BIFURCATION FROM HOMOCLINIC TO PERIODIC SOLUTIONS BY AN INCLINATION LEMMA WITH POINTWISE ESTIMATE

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Bifurcation from homoclinic to periodic orbits in two dimensions has been known for a long time [1,4]. L.P. Šil'nikov [8] obtained the first result for arbitrary finite dimension. His idea was to consider a point on the homoclinic trajectory as fixed point of a suitably constructed map so that continuation by the implicit function theorem yields fixed points which define periodic solutions. The difficulty involved is to show smoothness of Šil'nikov's map. This requires a careful investigation of trajectories close to a hyperbolic equilibrium. The underlying vectorfields have to be at least C^2-smooth [7].

In [9] we proved a result in infinite dimension: For the functional differential equation

$$\dot{x}(t) = af(x(t-1))$$

with periodic nonlinearity $f:\mathbb{R} \to \mathbb{R}$ in a certain class of functions, there exists a critical parameter $a = a_0$ with a heteroclinic solution, and for $a > a_0$, periodic solutions of the second kind bifurcate off. This was done without recourse to Šil'nikov's idea, and required only C^1-smoothness of f, but questions of uniqueness and stability for the bifurcating solutions remained unanswered.

Assuming more smoothness, M. Blazquez [2], S.N. Chow and B. Deng [3] and the author [11] recently obtained results for parabolic [2,3] and functional [3,11] differential equations which include uniqueness and stability. All these proofs employ modifications of Šil'nikov's map, but the crucial parts - how to derive smoothness - are different.

In [11] we tried to give a conceptually relatively simple proof of smoothness. The key is a sharpened inclination lemma, reals $\alpha < 1$, $\gamma \geq \beta > 1$ with (A.2) $|Lx| \leq \alpha|x|$ on Q, $\beta|x| \leq |Lx| \leq \gamma|x|$ on P, and with (A.3) $(\alpha\gamma)/\beta < 1$. Let (A.4) $\tilde{g}(\tilde{U} \cap P) \subset P$, $\tilde{g}(\tilde{U} \cap Q) \subset Q$.

Then there exist an open neighborhood $U \subset \tilde{U}$ of $0 \in B$ and $\bar{c} > 0$, $\tilde{\beta} \in (1,\beta)$ such that for all $p_2 \geq p_1 > 0$ there is a constant $\hat{c} > 0$ with the following property:

If a set $H \subset U$ satisfies

$$\left.\begin{array}{l} p_1 \leq |px| \leq p_2 \text{ on } H,\ q\chi \neq 0 \text{ and } \Lambda(\chi) := \dfrac{|p\chi|}{|q\chi|} \leq \bar{c} \\ \text{on } T_xH \setminus \{0\} \text{ for all } x \in H \end{array}\right\} \quad (A.5)$$

with the projections $p:B \to P$, $q:B \to Q$ given by (A.1), then

$$|px| \leq p_2\tilde{\beta}^{-k}, \quad (A.6)$$

$$q\chi \neq 0 \text{ and } \Lambda(\chi) \leq \hat{c}|px| \quad (A.7)$$

for all $k \in \mathbb{N}_0$, $x \in H_k := (\tilde{g}|U)^{-k}(H)$, $\chi \in T_xH_k \setminus \{0\}$.

<u>Remarks</u> <u>For</u> $\dim P = 1$, $|Lx| = |L||x|$ on P, and we may assume $\beta = |L| = \gamma$ so that (A.3) <u>is</u> <u>automatically</u> <u>satisfied</u>. – For an arbitrary set $Z \subset B$, the set T_xZ of tangent vectors at $x \in Z$ is defined as usual, by derivatives of differentiable curves which pass through x and have trace in Z. In general, T_xZ is not a vector space – but always, $0 \in T_xZ$. – (A.6) and (A.7) imply that inclinations $\Lambda(\chi)$, $\chi \in T_xH_k \setminus \{0\}$, tend to 0 uniformly with respect to $x \in H_k$ as $k \to +\infty$.

