let W_J be the subgroup of W generated by $\{\sigma_i\}_{i\in J}$, and let $W^J \subseteq W$ be a set of representatives of the cosets in W/W_J that have minimal length in the coset they define.

Lemma 3.1 (Bruhat decomposition). *One has* $G/P_J = \bigsqcup_{w\in W^J} BwP_J/P_J$.

Proof. This follows immediately from [1, Thm. 6.56, Rem. (1)]. □

Lemma 3.2. *Let* G *be a topological group and* $H_1 \leq H_2$ *subgroups of* G *and endow* G/H_i *with the quotient topology. Then the following hold:*

(a) *The projection map* $\pi : G \to G/H_1$ *is continuous and open.*

(b) *The canonical map* $\psi : G/H_1 \to G/H_2$ *is continuous and open.*

Proof. This is a standard exercise for topological groups. □

Definition and Remark 3.3. For $w \in W$, define the following restrictions of the canonical map $\psi : G/B \to G/P_J$:

- $\psi_w : BwB/B \to BwP_J/P_J$,
- $\psi_{\bar{w}} : \bigcup_{x\leq w} BxB/B \to \bigcup_{x\leq w} BxP_J/P_J$.

Since ψ is continuous, the same holds for the two restrictions. The space $\bigcup_{x\leq w} BxB/B$ is compact by [14, Cor. 3.10] and so $\psi_{\bar{w}}$ is a quotient map.

Lemma 3.4. *Let* G *be* 2*-spherical or symmetrizable and let* $w \in W^J$. *Then the canonical map* ψ_w *is a homeomorphism.*

Proof. By Remark 3.3, $\psi_{\bar{w}}$ is a quotient map. One has $\psi_{\bar{w}}^{-1}(BwP_J/P_J) = BwB/B$: Let $x \leq w$ such that $BxP_J/P_J = BwP_J/P_J$. Then $x \in BwP_J = BwW_JB$ where the equality holds since by definition of W^J one has $l(ww') = l(w) + l(w')$ for all $w' \in W_J$ which implies $Bww'B = BwBw'B$. The Bruhat decomposition of G yields $x \in wW_J$ and hence, $l(x) \geq l(w)$. This implies $x = w$.

Now, since BwB/B is open in its closure $\bigcup_{x\leq w} BxB/B$ in G/B (see [14, Prop. 5.9] plus Corollary B.8), the preceding observations yield that ψ_w is an injective quotient map and therefore a homeomorphism. □

Lemma 3.5. *Let* $\sigma_i \in S$. *Then each panel* $C_{\leq\sigma_i}(B)$ *is homeomorphic to the* 1*-sphere* $\mathbb{S}^1$.

Proof. The panel $C_{\leq\sigma_i}(B)$ is a subbuilding of G/B corresponding to the RGD system $\{G_i, U_{\alpha_i}, U_{-\alpha_i}, T \cap G_i\}$. By Remark 2.3, one has $G_i \cong \mathrm{SL}(2,\mathbb{R})$, $T \cap G_i \cong T_{\mathrm{SL}(2,\mathbb{R})}$ and $U_{\pm\alpha_i} \cong U_{\pm\alpha}$ where $T_{\mathrm{SL}(2,\mathbb{R})}$ denotes the subgroup of diagonal matrices and $U_{\pm\alpha}$ denote the canonical root subgroups of $\mathrm{SL}(2,\mathbb{R})$. This implies that $C_{\leq\sigma_i}(B)$ is homeomorphic to the building $\mathrm{SL}(2,\mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})} \simeq \mathbb{P}_1(\mathbb{R}) \simeq \mathbb{S}^1$. □

Definition 3.6. Following [35, Chap. 8], a *CW complex* is a triple (X, E, χ), where X is a Hausdorff space, E is a family of cells in X, and $\chi = \{\chi_e \mid e \in E\}$ is a family of maps, such that

(a) $X = \bigsqcup_{e\in E} E$.

(b) For $k \in \mathbb{N}$, let $X^{(k)} \subseteq X$ be the union of all cells of dimension $\leq k$. Then for each $(k+1)$-cell $e \in E$, the map $\chi_e : (D^{k+1}, S^k) \to (e \cup X^{(k)}, X^{(k)})$, is a *relative homeomorphism*, i.e., it is a continuous map and its restriction $D^{k+1} \setminus S^k \to e$ is a homeomorphism.

(c) If $e \in E$, then its closure $\mathrm{cl}\, e$ is contained in a finite union of cells in E.

(d) X has the weak topology determined by $\{\mathrm{cl}\, e \mid e \in E\}$, i.e., a subset A of X is closed if and only if $A \cap \mathrm{cl}\, e$ is closed in $\mathrm{cl}\, e$ for each $e \in E$.

For $k \in \mathbb{N}$, let Λ_k be an index set for the k-dimensional cells, so that $X^{(k)} \setminus X^{(k-1)} = \bigsqcup_{\lambda\in\Lambda_k} e_\lambda$ and set $\chi_\lambda := \chi_{e_\lambda}$. This map is called the *characteristic map* of e_λ.

Proposition 3.7. *Let G be 2-spherical or symmetrizable. Then for each $w \in W$, the set $C_w(B) = BwB/B$ is a cell of dimension $l(w)$ that is open in its compact closure $C_{\leq w}(B)$ in G/B. For each subset $J \subseteq I$, the Bruhat decomposition $G/P_J = \bigsqcup_{w \in W^J} BwP_J/P_J$ is a CW decomposition.*

Proof. The first statement is immediate by [14, Cor. 3.10 and Prop. 5.9] plus Corollary B.8; see also [24, pp. 170–171]. Furthermore, [14, Prop. 5.9] combined with Corollary B.8 states that the Bruhat decomposition of G/B is a CW decomposition. By Lemma 3.4, G/P_J is composed of cells that are homeomorphic to cells in G/B, so composing the characteristic maps of the latter cells with the canonical map $\psi : G/B \to G/P_J$ yields characteristic maps for the cells in G/P_J.

For the closure-finiteness, let BwP_J/P_J be a cell in G/P_J. Since ψ is continuous and restricts to a homeomorphism $BwB/B \to BwP_J/P_J$, it maps $\operatorname{cl} BwB/B$ surjectively onto $\operatorname{cl} BwP_J/P_J$. Now, $\operatorname{cl} BwB/B = \bigcup_{x \leq w} BxB/B$, which implies that

$$\operatorname{cl} BwP_J/P_J = \bigcup_{x \leq w} BxP_J/P_J = \bigcup_{\substack{x \leq w \\ x \in W^J}} BxP_J/P_J,$$

where the last equality holds since $W_J \subseteq P_J$. This proves that $\operatorname{cl} BwP_J/P_J$ is contained in a finite union of cells.

It remains to show that G/P_J has the weak topology determined by the cell closures.

For $w \in W$ and a representative $\widetilde{w} \in W^J$ of minimal length of wW_J, one has $BwP_J/P_J = B\widetilde{w}P_J/P_J$. Let $e_w := BwP_J/P_J = B\widetilde{w}P_J/P_J$ and $e'_w := BwB/B$. Let $\bar{e}_w = \operatorname{cl} e_w = \bigcup_{x \leq \widetilde{w}} BxP_J/P_J$ and $\bar{e}'_w := \operatorname{cl} e'_w = \bigcup_{x \leq w} BxB/B$.

Let A be a closed subset of G/P_J and let e_w, $w \in W^J$, be an arbitrary cell. Then $\psi^{-1}(A)$ is closed in G/B since ψ is continuous, so $\psi^{-1}(A) \cap \bar{e}'_w$ is closed in $\bar{e}'_w$ since G/B is a CW complex. Now,

$$\psi^{-1}(A) \cap \bar{e}'_w = \psi^{-1}(A) \cap \psi^{-1}(\bar{e}_w) = \psi^{-1}(A \cap \bar{e}_w) = \psi_{\bar{w}}^{-1}(A \cap \bar{e}_w).$$

Since $\psi_{\bar{w}}$ is a quotient map by Remark 3.3, this implies that $A \cap \bar{e}_w$ is closed in $\bar{e}_w$.

Now, let A be a subset of G/P_J such that $A \cap \bar{e}_w$ is closed in $\bar{e}_w$ for all $w \in W^J$. Since for each $w \in W$ one has $e_w = e_{\widetilde{w}}$ for any minimal-length representative $\widetilde{w} \in W^J$ of wW_J, in fact $A \cap \bar{e}_w$ is closed in $\bar{e}_w$ for all $w \in W$. Therefore, $\psi_{\bar{w}}^{-1}(A \cap \bar{e}_w)$ is closed in $\bar{e}'_w$ for all $w \in W$. Since $\psi_{\bar{w}}^{-1}(A \cap \bar{e}_w) = \psi^{-1}(A) \cap \bar{e}'_w$, the fact that G/B is a CW complex implies that $\psi^{-1}(A)$ is closed in G/B. Since ψ is open by Lemma 3.2, it follows that A is closed in G/P_J. This proves that G/P_J is a CW complex. □

The preceding result combined with the following lemma (which is a consequence of [29, Chap. 7, Thm. 2.1]) will allow us to efficiently compute the fundamental group of a generalized flag variety in Theorem 3.15 below.

Lemma 3.8. *Let X be a CW complex with only one 0-cell x_0. For each $\lambda \in \Lambda_2$, let $f_\lambda : [0,1] \to \mathbb{S}^1$ be a loop whose homotopy class generates $\pi_1(\mathbb{S}^1)$ and whose image $\gamma_\lambda := \chi_\lambda \circ f_\lambda$ under χ_λ is a loop in $X^{(1)}$ starting at x_0. Then*

$$\langle [\chi_\mu], \quad \mu \in \Lambda_1 \mid [\gamma_\lambda], \quad \lambda \in \Lambda_2 \rangle$$

is a presentation of $\pi_1(X, x_0)$, where the brackets denote the respective homotopy classes in $X^{(1)}$.

Next, we study the characteristic maps of the CW decomposition of a generalized flag variety explicitly.

Notation 3.9. Define $R : [0,1] \to \mathrm{SO}(2,\mathbb{R}), s \mapsto \begin{pmatrix} \cos(s\pi) & -\sin(s\pi) \\ \sin(s\pi) & \cos(s\pi) \end{pmatrix}$.

Lemma 3.10. *R induces a continuous, surjective map $\widetilde{R} : [0,1] \to \mathrm{SL}(2,\mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})}$ which maps the interior $(0,1)$ homeomorphically onto its image and maps the boundary $\{0,1\}$ surjectively onto its image.*

Proof. Let $\{x_0\} := \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}^\mathsf{T} \right\rangle \in \mathbb{P}^1$ where $\mathbb{P}^1$ denotes the real projective line, modelled as the subset of one-dimensional subspaces of $\mathbb{R}^2$. Since each one-dimensional subspace in $\mathbb{P}^1 \setminus \{x_0\}$ contains exactly one element in the upper half circle $R([0,1]) \cdot \begin{pmatrix} 1 & 0 \end{pmatrix}^\mathsf{T}$ while x_0 contains the two boundary points corresponding to $R(0)$ and $R(1)$, one has a surjection from $[0,1]$ onto $\mathbb{P}^1$ given by $t \mapsto \left\langle R(t) \cdot \begin{pmatrix} 1 & 0 \end{pmatrix}^\mathsf{T} \right\rangle$ which maps $(0,1)$ bijectively onto $\mathbb{P}^1 \setminus \{x_0\}$. Since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on the real projective line $\mathbb{P}^1$ with $B_{\mathrm{SL}(2,\mathbb{R})}$ being the stabilizer of $x_0 := \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}^\mathsf{T} \right\rangle$, one has a bijective correspondence $gB \mapsto gx_0$ between $\mathrm{SL}(2,\mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})}$ and $\mathbb{P}^1$. This yields the desired surjectivity and bijectivity properties of $\widetilde{R}$. Continuity is clear, as well as the fact that the restriction to the interior is a homeomorphism. □

Definition 3.11. Let $D^1 = [0,1]$ be the 1-dimensional unit disc and note that $D^2 \simeq D^1 \times D^1$. For $i,j \in I$, let γ_i, γ_j be as in Remark 2.3. Let $p : G \to G/B$ be the canonical projection. Define $\chi_i : D^1 \to G/B$ and $\chi_{(i,j)} : D^1 \times D^1 \to G/B$ by

- $\chi_i(s) := p(\gamma_i(R(s))) = \gamma_i(R(s)) \cdot B$,
- $\chi_{(i,j)}(s,t) := p(\gamma_i(R(s))\gamma_j(R(t))) = \gamma_i(R(s))\gamma_j(R(t)) \cdot B$.

