JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

Fundamental Groups of Split Real Kac–Moody Groups

Dissertation

zur Erlangung des akademischen Grades
Doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Fachbereich 07
der Justus-Liebig-Universität Gießen

von

Frau Paula Katrin Harring, M.Sc., geb. Müller

Betreuer: Prof. Dr. Ralf Köhl

Eingereicht am 21. Januar 2020

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor Ralf Köhl: for his invaluable advice and reassuring guidance, his tremendously helpful ideas and motivating enthusiasm, and for his ceaseless patience, support and encouragement when the going was tough. I would not be where I am today without him. Thank you so much, Ralf – for everything.

I would like to thank Tobias Hartnick for his helpful comments on the article that preceded this thesis and Stefan Witzel for his valuable ideas regarding the topological specifics in Chapter 3. Thanks to all members of the algebra research group, in particular to Bastian Christ and Sebastian Bischof for the good cooperation in our teaching assistance responsibilities, and to my fellow students Julius Grüning, Robin Lautenbacher, Nada Saleh and Christian Vock for being great colleagues.

My sincere thanks go to Finn, my loving husband, loyal friend and reliable partner whose support and care for our family have made this thesis possible. I would like to thank my marvellous children Matilda and Emil who brighten my life and are a constant motivation to improve myself. My special thanks go to my parents, for their unconditional love and faith in me and for being there whenever I need them. I am also grateful to my grandparents for their love and support.

Finally, I would like to acknowledge the DFG for the funding of my research project via the grant KO4323/11, "Generalized Real Flag Varieties and Buildings".

Contents

Introduction

The structure of maximal compact subgroups in semisimple Lie groups was investigated by Cartan and, later, Mostow: In [Mos49], Mostow gives a new proof of a Cartan's theorem stating that a connected semisimple Lie group G is a topological product of a maximal compact subgroup K and a Euclidean space, implying in particular that G and K have isomorphic fundamental groups. Subsequent case-by-case analysis provided the isomorphism types of these maximal compact subgroups and their fundamental groups; tables can be found in [Hel78, p 518] and [SBG$^+$95, 94.33].

Starting in the 1940's, Dynkin diagrams, introduced in [Dyn46], have been used to describe the structure of simple Lie groups. Dynkin diagrams correspond to Cartan matrices, and a generalization of this concept led to the theory of Kac–Moody algebras and, in particular, their associated Kac–Moody groups, developed by Kac in [Kac85] and Tits in [Tit87]. Kac–Moody groups endowed with the Kac–Peterson topology have been extensively investigated by Köhl and Hartnick, together with Glöckner in [GGH10] and with Mars in [HKM13].

The aim of this thesis is to determine the fundamental group of any algebraically simply-connected semisimple split real topological Kac–Moody group associated to a symmetrizable generalized Cartan matrix. We present a uniform result which makes it possible to determine the fundamental group of such a group – and, in particular, of any algebraically simply connected split real simple Lie group – directly from its Dynkin diagram.

In order to fix notation so as to state the three main theorems of this thesis, let A be a symmetrizable irreducible generalized Cartan matrix A, Π its Dynkin diagram and $G_{\mathbb{R}}(A)$ the algebraically simply-connected split real semisimple Kac–Moody group associated to A, endowed with the Kac–Peterson topology (for definitions, see Sections 3.1 - 3.3).

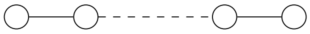
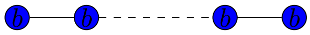
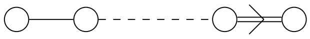
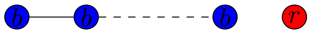
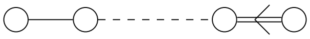
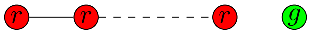
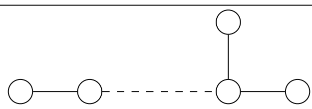
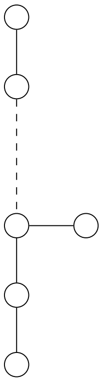
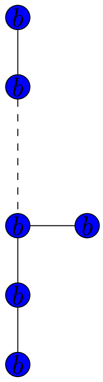
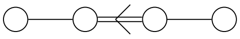
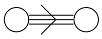
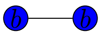
Given the Dynkin diagram $\Pi = (V, E)$ with a fixed labelling $\lambda : \{1, \ldots, n\} =: I \to V$, we define a modified diagram Π^{adm} with vertex set V and $\{i^\lambda, j^\lambda\} \in V \times V$ edge if and only if $\varepsilon(i,j) = \varepsilon(j,i) = -1$, where $\varepsilon(i,j)$ denotes the parity of the corresponding Cartan matrix entry. To each connected component $\overline{\Pi}^{\text{adm}}$ of Π^{adm} we then assign a colour as follows: Let $\overline{\Pi}^{\text{adm}}$ be coloured red (denoted by r) if it contains a vertex i^λ such that there exists a vertex $j^\lambda \in V$ satisfying $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$. Let $\overline{\Pi}^{\text{adm}}$ be coloured green (g) if it consists only of an isolated vertex, and blue (b) else.

One can then read off the isomorphism type of $\pi_1(G_{\mathbb{R}}(A))$ from the coloured diagram Π^{adm} as specified in the following theorem.

Theorem. *Let A be a symmetrizable irreducible generalized Cartan matrix A with Dynkin diagram Π. Let $n(g)$ and $n(b)$ be the number of connected components of Π^{adm} of colour g and b, respectively. Then*

$$\pi_1(G_{\mathbb{R}}(A)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b)}.$$

Example: Isomorphism types of $\pi_1(G_{\mathbb{R}}(A))$ for the spherical Dynkin diagrams [1].

Π	Π^{adm} coloured by γ	$\pi_1(G(\Pi))$
A_1 		$\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$
A_n 		$\pi_1(\mathrm{SL}_{n+1}(\mathbb{R})) \cong C_2 \quad (n \geq 2)$
B_n 		$\pi_1(\mathrm{Spin}(n, n+1)) \cong \begin{cases} \mathbb{Z} & \text{if } n \leq 2, \\ C_2 & \text{if } n > 2. \end{cases}$
C_n 		$\pi_1(\mathrm{Sp}(2n, \mathbb{R})) \cong \mathbb{Z}$
D_n 		$\pi_1(\mathrm{Spin}(n, n)) \cong C_2 \quad (n \geq 3)$
E_n $6 \leq n \leq 8$ 		$\pi_1(E_n) \cong C_2$
F_4 	F_4 	$\pi_1(F_4) \cong C_2$
G_2 		$\pi_1(G_{2,2}) \cong C_2$

A key to the proof is the computation of the fundamental groups of generalized flag varieties – that is, spaces of the form G/P_J for a parabolic subgroup P_J of G corresponding to an index subset $J \subseteq I$. We prove the following theorem:

Theorem. *Let $G = G_{\mathbb{R}}(A)$ for a symmetrizable irreducible generalized Cartan matrix A or let G be two-spherical. Then a presentation of $\pi_1(G/P_J)$ is given by*

$$\left\langle x_i; \quad i \in I \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i, \quad x_k = 1; \quad i, j \in I, k \in J \right\rangle.$$

We refer to [Wig98] for the analogous result in the finite-dimensional situation.

Furthermore, in [GHKW17, Section 16], the group $\mathrm{Spin}(\Pi, \kappa)$ – where κ denotes a so-called **admissible colouring** of the vertices of the Dynkin diagram Π – is defined as the canonical universal enveloping group of a Spin(2)-amalgam $\mathcal{A}(\Pi, \mathrm{Spin}(2)) = \{\tilde{G}_{ij}, \tilde{\phi}_{ij}^i \mid i \neq j \in I\}$ where the isomorphism type of $\tilde{G}_{ij}$ depends on the (i,j)- and (j,i)-entries of the Cartan matrix A as well as the values of κ on the corresponding vertices.

The fundamental group of $\mathrm{Spin}(\Pi, \kappa)$ is determined in a very similar way to the computation of

[1] Dynkin diagram LaTeX styles kindly provided by Max Horn at [Hor]

$\pi_1(G_{\mathbb{R}}(A))$, establishing the following theorem:

Theorem. *Let A be a symmetrizable irreducible generalized Cartan matrix A with Dynkin diagram Π. Let $n(g)$ be the number of connected components of Π^{adm} of colour g. Let $n(b, \kappa)$ be the number of connected components of Π^{adm} on which κ takes the value 1 and which have colour b. Then*

$$\pi_1(\mathrm{Spin}(\Pi, \kappa)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b,\kappa)}.$$

Overview

In Chapter 1, we collect topological results from various areas (and various sources) which will be of use later on. As we will use the CW structure of the generalized flag variety G/P_J to determine its fundamental group, we introduce CW complexes and state a result concerning the presentations of their fundamental groups. We provide some useful facts on mappings between topological spaces and also topological groups, such as the Open Mapping Theorem for locally compact groups, before giving a brief introduction to fibre bundles and stating a result based on works by Palais in [Pal61] and Hewitt/Ross in [HR63] on the existence of a long homotopy exact sequence for locally compact Lie subgroups of Hausdorff topological groups which we will use later to compute the fundamental group of G in the simply laced case. We introduce the k_ω property for topological spaces, as we will establish later that the Kac–Peterson topology is k_ω, and give some results on k_ω spaces by Glöckner/Gramlich/Hartnick from [GGH10]. Then the compact-open topology is defined, which will serve as a tool to prove that spherical subgroups carry the Lie topology. Finally, we turn to central extensions of topological groups and their realizations via 2-cocycles, citing a result by Neeb from [Nee02] which will be a helpful tool providing us with lifting arguments for the study of central quotients of Kac–Moody groups.

In Chapter 2, closely following Abramenko/Brown [AB08], we give an introduction to the theory of buildings, as the study of the buildings associated to Kac–Moody groups will provide us with much insight into the structure of these groups. We first define finite root systems before moving on to Coxeter groups and their associated root systems, generalized Cartan matrices, and Dynkin diagrams. Then we introduce buildings, twin buildings and their terminology via the W-metric space approach. We define topological twin buildings and strong topological twin buildings as these concepts are used by Hartnick/Köhl/Mars in [HKM13] for the investigation of Kac–Moody groups. The fact that the twin buildings associated to Kac–Moody groups are strong topological twin buildings gives rise to the CW decomposition of these buildings, a connection that will be of importance later on. Finally, we introduce BN-pairs and RGD systems, again following [AB08] and also Caprace/Remy [CR09].

In Chapter 3, we give an introduction to Kac–Moody groups and the Kac–Peterson topology. For Kac–Moody groups, we follow the functor definition by Tits from [Tit87]. We define the group that will be our main interest – the algebraically simply connected semisimple split real Kac–Moody group $G := G_{\mathbb{R}}(A)$ associated to a symmetrizable irreducible generalized Cartan matrix A, and state a result from [HKM13] that associates an RGD system with a Kac–Moody group which will enable us to apply methods from the theory of buildings to the study of Kac–Moody groups. We then define the Kac–Peterson topology τ_{KP}, thereby closely following [HKM13]. In establishing the topological properties of G, we deviate from this source in some places, as several of its proofs turned out to be somewhat lacking in accuracy. We first establish for the case of τ_{KP} being Hausdorff the properties of our main interest – such as the fact that τ_{KP} is a k_ω group topology, the universality of the topology with respect to the embeddings of $\mathrm{SL}_2(\mathbb{R})$ as fundamental rank one subgroups, and the fact that spherical subgroups carry the Lie topology. Then we prove that the Kac–Peterson topology on the adjoint group – that is, $G/Z(G)$ – is in fact Hausdorff, so that it carries all these properties. This information on the adjoint group is then used in combination with the tools from central extension theory to finally prove that the Kac–Peterson topology on the group G is Hausdorff as well and therefore has the desired properties. Finally, we introduce the subgroup K of G, the fixed point set of the Cartan-Chevalley involution, and give a result by Hartnick/Köhl published in their appendix to [HK19] stating that the fundamental groups of K and G are isomorphic which will allow us to reduce the computation of the fundamental group of G to that of K.

Chapter 4 contains mainly original results by the author and Ralf Köhl that were first published in [HK19]. Here, we set about the task of determining the fundamental group of G. We introduce the generalized flag variety G/P_J for a standard parabolic subgroup P_J of G and take note of the fact that the Bruhat decomposition of G/P_J is a CW decomposition since by results of Grüning/Köhl in an appendix to [HK19], the twin building associated to G is a strong topological twin building, another consequence of G being Hausdorff. The CW decomposition of topological twin buildings was first investigated by Kramer in [Kra01], then further developed in [HKM13]. We determine characteristic maps for the 1- and 2-cells of the CW complex and, employing a classical result from the theory of CW complexes, use these maps to give a presentation for the fundamental group of G/P_J. It turns out that all 2-cells are homeomorphic to either the Klein bottle or the torus $S^1 \times S^1$. Since it is our goal to determine the fundamental group of K, we state some results which relate the generalized flag variety G/P_J to quotient spaces of K. After these preparations, we first tackle the simply laced case, where the group G has a 2-fold central extension $\mathrm{Spin}(\Pi)$, as shown by Ghatei/Horn/Köhl/Weiss in [GHKW17]. By using our knowledge about the fundamental group of the generalized flag variety and a covering argument we prove that $\pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij}) \cong \pi_1(K/K_{ij}) \cong \{1\}$ for an A_2 subdiagram Π_{ij}. Then, via a homotopy exact sequence, we are able to establish that $\mathrm{Spin}(\Pi)$ is simply connected which yields that $\pi_1(G) \cong \pi_1(K) \cong C_2$. Finally, we move on to the general case. We introduce a modified Dynkin diagram Π^{adm} and a colouring γ of its connected component that depends on the edge types of the Dynkin diagram Π. We then prove that the fundamental group of the building G/B decomposes as a product of subgroups corresponding to the connected components of Π^{adm} whose isomorphism types depend on the respective colours. Using a covering argument, this allows us to determine the fundamental group of G. Finally, we introduce a generalized version of the spin group from the simply laced case and determine its fundamental group which is done in much the same way as in the case of $\pi_1(G)$.

Deutsche Einleitung

Die Struktur maximal kompakter Untergruppen in halbeinfachen Lie-Gruppen wurde durch Cartan und später Mostow untersucht: In [Mos49] führt Mostow einen neuen Beweis für den Satz von Cartan, welcher besagt, dass eine zusammenhängende halbeinfache Lie-Gruppe G das topologische Produkt einer maximal kompakten Untergruppe K und eines euklidischen Raumes ist. Dies impliziert insbesondere, dass G und K isomorphe Fundamentalgruppen haben.

Später wurden durch Fallunterscheidung die Isomorphietypen dieser maximal kompakten Untergruppen und ihrer Fundamentalgruppen ermittelt; entsprechende Tabellen finden sich in [Hel78, p 518] und [SBG$^+$95, 94.33].

Seit den 1940er Jahren werden Dynkindiagramme, die in [Dyn46] eingeführt wurden, verwendet, um die Struktur einfacher Lie-Gruppen zu beschreiben. Dynkin-Diagramme kodieren die in Cartan-Matrizen enthaltenen Informationen und eine Verallgemeinerung dieses Konzeptes mündete in die Theorie der Kac–Moody-Algebren und insbesondere der dazugehörigen Kac–Moody-Gruppen, die durch Kac in [Kac85] und Tits in [Tit87] entwickelt wurde. Kac–Moody-Gruppen, mit der Kac–Peterson-Topologie ausgestattet sind, sind umfangreich untersucht worden von Köhl und Hartnick, zusammen mit Glöckner in [GGH10] und mit Mars in [HKM13].

Ziel der vorliegenden Arbeit ist es, die Fundamentalgruppe der algebraisch einfach zusammenhängenden halbeinfachen zerfallenden reellen topologischen Kac–Moody-Gruppe zu einer beliebigen symmetrisierbaren vereinfachten Cartan-Matrix zu bestimmen. Wir stellen ein einheitliches Resultat vor, das es ermöglicht, die Fundamentalgruppe einer solchen Gruppe – und, insbesondere, die Fundamentalgruppe jeder algebraisch einfach zusammenhängenden zusammenhängenden zerfallenden reellen einfachen Lie-Gruppe zu bestimmen – direkt anhand ihres Dynkin-Diagrammes.

Wir legen zunächst die Notation fest, um die drei Hauptresultate dieser Arbeit anführen zu können. Sei dazu A eine symmetrisierbare irreduzible verallgemeinerte Cartan-Matrix, Π ihr Dynkin-Diagramm und $G_{\mathbb{R}}(A)$ die zu A gehörige algebraisch einfach zusammenhängenden halbeinfache zerfallenden reellen Kac–Moody-Gruppe, ausgestattet mit der Kac–Peterson-Topologie (für Definitionen sei auf Abschnitte 3.1 - 3.3 verwiesen).

Zum Dynkin-Diagramm $\Pi = (V, E)$ mit einer festgelegten Nummerierungsabbildung

$$\lambda : \{1, \ldots, n\} =: I \to V$$

definieren wir ein modifiziertes Diagramm Π^{adm} mit Knotenmenge V und $\{i^\lambda, j^\lambda\} \in V \times V$ Kante genau dann, wenn $\varepsilon(i,j) = \varepsilon(j,i) = -1$, wobei $\varepsilon(i,j)$ die Parität des entsprechenden Eintrages der Cartan-Matrix bezeichnet. Jeder Zusammenhangskomponente $\overline{\Pi}^{\text{adm}}$ von Π^{adm} weisen wir dann wie folgt eine Farbe zu: Sei die Zusammenhangskomponente $\overline{\Pi}^{\text{adm}}$ rot (bezeichnet durch r), wenn sie einen Knoten i^λ so enthält, dass ein Knoten $j^\lambda \in V$ existiert mit $\varepsilon(i,j) = 1$ und $\varepsilon(j,i) = -1$. Sei $\overline{\Pi}^{\text{adm}}$ grün (g), wenn sie nur aus einem isolierten Knoten besteht und blau (b) sonst.

Man kan nun den Isomorphietyp von $\pi_1(G_{\mathbb{R}}(A))$ dem gefärbten Diagramm Π^{adm} entnehmen, wie der folgende Satz beschreibt.

Satz. *Sei A eine symmetrisierbare irreduzible verallgemeinerte Cartan-Matrix und Π ihr Dynkin-Diagramm. Seien $n(g)$ und $n(b)$ die jeweilige Anzahl der Zusammenhangskomponenten mit Farbe g bzw. b. Dann gilt*

$$\pi_1(G_{\mathbb{R}}(A)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b)}.$$

Ein integraler Bestandteil des Beweises ist die Berechnung der Fundamentalgruppen von verallgemeinerten Fahnenvarietäten – das heißt, von Räumen der Form G/P_J für eine parabolische Untergruppe P_J zur Indexmenge $J \subseteq I$. Wir beweisen den folgenden Satz:

Satz. *Sei $G = G_{\mathbb{R}}(A)$ für eine symmetrisierbare irreduzible verallgemeinerte Cartan-Matrix A oder sei G zwei-sphärisch. Dann ist eine Präsentierung von $\pi_1(G/P_J)$ gegeben durch*

$$\left\langle x_i; \quad i \in I \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i, \quad x_k = 1; \quad i,j \in I, k \in J \right\rangle.$$

Für das analoge Resultat im endlichdimensionalen Fall sei verwiesen an [Wig98].

Des Weiteren wird in [GHKW17, Section 16] die Gruppe $\mathrm{Spin}(\Pi, \kappa)$ definiert, wobei κ eine sogenannte zulässige Färbung der Knoten des Dynkin-Diagrammes Π bezeichnet. Hierbei handelt es sich um die kanonische universelle einhüllende Gruppe eines Spin(2)-Amalgams $\mathcal{A}(\Pi, \mathrm{Spin}(2)) = \{\tilde{G}_{ij}, \tilde{\phi}_{ij}^i \mid i \neq j \in I\}$, wobei der Isomorphietyp von $\tilde{G}_{ij}$ abhängig ist sowohl von den (i,j)- und (j,i)-Eintrgën der Cartan-Matrix A als auch von den Werten, die κ auf den entsprechenden Knoten annimmt.

Die Fundamentalgruppe von $\mathrm{Spin}(\Pi, \kappa)$ lässt sich in ganz ähnlicher Weise zu der von $G_{\mathbb{R}}(A)$ bestimmen, was zum folgenden Resultat führt.

Satz. *Sei A eine symmetrisierbare irreduzible verallgemeinerte Cartan-Matrix und Π ihr Dynkin-Diagramm. Sei $n(g)$ die Anzahl der Zusammenhangskomponenten von Π^{adm} mit Farbe g. Sei $n(b, \kappa)$ die Anzahl der Zusammenhangskomponenten von Π^{adm}, auf denen κ den Wert 1 annimmt und die Farbe b haben. Dann gilt*

$$\pi_1(\mathrm{Spin}(\Pi, \kappa)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b,\kappa)}.$$

Chapter 1

Topological Prerequisites

1.1 CW Complexes

Definition 1.1.1. Following [Rot88, Chapter 8], a **CW complex** is an ordered triple (X, E, χ), where X is a Hausdorff space, E is a family of cells in X, and $\chi = \{\chi_e \mid e \in E\}$ is a family of maps such that

(a) $X = \bigsqcup_{e \in E} e$.

(b) For $k \in \mathbb{N}$, let $X^{(k)} \subseteq X$ be the union of all cells of dimension $\leq k$. Then for each $(k+1)$-cell $e \in E$, the map $\chi_e : (D^{k+1}, S^k) \to (e \cup X^{(k)}, X^{(k)})$, is a **relative homeomorphism**, i.e., it is a continuous map and its restriction $D^k \setminus S^{k-1} \to e$ is a homeomorphism.

(c) If $e \in E$, then its closure $\operatorname{cl} e$ is contained in a finite union of cells in E.

(d) X has the weak topology determined by $\{\operatorname{cl} e \mid e \in E\}$, i.e., a subset A of X is closed if and only if $A \cap \operatorname{cl} e$ is closed in $\operatorname{cl} e$ for each $e \in E$.

For $k \in \mathbb{N}$, let Λ_k be an index set for the k-dimensional cells, so that $X^{(k)} \setminus X^{(k-1)} = \bigsqcup_{\lambda \in \Lambda_k} e_\lambda$ and set $\chi_\lambda := \chi_{e_\lambda}$. This map is called the **characteristic map** of e_λ. □

The following Lemma is a consequence of [Mas77, Chapter 7, Theorem 2.1].

Lemma 1.1.2. *Let X be a CW complex with only one 0-cell x_0. For each $\lambda \in \Lambda_2$, let $f_\lambda : [0,1] \to S^1$ be a loop whose homotopy class generates $\pi_1(S^1)$ and whose image $\gamma_\lambda := \chi_\lambda \circ f_\lambda$ under χ_λ is a loop in $X^{(1)}$ starting at x_0. Then*

$$\langle [\chi_\mu], \quad \mu \in \Lambda_1 \mid [\gamma_\lambda], \quad \lambda \in \Lambda_2 \rangle$$

is a presentation of $\pi_1(X, x_0)$, where the brackets denote the respective homotopy classes in $X^{(1)}$. □

1.2 Some Facts on Maps and Fibre Bundles

A topological space is **σ-compact** if it is the union of countably many compact subspaces. By [Str06, Remark before 7.19], every connected locally compact topological group is σ-compact.

Proposition 1.2.1 (Open Mapping Theorem for locally compact groups, [Str06, 6.19])**.** *Let $f : G \to H$ be a surjective, continuous homomorphism between locally compact topological groups. If G is σ-compact and H is Hausdorff, then f is open.* □

Lemma 1.2.2 [Bou98, I.9.4, Corollary 2]. *Let $\varphi : X \to Y$ be a continuous, bijective map between topological spaces. If X is compact and Y is Hausdorff, then φ is a homeomorphism.* □

Lemma 1.2.3. *Let $\varphi : X \to Y$ be a continuous, open, surjective map between Hausdorff topological spaces. If all fibers are finite and of constant cardinality, then φ is a covering map.*

Proof. Let $y \in Y$ and let $\varphi^{-1}(y) = \{x_1, \dots, x_k\} \subseteq X$. Since X is Hausdorff, for $i = 1, \dots, k$ there exist neighborhoods U_i of x_i with $U_i \cap U_j = \emptyset$ for $i \neq j$. Let $V := \bigcap_{i=1}^k \varphi(U_i)$. Then V is open since φ is open and $V \neq \emptyset$ since $y \in V$. The preimage $\varphi^{-1}(V)$ is a disjoint union of open sets $\tilde{U}_i := \varphi^{-1}(V) \cap U_i$ and each $\tilde{U}_i$ is mapped bijectively to V: Surjectivity is clear; for the injectivity let $y' \in V$. Then each $\tilde{U}_i$ contains a preimage of y'. Since all fibers have constant cardinality k, it follows that $|\tilde{U}_i \cap \varphi^{-1}(y')| = 1$. This proves the assertion. ■

Lemma 1.2.4. *Let G be a topological group and $H_1 \leq H_2$ subgroups of G and endow G/H_i with the quotient topology. Then the following hold:*

(a) The projection map $\pi : G \to G/H_1$ is continuous and open.

(b) The canonical map $\psi : G/H_1 \to G/H_2$ is continuous and open.

(c) G/H_1 is Hausdorff if and only if H is closed in G.

Proof. (a): Let $U \subseteq G$ open. Since π is a quotient map, it suffices to show that $\pi^{-1}(\pi(U))$ is open. But this is true since $\pi^{-1}(\pi(U)) = UH_1 = \bigcup_{h \in H_1} Uh$ is a union of translates of open sets.

(b): This follows directly from (a) and the commutative diagram

$$\begin{array}{ccc} G & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\varphi} & \\ G/H_1 & \xrightarrow{\psi} & G/H_2 \end{array}.$$

(c): This follows from [Bou98, III.2.5, Proposition 13]. ■

Definition and Remark 1.2.5. A **fibre bundle** is a tuple (E, B, p, F) where E, B and F are topological spaces (E the **total space**, B the **base space** and F the **fibre**) and $p : E \to B$ is a continuous surjective map satisfying the following condition: For each $x \in B$ there exist a neighborhood $U_x \subseteq B$ and a homeomorphism $\phi_x : p^{-1}(U_x) \to U_x \times F$ such that the following diagram commutes (where π_1 denotes projection onto the first factor):

$$\begin{array}{ccc} p^{-1}(U_x) & \xrightarrow{\phi_x} & U_x \times F \\ \downarrow{\scriptstyle p} & \swarrow_{\pi_1} & \\ U_x & & \end{array}$$

We denote the fibre bundle by $F \longrightarrow E \xrightarrow{p} B$.