For other inclination lemmas in infinite-dimensional spaces, see [5,6].

<u>B</u>. <u>Preparations</u>. In order to avoid technicalities we present the core of our approach in the simplest nontrivial situation, without parameters. Consider a local C^2-flow $X:\Omega \to B$, $\Omega \subset \mathbb{R} \times B$, on a finite-dimensional space B, with stationary point $0 = X(t,0)$ for all $t \in \mathbb{R}$. We assume that the generator of the linearization $T:\mathbb{R} \times B \ni (t,x) \to T_tx \in B$ at $0 \in B$ has a simple positive eigenvalue u, and that there are constants

$$\left.\begin{array}{l} \lambda < -\mu < 0 \text{ with } \operatorname{Re} z < \lambda < 0 < u < \mu \text{ for all} \\ \text{eigenvalues } z \neq u. \end{array}\right\} \quad (B.1)$$

Then (B.2) $B = P \oplus Q$ with T_t-invariant spaces $P = \mathbb{R}\Phi$ and Q, where Φ is a unit eigenvector of the eigenvalue u, and there is a constant $c_1 > 0$ such that for all $t \geq 0$,

$$T_tx = e^{ut}x \text{ on } P,\ |T_tx| \leq c_1e^{\lambda t}|x| \text{ on } Q \quad (B.3)$$

We assume in addition that (B.4) P and Q are invariant under the nonlinear flow X, i.e. $(t,x) \in \Omega$ and $x \in P$ imply $X(t,x) \in P$, and analogously for Q. Write $X(t,x) = T_tx + R(t,x)$, with a remainder $R:\Omega \to B$ which is C^2-smooth. Note that (B.5) R leaves P and Q invariant. We have (B.6) $R(t,0) = 0$, $D_2R(t,0) = 0$, $D_1D_2R(t,0) = 0$ on $\mathbb{R}$.

We prepare both the construction of a shift Σ along trajectories close to $0 \in B$, and the application of the preceding lemma to a restriction of a time-N-map of X.

It is not hard to find a positive integer N, positive reals $\alpha < 1$ and $\gamma = \beta > 1$, and an open set $\tilde{U}$ such that the C^2-map

$$\tilde{g}:\tilde{U} \ni x \to X(N,x) \in B,\ L := D\tilde{g}(0) = T_N,$$

satisfies conditions (A.1) – (A.4), and furthermore

$$\alpha < e^{\lambda N},\ \beta < e^{\mu N},\ \alpha < 1/\beta \quad (B.7)$$

In particular, (B.8) $[0,N] \times \tilde{U} \subset \Omega$.

As in section A, write $x = px + qx$ with $px \in P$, $qx \in Q$, for all $x \in B$. We choose $c > 0$ and a convex open neighborhood $U \subset \tilde{U}$ of $0 \in B$ with $pU \subset U$, $qU \subset U$ so small that the lemma applies with constants $\bar{c}$ and $\tilde{\beta}$, and such that we have

$$|D_2pR(t,x)| + |D_2qR(t,x)| < c \text{ on } [0,N] \times U, \quad (B.9)$$

$$|D_2(D_1qR)(0,x)| + |D_2(D_1pR)(0,x)| < c \text{ on } U, \quad (B.10)$$

and (B.11) $\alpha + c < e^{\lambda N}$, (B.12) $c < u$, (B.13) $c < \beta$. – It follows that there is some $c^* > 0$ with (B.14) $|D_2X(t,x)| < c^*$ for t in $[0,N]$, $x \in U$. Set $g := \tilde{g}|U$. We have

$$\left.\begin{array}{l} (\beta-c)|px| \leq |pg(x)| \leq (\beta+c)|px| \text{ and} \\ |qg(x)| \leq (\alpha+c)|qx| \text{ for all } x \in U. \end{array}\right\} \quad (B.15)$$

Proof of the first estimate: Set $r := g - L$. Note $pr(qx) = 0$. Apply the mean value theorem to $pr(x) = pr(x) - pr(qx)$, use (B.9) and (A.2).