The following lemma was inspired by [34, Chap. 10, second Prop. of 6.8]; see also [20, §2.6, p. 198].

Lemma 3.12. *Let G be 2-spherical or symmetrizable. Then the maps defined above are characteristic maps for the following cells:*

(a) *χ_i for $C_{\sigma_i}(B) = Bs_iB/B$,*
(b) *$\chi_{(i,j)}$ for $C_{\sigma_i\sigma_j}(B) = Bs_is_jB/B$.*

Proof. (a) One has to show that $\chi_i([0,1]) \subseteq C_{\leq \sigma_i}(B)$ and that χ_i is a continuous map which maps $(0,1)$ homeomorphically to $C_{\sigma_i}(B)$. The first assertion is clear, since by Lemma 2.5 one has $C_{\leq \sigma_i} = G_iB/B$.

By Lemma 2.5, one has $C_{\sigma_i}(B) = \{kB \mid k \in K_i \setminus (K_i \cap B)\}$. Let $k \in K_i \setminus (K_i \cap B)$. Then $\gamma_i^{-1}(p^{-1}(kB)) = \gamma_i^{-1}(k) \cdot B_{\mathrm{SL}(2,\mathbb{R})} \in \mathrm{SL}(2,\mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})} \setminus B_{\mathrm{SL}(2,\mathbb{R})}$. By Lemma 3.10, there exists a unique $s \in (0,1)$ satisfying $R(s)B_{\mathrm{SL}(2,\mathbb{R})} = \gamma_i^{-1}(k)B_{\mathrm{SL}(2,\mathbb{R})}$. Hence, s is the unique preimage of kB under χ_i. This yields the desired bijectivity property. The continuity properties are clear.

(b) Since by Lemma 2.5 (b) one has $C_{\leq \sigma_i\sigma_j}(B) = K_iBK_jB/B$, it is clear that $\chi_{(i,j)}([0,1] \times [0,1]) \subseteq C_{\leq \sigma_i\sigma_j}(B)$. For the injectivity of the restriction, let $(s,t), (\widetilde{s},\widetilde{t}) \in (0,1)^2$ such that $\chi_{(i,j)}(s,t) = \chi_{(i,j)}(\widetilde{s},\widetilde{t})$. Then

$$\begin{aligned} & \gamma_i(R(s))\gamma_j(R(t))B = \gamma_i(R(\widetilde{s}))\gamma_j(R(\widetilde{t}))B, \\ \iff & (\gamma_i(R(\widetilde{s})))^{-1}\gamma_i(R(s))\gamma_j(R(t))B = \gamma_j(R(\widetilde{t}))B \in C_{\sigma_j}(B). \end{aligned}$$

This implies $R(\widetilde{s})^{-1}R(s) \in B_{\mathrm{SO}(2,\mathbb{R})}$, since otherwise the left expression is in $C_{\sigma_i\sigma_j}(B)$, contradicting $C_{\sigma_i\sigma_j}(B) \cap C_{\sigma_j}(B) = \varnothing$. Since $s, \widetilde{s} \in (0,1)$, one obtains $\widetilde{s} = s$. It follows that $\chi_j(t) = \chi_j(\widetilde{t})$, hence $t = \widetilde{t}$ by (a).

For the surjectivity, note that by Lemma 2.5(b), one has $C_{\sigma_i\sigma_j}(B) = Bs_is_jB/B = Bs_iBBs_jB/B$. Let x_ix_jB be an arbitrary element of $C_{\sigma_i\sigma_j}(B)$ with $x_i = b_1s_ib_2 \in Bs_iB$ and $x_j \in Bs_jB$. By (a), there exists an $s \in (0,1)$ with $\gamma_i(R(s))B = b_1s_iB \in C_{\sigma_i}(B)$. Hence, there exists a $b \in B$ with $(\gamma_i(R(s))b = b_1s_ib_2 = x_i$. Again by (a), there exists a $t \in (0,1)$ with $\gamma_j(R(t))B = bx_jB \in C_{\sigma_j}(B)$. This yields

$$\begin{aligned}\chi_{i,j}(s,t) &= \gamma_i(R(s)) \cdot \gamma_j(R(t))B \\ &= x_ib^{-1} \cdot bx_jB \\ &= x_ix_jB.\end{aligned}$$

This proves that $\chi_{i,j}$ maps $(0,1)\times(0,1)$ bijectively to $C_{\sigma_i\sigma_j}(B)$. The continuity properties are clear. $\square$

Notation 3.13. For $i,j \in I$, let $\varepsilon(i,j) := (-1)^{\alpha_j(\check{\alpha}_i)}$, where $\alpha_j(\check{\alpha}_i) = a_{ij}$ is the (i,j)-entry of the Cartan matrix $\mathbf{A}$ of Π.

Lemma 3.14 ([9, Rem. 15.4(1)])**.** *Let $e_i := \gamma_i(-I) \in G_i$ with γ_i as in Remark 2.3 and $k_j \in K_j$. Then $e_ik_je_i = k_j^{\varepsilon(i,j)}$.*

Theorem 3.15. *If the Bruhat decomposition satisfies the conclusion of Proposition 3.7, then a presentation of $\pi_1(G/P_J)$ is given by*

$$\langle x_i;\quad i \in I \mid x_ix_j^{\varepsilon(i,j)} = x_jx_i,\ x_k = 1;\ i,j \in I, k \in J\rangle.$$

In particular, this statement holds in the 2-spherical and the symmetrizable case.

Proof. By Lemma 3.4 and Proposition 3.7, the Bruhat decomposition

$$G/P_J = \bigsqcup_{w\in W^J} BwP_J/P_J$$

is a CW decomposition where each cell BwP_J/P_J has dimension $l(w)$. The characteristic maps of the 1-cells Bs_iP_J/P_J and 2-cells $Bs_is_jP_J/P_J$ are given by the compositions $\widetilde{\chi}_i := \psi_{s_i} \circ \chi_i$, respectively $\widetilde{\chi}_{(i,j)} := \psi_{s_is_j} \circ \chi_{(i,j)}$ (ψ_{s_i} and $\psi_{s_is_j}$ denoting the canonical homeomorphisms from Lemma 3.4).

Lemma 3.8 gives a presentation of $\pi_1(G/P_J)$. The generating elements are given by the homotopy classes $x_i := [\widetilde{\chi}_i]$ of the characteristic maps of the 1-cells — namely, the cells Bs_iP_J/P_J where $i \in I \setminus J$. For the homotopy classes x_k with $k \in J$, note that $\gamma_k(R(t)) \in G_k \subseteq P_J$, and so $\widetilde{\chi}_k(t) = \gamma_k(r(t)) \cdot P_J = P_J$ which implies $x_k = [\widetilde{\chi}_k] = 1_{\pi_1(G/P_J)}$. This yields the desired generating set as well as the trivial relation $x_k = 1$ for $i \in J$.

To obtain the set of relators, for $k = 1, \ldots, 4$ let $\varphi_k : [0,1] \to [0,1] \times [0,1]$ where

$$\begin{aligned}\varphi_1(t) &= (t,0), \\ \varphi_2(t) &= (1,t), \\ \varphi_3(t) &= (1-t,1), \\ \varphi_4(t) &= (0,1-t).\end{aligned}$$

Then the concatenation $\varphi := \varphi_1 * \varphi_2 * \varphi_3 * \varphi_4$ is a loop in the relative boundary $\partial([0,1] \times [0,1]) \simeq \mathbb{S}^1$ which generates its fundamental group. Moreover, for each characteristic map $\widetilde{\chi}_{(i,j)}$ of a 2-cell, one has $\widetilde{\chi}_{(i,j)}(\varphi(0)) = \widetilde{\chi}_{(i,j)}((0,0)) = \psi_{s_is_j}(\chi_{(i,j)}(0,0)) = \psi_{s_is_j}(B) =$

P_J where P_J is the unique 0-cell of the CW complex. Therefore, Lemma 3.8 implies that the set of relators is given by $\{[\widetilde{\chi}_{(i,j)} \circ \varphi] \mid \sigma_i\sigma_j \in W^J, l(\sigma_i\sigma_j) = 2\}$. Now,

$$[\widetilde{\chi}_{(i,j)} \circ \varphi] = [\widetilde{\chi}_{(i,j)} \circ \varphi_1] \cdot [\widetilde{\chi}_{(i,j)} \circ \varphi_2] \cdot [\widetilde{\chi}_{(i,j)} \circ \varphi_3] \cdot [\widetilde{\chi}_{(i,j)} \circ \varphi_4],$$

where $\widetilde{\chi}_{(i,j)}(s,t) = \alpha_i(R(s))\alpha_j(R(t))\cdot P_J$ with $R(0) = I_{\mathrm{SO}(2,\mathbb{R})}, R(1) = -I_{\mathrm{SO}(2,\mathbb{R})} \in B_{\mathrm{SO}(2,\mathbb{R})}$ which implies

$$\begin{aligned}
[\widetilde{\chi}_{(i,j)} \circ \varphi_1] &= x_i, \\
[\widetilde{\chi}_{(i,j)} \circ \varphi_3] &= x_i^{-1}, \\
[\widetilde{\chi}_{(i,j)} \circ \varphi_4] &= x_j^{-1}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\widetilde{\chi}_{(i,j)} \circ \varphi_2)(t) &= \alpha_i(-I)\alpha_j(R(t)) \cdot P_J \\
&= \alpha_i(-I)\alpha_j(R(t))\alpha_i(-I) \cdot P_J, \qquad \text{since } \alpha_i(-I) \in P_J \\
&= \alpha_j(R(t))^{\varepsilon(i,j)} \cdot P_J \qquad \text{by Lemma 3.14.}
\end{aligned}$$

Since $R(t)^{-1} = R(1-t)$, this yields $[\widetilde{\chi}_{(i,j)} \circ \varphi_2] = x_j^{\varepsilon(i,j)}$. One therefore obtains

$$[\widetilde{\chi}_{(i,j)} \circ \varphi] = x_i \cdot x_j^{\varepsilon(i,j)} \cdot x_i^{-1} \cdot x_j^{-1}.$$

This proves the assertion. $\square$

Lemma 3.16. *Let Π be irreducible simply-laced distinct from A_1 and $\varnothing \neq J \subset I = \{1, ..., n\}$. Then $\pi_1(G/P_J) \cong {C_2}^{n-|J|}$.*

Proof. For each generator x_h in the presentation of Theorem 3.15, one has $x_h^2 = 1$. Recall that λ denotes the labelling map $I \to V$ of the vertex set of Π. Since Π is connected, one has a minimal path $(i_1, \ldots, i_m = h)^\lambda$ in Π such that $i_1 \in J$. If $m = 1$, one has $x_h = 1$ by the presentation above. Let $x_{i_1}, \ldots, x_{i_{m-1}}$ have order ≤ 2. Since Π is simply-laced, $\varepsilon(m-1, h) = -1 = \varepsilon(h, m-1)$ which implies $x_h x_{i_{m-1}}^{-1} x_h^{-1} x_{i_{m-1}}^{-1} = 1$ and $x_{i_{m-1}} x_h^{-1} x_{i_{m-1}}^{-1} x_h^{-1} = 1$. Multiplying these expressions yields $x_h^2 = 1$.

