JUSTUS-LIEBIG-
UNIVERSITÄT
GIESSEN

Modular Groups Over Real Normed Division Algebras and Over-extended Hyperbolic Weyl Groups

Dissertation

zur Erlangung des akademischen Grades doctor rerum naturalium

vorgelegt von

Chengen Jiang

am

Mathematischen Institut
der Justus-Liebig-Universität Gießen

July 2018

Betreuer: Prof. Dr. Ralf Koehl

Declaration

I declare that I have completed this dissertation single-handedly without the unauthorized help of a second party and only with the assistance acknowledged therein. I have appropriately acknowledged and cited all text passages that are derived verbatim from or are based on the content of published work of others, and all information relating to verbal communications. I consent to the use of an anti-plagiarism software to check my thesis. I have abided by the principles of good scientific conduct laid down in the charter of the Justus Liebig University Giessen "Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" in carrying out the investigations described in the dissertation.

Date: 20/04/2018

Acknowledgment

Foremost, I would like to thank my supervisor, Prof. Dr. Ralf Köhl, for his support throughout my doctoral studies and for giving me the opportunity to complete my dissertation in University of Giessen. I truly appreciate his inexhaustible patience in listening to those little problems as well as his encouragement which helped me improve in my weaker areas, not only in the scientific area, but also on a personal level.

I am indebted to Prof. Dr. Max Horn for his helpful suggestions and critical comments, and especially, for his readiness and willingness for scientific discussion.

I would like to express my gratitude to Prof. Dr. Bernhard Mühlherr and Prof. Dr. Mohameden Ahmedou for serving as members of the dissertation committee.

I am also thankful to Ms. Carola Klein, the secretary of the Algebra Group for the help over years. Numerous thanks go to my colleagues Bastian Christ, Julius Grüning, and Paula Harring for all those group-chats and discussions during the "Algebra tea." Special thanks go to Robert Zeise for helping me get familiar with the facilities in the department.

I cannot forget friends who cheered me on and celebrated each accomplishment. I would like to thank my friends who I grew up with for their constant encouragement. I would also like to thank those amazing people in Giessen: Qianhui Dai, Björn Gebhard, Niclas Linne, Marius Scheld, Nicola Soave. The lunch riddles, Wednesday soccer, Friday evening movies, all the kind help and friendship were all greatly appreciated.

I am also thankful to the China Scholarship Council (CSC) and Deutsche Forschungsgemeinschaft (DFG) for the financial support that offered me the opportunity to pursue a doctoral degree in Germany.

Last, but by no means least, I would like to express my deepest gratitude to my family. My grandfather, the person I admire the most, has been consistently encouraging me in my choice of career. My parents always have more faith in me that I do in myself. My heartfelt thanks also go to my sister for her emotional support.

Contents

List of Symbols

$\mathbb{K}$: normed division algebra over $\mathbb{R}$

$\mathbb{R}$: algebra of real numbers

- $\mathbb{Z}$: ring of integers

$\mathbb{C}$: algebra of complex numbers

- G : ring of Gaussian integers
- E : ring of Eisensteinian integers

$\mathbb{H}$: algebra of quaternions

- L : ring of Lipschitzian integers
- H : ring of Hurwitzian integers

$\mathbb{O}$: algebra of octonions

- O : ring of Octavian integers

$\mathbb{KP}^n$: projective n-space over $\mathbb{K}$

$\mathfrak{h}_n(\mathbb{K})$: space of $n \times n$ hermitian matrices over $\mathbb{K}$

$\text{Möb}(\mathbb{K})$: Möbius group over $\mathbb{K}$

$\mathbb{M}(U, \tau)$: Moufang set

$\mathbb{M}(\mathbb{K})$: projective Moufang set over $\mathbb{K}$

$\mathcal{H}^n$: hyperbolic n-space

Φ^{++} : over-extension of some root system Φ

Introduction

Motivation

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra and let ω be the Cartan-Chevalley involution of $\mathfrak{g}$. Then $\mathfrak{g}$ decomposes as $\mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the +1 eigenspace and $\mathfrak{p}$ is the -1 eigenspace. It is clear that $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{g}$ and the commutator $[\mathfrak{k}, \mathfrak{p}]$ lies in $\mathfrak{p}$, i.e., there is an action of $\mathfrak{k}$ on $\mathfrak{p}$. In addition, let G be the adjoint group of $\mathfrak{g}$ and K_ω be the subgroup of G consisting of elements that commute with ω. Then both $\mathfrak{k}$ and $\mathfrak{p}$ are stable under the action of K_ω and in particular the latter defines a representation $K_\omega \to \mathtt{Aut}\mathfrak{p}$. One question is how one may interpret this representation and further study the orbits. This has been studied in [**?**], [**?**], etc. for the finite-dimensional case. However, for infinite-dimensional Kac-Moody Lie algebras, this question still remains unclear. As a particular case, we focus on the hyperbolic Kac-Moody Lie algebra $\mathfrak{e}_{10}$. We concentrate on $\mathfrak{e}_{10}$ for the following reasons.

(a) All simply-laced hyperbolic Kac-Moody algebras can be embedded into $\mathfrak{e}_{10}$ [**?**].

(b) $\mathfrak{e}_{10}$ is of significant importance in physics: it "knows all" about maximal supersymmetry. A coset model based on the hyperbolic Kac-Moody algebra $\mathfrak{e}_{10}$ has been conjectured to underly 11-dimensional supergravity and M theory [**?**] [**?**].

(c) There is a deep connection between supersymmetry and the four normed division algebras over $\mathbb{R}$. Most simply, the connection is visible from the fact that classical superstring theories and minimal super-Yang-Mills theories live in Minkowski spaces of dimension 3, 4, 6 and 10, which is isometric to $\mathfrak{h}_2(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, respectively.

In order to examine the infinite-dimensional Kac-Moody Lie algebra $\mathfrak{e}$, it suffices to study the hyperbolic root system E_{10}, which serves as the main object of this dissertation.

Methodologies, and objectives

In [FF83] Feingold and Frenkel came up with a very insightful way to study the structure of the rank 3 hyperbolic root system AE_3 by realizing the AE_3 root system as the set of 2×2 symmetric integral matrices X with $\det(X) \geqslant -1$. In this context, each element $M \in W(AE_3)$ acts on the root space $\mathfrak{h}_2(\mathbb{R})$ via $X \mapsto MXM^\top$. It was also mentioned in [FF83] that this methodology could be applied to two other (dual) rank 4 hyperbolic root systems whose Weyl groups both contain as an index 4 subgroup the Picard group $PSL_2(G)$. Moreover, in [KMW] Kac, Moody and Wakimoto generalized the structural results to the hyperbolic root system $E_{10} = E_8^{++}$. There has been very little new insight into the structure of the Weyl groups of hyperbolic root systems until Feingold, Kleinschmidt, and Nicolai presented in [FKN09] a coherent picture for many higher rank hyperbolic Kac-Moody root systems which was based on the relation to modular groups associated with lattices and subrings of the four normed division algebras over $\mathbb{R}$. Explicitly, their results are shown in the following table.

$\mathbb{K}$	Root system Φ	$W(\Phi)$	$W^+(\Phi^{++})$
$\mathbb{R}$	A_1	$2 \equiv \mathbb{Z}_2$	$PSL_2(\mathbb{Z})$
$\mathbb{C}$	A_2	$\mathbb{Z}_3 \rtimes 2$	$PSL_2(E)$
$\mathbb{C}$	$B_2 \equiv C_2$	$\mathbb{Z}_4 \rtimes 2$	$PSL_2(G) \rtimes 2$
$\mathbb{C}$	G_2	$\mathbb{Z}_6 \rtimes 2$	$PSL_2(E) \rtimes 2$
$\mathbb{H}$	A_4	$\mathcal{S}_5$	$PSL_2^{(0)}(I)$
$\mathbb{H}$	B_4	$2^4 \rtimes \mathcal{S}_4$	$PSL_2^{(0)}(H) \rtimes 2$
$\mathbb{H}$	C_4	$2^4 \rtimes \mathcal{S}_4$	$\widetilde{PSL}_2^{(0)}(H) \rtimes 2$
$\mathbb{H}$	D_4	$2^3 \rtimes \mathcal{S}_4$	$PSL_2^{(0)}(H)$
$\mathbb{H}$	F_4	$2^5 \rtimes (\mathcal{S}_3 \times \mathcal{S}_3)$	$PSL_2(H) \rtimes 2$
$\mathbb{O}$	D_8	$2^7 \rtimes \mathcal{S}_8$	$PSL_2^{(0)}(O)$
$\mathbb{O}$	B_8	$2^8 \rtimes \mathcal{S}_8$	$PSL_2^{(0)}(O) \rtimes 2$
$\mathbb{O}$	E_8	$2.O_8^+(2).2$	$PSL_2(O)$

In the table, $C = A.B$ means that the group C contains A as a normal subgroup with quotient C/A isomorphic to B. Such a group C is called an extension of A by B. It can happen that the extension is a semi-direct product, so that B is a subgroup of C which acts on A via conjugation as automorphisms, and in this case the product is denoted by $A \rtimes B$. Additionally, $\mathcal{S}_n$ denotes the symmetric group on n letters.

The results in [FKN09] are in analogy with the generators and relations description of the $W(E_{10})$ as products of fundamental reflections (and also in analogy with the description of the continuous Lorentz group $SO(9,1;\mathbb{R})$ via octonionic 2×2 matrices in [MS93]). However, it still remained an outstanding problem to find a more manageable realization of this group directly in terms of 2×2 matrices with Octavian entries.

About the dissertation

Outline and results

I will start by exploring the relationships between number systems $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ via Cayley-Dickson process (Chapter 1). In Chapter 2 I will include the idea in [Sud84] about how to define the Lie algebra $\mathfrak{sl}_2(\mathbb{O})$ such that it generalizes the lower dimensional cases $\mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{sl}_2(\mathbb{C})$, and $\mathfrak{sl}_2(\mathbb{H})$. There will be a description of Lie groups $GL_2(\mathbb{O})$ and $SL_2(\mathbb{O})$ in Chapter 3, which can be found in [MS93] and [MD99]. In particular, every matrix in $SL_2(\mathbb{O})$ is similar to some matrix of the form $\begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ or $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, which is called the *Jordan canonical form* and will be useful in classifying the conjugacy classes of $PSL_2(\mathbb{O})$.

In the second part, I will first specify a subgroup of the octonionic Möbius group that is isomorphic to $PSL_2(\mathbb{O})$ and then classify the conjugacy classes of $PSL_2(\mathbb{O})$.

[**Theorem** 4.3.2; Chapter 4] The conjugacy classes of $PSL_2(\mathbb{O})$ are given by

(i) Parabolic classes:

$$\{\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \mid \|a\| = 1\}$$

with uniqueness up to the similarity of a. Here $a, b \in \mathbb{O}$ are similar if there exists $h \in \mathbb{O}$ such that $a = hbh^{-1}$.

(ii) Elliptic classes:

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid \|a\| = 1\}$$

with uniqueness up to the similarity of a in $\mathbb{O}$.

(iii) Loxodromic classes:

$$\{\begin{bmatrix} \lambda a & 0 \\ 0 & \lambda^{-1} d \end{bmatrix} \mid \lambda > 1, \|a\| = \|d\| = 1, \lambda a \nsim \lambda^{-1} d\}$$

with uniqueness up to the similarity classes of λa and $\lambda^{-1} d$ and order of the diagonal entries.

(iv) Hyperbolic classes:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \mid \lambda > 1\}$$

with uniqueness up to the order of the diagonal entries.

In Chapter 5 and Chapter 6, I will explicitly give the generating sets for some modular groups defined over integral lattices inside those normed division algebras over $\mathbb{R}$. In particular, we have

$$\begin{aligned}
\mathrm{PSL}_2(\mathsf{E}) &= \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \rangle && \text{Equation 5.2} \\
\mathrm{PSL}_2(\mathsf{L}) &= \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix} \rangle && \text{Equation 5.1.2} \\
\mathrm{PSL}_2^*(\mathsf{H}) &= \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix} \rangle && \text{Equation 5.4} \\
\mathrm{PSL}_2^*(\mathsf{O}) &= \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon_i \\ 0 & 1 \end{bmatrix} \mid 1 \leqslant i \leqslant 8 \rangle && \text{Equation 5.6.}
\end{aligned}$$

Note that all these modular groups are generated by upper triangular matrices of the form $\begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix}$, plus a common generator $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

In the third part, I will first define projective lines using formally real Jordan algebras (Chpater 7), which will be used in Chapter 8 to prove that projective Moufang sets are local Moufang sets and are special. Thus, there will be a simplified expression of μ-maps, which would gives rise to Equation 8.8

$$\begin{bmatrix} 0 & -\bar{\mathbf{x}} \\ \mathbf{x} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \bar{\mathbf{x}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \bar{\mathbf{x}} \\ 0 & 1 \end{bmatrix}.$$

This will be significantly important in studying hyperbolic Weyl groups.

As for the last part, I will introduce in Chapter 9 some basics of root systems and Coxeter systems. Afterwards in Chapter 10, I will illustrate the relationships between modular groups previously defined and hyperbolic Weyl groups arising from over-extending finite root systems in those four normed division algebras over $\mathbb{R}$. Proposition 10.4.1 says that $W^+(\mathsf{D}_4^{++}) \cong \mathrm{PSL}_2^*(\mathsf{H})$, which follows that $\mathrm{PSL}_2^*(\mathsf{H}) = \mathrm{PSL}_2^{(0)}(\mathsf{H})$ since it has been shown in [FKN09] that $\mathrm{PSL}_2^*(\mathsf{H}) \cong W^+(\mathsf{D}_4^{++})$. More importantly, it is proved in Theorem 10.5.1 that $W^+(\mathsf{E}_{10}) \cong \mathrm{PSL}_2^*(\mathsf{O})$.

Open questions

1. It has been proved in Theorem 10.5.1 that $W^+(\mathsf{E}_{10})$ is isomorphic to the group $\mathrm{PSL}_2^*(\mathsf{O})$. This is different from the expression in [Equation 6.24; [FKN09]] which says that $W^+(\mathsf{E}_{10}) \cong \mathrm{PSL}_2(\mathsf{O})$. Thus, it seems like it should be the case that $\mathrm{PSL}_2^*(\mathbb{O}) = \mathrm{PSL}_2(\mathbb{O})$. However, this is kind of counter-intuitive because we have $\mathrm{PSL}_2(\mathbb{H})/\mathrm{PSL}_2^*(\mathbb{H}) \cong \mathbb{Z}_3$.

2. We have already found a concise description of the Weyl group $W(\mathsf{E}_{10})$. Then a natural question is to examine the action of $\mathrm{PSL}_2^*(\mathbb{O})$ on $\mathfrak{h}_2(\mathbb{O})$ and classify the orbits of this action.

PART I

PROJECTIVE SPECIAL LINEAR GROUPS OVER $\mathbb{K}$

CHAPTER 1

Normed Division Algebras Over $\mathbb{R}$

1.1 Normed division algebras

1.1.1 Quadratic spaces

Let V be a finite dimensional vector space over $\mathbb{R}$. A map $q : V \to \mathbb{R}$ is called a *quadratic form* on V if it satisfies:

(i) $q(t\mathbf{v}) = t^2 q(\mathbf{v})$ for all $t \in \mathbb{R}$, $\mathbf{v} \in V$; and

(ii) the symmetric pairing

$$B_q : V \times V \to \mathbb{R};\ (\mathbf{v}, \mathbf{w}) \mapsto \frac{1}{2}\left[q(\mathbf{v}+\mathbf{w}) - q(\mathbf{v}) - q(\mathbf{w})\right] \tag{1.1}$$

is bilinear.

The ordered pair (V, q) is then called a *quadratic space*.

It is easy to see that $B_q(\mathbf{v}, \mathbf{v}) = q(\mathbf{v})$. Actually, the map $q \to B_q$ gives rise to a one-to-one correspondence between quadratic forms on V and symmetric bilinear forms on V.

A vector $\mathbf{v} \in V$ such that $q(\mathbf{v}) = 0$ is called a null vector. Let $\mathcal{Q}$ denote the set of all null vectors in V, i.e.,

$$\mathcal{Q} = \{\mathbf{v} \in V \mid q(\mathbf{v}) = 0\}.$$

$\mathcal{Q}$ is called the *quadric* of q. When $\mathcal{Q} = \{0\}$, the quadratic form q is said to be anisotropic. Otherwise, q is isotropic, in which case the non-zero vectors in $\mathcal{Q}$ are also said to be isotropic.

Suppose that the quadratic space V is n-dimensional and has a basis $\{\mathbf{v}_i\}_{i=1}^n$. We write $a_{ij} := B_q(\mathbf{v}_i, \mathbf{v}_j)$ and define $A := (a_{ij}) \in \mathrm{Mat}_n(\mathbb{R})$. Since B_q is symmetric, we have $a_{ij} = a_{ji}$, and hence A is an symmetric matrix. For any $\mathbf{v} = \sum_{i=1}^n x_i \mathbf{v}_i \in V$, we have

$$\begin{aligned} q(\mathbf{v}) &= B_q(\mathbf{v}, \mathbf{v}) \\ &= B_q\Big(\sum_{i=1}^n x_i \mathbf{v}_i, \sum_{i=1}^n x_i \mathbf{v}_i\Big) \\ &= \sum_{i=1}^n B_q(\mathbf{v}_i, \mathbf{v}_i) x_i^2 + \sum_{i<j} \big[B_q(\mathbf{v}_i, \mathbf{v}_j) + B_q(\mathbf{v}_j, \mathbf{v}_i)\big] x_i x_j \\ &= \sum_{i=1}^n a_{ii} x_i^2 + \sum_{i<j} 2a_{ij} x_i x_j \qquad (1.2) \\ &= \mathbf{x}^\top A \mathbf{x}, \qquad (1.3) \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$. Expression 1.2 implies that q can be characterized as a homogeneous polynomial of degree two in n variables with coefficients in $\mathbb{R}$.

On the other hand, it follows from Expression 1.3 that the quadratic form q is determined by the matrix A. Thus, we define the positive or negative (semi)definiteness, or indefiniteness of q to be equivalent to the same property of the matrix A. In particular, (V, q) is called a *normed vector space* if q is positive-definite, i.e., A is a positive-definite matrix. In this case, q is a norm on V and $q(\mathbf{v})$ is usually written as $\|\mathbf{v}\|^2$ for all $\mathbf{v} \in V$.

1.1.2 Normed division algebras

An *algebra* is a finite dimensional real vector space A equipped with a bilinear map $m : A \times A \to A$ called *multiplication*. Unless otherwise specified, we always assume that A is unital, which means there exists a nonzero element $1_A \in A$ called *unit* such that $m(1, a) = m(a, 1) = a$. We do not require our algebras to be associative. The multiplication $m(a, b)$ is, as usual, abbreviated as ab.

Let $l_a : x \mapsto ax$ and $r_a : x \mapsto xa$ denote the left and right multiplication by $a \in A$, respectively. If, for all nonzero $a \in A$, the operations l_a and r_a are invertible, then A is called a *division algebra*. If A is a normed vector space with $\|ab\| = \|a\|\|b\|$ for all $a, b \in A$, then A is called a *normed division algebra*. Obviously, a normed division algebra is always a division algebra with $\|1_A\| = 1$.

1.2 Quaternions

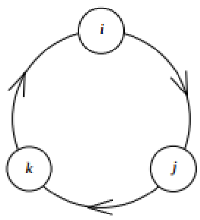
The quaternions $\mathbb{H}$ are a number system that extends the complex numbers. It is a 4-dimensional associative algebra with basis 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Formally, every quaternion could be expressed in the form

$$x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

where the coefficients x_0, x_1, x_2, and x_3 are all real numbers and the basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy

- $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$; and
- $\mathbf{ij} = \mathbf{k}, \quad \mathbf{ji} = -\mathbf{k}$.

This rule is better summarized in a picture as follows:



Moreover, given a quaternion $\mathbf{a} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, the conjugate of $\mathbf{a}$ is

$$\bar{\mathbf{a}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k},$$

and the norm of $\mathbf{a}$ is

$$\|\mathbf{a}\|^2 := x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

It is straightforward to check that $(\mathbb{H}, \|\cdot\|)$ is a normed division algebra.

1.3 Octonions

Let e_0, e_1, e_2, e_3, e_4, e_5, e_6, and e_7 denote the unit base octonions in $\mathbb{O}$, where $e_0 = 1$ is the scale element. That is, every octonion $\mathbf{x}$ can be written in the form

$$\mathbf{x} = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$$

with $x_i \in \mathbb{R}$ for all $i = 0, \ldots, 7$.

The algebra of octonions, $\mathbb{O}$, is neither commutative nor associative. The products of unit octonions can be summarized by the relations:

- $e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k$, where ε_{ijk} is a completely antisymmetric tensor with value +1 when $ijk = 123, 145, 176, 246, 257, 347, 365$;
- $e_i e_0 = e_0 e_i = e_i$ for $i = 1, \ldots, 7$.

We then obtain the following multiplication table.

Table 1.1: Unit Octonion Multiplication Table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

The above definition though is not unique, but is only one of 480 possible definitions for octonion multiplication with $e_0 = 1$. The others can be obtained by permuting and changing the signs of the non-scalar basis elements. The 480 different algebras are isomorphic to one another [Cox46]. Actually, any nontrivial product, say $e_1 e_5 = e_6$, together with the following two rules is enough to recover the whole multiplication table.

- Index cycling: $e_i e_j = e_k \quad \Rightarrow \quad e_{i+1} e_{j+1} = e_{k+1}$.
- Index doubling: $e_i e_j = e_k \quad \Rightarrow \quad e_{2i} e_{2j} = e_{2k}$.

Note that all indices are to be taken modulo 7.

The definition above does not seem very enlightening, especially when we are multiplying two octonions. Fortunately, we can use the Fano plane to remember the products of unit octonions more conveniently and comfortably.

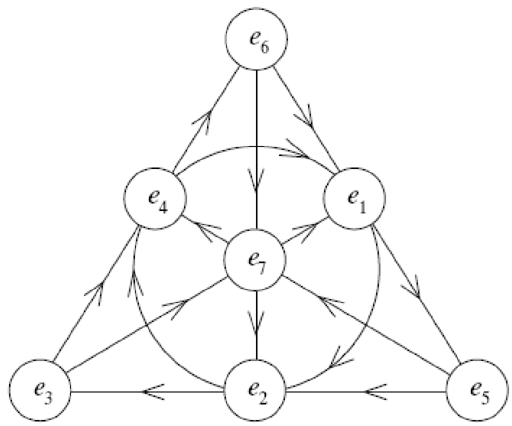


Figure 1.1: Fano plane

The "lines" of the Fano plane are the sides of the triangle, its altitudes, and the circle containing all the midpoints of the sides. The seven points correspond to the seven standard basis elements of $\mathrm{Im}\mathbb{O}$, the set of pure imaginary octonions. Each pair of distinct points lies on a unique line and each line runs through exactly three points. The lines are oriented as shown by the arrows. Explicitly, if (e_i, e_j, e_k) is an ordered triple lying on a given line with the order specified by the direction of the arrow, then we have

$$e_i e_j = e_k, \text{ and } e_j e_i = -e_k.$$

These rules together with

$$e_0 = 1,\ e_1^2 = \cdots = e_7^2 = -1$$

completely defines the algebra structure of the octonions. Moreover, each of the seven lines generated a subalgebra of $\mathbb{O}$ isomorphic to the quaternions $\mathbb{H}$.

Given an octonion $\mathbf{x} = x_0 + \sum_{i=1}^{7} x_i e_i \in \mathbb{O}$, the conjugate of $\mathbf{x}$ is defined as

$$\bar{\mathbf{x}} = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3 - x_4 e_4 - x_5 e_5 - x_6 e_6 - x_7 e_7.$$

Direct calculation shows that $\overline{\mathbf{xy}} = \bar{\mathbf{y}}\bar{\mathbf{x}}$. The norm of $\mathbf{x}$ is defined as

$$\|\mathbf{x}\|^2 = \mathbf{x}\bar{\mathbf{x}} = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2.$$

Clearly, the only octonion with norm zero is 0, and every nonzero octonion has a unique inverse, namely $\mathbf{x}^{-1} = \dfrac{\bar{\mathbf{x}}}{\|\mathbf{x}\|^2}$. It is then clear that $\mathbb{O}$ is a normed division algebra.

Even though the algebra $\mathbb{O}$ is not associative, it is alternative [CS03], that is, products involving no more than two independent octonions do associate. Moreover, we have the following Moufang identities as consequences of the alternativity [dra]:

$$\begin{aligned} (\mathbf{xyx})\mathbf{z} &= \mathbf{x}\big(\mathbf{y}(\mathbf{xz})\big), \\ \mathbf{z}(\mathbf{xyx}) &= \big((\mathbf{zx})\mathbf{y}\big)\mathbf{x}, \\ (\mathbf{xy})(\mathbf{zx}) &= \mathbf{x}(\mathbf{yz})\mathbf{x}. \end{aligned}$$

1.4 Cayley-Dickson process

1.4.1 Cayley-Dickson construction

An algebra A is said to be *conic* if there exists a quadratic form $\mathsf{q} : A \to \mathbb{R}$ such that

$$x^2 - 2B_{\mathsf{q}}(x, 1_A)x + \mathsf{q}(x)1_A = 0, \quad \forall x \in A.$$

Here B_{q} is the symmetric bilinear form associated to q. Actually, q is uniquely determined by the above condition [GP11] and (A, q) is a normed vector space. In addition, we define the conjugation map of A as

$$\bar{x} := 2B_{\mathsf{q}}(x, 1_A)1_A - x,$$

which has order 2 and is characterized by the conditions

$$\bar{1}_A = 1_A, \quad x\bar{x} = \mathsf{q}(x)1_A, \ \forall x \in A.$$

Let $A' = A \oplus Aj$ be the direct sum of two copies of A as vector spaces and $\epsilon \in \mathbb{R}\backslash\{0\}$ be a non-zero scalar. Then the following product gives rise to a conic algebra structure on A'

$$(u_1 + v_1 j)(u_2 + v_2 j) := (u_1 u_2 + \epsilon \bar{v}_2 v_1) + (v_2 u_1 + v_1 \bar{u}_2)j.$$

The norm and conjugation of A' are respectively given by

$$\begin{aligned} \mathsf{q}(u + vj) &= \mathsf{q}(u) - \epsilon \mathsf{q}(v), \\ \overline{u + vj} &= \bar{u} - vj. \end{aligned}$$

The resulted algebra is denoted $\mathtt{Cay}(A, \epsilon)$ and called the *Cayley-Dickson construction* from (A, ϵ). Note that A can be embedded into A' as a unital conic subalgebra through the first summand; we always identify $A \subseteq A'$ accordingly.