Similarly, (B.9) implies $|pR(t,x)| \leq c|px|$, $|qR(t,x)| \leq c|qx|$ on U. Using (B.3) and (B.15) one shows – <u>without</u> <u>the</u> <u>variation-of-constants</u> <u>formula</u> – that there exist $\rho > 0$, $c_2 > 0$, $c_3 > 0$ such that for all $x \in U$ with $X(s,x) \in U$ on $[0,t]$,

$$c_2e^{\rho t}|px| \leq |pX(t,x)| \leq c_3e^{\mu t}|px|, \quad (B.16)$$

$$|qX(t,x)| \leq c_3e^{\lambda t}|qx|. \quad (B.17)$$

Finally, there are $c_4 > 0$, $c_5 > 0$ such that for all $x \in U$,

$$c_4|px| \leq |pD_1X(0,x)| \leq c_5|px| \quad \text{and} \qquad \text{(B.18)}$$

$$|qD_1X(0,x)| \leq c_5|qx|. \qquad \text{(B.19)}$$

Proof of the first estimate in (B.18): (B.5) yields $pR(t,qx) = 0$ on $[0,N] \times U$, hence $D_1pR(0,qx) = 0$ on U. (B.10) and the mean value theorem for $D_1pR(0,x) - D_1pR(0,qx)$ give $|pD_1R(0,x)| \leq c|px|$ on U. $pT_tx = T_tpx = e^{ut}px$ on $\mathbb{R} \times B$ implies $D_1pT(0,x)(1) = u \cdot px$. Use $X(t,x) = T_tx + R(t,x)$ on Ω, and (B.12).

C. The map Σ. Observe first that (B.16) yields

$$\left.\begin{array}{l} pX(t,x) \in (0,\infty)\cdot\Phi \quad \text{for all } x \in U \text{ with } px \in (0,\infty)\cdot\Phi \\ \text{and } X(s,x) \in U \text{ on } [0,t]. \end{array}\right\} \text{(C.1)}$$

Fix $r > 0$ with $r\Phi \in U \cap P$. Set $H := r\Phi + Q$. Choose $\eta_1 > 0$ and $\delta_1 > 0$ so small that the open box

$$E^+ := E^+(\eta,\delta) := \{x \in B\colon px \in (0,\eta)\cdot\Phi, |qx| < \delta\}$$

with $\eta = \eta_1$, $\delta = \delta_1$ satisfies $E^+ \subset U$, $E^+ \cap H = \emptyset$, and that for every $x \in E^+$ there exists $\sigma = \sigma(x) > 0$ with $X(t,x) \in U$ on $[0,\sigma]$, $pX(t,x) \in (0,r)\cdot\Phi$ on $[0,\sigma)$, $pX(\sigma,x) = r\Phi$ (or $X(\sigma,x) \in H$), $D_1X(\sigma,x)(1) \notin Q$. Furthermore, we can achieve that the map $\sigma\colon E^+ \to (0,\infty)$ is continuously differentiable. – Let $x \in E^+$. By (B.16), $r = |pX(\sigma,x)| \leq c_3e^{\mu\sigma}|px|$, or

$$\frac{1}{\mu}\log\frac{1}{c_3|px|} \leq \sigma(x). \qquad \text{(C.2)}$$

With (B.17), we obtain

$$|qX(\sigma(x),x)| \leq c_3|qx|\left(\frac{r}{c_3}\right)^{\lambda/\mu}|px|^{-\lambda/\mu}. \qquad \text{(C.3)}$$