Since each generator has order ≤ 2, the relations show that the group is abelian. One concludes that $\pi_1(G/P_J) \cong {C_2}^{n-|J|}$. $\square$

4. The fundamental groups of $G(\Pi)$ and $\mathrm{Spin}(\Pi, \kappa)$

The Iwasawa decomposition $G = KAU_+$ implies that K acts transitively on the generalized flag varieties G/P_J. In this section, we describe the generalized flag varieties and suitable covering spaces as coset spaces of K and its various spin covers defined in [9]. This will then allow us to compute the fundamental group of K and its various spin covers via locally trivial fibre bundles and homotopy exact sequences.

Lemma 4.1. *The canonical map $\psi : K/(K \cap P_J) \to G/P_J$ is a homeomorphism. In particular, there exists a homeomorphism $G/P_J \to K/(K \cap T)K_J$.*

Proof. Bijectivity follows from the product formula for subgroups since $G = KP_J$. By Lemma 3.2, the map $\widetilde{\psi} : G/(K \cap P_J) \to G/P_J$ is continuous, so the same holds for its bijective restriction $\psi : K/(K \cap P_J) \to G/P_J$.

In order to show that ψ is closed, let $P := P_J$ and let $\widetilde{P} := P_J \cap K$. Consider the commutative diagram

$$\begin{tikzcd} K/\widetilde{P} \arrow[d, "\iota"] \arrow[dr, "\psi"] & \\ G/\widetilde{P} \arrow[r, "\varphi"] & G/P \end{tikzcd}$$

where ι denotes the canonical embedding and φ denotes the canonical map from $G/\widetilde{P}$ to G/P. Since K is closed in G by [8, Sect. 3F], the map ι is closed. By Lemma 3.2, φ is open.

Let $X\widetilde{P} \subseteq K/\widetilde{P}$ be a closed subset of $K/\widetilde{P}$ and suppose that $\psi(X\widetilde{P}) = XP$ is not closed in G/P. Then the complement $\mathsf{C}_{G/P}(XP)$ is not open in G/P, hence the complement $\mathsf{C}_{G/\widetilde{P}}(\varphi^{-1}(XP)) = \varphi^{-1}(\mathsf{C}_{G/P}(XP))$ is not open in $G/\widetilde{P}$. Therefore, $\varphi^{-1}(XP)$ is not closed in $G/\widetilde{P}$. This yields that $X\widetilde{P} = \psi^{-1}(XP) = \iota^{-1}(\varphi^{-1}(XP))$ is not closed in $K/\widetilde{P}$, a contradiction.

For the second claim, since $P_J = G_J B$ and $\theta(P_J) \cap P_J = G_J T$, one has $P_J \cap K = K_J(K \cap T)$. Furthermore, G_J is normal in $G_J T$ which implies $K_J(K\cap T) = (K\cap T)K_J$. The claim follows. $\square$

The key advantage of the description of a generalized flag variety as a K-coset space lies in the fact that $K \cap T$ is a finite group. It is therefore straightforward to write down covering spaces of generalized flag varieties via the following well-known basic observation from covering theory.

Lemma 4.2. *Let $\varphi : X \to Y$ be a continuous, open, surjective map between Hausdorff topological spaces. If all fibers are finite and of constant cardinality, then φ is a covering map.*

This readily applies in our setting.

Lemma 4.3. *The canonical map $\psi : K/K_J \to K/(K\cap T)K_J$ is a covering map of degree $2^{n-|J|}$.*

Proof. By Lemma 3.2, ψ is continuous, open and surjective.

By [8, Lem. 3.20 and the discussion after Prop 3.8], the group $\widetilde{T} := (K \cap T)$ has order 2^n. Note that one has $T_J \cap T_{I\setminus J} = \{1\}$, since the Kac–Moody group G being algebraically simply connected implies $T \cong T_J \times T_{I\setminus J}$. Now, for $k \in K$ one has $\psi^{-1}(k\widetilde{T}K_J) = \{ktK_J \mid t \in \widetilde{T}\}$, and since $T_J \cap T_{I\setminus J} = \{1\}$, one has $kt_iK_J \neq kt_jK_J$ for $t_i \neq t_j \in T \cap K_{I\setminus J}$. This yields

$$\begin{aligned} |\psi^{-1}(k\widetilde{T}K_J)| &= |\{ktK_J \mid t \in \widetilde{T}\}| = |\{ktK_J \mid t \in T \cap K_{I\setminus J}\}| \\ &= |T \cap K_{I\setminus J}| = |T_{I\setminus J} \cap K_{I\setminus J}| = 2^{n-|J|}. \end{aligned}$$

Lemma 4.2 now shows that ψ is a covering map. $\square$

Definition 4.4 ([9, Def. 16.2]**).** Let Π^{adm} be the graph on the vertex set V with edge set

$$\{\{i,j\} \in V \times V \mid i \neq j \in I, \varepsilon(i,j) = \varepsilon(j,i) = -1\},$$

where $\varepsilon(i,j)$ denotes the parity of the corresponding Cartan matrix entry, as defined in Notation 3.13.

An *admissible colouring* of Π is a map $\kappa : V \to \{1,2\}$ such that

(a) $\kappa(i^\lambda) = 1$ whenever there exists $j \in I \setminus \{i\}$ with $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$,

(b) the restriction of κ to any connected component of the graph Π^{adm} is a constant map.

Define $c(\Pi,\kappa)$ to be the number of connected components of Π^{adm} on which κ takes the value 2. For a subgraph Π_J^{adm} of Π^{adm} that is a union of connected components of Π^{adm} let κ_J be the corresponding restriction of κ.

Definition 4.5. Let be the colouring $\gamma : V \to \{r, g, b\}$ of Π^{adm} that to each connected component $\bar{\Pi}^{\mathrm{adm}}$ of Π^{adm} assigns a colour as follows. Let $\bar{\Pi}^{\mathrm{adm}}$ be coloured red (denoted by r) if it contains a vertex i^λ such that there exists a vertex $j^\lambda \in V$ satisfying $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$; let $\bar{\Pi}^{\mathrm{adm}}$ be coloured green (g) if it is not red and consists only of an isolated vertex; and blue (b) else.

We refer to the introduction for a discussion of various examples.

Definition and Remark 4.6. As recalled in the introduction, in [9, Def. 16.16] the *spin group* $\mathrm{Spin}(\Pi,\kappa)$ *with respect to* Π *and* κ is defined as the universal enveloping group of a particular $\mathrm{Spin}(2)$-amalgam $\{\widetilde{G}_{ij}, \widetilde{\phi}_{ij}^i \mid i \neq j \in I\}$ where the isomorphism type of $\widetilde{G}_{ij}$ depends on the (i,j)- and (j,i)-entries of the Cartan matrix of Π as well as the values of κ on the corresponding vertices. The group $K(\Pi)$ can be regarded as (being uniquely isomorphic to) the universal enveloping group of an $\mathrm{SO}(2,\mathbb{R})$-amalgam $\{G_{ij}, \phi_{ij}^i \mid i \neq j \in I\}$ where each $\widetilde{G}_{ij}$ covers G_{ij} via an epimorphism α_{ij}. By [9, Lem. 16.18] there exists a canonical central extension $\rho_{\Pi,\kappa} : \mathrm{Spin}(\Pi,\kappa) \to K(\Pi)$ that makes the following diagram commute for all $i \neq j \in I$:

$$\begin{tikzcd}
\widetilde{G}_{ij} \arrow[r, "\widetilde{\tau}_{ij}"] \arrow[d, "\alpha_{ij}"] & \mathrm{Spin}(\Pi,\kappa) \arrow[d, "\rho_{\Pi,\kappa}"] \\
G_{ij} \arrow[r, "\tau_{ij}"] & K(\Pi)
\end{tikzcd} .$$

Here, $\widetilde{\tau}_{ij}$ and τ_{ij} denote the respective canonical maps into the universal enveloping groups.

By [9, Prop. 3.9], one has

$$\ker(\rho_{\Pi,\kappa}) = \langle \widetilde{\tau}_{ij}(\ker(\alpha_{ij})) \mid i \neq j \in I \rangle_{\mathrm{Spin}(\Pi,\kappa)}.$$

Each connected component of Π^{adm} that admits a vertex i^λ with $\kappa(i^\lambda) = 2$ contributes a factor 2 to the order of $\ker(\rho_{\Pi,\kappa})$ so that $\mathrm{Spin}(\Pi,\kappa)$ is a $2^{c(\Pi,\kappa)}$-fold central extension of $K(\Pi)$.

In particular, this implies that the subspace topology on $K(\Pi)$ defines a unique topology on $\mathrm{Spin}(\Pi)$ that turns the extension into a covering map. The resulting group topology on $\mathrm{Spin}(\Pi,\kappa)$ is called the *Kac–Peterson topology* on $\mathrm{Spin}(\Pi,\kappa)$.

In the case of an irreducible simply-laced diagram Π, the only admissible colourings are the (trivial) constant colouring $V \to \{1\}$ (that every diagram admits) and the constant colouring $\kappa : V \to \{2\}$; we define the *spin group* $\mathrm{Spin}(\Pi)$ *with respect to* Π as $\mathrm{Spin}(\Pi) := \mathrm{Spin}(\Pi,\kappa)$.

Before turning to the general case, we will first consider the simply-laced case and formulate and prove the corresponding simplified versions of the main theorems.

Lemma 4.7. *Let* Π *be irreducible simply-laced distinct from* A_1 *and let* $\{i,j\} \subseteq I$ *be the index set of an* A_2*-subdiagram of* Π. *Then the spaces* $\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})$ *and* K/K_{ij} *are homeomorphic.*

Proof. From [9] (exact references below) it follows that the kernel of the covering map $\mathrm{Spin}(\Pi) \to K$ coincides with the kernel of the covering map $\mathrm{Spin}(\Pi_{ij}) \to K_{ij}$ and is equal to the group $Z := \{\pm 1_{\mathrm{Spin}(\Pi)}\}$ (for the definition of $-1_{\mathrm{Spin}(\Pi)}$, see below). This is a consequence of the following facts regarding an irreducible simply-laced diagram Π (all referring to [9]):

- There is an epimorphism $\mathrm{Spin}(2) \to \mathrm{SO}(2,\mathbb{R})$ with kernel $\{\pm 1_{\mathrm{Spin}(2)}\}$ (see [Thm. 6.8]).
- In $\mathrm{Spin}(\Pi)$, all elements $\widetilde{\tau}_{ij}(\widetilde{\phi}^i_{ij}(-1_{\mathrm{Spin}(2)}))$ coincide (see [Lem. 11.7]).
- Let $-1_{\mathrm{Spin}(\Pi)} := \widetilde{\tau}_{ij}(\widetilde{\phi}^i_{ij}(-1_{\mathrm{Spin}(2)}))$ for an arbitrary pair $i \neq j \in I$. Then $1_{\mathrm{Spin}(\Pi)} \neq -1_{\mathrm{Spin}(\Pi)}$ (see [Cor. 11.16]).
- $\mathrm{Spin}(\Pi)$ is a 2-fold central extension of $K(\Pi)$ (see [Thm. 11.17]).

Hence, the 2-fold covering map $\widetilde{\varphi} : \mathrm{Spin}(\Pi) \to K(\Pi)$ induces a continuous bijective map $\varphi : \mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij}) \to (\mathrm{Spin}(\Pi)/Z)/(\mathrm{Spin}(\Pi_{ij})/Z) \to K/K_{ij}$. One has a commutative diagram

$$\begin{tikzcd} \mathrm{Spin}(\Pi) \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi_1"] & K \arrow[d, "\pi_2"] \\ \mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij}) \arrow[r, "\varphi"] & K/K_{ij} \end{tikzcd} .$$

Since $\widetilde{\varphi}$ is open as a covering map and π_2 is open by Lemma 3.2, it follows that φ is a homeomorphism. □

Lemma 4.8. *Let* Π *be irreducible simply-laced distinct from* A_1 *and* $\varnothing \neq J \subset I = \{1, \ldots, n\}$. *Then* K/K_J *is simply connected.*

Proof. K/K_J is connected since K is generated by connected groups isomorphic to $\mathrm{SO}(2,\mathbb{R})$. Hence by Lemma 4.3 it is a nontrivial cover of $K/(K \cap T)K_J$ of degree $2^{n-|J|}$. The claim now follows from Corollary 3.16 and Corollary 4.1. □

The following proposition provides our main result in the simply laced case.