By [Hat02, Proposition 4.48 and Theorem 4.41], a fibre bundle with a path-connected base space B admits a long exact sequence

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \{1\}.$$ □

Definition 1.2.6. Let G be a topological group. A **principal G-bundle** is a tuple (G, p, X) where X is a topological space that admits a free G-action such that the translation function from the induced equivalence relation to G defined by $(x, x.g) \mapsto g$ is continuous. □

Lemma 1.2.7 [Hus94, Example 4.2.4]**.** *Let H be a closed subgroup of a topological group G. Then $(G, p, G/H)$ is a principal H-bundle, where p denotes the canonical projection.* □

Proposition 1.2.8 [Pal61, Corollary in Section 4.1]**.** *Let H be a closed Lie subgroup of a topological group G. Then the principal H-bundle $(G, p, G/H)$ is a fibre bundle.* □

Proposition 1.2.9 [HR63, Theorem 5.11]**.** *Let G be a Hausdorff topological group and H a subgroup of G that is locally compact in its relative topology. Then H is closed.* □

Corollary 1.2.10. *Let G be a Hausdorff topological group and H a subgroup of G that is a locally compact Lie group in its relative topology. Then there exists a long exact sequence*

$$\cdots \longrightarrow \pi_n(H) \longrightarrow \pi_n(G) \longrightarrow \pi_n(G/H) \longrightarrow \pi_{n-1}(H) \longrightarrow \cdots \longrightarrow \pi_0(G) \longrightarrow \{1\}. \qquad \square$$

1.3 Final Topologies

The main reference for this section is [Bou04, III.5, III.6].

Definition 1.3.1. Let G be a set and $(f_i : X_i \to G)_{i \in I}$ a family of maps from certain topological spaces X_i to G. The **final topology** on G with repect to $(f_i)_{i \in I}$ is the finest topology on G making each map f_i continuous. If G is a group, then the **final group topology** on G is the finest group topology with this property. □

Definition 1.3.2. Let I be a set with a partial order $\leq$ satisfying the following property: For each $\alpha, \beta \in I$ there exists a $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A **direct system** with respect to I is a family $(E_\alpha, f_{\beta\alpha})_{\substack{\alpha,\beta \in I \\ \alpha \leq \beta}}$ of sets E_α and mappings $f_{\beta\alpha} : E_\alpha \to E_\beta$ such that the following hold:

(a) If $\alpha \leq \beta \leq \gamma$, then $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$.

(b) For each $\alpha \in I$, $f_{\alpha\alpha} = \mathrm{id}_{E_\alpha}$.

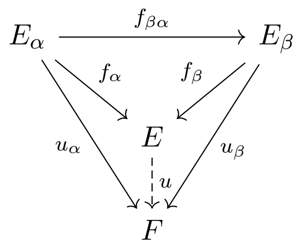
Let $\tilde{E}$ be the disjoint union of the sets E_α, and for each $x \in \tilde{E}$, let $\lambda(x) \in I$ be the unique index such that $x \in E_{\lambda(x)}$. Define an equivalence relation $\sim$ on $\tilde{E}$ by $x \sim y :\iff$ there exists a $\gamma \in I$ with $\gamma \geq \lambda(x) =: \alpha$ and $\gamma \geq \lambda(y) =: \beta$ such that $f_{\gamma\alpha}(x) = f_{\gamma\beta}(y)$.

The **direct limit** of the directed system is the set

$$E := \lim_{\rightarrow} E_\alpha := \tilde{E}/\sim .$$

If each E_α is equipped with a topology τ_α, then the **direct limit topology** on E with respect to the direct system $(E_\alpha, f_{\beta\alpha})$ is the final topology with repect to the family of canonical maps $f_\alpha : E_\alpha \to E$ sending each element to its equivalence class. □

By [Bou04, Proposition III.6.6], the direct limit E has the following universal property: For each set F that admits mappings $u_\alpha : E_\alpha \to F$ ($\alpha \in I$) such that $u_\beta \circ f_{\beta\alpha} = u_\alpha$ whenever $\alpha \leq \beta$, there exists a unique mapping $u : E \to F$ such that $u_\alpha = u \circ f_\alpha$. The following diagram illustrates this universal property:



We will be interested in direct limit topologies on groups arising from ascending sequences of topological spaces. In general, such a topology does not coincide with the corresponding final group topology, since the product $\lim_\rightarrow E_\alpha \times \lim_\rightarrow E_\alpha$ is in general not homeomorphic to $\lim_\rightarrow(E_\alpha \times E_\alpha)$. However, if the spaces involved are locally k_ω, then these limits can be exchanged, as Proposition 1.4.3 states.

1.4 k_ω-Spaces

Definition 1.4.1. Following [GGH10, Definition 4.1], a Hausdorff topological space X is a **k_ω-space** if there exists an ascending sequence of compact subsets $K_1 \subseteq K_2 \subseteq \cdots \subseteq X$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n for each $n \in \mathbb{N}$. A Hausdorff topological space is **locally** k_ω if each point has an open neighborhood which is k_ω in the induced topology. □

Proposition 1.4.2 [GGH10, Proposition 4.2]**.**

(a) σ-compact locally compact spaces are k_ω.

(b) Closed subsets of k_ω-spaces are k_ω in the induced topology.

(c) Finite products of k_ω-spaces are k_ω in the product topology.

(d) Hausdorff quotients of k_ω-spaces are k_ω.

(e) Countable disjoint unions of k_ω-spaces are k_ω.

(f) Every locally compact space is locally k_ω.

(g) Open subsets of locally k_ω spaces are locally k_ω.

(h) Closed subsets of locally k_ω-spaces are locally k_ω in the induced topology.

(i) Finite products of locally k_ω-spaces are locally k_ω in the product topology.

(j) Hausdorff quotients of locally k_ω-spaces are locally k_ω. □

Note that since every connected locally compact topological group is σ-compact, it is also k_ω.

Following [GGH10, description before Proposition 4.7], let $X_1 \subseteq X_2 \subseteq \ldots$ and $Y_1 \subseteq Y_2 \subseteq \ldots$ be ascending sequences of topological spaces with continuous inclusion maps, and let $X := \bigcup_{n\in\mathbb{N}} X_n = \lim_\rightarrow X_n$ and $Y := \bigcup_{n\in\mathbb{N}} Y_n = \lim_\rightarrow Y_n$, equipped with the respective direct limit topologies. Write $\lim_\rightarrow(X_n \times Y_n)$ for $\bigcup_{n\in\mathbb{N}}(X_n \times Y_n)$, equipped with the direct limit topology.

Proposition 1.4.3 [GGH10, Proposition 4.7]**.** *The natural map*

$$\beta : \lim_{\rightarrow}(X_n \times Y_n) \to (\lim_{\rightarrow} X_n) \times (\lim_{\rightarrow} Y_n) : (x, y) \mapsto (x, y)$$

is a continuous bijection. If each X_n and each Y_n is locally k_ω, then β is a homeomorphism. □

Proposition 1.4.4 [GGH10, Proposition 5.8]**.** *Let G be a group and $(f_i)_{i\in I}$ a countable family of maps $f_i : X_i \to G$ such that each X_i is a k_ω-space and $\bigcup_{i\in I} f_i(X_i)$ generates G. If the final group topology on G with respect to the family $(f_i)_{i\in I}$ is Hausdorff, then the topology is k_ω.* □

1.5 The Compact-Open Topology

We introduce the compact-open topology for sets of continuous functions. It will later be used as a tool to prove that spherical subgroups of Kac–Moody groups carry the Lie topology. The main reference for this section is [AGP02, 1.2], see also [Dug78, XII.1-2].

Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and denote by $M(X, Y)$ the set of continuous maps $X \to Y$.

Definition 1.5.1. A topology τ on $M(X, Y)$ is **admissible** if the evaluation map

$$e : M(X, Y) \times X \to Y : (f, x) \mapsto f(x)$$

is continuous with respect to the product topology $\tau \times \tau_X$. □

Definition 1.5.2. For two subsets $K \subseteq X$ and $U \subseteq Y$ let $U^K := \{f \in M(X, Y) \mid f(K) \subseteq U\}$. The **compact-open topology** τ_{co} on $M(X, Y)$ is the topology generated by the set

$$\{U^K \mid K \subseteq X \text{ compact}, U \subseteq Y \text{ open}\}.$$ □

Lemma 1.5.3. *Let Y be Hausdorff. Then τ_{co} is Hausdorff.*

Proof. Let $f \neq g \in M(X, Y)$ and let $x \in X$ with $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open neighborhoods U_f of $f(x)$ and U_g of $g(x)$. Then $U_f^{\{x\}}$ and $U_g^{\{x\}}$ are disjoint open neighborhoods of f and g, respectively. ■

Proposition 1.5.4 [AGP02, Proposition 1.2.2]**.** *Let τ be an admissible topology on $M(X, Y)$. Then $\tau_{co} \subseteq \tau$.* □

1.6 Central Extensions of Topological Groups

A **central extension** of a group G by a group Z is a group $\hat{G}$ with $Z \leq Z(\hat{G})$ such that there exists an epimorphism $\hat{G} \to G$ whose kernel is Z. Following [Nee02, Section 1], central extensions can be described using 2-cocycles: For a group Z and an abelian group G, a Z-valued **2-cocycle** is a map $G \times G \to Z$ satisfying $f(x, y)f(xy, z) = f(x, yz)f(y, z)$ and $f(1, x) = f(x, 1) = 1$ for all $x, y, z \in H$. Denote the group of Z-valued 2-cocycles by $Z^2(G, Z)$. For $f \in Z^2(G, Z)$ define the group

$$G \times_f Z := G \times Z \quad \text{where} \quad (g, z)(g', z') := (gg', zz' f(g, g')).$$

The group $G \times_f Z$ has neutral element $(1, 1)$ and inverse elements $(g, z)^{-1} = (g^{-1}, z^{-1} f(g, g^{-1})^{-1})$. The projection $G \times_f Z \to G$ is an epimorphism with kernel Z, so $G \times_f Z$ is a central extension of G by Z. It can be shown that every central extension can be realized in this way.

Proposition 1.6.1 [Nee02, Proposition 2.2]**.** *Let G and Z be topological groups where G is connected, and let $\hat{G}$ be a central extension of G by Z with quotient map $q : \hat{G} \to G$. Then the following are equivalent:*

(a) *$\hat{G}$ carries the structure of a topological group such that there exists an open 1-neighborhood $U \subseteq G$ and a continuous map $\sigma : U \to \hat{G}$ with $q \circ \sigma = \mathrm{id}_U$.*

(b) *The central extension can be described by a cocycle $f : G \times G \to Z$ which is continuous in a neighborhood of $(1, 1)$ in $G \times G$.* □

Corollary 1.6.2. *Let G, $\hat{G}$, Z and q be as in Proposition 1.6.1. If (b) holds and if Z is discrete, then the group topology in (a) turns q into a covering map.*

Proof. Since G is a topological group, for each $x \in G$ the set xU (with U as in (a) above) is an open neighborhood of x whose preimage $q^{-1}(xU)$ is homeomorphic to $Z \times U$. Since Z is discrete, it follows that q is a covering map. ■

Chapter 2

Buildings and Groups with RGD Systems

2.1 Finite Root Systems

We define finite root systems and Cartan matrices and Dynkin diagrams for crystallographic root systems. The main references for this section are [AB08, Appendix B] and [Hum78, Chapter III].

Let V be a Euclidean space with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$ and let $\alpha \in V$. Then the **reflection** s_α with respect to the hyperplane $\alpha^\perp$ is given by

$$s_\alpha(x) = x - \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

Define $\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

Definition 2.1.1. A subset $\Phi \subseteq V$ is called a **finite root system** if it satisfies the following conditions:

(a) Φ is finite, spans V and does not contain zero.

(b) If $\alpha \in \Phi$, the only scalar multiples of α in Φ are $\pm\alpha$.

(c) Φ is W_Φ-invariant where $W_\Phi := \langle s_\alpha \mid \alpha \in \Phi \rangle$.

If, in addition, $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$, then the finite root system Φ is called **crystallographic**. An element of Φ is called **root**. For a root α, the **length** $l(\alpha)$ of α is given by $l(\alpha) := \langle \alpha, \alpha \rangle^{1/2}$. The number $\dim V$ is called the **rank** of Φ. The group $W := W_\Phi$ is called the **Weyl group** of Φ. A finite root system is called **irreducible** if Φ cannot be partitioned into two proper orthogonal subsets. □

Fix a lexicographic ordering of V corresponding to an arbitrary ordered basis and consider an element $v \in V$ **positive** if $0 < \lambda$.

Definition 2.1.2. A subset $\Delta \subseteq \Phi$ is called a **simple system** if the following hold:

(a) Δ is a basis of V.

(b) Each $\alpha \in \Phi$ is a linear combination of Δ with all coefficients in $\mathbb{Z}$ and of the same sign.

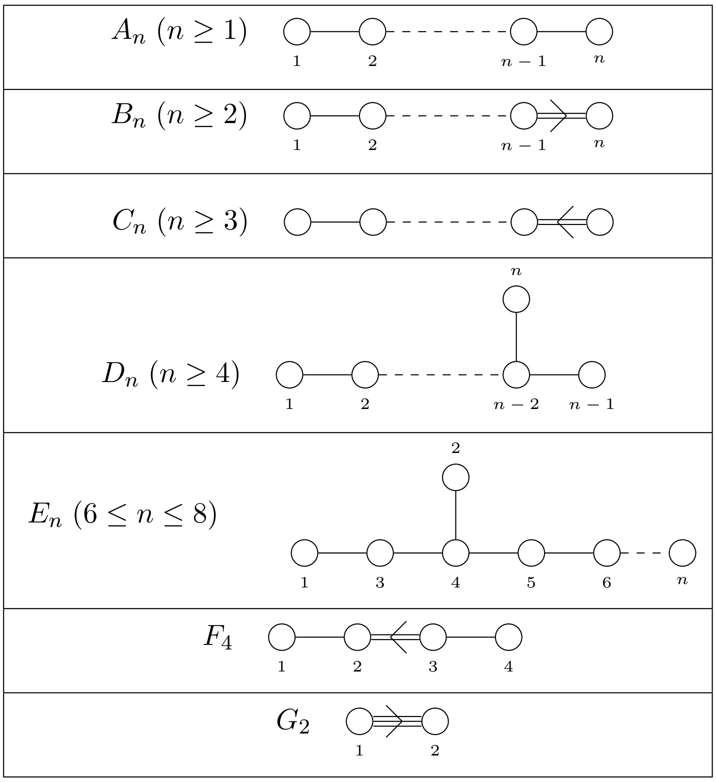
The elements of Δ are called **simple roots**. Simple systems always exists by [Hum78, 1.3, Theorem]. □

Definition 2.1.3. The **Cartan matrix** of a finite crystallographic root system Φ relative to a simple system Δ with a fixed ordering of elements $(\alpha_1, \ldots, \alpha_n)$ is given by $A = (a_{ij})_{1 \leq i,j \leq n}$ where $a_{ij} = \langle a_i^\vee, a_j \rangle$. The following hold (cf. [AB08, 8.11.1]):

- $a_{ii} = 2$ for all i.
- $a_{ij} \in \{0, -1, -2, -3\}$ for $i \neq j$.
- $a_{ij}a_{ji} \in \{0, 1, 2, 3\}$ for $i \neq j$.
- $a_{ij} = 0 \iff a_{ji} = 0$.

The **Dynkin diagram** of Φ is a graph with n labelled vertices where for $i \neq j$ the ith and jth vertex are joined by $a_{ij}a_{ji}$ edges with an arrow pointing to the vertex corresponding to the shorter root. □

Theorem 2.1.4 [Hum78, Theorem 11.4]**.** *If Φ is an irreducible crystallographic root system of rank 2, its Dynkin diagram is one of the following:*



□

Example 2.1.5. Let $V := \mathbb{R}^n$ with the standard bilinear form and let $\Phi := \{\alpha_{ij} := e_i - e_j \mid i,j = 1, \ldots, n,\ i \neq j\}$. Then Φ is a crystallographic root system. The reflection corresponding to α_{ij} is the transposition s_{ij} that interchanges the ith and jth coordinates. The Weyl group of Φ is the symmetric group on n letters. A simple system is given by $\{\alpha_i := \alpha_{i,i+1} \mid 1 \leq i \leq n+1\}$;

the corresponding Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

and the Dynkin diagram is of type A_{n-1}. □

2.2 The Root System Associated to a Coxeter Group

We introduce Coxeter groups, which are generalizations of finite reflection groups, and their associated root systems, generalized Cartan matrices, and Dynkin diagrams. The main references for this section are [Hum90, Chapter II.5] and [AB08], see also [CR09].

Definition 2.2.1. A **Coxeter group** is a group W with a presentation

$$\langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} \rangle,$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$; $m_{ij} = \infty$ is also permitted. A **Coxeter system** is a pair (W, S) where W is a Coxeter group and $S = \{s_1, \dots, s_n\}$ is a fixed set of generators for W. Let l_S be the **length function** that associates to each element the (unique) length of a corresponding reduced expression in S. Define the **strong Bruhat order** $\leq$ on W as follows: For $w_1, w_2 \in W$ let $w_1 \leq w_2$ if there exist reduced expressions $s_{i_1} \dots s_{i_{l_S(w_1)}}$ of w_1 and $s_{j_1} \dots s_{j_{l_S(w_2)}}$ such that the former is a (not necessarily consecutive) substring of the latter. □

Following [Hum90, II.5.3], every Coxeter group can be realized as a group of reflections of a real vector space V having a basis $\{\alpha_s \mid s \in S\}$ in one-to-one correspondence with S. This vector space is equipped with a symmetric bilinear form B defined by

$$B(\alpha_{s_i}, \alpha_{s_j}) := -\cos \frac{\pi}{m_{ij}}$$

with $B(\alpha_{s_i}, \alpha_{s_j}) := -1$ if $m_{ij} = \infty$. For each $s \in S$, a reflection $\sigma_s : V \to V$ is defined by

$$v \mapsto v - 2B(\alpha_s, v)\alpha_s.$$

By [CR09, Remark 1.2], this mapping extends to an injective homomorphism $\sigma : W \to \mathrm{GL}(V)$ and by [Hum90, II.5.3], the group $\sigma(W)$ preserves the form B on V. To simplify notation, we write $w(v)$ instead of $\sigma(w)(v)$ for $v \in V$.

Definition 2.2.2. Let (W, S) be a Coxeter system. The **root system** Φ of (W, S) is given by $\Phi := \{w(\alpha_s) \mid w \in W, s \in S\}$. The elements α_s are called **simple roots**. For each $\alpha \in \Phi$, there exists a unique expression

$$\alpha = \sum_{s \in S} c_s \alpha_s, \quad c_s \in \mathbb{R}.$$

Then α is **positive** (resp. **negative**) if all $c_s \geq 0$ (resp. all $c_s \leq 0$). Write Φ_+ (resp. Φ_-) for the set of positive (resp. negative) roots. By [CR09, Theorem 1.3(ii)], $\Phi = \Phi_+ \cup \Phi_-$.

For $\alpha = w(\alpha_s) \in \Phi$, the associated **reflection** $s_\alpha \in \mathrm{GL}(V)$ is given by $s_\alpha := \sigma(wsw^{-1})$. One has

$$s_\alpha(v) = v - 2B(v, \alpha)\alpha,$$

so s_α does not depend on the choice of w and s.

The notions of **length**, **rank**, and **irreducibility** are defined as for finite root systems in Definition 2.1.1. □

Note that if W is finite and if for each $\alpha \in \Phi$ the only scalar multiples of α in Φ are $\pm\alpha$, which can always be assumed without loss of generality, then Φ is a finite root system in the sense of Definition 2.1.1.

Example 2.2.3. Let Σ_n be the symmetric group on n letters and $S = \{s_1, \dots, s_{n-1}\}$ with $s_i = (i, i+1)$. Then $W = \langle S \mid s_i^2, (s_i s_{i+1})^3, (s_i s_j)^2; \quad |i-j| > 1\rangle$, so (Σ_n, S) is a Coxeter system. One has

$$B(\alpha_{s_i}, \alpha_{s_j}) = \begin{cases} 1, & j = i, \\ -1/2, & |j-i| = 1, \\ 0 & \text{else}, \end{cases}$$

so the associated root system is crystallographic and we obtain the same Cartan matrix as in Example 2.1.5 which shows that the root system given there is equivalent to the one constructed from the Coxeter system. □

Definition 2.2.4. A **generalized Cartan matrix** is a matrix $A = (a_{ij})_{1 \le i,j \le n} \in \mathbb{Z}^{n \times n}$ satisfying

- $a_{ii} = 2$,
- $a_{ij} \le 0$ if $i \ne j$,
- $a_{ij} = 0 \iff a_{ji} = 0$. □

To a generalized Cartan matrix $A = (a_{ij})_{1 \le i,j \le n}$ one associates a Coxeter system $(W(A) = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} \rangle, S)$ with $S = \{s_1, \dots, s_n\}$, where $m_{ii} = 1$ and m_{ij} is given by the following table (cf. [Kac85, Section 1]).

m_{ij}	2	3	4	6	∞
$a_{ij}a_{ji}$	0	1	2	3	≥ 4

The **Dynkin diagram** of a generalized Cartan matrix A is defined like the Dynkin diagram of a Cartan matrix of a finite root system with the additional requirement that for $a_{ij}a_{ji} \ge 4$, there is a single edge labelled by $|a_{ij}|, |a_{ji}|$ connecting the ith and jth vertex. The generalized Cartan matrix A is called **irreducible** if its Dynkin diagram is connected. A Dynkin diagram is called **simply laced** if $a_{ij}a_{ji} \in \{0, 1\}$ for all i, j. The **Coxeter diagram** of A is a graph with n labelled vertices where for $i \ne j$ the ith and jth vertex are joined by a single edge labelled by m_{ij}.

2.3 Buildings

We collect the necessary basics on buildings and twin buildings. This section follows [AB08, Chapter 5], see also [HKM13, Section 2.1]

Definition 2.3.1. Let (W, S) be a Coxeter system. Following [AB08, Definition 5.1], a **building** of type (W, S) is a pair $(\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$,whose elements are called chambers, together with a map $\delta\colon \mathcal{C} \times \mathcal{C} \to W$, called the **Weyl distance** function, such that for all $C, D \in \mathcal{C}$, the following conditions are satisfied:

(WD1) $\delta(C,D) = 1 \iff C = D$.

(WD2) If $\delta(C,D) = w$ and $C' \in \mathcal{C}$ satisfies $\delta(C',C) = s \in S$, then $\delta(C',D) = sw$ or w. If, in addition, $l_S(sw) = l_S(w) + 1$, then $\delta(C',D) = sw$.

(WD3) If $\delta(C,D) = w$, then for any $s \in S$ there is a chamber $C' \in \mathcal{C}$ such that $\delta(C',C) = s$ and $\delta(C',D) = sw$.

The Coxeter group W is called the **Weyl group** of $(\mathcal{C},\delta)$. For $I = \{1,\dots,n\}$ and $S = \{s_i \mid i \in I$ define W_J for a subset $J \subseteq I$ by $W_J := \langle s_i \mid i \in J \rangle$. □

Where the distance function does not need to be taken into explicit account, we will often omit it and simply denote by $\mathcal{C}$ the building $(\mathcal{C},\delta)$.

A building is called **spherical** if W is finite, and k-**spherical** if W_J is finite for each k-element subset $J \subseteq I$.

Given $s \in S$, two chambers $C, D \in \mathcal{C}$ are called s-**adjacent** if $\delta(C,D) = s$, and s-**equivalent** if $\delta(C,D) \in \{1,s\}$. s-equivalence is an equivalence relation whose equivalence classes are called s-**panels** and a **panel** is an s-panel for some $s \in S$. A building is called **thick** (resp **thin**) if each panel contains at least three (resp. exactly two) chambers. By (WD3), panels contain at least two chambers.

For $S = \{s_1,\dots,s_n\}$ and a subset $J \subseteq \{1,\dots,n\}$, two chambers $C, D \in \mathcal{C}$ are J-**equivalent** if $\delta(C,D) \in W_J$. One can show that J-equivalence is an equivalence relation; its equivalence classes are called J-**residues**. A J-residue is called **spherical** if W_J is finite. A **residue** is a subset $R \subseteq \mathcal{C}$ that is a J-residue for some $J \subseteq S$. For a chamber $C \in \mathcal{C}$, the J-residue containing C is denoted by $R_J(C)$. The collection of all J-residues in $\mathcal{C}$ is denoted by $\mathrm{Res}_J(\mathcal{C})$. The residues of co-rank one, i.e., the elements of $\mathcal{V}_s := \mathrm{Res}_{S\setminus\{s\}}(\mathcal{C})$ are called s-**vertices**. There is a canonical embedding

$$\iota : \mathcal{C} \hookrightarrow \prod_{s\in S} \mathcal{V}_s,$$
$$C \mapsto (R_{S\setminus\{s\}}(C))_{s\in S}$$

mapping a chamber onto the tuple consisting of its vertices.

Let (C,δ) and (C',δ') be two buildings of type (W,S) and (W',S'), respectively, and let $\mathcal{M} \subseteq \mathcal{C}$ and $\mathcal{M}' \subseteq \mathcal{C}'$. Let $\sigma : W \to W'$ be an isomorphism satisfying $\sigma(S) = S'$. A σ-**isometry** from $\mathcal{M}$ to $\mathcal{M}'$ is a map $\phi : \mathcal{M} \to \mathcal{M}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{M} & \xrightarrow{\delta} & W \\ \downarrow{\scriptstyle \phi\times\phi} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{M}' \times \mathcal{M}' & \xrightarrow{\delta'} & W' \end{array}$$

If $(W,S) = (W',S')$ and $\sigma = \mathrm{id}$, we call ϕ an **isometry** .