Consider the C^1-map $\Sigma\colon E^+ \ni x \to X(\sigma(x),x) \in H \subset B$ and its "Šil'nikov continuation" $\check{\Sigma}$ to the set

$$E := E(\eta_1,\delta_1) := \{x \in B\colon |px| < \eta_1, |qx| < \delta_1\}$$

defined by $\check{\Sigma}(x) := r\Phi$ on $E \setminus E^+$. $\check{\Sigma}$ is C^1-smooth on $E \setminus Q$. Proof of differentiability at points $x \in E \cap Q$, with $D\check{\Sigma}(x) = 0$: Suppose $\lim_{n\to\infty} x_n = x$, $x_n \in E \setminus \{x\}$ for all $n \in \mathbb{N}$, and $\bar{x}_k = x_{n_k} \in E^+$ for a subsequence $(n_k)_{k\in\mathbb{N}}$. Clearly $\check{\Sigma}(x_n) - \check{\Sigma}(x) = 0$ if $x_n \notin E^+$. For all $k \in \mathbb{N}$, $\check{\Sigma}(\bar{x}_k) - \check{\Sigma}(x) = X(\sigma(\bar{x}_k),\bar{x}_k) - r\Phi = qX(\sigma(\bar{x}_k),\bar{x}_k)$, and $|\bar{x}_k - x| \geq |p\bar{x}_k - 0| - |q||\bar{x}_k - x|$, or $(1 + |q|)|\bar{x}_k - x| \geq |p\bar{x}_k| > 0$. (C.3) and (B.1) show that difference quotients for $\check{\Sigma}$ tend to 0 as $k \to +\infty$.

In the next sections we shall show that there exists δ_3 in $(0,\delta_1)$ with

$$\sup_{x \in E^+(\eta,\delta_3)} |D\Sigma(x)| \to 0 \quad \text{as } \eta \to 0. \qquad \text{(C.4)}$$

This implies continuity of $D\check{\Sigma}$ at points $x \in Q$ with $|qx| < \delta_3$.

D. Discretization. The preimages $H_k^* := \sigma^{-1}(kN) \subset E^+$, $k \in \mathbb{N}$, are nonempty for k sufficiently large and satisfy $H_k^* \subset g^{-k}(H) =: H_k$ for all $k \in \mathbb{N}$, with

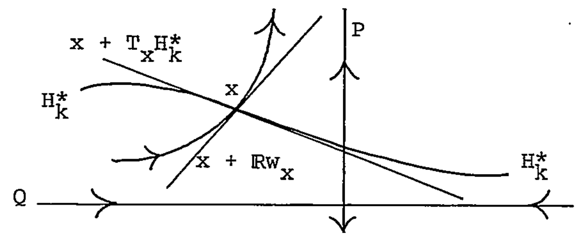
$$g^k(x) = X(kN,x) = \Sigma(x) \quad \text{on each } H_k^*. \qquad \text{(D.1)}$$

Let $x \in E^+$. The tangent vector $w_x := D_1X(0,x)(1)$ to the trajectory $X(\cdot,x)$ at $t = 0$ satisfies

$$D\sigma(x)w_x = -1, \qquad \text{(D.2)}$$

since $\sigma(X(t,x)) - \sigma(x) = -t$ for small $t > 0$. Therefore σ and H_k^* are transversal whenever $H_k^* \neq \emptyset$, and H_k^* is a submanifold of codimension 1. (D.2) and $T_{kN}\{kN\} = \{0\}$ imply $w_x \notin T_xH_k^*$ for all $x \in H_k^*$, $k \in \mathbb{N}$, so that

$$B = \mathbb{R}w_x \oplus T_xH_k^*. \qquad \text{(D.3)}$$



Σ is constant along trajectories. Therefore (D.4) $D\Sigma(x)w_x = 0$ on each $H_k^* \neq \emptyset$. (D.1) gives (D.5) $D\Sigma(x)\chi = Dg^k(x)\chi$ for all $\chi \in T_xH_k^*$, $x \in H_k^*$, $k \in \mathbb{N}$. The basic idea for the proof of (C.4) is to use (D.3) and (D.4) for an estimate of $|D\Sigma(x)|$, $x \in H_k^*$, in terms of $|(Dg^k(x)|T_xH_k^*)|$ and of the angle between the decomposing spaces in (D.3), and to apply (A.6) and the pointwise estimate (A.7) for the inclination of T_xH_k to the majorizing terms.