Inductively, we would obtain

$$A^{(m)} \triangleq \mathtt{Cay}(A; \epsilon_1, \cdots, \epsilon_m) := \mathtt{Cay}(\mathtt{Cay}(A; \epsilon_1, \cdots, \epsilon_{m-1}))$$

by iterating the Cayley-Dickson construction starting from A. It is a conic algebra of dimension $2^m \dim_{\mathbb{R}}(A)$. We say $A^{(m)}$ arises from A and $\epsilon_1, \cdots, \epsilon_m$ by means of the *Cayley-Dickson process*.

1.4.2 Normed division algebras

A conic algebra A is said to be *real* if $a = \bar{a}$ for all $a \in A$; is *nicely-normed* if $a + \bar{a} \in \mathbb{R}$ and $a\bar{a} > 0$ for all $a \in A \backslash \{0\}$. The following proposition shows the effect of repeatedly applying the Cayley-Dickson construction:

Proposition 1.4.1 ([Bae02]). *(i) A' is never real;*

(ii) A is real (and thus commutative) $\iff$ A' is commutative;

(iii) A is commutative and associative $\iff$ A' is associative;

(iv) A is associative and nicely-normed $\iff$ A' is alternative and nicely normed (which implies A' is a normed division algebra);

(v) A is nicely-normed $\iff$ A' is nicely normed.

It is clear that

$$\begin{aligned} \mathbb{C} &= \mathtt{Cay}(\mathbb{R}; -1), \\ \mathbb{H} &= \mathtt{Cay}(\mathbb{C}; -1) = \mathtt{Cay}(\mathbb{R}; -1, -1), \\ \mathbb{O} &= \mathtt{Cay}(\mathbb{H}; -1) = \mathtt{Cay}(\mathbb{C}; -1, -1) = \mathtt{Cay}(\mathbb{R}; -1, -1, -1). \end{aligned}$$

As a result of Proposition 1.4.1, we have

$\mathbb{R}$ is real, commutative, associative and nicely normed

$\Rightarrow \mathbb{C}$ is commutative, associative and nicely normed

$\Rightarrow \mathbb{H}$ is associative and nicely normed

$\Rightarrow \mathbb{O}$ is alternative and nicely normed

and, in particular, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are all normed division algebras.

Theorem 1.4.2 (Hurwitz Theorem; [CS03]). *Up to isomorphism, there are only four normed division algebras over $\mathbb{R}$ (which are also known as Euclidean Hurwitzian algebras): the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$.*

In the following, $\mathbb{K}$ is always understood to be one of the four normed division algebras over $\mathbb{R}$, and its dimension is $r := \dim_{\mathbb{R}} \mathbb{K}$.

1.5 Integral lattices of $\mathbb{K}$

1.5.1 Lattices and orders

A *lattice* Λ of rank n is a free abelian group isomorphic to $\mathbb{Z}^n$, equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. We may assign a matrix, called *Gram matrix*, to Λ; its entries are $\langle a_i, a_j \rangle$ with the elements a_i being a basis of Λ. Especially, the determinant of the Gram matrix is referred to as the determinant of the lattice.

The lattice Λ is

- integral if the bilinear form $\langle \cdot, \cdot \rangle$ takes values in $\mathbb{Z}$;
- unimodular if its determinant is 1 or -1;
- even or of type II if all norms $\langle a, a \rangle$ are even, otherwise odd or of type I.

Lattices are often embedded in a real vector space with a symmetric bilinear form. The signature of a lattice is the signature of the form on the vector space. Thus, the lattice is called positive definite, Lorentzian, etc. if the corresponding vector space is.

A subring $\mathcal{O}$ of a ring A is called an *order* if the following hold:

(i) the ring A is a finite-dimensional algebra over the rational number field $\mathbb{Q}$;

(ii) $\mathcal{O}$ is a lattice in A; and

(iii) $\mathcal{O}$ spans A over $\mathbb{Q}$.

The last two conditions can be stated in less formal terms: $\mathcal{O}$ is a free abelian group generated by a basis for A over $\mathbb{Q}$. An order $\mathcal{O}$ is said to be *maximal* if it is not properly contained in any other orders.

1.5.2 Integral lattices of $\mathbb{K}$

Clearly, $\mathbb{Z}$ is an integral lattice in $\mathbb{R}$ with respect to the standard norm of $\mathbb{R}$. If we restrict the Cayley-Dickson construction $\mathtt{Cay}(\mathbb{R}; -1) = \mathbb{C}$ to $\mathbb{Z}$, we would obtain $\mathtt{Cay}(\mathbb{Z}; -1) = \mathbb{Z}[\mathbf{i}]$, which is an order in $\mathbb{C}$. Let

$$G := \mathbb{Z}[\mathbf{i}] = \{m + n\mathbf{i} \mid m, n \in \mathbb{Z}\}.$$

The elements of G are commonly called *Gaussian integers*. Clearly, G has four units $\{\pm 1, \pm \mathbf{i}\}$.

Meanwhile, the *Eisensteinian integers* also form an order of $\mathbb{C}$,

$$\mathsf{E} := \{m + n\omega \mid m, n \in \mathbb{Z},\ \omega = \frac{-1+\sqrt{3}\mathbf{i}}{2}\}.$$

Similarly, restricting the Cayley-Dickson construction $\mathbb{H} = \mathtt{Cay}(\mathbb{C}; -1)$ to G gives rise to the *Lipschitzian integers*

$$\begin{aligned} \mathsf{L} &:= \mathtt{Cay}(\mathsf{G}; -1) \\ &= \{n_0 + n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \mid n_0,\ n_1,\ n_2,\ n_3\ \in \mathbb{Z}\} \\ &= \mathbb{Z}[\mathbf{i},\ \mathbf{j},\ \mathbf{k}] \end{aligned}$$

with units $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$.

In addition, restricting the previous Cayley-Dickson construction to E yields the *Eisensteinian quaternionic integers*

$$\begin{aligned} \mathsf{EisH} &:= \mathsf{Cay}(\mathsf{E}, -1) \\ &= \{n_0 + n_1\omega + n_2\mathbf{j} + n_3\omega\mathbf{j} \mid n_0,\ n_1,\ n_2,\ n_3\ \in \mathbb{Z}\} \\ &= \mathbb{Z}[\omega, \mathbf{j}]. \end{aligned}$$

It has 12 units:

$$\pm 1,\ \pm\omega,\ \pm\omega^2,\ \pm\mathbf{j},\ \pm\omega\mathbf{j},\ \omega^2\mathbf{j}.$$

Note that L is an order but not a maximal order in $\mathbb{H}_{\mathbb{Q}} = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_i \in \mathbb{Q}\}$ since Lipschitzian integers are contained in the ring of *Hurwitzian integers*

$$\mathsf{H} = \{n_0 + n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k} \mid n_0,\ n_1,\ n_2,\ n_3 \quad \begin{array}{r} \text{either all belong to } \mathbb{Z} \\ \text{or all belong to } \mathbb{Z} + \frac{1}{2}\}, \end{array}$$

which constitute a maximal order in $\mathbb{H}_{\mathbb{Q}}$. There are precisely 24 Hurwitzian units, namely the eight Lipschitzian units $\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, and the 16 others: $\pm\frac{1}{2} \pm \frac{1}{2}\mathbf{i} \pm \frac{1}{2}\mathbf{j} \pm \frac{1}{2}\mathbf{k}$. They form a subgroup in the unit quaternions.

Notice that if we write $\mathbf{h} = \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$. Then it is clear that

$$\mathsf{H} = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}.$$

Thus, H is a free abelian group and is isomorphic to the $\mathcal{F}_4$ lattice in $\mathbb{R}^4$.

We can generalize this process to the octonions and get the order of *Gravesian integers*:

$$\begin{aligned}\mathsf{Gra} &:= \mathtt{Cay}(\mathsf{L}; -1) \\ &= \{n_0 + \sum_{i=1}^{7} n_i e_i \mid n_i \in \mathbb{Z} \text{ for all } i = 0, 1, \ldots, 7\} \\ &= \mathbb{Z}[e_1, e_2, e_3, e_4, e_5, e_6, e_7],\end{aligned}$$

the *Eisensteinian octaves*:

$$\mathsf{EisO} := \mathtt{Cay}(\mathsf{EisH}; -1),$$

and the *Hurwitzian octaves*:

$$\mathsf{HurO} := \mathtt{Cay}(\mathsf{H}; -1).$$

Note that the Gravesian integers Gra is not a maximal order. As described in [Cox46], there are exactly seven maximal orders containing Gra. These seven maximal orders are all equivalent under automorphisms. Once a choice of one maximal order of $\mathbb{O}$ is specified, we will call it *octaves* and denote it by O. The elements of O are said to be Octavian. In this thesis, we fix O to be $\bigoplus_{i=1}^{8} \mathbb{Z}\epsilon_i$ with

$$\begin{aligned}\epsilon_1 &= \frac{1}{2}(1 - e_1 - e_5 - e_6), & \epsilon_2 &= e_1, \\ \epsilon_3 &= \frac{1}{2}(-e_1 - e_2 + e_6 + e_7), & \epsilon_4 &= e_2, \\ \epsilon_5 &= \frac{1}{2}(-e_2 - e_3 - e_4 - e_7), & \epsilon_6 &= e_3, \\ \epsilon_7 &= \frac{1}{2}(-e_3 + e_5 - e_6 + e_7), & \epsilon_8 &= e_4.\end{aligned}$$

The octaves O has some unusual properties [CS03]:

(1) Every ideal in O is 2-sided.

(2) Any 2-sided ideal Λ in O is the principal ideal $n\mathsf{O}$ generated by a rational integer n.

There are 240 Octavian units in O, which are listed in [CS03]. An Octavian unit ring is a subring of O generated by units. In particular, it is worth mentioning that

$$1 = 2\epsilon_i + 3\epsilon_2 + 4\epsilon_3 + 5\epsilon_4 + 6\epsilon_5 + 4\epsilon_6 + 2\epsilon_7 + 3\epsilon_8. \tag{1.4}$$

Theorem 1.5.1 (Theorem 5; [CS03]). *Up to isomorphism, there are precisely four types of integer rings generated by odd-order elements:* $\mathbb{Z}$, E, H, O, *from which all of the Octavian unit rings can be obtained by Cayley-Dickson process.*

$$
\begin{array}{ccccccc}
\mathbb{R} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{H} & \longrightarrow & \mathbb{O} \\
\mathbb{Z} & \longrightarrow & \mathsf{G} & \longrightarrow & \mathsf{L} & \longrightarrow & \mathsf{Gra} \\
 & & \mathsf{E} & \longrightarrow & \mathsf{EisH} & \longrightarrow & \mathsf{EisO} \\
 & & & & \mathsf{H} & \longrightarrow & \mathsf{HurO} \\
 & & & & & & \mathsf{O}
\end{array}
$$

Each arrow $(\mathrm{A} \to \mathrm{B})$ *refers to a Cayley-Dickson construction* $\mathrm{B} = \mathrm{Cay}(\mathrm{A}; -1)$.

CHAPTER 2

Special Linear Lie Algebra $\mathfrak{sl}_2(\mathbb{K})$

Let $\mathtt{Mat}_n(\mathbb{K})$ denote the set of $n \times n$ matrices over $\mathbb{K}$, which can be decomposed into

$$\mathtt{Mat}_n(\mathbb{K}) = \mathfrak{a}_n(\mathbb{K}) \oplus \mathfrak{h}_n(\mathbb{K}),$$

where

$$\mathfrak{a}_n(\mathbb{K}) := \{X \in \mathtt{Mat}_n(\mathbb{K}) \mid X^\dagger = -X\}$$

$$\mathfrak{h}_n(\mathbb{K}) := \{X \in \mathtt{Mat}_n(\mathbb{K}) \mid X^\dagger = X\}.$$

Here $X^\dagger := \bar{X}^\top$ is the conjugate transpose of X. The elements of $\mathfrak{a}_n(\mathbb{K})$ are called *skew-hermitian* and those of $\mathfrak{h}_n(\mathbb{K})$ are said to be *hermitian*.

When $\mathbb{K}$ is associative, both $\mathfrak{a}_n(\mathbb{K})$ and $\mathtt{Mat}_n(\mathbb{K})$ are Lie algebras with the Lie bracket given by the commutator. Conventionally, we denote by $\mathfrak{gl}_n(\mathbb{K})$ the Lie algebra $\mathtt{Mat}_n(\mathbb{K})$. The Lie algebras $\mathfrak{a}_n(\mathbb{K})$ and $\mathfrak{gl}_n(\mathbb{K})$ each have a center consisting of multiples of the identity matrix; the quotient by this center will be denoted by $\mathfrak{sa}_n(\mathbb{K})$ and $\mathfrak{sl}_n(\mathbb{K})$, respectively.

Unfortunately, the above process does not hold true for the non-associative case $\mathbb{K} = \mathbb{O}$. One way to handle this issue is, as suggested in [Sud84], to think of elements in $\mathfrak{sa}_n(\mathbb{K})$ and $\mathfrak{sl}_n(\mathbb{K})$ as derivations of Jordan algebras $\mathfrak{h}_n(\mathbb{K})$. From this perspective we may extend our definition to include the non-associative case when $n = 2$ or 3. We will focus on 2×2 matrices only because our purpose is to study modular groups.

2.1 Jordan algebras and Lie multiplication algebras

An algebra $\mathtt{J}$ is called a *Jordan algebra* if it is commutative and satisfies the Jordan identity

$$a \circ (b \circ (a \circ a)) = (a \circ b) \circ (a \circ a) \tag{2.1}$$

for all elements a and b. An ideal in the Jordan algebra is a subspace $\mathtt{I} \subseteq \mathtt{J}$ such that $b \in \mathtt{I}$ implies $a \circ b \in \mathtt{I}$ for all $a \in \mathtt{J}$. If a Jordan algebra has no nontrivial ideal, then it is said to be *simple*. If a Jordan algebra can be written as a direct sum of simple ones, then it is *semisimple*.

Given an associative algebra over $\mathbb{R}$, we may define a Jordan algebra structure via the quasi-multiplication:

$$x \circ y = \frac{1}{2}(xy + yx). \tag{2.2}$$

All such Jordan algebras, as well as their subalgebras, are called *special* Jordan algebras. A Jordan algebra that is not special is then said to be *exceptional*.

Let $\mathtt{J}$ be a Jordan algebra. If a linear transformation $D : \mathtt{J} \to \mathtt{J}$ satisfies

$$D(x \circ y) = (Dx) \circ y + x \circ (Dy),$$

then it is called a *derivation* of $\mathtt{J}$. Let $\mathtt{DerJ}$ denote the set of derivations of $\mathtt{J}$. It is straightforward to verify that $\mathtt{DerJ}$ is a Lie algebra with respect to the Lie bracket $[D_1, D_2] := D_1 D_2 - D_2 D_1$, where the multiplication is understood as the composition of derivations. Moreover, let R_x be the right multiplication $R_x : y \mapsto yx$. It is easy to check that for a linear map D :

$$D \in \mathtt{DerJ} \iff [R_x, D] = R_{D(x)}, \ \forall x \in \mathtt{J}. \tag{2.3}$$

On the other hand, using the Jordan identity 2.1 we can show that

$$[R_a, [R_b, R_c]] = R_{\mathcal{A}(b,a,c)}, \tag{2.4}$$

where $\mathcal{A}(b, a, c) = (ba)c - b(ac)$ is the associator. Note that the Jordan algebra $\mathtt{J}$ is commutative, which implies

$$\mathcal{A}(b, a, c) = (ab)c - (ac)b = a[R_b, R_c].$$

Hence, Equation 2.4 becomes

$$[R_a, [R_b, R_c]] = R_{a[R_b, R_c]}. \tag{2.5}$$

This, together with Equation 2.3, indicate that $[R_b, R_c]$, or, $x \mapsto \mathcal{A}(b, x, c)$ is a derivation. Such derivations are called *inner derivations* of $\mathtt{J}$. The set of inner derivations is denoted $\mathtt{Inn}(\mathtt{J})$. In particular, for semisimple Jordan algebras, we have

Proposition 2.1.1 (Theorem 2; [Jac49]). *Every derivation of a semisimple Jordan algebra with a finite basis over a field of characteristics 0 is inner.*

Consider the Lie subalgebra of $\mathfrak{gl}(\mathtt{J})$ generated by all the right multiplication maps R_x. (Of course, it can also be defined over left multiplication maps.) We call this enveloping Lie algebra the *Lie multiplication algebra* of $\mathtt{J}$ and denote it by $\mathfrak{L}(\mathtt{J})$. (It also appears in some papers under the name of *structure algebra* of $\mathtt{J}$.) Note that Equation 2.5 indicates that $\{R_a\}$ is a *Lie triple system* of linear transformations. From [Sud84] we obtain

$$\mathfrak{L}(\mathtt{J}) = R(\mathtt{J}) \oplus \mathtt{Inn}(\mathtt{J}).$$

Particularly, when $\mathtt{J}$ is semisimple, we obtain from Proposition 2.1.1 that $\mathtt{Der}\,\mathtt{J} = \mathtt{Inn}(\mathtt{J})$, and hence, $\mathfrak{L}(\mathtt{J}) = R(\mathtt{J}) \oplus \mathtt{Der}\,\mathtt{J}$.

As an example, from [JJ49] we know that the Jordan algebra $\mathfrak{h}_2(\mathbb{K})$ is semisimple. (Note that $\mathfrak{h}_2(\mathbb{O})$ is a spin factor, which will be explained later.) Therefore, we have

$$\mathfrak{L}(\mathfrak{h}_2(\mathbb{K})) = R(\mathfrak{h}_2(\mathbb{K})) \oplus \mathtt{Der}\,\mathfrak{h}_2(\mathbb{K}).$$

2.2 $\mathtt{Der}\,\mathfrak{h}_2(\mathbb{K})$ and $\mathfrak{sa}_2(\mathbb{K})$

When $\mathbb{K}$ is associative, it is known that every derivation $D \in \mathtt{Der}\,\mathfrak{h}_2(\mathbb{K})$ must be of the form $\mathtt{ad}(A)$ for some skew-hermitian matrix $A \in \mathfrak{a}_2(\mathbb{K})$. Here $\mathtt{ad}$ refers to the adjoint representation. Using the Jacobi identity

$$[A, [X, Y]] = [[A, X], Y] + [X, [A, Y]] \tag{2.6}$$

it is easy to see that $\mathtt{ad}(A) = 0$ if and only if $A = \lambda I_2$ with $\lambda \in \mathbb{K}$ and I_2 being the 2×2 identity matrix, or equivalently, $A \in Z(\mathfrak{a}_2(\mathbb{K}))$, the center of $\mathfrak{a}_2(\mathbb{K})$. This implies that, as Lie algebras,

$$\mathtt{Der}\,\mathfrak{h}_2(\mathbb{K}) \cong \mathfrak{a}_2(\mathbb{K})/Z(\mathfrak{a}_2(\mathbb{K})) = \mathfrak{sa}_2(\mathbb{K}). \tag{2.7}$$

We would like to define a Lie algebra structure on $\mathfrak{sa}_2(\mathbb{O})$ such that it generalizes Equation 2.7.

We first examine the Lie algebra $\mathtt{Der}\,\mathfrak{h}_2(\mathbb{O})$. Obviously, derivations of $\mathbb{O}$ act as derivations of $\mathfrak{h}_2(\mathbb{O})$ by acting on the entries in the matrices. Meanwhile, from [Sud84] we know that Equation 2.6 still holds true for $\mathbb{O}$ when $A \in \mathfrak{a}_2(\mathbb{O})$. Thus, $\mathtt{ad}(A)$ is a derivation of $\mathfrak{h}_2(\mathbb{O})$ as long as $A \in \mathfrak{a}_2(\mathbb{O})$. According to [Jac49], these are all the derivations of $\mathfrak{h}_2(\mathbb{O})$, which implies

$$\mathtt{Der}\,\mathfrak{h}_2(\mathbb{O}) = \mathtt{ad}(\mathfrak{a}_2(\mathbb{O})) + \mathtt{Der}\,\mathbb{O}. \tag{2.8}$$

We may further decompose $\mathfrak{a}_2(\mathbb{O})$ as

$$\mathfrak{a}_2(\mathbb{O}) = \mathfrak{a}_2'(\mathbb{O}) \oplus \mathbb{O}' I_2,$$

where $\mathfrak{a}_2'(\mathbb{O})$ is the subspace of traceless matrices, and $\mathbb{O}'$ is the subspace of $\mathbb{O}$ orthogonal to $\mathbb{R}$. Note that $\mathtt{ad}(\mathfrak{a}_2'(\mathbb{O})) \cong \mathfrak{a}_2'(\mathbb{O})$ since $\mathfrak{h}_2(\mathbb{O})$ is an irreducible set. For $x \in \mathbb{O}'$, $\mathtt{ad}(xI_2)$ acts on $\mathfrak{h}_2(\mathbb{O})$ by acting as C_x on each entry in the matrix; thus, $\mathtt{ad}(\mathbb{O}' I_2) \cong C(\mathbb{O}')$. Here C_x is the commutator map $C_x(y) = xy - yx$. Hence, we get

$$\mathtt{ad}(\mathfrak{a}_2(\mathbb{O})) \cong \mathfrak{a}_2'(\mathbb{O}) \oplus C(\mathbb{O}'),$$

where $C(\mathbb{O}')$ is the set of all commutator maps C_x. Replacing this into Equation 2.8 we obtain

$$\mathtt{Der}\,\mathfrak{h}_2(\mathbb{O}) = \mathfrak{a}_2'(\mathbb{O}) + C(\mathbb{O}') + \mathtt{Der}\,\mathbb{O}.$$

Note that, as illustrated in [Sud84], we have $\mathfrak{so}(\mathbb{O}') = \mathtt{Der}\,\mathbb{O} + C(\mathbb{O}')$, which leads to

$$\mathtt{Der}\,\mathfrak{h}_2(\mathbb{O}) \cong \mathfrak{a}_2'(\mathbb{O}) + \mathfrak{so}(\mathbb{O}').$$

Explicit calculations show that it is actually a direct sum, i.e.,

$$\mathtt{Der}\,\mathfrak{h}_2(\mathbb{O}) \cong \mathfrak{a}_2'(\mathbb{O}) \oplus \mathfrak{so}(\mathbb{O}'). \tag{2.9}$$

Define the vector space

$$\mathfrak{sa}_2(\mathbb{O}) := \mathfrak{a}_2'(\mathbb{O}) \oplus \mathfrak{so}(\mathbb{O}').$$

It remains to construct an appropriate Lie algebra structure on this space such that it is isomorphic to $\mathtt{Der}\,\mathfrak{h}_2(\mathbb{O})$ as Lie algebras. Recall that the Lie bracket of $\mathfrak{sa}_2(\mathbb{K})$ is given by the matrix commutator for associative $\mathbb{K}$; this is a consequence of the Jacobi identity (Equation 2.6). However, this identity is no longer true for the octonions due to the lack of associativity. Actually, we have

$$[[A,B],X] - [A,[B,X]] - [B,[A,X]] = \sum_{ij} \mathcal{A}(a_{ij}, b_{ji}, X), \tag{2.10}$$

which generally differs from 0. Nevertheless, we have the following result for some restricted classes of matrices.

Lemma 2.2.1 ([Sud84]). *When A, $B \in \mathfrak{a}_2'(\mathbb{O})$ and $X \in \mathfrak{h}_2(\mathbb{O})$, $\sum_{ij} \mathcal{A}(a_{ij}, b_{ji}, X)$ in Equation 2.10 can be written as $\Omega(A,B)X$ for some $\Omega(A,B) \in \mathfrak{so}(\mathbb{O}')$.*

This gives rise to a bilinear map

$$\mathfrak{a}_2'(\mathbb{O}) \times \mathfrak{a}'(\mathbb{O}) \to \mathfrak{so}(\mathbb{O}'), \quad (A,B) \mapsto \Omega(A,B), \tag{2.11}$$

which enables us to define a Lie algebra structure on $\mathfrak{a}_2'(\mathbb{K}) \oplus \mathfrak{so}(\mathbb{K}')$ in the following way:

- $\mathfrak{so}(\mathbb{O}')$ is contained as a subalgebra, that is, the Lie bracket on $\mathfrak{so}(\mathbb{O}')$ is retained;
- the Lie bracket of $T \in \mathfrak{so}(\mathbb{O}')$ and a matrix $A \in \mathfrak{a}_2'(\mathbb{O})$ is given by the action of T on the entries in A; and
- the Lie bracket between two matrices in $\mathfrak{a}_2'(\mathbb{O})$ is defined via

  $$[A,B] = (AB - BA - xI_2) \oplus (C_x + \Omega(A,B))$$

  where $x = \frac{1}{2}\mathrm{Tr}(AB - BA)$ and Ω comes from Equation 2.11.

It is straightforward to show that

Proposition 2.2.2. *Equipped with the Lie bracket defined above,*

$$\mathfrak{sa}_2(\mathbb{O}) \cong \mathtt{Der}\,\mathfrak{h}_2(\mathbb{O}) \text{ as Lie algebras.}$$

2.3 $\mathfrak{sl}_2(\mathbb{K})$ and $\mathfrak{L}(\mathfrak{h}_2(\mathbb{K}))$

Recall that the Lie multiplication algebra over the Jordan algebra $\mathfrak{h}_2(\mathbb{K})$ can be decomposed into

$$\begin{aligned} \mathfrak{L}(\mathfrak{h}_2(\mathbb{K})) &= R(\mathfrak{h}_2(\mathbb{K})) \oplus \mathtt{Der}\,\mathfrak{h}_2(\mathbb{K}) \\ &\cong \mathfrak{h}_2(\mathbb{K}) \oplus \mathtt{Der}\,\mathfrak{h}_2(\mathbb{K}). \end{aligned} \tag{2.12}$$

Obviously, the multiples of the identity element all belong to the center of $\mathfrak{L}(\mathfrak{h}_2(\mathbb{K}))$. We factor out this ideal and write the resulting algebra as $\mathfrak{L}'(\mathfrak{h}_2(\mathbb{K}))$. Then we have

$$\mathfrak{L}'(\mathfrak{h}_2(\mathbb{K})) \cong \mathfrak{sh}_2(\mathbb{K}) \oplus \mathtt{Der}\,\mathfrak{h}_2(\mathbb{K}),$$

where $\mathfrak{sh}_2(\mathbb{K}) = \mathfrak{h}_2(\mathbb{K})/\mathbb{K}I_2$.