For a Coxeter system (W,S), define $\delta_W : W \times W \to W, (w_1,w_2) \mapsto w_1^{-1}w_2$. Then it is straightforward to check that (W,δ_W) is a thin building, the **standard thin building** of type (W,S).

Using the characterisation from [AB08, Corollary 5.67], an **apartment** of a building $(\mathcal{C},\delta)$ is a subset $\Sigma \subseteq \mathcal{C}$ that is isometric to W where W is taken to be the set of chambers of the standard thin building. By [AB08, Corollary 5.74], any two chambers are contained in a common apartment. A **system of apartments** is a collection Ω of apartments such that any two chambers are contained in a common apartment in Ω.

Definition 2.3.2. Let (W, S) be a Coxeter system. Following [AB08, Definition 5.133], a **twin building** of type (W, S) is a triple $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ consisting of two buildings $(\mathcal{C}_+, \delta_+)$ and $(\mathcal{C}_-, \delta_-)$ of type (W, S) with disjoint $\mathcal{C}_+$ and $\mathcal{C}_-$, together with a **codistance** function

$$\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$$

such that the following conditions are satisfied for each $\epsilon \in \{+, -\}$, any $C \in \mathcal{C}_\epsilon$, and any $D \in \mathcal{C}_{-\epsilon}$, where $w := \delta^*(C, D)$:

(Tw1) $\delta^*(C, D) = \delta^*(D, C)^{-1}$.

(Tw2) If $C' \in \mathcal{C}_\epsilon$ satisfies $\delta_\epsilon(C, C') = s$ with $s \in S$ and $l(sw) < l(w)$, then $\delta^*(C', D) = sw$.

(Tw3) For any $s \in S$, there exists a chamber $C' \in \mathcal{C}_\epsilon$ with $\delta_\epsilon(C', C) = s$ and $\delta^*(C', D) = sw$. □

Two chambers $C \in \mathcal{C}_\epsilon$ and $D \in \mathcal{C}_{-\epsilon}$ are **opposite**, noted C op D, if $\delta^*(C, D) = 1$. A **twin apartment** of a twin building $\mathcal{C}$ is a pair $\Sigma = (\Sigma_+, \Sigma_-)$ such that for $\epsilon \in \{+, -\}$, Σ_ϵ is an apartment of $\mathcal{C}_\epsilon$, and every chamber in $\Sigma_+ \cup \Sigma_-$ is opposite precisely one chamber in $\Sigma_+ \cup \Sigma_-$. The twin building is called **thick** (resp. **thin**), if each of the buildings $\mathcal{C}_+$ and $\mathcal{C}_-$ is thick (resp. thin).

By [AB08, Lemma 5.149] (see also [HKM13, Section 2.1]), if R is a spherical residue of $\mathcal{C}_\pm$ and C is a chamber in $\mathcal{C}_\mp$, then there exists a unique chamber $D \in R$ such that $\delta^*(C, D)$ is maximal in the set $\{\delta^*(C, E) \mid E \in R\}$. This chamber is called the **co-projection of** C **onto** R and is denoted by $\mathrm{proj}^*_R(C)$.

For $C \in \mathcal{C}_\pm$ and $w \in W$ define

$$E_w(C) := \{D \in C_\pm \mid \delta_\pm(C, D) = w\},$$
$$E^*_w(C) := \{D \in C_\mp \mid \delta^*(C, D) = w\},$$

The sets $E_{\leq w}(C)$ and $E^*_{\leq w}(C)$ are defined accordingly. Moreover, define

$$\mathcal{C}_w := \{(C, D) \in (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \mid \delta^*(C, D) = w\}.$$

2.4 Topological Twin Buildings

Following [HKM13, Section 3.1], let $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ be a thick twin building of type (W, S) such that the Coxeter diagram of (W, S) has no isolated vertices. Then a **topology** τ on $\mathcal{C}$ is a pair of topologies $\tau_\pm$ on $\mathcal{C}_\pm$. The set $\mathcal{C}_+ \cup \mathcal{C}_-$ is equipped with the direct sum topology.

Definition 2.4.1. [HKM13, Definition 3.1], let $\mathcal{C}$ be a thick twin building of type (W, S) without isolated vertices in the Coxeter diagram and τ a topology on $\mathcal{C}$. Then the pair $(\mathcal{C}, \tau)$ is called a **topological twin building** if it satisfies the following axioms:

(TTB1) τ is a Hausdorff topology.

(TTB2) For each $C \in \mathcal{C}_\pm$ and each $s \in S$ the map

$$E^*_1(C) \to \mathcal{C}_+ \cup \mathcal{C}_-$$
$$D \mapsto \mathrm{proj}^*_{R_{\{s\}}(C)}(D)$$

is continuous, where $R_{\{s\}}(C)$ denotes the s-panel containing C.

(TTB3) There exist chambers $C_\pm \in \mathcal{C}_\pm$ such that $\mathcal{C}_\pm$ is the direct limit $\lim_\rightarrow E_{\leq w}(C_\pm)$ and τ is the direct limit topology.

(TTB4) For each $s \in S$ there exists a compact s-panel in $\mathcal{C}_\pm$.

□

Definition 2.4.2. A topological twin building $\mathcal{C}$ of type (W, S) is called a **strong topological twin building** if it satisfies the following additional axioms:

(TTB1+) The vertex sets $\mathcal{V}_s^\pm, s \in S$, are Hausdorff.

(TTB2+) For each $s \in S$ the map

$$\begin{aligned} \mathcal{C}_1 &\to \mathcal{C}_+ \cup \mathcal{C}_- \\ (C, D) &\mapsto \operatorname{proj}^*_{R_{\{s\}}(C)}(D) \end{aligned}$$

is continuous.

(TTB5) The set $\mathcal{C}_1 = \{(C, D) \in (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \mid \delta^*(C, D) = 1\}$ of opposite chambers is open.

(TTB6) For every $s \in S$, the canonical map $\mathcal{C}_\pm \to \mathcal{V}_s^\pm$ is open.

□

Lemma 2.4.3 [HKM13, Proposition 3.20]**.** *Let $\mathcal{C}$ be a strong topological twin building $\mathcal{C}$ of type (W, S). Then the following hold:*

(TTB6+) *For every $J \subseteq S$ the canonical map $\mathcal{C}_\pm \to \operatorname{Res}_J^\pm$ is open.*

(TTB7) *The embedding*

$$\begin{aligned} \iota : \mathcal{C} &\hookrightarrow \prod_{s \in S} \mathcal{V}_s, \\ C &\mapsto (R_{S \setminus \{s\}}(C))_{s \in S} \end{aligned}$$

is open and therefore a homeomorphism onto its image.

2.5 BN-Pairs

This section follows [AB08, Sections 6.1, 6.2]. Let $(\mathcal{C}, \delta)$ be a building. A group G **acts** on (W, δ) if it acts on the set $\mathcal{C}$ and preserves Weyl distance, that is, $\delta(C, D) = \delta(g.C, g.D)$ for all $C, D \in \mathcal{C}, g \in G$. Let $\mathcal{A}$ be a set of apartments of $(\mathcal{C}, \delta)$. The G-action is **strongly transitive** (with respect to $\mathcal{A}$) if G acts transitively on the set of pairs (C, Σ) where C is a chamber and $\Sigma \in \mathcal{A}$ is an apartment containing C.

Definition 2.5.1. Following [AB08, Definition 6.55], a BN**-pair** is a pair of subgroups B and N of a group G such that B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S such that the following two conditions hold:

(BN1) For $s \in S$ and $w \in W$,

$$sBw \subseteq BswB \cup BwB.$$

(BN2) For $s \in S$,
$$sBs^{-1} \nleq B.$$

The group W is called the **Weyl group** associated to the BN-pair. The group B is called a **Borel subgroup** and the quadruple (G, B, N, S) a **Tits system**. □

Example 2.5.2. [AB08, 6.5] Let B be the group of upper triangular matrices in $\mathrm{SL}_n(\mathbb{R})$ and let N be the group of monomial matrices in $\mathrm{SL}_n(\mathbb{R})$. Then B and N form a BN-Pair for $\mathrm{SL}_n(\mathbb{R})$. The intersection $T := B \cap N$ is the diagonal subgroup of $\mathrm{SL}_n(\mathbb{R})$, and $W := N/T$ can be identified with the symmetric group on n letters. The set S can be chosen as the standard set of generators $\{s_1, \ldots, s_{n-1}\}$ where s_i is the transposition that interchanges i and $i+1$. □

Theorem 2.5.3 [AB08, Theorem 6.56 (1)]**.** *Given a BN-pair in G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\Delta = \Delta(G, B)$ that admits a strongly transitive G-action such that B is the stabilizer of a fixed chamber C and N stabilizes a fixed apartment Σ containing C and is transitive on its chambers.* □

The **Bruhat decomposition** of G is given by

$$G = \bigsqcup_{w \in W} BwB := B\tilde{w}B$$

where $\tilde{w}$ is a representative of w in N. The **standard parabolic subgroups** of G are the subgroups $P_J := BW_JB$ $(J \subseteq \{1, \ldots, n\})$ where $S = \{s_1, \ldots, s_n\}$ and W_J is the subgroup of W generated by the corresponding subset of S. These are precisely the subgroups of G containing B. They are self-normalizing and no two of them are conjugate.

The building $\Delta(G, B) = (\mathcal{C}, \delta)$ can be described as follows: Let $\mathcal{C} := G/B$ and define $\delta : G/B \times G/B \to W$ by
$$\delta(gB, hB) = w :\iff Bg^{-1}hB = BwB.$$

Then G acts by left translation, and N stabilizes the **fundamental apartment** $\Sigma = \{wB \mid w \in W\}$ and is transitive on its chambers while B stabilizes the **fundamental chamber** B.

Definition 2.5.4. For $w \in W$ and a chamber $gB \in G/B$ define

$$C_w(gB) := \{hB \in G/B \mid \delta(gB, hB) = w\},$$

$$C_{\leq w}(gB) := \bigcup_{v \leq w} C_v(gB)$$

and
$$C_{<w}(gB) := C_{\leq w}(gB) \setminus C_w(gB).$$

In particular, one has $C_w(B) = BwB/B$ and $C_{\leq s}(B) = B\langle \tilde{s} \rangle B/B$ for $s \in S$ with representative $\tilde{s} \in N$. The sets $C_{\leq s}(gB)$ are exactly the s-panels as defined above. □

Let $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-)$ be a twin building of type (W, S). Following [AB08, Definition 6.67], a group G **acts** on $\mathcal{C}$ if it acts simultaneously on the two sets $\mathcal{C}_+$ and $\mathcal{C}_-$ and preserves Weyl distance and codistance. The G-action is **strongly transitive** if G acts transitively on $\{(C, C') \in \mathcal{C}_+ \times \mathcal{C}_- \mid C \text{ op } C'\}$.

Definition 2.5.5. Following [AB08, Definition 6.78], let B_+, B_-, and N be subgroups of a group G such that $B_+ \cap N = B_- \cap N =: T \trianglelefteq N$. Let $W = N/T$. The triple (B_+, B_-, N) is called a **twin** BN**-pair** with Weyl group W if it admits a set S of generators such that the following conditions hold for all $w \in W$ and $s \in S$ and $\epsilon \in \{+,-\}$:

(TBN0) (G, B_ϵ, N, S) is a Tits system.

(TBN1) If $l(sw) < l(w)$, then $B_\epsilon s B_\epsilon w B_{-\epsilon} = B_\epsilon s w B_{-\epsilon}$.

(TBN2) $B_+ s \cap B_- = \emptyset$.

The quintuple (G, B_+, B_-, N, S) is called a **twin Tits system**. □

Proposition 2.5.6. *Following [AB08, page 332], for a twin Tits system* (G, B_+, B_-, N, S) *and* $\epsilon = \pm$, *let* $(\mathcal{C}_\epsilon, \delta_\epsilon)$ *be the building associated to the BN pair* (B_ϵ, N). *Define*

$$\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$$

by

$$\delta^*(gB_\epsilon, hB_{-\epsilon}) = w :\Longleftrightarrow g^{-1}h \in B_\epsilon w B_{-\epsilon}.$$

Then $(\mathcal{C}_+, \mathcal{C}_-, \delta^*)$ *is a thick twin building.* □

2.6 RGD Systems

Let (W, S) be a Coxeter system with $S = \{s_1, \ldots, s_n\}$ and Φ the set of roots of (W, S) with positive (resp. negative) set of roots Φ_+ (resp. Φ_-). Following [CR09, Section 1.2.2], for a set of roots $\Psi \subseteq \Phi$ define

$$W_\pm(\Psi) := \{w \in W \mid w.\alpha \in \Phi_\pm \text{ for each } \alpha \in \Psi\}.$$

Moreover, define

$$\overline{\Psi} = \{\alpha \in \Phi \mid W_+(\Psi) \subseteq W_+(\alpha) \text{ and } W_-(\Psi) \subseteq W_-(\alpha)\}.$$

The set Ψ is called **prenilpotent** if $W_+(\Psi)$ and $W_-(\Psi)$ are both nonempty.

For a pair $\{\alpha, \beta\} \in \Phi$ set

$$[\alpha, \beta] := \overline{\{\alpha, \beta\}} \quad \text{and} \quad (\alpha, \beta) := [\alpha, \beta] \setminus \{\lambda\alpha, \mu\beta \mid \lambda, \mu \in \mathbb{R}_+\}.$$

Definition 2.6.1. An **RGD system of type** (W, S) is a triple $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ consisting of a group G, a family of subgroups U_α and a subgroup T satisfying the following conditions:

(RGD0) For all $\alpha \in \Phi$, $U_\alpha \neq \{1\}$.

(RGD1) For all $\alpha \neq \beta$ in Φ such that $\{\alpha, \beta\}$,

$$[U_\alpha, U_\beta] \leq U_{(\alpha,\beta)}$$

.

(RGD2) For every $s_i \in S$ there is a function $m_i : U_{\alpha_i}^* = U_{\alpha_i} \setminus \{1\} \to G$ such that for all $u \in U_{\alpha_i}^*$ and $\alpha \in \Phi$,

$$m_i(s_i) \in U_{-\alpha_i} u U_{-\alpha_i} \quad \text{and} \quad m_i(u) U_\alpha m_i(u)^{-1} = U_{s_i\alpha}.$$

Moreover, $m_i(u)^{-1} m_i(v) \in T$ for all $u, v \in U_{\alpha_i}^*$.

(RGD3) For all $s_i \in S$,

$$U_{-\alpha_i} \nleq U_+,$$

where $U_\pm := \langle U_\alpha \mid \alpha \in \Phi_\pm \rangle$.

(RGD4) $G = T\langle U_\alpha \mid \alpha \in \Phi \rangle$.

(RGD5) T normalizes U_α for each $\alpha \in \Phi$, i.e.,

$$T \leq \bigcap_{\alpha\in\Phi} N_G(U_\alpha).$$

The groups U_α are called **root subgroups** and the groups $G_\alpha := \langle U_\alpha, U_{-\alpha} \rangle$ are called **rank one subgroups**. The rank one subgroups $G_i := G_{\alpha_i}$ $(s_i \in S)$ corresponding to simple roots are called **fundamental rank one subgroups** . □

Let

$$\begin{aligned} N &:= T.\left\langle m_i(u) \mid u \in U_{\alpha_i} \setminus \{1\}, s_i \in S \right\rangle, \\ B_+ &:= T.U_+, \\ B_- &:= T.U_-. \end{aligned}$$

Theorem 2.6.2 [AB08, Proposition 8.54 (1)]**.** *Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be an RGD system. Then (G, B_+, B_-, N, S) is a twin Tits system with Weyl group $N/(B \cap N) = N/T \cong W$.* □

Corollary 2.6.3 [AB08, Corollaries 8.55 and 8.78]**.**

- $N_G(U_\pm) = B_\pm$.
- $B_+ \cap B_- = T$. □

In the following, we will only be interested in the Tits system (G, B_+, N, S) of an RGD system. We will therefore often denote B_+ by B.

Example 2.6.4 [AB08, Example 7.133]. Let $G = \mathrm{SL}_n(\mathbb{R})$, let T be the set of diagonal matrices in G, let W be the symmetric group on n letters with standard set of generators $S = \{s_1, \dots, s_n\}$. Recall that the root system of (W, S) is given by $\Phi := \{\alpha_{ij} := e_i - e_j \mid i, j = 1, \dots, n,\ i \neq j\}$ as shown in Example 2.2.3. For $1 \leq i, j \leq n$ with $i \neq j$, let $U_{\alpha_{ij}} := \{E_{ij}(\lambda) \mid \lambda \in \mathbb{R}\}$ where E_{ij} is the elementary matrix with λ in position (i, j) and 1's on the diagonal. Then $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is an RGD system.

The maps m_i are defined as follows: For $u = E_{i,i+1}(\lambda) \in U_{\alpha_i}$, the element $m_i(u)$ is the image of $\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ under the mapping that embeds $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{SL}_n(\mathbb{R})$ as the subgroup that acts on the space $e_i\mathbb{R} + e_j\mathbb{R}$ and fixes e_k for $k \neq i, i+1$.

The resulting BN-pair is the one given in Example 2.5.2. □

Lemma 2.6.5 [AB08, Exercise 7.126]. *For $S = \{s_i \mid i \in I\}$ and any subset $S_J = \{s_j \mid j \in J \subseteq I\}$, let $\Phi_J := \{w\alpha_s \mid w \in W_J, s \in S_J\}$ and $G_J := T\langle U_\alpha \mid \alpha \in \Phi_J \rangle$. Then $(G_J, (U_\alpha)_{\alpha\in\Phi_J}, T)$ is an RGD system of type (W_J, S_J).* □

Chapter 3

Kac-Moody Theory

3.1 Definitions

We introduce Kac–Moody groups, closely following [HKM13, 7.1] and [Mar18, 7.3].

Definition 3.1.1. Let $I = \{1, \ldots, n\}$ and $A = (a_{ij})_{i,j \in I}$ a generalized Cartan matrix. A quintuple $\mathcal{D}= (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ is called a **Kac–Moody root datum** if Λ is a free $\mathbb{Z}$-module, each c_i is an element of Λ, and each h_i is an element of the dual $\Lambda^\vee$ of Λ such that one has $h_i(c_j) = a_{ij}$ for all $i, j \in I$. We call $\mathcal{D}$ **free** if the c_i are $\mathbb{Z}$-linearly independent in Λ and **cofree** if the h_i are $\mathbb{Z}$-linearly independent in $\Lambda^\vee$. We call $\mathcal{D}$ **adjoint** if the c_i span Λ and **coadjoint** if the h_i span $\Lambda^\vee$. To any Kac–Moody root datum $\mathcal{D}$ one can associate an adjoint Kac–Moody root datum $\mathrm{ad}(\mathcal{D})$ by replacing Λ with its sublattice Λ^{ad} generated by the c_i and restricting the elements h_i accordingly. □

Definition 3.1.2. By [Mar18, Example 7.11], given a generalised Cartan matrix $A = (a_{ij})_{i,j\in I}$ there exists a unique Kac–Moody root datum associated to A that is both cofree and coadjoint, i.e. such that $\Lambda^\vee = \oplus_{i\in I}\mathbb{Z}h_i$. It is denoted by $\mathcal{D}^A_{\mathrm{sc}}$ and called the **simply connected** root datum associated to A. □

Definition 3.1.3. Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac–Moody root datum. The **Kac–Moody algebra $\mathfrak{g}_\mathcal{D}$ of type $\mathcal{D}$** is the complex Lie algebra with generators $\mathfrak{h}_\mathcal{D} := \Lambda^\vee \oplus_\mathbb{Z} \mathbb{C}$ (the **Cartan subalgebra** of $\mathfrak{g}_\mathcal{D}$), and the symbols $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$, together with the following defining relations:

$$\begin{aligned}
[\mathfrak{h}_\mathcal{D}, \mathfrak{h}_\mathcal{D}] &= 0,\\
[h, e_i] &= \langle c_i, h\rangle e_i \quad \text{and} \quad [h, f_i] = -\langle c_i, h\rangle f_i \quad \text{for } h \in \mathfrak{h}_\mathcal{D} \text{ and } i \in I,\\
[e_i, f_i] &= -\delta_{ij} h_i \quad \text{for } i, j \in I,\\
(\mathrm{ad}_{e_i})^{1-a_{ij}}.e_j &= (\mathrm{ad}_{f_i})^{1-a_{ij}}.f_j = 0 \quad \text{for } i, j \in I \text{ with } i \neq j.
\end{aligned}$$

For a generalized Cartan matrix A, we define $\mathfrak{g}_A := \mathfrak{g}_{\mathcal{D}^A_{\mathrm{sc}}}$ and call it the **simply connected Kac–Moody algebra of type A over $\mathbb{C}$**.

It follows from the last relation that e_i and f_i are ad-**locally nilpotent** (or, equivalently, ad_{e_i} and ad_{f_i} are **locally nilpotent**), which means that for each $g \in \mathfrak{g}_\mathcal{D}$ there exists an $m \in \mathbb{N}$ such that $(\mathrm{ad}_{e_i})^m.g = (\mathrm{ad}_{f_i})^m.g = 0$. □

Definition 3.1.4. In [Tit87, 3.6], Tits associates with each Kac–Moody root datum $\mathcal{D}$ a triple

$(\mathcal{G}, \{\varphi_i\}_{i\in I}, \eta)$ where $\mathcal{G}$ is a group functor on the category of commutative unital rings, the φ_i are homomorphisms $\mathrm{SL}_2(R) \to \mathcal{G}(R)$, and η is a natural transformation $\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], -) \to \mathcal{G}$ satisfying the following conditions:

(KMG1) If $\mathbb{K}$ is a field, then $\mathcal{G}(\mathbb{K})$ is generated by the images of the φ_i and of $\eta(\mathbb{K})$.

(KMG2) For every commutative unital ring R, the homomorphism $\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], R) \to \mathcal{G}(R)$ is injective.

(KMG3) Given a commutative unital ring R, $i \in I$ and $r \in R^\times$, one has $\varphi_i \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \eta(\lambda \mapsto r^{h_i(\lambda)})$.

(KMG4) If $\iota : R \to \mathbb{K}$ is an injective homoomorphism from a commutative unital ring R to a field $\mathbb{K}$, then $\mathcal{G}(\iota) : \mathcal{G}(R) \to \mathcal{G}(\mathbb{K})$ is injective.

(KMG5) There exists a homomorphism $\mathrm{Ad} : \mathcal{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ whose kernel is contained in $\eta(\mathbb{C})(\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{C}))$, such that for $z \in \mathbb{C}$ and $i \in I$ one has

$$\mathrm{Ad}\left(\varphi_i \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\, z e_i,$$
$$\mathrm{Ad}\left(\varphi_i \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \exp \mathrm{ad}(-z f_i).$$

Furthermore, for every homomorphism $\gamma \in \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{C})$, one has

$$\mathrm{Ad}(\eta(\mathbb{C})(\gamma))(e_i) = \gamma(c_i) \cdot e_i, \quad \mathrm{Ad}(\eta(\mathbb{C})(\gamma))(f_i) = \gamma(-c_i) \cdot f_i.$$

For a given Kac–Moody root datum $\mathcal{D}$ the group $G_{\mathcal{D}}(R) := \mathcal{G}(R)$ is called a **split Kac–Moody group of type $\mathcal{D}$ over R**. For a generalized Cartan matrix A, the group $G_{\mathbb{R}}(A) := G_{\mathcal{D}^A_{\mathrm{sc}}}(\mathbb{R})$ is called the **algebraically simply-connected semisimple split real Kac–Moody group of type A**. Denote by Z the centre of $G_{\mathbb{R}}(A)$ and define the **adjoint Kac–Moody group of type A** by $\mathrm{Ad}(G_{\mathbb{R}}(A)) := G_{\mathbb{R}}(A)/Z$. □

It is established in [Tit87, Theorem 1] that on the category of fields, the functor $\mathcal{G}$ is completely characterised by the axioms above, up to some non-degeneracy assumptions concerning the maps φ_i.

Definition 3.1.5. Following [FHHK, 3D], the **extended Weyl group** $\widetilde{W} \leq G_{\mathcal{D}}(R)$ is defined by $\widetilde{W} := \langle \overline{s}_1, \ldots, \overline{s}_n \rangle$ where $\overline{s}_i := \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By [Kac90, Proposition 2.1], the mapping $\overline{s}_i \to s_i$ induces an epimorphism from $\widetilde{W}$ to W. □

3.2 The RGD System of a Kac–Moody Group

We state a result of Rémy, reformulated by Hartnick, Köhl and Mars, which gives an RGD system for a split Kac–Moody group over a field.

Following [HKM13, Proposition 7.2] (see also [Rém02, Proposition 8.4.1], [Cap09, Lemma 1.4]), let $\mathbb{K}$ be a field, let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac–Moody root datum, and let $G_{\mathcal{D}}(\mathbb{K})$ be the corresponding split Kac–Moody group of type $\mathcal{D}$ over $\mathbb{K}$. Then $G_{\mathcal{D}}(\mathbb{K})$ admits an RGD system as follows. Let (W, S) be the Coxeter system associated to A and let Φ be its set of roots. Given $i \in I$, let U_{α_i} and $U_{-\alpha_i}$ be the respective images of the subgroups of strictly upper, resp. strictly lower triangular matrices of the matrix group $\mathrm{SL}_2(\mathbb{K})$ under the map φ_i. For an arbitrary root $\alpha = w.\alpha_i \in \Phi$ with $w = s_{i_1} \dots s_{i_k}$ define $U_\alpha := \tilde{w} U_{\alpha_i} \tilde{w}^{-1}$ where $\tilde{w} := \tilde{s}_{i_1} \dots \tilde{s}_{i_k}$. Denote by T the image of $\eta(\mathbb{K})$ in $G_{\mathcal{D}}(\mathbb{K})$.

Proposition 3.2.1 [HKM13, Proposition 7.2]**.** *One has $T = \bigcap_{\alpha \in \Phi} N_{G_{\mathcal{D}}(\mathbb{K})}(U_\alpha)$ and $W \cong N_{G_{\mathcal{D}}(\mathbb{K})}(T)/T$. Moreover, $(G_{\mathcal{D}}(\mathbb{K}), (U_\alpha)_{\alpha \in \Phi}, T)$ is an RGD system.* □

The group $G_{\mathcal{D}}(\mathbb{K})$ is called **(k-)spherical** if the corresponding building is (k-)spherical, and **simply laced** if the Dynkin diagram of its generalized Cartan matrix is simply laced. It is called **symmetrizable** if its generalized Cartan matrix A is symmetrizable, that is, if there exists a non-degenerate diagonal matrix D such that DA is symmetric.