E. Estimate of $Dg^k(x)$ on T_xH_k. Set $p_1 := p_2 := r$ and recall $\Lambda(\chi) = 0 < \bar{c}$ if $0 \neq \chi \in T_xH = Q$, $x \in H$. We have (A.6) and some $\hat{c} > 0$ so that (A.7) holds. Let us sketch how to find $c_6 > 0$ with

$$|Dg^k(x)\chi| \leq c_6 e^{\lambda k N}|\chi| \text{ for all } k \in \mathbb{N}, x \in H_k, \chi \in T_xH_k. \qquad (E.1)$$

The first inequality in (B.11), (A.6) and (A.7) permit to choose $j \in \mathbb{N}$ such that for all integers $k \geq j$, $x \in H_k$, $\bar{x} \in H_{k-1}$, $\chi \in T_xH_k \setminus \{0\}$, $\bar{\chi} \in T_{\bar{x}}H_{k-1} \setminus \{0\}$ we have

$$(\alpha + c + c\cdot\Lambda(\chi))\,\frac{1 + \Lambda(\bar{\chi})}{1 - \Lambda(\chi)} < e^{\lambda}. \qquad (E.2)$$

Consider $\chi \in T_xH_k \setminus \{0\}$, $x \in H_k$, $k \geq j$. Set $\bar{x} := g(x)$ and $\bar{\chi} := Dg(x)\chi \in T_{\bar{x}}H_{k-1} \setminus \{0\}$. We show (E.3) $|\bar{\chi}| \leq e^{\lambda N}|\chi|$: With $q\bar{\chi} \neq 0 \neq q\chi$,

$$\frac{|\bar{\chi}|}{|\chi|} = \frac{|q\bar{\chi}|}{|q\chi|}\;\frac{\left|\frac{1}{|q\bar{\chi}|}\,p\bar{\chi} + \frac{1}{|q\bar{\chi}|}\,q\bar{\chi}\right|}{\left|\frac{1}{|q\chi|}\,p\chi + \frac{1}{|q\chi|}\,q\chi\right|}.$$

(A.2) and (B.9) give $|q\bar{\chi}| = |qL\chi + qDr(x)\chi| \leq \alpha|q\chi| + c|\chi| \leq (\alpha+c)|q\chi| + c|p\chi|$. Using this and (E.2), we get (E.3). – Finally, iteration and an appropriate choice of c_6 yield (E.1).

For points $x \in H_k^*$, $k \in \mathbb{N}$, and vectors $\chi \in T_xH_k^*$, we combine (D.5), $T_xH_k^* \subset T_xH_k$, (E.1), $\sigma(x) = kN$ and (C.2) and infer

$$|D\Sigma(x)\chi| \leq c_7|px|^{-\lambda/\mu}|\chi| \quad \text{with } c_7 := c_6(c_3/r)^{-\lambda/\mu}. \qquad (E.4)$$

F. Estimate of $D\Sigma(x)$ at $x \in H_k^*$. We choose $\delta_2 \in (0,\delta_1)$ with (F.1) $c_5\delta_2 < c_4/(2\hat{c})$, $\varepsilon \in (0,1)$ with (F.2) $\varepsilon|q|/(1-\varepsilon) < 1/2$, and $j \in \mathbb{N}$ with (F.3) $|p\chi| \leq \varepsilon|\chi|$ for all integers $k \geq j$ and all $x \in H_k^*$, $\chi \in T_xH_k^*$. – The latter is possible because of (A.6) and (A.7).