In last section we have already defined the Lie algebra $\mathfrak{sa}_2(\mathbb{O})$ so that $\mathfrak{sa}_2(\mathbb{K}) \cong \mathtt{Der}\,\mathfrak{h}_2(\mathbb{K})$ for all normed division algebras $\mathbb{K}$. This means for all $\mathbb{K}$ we have

$$\mathfrak{L}'(\mathfrak{h}_2(\mathbb{K})) \cong \mathfrak{sh}_2(\mathbb{K}) \oplus \mathfrak{sa}_2(\mathbb{K}). \tag{2.13}$$

We define the special linear algebra as

$$\mathfrak{sl}_2(\mathbb{O}) := \mathfrak{L}'(\mathfrak{h}_2(\mathbb{O})).$$

This is compatible with the associative cases. In fact, when $\mathbb{K}$ is associative we have $\mathtt{Mat}_2(\mathbb{K}) = \mathfrak{a}_2(\mathbb{K}) \oplus \mathfrak{h}_2(\mathbb{K})$, and hence

$$\mathfrak{sl}_2(\mathbb{K}) = \mathfrak{sa}_2(\mathbb{K}) \oplus \mathfrak{sh}_2(\mathbb{K}) \cong \mathfrak{L}'(\mathfrak{h}_2(\mathbb{K})).$$

In order to define the Lie algebra structure on $\mathfrak{sl}_2(\mathbb{O})$, we recall that $\mathfrak{sa}_2(\mathbb{O}) = \mathfrak{a}_2'(\mathbb{O}) \oplus \mathfrak{so}(\mathbb{O}')$, which indicates that

$$\mathfrak{sl}_2(\mathbb{O}) = \mathfrak{L}'(\mathfrak{h}_2(\mathbb{O})) \cong \mathfrak{sh}_2(\mathbb{O}) \oplus \mathfrak{a}_2'(\mathbb{O}) \oplus \mathfrak{so}(\mathbb{O}').$$

Write $\mathfrak{gl}_2'(\mathbb{K}) \triangleq \mathfrak{sh}_2(\mathbb{K}) \oplus \mathfrak{a}_2'(\mathbb{K})$. It suffices to define Lie brackets on $\mathfrak{gl}_2'(\mathbb{K}) \oplus \mathfrak{so}(\mathbb{K}')$ as we did for $\mathfrak{sa}_2(\mathbb{K})$:

- $\mathfrak{so}(\mathbb{K}')$ remains to be a subalgebra of $\mathfrak{sl}_2(\mathbb{K})$;
- the Lie bracket of $T \in \mathfrak{so}(\mathbb{K}')$ and a matrix $A \in \mathfrak{gl}_2'(\mathbb{K})$ is again given by the action of T on the entries in A; and
- similar to Lemma 2.2.1 we define an analogous bilinear map Ω for $\mathfrak{gl}_2'(\mathbb{O})$, with which we may define the Lie bracket

  $$[A, B] = (AB - BA - xI) \oplus (C_x + \Omega(A, B))$$

  for matrices $A, B \in \mathfrak{gl}_2'(\mathbb{O})$. Here $x = \dfrac{1}{2}\mathrm{Tr}(AB - BA)$ as before.

It is straightforward to verify that $\mathfrak{sl}_2(\mathbb{O})$, equipped with the Lie bracket defined above, is a Lie algebra.

Furthermore, recall that for associative $\mathbb{K}$, we have

$$\mathfrak{sl}_2(\mathbb{K}) \simeq \mathfrak{so}(r + 1, 1)$$

with $r = \dim_{\mathbb{R}}\mathbb{K}$. This also holds for the Lie algebra $\mathfrak{sl}_2(\mathbb{O})$ [Sud84], that is

$$\mathfrak{sl}_2(\mathbb{O}) \simeq \mathfrak{so}(9, 1).$$

CHAPTER 3

(Projective) Special Linear Lie groups

3.1 Special linear groups over commutative $\mathbb{K}$

When $\mathbb{K}$ is commutative, i.e., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the special linear Lie group $\mathrm{SL}_n(\mathbb{K})$ can be characterized as the commutator subgroup:

$$\mathrm{SL}_n(\mathbb{K}) = [\mathrm{GL}_n(\mathbb{K}), \mathrm{GL}_n(\mathbb{K})],$$

which is a normal subgroup of $\mathrm{GL}_n(\mathbb{K})$.

On the other hand, the determinant map $\det : \mathrm{GL}_n(\mathbb{K}) \to \mathbb{K}^\times$ is a surjective group homomorphism, whose kernel is exactly $\mathrm{SL}_n(\mathbb{K})$, i.e., $\mathrm{SL}_n(\mathbb{K}) = \ker(\det)$. Here $\mathbb{K}^\times$ is referred to as the multiplicative group of $\mathbb{K}$. Therefore, we get

$$\mathrm{GL}_n(\mathbb{K})/\mathrm{SL}_n(\mathbb{K}) \cong \mathbb{K}^\times.$$

3.1.1 $\mathrm{SL}_2(\mathbb{R})$

It is known that the group $\mathrm{SL}_2(\mathbb{R})$ acts on its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ by conjugation, which induces a homomorphism from $\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{Aut}\big(\mathfrak{sl}_2(\mathbb{R})\big)$; it is called the adjoint representation and commonly denoted $\mathtt{Ad}$. Note that all elements in $\mathtt{Ad}\big(\mathrm{SL}_2(\mathbb{R})\big)$ preserve the Killing form, thus are of signature $(2,1)$ or $(1,2)$. Those two types actually yield the same group of isometries. As a consequence, we obtain a group homomorphism $\mathtt{Ad} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{O}(2,1)$. Moreover, since $\mathrm{SL}_2(\mathbb{R})$ is connected, this homomorphism actually maps $\mathrm{SL}_2(\mathbb{R})$ onto the connected component containing the identity in $\mathrm{O}(2,1)$, which is exactly the Lorentz group $\mathrm{SO}_0(2,1)$. It is easy to see that $\ker \mathtt{Ad} = \{\pm I_2\}$, which indicates that $\mathrm{SL}_2(\mathbb{R})$ is a double cover of $\mathrm{SO}_0(2,1)$.

On the other hand, it is known that the Lorentz group $SO_0(2,1)$ has a double cover, $Spin(2,1)$, which is called the *spin group* and has certain representations called spinor representations [Mei13]. Therefore, we obtain

$$SL_2(\mathbb{R}) \cong Spin(2,1).$$

3.1.2 $SL_2(\mathbb{C})$

Consider the action of $SL_2(\mathbb{C})$ on the Minkowski space-time that is isometric to $\mathfrak{h}_2(\mathbb{C})$:

$$SL_2(\mathbb{C}) \times \mathfrak{h}_2(\mathbb{C}) \to \mathfrak{h}_2(\mathbb{K}) : \quad (P, X) \mapsto PXP^\dagger.$$

This action preserves the determinant, i.e., $\det(PXP^\dagger) = \det(X)$. Especially, it yields a homomorphism, called the *spinor map*, from $SL_2(\mathbb{C})$ to $SO_0(3,1)$. The kernel of the map is the two-element subgroup $\{\pm I_2\}$. Therefore, the group $SL_2(\mathbb{C})$ is a double cover of $SO_0(3,1)$, that is,

$$SL_2(\mathbb{C}) \cong Spin(3,1).$$

3.2 Special linear group over quaternions

Things become more complicated for the quaternions $\mathbb{H}$.. The main obstacles, as expected, come from the non-commutative multiplication of quaternions: $\mathbb{H}$ is a skew-field.

Let $\mathtt{Mat}_n(\mathbb{H})$ be the set of $n \times n$ quaternionic matrices. A matrix $A \in \mathtt{Mat}_n(\mathbb{H})$ is invertible if there exists a matrix B such that $AB = I_n$ or $BA = I_n$. Even though $\mathbb{H}$ is non-commutative, it is still true that given any invertible matrix, its left inverse and right inverse coincide [Zha97]. We denote by $GL_n(\mathbb{H})$ the set of all invertible quaternionic matrices.

Let E_{ij} be the elementary matrix with 1 at the $(i,j)^{\text{th}}$ entry and 0 elsewhere. To every quaternion $\mathbf{x} \in \mathbb{H}$ we associate some quaternionic matrices

$$B_{ij}(\mathbf{x}) = I_n + \mathbf{x}E_{ij} \quad \text{for } i \neq j.$$

We define $SL_n(\mathbb{H})$ to be the subgroup of $GL_n(\mathbb{H})$ generated by these matrices

$$SL_n(\mathbb{H}) := \langle B_{ij}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{H}, 1 \leqslant i,j \leqslant n \text{ and } i \neq j \rangle. \tag{3.1}$$

The following lemma defends the definition of $SL_n(\mathbb{H})$.

Lemma 3.2.1 ([Asl96]). *(i)* $\mathrm{SL}_n(\mathbb{H})$ *is the commutator subgroup of* $\mathrm{GL}_n(\mathbb{H})$*, that is,*

$$\mathrm{SL}_n(\mathbb{H}) = [\mathrm{GL}_n(\mathbb{H}), \mathrm{GL}_n(\mathbb{H})].$$

(ii) Every matrix $A \in \mathrm{GL}_n(\mathbb{H})$ *can be written in the form* $A = D(\mathbf{x})B$ *with* $D(\mathbf{x}) = \mathrm{diag}(1, \cdots, 1, \mathbf{x})$ *for some* $\mathbf{x} \in \mathbb{H}$ *and* $B \in \mathrm{SL}_n(\mathbb{H})$*. Even though neither* $\mathbf{x}$ *nor* B *is unique, the coset* $\mathbf{x}[\mathbb{H}^\times, \mathbb{H}^\times] \in \mathbb{H}^\times/[\mathbb{H}^\times, \mathbb{H}^\times]$ *is uniquely determined.*

(iii) $D(\mathbf{x})$ *is a commutator in* $\mathrm{GL}_n(\mathbb{H})$ *if and only if* $\mathbf{x}$ *is a commutator in* $\mathbb{H}^\times$*, that is,*

$$D(\mathbf{x}) \in \mathrm{SL}_n(\mathbb{H}) \iff \mathbf{x} \in [\mathbb{H}^\times, \mathbb{H}^\times].$$

Here $\mathbb{H}^\times$ *is the multiplicative group of* $\mathbb{H}$.

Recall that for the real and complex cases, $\mathrm{SL}_n(\mathbb{K})$ is exactly the kernel of the determinant function. However, the universal notion of a determinant does not work well for non-commutative division rings. Actually, the question of the definition of a unique determinant of a square matrix in the general non-commutative case does not make sense if we consider determinants with values in the ring. Especially, there does not exist such a determinant for quaternionic matrices that extends the usual determinant of real and complex matrices. An alternative notion of quasi-determinants is used for non-commutative algebras, which can be found in [GGRW05].

For quaternions $\mathbb{H}$, the most commonly used "determinant" is the Dieudonné determinant [Art11].

Theorem 3.2.2 ([Die43]). *Let* F *be a skew-field and* $n \geqslant 2$*. Then the commutator subgroup* $\mathrm{SL}_n(F)$ *is normal. In addition, there exists a natural isomorphism*

$$\mathscr{D} : \mathrm{GL}_n(F)/\mathrm{SL}_n(F) \to F^\times/[F^\times, F^\times]$$

that is uniquely defined by the property that for any invertible diagonal matrix $X = \mathit{diag}(x_1, \cdots, x_n)$

$$\mathscr{D}(X) = \prod_{i=1}^{n} x_i \bmod [F^\times, F^\times].$$

Moreover, let $p : \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)/\mathrm{SL}_n(F)$ *denote the canonical projection. The* Dieudonné determinant *is defined as follows:*

$$\mathtt{Det} : \mathtt{Mat}_n(F) \to F^\times/[F^\times, F^\times] \cup \{0\}$$

$$\begin{cases} \mathtt{Det}(X) := \mathscr{D}\big(p(X)\big) & \text{when } X \text{ is invertible;} \\ \mathtt{Det}(X) := 0 & \text{when } X \text{ is not invertible.} \end{cases}$$

For $F = \mathbb{H}$, it follows from Lemma 3.2.1 and Theorem 3.2.2 that

$$\mathtt{Det}A = \mathbf{x}[\mathbb{H}^\times, \mathbb{H}^\times] \text{ for } A = D(\mathbf{x})B \in GL_n(\mathbb{H}).$$

In particular, it is obvious that

$$\ker \mathtt{Det} = SL_n(\mathbb{H}).$$

Furthermore, we have

Lemma 3.2.3 ([VP91]). *$[\mathbb{H}^\times, \mathbb{H}^\times] \simeq U(\mathbb{H})$, the set of quaternions of length one. This enables us to identify $\mathbb{H}^\times/[\mathbb{H}^\times, \mathbb{H}^\times]$ with the multiplicative group $\mathbb{R}_{>0}$ via*

$$\begin{array}{rrcl} \omega : & \mathbb{H}^\times/[\mathbb{H}^\times, \mathbb{H}^\times] & \to & \mathbb{R}_{>0} \\ & \mathbf{x}[\mathbb{H}^\times, \mathbb{H}^\times] & \mapsto & \|\mathbf{x}\| := \sqrt{\bar{\mathbf{x}}\mathbf{x}}. \end{array}$$

Thus, the (normalized) Dieudonné determinant for $\mathbb{H}$ becomes

$$\mathtt{Det} : \mathtt{Mat}_n(\mathbb{H}) \to \mathbb{R}_{\geqslant 0}$$

$$\begin{cases} \mathtt{Det}A = 0, & \text{when } A \text{ is not invertible;} \\ \mathtt{Det}A = \|\mathbf{x}\|, & \text{when } A \text{ is invertible and hence } A = D(\mathbf{x})B \\ & \text{with } \mathbf{x}[\mathbb{H}^\times, \mathbb{H}^\times] \text{ being uniquely determined.} \end{cases}$$

Because our focus is on modular groups, it is important to study 2×2 hermitian quaternionic matrices. Every matrix in $\mathfrak{h}_2(\mathbb{H})$ can be be written as $\begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix}$ for some $\mathbf{x} \in \mathbb{H}$ and $s, t \in \mathbb{R}$; hence we can define a quadratic form

$$\begin{array}{rrcl} \mathtt{M} : & \mathfrak{h}_2(\mathbb{H}) & \to & \mathbb{R} \\ & \begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix} & \mapsto & \mathbf{x}\bar{\mathbf{x}} - st. \end{array} \tag{3.2}$$

It is easy to see that $\mathtt{M}$ has signature $(5, 1)$ and can be negative, whereas the the Dieudonné determinant $\mathtt{Det}$ must be non-negative.

Lemma 3.2.4.

$$SL_2(\mathbb{H}) = \{A \in GL_2(\mathbb{H}) \mid \mathtt{M}(A^\dagger A) = -1\}.$$

Proof. For every 2×2 hermitian matrix $\begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix}$ we have

$$\begin{pmatrix} 1 & 0 \\ \bar{\mathbf{a}} & 1 \end{pmatrix} \begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix} \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s & s\mathbf{a}+\mathbf{x} \\ s\bar{\mathbf{a}}+\bar{\mathbf{x}} & s\bar{\mathbf{a}}\mathbf{a}+\bar{\mathbf{x}}\mathbf{a}+\bar{\mathbf{a}}\mathbf{x}+t \end{pmatrix},$$

which leads to

$$\begin{aligned} \mathtt{M}(\begin{pmatrix} 1 & 0 \\ \bar{\mathbf{a}} & 1 \end{pmatrix} \begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix} \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix}) &= (s\mathbf{a}+\mathbf{x})(s\bar{\mathbf{a}}+\bar{\mathbf{x}}) - s(s\bar{\mathbf{a}}\mathbf{a}+\bar{\mathbf{x}}\mathbf{a}+\bar{\mathbf{a}}\mathbf{x}+t) \\ &= \bar{\mathbf{x}}\mathbf{x} - st \\ &= \mathtt{M}(\begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix}) \end{aligned} \tag{3.3}$$

Similarly, we can prove that

$$\mathtt{M}(\begin{pmatrix} 1 & \bar{\mathbf{b}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathbf{b} & 1 \end{pmatrix}) = \mathtt{M}(\begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix}). \tag{3.4}$$

On the other hand, recall that every matrix $A \in GL_2(\mathbb{H})$ can be written as $A = D(\mathbf{x})B$, where $D(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{x} \end{pmatrix}$ and according to the definition 3.1 $B \in SL_2(\mathbb{H})$ is a product of matrices of the form $\begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ \mathbf{b} & 1 \end{pmatrix}$. Thus, we have

$$A^\dagger A = B^\dagger D(\bar{\mathbf{x}})D(\mathbf{x})B = B^\dagger D(\bar{\mathbf{x}}\mathbf{x})B. \tag{3.5}$$

Applying 3.3 and 3.4 to 3.5 gives rise to

$$\mathtt{M}(A^\dagger A) = \mathtt{M}(D(\bar{\mathbf{x}}\mathbf{x})) = \mathtt{M}(\begin{pmatrix} 1 & 0 \\ 0 & \bar{\mathbf{x}}\mathbf{x} \end{pmatrix} = -\bar{\mathbf{x}}\mathbf{x}.$$

Therefore, we see that

$$\mathtt{M}(A^\dagger A) = -1 \iff \bar{\mathbf{x}}\mathbf{x} = 1 \iff A \in \ker \mathtt{Det} = SL_2(\mathbb{H}).$$

□

Consider the action of $SL_2(\mathbb{H})$ on the 6-dimensional Minkowski space $\mathbb{R}^{5,1}$ that is isometric to $(\mathfrak{h}_2(\mathbb{H}), \mathtt{M})$:

$$SL_2(\mathbb{H}) \times \mathfrak{h}_2(\mathbb{H}), \quad (M, X) \mapsto MXM^\dagger.$$

Clearly, Lemma 3.2.4 guarantees that $\mathtt{Det}(MXM^\dagger) = \mathtt{Det}(X)$, that is, the action above preserves the Dieudonné determinant. Therefore, we have

Proposition 3.2.5.

$$\mathrm{SL}_2(\mathbb{H}) \cong \mathrm{Spin}(5,1),$$

which is a double cover of $\mathrm{SO}_0(5,1)$.

The following proposition will be useful later.

Proposition 3.2.6. *The group* $\mathrm{SL}_2(\mathbb{H})$ *is generated by*

$$\begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{b} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.6}$$

where $\mathbf{a}$, $\mathbf{b} \in \mathbb{H}$ *satisfying that* $\|\mathbf{ab}\| = 1$.

Proof. It is clear from Proposition 3.2.4 that the generating matrices in 3.6 all belong to $\mathrm{SL}_2(\mathbb{H})$. On the other hand, assuming $\mathbf{a} \neq 0$ with $\mathbf{a}^{1/2}$ being one square root of $\mathbf{a}$, we have

$$\begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{a}^{1/2} & 0 \\ 0 & \mathbf{a}^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}^{-1/2} & 0 \\ 0 & \mathbf{a}^{1/2} \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 \\ \mathbf{a} & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \text{ and } \begin{pmatrix} 1 & -\mathbf{a} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix}^{-1}.$$

Hence, matrices of the form $\begin{pmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ \mathbf{b} & 1 \end{pmatrix}$ can be expressed as products of generators in 3.6. Recall that, as demonstrated in 3.1, $\mathrm{SL}_2(\mathbb{H})$ is generated by those fundamental matrices. Thus, every matrix in $\mathrm{SL}_2(\mathbb{H})$ can be generated by those matrices in 3.6.

It is worth noting that negative real quaternions have infinitely many square roots while all others have just two (or one in the case of 0). For example, there are infinitely many square roots of -1: the quaternion solution for the square root of -1 is the unit sphere in $\mathbb{R}^3$. □

3.3 Special linear group over octonions

Due to the lack of commutativity and associativity, it is impossible to define the Lie group $GL_n(\mathbb{O})$ as we did early on. For example, given an "invertible" matrix, the left inverse is not necessarily equal to the right inverse. Since we are, as explained earlier, interested in modular groups, we will only consider 2×2 octonionic matrices.

To every matrix $M \in \mathtt{Mat}_2(\mathbb{O})$ we assign a linear transformation

$$\widetilde{M} : \mathfrak{h}_2(\mathbb{O}) \longrightarrow \mathfrak{h}_2(\mathbb{O}), \quad \widetilde{M}(X) = \frac{1}{2}\big[(MX)M^\dagger + M(XM^\dagger)\big]. \tag{3.7}$$

Obviously, the composition of such transformations is associative. Moreover, these transformations generate a free monoid. The subset of all invertible transformations form the largest group contained in the monoid, which is defined to be the group $GL_2(\mathbb{O})$. The product is understood via

$$\widetilde{MN}(X) = \widetilde{M}\big(\widetilde{N}(X)\big).$$

Consider Equation 3.7. When $(MX)M^\dagger = M(XM^\dagger)$ holds true, we simply write it as $MXM^\dagger$ without specifying the parentheses. In this case, we get $\widetilde{M}(X) = MXM^\dagger \in \mathfrak{h}_2(\mathbb{O})$. However, it is, in general, unlikely that $(MX)M^\dagger$ and $M(XM^\dagger)$ are equal, unless we impose some additional constraints on M.

Lemma 3.3.1 ([MS93]). *The following statements are equivalent:*

(i) $(MX)M^\dagger = M(XM^\dagger)$ *for all* $X \in \mathfrak{h}_2(\mathbb{O})$.

(ii) The imaginary part of M, $\mathtt{Im}M$, *contains only one octonionic direction.*

(iii) The columns of $\mathtt{Im}M$ *are real multiples of each other.*

Similar to the quaternion case, there does not exist a "determinant" on $\mathtt{Mat}_2(\mathbb{O})$ that generalizes the real and complex determinants. This requires us to find an alternative way to define the group $SL_2(\mathbb{O})$. First, we extend the quadratic form 3.2 to the space $\mathfrak{h}_2(\mathbb{O})$:

$$\begin{aligned} \mathtt{M} : \mathfrak{h}_2(\mathbb{O}) &\to \mathbb{R} \\ \begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix} &\mapsto \mathbf{x}\bar{\mathbf{x}} - st. \end{aligned}$$

It is clear that $(\mathfrak{h}_2(\mathbb{O}), \mathtt{M})$ has signature (9,1).

Recall that for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ the action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathfrak{h}_2(\mathbb{K})$ preserves the quadratic form $\mathtt{M}$. As for the octonionic case, given a matrix $A \in \mathrm{GL}_2(\mathbb{O})$ that satisfies any of the conditions in Lemma 3.3.1, in which case both $\widetilde{A}(X) = AXA^\dagger$ and $AA^\dagger$ are hermitian, we have

Lemma 3.3.2 ([Vei14]). *Write* $A = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$. *Then* $\mathtt{M}\big(AXA^\dagger\big) = \mathtt{M}(AA^\dagger)\mathtt{M}(X)$ *only for the following cases:*

(i) $\mathbf{a} = 0$, *and* $[\mathbf{b}, \mathbf{c}, \mathbf{x}] = 0$ *for all* $\mathbf{x} \in \mathbb{O}$;
Analogously, when $\mathbf{b} = 0$ *and* $[\mathbf{a}, \mathbf{d}, \mathbf{x}] = 0$; $\mathbf{c} = 0$ *and* $[\mathbf{a}, \mathbf{d}, \mathbf{x}] = 0$; *or* $\mathbf{d} = 0$ *and* $[\mathbf{b}, \mathbf{c}, \mathbf{x}] = 0$.

(ii) $[\mathbf{u}, \mathbf{v}, \mathbf{x}] = 0$ *for any* $\mathbf{u}, \mathbf{v} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$, $\mathbf{x} \in \mathbb{O}$.

Especially, in either of these cases, we have $\mathtt{M}(AA^\dagger) = \|\mathbf{ad} - \mathbf{bc}\|^2$.

It is then natural to define

$$\mathrm{SL}_2(\mathbb{O}) := \{\widetilde{A} \in \mathrm{GL}_2(\mathbb{O}) \mid \mathtt{M}(AA^\dagger) = 1;\ (AX)A^\dagger = A(XA^\dagger)\, \forall X \in \mathfrak{h}_2(\mathbb{O})\}$$

so that every $\widetilde{A} \in \mathrm{SL}_2(\mathbb{O})$ preserves the quadratic form $\mathtt{M}$:

$$\mathtt{M}\big(\widetilde{A}(X)\big) = \mathtt{M}(X),\ \forall X \in \mathfrak{h}_2(\mathbb{O}).$$

Moreover, we have

Theorem 3.3.3 ([Vei14]). $\mathrm{SL}_2(\mathbb{O})$ *is a Lie group and its algebra is exactly* $\mathfrak{sl}_2(\mathbb{O})$.

The following is a characterization of elements in $\mathrm{SL}_2(\mathbb{O})$, which can be derived from Lemma 3.3.1 and Lemma 3.3.2.

Proposition 3.3.4. *For every element* $\widetilde{A} \in \mathrm{SL}_2(\mathbb{O})$ *with* $A = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$, *there exists a pure imaginary unit* $\mathbf{q}$ *such that*

$$\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R} \oplus \mathbb{R}\mathbf{q}.$$

Here $\mathbf{q}$ *is a pure imaginary unit means that* $\mathbf{q} \in \mathtt{Im}\mathbb{O}$ *and* $\mathbf{q}^2 = -1$.

Note that $\mathbb{R} \oplus \mathbb{R}\mathbf{q} \cong \mathbb{C}$ is a commutative and associative subalgebra of $\mathbb{O}$.

On the other hand, for any $C \in SL_2(\mathbb{C})$ we can find some matrix $P \in SL_2(\mathbb{C})$ such that

$$PCP^{-1} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

which is actually the Jordan canonical form of C. Clearly, this can be generalized to $\mathbb{R} \oplus \mathbb{R}\mathbf{q}$. Explicitly, there exists a matrix $U \in SL_2(\mathbb{O})$ with all entries belonging to $\mathbb{R} \oplus \mathbb{R}\mathbf{q}$ such that

$$UAU^{-1} = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} \text{ or } \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \tag{3.8}$$

where $\alpha, \beta \in \mathbb{R} \oplus \mathbb{R}\mathbf{q}$ and satisfy $\|\alpha\beta\|^2 = 1$. Analogously, we call the resulting upper triangular matrix $\begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ or $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ the *Jordan canonical form* of A.