By [Car92, Section 6], the RGD system of the algebraically simply-connected semisimple split real Kac–Moody group $G_{\mathbb{R}}(A)$ is **centered**, that is, in (RGD4) one has the equality

$$G_{\mathbb{R}}(A) = \langle U_\alpha \mid \alpha \in \Phi \rangle.$$

Remark 3.2.2. Since the restriction to a subset of S and the corresponding root system yields again an RGD system by Lemma 2.6.5, the fundamental rank one subgroups $G_{\alpha_i} = \langle U_{\pm\alpha_i} \rangle$ admit an RGD system of type A_1 which implies that the φ_i are isomorphisms from $\mathrm{SL}_2(\mathbb{K})$ to G_{α_i}. □

Remark 3.2.3. The group T is called the **standard maximal torus** of $G_{\mathcal{D}}(\mathbb{K})$; it carries the structure of a Lie group. In the case of an algebraically simply-connected semisimple split real Kac–Moody group, it is generated by the images of the diagonal subgroup $T_0 \leq \mathrm{SL}_2(\mathbb{K})$ under the maps φ_i and is therefore isomorphic to $(\mathbb{K}^\times)^n$. In the case of $\mathbb{K} = \mathbb{R}$, approaching the torus via the theory of Kac–Moody algebras, it can be introduced as the intersection of $G_{\mathcal{D}}(\mathbb{R})$ with the exponential image of the Cartan subalgebra (which lives in $G_{\mathcal{D}}(\mathbb{C})$). □

Notation 3.2.4. From now on, G will always denote the group $G_{\mathbb{R}}(A)$ or a central quotient of this group, for a symmetrizable irreducible generalized Cartan matrix A, and Π the generalized Dynkin diagram of A.

Denote by $B := B_+$ the positive Borel subgroup of the twin BN-pair of G, by T the standard maximal torus and by W the Weyl group of G with generating set $S = \{s_i\}_{i \in I}$. For each $\sigma_i \in S$, take $\tilde{s}_i \in G$ to be a fixed representative for σ_i and recall that $Bs_iB := B\tilde{s}_iB$.

Unless specified more explicitly, the symbol J will always denote an arbitrary subset of the index set I, the symbol Π_J the subdiagram of Π corresponding to J, the symbol G_J the subgroup $G_{\mathbb{R}}(A_J)$ of G where A_J denotes the corresponding submatrix of A, and the symbols B_J and T_J the intersections $G_J \cap B$ and $G_J \cap T$, respectively. Recall that by Lemma 2.6.5, the tuple $(G_J, (U_\alpha)_{\alpha \in \Phi_J}, T)$ is an RGD system in its own right, and so is $(G_J, (U_\alpha)_{\alpha \in \Phi_J}, T_J)$, since $G_{\mathbb{R}}(A)$ is centered. The positive Borel subgroup of the latter RGD system is B_J.

For one-element subsets $\{i\} \subseteq I$ we use the convention $G_i := G_{\{i\}}$. This is consistent with the notation for the fundamental rank one subgroups: One has $G_{\mathbb{R}}(A_{\{i\}}) = G_i = G_{\alpha_i}$. □

3.3 The Kac–Peterson Topology

In this section, G denotes the group $G_{\mathbb{R}}(A)$, or a central quotient of this group, for a symmetrizable irreducible generalized Cartan matrix A.

We introduce the Kac–Peterson topology on the Kac–Moody group G, which will later turn out to coincide with the final group topology with respect to the maps φ_i. Rather than defining it to be the final group topology, it is constructed as a direct limit topology with respect to a system of products of the fundamental rank one subgroup. This makes it possible to establish certain properties, in particular the k_ω property. This section, as well as the following two, follow closely [HKM13, Section 7], see also [Mar11, 2.6,2.8,3.1]. However, some of the proofs from [HKM13] had to be revised which was done in collaboration with Ralf Köhl.

Let $\Delta := \{\alpha_i \mid i \in I\}$ be the set of simple roots of G. For a k-tuple $\overline{\beta} := (\beta_1, \dots, \beta_k) \in \Delta^k$ denote by $G_{\overline{\beta}} := G_{\beta_1} \dots G_{\beta_k}$ the subset of products of the form $g_1 \dots g_k$ where $g_j \in G_{\beta_j}$.

Note that on the set $L := \bigcup_{k \in \mathbb{N}} \Delta^k$, the subtuple relation defines a partial order $\leq$ and for each pair $\overline{\alpha} \leq \overline{\beta} \in L$, there is a canonical embedding $f_{\beta\alpha} : G_{\overline{\alpha}} \hookrightarrow G_{\overline{\beta}}$. Since G is centered, $G = \langle G_\alpha \mid \alpha \in \Phi \rangle$, so the direct limit of the direct system $(G_{\overline{\alpha}}, f_{\beta\alpha})_{\substack{\alpha,\beta \in L \\ \alpha \leq \beta}}$ coincides with the set underlying G.

Throughout this section, for any finite-dimensional real vector space V, we denote by $\mathcal{O}$ the canonical product topology on V induced from $\mathbb{R}$. Equip each fundamental rank one subgroup G_i with the corresponding topology induced by the isomorphism $\varphi_i : \mathrm{SL}_2(\mathbb{R}) \to G_i$ and denote this topology also by $\mathcal{O}$. Denote by $\tau_{\mathbb{R}}$ the Lie group topology on the torus T.

For each tuple of simple roots $\overline{\beta} = (\beta_1, \dots, \beta_k)$, denote by $\tau_{\overline{\beta}}$ the quotient topology on $TG_{\overline{\beta}}$ with respect to the surjective product map

$$p_{\overline{\beta}} : (T, \tau_{\mathbb{R}}) \times (G_{\beta_1}, \mathcal{O}) \times \dots \times (G_{\beta_k}, \mathcal{O}) \to TG_{\overline{\beta}}. \tag{3.1}$$

Definition 3.3.1. The **Kac–Peterson topology** τ_{KP} on the set G is the direct limit topology with respect to the direct system $(TG_{\overline{\alpha}}, f_{\beta\alpha})$ where each $TG_{\overline{\alpha}}$ is equipped with the topology $\tau_{\overline{\alpha}}$. □

We first make an observation on the structure of the standard maximal torus T of $G_{\mathbb{R}}(A)$. Since T is isomorphic to $(\mathbb{R}^\times)^n$ by Remark 3.2.3, it contains a unique maximal finite subgroup M of order 2^n. Denote by A the connected component of T with respect to the Lie topology.

Lemma 3.3.2 [FHHK, Proposition 3.17]**.** *Let $G = G_{\mathbb{R}}(A)$. Then multiplication induces a group isomorphism $M \times A \to T$. If the Kac–Peterson topology is a Hausdorff group topology, then A is also the connected component of T with respect to the Kac–Peterson topology and this map is an isomorphism of topological groups with respect to the Kac–Peterson topology.* □

To establish – first for the Hausdorff case in the following section and eventually for the general case – that the Kac–Peterson topology is actually a group topology, we will prove that it is k_ω on the subspaces $TG_{\overline{\alpha}}$ in order to employ Proposition 1.4.3.

To this end, we define an algebra $\mathcal{U}$ associated to G that admits a certain adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U})$ which for each $v \in \mathcal{U}$ admits a continous orbit map $TG_{\overline{\alpha}} \to V_{\overline{\alpha}}^v$ where $V_{\overline{\alpha}}^v$ is a vector space with a Hausdorff topology. The existence of this map establishes that $TG_{\overline{\alpha}}$ is Hausdorff and, consequently, k_ω.

Following the construction due to [Tit87], as exhibited in [HKM13, 7.2], let $\mathfrak{g}$ be the complex Kac–Moody algebra associated to the generalized Cartan matrix A, and let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra, that is, $\mathcal{U}(\mathfrak{g}) = \left(\bigotimes_{n \in \mathbb{N}} \mathfrak{g}^n\right) / I$ where $I := \langle [x, y] - xy + yx \rangle$.

For each $u \in \mathcal{U}(\mathfrak{g})$, let $u^{[n]} := (n!)^{-1} u^n$ and

$$\binom{u}{n} := (n!)^{-1} u(u-1) \dots (u-n+1).$$

Let $Q := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha$ be the free abelian group generated by the simple roots. As in [Rém02, 7.3.1], the algebras $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ admit an abstract Q-grading by declaring e_i of degree α_i and f_i of degree $-\alpha_i$ and extending linearly to $\mathcal{U}(\mathfrak{g})$, so that the elements of $\mathfrak{h}$ are of degree 0.

Let $\mathcal{U}_0(\mathfrak{g})$ be the subring of $\mathcal{U}(\mathfrak{g})$ generated by the degree-0-elements of the form $\binom{h}{n}$ where $h \in \mathfrak{h}$, and let $\mathcal{U}_{\alpha_i}(\mathfrak{g}) := \sum_{n\in\mathbb{N}} \mathbb{Z}e_i^{[n]}$ and $\mathcal{U}_{-\alpha_i}(\mathfrak{g}) := \sum_{n\in\mathbb{N}} \mathbb{Z}f_i^{[n]}$. Define $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$ to be the subring of $\mathcal{U}(\mathfrak{g})$ generated by $\mathcal{U}_0(\mathfrak{g})$ and $\{\mathcal{U}_{\pm\alpha}(\mathfrak{g}) \mid \alpha \in \Delta\}$.

Finally, define $\mathcal{U} := \mathcal{U}_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U})$ be the group of $\mathbb{R}$-linear automorphisms of $\mathcal{U}$ which preserve the given Q-grading.

Proposition 3.3.3 [HKM13, Proposition 7.6], see also [Rém02, Proposition 9.5.2]. *There exists a morphism of groups*

$$\mathrm{Ad} : G \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U})$$

which is characterised by the following axioms, where $i \in I$, $r \in \mathbb{R}$, and $h \in T$:

(a) $\mathrm{Ad}\left(\varphi_i \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\right) = \exp(\mathrm{ad}_{e_i} \otimes r) = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{e_i})^n}{n!} \otimes r^n.$

(b) $\mathrm{Ad}(T)$ *fixes* $\mathcal{U}_0$.

(c) $\mathrm{Ad}(h)(e_i \otimes r) = h^*(\alpha_i^\vee)(e_i \otimes r)$.

The kernel of this representation coincides with the centre of the group G. □

Proposition 3.3.4 [HKM13, Proposition 7.9]. *Let $v \in \mathcal{U}$ and $\overline{\beta} \in L$. Then there exists a finite-dimensional sub-vector space $V_{\overline{\beta}}^v$ of $\mathcal{U}$ with the following properties:*

(a) The image of the orbit map $TG_{\overline{\beta}} \to \mathcal{U} : g \mapsto g.v$ is contained in $V_{\overline{\beta}}^v$.

(b) If $\overline{\alpha} \leq \overline{\beta}$, then $V_{\overline{\alpha}}^v \leq V_{\overline{\beta}}^v$

(c) The orbit map $(TG_{\overline{\beta}}, \tau_{\overline{\beta}}) \to (V_{\overline{\beta}}^v, \mathcal{O}) : g \mapsto g.v$ is continuous.

Proof. By Proposition 3.3.3, we may disregard the finite-dimensional torus T.

(a) and (b): We argue by induction on k, the length of $\overline{\beta}$. For $k = 1$, let $G_{\overline{\beta}} = G_i$.

By the Gauss algorithm, $\mathrm{SL}_2(\mathbb{R}) = ULUL$, where U and L denote the subgroups of upper, respectively lower triangular matrices with 1's on the diagonal. Hence, $G_i.v$ is contained in

$$\{(\mathrm{Ad}(x_i(r_1))\,\mathrm{Ad}(x_{-i}(r_2))\,\mathrm{Ad}(x_i(r_3))\,\mathrm{Ad}(x_{-i}(r_4))).v \mid r_i \in \mathbb{R}\},$$

where

$$x_i(r) := \varphi_i \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{-i}(r) := \varphi_i \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

Using Proposition 3.3.3 (a), we therefore define

$$V_{\overline{\beta}}^v := \sum_{k,l,m,n} \left\langle \left(\frac{(\mathrm{ad}_{e_i})^k}{k!} \otimes 1\right) \left(\frac{(\mathrm{ad}_{f_i})^l}{l!} \otimes 1\right) \left(\frac{(\mathrm{ad}_{e_i})^m}{m!} \otimes 1\right) \left(\frac{(\mathrm{ad}_{f_i})^n}{n!} \otimes 1\right).v \right\rangle.$$

Since e_i and f_i are ad-locally nilpotent, the above sum is finite, so $V_{\overline{\beta}_i}^v$ is a finite-dimensional vector space containing $G_i.v$.

Now, let $\overline{\beta} = (\beta_1, \dots, \beta_k)$ and set $\overline{\beta}' := (\beta_2, \dots, \beta_k)$. By induction, $G_{\overline{\beta}'}.v$ is contained in a finite-dimensional vector space $V^v_{\overline{\beta}'}$. Let $\{b_1, \dots, b_n\}$ be a basis for $V^v_{\overline{\beta}'}$. Then for each j, there exists a finite-dimensional space $V^{b_j}_{\beta_1}$ that contains $G_{\beta_1}.b_j$. Therefore, the space $V^v_{\overline{\beta}} := \sum_{j=1}^n V^{b_j}_{\beta_1}$ has the desired properties.

(c): Let $\rho_{\beta_1} := \mathrm{Ad}|^{\mathrm{GL}(V^v_{\overline{\beta}})}_{G_{\beta_1}} : (G_{\beta_1}, \tau_{\beta_1}) \to (\mathrm{GL}(V^v_{\overline{\beta}}), \mathcal{O})$. By restricting the codomain to the Zariski closure of the image of ρ_{β_1}, one can apply the Borel-Tits theorem [BT18, Theorem 1.5] which yields that ρ_{β_1} is an isogeny and therefore continuous. Hence, the orbit map $(G_{\beta_1}, \tau_{\beta_1}) \to (V^v_{\overline{\beta}}, \mathcal{O}) : g \mapsto g.v$ is continuous. This proves (c) for 1-tuples $\overline{\beta} = (\beta_1)$.

Now, let $\overline{\beta}$ have length more than 1 and decompose it as $\overline{\beta} = (\beta_1, \overline{\gamma})$. By induction, the orbit map $(G_{\overline{\gamma}}, \tau_{\overline{\gamma}}) \to (V^v_{\overline{\gamma}}, \mathcal{O})$ is continuous, and since $V^v_{\overline{\gamma}} \leq V^v_{\overline{\beta}}$, so is the map $\varphi : (G_{\overline{\gamma}}, \tau_{\overline{\gamma}}) \to (V_{\overline{\beta}}, \mathcal{O}) : g \mapsto g.v$, obtained by extending the codomain.

Since the action $\varepsilon : (\mathrm{GL}(V^v_{\overline{\beta}}), \mathcal{O}) \times (V^v_{\overline{\beta}}, \mathcal{O}) \to (V^v_{\overline{\beta}}, \mathcal{O})$ is continuous, one obtains a continuous map

$$\begin{aligned} \varepsilon \circ (\rho_{\beta_1} \times \varphi) : (G_{\beta_1}, \tau_{\beta_1}) \times (G_{\overline{\gamma}}, \tau_{\overline{\gamma}}) \to (V^v_{\overline{\beta}}, \mathcal{O}) :& \\ (x_{\beta_1}, x_{\overline{\gamma}}) &\mapsto [\rho_{\beta_1}(x_{\beta_1})](x_{\overline{\gamma}}.v) \\ &= (x_{\beta_1} x_{\overline{\gamma}}).v. \end{aligned}$$

But this implies that the orbit map $(G_{\overline{\beta}}, \tau_{\overline{\beta}}) \to (V^v_{\overline{\beta}}, \mathcal{O}) : g \mapsto g.v$ is continuous as well, since $\tau_{\overline{\beta}}$ is defined as the quotient topology with respect to the product of the G_{β_i}'s. ■

Following [HKM13, Remark 7.12], the Kac–Peterson topology can alternatively (but equivalently) be described as another direct limit topology τ with respect to products of the root subgroups U_α. This will allow us to prove that B is closed by identifying it with the set of common zeroes of a family of functions that are continous with respect to τ. We obtain that the building G/B is Hausdorff.

As in the proof of Proposition 3.3.4, for a simple root α let

$$x_\alpha(r) := \varphi_i \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{-\alpha}(r) := \varphi_i \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

For $\overline{\beta} := (\beta_1, \dots, \beta_k) \in (\Delta \cup -\Delta)^k$, let

$$x_{\overline{\beta}} : \mathbb{R}^k \to G : \quad (t_1, \dots, t_n) \mapsto x_{\beta_1}(t_1) \cdot \dots \cdot x_{\beta_k}(t_k)$$

and denote by $U_{\overline{\beta}}$ the image of $x_{\overline{\beta}}$ in G.

Similarly to the definition of the Kac–Peterson topology, denote by τ the direct limit topology on G with respect to the directed system $(U_{\overline{\beta}}, f_{\overline{\beta}\overline{\alpha}})$, where $f_{\overline{\beta}\overline{\alpha}}$ denotes the natural embedding $U_{\overline{\alpha}} \hookrightarrow U_{\overline{\beta}}$ with respect to the subtuple relation, and where $U_{\overline{\beta}}$ is equipped with the topology $\tau_{\overline{\beta}}$ induced from the topology $\mathcal{O}$ on $\mathbb{R}^k$.

The direct limit topologies τ and τ_{KP} are determined by the respective topologies on the spaces $U_{\overline{\beta}}$ (where $\overline{\beta}$ is a tuple of positive or negative simple roots) and $TG_{\overline{\alpha}}$ (where $\overline{\alpha}$ is a tuple of simple roots). But G is centered, so the torus T can be recovered from the root groups. Therefore, the topologies on $U_{\overline{\beta}}$ and $TG_{\overline{\alpha}}$ are both induced by the maps φ_i; and since $G_\alpha = U_\alpha U_{-\alpha} U_\alpha U_{-\alpha}$ by the Gauss algorithm, the topologies τ and τ_{KP} in fact coincide.

Definition 3.3.5. A function $g : G \to \mathbb{R}$ is called **weakly regular** if $g \circ x_{\overline{\beta}} : \mathbb{R}^k \to \mathbb{R}$ is a polynomial function for all $k \in \mathbb{N}$ and all $\overline{\beta} \in (\Delta \cup -\Delta)^k$. □

Lemma 3.3.6 [HKM13, Lemma 7.14]. *Let $g : G \to \mathbb{R}$ be a weakly regular function. Then g is continuous with respect to the Kac–Peterson topology on G and the natural topology on $\mathbb{R}$.*

Proof. Let $B \subseteq \mathbb{R}$ be a closed subset. As polynomial functions are continuous, the preimage $(g \circ x_{\overline{\beta}})$ is closed in $(\mathbb{R}^k, \mathcal{O})$ for each $\overline{\beta} \in (\Delta \cup -\Delta)^k$. Since this set equals the preimage of $g^{-1}(B)$ under $x_{\overline{\beta}}$, its image in $\mathcal{U}_\beta$ is closed with respect to the induced topology, so the set $g^{-1}(B)$ is closed in G with respect to the direct limit topology τ introduced above. This proves the assertion, as τ and τ_{KP} coincide. ∎

Proposition 3.3.7 [HKM13, Proposition 7.15]. *The subgroup B is closed in G with respect to the Kac–Peterson topology. In particular, the building G/B is Hausdorff when equipped with the quotient topology.*

Proof. Denote by $\mathcal{U}^{\geq 0}$ the subspace of $\mathcal{U}$ consisting of the non-negative vectors with respect to the Q-grading introduced in the definition of $\mathcal{U}$. Since B is the stabilizer of $\mathcal{U}^{\geq 0}$ in G, as can be deduced from Proposition 3.3.3, for each $g \in G \setminus B$ one can choose a $v_g \in \mathcal{U}^{\geq 0}$ such that $g.v_g \notin \mathcal{U}^{\geq 0}$. Then one can choose a $v_g^* \in \mathrm{Ann}(\mathcal{U}^{\geq 0})$, the annihilator of $\mathcal{U}^{\geq 0}$ in U^*, such that $f_{v_g, v_g^*}(g) := v_g^*(g.v_g) \neq 0$. Now,

$$B = \bigcap_{g \in G \setminus B} \{h \in G \mid f_{v_g, v_g^*}(h) = 0\}.$$

Since ad_{e_i} and ad_{f_i} are locally nilpotent, the maps f_{v_g, v_g^*} are weakly regular, and so by Lemma 3.3.6, B is the intersection of the sets of zeroes of a family of continuous functions. This proves the assertion. ∎

3.4 Implications of τ_{KP} Being Hausdorff

As usual, G denotes the group $G_{\mathbb{R}}(A)$, or a central quotient of this group, for a symmetrizable irreducible generalized Cartan matrix A.

If the Kac–Peterson topology is Hausdorff, it has a variety of interesting and useful properties, as we will see in this section. Later we will show that the results established here all hold for adjoint split real Kac–Moody groups. Using the central extension $G \to \mathrm{Ad}(G)$ for a simply connected split real Kac–Moody group G, this will later enable us to prove that (G, τ_{KP}) is Hausdorff as well.

Lemma 3.4.1 [HKM13, Proposition 7.10]. *If $(TG_{\overline{\beta}}, \tau_{\overline{\beta}})$ is Hausdorff for each tuple $\overline{\beta}$ of simple roots, then the Kac–Peterson topology is a k_ω group topology on G. In particular, this holds if the Kac–Peterson topology is Hausdorff.*

Proof. Let $(TG_{\overline{\beta}}, \tau_{\overline{\beta}})$ be Hausdorff. Then it is a Hausdorff quotient of a finite product of k_ω spaces and therefore k_ω by Proposition 1.4.2.

Since the spaces $TG_{\overline{\alpha}}$ are k_ω, Proposition 1.4.3 implies that $G \times G = \lim_{\to}(TG_{\overline{\alpha}}) \times \lim_{\to}(TG_{\overline{\alpha}})$ is homeomorphic to $\lim_{\to}(TG_{\overline{\alpha}} \times TG_{\overline{\alpha}})$, so multiplication can be reduced to the $TG_{\overline{\alpha}}$ pieces.

Let $\overline{\beta} = (\beta_1, \dots, \beta_k), \overline{\gamma} = (\gamma_1, \dots, \gamma_l)$ be tuples of simple roots. By definition of $\tau_{(\beta_1, \dots, \gamma_l)}$, the multiplication map $G_{(\beta_1, \dots, \beta_k)} \times G_{(\gamma_1, \dots, \gamma_l)} \to G_{(\beta_1, \dots, \gamma_l)}$ is continuous, and so is the inversion map $G_{(\beta_1, \dots, \beta_k)} \to G_{(\beta_k, \dots, \beta_1)}$. This implies that (G, τ_{KP}) is a topological group. Since it is a direct limit of Hausdorff spaces, all singletons are closed in (G, τ_{KP}), so it is $T1$. But a $T1$ topological group is Hausdorff, so Proposition 1.4.4 implies that (G, τ_{KP}) is k_ω. This proves the first assertion.

The second assertion is clear, since a set $U \subseteq G$ is open in τ_{KP} iff each intersection $U \cap TG_{\overline{\beta}}$ is open in $\tau_{\overline{\beta}}$, so τ_{KP} being Hausdorff implies that each $(TG_{\overline{\beta}}, \tau_{\overline{\beta}})$ is Hausdorff. ■

If the Kac–Peterson topology is Hausdorff, it is a k_ω group topology by Lemma 3.4.1 which enables us to obtain a characterisation of τ_{KP} as the final group topology with respect to the maps φ_i. We define amalgams of groups and their universal enveloping groups and, for the two-spherical case, state an amalgamation result which gives a concrete presentation of G as the universal enveloping group of the amalgam of fundamental rank one and rank two subgroups in the categories of abstract, Hausdorff topological, and k_ω-groups.

Definition 3.4.2. The **universal topology** τ_{un} is the final group topology with respect to the maps

$$\varphi_i : \mathrm{SL}_2(\mathbb{R}) \to G, \quad \eta(\mathbb{R}) : \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{R}) \to G,$$

where $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{R}) \cong (\mathbb{R}^\times)$ are equipped with their Lie group topologies. □

The following proposition is Proposition 7.21 from [HKM13]. The alternative proof given here is an original result by the author.

Proposition 3.4.3. *Let τ_{KP} be a group topology. Then the universal topology τ_{un} and the Kac–Peterson topology τ_{KP} coincide. In particular, this holds if τ_{KP} is Hausdorff.*

Proof. It is clear that $\tau_{KP} \subseteq \tau_{un}$ since τ_{KP} is a group topology and the maps φ_i and $\eta(\mathbb{R})$ are continuous with respect to τ_{KP} by its definition, the topology $\tau_{\mathbb{R}}$ on T being induced by the map $\eta(\mathbb{R})$.

To prove that $\tau_{un} \subseteq \tau_{KP}$, let $\overline{\beta} = (\beta_1, \dots, \beta_k)$ be a k-tuple of simple roots and let τ_{un}^{k+1} be the product topology on G^{k+1}. We first show that the subspace topology $\overline{\tau}_{un}^{k+1}$ of τ_{un}^{k+1} on $T \times G_{\beta_1} \times \dots \times G_{\beta_k}$ is coarser than the product topology of the respective Lie topologies $\tau_{\mathbb{R}} \times \mathcal{O}^k$. For this, it suffices to show that each open 1-neighbourhood in $T \times G_{\beta_1} \times \dots \times G_{\beta_k}$ with respect to $\overline{\tau}_{un}^{k+1}$ contains an open 1-neighbourhood with respect to $\tau_{\mathbb{R}} \times \mathcal{O}^k$.