Proposition 4.9. *Let* Π *be irreducible simply-laced distinct from* A_1. *Then* $\mathrm{Spin}(\Pi)$ *is simply connected with respect to the Kac–Peterson topology. In particular,* $\pi_1(G) \cong C_2$.

Proof. By [19, 4.2.4], for a closed subgroup H of a topological group G, the projection $p : G \to G/H$ is a principal H-bundle. By Lemma A.13, this bundle is locally trivial if H is a (closed) Lie group (note that, by [17, Thm. 5.11], every locally compact subgroup of a topological group is closed). Since locally trivial bundles admit local cross sections, [37, Cor. in Sect. 7.4] implies that, if H is a closed Lie group, then $p : G \to G/H$ is a fibre bundle with fibre H. This yields a locally trivial fibre bundle

$$\mathrm{Spin}(\Pi_{ij}) \to \mathrm{Spin}(\Pi) \to \mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij}).$$

By [15, Chap. 4], this yields the homotopy long exact sequence

$$\begin{aligned} \pi_4(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \to \pi_3(\mathrm{Spin}(\Pi_{ij})) \to \pi_3(\mathrm{Spin}(\Pi)) \to \pi_3(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \\ \to \pi_2(\mathrm{Spin}(\Pi_{ij})) \to \pi_2(\mathrm{Spin}(\Pi)) \to \pi_2(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \\ \to \pi_1(\mathrm{Spin}(\Pi_{ij})) \to \pi_1(\mathrm{Spin}(\Pi)) \to \pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \end{aligned} \tag{2}$$

from which one extracts the exact sequence

$$\{1\} = \pi_1(\mathrm{Spin}(\Pi_{ij})) \to \pi_1(\mathrm{Spin}(\Pi)) \to \pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})).$$

By Lemmas 4.7 and 4.8, one has $\pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \cong \pi_1(K/K_{ij}) = \{1\}$ and so by exactness $\pi_1(\mathrm{Spin}(\Pi)) = \{1\}$.

The second assertion follows from the fact that $\pi_1(G) \cong \pi_1(K)$ by Corollary A.15 and the fact that $\mathrm{Spin}(\Pi)$ is a 2-fold central extension of K by [9, Thm. 11.17]. □

We will now return to the case of a general irreducible Dynkin diagram Π.

Notation 4.10. For a subset $J \subseteq I$ let

$$H_J := \langle x_i; \quad i \in J \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i, ; \quad i,j \in J \rangle.$$

Lemma 4.11. *Let $J \subseteq I$ be the index set of a connected component Π_J^{adm} of Π^{adm}. Then the following hold:*

(a) *If Π_J^{adm} has colour r, then $H_J \cong C_2^{|J|}$.*

(b) *If Π_J^{adm} has colour g, then $|J| = 1$ and $H_J \cong \mathbb{Z}$.*

(c) *If Π_J^{adm} has colour b, then $|H_J| = 2^{|J|+1}$.*

Proof. (a) If Π_J^{adm} has colour r, then there exist $i \in J, j \in I \setminus \{i\}$ with $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$. This implies $x_i x_j = x_j x_i$ and $x_j x_i^{-1} = x_i x_j$ which yields $x_i^2 = 1$. Now, if $\{i^\lambda, k^\lambda\}$ is an edge in Π^{adm}, then $x_i x_k^{-1} x_i^{-1} x_k^{-1} = 1 = x_k x_i^{-1} x_k^{-1} x_i^{-1}$. Multiplying these expressions shows that $x_i^2 = 1$ implies $x_k^2 = 1$. Since Π_J^{adm} is connected, this yields $x_k^2 = 1$ for each $k \in J$. Commutativity then follows from the relations of H_J.

(b) By definition, nodes of colour g are isolated in Π^{adm}.

(c) Let Π_J^{sl} be the simply laced Dynkin diagram with vertex set J^λ and edge set $\{\{i,j\} \in J \times J \mid \{i,j\} \text{ edge in } \Pi^{\mathrm{adm}}\}$. Let $\widetilde{T} := K(\Pi_J^{\mathrm{sl}}) \cap T(\Pi_J^{\mathrm{sl}})$ where $T(\Pi_J^{\mathrm{sl}})$ denotes the standard maximal torus of $G(\Pi_J^{\mathrm{sl}})$. Then by Lemma 4.3 and Proposition 4.9, $\mathrm{Spin}(\Pi_J^{\mathrm{sl}}) \to K(\Pi_J^{\mathrm{sl}}) \to K(\Pi_J^{\mathrm{sl}})/\widetilde{T}$ is a universal covering map where $K(\Pi_J^{\mathrm{sl}}) \to K(\Pi_J^{\mathrm{sl}})/\widetilde{T}$ has degree $2^{|J|}$ and $\mathrm{Spin}(\Pi_J^{\mathrm{sl}}) \to K(\Pi_J^{\mathrm{sl}})$ has degree 2 according to [9, Thm. 11.17]. Since we have $\pi_1(K(\Pi_J^{\mathrm{sl}})/\widetilde{T}) \cong H_J$ by Theorem 3.15 and Lemma 4.1, this implies $|H_J| = 2^{|J|+1}$. □

Proposition 4.12. *Let $J_1 \sqcup \cdots \sqcup J_k = I$ be the index sets of the connected components of Π^{adm}. If the Bruhat decomposition satisfies the conclusion of Proposition 3.7, then*

$$\pi_1(G/B) \cong H_{J_1} \times \cdots \times H_{J_k}.$$

Proof. By Theorem 3.15, $\pi_1(G/B) \cong H_I$ where

$$H_I = \langle x_i; \ i \in I \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i, ; \ i,j \in I \rangle$$

as defined in 4.10. For $J \subseteq I$, let

$$R_J := \{x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \mid i,j \in J\}, \tag{3}$$

the set of relators of H_J. Let

$$R^c := \bigcup_{\substack{i^\lambda, j^\lambda \text{in different} \\ \text{conn. components}}} \{x_i x_j x_i^{-1} x_j^{-1}\},$$

the set of commutators of pairs of generators from different connected components of Π^{adm}. Then

$$H_{J_1} \times \cdots \times H_{J_k} \cong \left\langle x_i;\ i \in I \;\middle|\; \bigcup_{l=1}^{k} R_{J_l} \cup R^c \right\rangle =: H.$$

Let π_{H_I} and π_H be the canonical homomorphisms from the free group $\langle x_i; \quad i \in I\rangle$ to H_I and H, respectively. It suffices to show that $\bigcup_{l=1}^{k} R_{J_l} \cup R^c \subseteq \ker \pi_{H_I}$ and $R_I \subseteq \ker \pi_H$. It is clear that a relator $x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \in R_I$ with i^λ and j^λ in a common connected component is contained in $\bigcup_{l=1}^{k} R_{J_l} \subseteq \ker \pi_H$, so let $x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \in R_I$ with i^λ and j^λ in different connected components of Π^{adm}. Then one has $(\varepsilon(i,j), \varepsilon(j,i)) \in \{(1,1), (1,-1), (-1,1)\}$. If $\varepsilon(i,j) = 1$, then $x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \in R^c \subseteq \ker \pi_H$, so let $\varepsilon(i,j) = -1$ and $\varepsilon(j,i) = 1$. Then j^λ is contained in a connected component $\Pi_{J_m}^{\mathrm{adm}}$ of colour r, and by Lemma 4.11, $\langle x_l; \quad l \in J_m \mid R_{J_m}\rangle = H_{J_m} \cong C_2^{|J_m|}$.

This implies that x_j has order 2 in H_{J_m}, hence $x_j^2 \in \langle\langle R_{J_m}\rangle\rangle_{\langle x_i; i \in I\rangle}$, the normal closure of R_{J_m} in the free group.

Since $\langle\langle R_{J_m}\rangle\rangle_{\langle x_i; i \in I\rangle} \subseteq \ker \pi_H$, one obtains $x_j^2 \in \ker \pi_H$. Since $x_j x_i x_j^{-1} x_i^{-1} \in R^c \subseteq \ker \pi_H$ and $\varepsilon(i,j) = -1$, one therefore has

$$\pi_H(x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1}) = \pi_H(x_j x_i x_j^{-1} x_i^{-1} \cdot x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1}) = 1_H.$$

Conversely, it is clear that $\bigcup_{l=1}^{k} R_{J_l} \subseteq R_I \subseteq \ker \pi_{H_I}$, so let $x_i x_j x_i^{-1} x_j^{-1} \in R^c$ with i^λ and j^λ in different connected components. As above, we can assume that $\varepsilon(i,j) = -1$ and $\varepsilon(j,i) = 1$. Since $x_j x_i^{\varepsilon(j,i)} x_j^{-1} x_i^{-1} \in \ker \pi_{H_I}$, this implies

$$\pi_{H_I}(x_i x_j x_i^{-1} x_j^{-1}) = \pi_{H_I}(x_j x_i^{\varepsilon(j,i)} x_j^{-1} x_i^{-1} \cdot x_i x_j x_i^{-1} x_j^{-1}) = 1_{H_I}.$$

This proves the assertion. $\square$

Theorem 4.13. *Let* Π *be an irreducible Dynkin diagram such that* $G(\Pi)$ *satisfies the conclusions of Proposition* 3.7 *and of Theorem* A.15*. Let* $n(g)$ *and* $n(b)$ *be the number of connected components of* Π^{adm} *of colour* g *and* b*, respectively. Then*

$$\pi_1(G(\Pi)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b)}.$$

In particular, this statement holds in the symmetrizable case.

Proof. By Theorem A.15, $\pi_1(G) \cong \pi_1(K)$, so it suffices to prove that $\pi_1(K)$ is of the given isomorphism type; note that Theorem A.15 has only been established in the symmetrizable case. Let $J \subseteq I$. The diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "p"] & K/K_J \arrow[d, "q"] \\ K/(K \cap T) \arrow[r, "\psi"] & K/(K \cap T)K_J \end{tikzcd},$$

with all maps being the respective canonical maps, commutes. Since the maps are continuous by Lemma 3.2, one obtains a commutative diagram of induced homomorphisms

$$\begin{tikzcd} \pi_1(K) \arrow[r, "\varphi_*"] \arrow[d, "p_*"] & \pi_1(K/K_J) \arrow[d, "q_*"] \\ \pi_1(K/(K \cap T)) \arrow[r, "\psi_*"] & \pi_1(K/(K \cap T)K_J) \end{tikzcd},$$

where p_* and q_* are injective, because p and q are covering maps (see Lemma 4.3). By Theorem 3.15 and Lemma 4.1, $\pi_1(K/(K\cap T))$ and $\pi_1(K/(K\cap T)K_J)$ can be identified with $H_I = \langle x_i;\ i \in I \mid R_I\rangle$ and $\langle x_i;\ i\in I \mid R_I \cup \{x_j \mid j \in J\}\rangle$, respectively ($R_I$ as in (3) in the above proof), where ψ_* corresponds to the canonical homomorphism between these groups as the proof of Theorem 3.15 shows.

For the index set J_m of a connected component of Π^{adm}, let $\bar{J}_m := I \setminus J_m$. Then by Proposition 4.12,

$$\langle x_i;\ i\in I \mid R_I \cup \{x_j \mid j \in \bar{J}_m\}\rangle \cong \left(\prod_{i=1}^{k} H_{J_i} \Big/ \prod_{\substack{i=1\\ i\neq m}}^{k} H_{J_i} \right) \cong H_{J_m}.$$

Summing up, one obtains a commutative diagram

$$\begin{array}{ccc} \pi_1(K) & \xrightarrow{\varphi_*} & \pi_1(K/K_{\bar{J}_m}) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle q_*} \\ \prod_{i=1}^k H_{J_i} & \xrightarrow{\pi_m} & H_{J_m} \end{array},$$

having replaced p_* and q_* from above with the corresponding monomorphisms.