Furthermore, according to the definition of $SL_2(\mathbb{O})$, the action

$$SL_2(\mathbb{O}) \times \mathfrak{h}_2(\mathbb{O}), \quad (M, X) \mapsto \widetilde{M}(X)$$

is obviously determinant-preserving. Analogous to previous cases, we have

Proposition 3.3.5 ([dra]).

$$SL_2(\mathbb{O}) \cong \mathrm{Spin}(9,1).$$

3.4 Projective Special Linear Groups

3.4.1 $PSL_2(\mathbb{K})$

It is well known that the center of $SL_2(\mathbb{K})$ is $\{\pm I_2\}$ when $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We claim that it is also true for the octonions $\mathbb{O}$. In fact, assume that $A \in Z\big(SL_2(\mathbb{O})\big)$, the center of $SL_2(\mathbb{O})$. Then for any $B \in SL_2(\mathbb{O})$

$$\widetilde{AB}(X) = \widetilde{BA}(X),\ \forall X \in \mathfrak{h}_2(\mathbb{O}).$$

That is,

$$(AB)X(AB)^\dagger = (BA)X(BA)^\dagger\ \forall X \in \mathfrak{h}_2(\mathbb{O}).$$

As a result of Proposition 3.3.4, the matrices A and B each contains one unique imaginary unit. Then the expression above is essentially quaternionic! Thus, we get $A \in \{\pm I_2\}$, and hence, $Z\big(SL_2(\mathbb{O})\big) = \{\pm I_2\}$.

The projective special linear group is then defined as

$$\mathrm{PSL}_2(\mathbb{K}) := \mathrm{SL}_2(\mathbb{K})/Z\big(\mathrm{SL}_2(\mathbb{K})\big) \equiv \mathrm{SL}_2(\mathbb{K})/\{\pm I_2\}.$$

We have already shown that

$$\mathrm{SL}_2(\mathbb{K}) \cong \mathrm{Spin}(r+1,1)$$

for all four normed division algebras over $\mathbb{R}$. Here r is the dimension of $\mathbb{K}$ over $\mathbb{R}$. Recall that the Spin group is a double cover of the Lorentz group $\mathrm{SO}_0(r+1,1)$. Thus, it follows that

$$\mathrm{PSL}_2(\mathbb{K}) \simeq \mathrm{SO}_0(r+1,1). \tag{3.9}$$

In particular, $\mathrm{PSL}_2(\mathbb{K})$ are all simple Lie groups because the Lorentz groups $\mathrm{SO}_0(n,1)$ are simple when $n \geqslant 2$.

3.4.2 Polar decomposition of Lorentz groups

Consider $\mathfrak{so}(n,1)$, the Lie algebra of the Lorentz group $\mathrm{SO}_0(n,1)$. Let κ be the *Cartan involution*, namely,

$$\kappa(A) = -A^\top,\ \forall A \in \mathfrak{so}(n,1).$$

Then κ has two eigenvalues, 1 and -1. Denote by $\mathfrak{k}_n$ and $\mathfrak{p}_n$ the eigenspace of 1 and -1, respectively. It is clear that

$$\mathfrak{so}(n,1) = \mathfrak{k}_n \oplus \mathfrak{p}_n.$$

This is called the *Cartan decomposition* of $\mathfrak{so}(n,1)$. Explicitly, we have

$$\begin{aligned}
\mathfrak{so}(n,1) &= \{\begin{pmatrix} B & \mathbf{u} \\ \mathbf{u}^\top & 0 \end{pmatrix} \in \mathtt{Mat}_{n+1}(\mathbb{R}) \mid \mathbf{u} \in \mathbb{R}^n,\ B^\top = -B\},\\
\mathfrak{k}_n &= \{\begin{pmatrix} B & 0 \\ 0^\top & 0 \end{pmatrix} \in \mathtt{Mat}_{n+1}(\mathbb{R}) \mid B^\top = -B\} \cong \mathfrak{so}(n),\\
\mathfrak{p}_n &= \{\begin{pmatrix} 0 & \mathbf{u} \\ \mathbf{u}^\top & 0 \end{pmatrix} \in \mathtt{Mat}_{n+1}(\mathbb{R}) \mid \mathbf{u} \in \mathbb{R}^n\}.
\end{aligned}$$

Let $\exp : \mathfrak{so}(n,1) \to \mathrm{SO}_0(n,1)$ denote the exponential map. It is surjective and $K_n \triangleq \exp \mathfrak{k}_n \cong \mathrm{SO}(n)$ is the maximal compact subgroup of $\mathrm{SO}_0(n,1)$ [Kna13]. Moreover, we have the following decomposition, which is called the *polar decomposition* of Lie group $\mathrm{SO}_0(n,1)$:

$$\mathrm{SO}_0(n,1) = K_n \exp \mathfrak{p}_n. \tag{3.10}$$

3.4.3 Relations between $\mathrm{PSL}_2(\mathbb{K})$

The polar decomposition 3.10 gives rise to a canonical map

$$\begin{aligned} \eta_n : \mathrm{SO}_0(n,1) = K_n \exp(\mathfrak{p}_n) &\to \mathrm{SO}(n) \times \mathbb{R}^n \\ \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \cdot \exp \begin{pmatrix} 0 & \mathbf{u} \\ \mathbf{u}^T & 0 \end{pmatrix} &\mapsto (Q, \mathbf{u}). \end{aligned}$$

This map enables us to embed $\mathrm{SO}_0(n,1)$ into $\mathrm{SO}_0(m,1)$ when $n \leqslant m$. Explicitly, we first define a map

$$\begin{aligned} \iota_{n,m} : \mathrm{SO}_0(n,1) &\to \mathrm{SO}_0(m,1) \\ A &\mapsto \begin{pmatrix} I_{m-n} & 0 \\ 0 & A \end{pmatrix}. \end{aligned}$$

Consider the following diagram

$$\begin{array}{ccc} \mathrm{SO}_0(n,1) & \xrightarrow{\eta_n} & \mathrm{SO}(n) \times \mathbb{R}^n \\ \big\downarrow \iota_{n,m} & & \big\downarrow \\ \mathrm{SO}_0(m,1) & \xrightarrow{\eta_m} & \mathrm{SO}(m) \times \mathbb{R}^m \end{array} \tag{3.11}$$

The embedding $\mathrm{SO}(n) \times \mathbb{R}^n \hookrightarrow \mathrm{SO}(m) \times \mathbb{R}^m$ is obvious:

$$(Q, \mathbf{u}) \mapsto \left(\begin{pmatrix} I_{m-n} & \\ & Q \end{pmatrix}, \begin{pmatrix} 0_{m-n} \\ \mathbf{u} \end{pmatrix} \right).$$

We claim that Diagram 3.11 commutes, or equivalently,

$$\eta_m\left(\begin{pmatrix} I_{m-n} & \\ & A \end{pmatrix} \right) = \left(\begin{pmatrix} I_{m-n} & \\ & Q \end{pmatrix}, \begin{pmatrix} 0_{(m-n)\times 1} \\ \mathbf{u} \end{pmatrix} \right).$$

It is sufficient to prove that

$$\begin{pmatrix} I_{m-n} & \\ & A \end{pmatrix} = \begin{pmatrix} I_{m-n} & & \\ & Q & \\ & & 1 \end{pmatrix} \cdot \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{u} \\ 0 & \mathbf{u}^T & 0 \end{pmatrix},$$

which is true because the polar decomposition for any non-singular matrix is unique.

Furthermore, we claim that the map $\iota_{n,m}$ is a group homomorphism. In fact, for any A, $B \in \mathrm{SO}_0(n,1)$,

$$\iota_{n,m}(AB) = \begin{pmatrix} I_{m-n} & \\ & AB \end{pmatrix} = \begin{pmatrix} I_{m-n} & \\ & A \end{pmatrix} \begin{pmatrix} I_{m-n} & \\ & B \end{pmatrix} = \iota_{n,m}(A)\iota_{n,m}(B).$$

Therefore, when $n \leqslant m$, the group $SO_0(n, 1)$ can be viewed as a subgroup of $SO_0(m, 1)$. Especially, following from

$$SO_0(2,1) \overset{\iota_{2,3}}{\hookrightarrow} SO_0(3,1) \overset{\iota_{3,5}}{\hookrightarrow} SO_0(5,1) \overset{\iota_{5,9}}{\hookrightarrow} SO_0(9,1),$$

we obtain

$$PSL_2(\mathbb{R}) \leqslant PSL_2(\mathbb{C}) \leqslant PSL_2(\mathbb{H}) \leqslant PSL_2(\mathbb{O}). \tag{3.12}$$

PART II

MÖBIUS TRANSFORMATIONS AND MODULAR GROUPS

CHAPTER 4

Möbius Transformations

In general, let $\hat{\mathbb{R}}^n \triangleq \mathbb{R}^n \cup \{\infty\} \simeq S^n$ be the one-point compactification of the n-dimensional Euclidean space. A *Möbius transformation* on $\hat{\mathbb{R}}^n$ is the composition of an even number of inversions through spheres or hyperplanes. In this chapter we will study the complex ($n = 2$), quaternionic ($n = 4$), and octonionic ($n = 8$) Möbius transformations.

Throughout the thesis, every matrix from $SL_2(\mathbb{K})$ is written as $\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ with parentheses around; its quotient image in $PSL_2(\mathbb{K})$ will be denoted $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ in brackets.

4.1 Complex Möbius transformations

4.1.1 Complex Möbius group

A complex Möbius transformation is an invertible function from the extended complex plane $\hat{\mathbb{C}} \triangleq \mathbb{C} \cup \{\infty\}$ to itself, defined by four complex numbers a, b, c, d with $ad - bc \neq 0$ as follows:

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} \text{ if } z \neq \infty \text{ and } cz+d \neq 0 \\ \dfrac{a}{c} \text{ if } z = \infty \\ \infty \text{ if } cz+d = 0, \end{cases} \tag{4.1}$$

where if $c = 0$ we use the convention $f(\infty) = \infty$.

Let $\text{Möb}(\mathbb{C})$ denote the group of complex Möbius transformations, which is called the *complex Möbius group*. Clearly, to every matrix $A \in GL_2(\mathbb{C})$ we may assign a complex Möbius transformation f_A as in Equation 4.1. In fact, the map $A \to f_A$ gives rise to a surjective homomorphism from $GL_2(\mathbb{C})$ to $\text{Möb}(\mathbb{C})$, whose kernel is $\mathbb{C}^* I_2$ with $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. Thus, we obtain

$$PGL_2(\mathbb{C}) \cong \text{Möb}(\mathbb{C}). \tag{4.2}$$

On the other hand, given any complex Möbius transformation f, let $A \in GL_2(\mathbb{C})$ be a representing matrix of f, that is, such that $f = f_A$. Let $D = \frac{1}{\sqrt{\det(A)}} I_2 \in Z(GL_2(\mathbb{C}))$, the center of $GL_2(\mathbb{C})$. Then it follows from the identification 4.2 that $f_{DA} = f_A$. Notice that $DA \in SL_2(\mathbb{C})$. This actually gives rise to a surjective homomorphism from $SL_2(\mathbb{C})$ to $\text{Möb}(\mathbb{C})$, whose kernel is $Z(SL_2(\mathbb{C})) = \{\pm I_2\}$. Therefore, we obtain

$$PSL_2(\mathbb{C}) \cong \text{Möb}(\mathbb{C}).$$

Especially, we have

$$PGL_2(\mathbb{C}) = PSL_2(\mathbb{C}).$$

Furthermore, it is well-known that the complex Möbius group $\text{Möb}(\mathbb{C})$ is finitely generated. Explicitly, every $f \in \text{Möb}(\mathbb{C})$ can be written as a composition of the following simple complex Möbius transformations:

(i) Translations: $t_x(z) = z + x$ for some $x \in \mathbb{C}$.

(ii) Dilations: $S_r(z) = rz$ with $r \in \mathbb{R}$.

(iii) Rotations: $R_\theta(z) = e^{i\theta} z$.

(iv) Inversion: $J(z) = \frac{1}{z}$.

4.1.2 Types of complex Möbius transformations

Let f be a non-trivial complex Möbius transformation. Then it is clear that f has at most two fixed points in $\hat{\mathbb{C}}$. Specifically,

- if f has a unique fixed point in $\hat{\mathbb{C}}$, which is exactly ∞, then it is called *parabolic*; in this case f is conjugate to the transformation $z \mapsto z + 1$.

- If f has two fixed points in $\hat{\mathbb{C}}$, then it must be conjugate to a transformation of the form $z \mapsto \lambda z$.
    - If $|\lambda| = 1$, then f is said to be *elliptic*. Note that if write $\lambda = e^{i\theta}$, then it is obvious to see that f is a rotation.
    - If $|\lambda| \neq 0, 1$, then f is called *loxodromic*. Especially, when $\lambda \in \mathbb{R}$ is positive, f is called *hyperbolic*.

Clearly, a loxodromic transformation can always be written as a composition of an elliptic transformation and a hyperbolic transformation: $z \mapsto |\lambda|\lambda_0 z$, where λ_0 is the directional unit of λ.

Since the Möbius group $\text{Möb}(\mathbb{C})$ can be identified as $\text{PSL}_2(\mathbb{C})$, it is sufficient to examine the representing matrices to classify Möbius transformations. We will say a matrix is parabolic, elliptic, loxodromic, or hyperbolic whenever the associated Möbius transformation is.

Note that the trace function is invariant under conjugation, that is, $\texttt{tr}(MAM^{-1}) = \texttt{tr}A$. Moreover, we have the following result.

Lemma 4.1.1 ([GY08]). *Two non-trivial matrices M, $N \in \text{PSL}_2(\mathbb{C})$ are conjugate if and only if $\texttt{tr}^2 M = \texttt{tr}^2 N$.*

The following lemma comes from explicit computations.

Lemma 4.1.2. *Consider a matrix $A \in \text{PSL}_2(\mathbb{C})$.*

(i) *A is parabolic when $\texttt{tr}^2(A) = 4$.*

(ii) *A is elliptic when $\texttt{tr} \in \mathbb{R}$ and $0 \leqslant \texttt{tr}^2 A < 4$.*

(iii) *A is loxodromic when $\texttt{tr}^2 A$ is not in range $[0, 4]$. In particular, A is hyperbolic when A is loxodromic and $\texttt{tr} \in \mathbb{R}$.*

It then follows that (for the details, see [GY08])

(a) Every parabolic matrix in $\text{PSL}_2(\mathbb{C})$ is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) Every non-parabolic matrix is conjugate to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ for some $\lambda \in \mathbb{C} \backslash \{0, \pm 1\}$.

Specifically,

- when $|\lambda| = 1$, it is elliptic, in which case it is common to write $\lambda = e^{i\theta}$;
- when $|\lambda| \neq 1$, it is loxodromic;
- when $\lambda \in \mathbb{R}^+$ and $\lambda \neq 1$, it is hyperbolic.

4.2 Quaternionic Möbius transformations

4.2.1 Quaternionic Möbius group

A quaternionic Möbius transformation is an inverse function from $\hat{\mathbb{H}} \triangleq \mathbb{H} \cup \{\infty\}$ to itself of the form

$$f(\mathbf{z}) = \frac{a\mathbf{z} + b}{c\mathbf{z} + d}, \text{ with } a, b, c, d \in \mathbb{H}.$$

Here f is subject to the same constraints as in the complex case in Equation 4.1. Note that $\frac{a\mathbf{z}+b}{c\mathbf{z}+d}$ is understood as $(a\mathbf{z} + b)(c\mathbf{z} + d)^{-1}$ with

$$(c\mathbf{z} + d)^{-1} = \frac{\overline{c\mathbf{z} + d}}{\|c\mathbf{z} + d\|^2}.$$

Let Möb($\mathbb{H}$) denote the group of quaternionic Möbius transformations. Similar to Möb($\mathbb{C}$), there exists a surjective homomorphism $GL_2(\mathbb{H}) \to$ Möb($\mathbb{H}$), whose kernel is exactly $Z\big(GL_2(\mathbb{H})\big)$, that is, $\mathbb{R}^* I_2$. Hence, we get

$$PGL_2(\mathbb{H}) \cong \text{Möb}(\mathbb{H}). \tag{4.3}$$

On the other hand, given any quaternionic Möbius transformation f, let $A \in GL_2(\mathbb{H})$ be a representing matrix of f such that $f = f_A$. Clearly, $\mathtt{Det}A > 0$, where $\mathtt{Det}$ is the Dieudonné determinant. Consider the matrix $\tilde{A} := \frac{1}{\sqrt{\mathtt{Det}}}A$. It is obvious that $\mathtt{Det}\tilde{A} = 1$, thus $\tilde{A} \in SL_2(\mathbb{H})$. At the same time, since $\frac{1}{\sqrt{\mathtt{Det}}} \in \mathbb{R}$, we conclude that $f_{\tilde{A}} = f_A$. Thus, we obtain a surjective homomorphism $SL_2(\mathbb{H}) \to \text{Möb}(\mathbb{H}) : \tilde{A} \mapsto f$, whose kernel is $Z(SL_2(\mathbb{H})) = \{\pm I_2\}$. Therefore, we get

$$PSL_2(\mathbb{H}) \cong \text{Möb}(\mathbb{H}). \tag{4.4}$$

In particular, the two isomorphisms 4.3 and 4.4 indicate that

$$PSL_2(\mathbb{H}) = PGL_2(\mathbb{H}).$$

4.2.2 Types of quaternionic Möbius transformations

Given a non-trivial quaternionic Möbius transformation f,

- f is said to be *parabolic* if it has exactly one fixed point in $\hat{\mathbb{H}}$;
- f is said to be *elliptic* if it has two fixed points in $\hat{\mathbb{H}}$ and is conjugate to a rotation, i.e., a transformation of the form $\mathbf{z} \mapsto \lambda\mathbf{z}$ with $\|\lambda\| = 1$;
- f is said to be *hyperbolic* if it has two fixed points in $\hat{\mathbb{H}}$ and is conjugate to a dilation, i.e., a transformation of the form $\mathbf{z} \mapsto k\mathbf{z}$, where $k \neq 1$ and $k \in \mathbb{R}_{>0}$;
- f is said to be *loxodromic* if it has exactly two fixed points in $\hat{\mathbb{H}}$ and is conjugate to a transformation $\mathbf{z} \mapsto \lambda\mathbf{z}$ with $\|\lambda\| \neq 0, 1$.

Clearly, a hyperbolic transformation must be loxodromic. Moreover, every loxodromic transformation can be written as a composition of elliptic and hyperbolic transformations.

Furthermore, a matrix $A \in \mathrm{PSL}_2(\mathbb{H})$ is parabolic, elliptic, hyperbolic, or loxodromic whenever the associated $f_A \in \mathrm{Möb}(\mathbb{H})$ is.

To every matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathtt{Mat}_2(\mathbb{H})$ we associate the following quantities

$$\begin{aligned} \beta_A &= \mathtt{Re}[(ad - bc)\bar{a} + (da - cb)\bar{d}], \\ \gamma_A &= |a + d|^2 + 2\mathtt{Re}[ad - bc], \\ \delta_A &= \mathtt{Re}[a + d]. \end{aligned}$$

It follows from [For04] that the functions β, γ and δ are constant on conjugacy classes in $\mathrm{SL}_2(\mathbb{H})$.

Lemma 4.2.1 (Theorem 1.3; [PS09]). *Two matrices $A, B \in \mathrm{PSL}_2(\mathbb{H})$ are conjugate if and only if the following two conditions hold:*

(i) either both of them or neither of them belong to $\mathrm{PSL}_2(\mathbb{R})$;

(ii) $\beta_A\delta_A = \beta_B\delta_B$, $\gamma_A = \gamma_B$, and $\delta_A^2 = \delta_B^2$.

Moreover, we define the following two functions that take the roles of "determinant" and "trace," respectively.

$$\sigma_A = \begin{cases} cac^{-1}d - cb, \text{ when } c \neq 0, \\ bdb^{-1}a, \text{ when } c = 0,\ b \neq 0, \\ (d-a)a(d-a)^{-1}d, \text{ when } b = c = 0,\ a \neq d, \\ a\bar{a}, \text{ when } b = c = 0,\ a = d; \end{cases}$$

$$\tau_A = \begin{cases} cac^{-1} + d, \text{ when } c \neq 0, \\ bdb^{-1} + a, \text{ when } c = 0,\ b \neq 0, \\ (d-a)a(d-a)^{-1} + d, \text{ when } b = c = 0,\ a \neq d, \\ a + \bar{a}, \text{ when } b = c = 0,\ a = d. \end{cases}$$

Proposition 4.2.2 (Theorem 1.4; [PS09]). *Consider a matrix* $A \in \mathrm{PSL}_2(\mathbb{H})$.

(a) *If* $\sigma_A = 1$ *and* $\tau_A \in \mathbb{R}$, *then* $\beta_A = \delta_A$, $\gamma_A = \delta_A^2 + 2$ *and the following trichotomy holds.*

- *If* $0 \leqslant \delta_A^2 < 4$, *then* A *is elliptic.*
- *If* $\delta_A^2 = 4$, *then* A *is parabolic.*
- *If* $\delta_A^2 > 4$, *then* A *is loxodromic.*

(b) *If* $\beta_A = \delta_A$ *and either* $\tau_A \notin \mathbb{R}$ *or* $\sigma_A \neq 1$, *then the following trichotomy holds.*

- *If* $\gamma_A - \delta_A^2 < 2$, *then* A *is elliptic.*
- *If* $\gamma_A - \delta_A^2 = 2$, *then* A *is parabolic.*
- *If* $\gamma_A - \delta_A^2 > 2$, *then* A *is loxodromic.*

(c) *If* $\beta_A \neq \delta_A$, *then* A *is loxodromic.*

By using these functions we can classify the conjugacy classes of $\mathrm{PSL}_2(\mathbb{H})$ as follows.

Proposition 4.2.3 ([For04]). *The conjugacy classes of* $\mathrm{PSL}_2(\mathbb{H})$ *are given by*

(i) *Parabolic classes:*

$$\left\{ \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \mid \|a\| = 1 \right\}$$

with uniqueness up to the similarity of a. *Note that two quaternions* a *and* b *are called similar, or* $a \sim b$, *if there exists* $q \in \mathbb{H}$ *such that* $a = qbq^{-1}$.

(ii) *Elliptic classes:*

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid \|a\| = 1\}$$

with uniqueness up to the similarity of a *in* $\mathbb{H}$.

(iii) *Loxodromic classes:*

$$\{\begin{bmatrix} \lambda a & 0 \\ 0 & \lambda^{-1} d \end{bmatrix} \mid \lambda \geqslant 1, \|a\| = \|d\| = 1, \lambda a \nsim \lambda^{-1} d\}$$

with uniqueness up to the similarity classes of λa *and* $\lambda^{-1} d$ *and order of the diagonal entries.*

(iv) *Hyperbolic classes:*

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \mid \lambda \geqslant 1\}$$

with uniqueness up to the order of the diagonal entries.

4.3 Octonionic Möbius transformations

4.3.1 Octonionic Möbius group

Analogous to the complex and quaternionic cases, we define an octonionic Möbius transformation as an inverse function from $\hat{\mathbb{O}} \triangleq \mathbb{O} \cup \{\infty\}$ to itself of the form

$$f(\mathbf{z}) = \frac{a\mathbf{z} + b}{c\mathbf{z} + d}, \text{ with } a,\ b,\ c,\ d\ \in \mathbb{O}.$$

Here we adopt the same conventions as in Equation 4.1. Also, $\dfrac{a\mathbf{z} + b}{c\mathbf{z} + d}$ is understood as $(a\mathbf{z} + b)(c\mathbf{z} + d)^{-1}$ with

$$(c\mathbf{z} + d)^{-1} = \frac{\overline{c\mathbf{z} + d}}{\|c\mathbf{z} + d\|^2}.$$

Let $\text{Möb}(\mathbb{O})$ be the group generated by octonionic Möbius transformations. It is tempting to claim that the map $A \to f_A$ gives rise to a homomorphism from $GL_2(\mathbb{O})$ to $\text{Möb}(\mathbb{O})$. Due to the non-associativity of $\mathbb{O}$, however, it is not at all obvious that

$f_A f_B = f_{AB}$ holds. Nevertheless, as illustrated in [MD99] and [MS93], the composition rule holds if we restrict ourselves to $SL_2(\mathbb{O})$, where the elements satisfy the "compatibility condition," i.e., Equation **??**. Hence, the map $A \to f_A$ does induce a group homomorphism from $SL_2(\mathbb{O})$ to $\text{Möb}(\mathbb{O})$. Clearly, the kernel of this map is $Z(SL_2(\mathbb{O})) = \{\pm I_2\}$. We then obtain an injective homomorphism

$$PSL_2(\mathbb{O}) \hookrightarrow \text{Möb}(\mathbb{O}).$$

It is not clear whether this homomorphism is surjective or not. In order to stay consistent with the complex and quaternionic cases, we consider the subgroup of the octonionic Möbius group

$$\text{Möb}^*(\mathbb{O}) := \{f_A \mid A \in PSL_2(\mathbb{O})\} \leqslant \text{Möb}(\mathbb{O})$$

such that

$$\text{Möb}^*(\mathbb{O}) \cong PSL_2(\mathbb{O}).$$

Note that the composition is closed within $\text{Möb}^*(\mathbb{O})$:

4.3.2 Types of octonionic Möbius transformations

Definition 4.3.1. *Let* $f \in \text{Möb}^*(\mathbb{O})$ *be an octonionic Möbius transformation. Then*

- f_A *is* parabolic *if it has exactly one fixed point in* $\hat{\mathbb{O}}$*;*
- f *is* elliptic *if it has two fixed points in* $\hat{\mathbb{O}}$ *and is conjugate to a rotation* $\mathbf{z} \mapsto \lambda\mathbf{z}$ *with* $\|\lambda\| = 1$*;*
- f *is* hyperbolic *if it has two fixed points in* $\hat{\mathbb{O}}$ *and is conjugate to a dilation* $\mathbf{z} \mapsto k\mathbf{z}$ *with* $k \in \mathbb{R}_{>0}$ *and* $k \neq 1$*;*
- f *is* loxodromic *if it has exactly two fixed points in* $\hat{\mathbb{H}}$ *and is conjugate to a transformation* $\mathbf{z} \mapsto \lambda\mathbf{z}$ *with* $\|\lambda\| \neq 0, 1$.