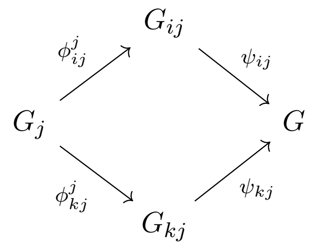
Let $U \subseteq T \times G_{\beta_1} \times \dots \times G_{\beta_k}$ be an open 1-neighbourhood in $\overline{\tau}_{un}^{k+1}$. Then there exists an open 1-neighbourhood $\tilde{U} \subseteq (G^{k+1}, \tau_{un}^{k+1})$ with $\tilde{U} \cap (T \times G_{\beta_1} \times \dots \times G_{\beta_k}) = U$. Hence, there exist open 1-neighbourhoods $\tilde{U}_0, \tilde{U}_1, \dots, \tilde{U}_k$ in (G, τ_{un}) with $\tilde{U}_0 \times \tilde{U}_1 \times \dots \times \tilde{U}_k \subseteq \tilde{U}$. Since $\tilde{U}_i$ is open in τ_{un}, its preimage $\varphi_i^{-1}(\tilde{U}_i) = \varphi_i^{-1}(\tilde{U}_i \cap G_i)$ is open in $\mathrm{SL}_2(\mathbb{R})$ for each $i = 1, \dots, n$ while the preimage $\eta(\mathbb{R})^{-1}(\tilde{U}_0) = \eta(\mathbb{R})^{-1}(\tilde{U}_0 \cap T)$ is open in $\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{R})$ with its Lie topology.

By definition of the Lie topologies $\mathcal{O}$ on G_i and $\tau_{\mathbb{R}}$ on T, this implies that $U_i := \tilde{U}_i \cap G_i$ is open in $(G_i, \mathcal{O})$for $i = 1, \dots, k$ while $U_0 := \tilde{U}_0 \cap T$ is open in $(T, \tau_{\mathbb{R}})$. Therefore, $U_0 \times U_1 \times \dots \times U_n$ is an open 1-neighbourhood in $\tau_{\mathbb{R}} \times \mathcal{O}^k$, and we have $U_0 \times U_1 \times \dots \times U_n \subseteq \tilde{U} \cap (T \times G_{\beta_1} \times \dots \times G_{\beta_k}) = U$. This proves $\overline{\tau}_{un}^{k+1} \subseteq \tau_{\mathbb{R}} \times \mathcal{O}^k$.

To prove that $\tau_{un} \subseteq \tau_{KP}$, let $U \subseteq G$ be open with respect to τ_{un}. Then for each $k \in \mathbb{N}$, the preimage $p_{k+1}^{-1}(U)$ under the product map $p_{k+1} : G^{k+1} \to G$ is open in $(G^{k+1}, \tau_{un}^{k+1})$, since (G, τ_{un}) is a topological group. Therefore, the intersection $p_{k+1}^{-1}(U) \cap (T \times G_{\beta_1} \times \dots \times G_{\beta_k})$ is open in $(T \times G_{\beta_1} \times \dots \times G_{\beta_k})$ with respect to the subspace topology $\overline{\tau}_{un}^{k+1}$ for each k-tuple $(\beta_1, \dots, \beta_k)$ of simple roots. Since $\overline{\tau}_{un}^{k+1} \subseteq \tau_{\mathbb{R}} \times \mathcal{O}^k$, the set $p_{k+1}^{-1}(U) \cap (T \times G_{\beta_1} \times \dots \times G_{\beta_k})$ is therefore open in $(T, \tau_{\mathbb{R}}) \times (G_{\beta_1}, \mathcal{O}) \times \dots \times (G_{\beta_k}, \mathcal{O})$. But $p_{k+1}^{-1}(U) \cap (T \times G_{\beta_1} \times \dots \times G_{\beta_k}) = p_{\overline{\beta}}^{-1}(U \cap TG_{\overline{\beta}})$, where $p_{\overline{\beta}} = p_{k+1}|_{(T \times G_{\beta_1} \times \dots \times G_{\beta_k})}$ is the product map given in 3.1. Since $\tau_{\overline{\beta}}$ is the quotient topology with respect to $p_{\overline{\beta}}$, we therefore obtain that $U \cap TG_{\overline{\beta}}$ is open in $(TG_{\overline{\beta}}, \tau_{\overline{\beta}})$ for each $\overline{\beta}$, which implies that U is open in τ_{KP}. This proves the first assertion. The second assertion follows directly from Lemma 3.4.1. ■

Definition 3.4.4. Let $\emptyset \neq J$ be a set. Following [GHKW17, Section 3], an **amalgam** over J is a set $\mathcal{A} = \{G_i, G_{ij}, \phi_{ij}^i \mid i \neq j \in J\}$ such that G_i and G_{ij} are groups and $\phi_{ij}^i : G_i \to G_{ij}$ is a monomorphism for all $i \neq j \in J$.

An **enveloping group** of $\mathcal{A}$ is a pair (G, ψ) where G is a group and ψ is a set $\psi = \{\psi_{ij} \mid i \neq j\}$ of **enveloping morphisms** $\psi_{ij} : G_{ij} \to G$ such that $G = \langle \psi_{ij}(G_{ij}) \mid i \neq j \in J \rangle$ and such that the following diagram commutes for $i \neq j \neq k \in J$:



A **universal enveloping group** of $\mathcal{A}$ is an enveloping group (G, ψ) such that for each enveloping group (H, ψ') of $\mathcal{A}$ there is a unique epimorphism $\pi : G \to H$ such that $\psi'_{ij} = \pi \circ \psi_{ij}$ for all $i \neq j \in J$. □

Theorem 3.4.5 (Topological Curtis-Tits Theorem) [HKM13, Theorem 7.22]**.** *Let G be two-spherical and τ_{KP} Hausdorff. For $i \neq j \in I$ let $\phi_{ij}^i : G_i \to G_{ij}$ and $\psi_{ij} : G_{ij} \to G$ be the canonical inclusion morphisms and let $\psi = \{\psi_{ij} \mid i \neq j \in I\}$. Then $((G, \tau_{KP}), \psi)$ is a universal enveloping group of the amalgam $\{G_i, G_{ij}, \phi_{ij}^i \mid i \neq j \in I\}$ in the categories of*

(a) abstract groups,

(b) Hausdorff topological groups and

(c) k_ω groups.

Proof. By Lemma 3.4.1, τ_{KP} is a k_ω group topology.

(a): This is the main result of [AM97].

(b): This holds since the Kac–Peterson topology coincides with the universal topology by Proposition 3.4.3.

(c): This follows directly from (b) by [GGH10, Corollary 5.10] ■

We are now in the position to prove that the restriction of the Kac–Peterson topology to a spherical subgroup coincides with its Lie group topology if τ_{KP} is a Hausdorff group topology; in particular, spherical subgroups are locally compact. To that end, we state an embedding result by Grüning and Köhl.

Remark 3.4.6. The following Proposition is a result by Grüning and Köhl, published in their appendix to [HK19]. It is proved there by employing the above Proposition 3.4.3 as given in [HKM13, Proposition 7.21]. In [HKM13], the proof of said Proposition 7.21 cites a result which we will only be able to prove at a later stage; namely the fact that spherical subgroups carry the Lie topology. As we have provided a different proof for Proposition 3.4.3, a circular argument is prevented and we can safely use [HK19, Proposition B.5] as well as some its corollaries which we will state later. □

Proposition 3.4.7 [HK19, Appendix B by Grüning and Köhl, Proposition B.5]**.** *Any symmetrizable topological Kac–Moody group endowed with the Kac–Peterson topology admits a continuous injective group homomorphism into a simply laced topological Kac–Moody group with closed image with respect to the Kac–Peterson topology.*

The proof of the following Proposition 7.16(iii)(iv) from [HKM13] was throroughly revised in collaboration with Ralf Köhl.

Proposition 3.4.8. *Let (G, τ_{KP}) be Hausdorff and let G_J be a spherical subgroup of G, that is, let $W_J = \langle s_i \mid i \in J\rangle$ be finite. Then the restriction of τ_{KP} to G_J coincides with its Lie group topology $\mathcal{O}$.*

Proof. Since the Lie group topology on G_J is the final group topology with respect to the maps φ_i $(i \in J)$, the restriction $\tau_{KP}|_{G_J}$ is coarser than $\mathcal{O}$, so the identity map $(G_J, \mathcal{O}) \to (G_J, \tau_{KP}|_{G_J})$ is continuous.

To prove the proposition, we will employ a theorem by Burns and Spatzier which requires that G_J be of rank at least two. If a rank one subgroup G_i is contained in a spherical subgroup of larger rank, this requirement is no obstruction. If G_i is contained in no such subgroup, we can use Proposition 3.4.7. The image of G_i will then be contained in a spherical subgroup of H of higher rank to which the Burns–Spatzier theorem can be applied, so that we eventually obtain continuous bijections $(G_i, \mathcal{O}) \to (G_i, \tau_{KP}|_{G_i}) \to (\iota(G_i), \mathcal{O})$, yielding the desired result. It therefore suffices to prove the statement for spherical subgroups of rank at least 2.

Since the identity map $(G_J, \mathcal{O}) \to (G_J, \tau_{KP}|_{G_J})$ is continuous, the group $(G_J, \mathcal{O})$ acts continuously on R_J, the spherical J-residue of B in the building G/B, endowed with the Kac–Peterson (quotient) topology $\tau_{KP}|_{R_J}$. Since the kernel of the action is $Z(G_J)$, we can consequently endow $G_J/Z(G_J)$ with the compact-open topology τ_{co} with respect to this action. Since the evaluation map $(G_J/Z(G_J), \tau_{KP}|_{G_J}) \times (R_J, \tau_{KP}|_{R_J}) \to (R_J, \tau_{KP}|_{R_J})$ is continuous, the (quotient) topology $\tau_{KP}|_{G_J}$ on $G_J/Z(G_J)$ is admissible, and so by Proposition 1.5.4, the identity map $(G_J/Z(G_J), \tau_{KP}|_{G_J}) \to (G_J/Z(G_J), \tau_{co})$ is continuous. We therefore obtain continuous identity maps

$$(G_J/Z(G_J), \mathcal{O}) \to (G_J/Z(G_J), \tau_{KP}|_{G_J}) \to (G_J/Z(G_J), \tau_{co}).$$

Our goal is to show that the first and the last topology coincide. This will imply that the quotient topology of $\tau_{KP}|_{G_J}$ on $G_J/Z(G_J)$ coincides with its Lie group topology, proving that the topologies coincide on G_J as well, since $G_J \to G_J/Z(G_J)$ is a covering map and covering groups of Lie groups are Lie groups (see, e.g., [HN11, Corollary 9.4.6]). To prove that τ_{co} and $\mathcal{O}$ coincide, we show that τ_{co} is locally compact and employ the Open Mapping Theorem for locally compact groups.

The action of G_J on R_J is transitive with stabilizer B_J, so we have a continuous surjection $(G_J/B_J, \mathcal{O}) \to (R_J, \tau_{KP}|_{R_J})$. But $(G_J/B_J, \mathcal{O})$ is compact by [HKM13, Corollary 3.13] and $(R_J, \tau_{KP}|_{R_J})$ is Hausdorff by Proposition 3.3.7, so the spaces are homeomorphic by Lemma 1.2.2. The compact-open topology τ_{co} on G_J with respect to the action of $(G_J, \mathcal{O})$ on $(R_J, \tau_{KP}|_{R_J})$ therefore coincides with the compact-open topology with respect to the action on the spherical building $(G_J/B_J, \mathcal{O})$.

The Burns–Spatzier theorem [BS87, Theorem 2.1] states that the topological automorphism group $\mathrm{Auttop}(\Delta)$ of a compact irreducible metric Tits building Δ of rank at least 2 is locally compact in the compact-open topology. Burns–Spatzier introduce the notion of a Tits building as a set of parabolic subgroups of different types with a corresponding induced topology. By [Tit74, Theorem 5.2] and Lemma 2.4.3 (TTB7), this approach is equivalent to the one taken here via BN-pairs and the quotient topology.

The group $(G_J/Z(G_J), \mathcal{O})$ acts via topological automorphisms on the building $(G_J/B_J, \mathcal{O})$. Our goal is to establish that $G_J/Z(G_J)$ is a closed subgroup of the locally compact group $(\mathrm{Auttop}(G_J/B_J), \tau_{co})$.

By [Tit74, Corollary 5.10], the automorphism group $\mathrm{Aut}(G_J/B_J)$ is given by $\mathrm{Aut}(G_J)(\mathbb{R})$, an algebraic group whose identity component in the Zariski topology is the group $\mathrm{Int}(G_J)(\mathbb{R})$ which is isomorphic to the adjoint form $\mathrm{Ad}(G_J)$ of G_J. The group $\mathrm{Int}(G_J)(\mathbb{R})$ is the group of type-preserving automorphisms of the building, which means that it induces the identity map on the corresponding Coxeter diagram and therefore preserves Weyl distance and codistance; in particular, it contains the group $G_J/Z(G_J)$. Being an algebraic group, $\mathrm{Aut}(G_J)(\mathbb{R})$ can be endowed with a Lie topology with which it acts as the group $\mathrm{Auttop}(G_J/B_J)$ of topological automorphisms of the building, while $\mathrm{Int}(G_J)(\mathbb{R})$ with its Lie topology acts as the group $\mathrm{Auttop}_S(G_J/B_J)$ of type-preserving topological automorphisms.

As $\mathrm{Auttop}(G_J/B_J)$ with its Lie topology is σ-compact, the Open Mapping Theorem 1.2.1 implies that the Lie topology coincides with the compact-open topology.

By [Tit74, page 80], each automorphism of G_J/B_J is a product of a type-preserving automorphism and a Coxeter diagram automorphism. Since the Coxeter diagram of G_J/B_J is finite, the group $\mathrm{Auttop}_S(G_J/B_J) \cong \mathrm{Int}(G_J)(\mathbb{R})$ has finite index in $\mathrm{Auttop}(G_J/B_J)$.

By [PR94, Proposition 3.6], any noncentral normal subgroup of $\mathrm{Ad}(G_J) \cong \mathrm{Int}(G_J)(\mathbb{R})$ has finite index. Since by [PR94, Section 2.2.3, (2.7)], there exists an exact sequence $\{1\} \to Z(G_J) \to G_J \to \mathrm{Ad}(G_J) \to \dots$, it follows that the image of G_J is a noncentral normal subgroup of $\mathrm{Ad}(G_J)$. This proves that $G_J/Z(G_J)$ has finite index in $\mathrm{Int}(G_J)(\mathbb{R})$ and, consequently, in $\mathrm{Auttop}(G_J/B_J)$. Since finite-index subgroups of Lie groups are closed and since the Lie topology on $\mathrm{Auttop}(G_J/B_J)$ coincides with the compact open topology, it follows from the Burns–Spatzier theorem that $G_J/Z(G_J)$ is locally compact with respect to the compact-open topology.

The Open Mapping Theorem 1.2.1 now yields that the compact-open topology on $G_J/Z(G_J)$ coincides with its Lie group topology $\mathcal{O}$. This proves the assertion. ■

The proof of the following Proposition 7.18 from [HKM13] was revised based on an idea by Stefan Witzel.

Proposition 3.4.9. *Let (G, τ_{KP}) be Hausdorff and let $\tau_{\mathbb{R}}$ be the Lie group topology on the standard maximal torus T. Then the map $(T, \tau_{\mathbb{R}}) \to (T, \tau_{KP}|_T)$ is a homeomorphism.*

Proof. We prove the statement for $G = G_{\mathbb{R}}(A)$, then it automatically follows for central quotients. Since G, being Hausdorff, is a topological group, Lemma 3.3.2 yields an isomorphism of topological groups $T \cong M \times A$, where M is finite. It therefore suffices to prove that the map $(A, \tau_{\mathbb{R}}) \to (A, \tau_{KP}|_A)$ is a homeomorphism. Here, $\tau_{\mathbb{R}}$ denotes the Lie topology on A induced from $(\mathbb{R}^\times)^n$, so that $(A, \tau_{\mathbb{R}})$ is homeomorphic to $\oplus_{j=1}^n (A_j, \tau_{\mathbb{R}})$. Since spherical subgroups carry the Lie topology by Proposition 3.4.8, τ_{KP} coincides with the Lie group topology on the subgroups A_i. We obtain the following commutative diagram for $i = 1, \dots, n$:

$$\begin{array}{ccccc}
\oplus_{j=1}^{n}(A_j, \tau_{\mathbb{R}}) & \xrightarrow{f_1} & (A, \tau_{KP}|_A) & \xrightarrow{f_2} & (\Pi_{j=1}^{n} A_j, \tau_{KP}|_{A_i} = \tau_{\mathbb{R}}) \\
f_4 \uparrow & & \swarrow f_3 & & \\
(A_i, \tau_{KP}|_{A_i} = \tau_{\mathbb{R}}) & & & &
\end{array}$$

All maps in the diagram are continuous: The map f_1 is continuous since its domain is homeomorphic to $(A, \tau_{\mathbb{R}})$ and since $\tau_{\mathbb{R}}$, being a Lie topology, is the final group topology on A with respect to the restrictions of the maps φ_i to the connected component of $\mathrm{SL}_2(\mathbb{R})$, while these restrictions are also continuous with respect to the group topology $\tau_{KP}|_A$. The map f_2 is continuous since the product topology on its codomain is the coarsest topology making the projection maps continuous; for the same reason, the projection map f_3 is continuous. Finally, the inclusion map f_4 is continuous since the box topology on its codomain is the final topology with respect to the inclusion maps. This proves that the map $(A, \tau_{\mathbb{R}}) \to (A, \tau_{KP}|_A)$ is a homeomorphism. ■

The following two results are given in [HK19, Appendix B by Grüning and Köhl] as corollaries to Proposition B.5, see Remark 3.4.6.

Lemma 3.4.10 [HK19, Appendix B by Grüning and Köhl, Corollary B.8]**.** *Let G be a split real Kac–Moody group such that the Kac–Peterson topology is Hausdorff. Then the multiplication map $m : U_+ \times T \times U_- \to B_+B_-$ is open.* □

Lemma 3.4.11 [HK19, Appendix B by Grüning and Köhl, Corollary B.9]**.** *Let G be a split real Kac–Moody group such that the Kac–Peterson topology is Hausdorff. Then the associated building endowed with the quotient topology is a strong topological twin building. In particular, the space B_+B_- is open.* □

The following result was established in collaboration with Ralf Köhl.

Lemma 3.4.12. *Let G be a split real Kac–Moody group and let Z be a finite central subgroup of G. Consider $\tilde{G} := G/Z$ as a split real Kac–Moody group in its own right, denote by $\tilde{U}_\pm$, $\tilde{T}$ and $\tilde{B}_\pm$ the respective subgroups corresponding to its RGD system, and endow $\tilde{G}$ with the Kac–Peterson topology $\tilde{\tau}_{KP}$. If $(\tilde{G}, \tilde{\tau}_{KP})$ is Hausdorff, then the Kac–Peterson topology on G is Hausdorff.*

Proof. To prove the statement, we will use Proposition 1.6.1 and its Corollary 1.6.2 to construct a topology τ_{cov} on G that turns $q : G \to G/Z$ into a covering map. We will then show that τ_{cov} is coarser than the Kac–Peterson topology on G which proves that (G, τ_{KP}) is Hausdorff since (G, τ_{cov}) is Hausdorff, being a covering space of a Hausdorff space.

Since Z is finite, the map $q : T \to \tilde{T}$ is a covering map when T and $\tilde{T}$ are endowed with their respective Lie topologies. Since by Proposition 3.4.9 the restriction of the Kac–Peterson topology to $\tilde{T}$ coincides with the Lie topology $\tilde{\tau}_{\mathbb{R}}$, there exists an open 1-neighbourhood $\tilde{U} \subseteq (\tilde{T}, \tilde{\tau}_{KP}|_{\tilde{T}} = \tilde{\tau}_{\mathbb{R}})$ whose preimage is a disjoint union of copies of $\tilde{U}$. Let $U \subseteq T$ be the copy containing 1_G, let $V \subseteq U$ be a 1-neighbourhood with $V^2 \subseteq U$ and let $\tilde{V} := q(V) \subseteq \tilde{U}$. Since q is a group homomorphism, $\tilde{V}$ is again a 1-neighborhood with $\tilde{V}^2 \subseteq \tilde{U}$. The restriction $q : U \to \tilde{U}$ is a homeomorphism, and so the section $\sigma : (\tilde{U}, \tilde{\tau}_{KP}) \to (T, \tau_{\mathbb{R}})$ mapping each element in $\tilde{U}$ to its preimage in U is continuous. Extend σ to a section $\sigma : \tilde{T} \to T$.

The map $q : G \to G/Z$ induces a bijection $b : U_\pm \to \tilde{U}_\pm$, so endowing $U_\pm$ with the Kac–Peterson topology $\tilde{\tau}_{KP}$ from $\tilde{U}_{\pm}$, one obtains a covering map

$$(U_+, \tilde{\tau}_{KP}) \times (T, \tau_{\mathbb{R}}) \times (U_-, \tilde{\tau}_{KP}) \to (\tilde{U}_+, \tilde{\tau}_{KP}) \times (\tilde{T}, \tilde{\tau}_{KP}) \times (\tilde{U}_-, \tilde{\tau}_{KP})$$

with a section

$$\overline{\sigma} := b^{-1} \times \sigma \times b^{-1} : (\tilde{U}_+, \tilde{\tau}_{KP}) \times (\tilde{T}, \tilde{\tau}_{\mathbb{R}}) \times (\tilde{U}_-, \tilde{\tau}_{KP}) \to (U_+, \tilde{\tau}_{KP}) \times (T, \tau_{\mathbb{R}}) \times (U_-, \tilde{\tau}_{KP})$$

that is continuous on the (by assumption) open 1-neighborhood $\tilde{W} = \tilde{U}_+ \times \tilde{V} \times \tilde{U}_-$.

Since $\tilde{V}^2 \subseteq \tilde{U}$ and $V^2 \subseteq U$, and since σ maps each element of $\tilde{U}$ to its unique representative in U, one has $\sigma(v_1)\sigma(v_2) = \sigma(v_1 v_2)$ for all $v_1, v_2 \in \tilde{V}$. Since b is induced by a group homomorphism, this implies that $\overline{\sigma}(x)\overline{\sigma}(y) = \overline{\sigma}(xy)$ for all $x, y \in \tilde{W}$. Extending $\overline{\sigma}$ to a section $\overline{\sigma}$ on G/Z therefore yields a 2-cocycle $f : G/Z \times G/Z \to Z$ defined by $f(x,y) := \overline{\sigma}(x)\overline{\sigma}(y)\overline{\sigma}(xy)^{-1}$ that is constant and therefore continuous on $\tilde{W}$. Now, Corollary 1.6.2 yields a group topology τ_{cov} on G induced from $G/Z \times_f Z$ such that $q : (G, \tau_{cov}) \to (G/Z, \tilde{\tau}_{KP})$ is a covering map.

To prove that τ_{cov} is coarser than τ_{KP}, we show that $\tau_{cov}|_{G_i}$ coincides with the Lie group topology $\mathcal{O}$ on G_i for each $i \in I$.

The fundamental rank one subgroups of G/Z are $G_i/(G_i \cap Z) = G_i Z/Z$. By Proposition 3.4.8, the restriction of the Kac–Peterson topology $\tilde{\tau}_{KP}$ to these groups coincides with the Lie group topology.

The restriction $f|_{G_i Z/Z}$ is a 2-cocycle $G_i Z/Z \times G_i Z/Z \to Z$ which is continuous on the set $\tilde{W} \cap G_i Z/Z$ which is an open 1-neighbourhood in the subspace topology $\tilde{\tau}_{KP}|_{G_i Z/Z}$. Therefore, applying Corollary 1.6.2 yields that the topology on G_i induced from $G_i Z/Z \times_{f|_{G_i Z/Z}} Z$ turns G_i into a topological group such that $G_i \to G_i Z/Z$ is a covering map. This implies that this topology coincides with the Lie topology $\mathcal{O}$, since $G_i Z/Z$ carries the Lie topology and covering groups of Lie groups are Lie groups (see, e.g., [HN11, Corollary 9.4.6]). But the topology induced from $G_i Z/Z \times_{f|_{G_i Z/Z}} Z$ coincides with $\tau_{cov}|_{G_i}$, since τ_{cov} is the topology on G induced from $G/Z \times_f Z$. Therefore, $\tau_{cov}|_{G_i} = \mathcal{O}$.

Since τ_{cov} is a group topology, this implies that for each tuple $\overline{\beta} = (\beta_1, \ldots, \beta_k)$ of simple roots, the multiplication map $(G_{\beta_1}, \tau_{cov}|_{G_{\beta_1}} = \mathcal{O}) \times \cdots \times (G_{\beta_k}, \tau_{cov}|_{G_{\beta_k}} = \mathcal{O}) \to (G, \tau_{cov})$ is continuous. By definition of τ_{KP}, it follows that $\tau_{cov} \subseteq \tau_{KP}$, proving that τ_{KP} is a Hausdorff topology. ■

3.5 The adjoint and the simply connected case

Here, $G = G_{\mathbb{R}}(A)$ for an irreducible symmetrizable generalized Cartan matrix A. We first show that for the adjoint form of G, the Kac–Peterson topology is a k_ω group topology. Using the central extension $G \to \mathrm{Ad}(G)$, this will in turn enable us to prove that (G, τ_{KP}) is Hausdorff as well, and therefore a k_ω topological group by Lemma 3.4.1.

Proposition 3.5.1 [GGH10, Proposition 7.10]**.** *Let $H = Ad(G)$. Then the Kac–Peterson topology is Hausdorff. In particular, it is a k_ω group topology on H.*

Proof. Let $\overline{\beta} = (\beta_1, \ldots, \beta_k)$ be a tuple of simple roots. We prove that $(TH_{\overline{\beta}}, \tau_{\overline{\beta}})$ is Hausdorff. Let $g \neq h \in TH_{\overline{\beta}}$. Then by Proposition 3.3.3, H being adjoint and therefore centre-free implies that there exists a $v \in \mathcal{U}$ with $g.v \neq h.v$. Since the orbit map $(TH_{\overline{\beta}}, \tau_{\overline{\beta}}) \to (V^v_{\overline{\beta}}, \mathcal{O})$ is continuous by Lemma 3.3.4, taking preimages of disjoint open neighborhoods of $g.v$ and $h.v$ yields disjoint open neighborhoods of g and h, implying that $TH_{\overline{\beta}}$ is Hausdorff. The assertion now follows from Lemma 3.4.1. ■

The following result was established in collaboration with Ralf Köhl.