Let $x \in H_k^*$, $k \geq j$, $|qx| < \delta_2$. (F.3) and $|p\Phi| = |\Phi| = 1 > \varepsilon$ imply $\Phi \notin T_xH_k^*$ so that we have another decomposition

$$B = P \oplus T_xH_k^* \qquad (F.4)$$

The associated projections p_x onto P and q_x onto $T_xH_k^*$ satisfy

$$\left.\begin{array}{l} p_x\chi = p\chi \text{ if } \chi \in P,\ p_x\chi = p\chi - (|q\chi|/|qq_x\chi|)\cdot pq_x\chi \\ \text{if } \chi \in B \setminus P. \end{array}\right\} \qquad (F.5)$$

(The equation for $\chi \in B \setminus P$ follows from $p_x\chi = pp_x\chi = p(\chi - q_x\chi)$, $0 \neq q\chi = qq_x\chi + qp_x\chi = qq_x\chi$.) Proof of

$$|p_x - p| < 1/2,\ |q_x - q| = |id - p_x - (id - p)| < 1/2 \qquad (F.6)$$

– consider $\chi = y\Phi + \bar{\chi}$ with $|\chi| = 1$, $y \in \mathbb{R}$, $\bar{\chi} \in T_xH_k^*$. Then $|(p_x - p)\chi| = |y\Phi - p(y\Phi + \bar{\chi})| = |p\bar{\chi}|$. By (F.3), $|p\bar{\chi}| \leq \varepsilon|\bar{\chi}| \leq \varepsilon(|p\bar{\chi}| + |q\bar{\chi}|)$. Note $|q\bar{\chi}| = |q(y\Phi + \bar{\chi})| = |q\chi| \leq |q|$. Hence $|p\bar{\chi}| \leq \varepsilon(|p\bar{\chi}| + |q|)$, $|p\bar{\chi}| \leq \varepsilon|q|/(1-\varepsilon) < 1/2$, with (F.2).

It follows that

$$|\tilde{L}| \leq c_8\cdot\sup_{-1\leq y\leq 1,\,\chi \in T_xH_k^*,\,|\chi|=1}|\tilde{L}(y\Phi + \chi)| \qquad (F.7)$$

for all continuous linear maps $\tilde{L}: B \to B$, where $c_8 := 1 + |p| + |q|$. – (D.2) gives $p_xw_x \neq 0$. Next, we show

$$|D\Sigma(x)| \leq c_9|px|^{-\lambda/\mu}\left(1 + \frac{1}{|p_xw_x|}\right) \qquad (F.8)$$

with $c_9 := c_8c_7(1 + (1 + |q|)\cdot c_5\cdot(\eta_1 + \delta_1))$:
(F.7) implies $|D\Sigma(x)| \leq c_8(|D\Sigma(x)\Phi| + \sup_{\chi \in T_xH_k^*,\,|\chi|=1}|D\Sigma(x)\chi|)$.
From (D.4), $|D\Sigma(x)\Phi| = |D\Sigma(x)p_xw_x|/|p_xw_x| = |-D\Sigma(x)q_xw_x|/|p_xw_x|$. Using (E.4), we arrive at $|D\Sigma(x)| \leq c_8(c_7|px|^{-\lambda/\mu}|q_xw_x|\cdot|p_xw_x|^{-1} + c_7|px|^{-\lambda/\mu})$. (F.6), (B.18) and (B.19) yield $|q_xw_x| \leq (1 + |q|)\cdot(|pw_x| + |qw_x|) \leq (1 + |q|)c_5(|px| + |qx|)$, and (F.8) becomes obvious.