Also, a matrix in $PSL_2(\mathbb{O})$ is parabolic, elliptic, hyperbolic, or loxodromic whenever the associated octonionic Möbius transformation is. Therefore, in order to classify octonionic Möbius transformations, it is sufficient to examine the conjugacy classes of group $PSL_2(\mathbb{O})$.

Theorem 4.3.2. *The conjugacy classes of* $PSL_2(\mathbb{O})$ *are given by* *(i) Parabolic classes:*

$$\{\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \mid \|a\| = 1\}$$

with uniqueness up to the similarity of a. *Here* $a, b \in \mathbb{O}$ *are similar if there exists* $h \in \mathbb{O}$ *such that* $a = hbh^{-1}$.

(ii) Elliptic classes:

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid \|a\| = 1\}$$

with uniqueness up to the similarity of a *in* $\mathbb{O}$.

(iii) Loxodromic classes:

$$\{\begin{bmatrix} \lambda a & 0 \\ 0 & \lambda^{-1} d \end{bmatrix} \mid \lambda > 1, \|a\| = \|d\| = 1, \lambda a \nsim \lambda^{-1} d\}$$

with uniqueness up to the similarity classes of λa *and* $\lambda^{-1} d$ *and order of the diagonal entries.*

(iv) Hyperbolic classes:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \mid \lambda > 1\}$$

with uniqueness up to the order of the diagonal entries.

Proof. Given any matrix $A \in PSL_2(\mathbb{O})$, it suffices to consider the Jordan canonical form of A. Following from Equation 3.8, we may simply assume that $A = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}$ or $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, where $\alpha, \beta \in \mathbb{R} \oplus \mathbb{R}\mathbf{q}$ satisfy $\|\alpha\beta\|^2 = 1$ and $\mathbf{q} \in \mathbb{O}$ is a pure imaginary unit.

Write $\alpha = u + v\mathbf{q}$, $\beta = m + n\mathbf{q}$ with $u, v, m, n \in \mathbb{R}$ and let $k = 0$ or 1. Then determining fixed points of matrix $\begin{bmatrix} \alpha & k \\ 0 & \beta \end{bmatrix}$ is identical to solving the equation

$$\frac{\alpha\mathbf{x} + k}{\beta} = (\alpha\mathbf{x} + k)\beta^{-1} = \mathbf{x},$$

or equivalently

$$\alpha\mathbf{x} + k = \mathbf{x}\beta.$$

Write $\mathbf{x} = s + t\mathbf{l}$ with $s, t \in \mathbb{R}$ and $\mathbf{l}$ being the imaginary unit of $\mathbf{x}$. Then the above equation can be rewritten as

$$(u + v\mathbf{q})(s + t\mathbf{l}) + k = (s + t\mathbf{l})(m + n\mathbf{q}),$$

that is,

$$s(u - m) + k + s(v - n)\mathbf{q} + t(u - m)\mathbf{l} + tv\mathbf{q}\mathbf{l} - tn\mathbf{l}\mathbf{q} = 0. \tag{4.5}$$

(i) If A is parabolic, then f_A, the associated Möbius transformation, has a unique fixed point in $\hat{\mathbb{O}}$ that is exactly ∞. This indicates that $k = 1$ and $A = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}$. Additionally, we claim that $\alpha = \beta$. Otherwise, let $\mathbf{l} = \mathbf{q}$, in which case Equation 4.5 becomes

$$s(u - m) - t(v - n) + 1 + [s(v - n) + t(u - m)]\mathbf{l} = 0,$$

which implies

$$\begin{cases} s(u - m) - t(v - n) = -1 \\ s(v - n) + t(u - m) = 0, \end{cases}$$

or

$$\begin{pmatrix} u - m & n - v \\ v - n & u - m \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{4.6}$$

Since $\alpha \neq \beta$ we see that $\det \begin{pmatrix} u - m & n - v \\ v - n & u - m \end{pmatrix} = \|\alpha - \beta\|^2 \neq 0$. Thus Equation 4.6 has a solution

$$\begin{pmatrix} \tilde{s} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} u - m & n - v \\ v - n & u - m \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Obviously, $\tilde{s} + \tilde{t}\mathbf{q}$ is a fixed point of f_A. This contradicts with the assumption that f_A is parabolic. Hence, we have $\alpha = \beta$ and

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \text{ with } \|\alpha\|^2 = 1.$$

(ii) If A is elliptic, then f_A fixes 0 and ∞. Thus, we have $k = 0$ and $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$. In addition, f_A is conjugate to a rotation $\mathbf{z} \mapsto e^{\mathbf{q}\theta}\mathbf{z}$, which indicates that $A =$

$\begin{bmatrix} e^{\mathbf{q}\theta/2} & 0 \\ 0 & e^{-\mathbf{q}\theta/2} \end{bmatrix}$. Therefore, we conclude that

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \text{ with } \|\alpha\| = 1.$$

(iii) If A is hyperbolic, then similar to the elliptic case we may obtain $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda \geqslant 1$.

(iv) As for the loxodromic case, keep in mind that the transformation $\mathbf{z} \mapsto a\mathbf{z}$ is a product of a hyperbolic transformation $\mathbf{z} \mapsto \|a\|\mathbf{z}$ and an elliptic transformation $\mathbf{z} \mapsto a_0\mathbf{z}$. Here $a = \|a\|a_0$ with a_0 being the imaginary unit of a.

□

CHAPTER 5

Modular Groups

A *modular group* is a group of linear fractional transformations whose coefficients are integers in some basic system. In this chapter we will examine the modular groups defined over those integral lattices listed in Diagram 1.5.1, which are discrete subgroups of the projective special linear groups $\mathrm{PSL}_2(\mathbb{K})$.

5.1 Generators of modular groups

5.1.1 Real modular groups

The *classical modular group* $\mathrm{PSL}_2(\mathbb{Z})$ is a Fuchsian group, that is, a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ with respect to the standard topology of $\mathrm{PSL}_2(\mathbb{C})$. It is well known that

$$\mathrm{PSL}_2(\mathbb{Z}) = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle.$$

5.1.2 Complex modular groups

Recall that in Diagram 1.5.1 there are two integral lattices inside $\mathbb{C}$:

- Gaussian integers: $\mathsf{G} = \mathbb{Z} \oplus \mathbb{Z}\mathbf{i}$; and
- Eisensteinian integers $\mathsf{E} = \mathbb{Z} \oplus \mathbb{Z}\omega$, where $\omega = \dfrac{-1+\sqrt{3}\mathbf{i}}{2}$.

Gaussian modular group

$PSL_2(\mathsf{G})$, the modular group defined over G, is commonly called *Picard group* and is the most widely studied Bianchi group. (A *Bianchi group* is a group of the form $PSL_2(\mathcal{O}_d)$ where d is a positive square-free integer and $\mathcal{O}_d = \mathbb{Q}(\sqrt{-d})$.) Additionally, the modular group $PSL_2(\mathsf{G})$ is generated by the following elements [Fin89]:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}.$$

Note that if we apply Equation 8.8 to $\mathbf{x} = \mathbf{i}$, we would get

$$\begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}.$$

This implies that

$$PSL_2(\mathsf{G}) = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix} \rangle. \tag{5.1}$$

The Eisensteinian modular group

It follows from [Swa71] that the group $SL_2(\mathsf{E})$ is generated by the following matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{pmatrix}.$$

As a result, the Eisensteinian modular group $PSL_2(\mathsf{E})$ is generated by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix}.$$

Proposition 5.1.1.

$$PSL_2(\mathsf{E}) = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \rangle. \tag{5.2}$$

Proof. Denote by K the group on the right hand side of 5.2. Applying Equation 8.8 to $\mathbf{x} = \omega$ gives rise to

$$\begin{bmatrix} 0 & -\omega \\ \bar{\omega} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \bar{\omega} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}.$$

Notice that

$$\begin{aligned}\begin{bmatrix}1 & \bar{\omega}\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1-\omega\\ 0 & 1\end{bmatrix} &= \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -\omega\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & \omega\\ 0 & 1\end{bmatrix}^{-1} \in \mathsf{K},\end{aligned}$$

which implies $\begin{bmatrix}0 & -\omega\\ \bar{\omega} & 0\end{bmatrix} \in \mathsf{K}$, and hence

$$\begin{bmatrix}\bar{\omega} & 0\\ 0 & \omega\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -\omega\\ \bar{\omega} & 0\end{bmatrix} \in \mathsf{K}.$$

□

5.1.3 Quaternionic modular groups

As for quaternions, it has three integral lattices in Diagram 1.5.1:

- Lipschitzian integers: $\mathsf{L} = \mathbb{Z} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$.
- Eisensteinian quaternionic integers: $\mathsf{EisH} = \mathbb{Z} \oplus \mathbb{Z}\omega \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\omega\mathbf{j}$.
- Hurwitzian integers: $\mathsf{H} = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$.

Lipschitzian modular group

It has been shown in [Theorem 6.4; [JW99b]] that $\mathrm{PSL}_2(\mathsf{L})$ is generated by

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & \mathbf{i}\end{bmatrix}, \text{ and } \begin{bmatrix}1 & 0\\ 0 & \mathbf{j}\end{bmatrix}.$$

Alternatively, it could also be generated by parabolic matrices together with $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$, as illustrated in the following proposition.

Proposition 5.1.2.

$$\mathrm{PSL}_2(\mathsf{L}) = \langle \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & \mathbf{i}\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & \mathbf{j}\\ 0 & 1\end{bmatrix} \rangle.$$

Proof. Write $\mathsf{K} = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix} \rangle$. It is clear that

$$\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix} \in \mathrm{PSL}_2(\mathsf{G}) \subset \mathsf{K}.$$

Analogously, we have $\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{j} \end{bmatrix} \in \mathsf{K}$. Thus, $\mathrm{PSL}_2(\mathsf{L}) \leqslant \mathsf{K}$.

On the other hand, we have

$$\begin{aligned} \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix}^{-1} \in \mathrm{PSL}_2(\mathsf{L}). \end{aligned}$$

Similarly,

$$\begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{j} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{j} \end{bmatrix}^{-1} \in \mathrm{PSL}_2(\mathsf{L}).$$

Then we get $\mathsf{K} \leqslant \mathrm{PSL}_2(\mathsf{L})$, and hence, $\mathrm{PSL}_2(\mathsf{L}) = \mathsf{K}$. □

Hurwitzian modular group

Speaking of the modular group over Hurwitzian integers, we have

Proposition 5.1.3 (Theorem 8.2; [JW99b])**.** $\mathrm{PSL}_2(\mathsf{H})$ *is generated by the following matrices*

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{u} \end{bmatrix}, \textit{ and } \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{v} \end{bmatrix},$$

where

$$\mathbf{u} = \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} + \mathbf{k}), \quad \mathbf{v} = \frac{1}{2}(1 + \mathbf{i} - \mathbf{j} - \mathbf{k}).$$

This group has yet another set of generators:

Proposition 5.1.4.

$$\mathrm{PSL}_2(\mathsf{H}) = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{h} \end{bmatrix} \rangle. \tag{5.3}$$

Proof. Denote by K the group on the right side. According to the definition of $SL_2(\mathbb{H})$ in Lemma 3.2.4, it is easy to check that $\begin{bmatrix}1 & \mathbf{i}\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & \mathbf{j}\\ 0 & 1\end{bmatrix}$, and $\begin{bmatrix}1 & 0\\ 0 & \mathbf{h}\end{bmatrix} \in PSL_2(H)$, and hence $K \leqslant PSL_2(H)$.

Conversely, we first observe that

$$\begin{aligned}\begin{bmatrix}1 & \mathbf{h}\\ 0 & 1\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 0 & \bar{\mathbf{h}}\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \mathbf{h}\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ 0 & \mathbf{h}\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \mathbf{h}\end{bmatrix} \in K.\end{aligned}$$

Notice that $\mathbf{u} = \mathbf{h} - \mathbf{i} - \mathbf{j}$, which implies

$$\begin{bmatrix}1 & \mathbf{u}\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & \mathbf{h}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & \mathbf{i}\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & \mathbf{j}\\ 0 & 1\end{bmatrix}^{-1} \in K$$

and then

$$\begin{bmatrix}1 & \bar{\mathbf{u}}\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1-\mathbf{u}\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & \mathbf{u}\\ 0 & 1\end{bmatrix}^{-1} \in K.$$

This, together with Equation 8.8, yields

$$\begin{bmatrix}\mathbf{u} & 0\\ 0 & \bar{\mathbf{u}}\end{bmatrix} = \begin{bmatrix}1 & \mathbf{u}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & \bar{\mathbf{u}}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & \mathbf{u}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \in K.$$

Meanwhile, it is clear that

$$\begin{aligned}\begin{bmatrix}\mathbf{i} & 0\\ 0 & 1\end{bmatrix} &= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \mathbf{i}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \in K, \text{ and}\\ \begin{bmatrix}\bar{\mathbf{h}} & 0\\ 0 & \bar{\mathbf{h}}\end{bmatrix} &= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \mathbf{h}\end{bmatrix}^{-1}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \mathbf{h}\end{bmatrix}^{-1} \in K.\end{aligned}$$

Therefore, we obtain

$$\begin{bmatrix}1 & 0\\ 0 & \mathbf{v}\end{bmatrix} = \begin{bmatrix}\mathbf{i} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\bar{\mathbf{h}} & 0\\ 0 & \bar{\mathbf{h}}\end{bmatrix}\begin{bmatrix}\mathbf{u} & 0\\ 0 & \bar{\mathbf{u}}\end{bmatrix} \in K.$$

This last identity holds because

$$\begin{bmatrix}\mathbf{i} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\bar{\mathbf{h}} & 0\\ 0 & \bar{\mathbf{h}}\end{bmatrix}\begin{bmatrix}\mathbf{u} & 0\\ 0 & \bar{\mathbf{u}}\end{bmatrix} \overset{\mathbf{h}=\bar{\mathbf{u}}\bar{\mathbf{v}}}{=\!=} \begin{bmatrix}\mathbf{i} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{vu} & 0\\ 0 & \mathbf{vu}\end{bmatrix}\begin{bmatrix}\mathbf{u} & 0\\ 0 & \bar{\mathbf{u}}\end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \mathbf{i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}\mathbf{u}^2 & 0 \\ 0 & \mathbf{v} \end{bmatrix} \\
&\underline{\underline{\mathbf{u}^2=-\bar{\mathbf{u}}}} \begin{bmatrix} \mathbf{i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\mathbf{v}\bar{\mathbf{u}} & 0 \\ 0 & \mathbf{v} \end{bmatrix} \\
&\underline{\underline{\mathbf{v}\bar{\mathbf{u}}=\mathbf{i}}} \begin{bmatrix} \mathbf{i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{v} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{v} \end{bmatrix}
\end{aligned}
$$

Similarly, we can prove that $\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{u} \end{bmatrix} \in \mathsf{K}$. Therefore, we have $\mathrm{PSL}_2(\mathsf{H}) \leqslant \mathsf{K}$, which completes the proof. □

Proposition 5.1.5 (Theorem 9.2;[JW99b]). *$\mathrm{PSL}_2(\mathsf{H})$ contains $\mathrm{PSL}_2(\mathsf{L})$ as a subgroup of index 30.*

Next we consider the subgroup of $\mathrm{PSL}_2(\mathsf{H})$ generated by all the parabolic matrices $\begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix}$ with $\mathbf{x} \in \mathsf{H}$, plus $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. It is called the *Hurwitzian modular group* and denoted $\mathrm{PSL}_2^*(\mathsf{H})$. Recall that we have $\mathsf{H} = \mathbb{Z}\mathbf{h} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$, it is then obvious that the group $\mathrm{PSL}_2^*(\mathsf{H})$ is generated by

$$
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{k} \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix}.
$$

Note that

$$
\begin{bmatrix} 1 & \mathbf{k} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{j} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{j} \end{bmatrix},
$$

which implies

$$
\mathrm{PSL}_2^*(\mathsf{H}) = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix} \rangle. \tag{5.4}
$$

Meanwhile, let $\mathsf{C} := \mathsf{H}[\mathsf{H},\mathsf{H}]\mathsf{H}$ be the two-sided ideal in H generated by the commutators $[a, b] = ab - ba$ for all a, $b \in \mathsf{H}$. Then H/C consists of the following elements

$$
0,\ 1,\ -\mathbf{h},\ \text{and}\ \mathbf{h} - 1,
$$

which form the finite field $\mathbb{F}_4$ whose nonzero elements form the cyclic group of order 3 [FKN09]. Define

$$\mathrm{PSL}_2^{(0)}(\mathsf{H}) := \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathsf{H}) \mid ad - bc \equiv 1 \,(\mathrm{mod}\, \mathsf{C})\}. \tag{5.5}$$

It follows from [FKN09] that $\mathrm{PSL}_2(\mathsf{H})/\mathrm{PSL}_2^{(0)}(\mathsf{H}) \simeq \mathbb{Z}_3$ and $\mathrm{PSL}_2^{(0)}(\mathsf{H})$ is generated by the following matrices

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where a, b are both Hurwitzian units satisfying $ab \equiv 1 \,(\mathrm{mod}\, \mathsf{C})$.

Proposition 5.1.6.

$$\mathrm{PSL}_2^{(0)}(\mathsf{H}) = \mathrm{PSL}_2^*(\mathsf{H}).$$

Proof. From the definitions in Equation 5.4 and Equation 5.5 it is obvious that

$$\mathrm{PSL}_2^*(\mathsf{H}) \leqslant \mathrm{PSL}_2^{(0)}(\mathsf{H}).$$

Meanwhile, we have

$$\mathrm{PSL}_2^{(0)}(\mathsf{H}) \cong W^+(D_4^{++}) \cong \mathrm{PSL}_2^*(\mathsf{H}).$$

The first isomorphism can be found in [FKN09] while the second one comes from Proposition 10.4.1. We thus obtain

$$\mathrm{PSL}_2^{(0)}(\mathsf{H}) = \mathrm{PSL}_2^*(\mathsf{H}).$$

□

Eisensteinian quaternionic modular group

$\mathrm{PSL}_2(\mathsf{EisH})$ is isomorphic to a subgroup of a hypercompact Coxeter group operating in $\mathcal{H}^5$:

$$\mathrm{PSL}_2(\mathsf{EisH}) \simeq 4[1^+, 6, (3,3,3,3)^+, 6, 1^+].$$

The notation on the right hand side will be explained at the end of this chapter.

5.1.4 Octonionic modular groups

Recall that in the context of this thesis we fix the subring of octaves to be $\mathcal{O} = \sum_{i=1}^{8} \mathbb{Z}\epsilon_i$ with

$$\begin{aligned} \epsilon_1 &= \frac{1}{2}(1 - e_1 - e_5 - e_6), & \epsilon_2 &= e_1, \\ \epsilon_3 &= \frac{1}{2}(-e_1 - e_2 + e_6 + e_7), & \epsilon_4 &= e_2, \\ \epsilon_5 &= \frac{1}{2}(-e_2 - e_3 - e_4 - e_7), & \epsilon_6 &= e_3, \\ \epsilon_7 &= \frac{1}{2}(-e_3 + e_5 - e_6 + e_7), & \epsilon_8 &= e_4. \end{aligned}$$

Due to non-associativity of $\mathbb{O}$, it is too difficult to describe the modular group $\mathrm{PSL}_2(\mathcal{O})$. But, in accordance with the real, complex and quaternionic cases, we define the *Octavian modular group* as the group

$$\mathrm{PSL}_2^*(\mathcal{O}) := \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon_i \\ 0 & 1 \end{bmatrix} \mid 1 \leqslant i \leqslant 8 \rangle.$$

Recall that Equation 1.4 says that

$$2\epsilon_i + 3\epsilon_2 + 4\epsilon_3 + 5\epsilon_4 + 6\epsilon_5 + 4\epsilon_6 + 2\epsilon_7 + 3\epsilon_8 = 1,$$

which implies that

$$\begin{bmatrix} 1 & \epsilon_1 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & \epsilon_2 \\ 0 & 1 \end{bmatrix}^3 \begin{bmatrix} 1 & \epsilon_3 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 1 & \epsilon_4 \\ 0 & 1 \end{bmatrix}^5 \begin{bmatrix} 1 & \epsilon_5 \\ 0 & 1 \end{bmatrix}^6 \begin{bmatrix} 1 & \epsilon_6 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 1 & \epsilon_7 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & \epsilon_8 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we obtain

$$\mathrm{PSL}_2^*(\mathcal{O}) = \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon_i \\ 0 & 1 \end{bmatrix} \mid 1 \leqslant i \leqslant 8 \rangle. \tag{5.6}$$

5.2 Construction via quadratic forms

An *algebraic group* is a group that is a variety as well such that the group operations are morphisms between varieties. Recall that an affine variety $\mathscr{V}$ is a subspace of $\mathbb{R}^n$ defined by a set of polynomial equations:

$$\mathscr{V} = \{x \in \mathbb{R}^n \mid P_i(x) = 0,\ i \in I\}.$$

If all the polynomials P_i, $i \in I$ are rational polynomials, then $\mathscr{V}$ is said to be defined over $\mathbb{Q}$. An algebraic group is *defined over* $\mathbb{Q}$ if the variety is defined over $\mathbb{Q}$ and the group operations are morphisms defined over $\mathbb{Q}$.

Assume that $H \subseteq GL_n(\mathbb{R})$ is a linear semisimple Lie group. According to [Theorem 8.23; [Kna02]], there exists a left-invariant Haar measure μ on H, which is induced from a left-invariant n-form and is unique up to strictly positive scalar multiples. Let Γ be a discrete subgroup of H. Then from [Proposition 4.1.3;[Mor15]] we know that the Haar measure μ induces a unique (up to a scalar multiple) σ-finite, H-invariant Borel measure ν on H/Γ. Especially, if $\nu(H/\Gamma) < \infty$, we say H/Γ has finite volume; in this case, the subgroup Γ is called a *lattice* in H.

Let G be an algebraic group. Then we may embed G into the linear Lie group $GL_n(\mathbb{R})$. Define

$$G(\mathbb{Z}) = G \cap GL_n(\mathbb{Z}).$$

Lemma 5.2.1 (Theorem 5.1.11; [Mor15])**.** *If the algebraic group* G *is defined over* $\mathbb{Q}$*, then* $G(\mathbb{Z})$ *is a lattice in* G.

A typical example is that $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$.

Definition 5.2.2. *Let* G *be an algebraic group. A subgroup* Γ *of* G *called an* arithmetic subgroup *if it is commensurable with* $G(\mathbb{Z})$*, i.e., the intersection* $\Gamma \cap G(\mathbb{Z})$ *has finite index in both* Γ *and* $G(\mathbb{Z})$*. In particular,* $G(\mathbb{Z})$ *and its subgroups of finite index are arithmetic groups.*

We have shown in Equation 3.9 that $PSL_2(\mathbb{K}) \simeq SO_0(r+1,1)$ with $r = \dim_{\mathbb{R}} \mathbb{K}$. Thus, we may embed modular groups as discrete subgroups of the Lorentz groups, which can be constructed by taking the integral points in $SO_0(r+1,1)$ associated to the quadratic form defined over $\mathbb{K}$. From [Section 6.4; [Mor15]] we know that those subgroups obtained in this way are arithmetic subgroups of $SO_0(r+1,1)$. It thus follows that

Proposition 5.2.3. *The previously defined modular groups are arithmetic subgroups of the Möbius groups.*

5.3 Realization of modular groups via reflection groups

A group of symmetries is called a *reflection group* if it can be generated by finitely many reflections in some Euclidean space. A *Coxeter group* P is is a reflection group generated by reflections ρ_0, $\rho_1, \cdots, \rho_n$ in the facets of a polytope $\mathcal{P}$, each of those dihedral angles is a submultiple of π. If the angle between the i^{th} and j^{th} facets is π/p_{ij}, the product of reflections ρ_i and ρ_j is a rotation of period p_{ij}. The Coxeter group P is thus defined by the relations

$$(\rho_i \rho_j)^{p_{ij}} = 1, \quad i, j = 0, 1, \dots, n.$$

In particular, we have $p_{ii} = 1$ for all i.

When $\mathcal{P}$ is a *orthoscheme*, that is, a simplex whose facets may be ordered so that any two that are not consecutive are orthogonal, we have

$$p_{ij} = 2 \text{ for } j - i > 1.$$

If we abbreviate $p_{j-1,j}$ as p_j, then the group P can be characterized by those integers p_js. Thus, we may denote the Coxeter group P by the *Coxeter symbol*

$$[p_1, \dots, p_n].$$

When $\mathcal{P}$ is not an orthoscheme, the group P may likewise be given a Coxeter symbol. For instance, the group whose fundamental domain is the closure of a triangle (p, q, r) with acute angles π/p, π/q, π/r is denoted by the symbol $[(p, q, r)]$ or, if $p = q = r$, simply by $[p^{[3]}]$.

A Coxeter group is *compact* if each subgroup generated by all but one of the reflections is spherical (see Chapter 10 for "spherical subgroups"). If each such subgroup is either spherical or Euclidean, including at least one of the latter, it is *paracompact*. If at least one such subgroup is hyperbolic, it is *hypercompact*. A Coxeter group is *crystallographic* if it leaves invariant some $(n+1)$-dimensional lattice.

Denote by P^+ the *direct subgroup* of P, that is, the index 2 subgroup of P whose elements are products of an even number of reflections. Let the generators $\rho_0, \rho_1, \dots, \rho_n$ of P (relabeled if necessary) be partitioned into sets of $k+1$ and $n-k$, where $0 \leqslant k \leqslant n$, so that for each pair of ρ_j and ρ_l with $0 \leqslant j \leqslant k < l \leqslant n$ the period p_{jl} is even (or infinite). Let Q be the distinguished subgroup of P generated by reflections ρ_0 through ρ_k. Then Q has a direct subgroup Q^+ generated, if $k \geqslant 1$, by even transformations $\tau_{ij} = \rho_i\rho_j$ $(0 \leqslant i < j \leqslant k)$. Then the group P has an index 2 subgroup generated by the even transformations τ_{ij} $(0 \leqslant i < j \leqslant k)$, the reflections ρ_l $(k < l \leqslant n)$, and the conjugate reflections $\rho_{jlj} = \rho_j\rho_l\rho_j$ $(0 \leqslant j \leqslant k < l \leqslant n)$. This is a halving subgroup if $k = 0$, a semidirect subgroup if $0 < k < n$, or the direct subgroup P^+ if $k = n$. Such a subgroup is denoted by affixing a superscript plus sign to the Coxeter symbol for P so that the resulting symbol contains the symbol for Q^+, minus the enclosing brackets.