By Lemma 4.3, the covering $K/K_{\bar{J}_m} \to K/K_{\bar{J}_m}(K\cap T)$ has degree $2^{n-|\bar{J}_m|} = 2^{|J_m|}$. This implies that $\widetilde{H}_m := q_*(\pi_1(K/K_{\bar{J}_m}))$ is a subgroup of H_{J_m} of index $2^{|J_m|}$. The isomorphism type of $\widetilde{H}_m$ is uniquely determined by this index and Lemma 4.11. One has

$$\widetilde{H}_m \cong \begin{cases} \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } r,\\ 2\mathbb{Z} \cong \mathbb{Z}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } g,\\ C_2 & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b. \end{cases}$$

Again by Lemma 4.3, the covering $K \to K/(K\cap T)$ has degree 2^n, so $p_*(\pi_1(K))$ is a subgroup of index 2^n of $\prod_{i=1}^k H_{J_i}$. The commutative diagram above implies that $\pi_1(K) \cong p_*(\pi_1(K)) \subseteq \pi_m^{-1}(\widetilde{H}_m)$. Since this holds for the index set of every connected component of Π^{adm}, one has $p_*(\pi_1(K)) \subseteq \widetilde{H}_1 \times \cdots \times \widetilde{H}_m$. But the latter is a subgroup of index $2^{|J_1|}\cdot \cdots \cdot 2^{|J_m|} = 2^n$ of $\prod_{i=1}^k H_{J_i}$, so equality holds. This proves the assertion. □

Theorem 4.14. *Let* Π *be an irreducible Dynkin diagram such that* $G(\Pi)$ *satisfies the conclusion of Proposition* 3.7*. Let* $n(g)$ *be the number of connected components of* Π^{adm} *of colour* g*. Let* $n(b,\kappa)$ *be the number of connected components of* Π^{adm} *on which* κ *takes the value 1 and which have colour* b*. Then*

$$\pi_1(\mathrm{Spin}(\Pi,\kappa)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b,\kappa)}.$$

In particular, this statement holds in the 2*-spherical and the symmetrizable case.*

Proof. By [9, Thm. 17.1], the map $\rho_{\Pi,\kappa} : \mathrm{Spin}(\Pi,\kappa) \to K$ is a $2^{c(\Pi,\kappa)}$-fold central extension. Let J be the index set of a connected component of Π^{adm} and let $\bar{J} := I \setminus \bar{J}$. Let $U_{\bar{J}} := \langle \widetilde{G}_{ij} \mid i \neq j \in J\rangle_{\mathrm{Spin}(\Pi,\kappa)}$.

Since $\rho_{\Pi,\kappa}(U_{\bar{J}}) \subseteq K_{\bar{J}}$, one has a continuous induced map

$$\rho^J_{\Pi,\kappa} : \mathrm{Spin}(\Pi,\kappa)/\mathrm{Spin}(\Pi_{\bar{J}},\kappa_{\bar{J}}) \to K/K_{\bar{J}}$$

making the following diagram commute, where $\widetilde{\varphi}$ and φ denote the respective canonical maps:

$$\begin{tikzcd} \mathrm{Spin}(\Pi,\kappa) \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\rho_{\Pi,\kappa}"] & \mathrm{Spin}(\Pi,\kappa)/U_{\bar{J}} \arrow[d, "\rho^{J}_{\Pi,\kappa}"] \\ K \arrow[r, "\varphi"] & K/K_{\bar{J}} \end{tikzcd}.$$

Each fiber of $\rho^{J}_{\Pi,\kappa}$ has cardinality

$$\begin{aligned} |\{xU_{\bar{J}} \mid x \in \ker \rho_{\Pi,\kappa}\}| &= |\ker(\rho_{\Pi,\kappa})/(U_{\bar{J}} \cap \ker(\rho_{\Pi,\kappa}))| \\ &= 2^{c(\Pi,\kappa)-c(\Pi_{\bar{J}},\kappa_{\bar{J}})} \text{ by Remark 4.6.} \end{aligned}$$

Since $\rho_{\Pi,\kappa}$ is open as a covering map and φ is open by Lemma 3.2, it follows from Lemma 4.2 that $\rho^{J}_{\Pi,\kappa}$ is a covering map.

From here the proof is analogous to the proof of Theorem 4.13, after extending the commutative diagram at the beginning of the latter proof:

$$\begin{tikzcd} \mathrm{Spin}(\Pi,\kappa) \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\rho_{\Pi,\kappa}"] & \mathrm{Spin}(\Pi,\kappa)/U_{\bar{J}} \arrow[d, "\rho^{J}_{\Pi,\kappa}"] \\ K \arrow[r, "\varphi"] \arrow[d, "p"] & K/K_{J} \arrow[d, "q"] \\ K/(K\cap T) \arrow[r, "\psi"] & K/(K\cap T)K_{J} \end{tikzcd}.$$

One obtains that $\pi_1(\mathrm{Spin}(\Pi,\kappa)) \cong \prod_{i=1}^{k} H'_{J_i}$ where each H'_{J_m} is a subgroup of index $2^{c(\Pi,\kappa)-c(\Pi_{\bar{J}_m},\kappa_{\bar{J}_m})}$ of

$$\widetilde{H}_m \cong \begin{cases} \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } r, \\ 2\mathbb{Z} \cong \mathbb{Z}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } g, \\ C_2 & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b. \end{cases}$$

Since $\Pi^{\mathrm{adm}}_{\bar{J}_m}$ is the union of all connected components except $\Pi^{\mathrm{adm}}_{J_m}$, one has $c(\Pi,\kappa) - c(\Pi_{\bar{J}_m},\kappa_{\bar{J}_m}) \in \{0,1\}$, depending on whether κ is constant 1 or 2 on $\Pi^{\mathrm{adm}}_{J_m}$. This implies

$$H'_m \cong \begin{cases} \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } r, \\ \mathbb{Z}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } g, \\ C_2 & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b \text{ and } \kappa \equiv 1 \text{ on } \Pi^{\mathrm{adm}}_{J_m}, \\ \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b \text{ and } \kappa \equiv 2 \text{ on } \Pi^{\mathrm{adm}}_{J_m}. \end{cases}$$

This proves the assertion. □

Now all theorems from the introduction have been proved.

A. Maximal unipotent subgroups of Kac–Moody groups and applications to Kac–Moody symmetric spaces

T. HARTNICK, R. KÖHL

Throughout this appendix, we fix a symmetrizable generalized Cartan matrix $\mathbf{A}$ with underlying diagram Π. We consider the corresponding algebraically simply-connected semisimple split real Kac–Moody group $G := G(\Pi) = [G_{\mathbf{A}}(\mathbb{R}), G_{\mathbf{A}}(\mathbb{R})]$ as given by Definition 1.1. As in Section 2, we also denote by $K_{\mathbf{A}}(\mathbb{R}) \leq G_{\mathbf{A}}(\mathbb{R})$ the fixed point subgroup of the Cartan–Chevalley involution θ and set $K := K(\Pi) = K_{\mathbf{A}}(\mathbb{R}) \cap G$. We equip all of these groups with the restrictions of the Kac–Peterson topology.

The goal of this appendix is to relate the topology of G to the topology of K. Our main result (see Theorem A.15 below) asserts that the inclusion $K \hookrightarrow G$ is a weak homotopy equivalence. This implies in particular that $\pi_1(G) \cong \pi_1(K)$ and thus allows the computation of $\pi_1(G)$ by the methods presented in the main part of the article.

In the spherical case, the subgroup $K < G$ is even a deformation retract and hence the inclusion $K \hookrightarrow G$ is a homotopy equivalence, as a consequence of the topological Iwasawa decomposition of G. This decomposition also implies that the associated Riemannian symmetric space G/K is contractible.

While real Kac–Moody groups also possess an Iwasawa decomposition, it is currently unknown whether this decomposition is topological. To establish our main result, we thus have to work with a certain central quotient $\overline{G}$ of G, for which the topological Iwasawa decomposition was established in [8]. We will show that the image $\overline{K}$ of K in $\overline{G}$ is a strong deformation retract and that the reduced Kac–Moody symmetric space $\overline{G}/\overline{K}$ is contractible. Since the finite-dimensional central extension $G \to \overline{G}$ is a Serre fibration by a classical result of Palais [32], this will allow us to deduce the desired result about G and K.

A.1. The topological Iwasawa decomposition

Let us denote by $\mathrm{Ad} : G_{\mathbf{A}}(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}(\mathbf{A}))$ and $\mathrm{Ad} : G(\Pi) \to \mathrm{Aut}(\mathfrak{g}'_{\mathbb{R}}(\mathbf{A}))$ the adjoint representations of $G_{\mathbf{A}}(\mathbb{R})$ and $G = G(\Pi)$, respectively. We recall from [8] that the quotient map $G \to \mathrm{Ad}(G)$ factors as

$$G \xrightarrow{p_1} \overline{G} \xrightarrow{p_2} \mathrm{Ad}(G), \tag{4}$$

where $\overline{G}$ is uniquely determined by the fact that $\overline{T} := p_1(T) \cong (\mathbb{R}^\times)^{\mathrm{rk}(\mathbf{A})}$ is a torus and p_2 has finite kernel. The group $\overline{G}$ is referred to as the *semisimple adjoint quotient* of G, and we equip it with the quotient topology with respect to the Kac–Peterson topology on G. We will denote by $U^\pm$ the positive, respectively negative maximal unipotent subgroup of $G(\Pi)$ as introduced in Section 2. Also recall from Section 2 that $A_{\mathbb{R}} := \exp(\mathfrak{h}_{\mathbb{R}}(\mathbf{A})) \leq G_{\mathbf{A}}(\mathbb{R})$ and set $A := A_{\mathbb{R}} \cap G$.

Lemma A.1 (Iwasawa decomposition). *Multiplication induces continuous bijections*

$$K_{\mathbf{A}}(\mathbb{R}) \times A_{\mathbb{R}} \times U^+ \to G_{\mathbf{A}}(\mathbb{R}) \quad \textit{and} \quad K \times A \times U^+ \to G.$$

Proof. This follows from [23, Prop. 5.1(a)]. □

A more refined statement has been established in [8] for the semisimple adjoint quotient $\overline{G}$ of G. To state this result, denote by

$$G \xrightarrow{p_1} \overline{G} \xrightarrow{p_2} \mathrm{Ad}(G)$$

the canonical quotient maps from (4) and set $\overline{K} := p_1(K)$, $\overline{T} := p_1(T) \cong (\mathbb{R}^\times)^{\mathrm{rk}(\mathbf{A})}$, $\overline{A} := p_1(A) = \overline{T}^o$ and $\overline{U^+} := p_1(U^+)$. Equip these groups with their respective quotient topologies and note that p_1 restricts to a bijection between U^+ and $\overline{U^+}$.

Theorem A.2 (Topological Iwasawa decomposition, [8, Thm. 3.23]). *Multiplication induces homeomorphisms*

$$\overline{K} \times \overline{A} \times \overline{U^+} \to \overline{G} \quad \text{and} \quad \overline{U^+} \times \overline{A} \times \overline{K} \to \overline{G}.$$

Since $\overline{A}$ is contractible, in order to show that $\overline{K}$ is a deformation retract of $\overline{G}$ it will suffice to show that $\overline{U^+}$ is contracible. We thus need to understand the topology induced by the Kac–Peterson topology on the standard unipotent subgroups.