Proposition 3.5.2. *There exists a finite central subgroup Z of G such that G/Z is Hausdorff when endowed with the quotient topology. In particular, the Kac–Peterson topology on G is Hausdorff.*

Proof. Let B be an invertible symmetrizable generalized Cartan matrix that contains A as a principal submatrix and let $H := \mathrm{Ad}(G_{\mathbb{R}}(B))$. Then there exists a surjective group homomorphism f from G to the corresponding subgroup $\tilde{G}$ of H. If a subset $U \subseteq \tilde{G}$ is open in the subspace topology induced by the Kac–Peterson topology on H, then in particular each intersection $U \cap \tilde{G}_{\overline{\beta}}$ is open in $\tilde{\tau}_{\overline{\beta}}$. Moreover, the following diagram commutes for each k-tuple $\overline{\beta} = (\beta_1, \ldots, \beta_k)$ of simple roots of G:

$$\begin{array}{ccccc} (T,\tau_{\mathbb{R}}) \times (\mathrm{SL}_2(\mathbb{R}),\mathcal{O})^k & \longrightarrow & (T,\tau_{\mathbb{R}}) \times (G_{\beta_1},\mathcal{O}) \times \cdots \times (G_{\beta_k},\mathcal{O}) & \longrightarrow & (G_{\overline{\beta}},\tau_{\overline{\beta}}) \\ \downarrow & & & & \downarrow f \\ (\tilde{T},\tau_{\mathbb{R}}) \times (\mathrm{SL}_2(\mathbb{R}),\mathcal{O})^k & \longrightarrow & (\tilde{T},\tau_{\mathbb{R}}) \times (\tilde{G}_{\beta_1},\mathcal{O}) \times \cdots \times (\tilde{G}_{\beta_k},\mathcal{O}) & \longrightarrow & (\tilde{G}_{\overline{\beta}},\tilde{\tau}_{\overline{\beta}}) \end{array}$$

It follows that f is continuous, since the other maps in the diagram are all continuous and open. This implies that $G/\ker f$ is Hausdorff when endowed with the quotient topology.

We show that $\ker f$ is finite and central. Since G and $\tilde{G}$ act on the same building with the respective centres as kernels, $\ker f$ is contained in $Z(G)$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Kac–Moody algebras of G and H, respectively, and let $\overline{f}$ be the injection $\mathfrak{g} \to \mathfrak{h}$ corresponding to f. There exists a choice of Cartan subalgebra $\overline{\mathfrak{g}}$ of $\mathfrak{g}$ that corresponds to T, and the group $(\overline{\mathfrak{g}}, +)$ contains a subgroup $\mathfrak{a}$ that can be identified with A. Moreover, there exists a choice of Cartan subalgebra $\overline{\mathfrak{h}}$ of $\mathfrak{h}$ such that $\overline{f}$ maps $\overline{\mathfrak{g}}$ into $\overline{\mathfrak{h}}$. Since B is invertible, the connected component of its torus is trivial, as is pointed out in [FHHK, Section 3.7]. This implies that there is an injection $\overline{\mathfrak{h}} \to T_H$. Summing up, this yields an injection

$$A \hookrightarrow \overline{\mathfrak{g}} \hookrightarrow \overline{\mathfrak{h}} \hookrightarrow T_H \hookrightarrow H.$$

Since A therefore intersects trivially with $\ker f$, it follows that $\ker f$ is finite. This proves the first assertion.

To prove the second assertion, note that since $Z(G) \subseteq T \subseteq TG_{\overline{\beta}}$ for all tuples $\overline{\beta}$ of simple roots, one can check easily that the quotient topology on G/Z induced by the Kac–Moody topology on G coincides with the Kac–Moody topology on G/Z, where G/Z is viewed as a Kac–Moody group in its own right. Therefore, Lemma 3.4.12 yields that G is Hausdorff. ■

We collect the consequences of τ_{KP} being Hausdorff established in the previous section.

Corollary 3.5.3. *Let A be symmetrizable and irreducible and let $G = G_{\mathbb{R}}(A)$, the algebraically simply-connected semisimple split real Kac–Moody group of type A, endowed with the Kac–Peterson topology. The following hold:*

(a) τ_{KP} is a k_ω group topology.

(b) The universal topology τ_{un} and the Kac–Peterson topology τ_{KP} coincide.

(c) G satisfies the Topological Curtis-Tits Theorem 3.4.5.

(d) The building G/B is Hausdorff when equipped with the quotient topology.

(e) *The restriction of τ_{KP} to a spherical subgroup G_J coincides with its Lie group topology.*

(f) *The restriction of τ_{KP} to the standard maximal torus T coincides with its Lie group topology.*

(g) *The twin building associated to G, endowed with the quotient topology, is a strong topological twin building.* □

Notation 3.5.4. In what follows, G will always denote the group $G_{\mathbb{R}}(A)$ for a symmetrizable irreducible generalized Cartan matrix A. Moreover, G will always be endowed with the Kac–Peterson topology, the building G/B with the corresponding Hausdorff quotient topology and the spaces $C_w(gB)$ and $C_{\leq w}(gB)$ with the respective subspace topologies induced by G/B. □

3.6 The Group K and the Iwasawa Decomposition

Let A be symmetrizable and irreducible and let $G = G_{\mathbb{R}}(A)$. We introduce the group K, the fixed point set of the Cartan-Chevalley involution, and give a result by Hartnick and Köhl which states that the fundamental groups of K and G are isomorphic, which will allow us to reduce the computation of the fundamental group of G to that of K.

Definition 3.6.1. By [Mar18, Exercise 7.61] (see also [KP85, Section 2]), for suitable choices of the isomorphisms $\varphi_i : \mathrm{SL}_2(\mathbb{R}) \to G_i$ there exists a unique involution θ: $G \to G$ satisfying $\varphi_i((x^{-1})^T) = \theta(\varphi_i(x))$ for all $i \in I$. This involution is called the **Cartan-Chevalley involution** of G. Its fixed point set is denoted by

$$K := G^\theta = \{g \in G \mid g^\theta = g.\}$$

Since θ is continuous by definition of the Kac–Peterson topology, K is closed and therefore k_ω by Proposition 1.4.2. □

As in Notation 3.2.4, let $K_J := K \cap G_J$.

Recall that M denotes the unique maximal finite subgroup M of order 2^n and A the connected component of T with respect to its Lie topology, which by 3.5.3 coincides with the restricted Kac–Peterson topology.

Lemma 3.6.2 [FHHK, Proposition 3.12], [FHHK, Lemma 3.20]**.**

(a) *Multiplication induces a homeomorphism $M \times A \times U_+ \to B$.*

(b) *$B \cap K = T \cap K = M$.* □

Theorem 3.6.3 (Iwasawa Decomposition)[FHHK, Theorem 3.24]**.** *Multiplication induces a continuous bijection $K \times A \times U_+ \to G$.* □

Corollary 3.6.4. *Multiplication induces a continuous surjection $K \times B \to G$.*

Proof. This follows directly from Lemma 3.6.2 and the Iwasawa decomposition. ■

Theorem 3.6.5 [HK19, Appendix A by Hartnick and Köhl, Theorem A.15]**.** *For every $n \geq 0$ the inclusion $K \hookrightarrow G$ induces isomorphisms*

$$\pi_n(K) \hookrightarrow \pi_n(G),$$

hence is a weak homotopy equivalence. In particular, $\pi_1(G) \cong \pi_1(K)$. □

Lemma 3.6.6. *Let* $s_i \neq s_j \in S$. *Then the following hold:*

(a) $P_i = G_i B = K_i B$. *In particular,* $C_{\leq s_i}(B) = K_i B/B$.

(b) $Bs_i s_j B = Bs_i BBs_j B$. *In particular,* $C_{\leq s_i s_j}(B) = K_i BK_j B/B$

Proof. Assertions (a) and (b) follow from [AB08, Remark 8.51] and [AB08, Remark (2) after Theorem 6.56], respectively, and the Iwasawa decomposition $G_i = K_i B_i$. ■

Lemma 3.6.7. *The isomorphisms* $\varphi_i : \mathrm{SL}(2,\mathbb{R}) \to G_i$ *are homeomorphisms. Moreover, let* $B_{\mathrm{SL}(2,\mathbb{R})}$ *be the group of upper triangular matrices in* $\mathrm{SL}(2,\mathbb{R})$*, let* $T_{\mathrm{SL}(2,\mathbb{R})}$ *be the subgroup of diagonal matrices and let* $U_{\pm\beta}$ *denote the canonical root subgroups of* $\mathrm{SL}(2,\mathbb{R})$. *Then*

(a) $\varphi_i(U_{\pm\beta}) = U_{\pm\alpha_i}$.

(b) $\varphi_i(B_{\mathrm{SL}(2,\mathbb{R})}) = B_i$.

(c) $\varphi_i(T_{\mathrm{SL}(2,\mathbb{R})}) = T_i$.

(d) $\varphi_i(\mathrm{SO}(2)) = K_i$.

Proof. By definition of the Kac–Peterson topology, the maps φ_i are continuous, therefore the Open Mapping Theorem 1.2.1 implies that they are homeomorphisms, since the groups G_i carry the Lie topology by Proposition 3.4.8.

(a) holds by definition of U_{α_i}. By Lemma 2.6.3, one has $B_{\mathrm{SL}(2,\mathbb{R})} = N_{\mathrm{SL}(2,\mathbb{R})}(U_\beta)$ and $T_{\mathrm{SL}(2,\mathbb{R})} = N_{\mathrm{SL}(2,\mathbb{R})}(U_\beta) \cap N_{\mathrm{SL}(2,\mathbb{R})}(U_{-\beta})$ which implies (b) and (c), since $B_i = N_{G_i}(U_{\alpha_i})$ and $T_i = N_{G_i}(U_{\alpha_i}) \cap N_{G_i}(U_{-\alpha_i})$. (d) is clear from the definition of K, since $\mathrm{SO}(2)$ is the fixed point set of the transpose inverse map on $\mathrm{SL}(2,\mathbb{R})$. ■

Chapter 4

Computing the Fundamental Groups

Throughout this chapter, $G = G_{\mathbb{R}}(A)$ for a symmetrizable irreducible generalized Cartan matrix A (unless stated otherwise). We determine the fundamental group of G and of its covering spaces $\mathrm{Spin}(\Pi, \kappa)$, introduced in Section 4.3. An important tool is the presentation of the fundamental groups of the generalized flag varieties G/P_J which we determine by way of a CW decomposition.

This chapter is mainly comprised of original results by the author and Ralf Köhl which first appeared in [HK19].

4.1 On the Topology of the Generalized Flag Varieties

Throughout this section, let $J \subseteq I$, and let $W^J \subseteq W$ be a set of representatives of the cosets in W/W_J that have minimal length. The space G/P_J is called **generalized flag variety**. Generalized flag varieties admit Bruhat decompositions which, in the two-spherical case, carry a CW structure. This makes it possible to give a presentation for their fundamental groups by determining characteristic maps for the 1- and 2-cells.

Lemma 4.1.1 (Bruhat decomposition). *One has* $G/P_J = \bigsqcup_{w \in W^J} BwP_J/P_J$.

Proof. By [AB08, Theorem 6.56, Remark (1)], the group G has a Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB = \bigcup_{w \in W} BwBW_JB = \bigcup_{w \in W} BwP_J.$$

Since double cosets partition G, one has $w_1W_J = w_2W_J$ if and only if $Bw_1P_J = Bw_2P_J$ for $w_1, w_2 \in W$. This yields the desired disjoint decomposition of G/P_J. ∎

Definition and Remark 4.1.2. For $w \in W$, define the following restrictions of the canonical map $\psi : G/B \to G/P_J$:

- $\psi_w : BwB/B \to BwP_J/P_J$,
- $\psi_{\overline{w}} : \bigcup_{x \le w} BxB/B \to \bigcup_{x \le w} BxP_J/P_J$.

Since ψ is continuous by Lemma 1.2.4, the same holds for the two restrictions. The space $\bigcup_{x \le w} BxB/B$ is compact by [HKM13, Corollary 3.10] and so $\psi_{\overline{w}}$ is a quotient map. □

Remark 4.1.3. In [HKM13, Proposition 5.9], based on a result by Kramer (see [Kra01, Proposition 7.9]), it is shown that the Bruhat decomposition $G/B = \bigsqcup_{w \in W} BwB/B$ is a CW decomposition provided the twin building associated to G is a smooth strong topological twin building.

Smooth means here that its panels are finite-dimensional real manifolds. These conditions are satisfied since by Corollary 3.5.3 spherical subgroups carry the Lie topology and the twin building of G is a strong topological twin building. By [HKM13, Theorem 1], this also holds in the case of a possibly non-symmetrizable, but two-spherical group G. The following assertions regarding the CW structure of G/P_J are therefore also valid in the two-spherical case; this holds in particular for Theorem 4.1.13 which gives a presentation for the fundamental group of G/P_J. □

Lemma 4.1.4. *Let $w \in W^J$. Then the canonical map ψ_w is a homeomorphism.*

Proof. By Remark 4.1.2, $\psi_{\overline{w}}$ is a quotient map. One has $\psi_{\overline{w}}^{-1}(BwP_J/P_J) = BwB/B$: Let $x \leq w$ such that $BxP_J/P_J = BwP_J/P_J$. Then $x \in BwP_J = BwW_JB$ where the equality holds since by definition of W^J one has $l(ww') = l(w)+l(w')$ for all $w' \in W_J$ which implies $Bww'B = BwBw'B$. The Bruhat decomposition of G yields $x \in wW_J$ and hence, $l(x) \geq l(w)$. This implies $x = w$.

Now, since the Bruhat decomposition $G/B = \bigsqcup_{w\in W} BwB/B$ is a CW decomposition by Remark 4.1.3, the set BwB/B is open in its closure $\bigcup_{x\leq w} BxB/B$ in G/B, and so the preceding observations yield that ψ_w is an injective quotient map and therefore a homeomorphism. ■

Lemma 4.1.5. *Let $s_i \in S$. Then the panel $C_{\leq s_i}(B)$ is homeomorphic to $S^1(\mathbb{R})$.*

Proof. The panel $C_{\leq s_i}(B)$ is a subbuilding of G/B corresponding to the RGD system

$$\{G_i, U_{\alpha_i}, U_{-\alpha_i}, T \cap G_i\}.$$

By Lemma 3.6.7 one has homeomorphisms $G_i \simeq \mathrm{SL}(2,\mathbb{R})$, $T_i \simeq T_{\mathrm{SL}(2,\mathbb{R})}$ and $U_{\pm\alpha_i} \simeq U_{\pm\alpha}$ where $T_{\mathrm{SL}(2,\mathbb{R})}$ denotes the subgroup of diagonal matrices and $U_{\pm\alpha}$ denote the canonical root subgroups of $\mathrm{SL}(2,\mathbb{R})$. This implies that $C_{\leq s_i}(B)$ is homeomorphic to the building $\mathrm{SL}(2,\mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})} \simeq \mathbb{P}_1(\mathbb{R}) \simeq S^1(\mathbb{R})$. ■

Proposition 4.1.6. *For each $w \in W$, the set $C_w(B) = BwB/B$ is a cell of dimension $l(w)$ that is open in its compact closure $C_{\leq w}(B)$ in G/B. For each subset $J \subseteq I$, the Bruhat decomposition $G/P_J = \bigsqcup_{w\in W^J} BwP_J/P_J$ is a CW decomposition.*

Proof. The first statement is immediate by [HKM13, Corollary 3.10 and Proposition 5.9], see also [Kra01, p. 170, 171]. Furthermore, [HKM13, Proposition 5.9] states that the Bruhat decomposition of G/B is a CW decomposition (see Remark 4.1.3). By Lemma 4.1.4, G/P_J is composed of cells that are homeomorphic to cells in G/B, so composing the characteristic maps of the latter cells with the canonical map $\psi : G/B \to G/P_J$ yields characteristic maps for the cells in G/P_J.

For the closure-finiteness, let BwP_J/P_J be a cell in G/P_J. Since ψ is continuous and restricts to a homeomorphism $BwB/B \to BwP_J/P_J$, it maps $\mathrm{cl}\, BwB/B$ surjectively onto $\mathrm{cl}\, BwP_J/P_J$. Now, $\mathrm{cl}\, BwB/B = \bigcup_{x\leq w} BxB/B$, which implies that

$$\mathrm{cl}\, BwP_J/P_J = \bigcup_{x\leq w} BxP_J/P_J = \bigcup_{\substack{x\leq w\\ x\in W^J}} BxP_J/P_J,$$

where the last equality holds since $W_J \subseteq P_J$. This proves that $\mathrm{cl}\, BwP_J/P_J$ is contained in a finite union of cells.

It remains to show that G/P_J has the weak topology determined by the cell closures.

First note that for any $v \in W$ and any minimal-length representative $\tilde{v} \in W^J$ of vW_J, one has $BvP_J/P_J = B\tilde{v}P_J/P_J$. Let $e_v := BvP_J/P_J = B\tilde{v}P_J/P_J$ and $e'_v := BwB/B$. Let $\overline{e}_v = \mathrm{cl}\, e_v = \bigcup_{x\leq\tilde{v}} BxP_J/P_J$ and $\overline{e}'_v := \mathrm{cl}\, e'_v = \bigcup_{x\leq v} BxB/B$.

Let A be a closed subset of G/P_J and let e_w, $w \in W^J$, be an arbitrary cell. Then $\psi^{-1}(A)$ is closed in G/B since ψ is continuous, so $\psi^{-1}(A) \cap \overline{e}'_w$ is closed in $\overline{e}'_w$ since G/B is a CW complex. Now,

$$\psi^{-1}(A) \cap \overline{e}'_w = \psi^{-1}(A) \cap \psi^{-1}(\overline{e}_w) = \psi^{-1}(A \cap \overline{e}_w) = \psi_{\overline{w}}^{-1}(A \cap \overline{e}_w).$$

Since $\psi_{\overline{w}}$ is a quotient map by Remark 4.1.2, this implies that $A \cap \overline{e}_w$ is closed in $\overline{e}_w$.

Now, let A be a subset of G/P_J such that $A \cap \overline{e}_w$ is closed in $\overline{e}_w$ for all $w \in W^J$. Since for each $w \in W$ one has $e_w = e_{\tilde{w}}$ for any minimal-length representative $\tilde{w} \in W^J$ of wW_J, in fact $A \cap \overline{e}_w$ is closed in $\overline{e}_w$ for all $w \in W$. Therefore $\psi_{\overline{w}}^{-1}(A \cap \overline{e}_w)$ is closed in $\overline{e}'_w$ for all $w \in W$. Since $\psi_{\overline{w}}^{-1}(A \cap \overline{e}_w) = \psi^{-1}(A) \cap \overline{e}'_w$, the fact that G/B is a CW complex implies that $\psi^{-1}(A)$ is closed in G/B. Since ψ is open by Lemma 1.2.4, it follows that A is closed in G/P_J. This proves that G/P_J is a CW complex. ∎

Notation 4.1.7. Define $R : [0, 1] \to \mathrm{SO}(2), s \mapsto \begin{pmatrix} \cos(s\pi) & -\sin(s\pi) \\ \sin(s\pi) & \cos(s\pi) \end{pmatrix}$. □

Lemma 4.1.8. *R induces a continuous, surjective map $\tilde{R} : [0, 1] \to \mathrm{SL}(2, \mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})}$ which maps the interior $(0, 1)$ homeomorphically onto its image and maps the boundary $\{0, 1\}$ surjectively onto its image.*

Proof. Let $\{x_0\} := \langle (1 \;\; 0)^\intercal \rangle \in \mathbb{P}^1$ where $\mathbb{P}^1$ denotes the real projective line, modelled as the subset of one-dimensional subspaces of $\mathbb{R}^2$. Since each one-dimensional subspace in $\mathbb{P}^1 \setminus \{x_0\}$ contains exactly one element in the upper half circle $R([0, 1]) \cdot (1 \;\; 0)^\intercal$ while x_0 contains the two boundary points corresponding to $R(0)$ and $R(1)$, one has a surjection from $[0, 1]$ onto $\mathbb{P}^1$ given by $t \mapsto \langle R(t) \cdot (1 \;\; 0)^\intercal \rangle$ which maps $(0, 1)$ bijectively onto $\mathbb{P}^1 \setminus \{x_0\}$. Since $\mathrm{SL}(2, \mathbb{R})$ acts transitively on the real projective line $\mathbb{P}^1$ with $B_{\mathrm{SL}(2,\mathbb{R})}$ being the stabilizer of $x_0 := \langle (1 \;\; 0)^\intercal \rangle$, one has a bijective correspondence $gB \mapsto gx_0$ between $\mathrm{SL}(2, \mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})}$ and $\mathbb{P}^1$. This yields the desired surjectivity and bijectivity properties of $\tilde{R}$. Continuity is clear, as well as the fact that the restriction to the interior is a homeomorphism. ∎

Definition 4.1.9. Let $D^1 = [0, 1]$ be the one-dimensional unit disc and note that D^2 is homeomorphic to $D^1 \times D^1$. Let $p : G \to G/B$ be the canonical projection. Define $\chi_i : D^1 \to G/B$ and $\chi_{(i,j)} : D^1 \times D^1 \to G/B$ by

- $\chi_i(s) := p(\varphi_i(R(s))) = \varphi_i(R(s)) \cdot B$,
- $\chi_{(i,j)}(s, t) := p(\varphi_i(R(s))\varphi_j(R(t))) = \varphi_i(R(s))\varphi_j(R(t)) \cdot B$. □

The following lemma was inspired by [Pro07, Ch. 10, second Proposition of 6.8], see also [Kac85, §2.6, p.198].

Lemma 4.1.10. *The maps defined above are characteristic maps for the following cells:*

(a) χ_i *for* $C_{s_i}(B) = Bs_iB/B$,

(b) $\chi_{(i,j)}$ *for* $C_{s_is_j}(B) = Bs_is_jB/B$.

Proof. By Lemma 3.6.7, the map φ_i is a homeomorphism which maps $\mathrm{SO}(2)$ to K_i.

(a): One has to show that $\chi_i([0, 1]) \subseteq C_{\leq s_i}(B)$ and that χ_i is a continuous map which maps $(0, 1)$ homeomorphically to $C_{s_i}(B)$. The first assertion is clear, since by Lemma 3.6.6 one has $C_{\leq s_i} = G_iB/B$.

By Lemma 3.6.6, one has $C_{s_i}(B) = \{kB \mid k \in K_i \setminus (K_i \cap B)\}$. Let $k \in K_i \setminus (K_i \cap B)$. Then $\varphi_i^{-1}(p^{-1}(kB)) = \varphi_i^{-1}(k) \cdot B_{\mathrm{SL}(2,\mathbb{R})} \in \mathrm{SL}(2,\mathbb{R})/B_{\mathrm{SL}(2,\mathbb{R})} \setminus B_{\mathrm{SL}(2,\mathbb{R})}$. By Lemma 4.1.8, there exists a unique $s \in (0,1)$ satisfying $R(s)B_{\mathrm{SL}(2,\mathbb{R})} = \varphi_i^{-1}(k)B_{\mathrm{SL}(2,\mathbb{R})}$. Hence, s is the unique preimage of kB under χ_i. This yields the desired bijectivity property. The continuity properties are clear.

(b): Since by Lemma 3.6.6 (c) one has $C_{\leq s_is_j}(B) = K_iBK_jB/B$, it is clear that $\chi_{(i,j)}([0,1] \times [0,1]) \subseteq C_{\leq s_is_j}(B)$. For the injectivity of the restriction, let $(s,t), (\tilde{s},\tilde{t}) \in (0,1)^2$ such that $\chi_{(i,j)}(s,t) = \chi_{(i,j)}(\tilde{s},\tilde{t})$. Then

$$\begin{aligned} &\varphi_i(R(s))\varphi_j(R(t))B = \varphi_i(R(\tilde{s}))\varphi_j(R(\tilde{t}))B \\ \iff &(\varphi_i(R(\tilde{s})))^{-1}\varphi_i(R(s))\varphi_j(R(t))B = \varphi_j(R(\tilde{t}))B \in C_{s_j}(B). \end{aligned}$$

This implies $R(\tilde{s})^{-1}R(s) \in B_{\mathrm{SO}(2,\mathbb{R})}$, since otherwise the left expression is in $C_{s_is_j}(B)$, contradicting $C_{s_is_j}(B) \cap C_{s_j}(B) = \emptyset$. Since $s, \tilde{s} \in (0,1)$, one obtains $\tilde{s} = s$. It follows that $\chi_j(t) = \chi_j(\tilde{t})$, hence $t = \tilde{t}$ by (a).

For the surjectivity, note that by Lemma 3.6.6 (c), one has $C_{s_is_j}(B) = Bs_is_jB/B = Bs_iBBs_jB/B$. Let x_ix_jB be an arbitrary element of $C_{s_is_j}(B)$ with $x_i = b_1s_ib_2 \in Bs_iB$ and $x_j \in Bs_jB$. By (a), there exists an $s \in (0,1)$ with $\varphi_i(R(s))B = b_1s_iB \in C_{s_i}(B)$. Hence, there exists a $b \in B$ with $(\varphi_i(R(s))b = b_1s_ib_2 = x_i$. Again by (a), there exists a $t \in (0,1)$ with $\varphi_j(R(t))B = bx_jB \in C_{s_j}(B)$. This yields

$$\begin{aligned} \chi_{i,j}(s,t) &= \varphi_i(R(s)) \cdot \varphi_j(R(t))B \\ &= x_ib^{-1} \cdot bx_jB \\ &= x_ix_jB. \end{aligned}$$

This proves that $\chi_{i,j}$ maps $(0,1) \times (0,1)$ bijectively to $C_{s_is_j}(B)$ The continuity properties are clear. ■

Notation 4.1.11. For $i,j \in I$, let $\varepsilon(i,j) := (-1)^{a_{ij}}$, where a_{ij} is the (i,j)-entry of the generalized Cartan matrix A. □

Lemma 4.1.12 [GHKW17, Remark 15.4(1)]. *Let $e_i := \varphi_i(-I) \in G_i$ and $k_j \in K_j$. Then $e_ik_je_i = k_j^{\varepsilon(i,j)}$.* □

The following theorem gives a presentation for the fundamental group of G/P_J. We refer to [Wig98] for the analogous result in the finite-dimensional situation.