Now the pointwise estimate (A.7) becomes crucial. We derive

$$|p_xw_x|^{-1} \leq (2/c_4)|px|^{-1} \qquad (F.9)$$

– in case $w_x \in P$, this is a trivial consequence of $p_xw_x = pw_x$ and of the lower estimate in (B.18). For $w_x \notin P$, (F.5) gives $|p_xw_x| \geq |pw_x| - |qw_x|\Lambda(q_xw_x)$. Using (B.18) as before, (B.19) for $|qw_x|$, $|qx| < \delta_2$, (F.1) and finally (A.7), we get $|p_xw_x| \geq c_4|px| - (c_4/2\hat{c})\cdot\hat{c}\cdot|px|$.

Altogether, we have shown in this section that for all integers $k \geq j$ and all $x \in H_k^*$ with $|qx| < \delta_2$,

$$|D\Sigma(x)| \leq c_{10}(|px|^{-\lambda/\mu} + |px|^{(-\lambda/\mu) - 1}) \qquad (F.10)$$

where $c_{10} := c_9(1 + (2/c_4))$.

G. Estimate at arbitrary points $x \in E^+(\eta_2,\delta_3)$. Choose positive reals $\delta_3 < \delta_2$ and $\eta_2 < \eta_1$ so small that

$$\left.\begin{array}{l} \text{for all } x\in E^+(\eta_2,\delta_3),\ \sigma(x) > jN \text{ and} \\ X(t,x)\in E^+(\eta_1,\delta_2) \text{ on } [0,N]. \end{array}\right\} \qquad (G.1)$$

Let $x \in E^+(\eta_2,\delta_3)$. The largest integer k with $kN \le \sigma(x) < kN + N$ satisfies $k \geqq j$. Set $\bar{x} := X(\sigma(x) - kN, x)$. Then $\bar{x} \in E^+(\eta_1,\delta_2)$ and $\sigma(\bar{x}) = kN$, or $\bar{x} \in H_k^*$. We have $\Sigma(x) = \Sigma(\bar{x}) = \Sigma(X(\sigma(x) - kN,x))$, and there is a neighborhood $W \subset E^+(\eta_2,\delta_3)$ of x such that for all $y \in W$, $\Sigma(y) = \Sigma(X(\sigma(x) - kN,y))$. Hence $|D\Sigma(x)| \le |D\Sigma(\bar{x})|\cdot|D_2X(\sigma(x) - kN,x)|$. (B.14) for $t = \sigma(x) - kN < N$, x in $E^+(\eta_2,\delta_3) \subset U$, and (F.10) yield $|D\Sigma(x)| \le c_{10}(|p\bar{x}|^{-\lambda/\mu} + |p\bar{x}|^{(-\lambda/\mu)-1})\cdot c^*$. With (G.1) and (B.16) - and with (B.1) - we obtain

$$|D\Sigma(x)| \le c_{11}(|px|^{-\lambda/\mu} + |px|^{(-\lambda/\mu)-1}) \qquad (G.2)$$

where $c_{11} := c_{10}c^*((c_3e^{\mu N})^{-\lambda/\mu} + (c_3e^{\mu N})^{(-\lambda/\mu)-1})$. Finally, (G.2) and the hypothesis (B.1) on the spectrum imply (C.4).

H. Bifurcation. The simplest nontrivial situation with parameters occurs for a local flow $(t,x,a) \to X(t,x,a)$ of class C^2 in a finite-dimensional space B, with parameters in an open interval $A \ni 0$. Suppose (1) $0 \in B$ is a stationary point, the spectral hypothesis (3) is satisfied, and X is locally normalized (4). Then one can make the previous considerations locally uniform with respect to the parameter. The result is that there exist $\eta_2 > 0$ and $\delta_3 > 0$ and an open interval $A_1 \ni 0$ with

$$\sup_{x \in E^+(\eta,\delta_3,a),\, a \in A_1} |D_1\Sigma(x,a)| \to 0 \quad \text{as } \eta \to 0 \qquad (H.1)$$

where $\Sigma(x,a) = X(\sigma(x,a),x,a) \in H_a \subset B$ for $x \in E^+(\eta_2,\delta_3,a)$, a in A_1.