A.2. The Kac–Peterson topology on $U^{\pm}$

We now turn to the study of the restriction of the Kac–Peterson topology to the standard maximal unipotent subgroups U^- and U^+. Recall from Section 2 that the Weyl group W is a Coxeter group, so elements of W can be represented by reduced words in the generators $s_1, \dots, s_r$. Given such a reduced word $w = (s_{i_1}, \dots, s_{i_r})$ in W with corresponding simple roots $\alpha_{i_1}, \dots \alpha_{i_r}$ we define positive roots $\beta_1, \dots, \beta_r$ by

$$\beta_1 := \alpha_{i_1}, \quad \beta_2 := s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_r := s_{i_1} s_{i_2} \cdots s_{i_{r-1}}(\alpha_{i_r}). \tag{5}$$

We then set $U_w := U_{\beta_1} \cdots U_{\beta r} \subset U^+$ and define a map

$$\mu_w : U_{\beta_1} \times \cdots \times U_{\beta_r} \to U_w, \quad (x_1, \dots, x_r) \mapsto x_1 \cdots x_r.$$

It is established in [4, Sect. 5.5, Lem.] that the map μ_w is a bijection for every reduced word w, and that its image U_w depends only on the Weyl group element represented by w, but not on the chosen reduced expression. Since $G_{\mathbf{A}}(\mathbb{R})$ is a topological group, the bijection μ_w is continuous. In fact, one can show that μ_w is a homeomorphism. A proof of this fact was sketched in [14, Lem. 7.25]; since openness of the maps μ_w is crucial for everything that follows, we fill in the details of this sketch here.

Lemma A.3. *For every reduced word w the map μ_w is a homeomorphism onto its image.*

Proof. We argue by induction on the length m of w and observe that the case $m = 1$ holds by definition. Since the linear functionals $\alpha_1, \dots, \alpha_r$ are linearly independent, there exists an element $X \in \mathfrak{h}_{\mathbb{R}}(\mathbf{A})$ (see Section 2) such that $\alpha_{i_1}(X) = 0$ and $\alpha_j(X) < 0$ for all $j \in \{1, \dots, \widehat{i_1}, \dots, r\}$. It follows that $\beta_1(X) = \alpha_{i_1}(X) = 0$ and $\beta_k(X) < 0$ for all $k = 2, \dots, m$. Indeed, since the word w is reduced, none of the positive real roots $\beta_2, \dots, \beta_m$ equals α_{i_1}, and since $n\alpha_{i_1}$ is not a root for any $n \geq 2$ (cf. [21, Prop. 5.1]), each of them contains at least one other positive simple root as a summand. Now for $j \in \{1, \dots, m\}$ and $Y \in \mathfrak{g}_{\beta_j}$ we have $\mathrm{ad}(X)(Y) = \beta_j(X)(Y)$, and thus

$$\lim_{t \to \infty} \mathrm{Ad}(\exp(tX))(Y) = \begin{cases} Y, & j = 1, \\ 0, & j > 1. \end{cases}$$

We conclude that if $x_j \in U_{\beta_j}$, then

$$\lim_{t \to \infty} \exp(tX)(x_1 \cdots x_m) \exp(-tX) = x_1,$$

where the convergence is uniform on compacta. This shows that the map

$$\pi_1 : U_w \to U_{\beta_1}, \quad x_1 \cdots x_m \mapsto x_1$$

is continuous, and hence the map

$$U_w \to U_{\beta_1} \times U_{\beta_2} \cdots U_{\beta_m}, \quad x_1 \cdots x_m \mapsto (x_1, x_2 \cdots x_m) \tag{6}$$

is continuous. Now let $w' = (r_{i_2}, \ldots, r_{i_m})$ and let $\beta'_2 = s_{i_1}(\beta_2), \ldots, \beta'_m := s_{i_1}(\beta_m)$. Now by Axiom (RGD2) of an RGD system (see [1, Chap. 8]) there exists an element $g \in G_{\mathbf{A}}(\mathbb{R})$ such that $gU_{\beta_j}g^{-1} = U_{\beta'_j}$ for all $j = 2, \cdots, m$, and by induction hypothesis we have a homeomorphism

$$\mu_{w'} : U_{\beta'_2} \times \cdots \times U_{\beta'_m} \to U_{w'}, \quad (x_2, \ldots, x_m) \mapsto x_2 \cdots x_m.$$

Conjugating the inverse of this homeomorphism by g^{-1} we obtain a homeomorphism

$$U_{\beta_2} \cdots U_{\beta_m} \to U_{\beta_2} \times \cdots \times U_{\beta_m}.$$

Composing this homeomorphism with the map (6) now provides the desired continuous inverse to μ_w. $\square$

To describe the topology on U^+ we recall that there exist several distinct but related partial orders on W which in different places in the literature are referred to as the *Bruhat order* on W. In the sequel we will consider the following version; here ℓ denotes the length function with respect to the generating set $\{s_1, \ldots, s_r\}$.

Definition A.4. The *weak right Bruhat order* on W is the partial order $\leq_w$ defined as

$$w_1 \leq_w w_2 \quad :\Longleftrightarrow \quad \ell(w_2) = \ell(w_1) + \ell(w_1^{-1}w_2), \quad (w_1, w_2 \in W).$$

According to [4, p. 44] we have $w_1 \leq_w w_2$ if and only if there exists a reduced word $(r_{i_1}, \ldots, r_{i_{\ell(w_2)}})$ for w_2 such that $w_1 = r_{i_1} \cdots r_{i_{\ell(w_1)}}$.

Recall that for the strong Bruhat order $\leq$ one has $w_1 \leq w_2$ if there exists a reduced word $(r_{i_1}, \ldots, r_{i_m})$ for w_2 and a reduced word $(r_{j_1}, \ldots, r_{j_l})$ for w_1 such that $(r_{j_1}, \ldots, r_{j_l})$ is a substring of $(r_{i_1}, \ldots, r_{i_m})$ (not necessarily consecutive). By definition,

$$w_1 \leq_w w_2 \quad \Longrightarrow \quad w_1 \leq w_2,$$

but the converse is not true. An important difference between the weak right Bruhat order and the strong Bruhat order is that $(W, \leq)$ contains a cofinal chain, i.e., a totally ordered subset $T \subset W$ such that for every $w \in W$ there exists $t \in T$ such that $w \leq t$, whereas for the weak right Bruhat order, such a cofinal chain does not exist. In fact, given $w_1, w_2 \in W$ there will in general not exist an element $w_3 \in W$ with $w_1 \leq_w w_3$ and $w_2 \leq_w w_3$.

Note that if $w_1 \leq_w w_2$, then we can choose a reduced word $(r_{i_1}, \ldots, r_{i_{\ell(w_2)}})$ for w_2 such that $w_1 = r_{i_1} \cdots r_{i_{\ell(w_1)}}$. Thus if we define $\beta_1, \ldots, \beta_{\ell(w_2)}$ as above then we have a commuting diagram

$$\begin{tikzcd}
U_{\beta_1} \times \cdots \times U_{\beta_{\ell(w_1)}} \arrow[r] \arrow[d] & U_{\beta_1} \times \cdots \times U_{\beta_{\ell(w_2)}} \arrow[d] \\
U_{w_1} \arrow[r] & U_{w_2}
\end{tikzcd},$$

where the horizontal maps are inclusions, and the vertical maps are homeomorphisms. In particular, we have a continuous inclusion $\iota_{w_1}^{w_2} : U_{w_1} \hookrightarrow U_{w_2}$, hence we may form the colimit

$$\varinjlim((U_w)_{w \in W}, (\iota_{w_1}^{w_2})_{w_1 \leq_w w_2})$$

in the category of topological spaces. We emphasize that in view of the previous remark the system $((U_w)_{w \in W}, (\iota_{w_1}^{w_2})_{w_1 \leq_w w_2})$ is *not* directed, hence this colimit is not a direct limit.

Proposition A.5. *The k_ω-space U^+ is given by the colimit*

$$U^+ = \lim_{\rightarrow}((U_w)_{w\in W}, (\iota_{w_1}^{w_2})_{w_1 \leq_w w_2})$$

both in the category of topological spaces and in the category of k_ω-topological spaces.

Proof. The corresponding statement in the category of sets is established in [4, Thm. 5.3]. For the topological statement see [14, Prop. 7.27]. □

In view of the applications to Kac–Moody symmetric spaces that we have in mind we recall that $U^\pm$ are subgroups of the commutator subgroup G of $G_\mathbb{R}(\mathbf{A})$; in particular we can consider their images $\overline{U}^\pm := p_1(U^\pm)$ under the map $p_1 : G \to \overline{G}$ from (4). In this context, we will need the following fact.

Proposition A.6. *The map p_1 induces homeomorphisms $U^\pm \to \overline{U}^\pm$.*

Proof. By [14, Prop. 7.27] the map $T \times U^+ \to TU^+$ is a homeomorphism and the kernel of p_1 is contained in T. The latter implies that p_1 restricts to a continuous bijection $U^+ \to \overline{U^+}$, and the former implies that this bijection is open. □

A.3. Dilation structures on $U^\pm$

Definition A.7. Let U be a topological group. By a *dilation structure* on U, we mean a family of maps $(\Phi_t : U \to U)_{t\in\mathbb{R}}$ with the following properties:

(a) Each Φ_t is a continuous automorphisms of the topological group U.

(b) $(\Phi_t)_{t\in\mathbb{R}}$ is a 1-parameter group, i.e. $\Phi_0 = \mathrm{Id}$ and $\Phi_{s+t} = \Phi_s \circ \Phi_t$ for all $s, t \in \mathbb{R}$.

(c) If we define $\Phi_{-\infty} : U \to U$ by $\Phi_{-\infty}(u) := e$, then the map

$$[-\infty, \infty) \times U \to U, \quad (t, u) \mapsto \Phi_t(u)$$

is continuous.

Remark A.8. Note that if a topological group U admits a dilation structure, then it is in particular contractible. Indeed, if we define $\Psi_t := \Phi_{t/(t-1)}$, then

$$\Psi : [0, 1] \times U \to U, \quad (t, u) \mapsto \Psi_t(u)$$

is continuous with $\Psi_0 = \Phi_0 = \mathrm{Id}$ and $\Psi_1 = \Phi_{-\infty}$, hence a contraction to the identity.

Dilation structures on finite-dimensional simply-connected nilpotent Lie groups play a major role in conducting analysis on such groups, see, e.g., [11]. Not every finite-dimensional simply-connected nilpotent Lie group admits a dilation structure, but if U is the unipotent radical of a minimal parabolic subgroup of a semisimple Lie group, then such a dilation structure always exists. The methods of [25] allow one to extend this result to the Kac–Moody setting.

Following [21, §3.12], we define the *fundamental chamber* of $\mathfrak{h}_\mathbb{R}(\mathbf{A})$ as

$$C := \{h \in \mathfrak{h}_\mathbb{R}(\mathbf{A}) \mid \forall 1 \leq i \leq n : \alpha_i(h) \geq 0\} \subset \mathfrak{h}_\mathbb{R}(\mathbf{A}).$$

Since the family $(\alpha_i)_{1\leq i\leq n}$ is linearly independent, there exists

$$X_0 \in C \text{ such that } \alpha_i(X_0) = 1 \text{ for all } 1 \leq i \leq n.$$

Indeed, by the linear independence of $(\alpha_i)_{1\leq i\leq n}$ the solution space for the system of $n-1$ linear equations $\forall 2\leq i\leq n : \alpha_1(x)-\alpha_i(x)=0$ has strictly larger dimension than the solution space for the system of n linear equations $\forall 1\leq i\leq n : \alpha_i(x)=0$.

We now define a 1-parameter subgroup of $A_{\mathbb{R}}$ by $a_t := \exp(tX_0)$ and denote by

$$\varphi_t := \mathrm{Ad}(a_t) \in \mathrm{Aut}(\mathfrak{u}^+)$$

the associated automorphism of the Lie algebra $\mathfrak{u}^+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha^k$. Similarly we denote by

$$\Phi_t := c_{a_t}|_{U^+} \in \mathrm{Aut}(U^+)$$

the restriction of the conjugation-action of a_t on $G_{\mathbf{A}}(\mathbb{R})$ to U^+. Note that if $X\in\mathfrak{u}^+$ is ad-locally finite, then

$$\Phi_t(\exp(X)) = \exp(\varphi_t(X)).$$

From (1) and the defining property of X_0 one deduces that for every positive root α with height $|\alpha|$

$$\forall Y\in\mathfrak{g}_\alpha :\ \varphi_t(Y) = e^{t|\alpha|}Y.$$

It follows that for all positive roots α, one has

$$\Phi_t(x_\alpha(s)) = \exp(tX_0)\cdot x_\alpha(s)\cdot \exp(-tX_0) = x_\alpha(e^{t|\alpha|}s) \tag{7}$$

(see [39, (4), p. 549]), where $\{x_\alpha(s) \mid s\in\mathbb{R}\}\cong(\mathbb{R},+)$ is the root subgroup of $G_{\mathbf{A}}(\mathbb{R})$ corresponding to the root space $\mathfrak{g}_\alpha$. As a consequence, if one endows each of the root subgroups $\{x_\alpha(s) \mid s\in\mathbb{R}\}$ with the natural topology of $\mathbb{R}$, then Φ_t contracts each of them. We are now in a position to reproduce the following result and proof by Kumar.