Theorem 4.1.13. *Let $G = G_{\mathbb{R}}(A)$ for a symmetrizable irreducible generalized Cartan matrix A or let G be two-spherical. Then a presentation of $\pi_1(G/P_J)$ is given by*

$$\left\langle x_i; \quad i \in I \mid x_ix_j^{\varepsilon(i,j)} = x_jx_i, \quad x_k = 1; \quad i,j \in I, k \in J \right\rangle.$$

Proof. By Lemma 4.1.4 and Proposition 4.1.6, the Bruhat decomposition

$$G/P_J = \bigsqcup_{w \in W^J} BwP_J/P_J$$

is a CW decomposition where each cell BwP_J/P_J has dimension $l(w)$. For each 1-cell Bs_iP_J/P_J and 2-cell $Bs_is_jP_J/P_J$, the compositions $\tilde{\chi}_i := \psi_{s_i} \circ \chi_i$ and $\tilde{\chi}_{(i,j)} := \psi_{s_is_j} \circ \chi_{(i,j)}$ are, respectively, characteristic maps (ψ_{s_i} and $\psi_{s_is_j}$ denoting the canonical homeomorphisms from Lemma 4.1.4).

Lemma 1.1.2 gives a presentation of $\pi_1(G/P_J)$. The generating elements are given by the homotopy classes $x_i := [\tilde{\chi}_i]$ of the characteristic maps of the 1-cells – namely, the cells Bs_iP_J/P_J where $i \in I \setminus J$. For the homotopy classes x_k with $k \in J$, note that $\varphi_k(R(t)) \in G_k \subseteq P_J$, and so $\tilde{\chi}_k(t) = \varphi_k(r(t)) \cdot P_J = P_J$ which implies $x_k = [\tilde{\chi}_k] = 1_{\pi_1(G/P_J)}$. This yields the desired generating set as well as the trivial relation $x_k = 1$ for $i \in J$.

To obtain the set of relators, for $k = 1, \ldots, 4$ let $\gamma_k : [0,1] \to [0,1] \times [0,1]$ where

$$\begin{aligned}\gamma_1(t) &= (t, 0),\\ \gamma_2(t) &= (1, t),\\ \gamma_3(t) &= (1-t, 1),\\ \gamma_4(t) &= (0, 1-t).\end{aligned}$$

Then the concatenation $\gamma := \gamma_1 * \gamma_2 * \gamma_3 * \gamma_4$ is a loop in the relative boundary $\partial([0,1]\times[0,1]) \simeq S^1$ which generates its fundamental group. Moreover, for each characteristic map $\tilde{\chi}_{(i,j)}$ of a 2-cell, one has $\tilde{\chi}_{(i,j)}(\gamma(0)) = \tilde{\chi}_{(i,j)}((0,0)) = \psi_{s_is_j}(\chi_{(i,j)}(0,0)) = \psi_{s_is_j}(B) = P_J$ where P_J is the unique 0-cell of the CW complex. Therefore, Lemma 1.1.2 implies that the set of relators is given by $\{[\tilde{\chi}_{(i,j)} \circ \gamma] \mid \sigma_i\sigma_j \in W^J, l(\sigma_i\sigma_j) = 2\}$. Now,

$$[\tilde{\chi}_{(i,j)} \circ \gamma] = [\tilde{\chi}_{(i,j)} \circ \gamma_1] \cdot [\tilde{\chi}_{(i,j)} \circ \gamma_2] \cdot [\tilde{\chi}_{(i,j)} \circ \gamma_3] \cdot [\tilde{\chi}_{(i,j)} \circ \gamma_4],$$

where

$\tilde{\chi}_{(i,j)}(s,t) = \alpha_i(R(s))\alpha_j(R(t)) \cdot P_J$ with $R(0) = I_{\mathrm{SO}(2,\mathbb{R})}, R(1) = -I_{\mathrm{SO}(2,\mathbb{R})} \in B_{\mathrm{SO}(2,\mathbb{R})}$ which implies

$$\begin{aligned}[\tilde{\chi}_{(i,j)} \circ \gamma_1] &= x_i,\\ [\tilde{\chi}_{(i,j)} \circ \gamma_3] &= x_i^{-1},\\ [\tilde{\chi}_{(i,j)} \circ \gamma_4] &= x_j^{-1}.\end{aligned}$$

Moreover,

$$\begin{aligned}(\tilde{\chi}_{(i,j)} \circ \gamma_2)(t) &= \alpha_i(-I)\alpha_j(R(t)) \cdot P_J\\ &= \alpha_i(-I)\alpha_j(R(t))\alpha_i(-I) \cdot P_J, \quad \text{since } \alpha_i(-I) \in P_J\\ &= \alpha_j(R(t))^{\varepsilon(i,j)} \cdot P_J \quad \text{by Lemma 4.1.12.}\end{aligned}$$

Since $R(t)^{-1} = R(1-t)$, this yields $[\tilde{\chi}_{(i,j)} \circ \gamma_2] = x_j^{\varepsilon(i,j)}$. One therefore obtains $[\tilde{\chi}_{(i,j)} \circ \gamma] = x_i \cdot x_j^{\varepsilon(i,j)} \cdot x_i^{-1} \cdot x_j^{-1}$. This proves the assertion. ■

By [HKM13, Proposition 3.8] (see also [Kra01, p. 170, 171]), there exists a fibre bundle $(C_{\leq\sigma_i\sigma_j}, C_{\leq\sigma_i}, p, C_{\leq\sigma_j})$. Since the panels $C_{\leq\sigma_i}$ are homeomorphic to $S^1(\mathbb{R})$ by Lemma 4.1.5, it is a so-called circle bundle over a circle, that is, both base space and fibre are circles. By [Ste99, Section 26.1], there are two types of circle bundles over circles: The torus $S^1 \times S^1$, which is the trivial bundle, and the Klein bottle. Hence, each 2-cell $C_{\leq\sigma_i\sigma_j}(B)$ must be homeomorphic to one of those spaces. This observation can be specified using Lemma 1.1.2 and the computations in the above proof: One obtains the presentation $\pi_1(C_{\sigma_i\sigma_j}(B)) \cong \left\langle x_i, x_j \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i \right\rangle$ which yields the following result.

Proposition 4.1.14. *Let $\sigma_i, \sigma_j \in W$. Then the following hold:*

- *If $\varepsilon(i,j) = -1$, then $C_{\sigma_i\sigma_j}(B)$ is homeomorphic to a Klein bottle.*

- *If* $\varepsilon(i, j) = 1$*, then* $C_{\sigma_i \sigma_j}(B)$ *is homeomorphic to a torus.* □

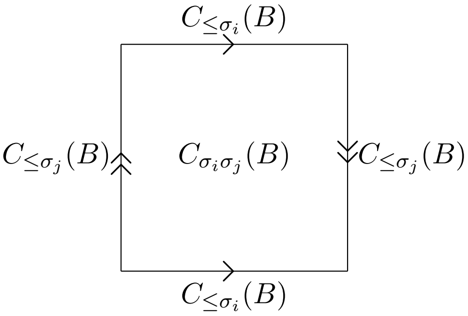


Figure 4.1: Klein Bottle

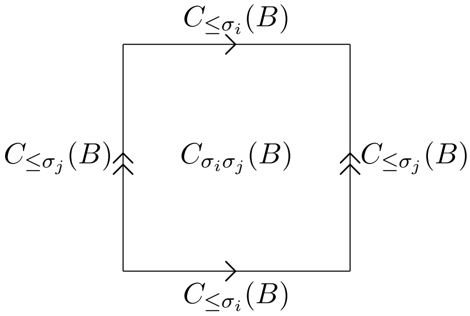


Figure 4.2: Torus

Lemma 4.1.15. *Let* G *be simply laced and* $J \neq \emptyset$*. Then* $\pi_1(G/P_J) \cong {C_2}^{n-|J|}$*.*

Proof. For each generator x_h in the presentation of Theorem 4.1.13, one has $x_h^2 = 1$: Let Π be the Dynkin diagram of G and λ the labelling map $I \to V$ of the vertex set of Π. Since Π is connected, one has a minimal path $(i_1, \ldots, i_m = h)^\lambda$ in Π such that $i_1 \in J$. If $m = 1$, one has $x_h = 1$ by the presentation above. Let $x_{i_1}, \ldots, x_{i_{m-1}}$ have order ≤ 2. Since Π is simply laced, $\varepsilon(m-1, h) = -1 = \varepsilon(h, m-1)$ which implies $x_h x_{i_{m-1}}^{-1} x_h^{-1} x_{i_{m-1}}^{-1} = 1$ and $x_{i_{m-1}} x_h^{-1} x_{i_{m-1}}^{-1} x_h^{-1} = 1$. Multiplying these expressions yields $x_h^2 = 1$.

Since each generator has order ≤ 2, the relations show that the group is abelian. One concludes that $\pi_1(G/P_J) \cong {C_2}^{n-|J|}$. ■

The fundamental group of the generalized flag variety will be used in the next sections as a tool to compute the fundamental group of K which is isomorphic to the fundamental group of G by Theorem 3.6.5. To this end, we state some results which relate the generalized flag variety to quotient spaces of K.

Lemma 4.1.16. *The canonical map* $\psi : K/(K \cap P_J) \to G/P_J$ *is a homeomorphism.*

Proof. Bijectivity follows from the product formula for subgroups since $G = KP_J$ by Corollary 3.6.4. By Lemma 1.2.4, the map $\tilde{\psi} : G/(K \cap P_J) \to G/P_J$ is continuous, so the same holds for its bijective restriction $\psi : K/(K \cap P_J) \to G/P_J$.

In order to show that ψ is closed, let $P := P_J$ and let $\tilde{P} := P_J \cap K$. Consider the commutative diagram

$$\begin{array}{ccc} K/\tilde{P} & & \\ \big\downarrow \iota & \searrow^{\psi} & \\ G/\tilde{P} & \xrightarrow{\varphi} & G/P \end{array}$$

where ι denotes the canonical embedding and φ denotes the canonical map from $G/\tilde{P}$ to G/P. Since K is closed in G, the map ι is closed. By Lemma 1.2.4, φ is open.

Let $X\tilde{P} \subseteq K/\tilde{P}$ be a closed subset of $K/\tilde{P}$ and suppose that $\psi(X\tilde{P}) = XP$ is not closed in G/P. Then the complement $\complement_{G/P}(XP)$ is not open in G/P, hence the complement $\complement_{G/\tilde{P}}(\varphi^{-1}(XP)) = \varphi^{-1}(\complement_{G/P}(XP))$ is not open in $G/\tilde{P}$. Therefore, $\varphi^{-1}(XP)$ is not closed in $G/\tilde{P}$. This yields that $X\tilde{P} = \psi^{-1}(XP) = \iota^{-1}(\varphi^{-1}(XP))$ is not closed in $K/\tilde{P}$, a contradiction. ■

Corollary 4.1.17. *There exists a homeomorphism $G/P_J \to K/(K \cap T)K_J$.*

Proof. Since $P_J = G_J B$ and $\theta(P_J) \cap P_J = G_J T$, one has $P_J \cap K = K_J(K \cap T)$. Furthermore, G_J is normal in $G_J T$ which implies $K_J(K \cap T) = (K \cap T)K_J$. The claim now follows from Lemma 4.1.16. ■

Lemma 4.1.18. *The canonical map $\psi : K/K_J \to K/(K \cap T)K_J$ is a covering map of degree $2^{n-|J|}$.*

Proof. By Lemma 1.2.4, ψ is continuous, open and surjective. The group $M = (K \cap T)$ has order 2^n, as is pointed out in section 3.6.

Note that one has $T_J \cap T_{I\setminus J} = \{1\}$, since T is isomorphic to $(\mathbb{R}^\times)^n$. Now, for $k \in K$ one has $\psi^{-1}(k(K \cap T)K_J) = \{ktK_J \mid t \in (K \cap T)\}$, and since $T_J \cap T_{I\setminus J} = \{1\}$, one has $kt_iK_J \neq kt_jK_J$ for $t_i \neq t_j \in T \cap K_{I\setminus J}$. This yields $|\psi^{-1}(k(K \cap T)K_J)| = |\{ktK_J \mid t \in (K \cap T)\}| = |\{ktK_J \mid t \in T \cap K_{I\setminus J}\}| = |T \cap K_{I\setminus J}| = |T_{I\setminus J} \cap K_{I\setminus J}| = 2^{n-|J|}$. Lemma 1.2.3 now shows that ψ is a covering map. ■

4.2 The Simply Laced Case

Throughout this section, let G be simply laced. Denote by E the edge set of the simply laced Dynkin diagram Π of G. As shown in a paper by Ghatei, Horn, Köhl and Weiss, K has a 2-fold central extension $\mathrm{Spin}(\Pi)$. By proving that $\pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij}) \cong \pi_1(K/K_{ij}) \cong \{1\}$ for an A_2 subdiagram Π_{ij}, we establish via a homotopy exact sequence that $\mathrm{Spin}(\Pi)$ is simply connected which yields that $\pi_1(G) \cong \pi_1(K) \cong C_2$.

In [GHKW17, Section 11], it is shown that in the simply laced case K is isomorphic to the canonical universal enveloping group of an SO(2)-amalgam $\mathcal{A}(\Pi, \mathrm{SO}(2)) = \{H_{ij}, \phi^i_{ij} \mid i \neq j \in I\}$ where

$$H_{ij} = \begin{cases} \mathrm{SO}(3), & \text{if } \{i,j\} \in E, \\ \mathrm{SO}(2) \times \mathrm{SO}(2), & \text{if } \{i,j\} \notin E, \end{cases}$$

and where for $i < j$, the map ϕ^i_{ij} embeds SO(2) into H_{ij} as upper left diagonal block if $(i,j) \in E$ and as first SO(2)-factor if $(i,j) \notin E$ (for $i > j$ replace upper left by lower right and first factor by second factor). The corresponding SO(3) and SO(2)×SO(2) subgroups of K are the intersections of K with the fundamental rank two subgroups G_{ij} of G, i.e., the groups $K_{ij} := G^\theta_{ij}$.

Furthermore, again in [GHKW17, Section 11], the group $\mathrm{Spin}(\Pi)$ is defined as the canonical universal enveloping group of the Spin(2)-amalgam $\mathcal{A}(\Pi, \mathrm{Spin}(2)) = \{\hat{H}_{ij}, \hat{\phi}^i_{ij} \mid i \neq j \in I\}$ which is a lift of the amalgam above in the sense that the groups $\hat{H}_{ij}$ are equal to either Spin(3) or to Spin(2) × Spin(2) and the connecting homomorphisms $\hat{\phi}^i_{ij}$ are lifts of the connecting homomorphisms ϕ^i_{ij} above.

Theorem 4.2.1 [GHKW17, Theorem 11.17]**.** $\mathrm{Spin}(\Pi)$ *is a 2-fold central extension of K.* □

Since the extension $\mathrm{Spin}(\Pi) \to K$ has finite kernel, the Kac–Peterson topology on K induces a unique topology on $\mathrm{Spin}(\Pi)$ that turns the extension into a covering map. The resulting group topology on $\mathrm{Spin}(\Pi)$ is called the **Kac–Peterson topology** on $\mathrm{Spin}(\Pi)$.

Lemma 4.2.2. *Let Π be simply laced and let $\{i, j\} \subseteq I$ be the index set of an A_2-subdiagram of Π. Then the spaces $\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})$ and K/K_{ij} are homeomorphic.*

Proof. From [GHKW17] (exact references below) it follows that the kernel of the covering map $\mathrm{Spin}(\Pi) \to K$ coincides with the kernel of the covering map $\mathrm{Spin}(\Pi_{ij}) \to K_{ij}$ and is equal to the group $Z := \{\pm 1_{\mathrm{Spin}(\Pi)}\}$ (for the definition of $-1_{\mathrm{Spin}(\Pi)}$, see below). This is a consequence of the following facts regarding an irreducible simply laced diagram Π (all referring to [GHKW17]):

- There is an epimorphism $\mathrm{Spin}(2) \to \mathrm{SO}(2, \mathbb{R})$ with kernel $\{\pm 1_{\mathrm{Spin}(2)}\}$ (see [Theorem 6.8]).
- In $\mathrm{Spin}(\Pi)$, all elements $\tilde{\tau}_{ij}(\tilde{\phi}^i_{ij}(-1_{\mathrm{Spin}(2)}))$ coincide (see [Lemma 11.7]).
- Let $-1_{\mathrm{Spin}(\Pi)} := \tilde{\tau}_{ij}(\tilde{\phi}^i_{ij}(-1_{\mathrm{Spin}(2)}))$ for an arbitrary pair $i \neq j \in I$. Then $1_{\mathrm{Spin}(\Pi)} \neq -1_{\mathrm{Spin}(\Pi)}$ (see [Corollary 11.16]).
- $\mathrm{Spin}(\Pi)$ is a 2-fold central extension of $K(\Pi)$ (see [Theorem 11.17]).

Hence, the 2-fold covering map $\tilde{\varphi} : \mathrm{Spin}(\Pi) \to K(\Pi)$ induces a continuous bijective map $\varphi : \mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij}) \to (\mathrm{Spin}(\Pi)/Z)/(\mathrm{Spin}(\Pi_{ij})/Z) \to K/K_{ij}$. One has a commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}(\Pi) & \xrightarrow{\quad\tilde{\varphi}\quad} & K \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_2} \\ \mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij}) & \xrightarrow{\ \varphi\ } & K/K_{ij} \end{array} .$$

Since $\tilde{\varphi}$ is open as a covering map and π_2 is open by Lemma 1.2.4, it follows that φ is a homeomorphism. ■

Corollary 4.2.3. *Let Π be simply laced. Then K/K_J is simply connected.*

Proof. K/K_J is connected since K is generated by connected groups isomorphic to $\mathrm{SO}(2, \mathbb{R})$. Hence by Lemma 4.1.18 it is a non-trivial cover of $K/(K \cap T)K_J$ of degree $2^{n-|J|}$. The claim now follows from Corollary 4.1.15 and Corollary 4.1.17. ■

Proposition 4.2.4. *Let Π be simply laced. Then $\mathrm{Spin}(\Pi)$ is simply connected with respect to the Kac–Peterson topology. In particular, $\pi_1(G) \cong C_2$.*

Proof. By Corollary 1.2.10, there exists a homotopy long exact sequence

$$\begin{aligned} \pi_4(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) &\longrightarrow \pi_3(\mathrm{Spin}(\Pi_{ij})) \longrightarrow \pi_3(\mathrm{Spin}(\Pi)) \longrightarrow \pi_3(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \\ &\longrightarrow \pi_2(\mathrm{Spin}(\Pi_{ij})) \longrightarrow \pi_2(\mathrm{Spin}(\Pi)) \longrightarrow \pi_2(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \\ &\longrightarrow \pi_1(\mathrm{Spin}(\Pi_{ij})) \longrightarrow \pi_1(\mathrm{Spin}(\Pi)) \longrightarrow \pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \end{aligned}$$

from which one extracts the exact sequence

$$\{1\} = \pi_1(\mathrm{Spin}(\Pi_{ij})) \longrightarrow \pi_1(\mathrm{Spin}(\Pi)) \longrightarrow \pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})).$$

By Corollary 4.2.3 and Lemma 4.2.2 one has $\pi_1(\mathrm{Spin}(\Pi)/\mathrm{Spin}(\Pi_{ij})) \cong \pi_1(K/K_{ij}) = \{1\}$ and so by exactness $\pi_1(\mathrm{Spin}(\Pi)) = \{1\}$.

The second assertion follows from the fact that $\pi_1(G) \cong \pi_1(K)$ by Theorem 3.6.5 and the fact that $\mathrm{Spin}(\Pi)$ is a 2-fold central extension of K by Theorem 4.2.1. ■

4.3 The General Case

Once more, we remind the reader that $G = G_{\mathbb{R}}(A)$ for a symmetrizable irreducible generalized Cartan matrix A.

We introduce a modified Dynkin diagram Π^{adm} and a colouring of the connected components of Π^{adm} which depends on the edge types of Π. We then show that $\pi_1(G/B)$ is isomorphic to a direct product of groups H_J that correspond to the connected components Π^{adm}_J of Π^{adm} and depend on their respective colour. For each connected component Π^{adm}_J where $\overline{J} := I \setminus J$, we obtain a commutative diagram

$$\begin{array}{ccccccc} & \pi_1(G) & \cong & \pi_1(K) & \longrightarrow & \pi_1(K/K_{\overline{J}}) & \\ & & & \downarrow & & \downarrow & \\ \prod\limits_{\substack{\text{Connected} \\ \text{components}}} & H_{J_i} \cong \pi_1(G/B) & \cong & \pi_1(K/(K\cap T)) & \longrightarrow & \pi_1(K/(K\cap T)K_{\overline{J}}) & \cong H_J \end{array}$$

This allows us to piece together the group $\pi_1(G)$ as a product of subgroups of the H_J whose respective isomorphism types can be exactly determined by the degree of the covering $K/K_{\overline{J}} \to K/K_{\overline{J}}(K \cap T)$.

Definition 4.3.1 [GHKW17, Definition 16.2]. For the generalized Dynkin diagram $\Pi = (V, E)$ of A, let Π^{adm} be the graph on the vertex set V with edge set

$$\{\{i,j\} \in V \times V \mid i \neq j \in I, \varepsilon(i,j) = \varepsilon(j,i) = -1\},$$

where $\varepsilon(i,j)$ denotes the parity of the corresponding Cartan matrix entry, as defined in Notation 4.1.11.

An **admissible colouring** of Π is a map $\kappa : V \to \{1, 2\}$ such that

(a) $\kappa(i^\lambda) = 1$ whenever there exists $j \in I \setminus \{i\}$ with $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$.

(b) the restriction of κ to any connected component of the graph Π^{adm} is a constant map.

Define $c(\Pi, \kappa)$ to be the number of connected components of Π^{adm} on which κ takes the value 2. For a subgraph Π^{adm}_J of Π^{adm} that is a union of connected components of Π^{adm} let κ_J be the corresponding restriction of κ. □

Definition 4.3.2. For the generalized Dynkin diagram $\Pi = (I, E)$ of A, let Π^{adm} be the graph on the vertex set I with edge set

$$\{\{i,j\} \in I \times I \mid i \neq j, \varepsilon(i,j) = \varepsilon(j,i) = -1\}.$$

Let the colouring $\gamma : I \to \{r, g, b\}$ of Π^{adm} be defined as follows:

(a) The restriction of γ to any connected component of the graph Π^{adm} is a constant map. □

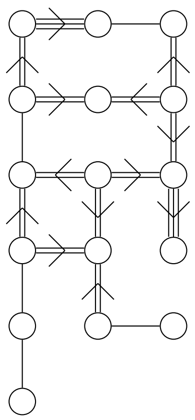
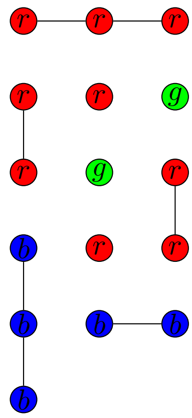
(b) $\gamma(i) = r$ whenever there exists $j \in I \setminus \{i\}$ with $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$.

(c) Else, $\gamma(i) = g$ whenever the vertex i is isolated in Π^{adm}.

(d) $\gamma(i) = b$ whenever the vertex i is not isolated in Π^{adm} and case (1) does not apply to any of the vertices in its connected component.

The notion of an admissible colouring κ will not come into play until we introduce the so-called spin group with respect to Π and κ.

Example 4.3.3. A sample Dynkin diagram Π with the corresponding diagram Π^{adm}, coloured by the map γ.

Π	Π^{adm} coloured by γ
	

Notation 4.3.4. For a subset $J \subseteq I$ let

$$H_J := \left\langle x_i; \quad i \in J \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i, ; \quad i,j \in J \right\rangle. \qquad \square$$

Lemma 4.3.5. *Let $J \subseteq I$ be the index set of a connected component Π_J^{adm} of Π^{adm}. Then the following hold:*

(a) If Π_J^{adm} has colour r, then $H_J \cong C_2^{|J|}$.

(b) If Π_J^{adm} has colour g, then $|J| = 1$ and $H_J \cong \mathbb{Z}$.

(c) If Π_J^{adm} has colour b, then $|H_J| = 2^{|J|+1}$.

Proof. (a): If Π_J^{adm} has colour r, then there exist $i \in J, j \in I \backslash \{i\}$ with $\varepsilon(i,j) = 1$ and $\varepsilon(j,i) = -1$. This implies $x_i x_j = x_j x_i$ and $x_j x_i^{-1} = x_i x_j$ which yields $x_i^2 = 1$. Now, if $\{i^\lambda, k^\lambda\}$ is an edge in Π^{adm}, then $x_i x_k^{-1} x_i^{-1} x_k^{-1} = 1 = x_k x_i^{-1} x_k^{-1} x_i^{-1}$. Multiplying these expressions shows that $x_i^2 = 1$ implies $x_k^2 = 1$. Since Π_J^{adm} is connected, this yields $x_k^2 = 1$ for each $k \in J$. Commutativity then follows from the relations of H_J.

(b): By definition, two nodes i^λ and j^λ in Π^{adm} are connected if and only if $\varepsilon(i,j) = \varepsilon(j,i) = -1$, and by definition a node i^λ has color g if and only if for each node j^λ this is not the case – namely, one of $\varepsilon(i,j)$ and $\varepsilon(j,i)$ is 1. Nodes of colour g are therefore isolated in Π^{adm}.

(c): Let Π_J^{sl} be the simply laced Dynkin diagram with vertex set J^λ and edge set $\{\{i,j\} \in J \times J \mid \{i,j\}$ edge in $\Pi^{\text{adm}}\}$.