For example, the group $[p, q, r]$, generated by reflections ρ_0, ρ_1, ρ_2, ρ_3, satisfying the relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_2\rho_3)^r = 1,$$

has a direct subgroup $[p, q, r]^+$, generated by the rotations $\sigma_1 = \tau_{01}$, $\sigma_2 = \tau_{12}$, and $\sigma_3 = \tau_{23}$, with the defining relations

$$\sigma_1^p = \sigma_2^q = \sigma_3^r = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1.$$

If r is even, the semidirect subgroup $[(p, q)^+, r]$, generated by the rotations σ_1 and σ_2 and the reflection ρ_3, is defined by the relations

$$\sigma_1^p = \sigma_2^q = \rho_3^2 = (\sigma_1\sigma_2)^2 = (\sigma_2^{-1}\rho_3\sigma_2\rho_3)^{r/2} = 1.$$

Applying the Coxeter symbols to those modular groups defined earlier shows that

Theorem 5.3.1 ([JW99a]; [JW99b]).

$$\mathrm{PSL}_2(\mathbb{Z}) \cong [3, \infty]^+,$$

$$\mathrm{PSL}_2(\mathsf{G}) \cong [3, 4, 1^+, 4]^+, \quad \mathrm{PSL}_2(\mathsf{E}) \cong [(3, 3)^+, 6, 1^+],$$

$$\mathrm{PSL}_2(\mathsf{L}) \cong [3, 4, (3, 3)^+, 4]^+, \quad \mathrm{PSL}_2(\mathsf{H}) \cong [(3, 3, 3)^+, 4, 3^+],$$

$$\mathrm{PSL}_2(\mathsf{EisH}) \cong 4[1^+, 6, (3, 3, 3, 3)^+, 6, 1^+].$$

CHAPTER 6

Actions of Modular Groups

6.1 Minkowski spaces and hyperbolic n-spaces

6.1.1 Minkowski spaces

Consider the indefinite nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+1}$ given by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1},$$

where $\mathbf{x} = (x_1, \cdots, x_{n+1})$. We call $\langle \cdot, \cdot \rangle$ a Minkowski bilinear form, and the pair $\mathbb{R}^{n,1} \triangleq (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ is called the $(n+1)$-dimensional *Minkowski space*. We will occasionally use the physical terminology and say that a vector $\mathbf{x}$ is *lightlike* if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, is *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, and is *spacelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$. In addition, the subset of lightlike vectors is the *light-cone*

$$\mathscr{L}^n := \{\mathbf{x} \in \mathbb{R}^{n,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\}.$$

The following subspace is a two-sheeted hyperboloid and is geometrically significant:

$$\mathscr{H}^n := \{\mathbf{x} \in \mathbb{R}^{n,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\},$$

Consider its upper sheet

$$\mathscr{H}^n_+ := \{\mathbf{x} \in \mathbb{R}^{n,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_{n+1} > 0\}. \tag{6.1}$$

The hypersurface $\mathscr{H}^n_+$ is smooth, connected, and simply-connected. It is actually the hyperboloid model of the hyperbolice n-space.

Recall that $\mathfrak{h}_2(\mathbb{K})$ is $(r+2)$-dimensional and the quadratic form $\mathtt{M}$ has signature $(r+1,1)$. Thus, we can identify $(\mathfrak{h}_2(\mathbb{K}),\mathtt{M})$ with the Minkowski space $\mathbb{R}^{r+1,1}$. In this case, we have

$$\begin{aligned}\mathscr{H}^{r+1} &= \{A \in \mathfrak{h}_2(\mathbb{K}) \mid \mathtt{M}(A) = -1\},\\ \mathscr{H}^{r+1}_{+} &= \{A \in \mathfrak{h}_2(\mathbb{K}) \mid \mathtt{M}(A) = -1 \text{ and } s > 0, t > 0\}.\end{aligned}$$

6.1.2 Hyperbolic spaces

The *hyperbolic* n*-space*, denoted $\mathcal{H}^n$, is the maximally symmetric, simply connected, n-dimensional Riemannian manifold with a constant negative sectional curvature. It can be constructed using different models.

If we use the hyperboloid model, then the hyperbolic n-space is exactly $\mathscr{H}^n_+$ as defined in Equation 6.1. Especially, from [Esc97] we know that $\mathscr{H}^n_+$ is homeomorphic to the symmetric space $SO(n+1,1)/SO(n+1)$. Another model for describing hyperbolic spaces is the half-plane model, which will be explained below.

For more about hyperbolic geometry and more details on different analytic models of hyperbolic spaces, see [CFK⁺97].

6.1.3 Hyperbolic orbifolds

A *hyperbolic manifold* is an n-dimensional manifold equipped with a complete Riemannian metric of constant sectional curvature -1. Denote by $\mathtt{Isom}(\mathcal{H}^n)$ the group of hyperbolic isometries of $\mathcal{H}^n$. Then any hyperbolic manifold $\mathcal{M}$ can be obtained as the quotient of $\mathcal{H}^n$ by a torsion-free discrete subgroup Γ of $\mathtt{Isom}(\mathcal{H}^n)$:

$$\mathcal{M} = \mathcal{H}^n/\Gamma.$$

If we allow more generally the discrete group to have elements of finite order, then the resulting quotient space $\mathcal{O} = \mathcal{H}^n/\Gamma$ is called a *hyperbolic* n*-orbifold*. A hyperbolic n-orbifold $\mathcal{H}^n/\Gamma$ is said to be *cusped* if Γ contains at least one parabolic element. Recall that an isometry is called parabolic if it has exactly one fixed point.

We can descend the volume form from $\mathcal{H}^n$ to $\mathcal{O}$ and integrate it over the quotient space. This defines the hyperbolic volume of $\mathcal{O}$. It follows from [KM68] that the spectrum of the volumes of cusped complete hyperbolic n-orbifolds has a minimal element. In Chapter 10 we will find out the discrete subgroup Γ_0 of $\mathtt{Isom}(\mathcal{H}^{r+1})$ such that the corresponding hyperbolic $(r+1)$-orbifold $\mathcal{H}^{r+1}/\Gamma_0$ has the minimal volume.

6.2 Generalized upper half planes

The *generalized upper half plane* associated with $\mathbb{K}$ is defined as

$$\mathcal{H}(\mathbb{K}) := \{\mathbf{u} + t\mathbf{e} \mid \mathbf{u} \in \mathbb{K} \text{ and } t \in \mathbb{R}_{>0}\}.$$

Here $\mathbf{e}$ is a new imaginary unit that is not contained in $\mathbb{K}$. Clearly, $\mathcal{H}(\mathbb{K})$ is contained in a hyperplane in the Cayley-Dickson double $\mathbb{K} \oplus \mathbf{e}\mathbb{K}$ and has real dimension $r+1$. By applying the conjugation map inherited from the Cayley-Dickson construction we get $\overline{\mathbf{u} + t\mathbf{e}} = \bar{\mathbf{u}} - t\mathbf{e}$, which indicates that $\overline{\mathbf{u} + t\mathbf{e}}$ parametrizes the corresponding "lower half plane" $\overline{\mathcal{H}}$.

The line element (or arclength element) in $\mathcal{H}(\mathbb{K})$ is $ds^2 = \dfrac{|d\mathbf{u}|^2 + dt^2}{t^2}$. The geodesics are given by straight lines parallel to the "imaginary" $(= t)$ axis, or by half circles whose starting point and ending point both lie on the boundary $t = 0$ of $\mathcal{H}(\mathbb{K})$ [KNP+12].

The generalized upper half plane $\mathcal{H}(\mathbb{K})$ is isometric to $\mathscr{H}_+^{r+1}$, the hyperboloid model of hyperbolic $(r+1)$-space. Explicitly, define

$$\begin{aligned} \mathscr{S} : \mathcal{H}(\mathbb{K}) &\to \mathscr{H}_+^{r+1} \\ \mathbf{u} + t\mathbf{e} &\mapsto \begin{pmatrix} t + t^{-1}\|\mathbf{u}\|^2 & t^{-1}\mathbf{u} \\ t^{-1}\bar{\mathbf{u}} & t^{-1} \end{pmatrix}. \end{aligned}$$

It is inverse; the inverse map is given by

$$\begin{aligned} \mathscr{S}^{-1} : \mathscr{H}_+^{r+1} &\to \mathcal{H}(\mathbb{K}) \\ \begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix} &\mapsto t^{-1}\mathbf{x} + t^{-1}\mathbf{e}. \end{aligned}$$

It is easy to show that $\mathscr{S}$ is an isometry.

6.3 Action of modular groups on $\mathcal{H}(\mathbb{K})$

Given any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{K})$, let $f_A \in \mathrm{M\ddot{o}b}(\mathbb{K})$ be the associated Möbius transformation, which can be obtained from the isomorphism $\mathrm{PSL}_2(\mathbb{K}) \simeq$

Möb($\mathbb{K}$). Recall that f_A is essentially an inverse map from the extended plane $\hat{\mathbb{K}}$ to itself. We can extend f_A to be an isometry of $\mathcal{H}(\mathbb{K})$. Explicitly, define $\tilde{f}_A : \mathcal{H}(\mathbb{K}) \to \mathcal{H}(\mathbb{K})$ by requiring

$$\mathbf{z} = \mathbf{u} + t\mathbf{e} \mapsto \frac{a\mathbf{z}+b}{c\mathbf{z}+d} = \frac{a\mathbf{u}+b+at\mathbf{e}}{c\mathbf{u}+d+ct\mathbf{e}},$$

where $\frac{a\mathbf{u}+b+at\mathbf{e}}{c\mathbf{u}+d+ct\mathbf{e}}$ is understood as $(a\mathbf{u}+b+at\mathbf{e})(c\mathbf{u}+d+ct\mathbf{e})^{-1}$ and

$$(c\mathbf{u}+d+ct\mathbf{e})^{-1} = \frac{\overline{c\mathbf{u}+d+ct\mathbf{e}}}{\|c\mathbf{u}+d+ct\mathbf{e}\|^2}.$$

It is tedious but straightforward to show that $\tilde{f}_A \in \mathtt{Isom}(\mathcal{H}(\mathbb{K}))$ and is orientation-preserving. Therefore, we obtain an action of $\mathrm{PSL}_2(\mathbb{K})$ on $\mathcal{H}(\mathbb{K})$:

$$A(\mathbf{z}) := \tilde{f}_A(\mathbf{z}). \tag{6.2}$$

Next we will examine the actions of those modular groups on the generalized upper half plane.

6.3.1 Action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{H}(\mathbb{R})$

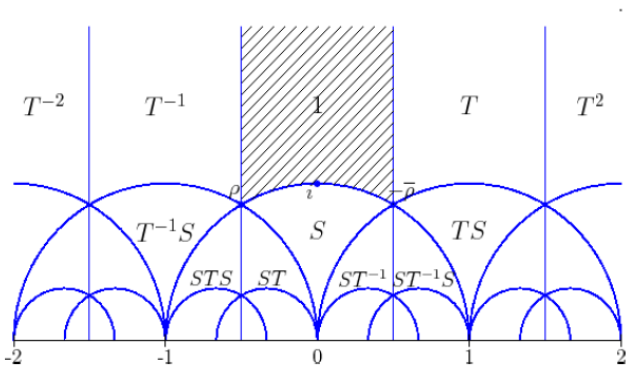
The classical modular group $\mathrm{PSL}_2(\mathbb{Z})$ is generated by $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It acts faithfully on the upper half-plane $\mathcal{H}(\mathbb{R})$ by linear transformations, as given in Equation 6.2. Specifically, for any $\mathbf{z} \in \mathcal{H}(\mathbb{R})$, we have

$$S(\mathbf{z}) = -\frac{1}{\mathbf{z}}, \ T(\mathbf{z}) = \mathbf{z}+1.$$

It is well-known that the typical fundamental domain for this action is the hyperbolic triangle

$$\begin{aligned} ⌑ &= \{\mathbf{z} \in \mathcal{H}(\mathbb{R}) \ : \ |\mathbf{z}| \geqslant 1 \text{ and } |\mathrm{Re}(\mathbf{z})| \leqslant \frac{1}{2}\} \\ &= \{u + t\mathbf{e} \mid u^2 + t^2 \geqslant 1, |u| \leqslant \frac{1}{2}, t > 0\}, \end{aligned}$$

which is the shaded area in the picture below.



6.3.2 Actions of complex modular groups on $\mathcal{H}(\mathbb{C})$

Action of $\mathrm{PSL}_2(G)$ **on** $\mathcal{H}(\mathbb{C})$

It follows from Equation 5.1 that the Gaussian modular group

$$\mathrm{PSL}_2(G) = \langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, U_{\mathbf{i}} = \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix} \rangle.$$

Concerning the action of $\mathrm{PSL}_2(G)$ on $\mathcal{H}(\mathbb{C})$, it is clear that

$$S(\mathbf{z}) = -\frac{1}{\mathbf{z}}, \ T(\mathbf{z}) = \mathbf{z} + 1, \text{ and } U_{\mathbf{i}}(\mathbf{z}) = \mathbf{z} + \mathbf{i}.$$

Let $\mathcal{B}_G$ denote the set of elements $\mathbf{u} + t\mathbf{e} \in \mathcal{H}(\mathbb{C})$ such that

$$\|\alpha\mathbf{u} + \beta\|^2 + t^2\|\alpha\|^2 \geqslant 1$$

for all $\alpha, \beta \in G$ satisfying $\alpha G + \beta G = G$. Then it can be derived from [Swa71] that the set

$$\mathscr{D}_G := \{\mathbf{u} + t\mathbf{e} \in \mathcal{B}_G \ : \ |\mathrm{Re}(\mathbf{u})| \leqslant \frac{1}{2}, 0 \leqslant \mathrm{Im}(\mathbf{u}) \leqslant \frac{1}{2}, t > 0\}$$

is a fundamental domain for the action of $\mathrm{PSL}_2(G)$ on $\mathcal{H}(\mathbb{C})$.

Note that $\mathscr{D}_G$ can be rewritten as

$$\{\mathbf{u} + t\mathbf{e} \in \mathcal{H}(\mathbb{C}) \ : \ \mathbf{u}\bar{\mathbf{u}} + t^2 \geqslant 1, |\mathrm{Re}(\mathbf{u})| \leqslant \frac{1}{2}, 0 \leqslant \mathrm{Im}(\mathbf{u}) \leqslant \frac{1}{2}, t > 0\}. \tag{6.3}$$

Action of $\mathrm{PSL}_2(E)$ **on** $\mathcal{H}(\mathbb{C})$

According to Equation 5.2, we have

$$\mathrm{PSL}_2(E) = \langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, U_{\omega} = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \rangle.$$

The generators act on $\mathcal{H}(\mathbb{C})$ via

$$S(\mathbf{z}) = -\frac{1}{\mathbf{z}},\ T(\mathbf{z}) = \mathbf{z}+1,\ \text{and}\ U_\omega(\mathbf{z}) = \mathbf{z}+\omega.$$

Let $\mathcal{B}_E$ denote the subset consisting of elements $\mathbf{u}+t\mathbf{e}$ such that

$$\|\alpha\mathbf{u}+\beta\|^2 + t^2\|\alpha\|^2 \geqslant 1$$

for all $\alpha, \beta \in E$ satisfying $\alpha E + \beta E = E$.

Proposition 6.3.1 ([EGM13]). *The set*

$$\mathscr{D}_E := \{\mathbf{u}+t\mathbf{e} \in \mathcal{B}_E \ :\ t>0,\quad 0 \leqslant \mathrm{Re}(\mathbf{u}) \leqslant \frac{1}{2}, \\ -\frac{\sqrt{3}}{3}\mathrm{Re}(\mathbf{u}) \leqslant \mathrm{Im}(\mathbf{u}) \leqslant \frac{\sqrt{3}}{3}(1-\mathrm{Re}(\mathbf{u}))\}$$

is a fundamental domain for the action of $\mathrm{PSL}_2(E)$ *on* $\mathcal{H}(\mathbb{C})$.

6.3.3 Actions of quaternionic modular groups on $\mathcal{H}(\mathbb{H})$

Action of $\mathrm{PSL}_2(L)$ **on** $\mathcal{H}(\mathbb{H})$

As shown in Proposition 5.1.2, the Lipschitzian modular group $\mathrm{PSL}_2(L)$ is generated by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ U_{\mathbf{i}} = \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix},\ \text{and}\ U_{\mathbf{j}} = \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix}.$$

Speaking of the action of $\mathrm{PSL}_2(L)$ on the upper half-plane $\mathcal{H}(\mathbb{H})$, it is obvious that

$$S(\mathbf{z}) = -\frac{1}{\mathbf{z}},\ T(\mathbf{z}) = \mathbf{z}+1,\ U_{\mathbf{i}}(\mathbf{z}) = \mathbf{z}+\mathbf{i},\ \text{and}\ U_{\mathbf{j}}(\mathbf{z}) = \mathbf{z}+\mathbf{j}.$$

Analogous to Equation 6.3, we have

Proposition 6.3.2. *For an vector* $\mathbf{u} \in \mathbb{H}$ *we write* $\mathbf{u} = u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$. *The set*

$$\mathcal{D}_L := \{\mathbf{u}+t\mathbf{e} \in \mathcal{H}(\mathbb{H}) \ :\ \mathbf{u}\bar{\mathbf{u}} + t^2 \geqslant 1,\ |u_0| \leqslant \frac{1}{2},\ 0 \leqslant u_, u_2, u_3 \leqslant \frac{1}{2},\ t>0\}$$

is a fundamental domain for the action of $\mathrm{PSL}_2(L)$ *on* $\mathcal{H}(\mathbb{H})$.

Action of $\mathrm{PSL}_2^*(\mathsf{H})$ on $\mathcal{H}(\mathbb{H})$

We have proved in Equation 5.4 that the Hurwitzian modular group $\mathrm{PSL}_2^*(\mathsf{H})$ is generated by the following matrices:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, U_{\mathbf{i}} = \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix}, U_{\mathbf{j}} = \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix}, \text{ and } U_{\mathbf{h}} = \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix}.$$

In particular, we have $U_{\mathbf{h}}(\mathbf{z}) = \mathbf{z} + \mathbf{h}$ for all $\mathbf{z} \in \mathcal{H}(\mathbb{H})$.

On the other hand, notice that 1, **i**, **j**, and **h** form a basis of the space $\mathbb{H}$. Then each element $\mathbf{u} \in \mathbb{H}$ could be written in the form $\mathbf{u} = \tilde{u}_0 + \tilde{u}_1\mathbf{i} + \tilde{u}_2\mathbf{j} + \tilde{u}_3\mathbf{h}$.

Proposition 6.3.3. *The set*

$$\mathcal{D}_{\mathsf{H}}^* := \{\mathbf{u} + t\mathbf{e} \in \mathcal{H}(\mathbb{H}) \,:\, \mathbf{u}\bar{\mathbf{u}} + t^2 \geqslant 1, |\tilde{u}_0| \leqslant \frac{1}{2}, 0 \leqslant \tilde{u}_1, \tilde{u}_2, \tilde{u}_3 \leqslant \frac{1}{2}, t > 0\}$$

is a fundamental domain for the action of $\mathrm{PSL}_2^*(\mathsf{H})$ *on* $\mathcal{H}(\mathbb{H})$.

6.3.4 Action of octonionic modular groups on $\mathcal{H}(\mathbb{O})$

As illustrated in Equation 5.6, the Octavian modular group $\mathrm{PSL}_2^*(\mathsf{O})$ is generated by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } U_{\epsilon_i} = \begin{bmatrix} 1 & \epsilon_i \\ 0 & 1 \end{bmatrix} \text{ for } 1 \leqslant i \leqslant 8.$$

Regarding the action of $\mathrm{PSL}_2^*(\mathsf{O})$ on the upper half-plane $\mathcal{H}(\mathbb{O})$, it follows from Equation 6.2 that $U_{\epsilon_i}(\mathbf{z}) = \mathbf{z} + \epsilon_i$ for all elements $\mathbf{z} \in \mathcal{H}(\mathbb{O})$.

It is clear that $\{\epsilon_i\}_{i=1}^8$ form a basis of $\mathbb{O}$. Thus, every element $\mathbf{u} \in \mathbb{O}$ can be written in the form $\mathbf{u} = \sum_{i=1}^8 u_i\epsilon_i$ with coefficients $u_i \in \mathbb{R}$.

Proposition 6.3.4. *The set*

$$\mathcal{D}_{\mathsf{O}}^* := \{\mathbf{u} + t\mathbf{e} \in \mathcal{H}(\mathbb{O}) \,:\, \mathbf{u}\bar{\mathbf{u}} + t^2 \geqslant 1, |u_0| \leqslant \frac{1}{2}, 0 \leqslant u_1, \dots, u_7 \leqslant \frac{1}{2}, t > 0\}$$

is a fundamental domain for the action of $\mathrm{PSL}_2^*(\mathsf{O})$ *on* $\mathcal{H}(\mathbb{O})$.

PART III

PROJECTIVE GEOMETRY AND MOUFANG SETS

CHAPTER 7

Projective Geometry

7.1 Projective spaces

A *projective space* can be defined axiomatically as a set P (the set of points), together with a set L of subsets of P (the set of lines), satisfying the following axioms:

(a) Each two distinct points p and q are in exactly one line.

(b) Any line has at least three points on it.

(c) If a, b, c, d are distinct points and the lines through ab and cd meet, then so do the lines through ac and bd.

Given a subset S of P, denote by $\mathtt{Span}(S)$ the smallest subset $T \subseteq P$ containing S such that if a and b lie in T, so do all points on the line ab. Then we can define the dimension of the projective space P as

$$\min\{|S| - 1 : \mathtt{Span}(S) = P\}.$$

Here $|S|$ refers to the cardinality of S.

As an example, given any field k of characteristic 0, we can construct an n-dimensional projective space where the points are lines through the origin in k^{n+1}, the lines are planes through the origin in k^{n+1}, and the relation of 'lying on' is *inclusion*. This construction works even when k is a skew field. In fact, as illustrated in [Bae02], every projective n-space ($n > 2$) can be constructed in this way for some skew field k.

In particular, the real projective space of dimension n, $\mathbb{RP}^n$, is essentially

$$\mathbb{RP}^n := (\mathbb{R}^{n+1}\backslash\{0\})/\sim,$$

where $\sim$ is the equivalence relation defined by:

$$\begin{aligned}(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n) \quad & \iff \\ (x_0, x_1, \dots, x_n) \quad & = \quad (\lambda y_0, \lambda y_1, \dots, \lambda y_n) \text{ for some } \lambda \neq 0.\end{aligned}$$

Analogously, we can define the complex projective space $\mathbb{CP}^n$ and the quaternionic projective space $\mathbb{HP}^n$. As for the non-associative normed division algebra $\mathbb{O}$, we need to be careful because it is impossible to produce any kind of $\mathbb{OP}^n$ for $n \geqslant 3$. The octonionic projective line $\mathbb{OP}^1$ will be introduced latter in this section while the detailed construction of $\mathbb{OP}^2$ (which is called the Cayley plane) can be found in [dra].

7.2 Projective line over $\mathbb{K}$

A one-dimensional projective space is commonly called a *projective line*. Let $\mathbb{K}$ be one of the four normed division algebras over $\mathbb{R}$. When $\mathbb{K}$ is associative, the projective line $\mathbb{KP}^1$, as described earlier, can be defined as the set of lines in $\mathbb{K}^2$ passing through the origin. Note that such lines are always of the form

$$[\mathbf{x}, \mathbf{y}] := \{(\alpha\mathbf{x}, \alpha\mathbf{y}) \mid \alpha \in \mathbb{K}\}. \tag{7.1}$$

Thus, it is easy to observe that

$$\mathbb{KP}^1 = \{[1, \mathbf{x}] \mid \mathbf{x} \in \mathbb{K}\} \cup \{[0, 1]\} \text{ for } \mathbb{K} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H}. \tag{7.2}$$

However, for the non-associative case $\mathbb{K} = \mathbb{O}$, Equation 7.1 does not always hold true. Actually, the line in $\mathbb{O}^2$ through the origin containing $(\mathbf{x}, \mathbf{y})$ can be expressed as:

$$\begin{cases} \{(\alpha(\mathbf{y}^{-1}\mathbf{x}), \alpha) \ : \ \alpha \in \mathbb{K}\}, & \text{when } \mathbf{x} \neq 0 \\ \{(\alpha, \alpha(\mathbf{x}^{-1}\mathbf{y})) \ : \ \alpha \in \mathbb{K}\}, & \text{when } \mathbf{y} \neq 0. \end{cases}$$

Nevertheless, such a line is always a real vector space isomorphic to $\mathbb{O}$. Moreover, similar to Equation 7.2, we still have

$$\mathbb{OP}^1 = \{[1, \mathbf{x}] \mid \mathbf{x} \in \mathbb{O}\} \cup \{[0, 1]\}. \tag{7.3}$$

As a result of Equation 7.2 and Equation 7.3, we have

Theorem 7.2.1. $\mathbb{KP}^1$ *is a smooth manifold and is homeomorphic to* $\mathbb{K} \cup \{\infty\}$*, the one-point compactification of* $\mathbb{K}$.

Additionally, $\mathbb{K} \cup \{\infty\}$ is homeomorphic to the r-sphere S^r, where $r = \dim_{\mathbb{R}} \mathbb{K}$. Consequently, we obtain

$$\mathbb{KP}^1 \simeq S^r.$$

7.3 Constructing projective spaces from formally real Jordan algebras

Let A be a Jordan algebra, i.e., A is commutative and satisfies the Jordan identity

$$a \circ (b \circ (a \circ a)) = (a \circ b) \circ (a \circ a).$$

If, in addition, it satisfies

$$a_1 \circ a_1 + \cdots a_k \circ a_k = 0 \iff a_1 = \cdots = a_k = 0$$

for all $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in A$, then A is called a *formally real Jordan algebra*. Every formally real Jordan algebra is a direct sum of simple ones; and simple formally real Jordan algebras are classified as follows.