As a consequence one obtains that $D_1\check{\Sigma}$ exists on the whole domain of $\check{\Sigma}$, and is continuous, now with respect to (x,a). As in section C, $D_1\check{\Sigma}(r\Phi_0,0) = 0$.

We show that existence of the partial derivative $D_2\check{\Sigma}$ follows from the analogue of (C.3), asserting that there is $c_3' > 0$ with

$$|q_aX(\sigma(x,a),x,a)| \le c_3'|q_ax|(r/c_3')^{\lambda/\mu}|p_ax|^{-\lambda/\mu} \qquad (C.3')$$

for all $x \in E^+(\eta_2,\delta_3,a)$ and all $a \in A_1$: Consider $x \in E(\eta_2,\delta_3,a)$ with $p_ax = 0$, $a \in A_1$, so that $\Sigma(x,a) = r\Phi_a$, and sequences of points $a_n \in A_1 \setminus \{a\}$ with $\lim_{n\to\infty} a_n = a$. Differentiability of $a^* \to \Phi_{a^*}$ shows that in case $p_{a_n}x \in (-\eta_2,0]\cdot\Phi_{a_n}$ for all n, difference quotients $D_n := (a_n - a)^{-1}(\check{\Sigma}(x,a_n) - \check{\Sigma}(x,a))$ tend to $r\cdot\lim_{h\to 0} h^{-1}(\Phi_{a+h} - \Phi_a)$. - If $p_{a_n}x \in (0,\eta_2)\cdot\Phi_{a_n}$ for all n, write $D_n = r(a_n - a)^{-1}(\Phi_{a_n} - \Phi_a) + d_n$ with $d_n := (a_n - a)^{-1}(\check{\Sigma}(x,a_n) - r\Phi_{a_n}) = (a_n - a)^{-1}q_{a_n}\Sigma(x,a_n)$. By C^1-smoothness of $a^* \to p_{a^*}$, there is $c_p > 0$ with $|p_{a_n}x| = |(p_{a_n} - p_a)x| \le |p_{a_n} - p_a||x| \le c_p(\eta_2 + \delta_3)|a_n - a|$. Using this and (C.3') and $\lambda < -\mu$, we obtain $\lim_{n\to\infty} d_n = 0$. - Now it is easy to complete the argument.

We return to bifurcation. Suppose in addition that (2) there is a homoclinic trajectory x^0 for $a = 0$. Then the map

$$\check{S}: \tilde{B} \times \tilde{A} \ni (x,a) \to \check{\Sigma}(X(\theta,x,a),a) \in B$$

is defined, with some fixed $\theta > 0$ so that $X(t,r\Phi_0,0)$ is in $E(\eta_2,\delta_3,0)$ for all $t \geqq \theta$. $\check{S}$ is as smooth as $\check{\Sigma}$. We have $\check{S}(r\Phi_0,0) = r\Phi_0 = x^0(0)$, and (H.2) $D_1\check{S}(r\Phi_0,0) = 0$. Altogether, we can use a version of the implicit function theorem which guarantees existence of a locally unique differentiable curve $a \to \psi_a$ of solutions to an equation $F(\psi,a) = 0$ through a given solution ψ^* at $a = 0$, provided both derivatives D_1F and D_2F exist, D_1F is continuous, and $D_1F(\psi^*,0)$ is an isomorphism.

We obtain a differentiable curve of fixed points x_a^* of $\check{S}(\cdot,a)$ through $x^0(0)$, which are all stable and attractive, due to (H.2).

If finally (5) $X(\theta,r\Phi_a,a) \in E^+(\eta_2,\delta_3,a)$ for $a > 0$, then it can be shown that for $a > 0$ these fixed points define periodic trajectories which are unique in a neighborhood of the homoclinic orbit, and stable and attractive with asymptotic phase.

Precise statements and complete proofs, in a more difficult situation with semiflows in an infinite-dimensional space, are contained in [11].

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