Theorem A.9 ([25, Prop. 7.4.17]). *The family $(\Phi_t)_{t\in\mathbb{R}}$ defines a dilation structure on U^+.*

Proof. Let w be a reduced word and write $w = s_{i_1}\cdots s_{i_r}\in W$ with corresponding simple roots $\alpha_{i_1},\dots\alpha_{i_r}$. Recall that multiplication induces a homeomorphism

$$U_{\beta_1}\times\cdots\times U_{\beta_r}\to U_w,$$

where the roots $\beta_1,\dots,\beta_r$ are given by

$$\beta_1 := \alpha_{i_1},\quad \beta_2 := s_{i_1}(\alpha_{i_2}),\quad \dots,\quad \beta_r := s_{i_1}s_{i_2}\cdots s_{i_{r-1}}(\alpha_{i_r}).$$

Given an element $x_{\beta_1}(y_1)x_{\beta_2}(y_2)\cdots x_{\beta_r}(y_r)\in U_w$ by (7) one has

$$\Phi_t(x_{\beta_1}(y_1)x_{\beta_2}(y_2)\cdots x_{\beta_r}(y_r)) = x_{\beta_1}(e^{t|\beta_1|}y_1)x_{\beta_2}(e^{t|\beta_2|}y_2)\cdots x_{\beta_r}(e^{t|\beta_r|}y_r).$$

Setting $\Phi_{-\infty}(u) := e$ for all $u\in U^+$, we deduce that the map

$$\Phi|_{U_w} : [-\infty,\infty)\times U_w\to U_w,\quad (t,u)\mapsto\Phi_t(u)$$

is continuous and that $\Phi_0 = \mathrm{Id}_{U_w}$. Combining this with Proposition A.5, one deduces that the map

$$\Phi : [-\infty,\infty)\times U^+\to U^+,\quad (t,u)\mapsto\Phi_t(u)$$

is continuous, hence a dilation structure. □

Recall that U^+ is isomorphic to U^- under the Cartan–Chevalley involution of $G_{\mathbf{A}}(\mathbb{R})$, which maps a_t to a_{-t}. Thus if we define $\Phi_t^- := c_{a_{-t}}|_{U^-}$ then we obtain the following.

Corollary A.10. *The family $(\Phi_t^-)_{t\in\mathbb{R}}$ defines a dilation structure on U^-.*

Combining this with Remark A.8 and Proposition A.6, we can record the following.

Corollary A.11. *The topological groups U^+ and U^- are contractible. Consequently, the groups $\overline{U}^+$ and $\overline{U}^-$ are contractible.*

A.4. Homotopy groups of real-split semisimple Kac–Moody groups

Corollary A.12. *The subgroup $\overline{K} < \overline{G}$ is a deformation retract. In particular, the inclusion $i_K : K_{\mathbf{A}}(\mathbb{R}) \hookrightarrow G_{\mathbf{A}}(\mathbb{R})$ is a homotopy equivalence and thus induces isomorphisms $(i_K)_* : \pi_n(\overline{K}) \to \pi_n(\overline{G})$ for all $n \geq 0$.*

Proof. We have established in Corollary A.11 that $\overline{U^+}$ is contractible, and $\overline{A}$ is contractible since it is homeomorphic to $\mathbb{R}^{\mathrm{rk}(\mathbf{A})}$. The assertion now follows from Theorem A.2. □

Since it is currently unknown whether the Iwasawa decomposition of G is also a topological decomposition, the strategy of the above proof can not be applied to G. However, using the following result of Palais [32, Sect. 4.1, Cor.], one can still obtain an isomorphism between the fundamental groups of G and K.

Proposition A.13 (Palais). *Let G be a topological group and let $H < G$ be a subgroup which is homeomorphic to a Lie group. Then the fibration $H \hookrightarrow G \to G/H$ is locally trivial, in particular a Hurewicz fibration, hence there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_2(H) \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G).$$

Recall that the kernel of the quotient map $G \to \overline{G}$ is homeomorphic to $(\mathbb{R}^\times)^{\mathrm{cork}(\mathbf{A})}$. In particular, it has $2^{\mathrm{cork}(\mathbf{A})}$ connected components, whereas its higher homotopy groups vanish. Applying Proposition A.13 to the diagram of fibrations

$$\begin{tikzcd}
(\mathbb{R}^\times)^{\mathrm{cork}(\mathbf{A})} \arrow[r] & G \arrow[r] & \overline{G} \\
(\mathbb{Z}/2\mathbb{Z})^{\mathrm{cork}(\mathbf{A})} \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & \overline{K} \arrow[u]
\end{tikzcd}$$

we thus obtain the following.

Corollary A.14. *There is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(G) \arrow[r] & \pi_1(\overline{G}) \arrow[r] & (\mathbb{Z}/2\mathbb{Z})^{\mathrm{cork}(\mathbf{A})} \arrow[r] & 0 \\
0 \arrow[r] & \pi_1(K) \arrow[r] \arrow[u] & \pi_1(\overline{K}) \arrow[r] \arrow[u] & (\mathbb{Z}/2\mathbb{Z})^{\mathrm{cork}(\mathbf{A})} \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}.$$

Moreover, for $n \geq 2$ there are isomorphisms $\pi_n(G_{\mathbf{A}}(\mathbb{R})) \cong \pi_n(G)$ and $\pi_n(K_{\mathbf{A}}(\mathbb{R})) \cong \pi_n(K)$.

Combining this with Corollary A.12 we deduce the following.

Theorem A.15. *For every $n \geq 0$ the inclusion $K \hookrightarrow G$ induces isomorphisms*

$$\pi_n(K) \hookrightarrow \pi_n(G),$$

hence is a weak homotopy equivalence. In particular, $\pi_1(G) \cong \pi_1(K)$.

A.5. Kac–Moody symmetric spaces and causal contractions

We conclude this appendix with an application to the results obtained so far to Kac–Moody symmetric spaces. By [8], the homogeneous spaces $G_{\mathbf{A}}(\mathbb{R})/K_{\mathbf{A}}(\mathbb{R})$ and G/K carry the natural structure of topological reflection spaces, and the same holds for their quotients $\mathrm{Ad}(G_{\mathbf{A}}(\mathbb{R}))/\mathrm{Ad}(K_{\mathbf{A}}(\mathbb{R}))$ and $\mathrm{Ad}(G)/\mathrm{Ad}(K)$. The topological reflection space $\mathcal{X} = G/K$ is called the *unreduced Kac–Moody symmetric space* of type $\mathbf{A}$, and the topological reflection space $\overline{\mathcal{X}} = \mathrm{Ad}(G)/\mathrm{Ad}(K) = \overline{G}/\overline{K}$ is called the *reduced Kac–Moody symmetric space* of type $\mathbf{A}$.

Corollary A.16. *The reduced symmetric space $\overline{\mathcal{X}}$ is contractible.*

Proof. In view of the topological Iwasawa decomposition, the orbit map at the basepoint $o = e\overline{K}$

$$\overline{U^+} \times \overline{A} \to \overline{\mathcal{X}}, \quad (u, a) \mapsto ua.o$$

is a homeomorphism. Since $\overline{U^+}$ and $\overline{A}$ are contractible, this implies contractability of $\overline{\mathcal{X}}$. $\square$

The proof of Theorem A.9 can be used to provide an explicit contraction for $\overline{\mathcal{X}} \simeq \overline{U^+} \times \overline{A}$, using the contraction by conjugation with suitable elements of the torus $T_{\mathbb{R}}$ on the group $\overline{U^+}$ and the standard contraction on the finite-dimensional real vector space A. It turns out that this contraction has interesting additional properties. Recall from [8, Sect. 7] that the symmetric space $\overline{\mathcal{X}}$ admits future and past boundaries $\Delta_{\|}^{+}$ and $\Delta_{\|}^{-}$ that both carry a simplicial structure which turns them in the geometric realizations of the positive and negative halves of the twin building of $G_{\mathbf{A}}(\mathbb{R})$. Following [8, Sect. 7], a *causal ray* is a geodesic ray of $\overline{\mathcal{X}}$ whose parallelity class equals a point in $\Delta_{\|}^{+}$ and a *piecewise geodesic causal curve* is the concatenation of a finite set of segments of causal rays that can be parametrized in such a way that the walking direction always points towards the future boundary. Given $x, y \in \overline{\mathcal{X}}$, we say that x *causally preceeds* y (in symbols $x \preceq y$) if there exists a piecewise geodesic causal curve from x to y.

Since both conjugation by elements of $T_{\mathbb{R}}$ and the standard contraction of the vector space A preserve geodesic rays and the future and past boundaries (cf. [8, Sect. 7]), the set of piecewise geodesic causal curves of $\overline{\mathcal{X}}$, and hence the causal pre-order $\preceq$, are invariant under the given contraction.

Corollary A.17. *The reduced symmetric space $\overline{\mathcal{X}}$ is causally contractible, i.e., it admits a contraction that preserves $\preceq$.*

B. The Bruhat decomposition is a CW decomposition

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Let G be a Kac–Moody group endowed with the Kac–Peterson topology and let T be the standard maximal torus and U^+, U^- the standard unipotent subgroups. [22, Thm. 4(a)] asserts without proof that the multiplication map

$$U^+ \times T \times U^- \to U^+TU^-$$

is a homeomorphism with respect to the Kac–Peterson topology. In this note, we provide a proof in the symmetrizable case that makes use of this fact in the two-spherical case ([14, Prop. 7.31]), of the embedding of Kac–Moody groups constructed in [28, Thm. 3.15(2)], and of the fact that the Kac–Peterson topology is k_ω. Among the various consequences of this result is that the Bruhat decomposition of a symmetrizable topological Kac–Moody group is a CW decomposition.

Recall that a *k-space* (alternatively, *compactly generated space*) is a topological space X in which a set $C \subset X$ is closed if and only if its intersection $C \cap K$ with any compact subset K of X is compact. That is, a k-space is a topological space X whose topology is coherent with the family of all compact subspaces of X. A k_ω-space is a topological space X whose topology is coherent with respect to a countable ascending family of compact subspaces. By (3) of [7] any k_ω-space is a k-space.

Proposition B.1 ([33, Cor.]). *A continuous proper map $f : X \to Y$ from a topological space X to a k-space Y is closed. In particular, a continuous injection $\iota : X \to Y$ into a k_ω-space $Y = \bigcup_{n\in\mathbb{N}} Y_m$ with compact Y_m such that for each $m \in \mathbb{N}$ the pre-image $\iota^{-1}(Y_m)$ is also compact is a topological embedding, i.e., it is a homeomorphism onto its image.*

Proof. The first statement is exactly [33, Cor.]. The second statement is an immediate consequence of the first, since a k_ω-space is a k-space in which any compact subset K of Y is contained in some Y_m of the ascending family $(Y_m)_{m\in\mathbb{N}}$ of compact subsets (statement (3) of [7]). $\square$

Remark B.2. The authors thank T. Hartnick and S. Witzel for various lively discussions concerning the correct formulation and application of Proposition B.1. Moreover, they thank S. Witzel for suggesting to make use of the concept of proper maps.

A subgroup of a Kac–Moody group is *bounded* if it lies in the intersection of two spherical parabolic subgroups of opposite signs. In other words, it is bounded if and only if it stabilises a point the Davis CAT(0) realization of each half of its twin building. The maximal bounded subgroups of a Kac–Moody group have been determined in [3, Thm. 4.1].