Theorem 7.3.1 ([JvNW93]). *The simple formally real Jordan algebras consist of four infinite families and one exception:*

(i) $\mathfrak{h}_n(\mathbb{R})$ *with the product* $A \circ B = \frac{1}{2}(AB + BA)$

(ii) $\mathfrak{h}_n(\mathbb{C})$ *with the product* $A \circ B = \frac{1}{2}(AB + BA)$

(iii) $\mathfrak{h}_n(\mathbb{H})$ *with the product* $A \circ B = \frac{1}{2}(AB + BA)$

(iv) $\mathbb{R}^n \oplus \mathbb{R}$ *with the product* $(\mathbf{v}, s) \circ (\mathbf{w}, t) = (s\mathbf{w} + t\mathbf{v}, \langle \mathbf{v}, \mathbf{w} \rangle + st)$

(v) $\mathfrak{h}_3(\mathbb{O})$ *with the product* $A \circ B = \frac{1}{2}(AB + BA)$.

Those in the fourth family are called *spin factors* or Jordan algebras of Clifford type, and $(\mathfrak{h}_3(\mathbb{O}), \circ)$ is exceptional and is usually called the *Albert algebra*.

It is worth noting that $\mathfrak{h}_2(\mathbb{O})$ is also a simple formally real Jordan algebra. Actually, it is isomorphic to the spin factor $\mathbb{R}^9 \oplus \mathbb{R}$ because $\big(\mathfrak{h}_2(\mathbb{O}), \mathtt{M}\big)$ has signature (9,1).

Let $\mathcal{J}$ be a formally real Jordan algebra. An element $p \in \mathcal{J}$ is called a *projection* if $p^2 = p$. Denote by $\mathtt{P}(\mathcal{J})$ the set of projections of $\mathcal{J}$. We may define a relation in $\mathtt{P}(\mathcal{J})$:

$$p \leqslant q \quad \Longleftrightarrow \quad \mathtt{Ran}(p) \subseteq \mathtt{Ran}(q),$$

where $\mathtt{Ran}(p)$ means the range of p, which is a closed subspace of $\mathcal{J}$. Since the set of closed subspaces is partially ordered with respect to inclusion, the above relation $\leqslant$ is obviously a partial order. Especially, we can always construct a chain of inequalities of projections:

$$0 = p_0 < p_1 < \cdots < p_m = p.$$

The largest possible such m is called the rank of p in $\mathcal{J}$.

We may define a projective space using projections in $\mathtt{P}(\mathcal{J})$. Explicitly, the points are rank-1 projections, the lines are rank-2 projections, and the inclusion is derived from the partial order $\leqslant$. That is, given a rank-1 projection p_1 and a rank-2 projection p_2, we say the point p_1 is on the line p_2 provided $p_1 < p_2$.

Applying this construction to $\mathfrak{h}_n(\mathbb{R})$, $\mathfrak{h}_n(\mathbb{C})$, and $\mathfrak{h}_n(\mathbb{H})$ with $n \geqslant 2$ yields the projective spaces $\mathbb{RP}^{n-1}$, $\mathbb{CP}^{n-1}$ and $\mathbb{HP}^{n-1}$, respectively. Carried out in the spin factors $\mathbb{R}^n \oplus \mathbb{R}$ with $n \geqslant 2$, one obtains a series of 1-dimensional projective spaces related to Lorentzian geometry. The exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$ produces the Cayley plane $\mathbb{OP}^2$, which was first discovered by R. Moufang [Mou33].

As an example, consider the application to $\mathfrak{h}_2(\mathbb{K})$. Note that $\mathbb{K} = \mathbb{O}$ is also included here. It is easy to see that all the nontrivial projections in $\mathtt{P}\big(\mathfrak{h}_2(\mathbb{K})\big)$ are of the form

$$\begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{y}} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{x}}\mathbf{x} & \bar{\mathbf{x}}\mathbf{y} \\ \bar{\mathbf{y}}\mathbf{x} & \bar{\mathbf{y}}\mathbf{y} \end{pmatrix} \tag{7.4}$$

where $(\mathbf{x}, \mathbf{y}) \in \mathbb{K}^2$ satisfies $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = 1$. These nontrivial projections all have rank 1, so they are the points of the projective line $\mathbb{KP}^1$. Conversely, given any nonzero element $(\mathbf{x}, \mathbf{y}) \in \mathbb{K}^2$, we can normalize it and set $(\mathbf{x}', \mathbf{y}') = \dfrac{1}{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}(\mathbf{x}, \mathbf{y})$. Applying Equation 7.4 to $(\mathbf{x}', \mathbf{y}')$ we then obtain a rank-1 projection, and thus a point in $\mathbb{KP}^1$. We denote such a point as $[\mathbf{x}, \mathbf{y}]$. This gives rise to an equivalence relation on nonzero elements of $\mathbb{K}^2$:

$$(\mathbf{x}, \mathbf{y}) \sim (\mathbf{x}', \mathbf{y}') \;\Leftrightarrow\; [\mathbf{x}, \mathbf{y}] = [\mathbf{x}', \mathbf{y}'].$$

Each equivalence class will be called a line through the origin in $\mathbb{K}^2$. The set of lines through the origin in $\mathbb{K}^2$ is then identical to the set of equivalence classes. This coincides with Equation 7.2 and Equation 7.3, which demonstrate that for all four normed division algebras over $\mathbb{R}$ we have

$$\mathbb{KP}^1 = \{[1, \mathbf{x}] \mid \mathbf{x} \in \mathbb{K}\} \cup \{[0, 1]\}. \tag{7.5}$$

CHAPTER 8

Projective Moufang Sets

8.1 Moufang sets

Definition 8.1.1. *A* Moufang set *is a set* X *(with* $|X| \geqslant 3$*) together with a collection of groups* $\{U_x | x \in X\}$ *satisfying the following two properties:*

M$_1$) *For each* $x \in X$, $U_x \leqslant \mathrm{Sym}(X)$ *fixes* x *and acts regularly (i.e., sharply transitively) on* $X \backslash \{x\}$.

M$_2$) *For all* $x \in X$, U_x *permutes the set* $\{U_y | y \in X\}$ *by conjugation.*

The Moufang set is then denoted $\mathbb{M} = (X, (U_x)_{x \in X})$*; the groups* U_x *are called the* root groups *of* $\mathbb{M}$*, and the group* $G := \langle U_x \mid x \in X \rangle$ *is called the* little projective group *of* $\mathbb{M}$.

Note that G acts doubly transitively on X, and that $U_x^{\phi} = U_{x\phi}$ for all $x \in X$ and all $\phi \in G$. Here we adopt the convention in [DMS09] that permutations of X act from the right side and $x^y := y^{-1}xy$ for $x, y \in G$.

Let U be a group with composition $+$ and identity 0. The operation $+$ is not necessarily commutative. Let $X = U \dot\cup \{\infty\}$ denote the disjoint union of U with a new symbol ∞ (i.e., $\infty \notin X$). For each element $a \in U$, we may define a permutation of X as follows:

$$t_a : X \to X \quad \begin{cases} x \mapsto x + a \text{ when } x \in U, \\ \infty \mapsto \infty. \end{cases}$$

Clearly, the map $a \mapsto t_a$ is essentially the right regular representation of the group U. Set

$$U_\infty := \{t_a \mid a \in U\}.$$

Let τ be a permutation of U^*, where $U^* := U\backslash\{0\}$ consists of nontrivial elements in U. We can extend τ to be an element of $\mathrm{Sym}(X)$ (which is still denoted τ) by requiring $0\tau = \infty$ and $\infty\tau = 0$. Define

$$U_0 := U_\infty^\tau, \tag{8.1}$$

$$U_a := U_0^{t_a} \quad \forall a \in U. \tag{8.2}$$

In particular, we define

$$\mathbb{M}(U,\tau) := \big(X, (U_x)_{x\in X}\big) \tag{8.3}$$

$$G := \langle U_\infty, U_0\rangle \equiv \langle U_x \mid x \in X\rangle.$$

Note that this construction does not always give rise to a Moufang set, but every Moufang set can be obtained in this way [DMS09].

In order to determine when $\mathbb{M}(U,\tau)$ defined in 8.3 forms a Moufang set, we need to consider the following maps:

$$h_a := \tau t_a \tau^{-1} t_{-(a\tau^{-1})} \tau t_{-(-(a\tau^{-1}))\tau} \in \mathrm{Sym}(X),\ \forall a \in U^*.$$

Such a map h_a is called a *Hua map*. It is convenient to agree that $h_0 := 0$.

Theorem 8.1.2 ([DMW06]). *The $\mathbb{M}(U,\tau)$ defined in Equation 8.3 is a Moufang set if and only if the restriction of each Hua map to U is contained in* $\mathrm{Aut}(U)$, *i.e.,*

$$(a+b)h_c = ah_c + bh_c,\ \forall a,b \in U,\ c \in U^*.$$

Furthermore, for any $a \in U^*$ we define a μ-*map*

$$\mu_a := \tau^{-1}h_a = t_a\tau^{-1}t_{-(a\tau^{-1})}\tau t_{-(-(a\tau^{-1}))\tau},$$

which is the unique permutation in $U_0^* t_a U_0^*$ that interchanges 0 and ∞ [DMS09]. It is easy to see that

$$\mu_a^{-1} = \mu_{-a}.$$

In particular, we have the following fact from [DMW06]:

$$\mathbb{M}(U,\tau) = \mathbb{M}(U,\mu_a), \quad \forall a \in U^*.$$

Additionally, the group

$$H := \langle \mu_a\mu_b \mid a,b \in U^*\rangle$$

is called the *Hua subgroup* of the Moufang set $\mathbb{M}(U, \tau)$. This group coincides with the point-wise stabilizer of 0 and ∞, that is,

$$H = G_{0,\infty}.$$

8.2 Local Moufang sets

Suppose $(X, \sim)$ is a set with an equivalence relation $\sim$. Let $\mathrm{Sym}(X, \sim)$ be the group of equivalence-preserving permutations of $(X, \sim)$. We denote the equivalence class of $x \in X$ by $\bar{x}$, and the set of such equivalence classes by $\bar{X}$. In addition, for any element $g \in \mathrm{Sym}(X, \sim)$, denote by $\bar{g}$ the corresponding element in $\mathrm{Sym}(\bar{X})$.

Definition 8.2.1 ([DMW06]). *A* local Moufang set *consists of a set with an equivalence relation* $(X, \sim)$ *such that* $|\bar{X}| > 2$, *and a family of root groups* $U_x \leqslant \mathrm{Sym}(X, \sim)$ *for all* $x \in X$. *We denote* $U_{\bar{x}} := \overline{U_x} = \mathrm{Im}\big(U_x \to \mathrm{Sym}(\bar{X})\big)$ *for the permutation group induced by the action of* U_x *on the set of equivalence classes. The group generated by the root groups is called the little projective group, and is usually denoted* $G := \langle U_x | x \in X\rangle$. *Furthermore, we demand the following:*

(LM1) *If* $x \sim y$ *for* $x, y \in X$, *then* $U_{\bar{x}} = U_{\bar{y}}$.

(LM2) *For* $x \in X$, U_x *fixes* x *and acts sharply transitively on* $X \backslash \bar{x}$.

(LM2′) *For* $\bar{x} \in \bar{X}$, $U_{\bar{x}}$ *fixes* $\bar{x}$ *and acts sharply transitively on* $\bar{X} \backslash \{\bar{x}\}$.

(LM3) *For* $x \in X$ *and* $g \in G$, *we have* $U_x^g = U_{xg}$.

(LM3′) *For* $\bar{x} \in \bar{X}$ *and* $g \in G$, *we have* $U_{\bar{x}}^g = U_{\overline{xg}}$.

Note that (LM2′) and (LM3′) precisely state that $(\bar{X}, \{U_{\bar{x}}\}_{\bar{x} \in \bar{X}})$ is a Moufang set.

A natural question is: given a group U and a permutation τ, both acting faithfully on a set with an equivalence relation, is it possible to construct a local Moufang set like we did in the previous section? Obviously, we need some additional conditions on U and τ.

Suppose $(X, \sim)$ is a set with an equivalence relation satisfying $|\bar{X}| > 2$, $U \leqslant \mathrm{Sym}(X, \sim)$ is a subgroup and $\tau \in \mathrm{Sym}(X, \sim)$. We require that

(C1): U has a fixed point, denoted by ∞, and acts sharply transitively on $X \backslash \overline{\infty}$.

(C1′): The induced action of U on $\bar{X}$ is sharply transitive on $\bar{X}\backslash\overline{\infty}$.

(C2): $\infty\tau \nsim \infty$ and $\infty\tau^2 = \infty$. We write $0 := \infty\tau$.

For $x \nsim \infty$, let α_x be the unique element of U that maps 0 to x. We generally write $-x := 0\alpha_x^{-1}$ and define

$$U_\infty := U, \quad U_0 := U_\infty^\tau, \quad U_x := \begin{cases} U_0^{\alpha_x} & \text{for } x \nsim \infty, \\ U_\infty^{\gamma_{x\tau^{-1}}} & \text{for } x \sim \infty. \end{cases}$$

Here $\gamma_x := \alpha_x^\tau$ maps ∞ to $x\tau$. Moreover, write $U_{\bar{x}}$ for the induced action of U_x on $\bar{X}$, i.e.,

$$U_{\bar{x}} := \operatorname{Im}(U_x \to \operatorname{Sym}(\bar{X})).$$

For any element $x \in X$, we call x a *unit* if $x \nsim 0$ and $x \nsim \infty$. For such a unit, we can define the μ-map as

$$\mu_x := \gamma_{(-x)\tau^{-1}}\alpha_x\gamma_{-(x\tau^{-1})}.$$

Theorem 8.2.2 ([DMR16]). $\mathbb{M}(U,\tau)$ *as defined above is a local Moufang set if and only if one of the following equivalent conditions holds:*

(1) $U_\infty^{\gamma_{x\tau^{-1}}} = U_x$ *for all units* $x \in X$;

(2) $U_0^{\mu_x} = U_\infty$ *for all units* $x \in X$;

(3) $U_0 = U_\infty^{\mu_x}$ *for all units* $x \in X$.

Given a local Moufang set $\mathbb{M}$, let x be a unit in X. Define

$$\sim x := \big(-(x\tau^{-1})\big)\tau.$$

$\mathbb{M}$ is said to be *special* if $\sim x = -x$ for all units $x \in X$, or equivalently, if

$$(-x)\tau = -(x\tau)$$

for all units $x \in X$.

Lemma 8.2.3 ([DMS09]). *Then the following statements are equivalent:*

(i) $\mathbb{M}$ *is special;*

(ii) $\sim x = -x$ *for all units* $x \in X$.

(iii) $(-x)\mu_x = x$ *for all units* $x \in X$.

8.3 Projective Moufang sets over $\mathbb{K}$

Obviously, $\mathbb{K}$ is an abelian group with respect to the addition operation. Let $\hat{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$ and $\mathbb{K}^* = \mathbb{K} \backslash \{0\}$. First we define

$$\tau : \mathbb{K}^* \to \mathbb{K}^* : \quad \mathbf{x} \mapsto -\mathbf{x}^{-1}$$

and then extend it to $\mathrm{Sym}(\hat{\mathbb{K}})$ by requiring that

$$0\tau = \infty \text{ and } \infty\tau = 0.$$

In addition, we define $\mathbb{K}_\infty$ to be the group of translations $t_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{K}$:

$$t_{\mathbf{a}} : X \to X \quad \begin{cases} \mathbf{x} \mapsto \mathbf{x} + \mathbf{a} \\ \infty \mapsto \infty. \end{cases}$$

Similar to Equation 8.1, we set

$$\begin{aligned} \mathbb{K}_0 &:= \mathbb{K}_\infty^\tau, \\ \mathbb{K}_{\mathbf{x}} &:= \mathbb{K}_0^{t_{\mathbf{x}}} \quad \forall\, \mathbf{x} \in \mathbb{K}^*. \end{aligned}$$

We thus get the following construction in the same way as in Equation 8.3:

$$\mathbb{M}(\mathbb{K}, \tau) := \big(\hat{\mathbb{K}}, (\mathbb{K}_{\mathbf{x}})_{\mathbf{x} \in \hat{\mathbb{K}}}\big).$$

By using Theorem 8.1.2 it is straightforward to check that $\mathbb{M}(\mathbb{K}, \tau)$ is a Moufang set. It is called the *projective Moufang set* over $\mathbb{K}$ and simply denoted $\mathbb{M}(\mathbb{K})$ when there is no ambiguity about τ.

Additionally, let $G(\mathbb{K})$ be the little projective group of the projective Moufang set $\mathbb{M}(\mathbb{K})$. We have

Proposition 8.3.1.

$$G(\mathbb{K}) \simeq \mathrm{PSL}_2(\mathbb{K}).$$

Proof. Recall that the extended plane $\hat{\mathbb{K}}$ can be viewed as the projective line $\mathbb{KP}^1$ through the bijection

$$\xi : \ \hat{\mathbb{K}} \to \mathbb{KP}^1, \quad \begin{cases} \mathbf{x} \mapsto [1, \mathbf{x}] \\ \infty \mapsto [0, 1]. \end{cases}$$

This map induces an isomorphism

$$\Xi : \operatorname{Sym}(\hat{\mathbb{K}}) \to \operatorname{Sym}(\mathbb{KP}^1) : \quad \rho \mapsto \rho^{\xi}.$$

Consider the restriction of Ξ to $G(\mathbb{K}) \leqslant \operatorname{Sym}(\hat{\mathbb{K}})$. Direct calculations show that

- $\Xi(t_{\mathbf{a}}) \in \text{Möb}(\mathbb{K})$ with representing matrix $\begin{bmatrix} 1 & \mathbf{a} \\ 0 & 1 \end{bmatrix} \in PSL_2(\mathbb{K})$.
- $\Xi(\tau) \in \text{Möb}(\mathbb{K})$ with representing matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in PSL_2(\mathbb{K})$.

Thus, Ξ maps $G(\mathbb{K})$ to $\text{Möb}(\mathbb{K})$ when $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and to $\text{Möb}^*(\mathbb{O})$ when $\mathbb{K} = \mathbb{O}$. Recall that the Möbius group $\text{Möb}(\mathbb{K})$ ($\text{Möb}^*(\mathbb{O})$ in the case $\mathbb{K} = \mathbb{O}$) can be identified with $PSL_2(\mathbb{K})$. We then obtain a map, still denoted Ξ, from $G(\mathbb{K})$ to $PSL_2(\mathbb{K})$. From the calculations above it is clear that Ξ is a group isomorphism. □

Furthermore, there is a canonical equivalence relation within $\mathbb{KP}^1$ defined as:

$$[1, \mathbf{x}] \sim [1, \mathbf{y}] \iff \mathbf{x} = \mathbf{y},$$

and

$$[0, 1] \nsim [1, \mathbf{x}] \text{ for any } \mathbf{x} \in \mathbb{K}.$$

Theorem 8.3.2. *The projective Moufang set $\mathbb{M}(\mathbb{K})$, equipped with the equivalence defined above, is a local Moufang set.*

Proof. According to Theorem 8.2.2, it suffices to prove that $\mathbb{K}_0^{\mu_x} = \mathbb{K}_\infty$ for all $\mathbf{x} \in \mathbb{K}^*$. Let $V_0 := \Xi(\mathbb{K}_0)$ and $V_\infty := \Xi(\mathbb{K}_\infty)$. Then it is sufficient to show that

$$V_0^{\Xi(\mu_x)} = V_\infty, \quad \forall \mathbf{x} \in \mathbb{K}^*.$$

It is easy to see that

$$V_\infty = \{ \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix} : \mathbf{x} \in \mathbb{K}\} \text{ and } V_0 = \{ \begin{bmatrix} 1 & 0 \\ \mathbf{x} & 1 \end{bmatrix} : \mathbf{x} \in \mathbb{K}\}.$$

In addition, we have $\Xi(t_{\mathbf{x}}) = \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix}$ and $\Xi(\tau) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then from $\mu_{\mathbf{x}} = t_{\mathbf{x}}\tau^{-1}t_{-(\mathbf{x}\tau^{-1})}\tau t_{-(-(\mathbf{x}\tau^{-1}))\tau}$ we calculate that

$$\Xi(\mu_{\mathbf{x}}) = \begin{bmatrix} 0 & -\mathbf{x} \\ \mathbf{x}^{-1} & 0 \end{bmatrix}.$$

In order to prove $V_0^{\Xi(\mu_x)} = V_\infty$, we first notice that

$$\begin{bmatrix} 0 & -\mathbf{x} \\ \mathbf{x}^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{y} & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{x} \\ -\mathbf{x}^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\mathbf{xyx} \\ 0 & 1 \end{bmatrix}, \tag{8.4}$$

which implies $V_0^{\Xi(\mu_x)} \subseteq V_\infty$.

On the other hand, for every element $\mathbf{z} \in \mathbb{K}$, we have

$$\begin{bmatrix} 1 & \mathbf{z} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{x} \\ \mathbf{x}^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mathbf{x}^{-1}\mathbf{z}\mathbf{x}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{x} \\ -\mathbf{x}^{-1} & 0 \end{bmatrix} \in V_0^{\nu_x}, \tag{8.5}$$

which indicates that $V_\infty \subseteq V_0^{\Xi(\mu_x)}$. Note that Equation 8.4 and Equation 8.5 also hold for $\mathbb{O}$ because of the alternativity of $\mathbb{O}$. □

Proposition 8.3.3. *The projective Moufang set* $\mathbb{M}(\mathbb{K})$ *is special.*

Proof. It is sufficient to prove that

$$(-\mathbf{x})\mu_{\mathbf{x}} = \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{K}^*,$$

or equivalently,

$$(-[1,\mathbf{x}])\Xi(\mu_{\mathbf{x}}) = [1,\mathbf{x}], \quad \forall \mathbf{x} \in \mathbb{K}^*.$$

Note that $-[1,\mathbf{x}] = [1,-\mathbf{x}]$ and $\Xi(\mu_{\mathbf{x}}) = \begin{bmatrix} 0 & -\mathbf{x} \\ \mathbf{x}^{-1} & 0 \end{bmatrix}$. Hence, we have

$$(-[1,\mathbf{x}])\Xi(\mu_{\mathbf{x}}) = [1,-\mathbf{x}] \begin{bmatrix} 0 & -\mathbf{x} \\ \mathbf{x}^{-1} & 0 \end{bmatrix} = [-1,-\mathbf{x}] = [1,\mathbf{x}],$$

which completes the proof. □

Since the projective Moufang sets are local Moufang sets and are special, the μ-maps can be simplified as

$$\begin{aligned} \mu_x &= t_x\tau^{-1}t_{-(x\tau^{-1})}\tau t_{-(-(x\tau^{-1}))\tau} \\ &= t_x\tau^{-1}t_{-(x\tau^{-1})}\tau t_x. \end{aligned}$$

Applying the isomorphism Ξ to both sides gives rise to

$$\begin{bmatrix} 0 & -\mathbf{x} \\ \mathbf{x}^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{x}^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix}. \tag{8.6}$$

In particular, when $\mathbf{x}$ is of norm one, we have $\mathbf{x}^{-1} = \dfrac{\bar{\mathbf{x}}}{||\mathbf{x}||^2} = \bar{\mathbf{x}}$. In this case, Equation 8.6 becomes

$$\begin{bmatrix} 0 & -\mathbf{x} \\ \bar{\mathbf{x}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \bar{\mathbf{x}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix}. \tag{8.7}$$

Alternatively, we have

$$\begin{bmatrix} 0 & -\bar{\mathbf{x}} \\ \mathbf{x} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \bar{\mathbf{x}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \bar{\mathbf{x}} \\ 0 & 1 \end{bmatrix}. \tag{8.8}$$

Note that $\Xi(\tau) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \Xi(\mu_1)$, which implies $\tau = \mu_1$ is a μ-map. Thus, the Hua subgroup of $\mathbb{M}(\mathbb{K})$ becomes

$$H = \langle \mu_{\mathbf{a}}\mu_{\mathbf{b}} \ : \ \mathbf{a}, \mathbf{b} \in \mathbb{K}^* \rangle = \langle \tau\mu_{\mathbf{a}} \ : \ \mathbf{a} \in \mathbb{K}^* \rangle.$$

In particular, we have

$$\Xi(H) = \langle \begin{bmatrix} -\mathbf{x} & 0 \\ 0 & \mathbf{x}^{-1} \end{bmatrix} \mid \mathbf{x} \in \mathbb{K}^* \rangle.$$

PART IV

OVER-EXTENDED ROOT SYSTEMS AND HYPERBOLIC WEYL GROUPS

CHAPTER 9

Root Systems and Weyl Groups

9.1 Root systems

9.1.1 Finite root systems

Let V be a finite dimensional *Euclidean space*, i.e., a finite dimensional real vector space with inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ which is bilinear, symmetric, and positive definite. For any non-zero element $v \in V$, the hyperplane perpendicular to v is

$$H_v = \{w \in V \mid \langle w, v \rangle = 0\}.$$

Define the reflection in H_v by

$$s_v : V \to V, \quad w \mapsto w - \frac{2\langle w, v \rangle}{\langle v, v \rangle} v.$$

An important observation is that each s_v leaves the inner product invariant, that is,

$$\langle s_v(w), s_v(w') \rangle = \langle w, w' \rangle, \ \forall w, w' \in V.$$

For the purpose of simplicity, we write $v^\vee := \dfrac{2}{\langle v, v \rangle} v$ for all $v \in V$. Thus, s_v becomes $s_v(w) = w - \langle v^\vee, w \rangle v$ for all $w \in V$.

Definition 9.1.1. *A finite subset $\Phi \subseteq V$ is a* root system *for V if the following hold:*

(a) $0 \notin \Phi$ *and* $\mathrm{Span}_{\mathbb{R}}(\Phi) = V$;

(b) if $\alpha \in \Phi$ *and* $\lambda\alpha \in \Phi$ *with* $\lambda \in \mathbb{R}$, *then* $\lambda \in \{\pm 1\}$;

(c) $s_\alpha(\beta) \in \Phi$ *for all* $\alpha, \beta \in \Phi$;

(d) $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Phi$.

The dimension of V, $\dim_{\mathbb{R}} V$, is called the rank of Φ. Note that each s_α permutes Φ, and $-\Phi = \Phi$.