Let $\tilde{T} := K(\Pi_J^{\text{sl}}) \cap T(\Pi_J^{\text{sl}})$ where $T(\Pi_J^{\text{sl}})$ denotes the standard maximal torus of $G(\Pi_J^{\text{sl}})$. Then by Lemma 4.1.18 and Proposition 4.2.4, $\text{Spin}(\Pi_J^{\text{sl}}) \to K(\Pi_J^{\text{sl}}) \to K(\Pi_J^{\text{sl}})/\tilde{T}$ is a universal covering map where $K(\Pi_J^{\text{sl}}) \to K(\Pi_J^{\text{sl}})/\tilde{T}$ has degree $2^{|J|}$ and $\text{Spin}(\Pi_J^{\text{sl}}) \to K(\Pi_J^{\text{sl}})$ has degree 2 according to [GHKW17, Theorem 11.17]. Since $\pi_1(K(\Pi_J^{\text{sl}})/\tilde{T}) \cong H_J$ by Theorem 4.1.13 and Lemma 4.1.16, this implies $|H_J| = 2^{|J|+1}$. ■

Proposition 4.3.6. *Let $J_1 \sqcup \cdots \sqcup J_k = I$ be the index sets of the connected components of Π^{adm}. Then*

$$\pi_1(G/B) \cong H_{J_1} \times \cdots \times H_{J_K}.$$

Proof. By Theorem 4.1.13, $\pi_1(G/B) \cong H_I$ where

$$H_I = \left\langle x_i; \quad i \in I \mid x_i x_j^{\varepsilon(i,j)} = x_j x_i, ; \quad i,j \in I \right\rangle$$

as defined in Notation 4.3.4. For $J \subseteq I$, let

$$R_J := \{x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \mid i,j \in J\}, \tag{4.1}$$

the set of relators of H_J. Let

$$R^c := \bigcup_{\substack{i^\lambda, j^\lambda \text{ in different} \\ \text{conn. components}}} \{x_i x_j x_i^{-1} x_j^{-1}\},$$

the set of commutators of pairs of generators from different connected components. Then

$$H_{J_1} \times \cdots \times H_{J_k} \cong \left\langle x_i; \quad i \in I \mid \bigcup_{l=1}^{k} R_{J_l} \cup R^c \right\rangle =: H.$$

Let π_{H_I} and π_H be the canonical homomorphisms from the free group $\langle x_i; \quad i \in I\rangle$ to H_I and H, respectively. It suffices to show that $\bigcup_{l=1}^{k} R_{J_l} \cup R^c \subseteq \ker \pi_{H_I}$ and $R_I \subseteq \ker \pi_H$. It is clear that a relator $x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \in R_I$ with i^λ and j^λ in a common connected component is contained in $\bigcup_{l=1}^{k} R_{J_l} \subseteq \ker \pi_H$, so let $x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \in R_I$ with i^λ and j^λ in different connected components. Then one has $(\varepsilon(i,j), \varepsilon(j,i)) \in \{(1,1),(1,-1),(-1,1)\}$. If $\varepsilon(i,j) = 1$, then $x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1} \in R^c \subseteq \ker \pi_H$, so let $\varepsilon(i,j) = -1$ and $\varepsilon(j,i) = 1$. Then j^λ is contained in a connected component $\Pi^{\text{adm}}_{J_m}$ of colour r, and by Lemma 4.3.5, $\langle x_l; \quad l \in J_m \mid R_{J_m}\rangle = H_{J_m} \cong C_2^{|J_l|}$. This implies that x_j has order 2 in H_{J_m}, hence $x_j^2 \in \langle\langle R_{J_m}\rangle\rangle_{\langle x_i; i \in I\rangle}$, the normal closure of R_{J_m} in the free group.

Since $\langle\langle R_{J_m}\rangle\rangle_{\langle x_i; i \in I\rangle} \subseteq \ker \pi_H$, one obtains $x_j^2 \in \ker \pi_H$. Since $x_j x_i x_j^{-1} x_i^{-1} \in R^c \subseteq \ker \pi_H$ and $\varepsilon(i,j) = -1$, one therefore has

$$\pi_H(x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1}) = \pi_H(x_j x_i x_j^{-1} x_i^{-1} \cdot x_i x_j^{\varepsilon(i,j)} x_i^{-1} x_j^{-1}) = 1_H.$$

Conversely, it is clear that $\bigcup_{l=1}^{k} R_{J_l} \subseteq R_I \subseteq \ker \pi_{H_I}$, so let $x_i x_j x_i^{-1} x_j^{-1} \in R^c$ with i^λ and j^λ in different connected components. As above, we can assume that $\varepsilon(i,j) = -1$ and $\varepsilon(j,i) = 1$. Since $x_j x_i^{\varepsilon(j,i)} x_j^{-1} x_i^{-1} \in \ker \pi_{H_I}$, this implies

$$\pi_{H_I}(x_i x_j x_i^{-1} x_j^{-1}) = \pi_{H_I}(x_j x_i^{\varepsilon(j,i)} x_j^{-1} x_i^{-1} \cdot x_i x_j x_i^{-1} x_j^{-1}) = 1_{H_I}.$$

This proves the assertion. ■

Theorem 4.3.7. *Let A be a symmetrizable irreducible generalized Cartan matrix A with Dynkin diagram Π. Let $n(g)$ and $n(b)$ be the number of connected components of Π^{adm} of colour g and b, respectively. Then*

$$\pi_1(G_{\mathbb{R}}(A))) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b)}.$$

Proof. As usual, let $G := G_{\mathbb{R}}(A)$. By Theorem 3.6.5, $\pi_1(G) \cong \pi_1(K)$, so it suffices to prove that $\pi_1(K)$ is of the given isomorphism type. Let $J \subseteq I$. The diagram

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & K/K_J \\ \downarrow p & & \downarrow q \\ K/(K\cap T) & \xrightarrow{\psi} & K/(K\cap T)K_J \end{array},$$

with all maps being the respective canonical maps, commutes. Since the maps are continuous by Lemma 1.2.4, one obtains a commutative diagram of induced homomorphisms

$$\begin{array}{ccc} \pi_1(K) & \xrightarrow{\quad\varphi_*\quad} & \pi_1(K/K_J) \\ \big\downarrow{\scriptstyle p_*} & & \big\downarrow{\scriptstyle q_*} \\ \pi_1(K/(K\cap T)) & \xrightarrow{\psi_*} & \pi_1(K/(K\cap T)K_J) \end{array},$$

where p_* and q_* are injective. By Theorem 4.1.13 and Lemma 4.1.16, $\pi_1(K/(K\cap T))$ and $\pi_1(K/(K\cap T)K_J)$ can be identified with $H_I = \langle x_i;\quad i\in I \mid R_I\rangle$ and $\langle x_i;\quad i\in I\mid R_I\cup\{x_J\mid j\in J\}\rangle$, respectively ($R_I$ as in (4.1) in the above proof), where ψ_* corresponds to the canonical homomorphism between these groups as the proof of Theorem 4.1.13 shows.

For the index set J_m of a connected component of Π^{adm}, let $\overline{J}_m := I\setminus J_m$. Then by Proposition 4.3.6,

$$\big\langle x_i;\quad i\in I\mid R_I\cup\{x_J\mid j\in\overline{J}_m\}\big\rangle \cong \left(\prod_{i=1}^{k} H_{J_i}\Big/\prod_{\substack{i=1\\ i\neq m}}^{k} H_{J_i}\right)\cong H_{J_m}.$$

Summing up, one obtains a commutative diagram

$$\begin{array}{ccc} \pi_1(K) & \xrightarrow{\varphi_*} & \pi_1(K/K_{\overline{J}_m}) \\ \big\downarrow{\scriptstyle p_*} & & \big\downarrow{\scriptstyle q_*} \\ \prod_{i=1}^{k} H_{J_i} & \xrightarrow{\quad\pi_m\quad} & H_{J_m} \end{array},$$

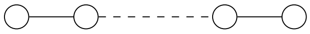
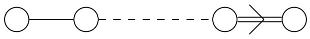
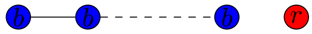
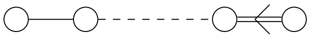
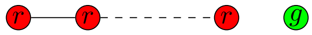
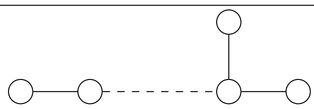
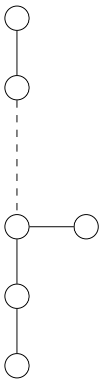
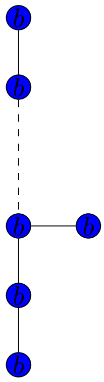
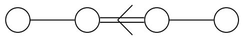
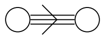
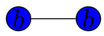
having replaced p_* and q_* from above with the correponding monomorphisms.

By Lemma 4.1.18, the covering $K/K_{\overline{J}_m}\to K/K_{\overline{J}_m}(K\cap T)$ has degree $2^{n-|\overline{J}_m|}=2^{|J_m|}$. This implies that $\tilde{H}_m := q_*(\pi_1(K/K_{\overline{J}_m}))$ is a subgroup of H_{J_m} of index $2^{|J_m|}$. The isomorphism type of $\tilde{H}_m$ is uniquely determined by this index and the isomorphism type of H_{J_m}, given in Lemma 4.3.5: One has

$$\tilde{H}_m \cong \begin{cases} \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } r,\\ 2\mathbb{Z}\cong\mathbb{Z}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } g,\\ C_2 & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b.\end{cases}$$

Again by Lemma 4.1.18, the covering $K\to K/(K\cap T)$ has degree 2^n, so $p_*(\pi_1(K))$ is a subgroup of index 2^n of $\prod_{i=1}^k H_{J_i}$. The commutative diagram above implies that $\pi_1(K)\cong p_*(\pi_1(K))\subseteq \pi_m^{-1}(\tilde{H}_m)$. Since this holds for the index set of every connected component of Π^{adm}, one has $p_*(\pi_1(K))\subseteq \tilde{H}_1\times\cdots\times\tilde{H}_m$. But the latter is a subgroup of index $2^{|J_1|}\cdot\dots\cdot 2^{|J_m|}=2^n$ of $\prod_{i=1}^k H_{J_i}$, so equality holds. This proves the assertion. ■

Example 4.3.8 Isomorphism types of $\pi_1(G_{\mathbb{R}}(A))$ for the spherical Dynkin diagrams.

Π	Π^{adm} coloured by γ	$\pi_1(G(\Pi))$
A_1 		$\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$
A_n 		$\pi_1(\mathrm{SL}_{n+1}(\mathbb{R})) \cong C_2 \quad (n \geq 2)$
B_n 		$\pi_1(\mathrm{Spin}(n, n+1)) \cong \begin{cases} \mathbb{Z} & \text{if } n \leq 2, \\ C_2 & \text{if } n > 2. \end{cases}$
C_n 		$\pi_1(\mathrm{Sp}(2n, \mathbb{R})) \cong \mathbb{Z}$
D_n 		$\pi_1(\mathrm{Spin}(n, n)) \cong C_2 \quad (n \geq 3)$
E_n $6 \leq n \leq 8$ 		$\pi_1(E_n) \cong C_2$
F_4 	F_4 	$\pi_1(F_4) \cong C_2$
G_2 		$\pi_1(G_{2,2}) \cong C_2$

□

Definition and Remark 4.3.9. In [GHKW17, Definition 16.16], the **spin group** $\mathrm{Spin}(\Pi, \kappa)$ **with respect to** Π **and** κ is defined as the universal enveloping group of a particular $\mathrm{Spin}(2)$-amalgam $\{\tilde{G}_{ij}, \tilde{\phi}^i_{ij} \mid i \neq j \in I\}$ where the isomorphism type of $\tilde{G}_{ij}$ depends on the (i,j)- and (j,i)-entries of the Cartan matrix of Π as well as the values of κ on the corresponding vertices. The group K can be regarded as (being uniquely isomorphic to) the universal enveloping group of an $\mathrm{SO}(2,\mathbb{R})$-amalgam $\{G_{ij}, \phi^i_{ij} \mid i \neq j \in I\}$ where each $\tilde{G}_{ij}$ covers G_{ij} via an epimorphism α_{ij}. By [GHKW17, Lemma 16.18] there exists a canonical central extension $\rho_{\Pi,\kappa} : \mathrm{Spin}(\Pi, \kappa) \to K$ that makes the following diagram commute for all $i \neq j \in I$:

$$\begin{array}{ccc} \tilde{G}_{ij} & \xrightarrow{\tilde{\tau}_{ij}} & \mathrm{Spin}(\Pi, \kappa) \\ \downarrow{\scriptstyle \alpha_{ij}} & & \downarrow{\scriptstyle \rho_{\Pi,\kappa}} \\ G_{ij} & \xrightarrow{\tau_{ij}} & K \end{array}$$

Here, $\tilde{\tau}_{ij}$ and τ_{ij} denote the respective canonical maps into the universal enveloping groups.

By [GHKW17, Proposition 3.9], one has

$$\ker(\rho_{\Pi,\kappa}) = \langle \tilde{\tau}_{ij}(\ker(\alpha_{ij})) \mid i \neq j \in I \rangle_{\mathrm{Spin}(\Pi,\kappa)}.$$

Each connected component of Π^{adm} that admits a vertex i^λ with $\kappa(i^\lambda) = 2$ contributes a factor

2 to the order of $\ker(\rho_{\Pi,\kappa})$ so that $\mathrm{Spin}(\Pi,\kappa)$ is a $2^{c(\Pi,\kappa)}$-fold central extension of K.

In particular, this implies that the subspace topology on K defines a unique topology on $\mathrm{Spin}(\Pi)$ that turns the extension into a covering map. The resulting group topology on $\mathrm{Spin}(\Pi,\kappa)$ is called the **Kac–Peterson topology** on $\mathrm{Spin}(\Pi,\kappa)$.

In the case of a simply laced diagram Π, the only admissible colourings are the trivial colouring and the constant colouring $\kappa : V \to \{2\}$ and $\mathrm{Spin}(\Pi,\kappa) = \mathrm{Spin}(\Pi)$ with $\mathrm{Spin}(\Pi)$ as in Section 4.2. □

Theorem 4.3.10. *Let A be a symmetrizable irreducible generalized Cartan matrix A with Dynkin diagram Π. Let $n(g)$ be the number of connected components of Π^{adm} of colour g. Let $n(b,\kappa)$ be the number of connected components of Π^{adm} on which κ takes the value 1 and which have colour b. Then*

$$\pi_1(\mathrm{Spin}(\Pi,\kappa)) \cong \mathbb{Z}^{n(g)} \times C_2^{n(b,\kappa)}.$$

Proof. By [GHKW17, Theorem 17.1], the map $\rho_{\Pi,\kappa} : \mathrm{Spin}(\Pi,\kappa) \to K$ is a $2^{c(\Pi,\kappa)}$-fold central extension. Let J be the index set of a connected component of Π^{adm} and let $\overline{J} := I \setminus \overline{J}$. Let $U_{\overline{J}} := \langle \tilde{G}_{ij} \mid i \neq j \in J \rangle_{\mathrm{Spin}(\Pi,\kappa)}$.

Since $\rho_{\Pi,\kappa}(U_{\overline{J}}) \subseteq K_J$, one has a continuous induced map $\rho^J_{\Pi,\kappa} : \mathrm{Spin}(\Pi,\kappa)/\mathrm{Spin}(\Pi_{\overline{J}},\kappa_{\overline{J}}) \to K/K_{\overline{J}}$ making the following diagram commute, where $\tilde{\varphi}$ and φ denote the respective canonical maps:

$$\begin{array}{ccc}
\mathrm{Spin}(\Pi,\kappa) & \xrightarrow{\tilde{\varphi}} & \mathrm{Spin}(\Pi,\kappa)/U_{\overline{J}} \\
\big\downarrow{\scriptstyle \rho_{\Pi,\kappa}} & & \big\downarrow{\scriptstyle \rho^J_{\Pi,\kappa}} \\
K & \xrightarrow{\quad\varphi\quad} & K/K_{\overline{J}}
\end{array}$$

Each fiber of $\rho^J_{\Pi,\kappa}$ has cardinality

$$\begin{aligned}
|\{xU_{\overline{J}} \mid x \in \ker\rho_{\Pi,\kappa}\}| &= |\ker(\rho_{\Pi,\kappa})/(U_{\overline{J}} \cap \ker(\rho_{\Pi,\kappa}))| \\
&= 2^{c(\Pi,\kappa)-c(\Pi_{\overline{J}},\kappa_{\overline{J}})} \text{ by Remark 4.3.9}
\end{aligned}$$

Since $\rho_{\Pi,\kappa}$ is open as a covering map and φ is open by Lemma 1.2.4, it follows from Lemma 1.2.3 that $\rho^J_{\Pi,\kappa}$ is a covering map.

From here the proof is analogous to the proof of Theorem 4.3.7, after extending the commutative diagram at the beginning of the latter proof:

$$\begin{array}{ccc}
\mathrm{Spin}(\Pi,\kappa) & \xrightarrow{\tilde{\varphi}} & \mathrm{Spin}(\Pi,\kappa)/U_{\overline{J}} \\
\big\downarrow{\scriptstyle \rho_{\Pi,\kappa}} & & \big\downarrow{\scriptstyle \rho^J_{\Pi,\kappa}} \\
K & \xrightarrow{\quad\varphi\quad} & K/K_J \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\
K/(K\cap T) & \xrightarrow{\psi} & K/(K\cap T)K_J
\end{array} .$$

One obtains that $\pi_1(\mathrm{Spin}(\Pi,\kappa)) \cong \prod_{i=1}^k H'_{J_i}$ where each H'_{J_m} is a subgroup of index $2^{c(\Pi,\kappa)-c(\Pi_{\overline{J}_m},\kappa_{\overline{J}_m})}$ of

$$\tilde{H}_m \cong \begin{cases} \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } r, \\ 2\mathbb{Z} \cong \mathbb{Z}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } g, \\ C_2 & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b. \end{cases}$$

Since $\Pi^{\mathrm{adm}}_{\overline{J}_m}$ is the union of all connected components except $\Pi^{\mathrm{adm}}_{J_m}$, one has $c(\Pi, \kappa) - c(\Pi_{\overline{J}_m}, \kappa_{\overline{J}_m}) \in \{0, 1\}$, depending on whether κ is constant 1 or 2 on $\Pi^{\mathrm{adm}}_{J_m}$. This implies

$$H'_m \cong \begin{cases} \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } r, \\ \mathbb{Z}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } g, \\ C_2 & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b \text{ and } \kappa \equiv 1 \text{ on } \Pi^{\mathrm{adm}}_{J_m}, \\ \{1\}, & \text{if } \Pi^{\mathrm{adm}}_{J_m} \text{ has colour } b \text{ and } \kappa \equiv 2 \text{ on } \Pi^{\mathrm{adm}}_{J_m}. \end{cases}$$

This proves the assertion. ■

Index

Bibliography

[AB08] Peter Abramenko and Kenneth S. Brown, **Buildings. Theory and applications**, Springer, New York, 2008.

[AGP02] Marcelo Aguilar, Samuel Gitler, and Carlos Prieto, **Algebraic topology from a homotopical viewpoint**, Universitext, Springer-Verlag, New York, 2002, Translated from the Spanish by Stephen Bruce Sontz. MR 1908260

[AM97] Peter Abramenko and Bernhard Mühlherr, **Présentations de certaines BN-paires jumelées comme sommes amalgamées**, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 7, 701–706. MR 1483702

[Bou98] Nicolas Bourbaki, **General topology. Chapters 1–4**, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation. MR 1726779

[Bou04] ———, **Theory of sets**, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2004, Reprint of the 1968 English translation [Hermann, Paris; MR0237342]. MR 2102219

[BS87] Keith Burns and Ralf Spatzier, **On topological Tits buildings and their classification**, Inst. Hautes Études Sci. Publ. Math. (1987), no. 65, 5–34. MR 908214

[BT18] Armand Borel and Jacques Tits, **On "abstract" homomorphisms of simple algebraic groups**, Innov. Incidence Geom. Algebr. Topol. Comb. **16** (2018), no. 1, 225–234.

[Cap09] Pierre-Emmanuel Caprace, **"Abstract" homomorphisms of split Kac-Moody groups**, Mem. Amer. Math. Soc. **198** (2009), no. 924, xvi+84. MR 2499773

[Car92] R. W. Carter, **Kac-Moody groups and their automorphisms**, Groups, combinatorics & geometry (Durham, 1990), London Math. Soc. Lecture Note Ser., vol. 165, Cambridge Univ. Press, Cambridge, 1992, pp. 218–228. MR 1200262

[CR09] Pierre-Emmanuel Caprace and Bertrand Rémy, **Groups with a root group datum**, Innov. Incidence Geom. **9** (2009), 5–77. MR 2658894

[Dug78] James Dugundji, **Topology**, Allyn and Bacon, Inc., Boston, Mass.-London-Sydney, 1978, Reprinting of the 1966 original, Allyn and Bacon Series in Advanced Mathematics. MR 0478089

[Dyn46] E. Dynkin, **Classification of the simple Lie groups**, Rec. Math. [Mat. Sbornik] N.S. **18(60)** (1946), 347–352.

[FHHK] Walter Freyn, Tobias Hartnick, Max Horn, and Ralf Köhl, **Kac-Moody symmetric spaces**, Münster J. Math., to appear, `https://www.uni-muenster.de/FB10/mjm/acc/mjm-Koehl.pdf`.

[GGH10] Helge Glöckner, Ralf Gramlich, and Tobias Hartnick, **Final group topologies, Kac-Moody groups and Pontryagin duality**, Israel J. Math. **177** (2010), 49–101. MR 2684413

[GHKW17] David Ghatei, Max Horn, Ralf Köhl, and Sebastian Weiß, **Spin covers of maximal compact subgroups of Kac–Moody groups and spin-extended Weyl groups**, J. Group Theory **20** (2017), 401–504.

[Hat02] Allen Hatcher, **Algebraic topology**, Cambridge University Press, 2002.

[Hel78] S. Helgason, **Differential geometry, Lie groups, and symmetric spaces**, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR 514561

[HK19] Paula Harring and Ralf Köhl, **Fundamental groups of split real Kac-Moody groups and generalized real flag manifolds. With appendices by Tobias Hartnick and Ralf Köhl and by Julius Grüning and Ralf Köhl. Version v.2**, October 2019, arXiv: 1905.13444.

[HKM13] Tobias Hartnick, Ralf Köhl, and Andreas Mars, **On topological twin buildings and topological split Kac–Moody groups**, Innov. Incidence Geom. **13** (2013), 1–71.

[HN11] J. Hilgert and K.H. Neeb, **Structure and geometry of Lie groups**, Springer Monographs in Mathematics, Springer New York, 2011.

[Hor] Max Horn, **Tikz styles for dynkin diagrams**, `https://github.com/fingolfin/tikz-dynkin/blob/master/dynkin-diagrams.tex`, Accessed: 2019-02-22.

[HR63] Edwin Hewitt and Kenneth A. Ross, **Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations**, Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[Hum78] James E. Humphreys, **Introduction to Lie algebras and representation theory**, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978, Second printing, revised. MR 499562

[Hum90] ______, **Reflection groups and Coxeter groups**, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR 1066460

[Hus94] Dale Husemoller, **Fibre bundles**, third ed., Springer-Verlag, New York, 1994.

[Kac85] Victor G. Kac, **Constructing groups associated to infinite-dimensional Lie algebras**, Infinite-dimensional groups with applications (Berkeley, Calif., 1984), Math. Sci. Res. Inst. Publ., vol. 4, Springer, New York, 1985, pp. 167–216.

[Kac90] ______, **Infinite-dimensional Lie algebras**, third ed., Cambridge University Press, Cambridge, 1990. MR 1104219

[KP85] V. G. Kac and D. H. Peterson, **Defining relations of certain infinite-dimensional groups**, Astérisque (1985), no. Numéro Hors Série, 165–208, The mathematical heritage of Élie Cartan (Lyon, 1984). MR 837201

[Kra01] Linus Kramer, **Loop groups and twin buildings**, Geometriae Dedicata **92** (2001), 145–178.

[Mar11] Andreas Mars, **On the topology and geometry of Kac-moody groups**, Ph.D. thesis, Technische Universität, Darmstadt, März 2011.

[Mar18] Timothée Marquis, **An introduction to Kac-Moody groups over fields**, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2018. MR 3838421

[Mas77] William S. Massey, **Algebraic topology: An introduction**, 4th corrected printing ed., Graduate texts in mathematics 56, Springer-Verlag, 1977.

[Mos49] George Daniel Mostow, **A new proof of E. Cartan's theorem on the topology of semi-simple groups**, Bull. Amer. Math. Soc. **55** (1949), no. 10, 969–980.

[Nee02] Karl-Hermann Neeb, **Central extensions of infinite-dimensional Lie groups**, Annales de l'Institut Fourier **52** (2002), no. 5, 1365–1442 (en). MR 1935553

[Pal61] Richard S. Palais, **On the existence of slices for actions of non-compact Lie groups**, Ann. of Math. (2) **73** (1961), 295–323.

[PR94] Vladimir Platonov and Andrei Rapinchuk, **Algebraic groups and number theory**, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR 1278263

[Pro07] Claudio Procesi, **Lie groups**, Springer, New York, 2007, An approach through invariants and representations.

[Rém02] Bertrand Rémy, **Groupes de Kac-Moody déployés et presque déployés**, Astérisque (2002), no. 277, viii+348. MR 1909671

[Rot88] Joseph J. Rotman, **An introduction to algebraic topology**, Springer-Verlag, New York, 1988.

[SBG+95] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, and M. Stroppel, **Compact projective planes: With an introduction to octonion geometry**, De Gruyter expositions in mathematics, vol. 21, Walter de Gruyter, 1995. MR 1384300

[Ste99] Norman E. Steenrod, **The topology of fibre bundles**, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, Reprint of the 1957 edition, Princeton Paperbacks.

[Str06] Markus Stroppel, **Locally compact groups**, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006. MR 2226087

[Tit74] Jacques Tits, **Buildings of spherical type and finite BN-pairs**, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York, 1974. MR 0470099

[Tit87] ———, **Uniqueness and presentation of Kac-Moody groups over fields**, J. Algebra **105** (1987), no. 2, 542–573. MR 873684

[Wig98] Mark Wiggerman, **The fundamental group of a real flag manifold**, Indag. Math. **9** (1998), no. 1, 141–153.

Selbstständigkeitserklärung

Ich erkläre: Ich habe die vorgelegte Dissertation selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Ich stimme einer evtl. Überprüfung meiner Dissertation durch eine Antiplagiat-Software zu. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der "Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" niedergelegt sind, eingehalten.

Gießen, den 21.01.2020

Paula Katrin Harring