Proposition B.3. *Let G be a split real Kac–Moody group. Then the Kac–Peterson topology τ_{KP} on G equals the finest group topology τ_{MB} on G such that the embeddings of the maximal bounded subgroups, each endowed with its Lie group topology, are continuous.*

Proof. By [26, Lem. 4.3], the Kac–Peterson topology τ_{KP} on G induces the Lie group topology on its maximal bounded subgroups. A fundamental $\mathrm{SL}_2(\mathbb{R})$ is bounded and, in particular, embeds as a closed subgroup into a maximal bounded subgroup. Therefore, its subspace topology equals its Lie group topology; by [14, Prop. 7.21] the topology τ_{KP} equals the finest group topology on G such that the embeddings of the fundamental $\mathrm{SL}_2(\mathbb{R})$ Lie subgroups is continuous, whence τ_{KP} is finer than or equal to the final group topology τ_{MB} with respect to the embedded maximal bounded subgroups. Again, since by [26, Lem. 4.3] the Kac–Peterson topology on G induces the Lie group topology on its maximal bounded subgroups, the two described topologies actually coincide. $\square$

Corollary B.4. *Let G be a split real Kac–Moody group endowed with the Kac–Peterson topology and let $(G_i)_{i\in I}$ be a finite family of Lie-subgroups of G such that each fundamental $\mathrm{SL}_2(\mathbb{R})$ is contained in at least one of the G_i. Then the Kac–Peterson topology on G equals the finest group topology on G such that the embeddings of the $(G_i)_i$, each endowed with its Lie group topology, are continuous.*

Proposition B.5 (cf. [13, 1.5, 1.10], [28, Thm. 3.15(2)])**.** *Any symmetrizable topological Kac–Moody group endowed with the Kac–Peterson topology admits a continuous injective group homomorphism into a simply laced topological Kac–Moody group with closed image with respect to the Kac–Peterson topology.*

Proof. By [28, Thm. 3.15(2)] for any symmetrizable Kac–Moody group G there is an injective group homomorphism $\iota : G \to H$ into a simply laced Kac–Moody group H embedding each fundamental rank-1 subgroup $G_{\alpha_i} \cong \mathrm{SL}_2(\mathbb{R})$ diagonally into the direct product

$$\prod_{j=1}^{n_i} H_{\alpha_{i,j}} \cong \mathrm{SL}_2(\mathbb{R})^{n_i}$$

of a suitable (finite) family of fundamental rank-1 subgroups $H_{\alpha_{i,j}}$ of H.

The restriction of this map to any fundamental rank-1 subgroup G_α of G is continuous with respect to the Lie group topology on G_α and the Kac–Peterson topology on H. Hence, by universality (see [14, Prop. 7.21]), the map $\iota : G \to H$ is continuous with respect to the Kac–Peterson topology on both G and H.

One has

$$\iota(G) = \bigcap_\sigma \mathrm{Fix}(\varphi_\sigma),$$

where φ_σ is the automorphism of H given by

$$H_{\alpha_{i,j}} \to H_{\alpha_{i,\sigma(j)}}$$

for some $\sigma = (\sigma_1, \ldots, \sigma_N)$, where $\sigma_i \in \mathrm{Sym}(n_i)$ acts by permuting the factors of the direct product $\prod_{j=1}^{n_i} H_{\alpha_{i,j}} \cong \mathrm{SL}_2(\mathbb{R})^{n_i}$. Since the automorphisms φ_σ are continuous with respect to the Kac–Peterson topology on H, the group $\iota(G)$ is a closed subgroup of H. □

The embedding $\iota : G \to H$ corresponds to an embedding of the twin building Δ_G of G into the twin building Δ_H of H such that $\Delta_G = \bigcap_\sigma \mathrm{Fix}(\varphi_\sigma)$ (with the φ_σ now considered as twin building automorphisms) and the additional property that two chambers of Δ_G are opposite in Δ_G if and only if they are opposite in Δ_H.

Indeed, this is immediate from an argument along the lines of descent in buildings (cf. [31]). The automorphisms φ_σ act on the twin apartment defined by the fundamental chambers c_+, c_- of Δ_H and, by definition, the fixed substructure is isometric to a twin apartment of Δ_G. The claim then follows from the fact that G acts transitively on the twin apartments of Δ_G.

In particular, this embedding

$$\Delta_G = (\Delta_G^+, \Delta_G^-, \delta_G^*) \to \Delta_H = (\Delta_H^+, \Delta_H^-, \delta_H^*)$$

of twin buildings induces an embedding of opposite geometries

$$\begin{aligned} \mathrm{Opp}(\Delta_G) = \{(c,d) \in \Delta_G^+ \times \Delta_G^- \mid \delta_G^*(c,d) = 1\} \\ \to \mathrm{Opp}(\Delta_H) = \{(c,d) \in \Delta_H^+ \times \Delta_H^- \mid \delta_H^*(c,d) = 1\}. \end{aligned}$$

Specialising to the embedding of a fundamental rank-1 subgroup $G_{\alpha_i} \cong \mathrm{SL}_2(\mathbb{R})$ of G diagonally into the direct product

$$\prod_{j=1}^{n_i} H_{\alpha_{i,j}} \cong \mathrm{SL}_2(\mathbb{R})^{n_i}$$

of a suitable (finite) family of fundamental rank-1 subgroups $H_{\alpha_{i,j}}$ of H, one obtains an embedding of the real projective line $\mathbb{S}^1$ (the building of type A_1) diagonally into a suitable product $\left(\mathbb{S}^1\right)^{n_i}$ of real projective lines (the building of type ${\mathsf{A}_1}^{n_i} = \underbrace{\mathsf{A}_1 \oplus \cdots \oplus \mathsf{A}_1}_{n_i}$).

This in turn yields an embedding of the corresponding opposites geometries of pairs of distinct points of $\mathbb{S}^1$ with adjacency relation given by the complete relation (the opposite geometry of type A_1 of diameter 1), respectively, of n_i-tuples of pairs of distinct points of $\mathbb{S}^1$ with adjacency relation given by equality in all up to at most one component (the opposite geometry of type ${\mathsf{A}_1}^{n_i}$ of diameter n_i).

Refer to [12, Sect. 4.3] for more details, some examples, and applications of the opposite geometry. The most striking application of the opposite geometry is a proof of [38, Thm. 13.32] via its simple connectedness and Mühlherr's generalization to Kac–Moody groups[3]; see also [2].

The following result follows immediately from the preceding discussion.

Proposition B.6. *Let $\iota : G \to H$ be the injective group homomorphism from Proposition* B.5*, let $\Delta_G = (\Delta_G^+, \Delta_G^-, \delta_G^*) \to \Delta_H = (\Delta_H^+, \Delta_H^-, \delta_H^*)$ be the induced embedding of twin buildings, and* $\mathrm{Opp}(\Delta_G) = \{(c,d) \in \Delta_G^+ \times \Delta_G^- \mid \delta_G^*(c,d) = 1\} \to \mathrm{Opp}(\Delta_H) = \{(c,d) \in \Delta_H^+ \times \Delta_H^- \mid \delta_H^*(c,d) = 1\}$ *the resulting embedding of opposite geometries. Given $(c_+, c_-) \in \mathrm{Opp}(\Delta_G)$, for all $n \in \mathbb{N}$ exists $m \in \mathbb{N}$ such that the intersection of $\mathrm{Opp}(\Delta_G)$ with the ball of radius n in $\mathrm{Opp}(\Delta_H)$ around (c_+, c_-) is contained in the ball of radius m in $\mathrm{Opp}(\Delta_H)$ around (c_+, c_-).*

Corollary B.7. *Let G be a topological Kac–Moody group endowed with the Kac–Peterson topology. If it is two-spherical or symmetrizable, then the multiplication map $\varphi : U^+ \times T \times U^- \to G$ is a homeomorphism onto its image.*

Proof. The two-spherical case is [14, Prop. 7.31]. In the symmetrizable case, note that Proposition B.1 is applicable since the Kac–Peterson topology is k_ω by [14, Prop. 7.10]. Consequently, the injection from Proposition B.5 yields a topological embedding $\iota : G \to H$, provided one can find k_ω-decompositions $G = \bigcup_n G_n$ and $H = \bigcup_m H_m$ such that each intersection $H_m \cap \iota(G)$ lies in some $\iota(G_n)$. (Indeed, $\iota^{-1}(H_m)$ is closed by continuity of ι, so it is compact once it lies inside some compact set G_n, which is equivalent to $H_m \cap \iota(G) \subset \iota(G_n)$.)

For G and H, choose k_ω-decompositions making use of Corollary B.4 and k_ω-decompositions of the fundamental subgroups $G_{\alpha_i} \cong \mathrm{SL}_2(\mathbb{R})$ of G and the corresponding subgroups $\prod_{j=1}^{n_i} H_{\alpha_{i,j}} \cong \mathrm{SL}_2(\mathbb{R})^{n_i}$ of H into which the G_{α_i} embed diagonally, endowed with their Lie group topology. That is,

$$X_1 := X_1^1, \quad X_2 := X_1^1 X_2^2, \quad X_3 := X_1^1 X_2^2 X_3^3, \; \ldots, \; X_t := X_1^1 \cdots X_t^t, \; \ldots$$

where each of the X_t^t is the ball of radius t around 1 of the maximal bounded subgroup X_t endowed with some suitable metric inducing its Lie group topology, with $X \in \{G, H\}$ and lower index t taken modulo the total number of maximal bounded subgroups.

By construction, each H_j^j intersects $\iota(G)$ in some compact subset of a fundamental subgroup G_{α_i} of G with respect to the Lie group topology. In other words, each $H_j^j \cap \iota(G)$ lies in some $\iota(G_k^k)$. Forming finite products of such sets and using Proposition B.6 one concludes that $H_t \cap \iota(G) = (H_1^1 \cdots H_t^t) \cap \iota(G)$ lies in some suitable product $G_t = G_1^1 \cdots G_t^t$; that is, the injective homomorphism $\iota : G \to H$ indeed is a topological embedding.

[3] A manuscript that has never been published and unfortunately seems to be lost. To the second author's dismay he has lost his copy that he once owned.

Since ι restricts to maps $\iota|_{U_+^G} : U_+^G \to U_+^H$, $\iota|_{U_-^G} : U_-^G \to U_-^H$, $\iota|_{T^G} : T^G \to T^H$, one can conclude that the diagram

$$\begin{array}{ccc} U_+^G \times T^G \times U_-^G & \xrightarrow{\varphi^G} & G \\ {\scriptstyle \iota|_{U_+^G} \times \iota|_{T^G} \times \iota|_{U_-^G}}\downarrow & & \downarrow{\scriptstyle \iota} \\ U_+^H \times T^H \times U_-^H & \xrightarrow{\varphi^H} & H \end{array}$$

commutes, which proves that the map φ^G is a homeomorphism onto its image, since φ^H is a homeomorphism onto its image by [14, Prop. 7.31]. □

Corollary B.8. *Let G be a topological Kac–Moody group endowed with the Kac–Peterson topology. If it is two-spherical or symmetrizable, then the associated twin building with the quotient topology is a strong topological twin building.*

Proof. The two-spherical case is [14, Thm. 1]. In the symmetrizable case it follows by replacing [14, Prop. 7.31] with Corollary B.7; cf. the discussion after [14, Thm. 1]. □

Corollary B.9. *Let G be a topological Kac–Moody group endowed with the Kac–Peterson topology. If it is two-spherical or symmetrizable, then the Bruhat decomposition of a symmetrizable Kac–Moody group is a CW decomposition.*

Proof. This is a restatement of Proposition 3.7 from the main text. Its proof heavily relies on Corollary B.8. □

Corollary B.10. *Let G be a topological Kac–Moody group endowed with the Kac–Peterson topology. If it is two-spherical or symmetrizable, then the coset model, the group model, and the involution model of the reduced Kac–Moody symmetric space are pairwise homeomorphic with respect to their internal topologies.*

Proof. The two-spherical case is [8, Prop. 4.19]. In the symmetrizable case, it follows from [8, Prop. 4.19] and Corollary B.7. □

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