A root system Φ is *reduced* if

$$\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi,$$

and *irreducible* if there is not a non-trivial partition $\Phi = \Phi_1 \cup \Phi_2$ with

$$\langle \alpha, \beta \rangle = 0, \quad \forall \alpha \in \Phi_1, \beta \in \Phi_2.$$

Definition 9.1.2. *Let* Φ *be a root system for* V*. A subset* $\Pi \subseteq \Phi$ *is a* root basis *of* Φ *if the following hold:*

(a) Π *is a vector space basis for* V*; and*

(b) *every* $\alpha \in \Phi$ *can be written as* $\alpha = \sum_{\beta \in \Pi} k_\beta \beta$*, where* k_β *are integers all of the same sign.*

The roots in Π are called *simple roots* of Φ and their corresponding reflections are called *simple reflections*. Clearly, $Q = \mathtt{Span}_{\mathbb{Z}}(\Pi)$ is a free abelian group; it is called the *root lattice* of Φ. In addition, let

$$\Phi^+ := \{\alpha = \sum_{\beta \in \Pi} k_\beta \beta \in \Phi \mid k_\beta \in \mathbb{N} \ \forall \beta \in \Pi\}$$
$$\Phi^- := \{\alpha = \sum_{\beta \in \Pi} k_\beta \beta \in \Phi \mid -k_\beta \in \mathbb{N} \ \forall \beta \in \Pi\}.$$

Then elements in Φ^+ are called positive roots and those in Φ^- are called negative roots. Especially, we have $-\Phi^- = \Phi^+$ and $\Phi = \Phi^+ \cup \Phi^-$.

Moreover, there exists a partial order $\leqslant$ on the root system Φ, called the *dominance order* and defined by

$$w \leqslant v \iff v - w = \sum_{\beta \in \Pi} k_\beta \beta \text{ with all } k_\beta \in \mathbb{N}.$$

A maximal element in Φ with respect to the dominance order is called the *highest root* of Φ. It follows from [Bou02] that every irreducible root system has a unique highest root.

Definition 9.1.3. *The* Weyl group *of the root system* Φ *is the subgroup of* $GL(V)$ *generated by all the simple reflections, i.e.,*

$$W(\Phi) := \langle s_\alpha \mid \alpha \in \Pi \rangle.$$

9.1.2 Affine root systems

Let V be a finite dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$. A function $f : V \to \mathbb{R}$ is said to be *affine linear* if there exist $\alpha \in V$ and a constant $m \in \mathbb{R}$ such that

$$f(x) = \langle \alpha, x \rangle + m, \quad \forall x \in V.$$

Let F be the set of all affine linear functions on V. Clearly, we have the direct sum of vector spaces $F = V \oplus \mathbb{R}\delta$, where $\delta \in F$ is the constant function with value 1. Let D be the projection of F into V. Then $Df = \alpha$ is the gradient of f such that $f = Df + f(0)$.

We extend the inner product $\langle \cdot, \cdot \rangle$ to F by defining

$$\langle Df + c\delta, Dg + d\delta \rangle := \langle Df, Dg \rangle.$$

This bilinear form is symmetric and degenerate with kernel $\mathbb{R}\delta$.

Moreover, it is easy to see that $H_f = \{x \in V \,|\, f(x) = 0\}$ is an affine hyperplane. Let s_f denote the orthogonal reflection through H_f. That is,

$$s_f(x) = x - f^\vee(x) Df = x - f(x) Df^\vee,$$

where $f^\vee := \dfrac{2f}{\langle f, f \rangle}$. Notice that s_f is an affine linear isometry of V depending only on the hyperplane $H = H_a$ and not on the choice of a.

Let F^* be the group of invertible affine linear transformations in F. F^* acts on F via the *transposition*:

$$F^* \times F \to F, \quad (g, f) \mapsto fg^{-1}.$$

Applying to s_f gives rise to

$$s_a \cdot f = f - \langle a^\vee, f \rangle a.$$

Naturally, a translation $t_v : V \to V,\ x \mapsto x + v$, for some $v \in V$, is an affine transformation of V. We have

$$t_v \cdot f = f - \langle v, Df \rangle \delta.$$

Definition 9.1.4. *An* affine root system *in* V *is a subset* $S \subseteq F$ *such that*

(a) S *consists of non-constant functions and spans* F;

(b) $s_a \cdot S \subseteq S$ *for all* $a \in S$;

(c) $\langle a^\vee, b \rangle \in \mathbb{Z}$ *for all* $a, b \in S$;

(d) *the group* $W(S) := \langle s_a \mid a \in S \rangle$ *acts properly on* V.

The group $W(S)$ *is called the Weyl group of the affine root system* S.

An affine root system is *reduced* if

$$a \in S \Rightarrow \frac{1}{2}a \notin S,$$

and *irreducible* if there is not a non-trivial partition $S = S^1 \cup S^2$ such that

$$\langle a, b \rangle = 0, \quad \forall a \in S^1, b \in S^2.$$

The following proposition provides examples of affine root systems.

Proposition 9.1.5 ([Bou02];[Kac94]). *Let* Φ *be an irreducible finite root system spanning a real finite-dimensional vector space* V, *and let* $\langle \cdot, \cdot \rangle$ *be a positive-definite symmetric bilinear form on* V, *invariant under* $W(\Phi)$, *the Weyl group of* Φ. *Then*

$$S(\Phi) := \{f_{\alpha,k} = \langle \alpha, - \rangle + k \mid \alpha \in \Phi, k \in \mathbb{Z} \text{ such that } \frac{1}{2}\alpha \notin \Phi\}$$

is a reduced irreducible affine root system on V.

This proposition has the following implications:

Corollary 9.1.6. (i) *The Weyl group of* $S(\Phi)$

$$W(S(\Phi)) \simeq \Gamma \rtimes W(\Phi),$$

where Γ *is the group of translations generated by all* $t_{\alpha^\vee}$ *for* $\alpha \in \Phi$.

(ii) *Let* Π *denote the root basis of* Φ *and* θ *be the highest root of* Φ. *Then* $\{1 - \theta\} \cup \Pi$ *is a root basis of* $S(\Phi)$.

9.2 Coxeter systems

9.2.1 Coxeter systems and Coxeter diagrams

A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$, subject to relations

$$(ss')^{m(s,s')} = 1,$$

where

$$\begin{cases} m(s,\, s) = 1, \\ m(s,\, s') = m(s',\, s) \geqslant 2, \quad \text{if } s \neq s'. \end{cases} \tag{9.1}$$

Conventionally, $m(s, s') = \infty$ if no relation occurs for a pair $(s,\, s')$. The cardinality of S, $|S|$, is called the rank of (W, S). The group W is a *Coxeter group*.

Let M be a symmetric matrix indexed by S with entires $m(s, s') \in \mathbb{N} \cup \{\infty\}$ subject to the relations 9.1. Obviously, the Coxeter system (W, S) can be characterized by the finite set S and such a symmetric matrix M. The matrix M is called the *Coxeter matrix*, which can be conveniently encoded by a *Coxeter diagram*. Explicitly, given a Coxeter system (W, S), the corresponding Coxeter diagram is a undirected graph Γ with S as its vertex set, joining vertices s and s' by an edge labeled $m(s, s')$ whenever this number (∞ allowed) is at least 3. If distinct vertices s and s' are not joined, it is understood that $m(s, s') = 2$. Conventionally, the label $m(s, s') = 3$ may be omitted. The Coxeter system (W, S) is said to be *irreducible* if the underlying Coxeter diagram Γ is connected. We also call the corresponding Coxeter group W irreducible. Throughout this thesis, we only focus on irreducible Coxeter groups.

Let W be an irreducible Coxeter group. Then W is *finite* (or *spherical*) when the Coxeter matrix M is positive definite, is *affine* when M is semi-positive definite of rank $n-1$, and is *hyperbolic* when M has signature $(n-1, 1)$. Here $n = |S|$ is the rank of (W, S).

A Coxeter group W is said to be *simply-laced* if the entries of the Coxeter matrix M are either 2 or 3.

9.2.2 Geometric representation of Coxeter groups

Let $(W,\, S)$ be a Coxeter system and V be a vector space over $\mathbb{R}$ of dimension $n := |S|$. Denote by $\{\alpha_s \mid s \in S\}$ a basis of V that is in one-to-one correspondence with the set S. We define a bilinear form on V by requiring

$$\langle \alpha_s, \alpha_{s'} \rangle = -\cos\left(\frac{\pi}{m(s, s')}\right), \tag{9.2}$$

where by convention $\dfrac{\pi}{m(s, s')} = 0$ if $m(s, s') = \infty$.

For each $s \in S$ we may define a reflection of V with respect to this bilinear form:

$$\sigma_s \;:\; V \;\to\; V \tag{9.3}$$

$$x \mapsto x - 2\langle x, \alpha_s \rangle \alpha_s.$$

According to [Hum92], σ_s preserves the bilinear form $\langle -, - \rangle$ on V, and the group homomorphism $\sigma : W \to O(V), \quad s \mapsto \sigma_s$ is unique. We call (V, σ) the *geometric representation* of W. Note that W acts on V in the natural way: $w(x) = \sigma(w)(x), \ \forall x \in V$.

9.2.3 Root systems of Coxeter groups

Let (W, S) be a Coxeter system and V be the geometric representation of W with basis $\{\alpha_s\}_{s \in S}$. Then it is straightforward to show the set

$$\Phi := \{\omega(\alpha_s) \mid w \in W, \ s \in S\}$$

is a finite root system with root basis $\{\alpha_s\}_{s \in S}$. Obviously, the root basis is in one-to-one correspondence with S. Moreover, the Weyl group of Φ is generated by all simple reflections σ_s defined in Equation 9.3. It is isomorphic to the Coxeter group W under σ. Especially, generators of $W(\Phi)$ are subject to the same relations in Equation 9.1. Therefore, the Coxeter diagram of (W, S) could also be used to describe the root system Φ, hence is also referred to as the Coxeter diagram of Φ. In particular, if Φ is irreducible, then the root system is finite when the Coxeter matrix M is positive definite, is affine when M is semi-positive definite of rank $n - 1$, and is hyperbolic when M has signature $(n - 1, 1)$. Here n refers to the rank of Φ.

CHAPTER 10

Hyperbolic Weyl Groups of Rank $\dim_{\mathbb{R}} \mathbb{K}$

10.1 Over-extension of root systems in $\mathbb{K}$

10.1.1 Root systems in $\mathbb{K}$

Clearly, the norm $\|\cdot\|$ of $\mathbb{K}$ gives rise to a positive-definite inner product:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{2}(\mathbf{a}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{a}}).$$

Equipped with this inner product, $\mathbb{K}$ can be identified with the root space of some finite root system Φ_r of rank $r = \dim_{\mathbb{R}} \mathbb{K}$. Let α_i $(i = 1, \ldots, r)$ be the simple roots of Φ_r. Then the associated simple reflections can be expressed as

$$\begin{aligned} s_{\alpha_i}(\mathbf{x}) &= \mathbf{x} - \frac{2\langle \mathbf{x}, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \\ &= \mathbf{x} - \frac{\mathbf{x}\bar{\alpha}_i + \alpha_i\bar{\mathbf{x}}}{\|\alpha_i\|^2}\alpha_i \\ &= -\frac{\alpha_i\bar{\mathbf{x}}\alpha_i}{\|\alpha_i\|^2}. \end{aligned} \tag{10.1}$$

Here the last "=" also holds for octonions because $\mathbb{O}$ is alternative.

10.1.2 Over-extension of Φ_r

To any finite root system Λ of rank n one may associate an indefinite root system of rank $n + 2$ in the following way. The first step is to construct the non-twisted affine extension, thereby increasing the rank by one. Then one adds an additional node with a single line to the affine node in the Coxeter diagram. Explicitly, we first notice that the root space of the affine extension is $\mathbb{R}^{n+1}$, which can be embedded into the $(n+1,1)$-Minkowski space $\mathbb{R}^{n+1}\oplus\mathbb{R}$ via the natural embedding $\mathbf{x}\mapsto(\mathbf{x},0)$. Let $\{\alpha\}_{1\leqslant i\leqslant n}$ be the set of simple roots of Λ and α_0 be the affine root. Especially, We write the image of α_i in $\mathbb{R}^{n+1}\oplus\mathbb{R}$ as A_i for all $i=0,1,\dots,n$. Denote by A_{-1} the over-extension node. Then A_{-1} can be obtained from solving the following equations

$$\langle A_{-1},A_0\rangle=-\frac{1}{2},\quad \text{and } \langle A_{-1},A_k\rangle=0 \text{ for all } 1\leqslant k\leqslant n.$$

This results in a root system which is called the *over-extension* of Λ and denoted Λ^{++}. Λ^{++} has a root basis $\{A_{-1},A_0,A_1,\dots,A_n\}$ and in many cases turns out to be hyperbolic.

As an example, consider the finite root system Φ_r in $\mathbb{K}$. Recall that the Minkowski space $\mathbb{R}^{r+1}\oplus\mathbb{R}$ can be identified as $\big(\mathfrak{h}_2(\mathbb{K}),\langle\cdot,\cdot\rangle\big)$, where the bilinear form $\langle\cdot,\cdot\rangle$ arises from the quadratic form $\mathtt{M}$:

$$\langle X,Y\rangle=\frac{1}{2}[\mathtt{M}(X+Y)-\mathtt{M}(X)-\mathtt{M}(Y)]$$

with

$$\mathtt{M}:\mathfrak{h}_2(\mathbb{K})\to\mathbb{R},\qquad \begin{pmatrix} s & \mathbf{x}\\ \bar{\mathbf{x}} & t\end{pmatrix}\mapsto\bar{\mathbf{x}}\mathbf{x}-st.$$

Let θ_r be the highest root of Φ_r. It follows from Corollary 9.1.6 that α_i, $1\leqslant i\leqslant r$, together with $\alpha_0=(-\theta_r,1)\in\mathbb{R}^{r+1}$ form a root basis of the affine extension of Φ_r. When considered in the context of $\mathfrak{h}_2(\mathbb{K})$, we obtain

$$A_0:=\begin{pmatrix} 1 & -\theta\\ -\bar{\theta} & 0\end{pmatrix}\text{ and } A_i=\begin{pmatrix} 0 & \alpha_i\\ \bar{\alpha}_i & 0\end{pmatrix},\ 1\leqslant i\leqslant r.$$

The over-extension node A_{-1} can be specified via solving

$$\langle A_{-1},A_0\rangle=-\frac{1}{2},\quad \langle A_{-1},A_k\rangle=0 \text{ for } 1\leqslant k\leqslant r.$$

It is easy to see that $A_{-1}=\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}$ is one solution to the above series of equations. Therefore, we obtain a root basis for the over-extension Φ_r^{++} :

$$A_{-1}=\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix},A_0=\begin{pmatrix} 1 & -\theta\\ -\bar{\theta} & 0\end{pmatrix},A_i=\begin{pmatrix} 0 & \alpha_i\\ \bar{\alpha}_i & 0\end{pmatrix},1\leqslant i\leqslant r. \tag{10.2}$$

Note that for simply-laced finite root systems, we may normalize the simple roots such that they all have unit norm and that the highest root θ_r equals 1. In this case, we have $\langle A_i, A_i\rangle = \mathtt{M}(A_i) = 1$ for all $i = -1, 0, 1, \ldots, r$. After the normalization, the norm in the root space of Φ_r^{++} would coincide with the standard norm $\|\cdot\|$ in the normed division algebra $\mathbb{K}$, as soon as we restrict the root lattice to the finite subalgebra. (For some twisted cases we may instead choose θ_r to be the lowest root, but we still assume that $\theta_r\bar{\theta}_r = 1$ [KNP+12].)

Next we consider the simple reflection associated with A_i. When $i = 1, \ldots, r$, it follows from Equation 10.1 that, given $X = \begin{pmatrix} s & \mathbf{x} \\ \bar{\mathbf{x}} & t \end{pmatrix} \in \mathfrak{h}_2(\mathbb{K})$,

$$\begin{aligned} s_{A_i}(X) &= \begin{pmatrix} s & -\dfrac{\alpha_i\bar{\mathbf{x}}\alpha_i}{\|\alpha_i\|^2} \\ -\dfrac{\bar{\alpha}_i\mathbf{x}\bar{\alpha}_i}{\|\bar{\alpha}_i\|^2} & t \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\alpha_i}{\|\alpha_i\|} & 0 \\ 0 & -\dfrac{\bar{\alpha}_i}{\|\bar{\alpha}_i\|} \end{pmatrix} \begin{pmatrix} s & \bar{\mathbf{x}} \\ \mathbf{x} & t \end{pmatrix} \begin{pmatrix} \dfrac{\bar{\alpha}_i}{\|\bar{\alpha}_i\|} & 0 \\ 0 & \dfrac{-\alpha_i}{\|\alpha_i\|} \end{pmatrix}. \end{aligned}$$

Meanwhile, for $s_{A_{-1}}$ we have

$$\mathtt{M}(A_{-1} + X) = \mathtt{M}\left(\begin{pmatrix} s-1 & \mathbf{x} \\ \bar{\mathbf{x}} & t+1 \end{pmatrix}\right) = \mathbf{x}\bar{\mathbf{x}} - (s-1)(t+1),$$

thus,

$$\mathtt{M}(A_{-1} + X) - \mathtt{M}(A_{-1}) - \mathtt{M}(X) = -s + t,$$

which implies

$$\begin{aligned} s_{A_{-1}}(X) &= X - \frac{2\langle A_{-1}, X\rangle}{\langle A_{-1}, A_{-1}\rangle} A_{-1} \\ &= X - (t-s)A_{-1} \\ &= \begin{pmatrix} t & \bar{\mathbf{x}} \\ \mathbf{x} & s \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s & \bar{\mathbf{x}} \\ \mathbf{x} & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} s_{A_0}(X) &= X - \frac{2\langle A_0, X\rangle}{\langle A_0, A_0\rangle} A_0 \\ &= \begin{pmatrix} -\dfrac{\theta}{\|\theta\|} & 1 \\ 0 & \dfrac{\bar{\theta}}{\|\bar{\theta}\|} \end{pmatrix} \begin{pmatrix} s & \bar{\mathbf{x}} \\ \mathbf{x} & t \end{pmatrix} \begin{pmatrix} -\dfrac{\bar{\theta}}{\|\bar{\theta}\|} & 0 \\ 1 & \dfrac{\theta}{\|\theta\|} \end{pmatrix}. \end{aligned}$$

Therefore, the simple reflections in $W(\Phi_n^{++})$ can be expressed as

$$s_I:\ X \mapsto M_I \bar{X} M_I^\dagger$$

with

$$M_{-1}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ M_0=\begin{pmatrix}-\dfrac{\theta}{\|\theta\|}&1\\0&\dfrac{\bar\theta}{\|\bar\theta\|}\end{pmatrix},\ M_I=\begin{pmatrix}\dfrac{\alpha_I}{\|\alpha_I\|}&0\\0&-\dfrac{\bar\alpha_I}{\|\bar\alpha_I\|}\end{pmatrix},\ 1\leqslant I\leqslant n.$$

Obviously, $(-M_I)\bar{X}(-M_I)^\dagger = M_I\bar{X}M_I^\dagger$. So we do not have to distinguish between M_I and $-M_I$. Therefore, it is sufficient to consider

$$s_I:\ X \mapsto M_I \bar{X} M_I^\dagger \tag{10.3}$$

with

$$M_{-1}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ M_0=\begin{bmatrix}-\dfrac{\theta}{\|\theta\|}&1\\0&\dfrac{\bar\theta}{\|\bar\theta\|}\end{bmatrix},\ M_I=\begin{bmatrix}\dfrac{\alpha_I}{\|\alpha_I\|}&0\\0&-\dfrac{\bar\alpha_I}{\|\bar\alpha_I\|}\end{bmatrix},\ 1\leqslant I\leqslant n.$$

Note that the products of matrices $M_I \in \mathrm{PSL}_2(\mathbb{K})$ are well-defined and associative for all the four normed division algebras over $\mathbb{R}$.

The formula 10.3 involves complex conjugation of X. Clearly, if we consider the even part of the Weyl group $W(\Phi_n^{++})$, which is denoted $W^+(\Phi_n^{++})$ and called *direct subgroup* of $W(\Phi_n^{++})$, the generators can be represented without complex conjugation. Specifically, $W^+(\Phi_n^{++})$ is an index 2 normal subgroup of $W(\Phi_n^{++})$ and consists of those elements which can be expressed as the product of an even number of simple reflections. There are more than one sets of generating elements. We will use the following list of generators:

$$s_{-1}s_i,\ i=0,1,\dots,n.$$

Especially, we have

$$\begin{aligned}
s_{-1}s_i(X)=s_{-1}\big(s_i(X)\big) &= s_{-1}\big(M_i\bar{X}M_i^\dagger\big)\\
&= M_{-1}\overline{M_i\bar{X}M_i^\dagger}M_{-1}^\dagger\\
&= M_{-1}(M_i^\top X\overline{M_i})M_{-1}^\dagger\\
&= (M_{-1}M_i^\top)X(M_{-1}M_i^\top)^\dagger
\end{aligned}$$

Let $S_I = M_{-1}M_I^\top$ for $I = 0, 1, \ldots, n$. Then we get

$$S_0 = \begin{bmatrix} 1 & \frac{\bar{\theta}}{\|\bar{\theta}\|} \\ -\frac{\theta}{\|\theta\|} & 0 \end{bmatrix}, \quad S_I = \begin{bmatrix} 0 & -\frac{\bar{\alpha}_I}{\|\bar{\alpha}_I\|} \\ \frac{\alpha_I}{\|\alpha_I\|} & 0 \end{bmatrix}, 1 \leqslant I \leqslant n.$$

Theorem 10.1.1 ([FKN09]).

$$W^+(\Phi_n^{++}) \cong \langle S_I \mid I = 0, \ldots, n \rangle \leqslant \mathrm{PSL}_2(\mathbb{K}).$$

10.2 $\mathbb{K} = \mathbb{R}$ and the A_1 root system

The finite root system A_1 has root lattice $\Lambda_{A_1} = \mathbb{Z}$. The over-extension of A_1 is the hyperbolic root system A_1^{++}, which is commonly denoted AE_3 and characterized by the following Coxeter diagram

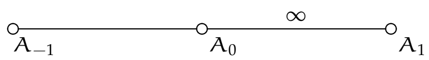


Figure 10.1: Coxeter diagram of AE_3

Inside $\Lambda_{A_1} = \mathbb{Z}$ the only simple root and the highest root are both equal to one, i.e., $\alpha = \theta = 1$. Hence, as a result in Equation 10.2, the simple roots of AE_3 are

$$A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \text{ and } A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In addition, we have

$$S_0 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus, as a consequence of Theorem 10.1.1, the direct subgroup of $W(AE_3)$

$$W^+(AE_3) \cong \langle \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rangle.$$

Note that

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}.$$

It follows that

$$W^+(AE_3) \cong \langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle = \mathrm{PSL}_2(\mathbb{Z}). \tag{10.4}$$

Especially, we have

$$W(AE_3) \cong \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle = \mathrm{PGL}_2(\mathbb{Z}).$$

Furthermore, consider the action of $\mathrm{PGL}_2(\mathbb{Z})$ on the upper half plane $\mathcal{H}(\mathbb{R})$ as defined in Equation 6.2. Clearly, the group $\mathrm{PGL}_2(\mathbb{Z})$ can then be identified as a discrete subgroup of $\mathtt{Isom}(\mathcal{H}(\mathbb{R}))$. In particular, $\mathcal{H}(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$ is a cusped complete hyperbolic 2-orbifold and its volume is minimal among all such cusped hyperbolic 2-orbifolds. Actually, this is a direct consequence of [Main Theorem; [Hil07]], which says that $\mathcal{H}^2/W(AE_3)$ is a cusped complete hyperbolic 2-orbifold whose volume is minimal among all such cusped hyperbolic 2-orbifolds.

10.3 $\mathbb{K} = \mathbb{C}$

For $\mathbb{K} = \mathbb{C}$ there are different choices of simple finite root systems which will be discussed separately.

10.3.1 Type A_2

The simple root system of type A_2 is simply-laced and has simple roots

$$\alpha_1 = 1, \quad \alpha_2 = \frac{-1 + \mathbf{i}\sqrt{3}}{2} = \omega.$$

The root lattice of A_2 is then $\Lambda_{A_2} = \mathbb{Z}[\omega] = \mathsf{E}$ and the highest root in Λ_{A_2} is $\theta = -\bar{\omega}$.

Over-extending the A_2 root system results in the hyperbolic root system A_2^{++}, whose simple roots in the root space $\mathfrak{h}_2(\mathbb{C})$ are given by

$$A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, A_0 = \begin{pmatrix} 1 & \bar{\omega} \\ \omega & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{pmatrix}.$$

Moreover, as a result of Theorem 10.1.1, the direct subgroup

$$W^+(A_2^{++}) \cong \langle S_0 = \begin{bmatrix} 1 & -\omega \\ \bar{\omega} & 0 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & -\bar{\omega} \\ \omega & 0 \end{bmatrix} \rangle. \tag{10.5}$$

Proposition 10.3.1.

$$W^+(A_2^{++}) \cong \mathrm{PSL}_2(\mathsf{E}).$$

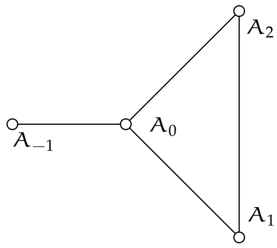


Figure 10.2: Coxeter diagram of A_2^{++} with numbering of nodes

Proof. As illustrated in Proposition 5.2, $\mathrm{PSL}_2(\mathsf{E})$ is generated by

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } U_\omega = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}.$$

Applying Equation 8.8 to $\mathbf{x} = \omega$ gives rise to

$$\begin{bmatrix} 0 & -\bar{\omega} \\ \omega & 0 \end{bmatrix} = \begin{bmatrix} 1 & \bar{\omega} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \bar{\omega} \\ 0 & 1 \end{bmatrix}.$$

As $\begin{bmatrix} 1 & \bar{\omega} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1-\omega \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}^{-1} = T^{-1}U_\omega^{-1}$, we then get

$$S_2 = T^{-1}U_\omega^{-1}SU_\omega ST^{-1}U_\omega^{-1} \in \mathrm{PSL}_2(\mathsf{E}).$$

Moreover, from

$$\begin{bmatrix} 1 & -\omega \\ \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\bar{\omega} \\ \omega & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}$$

we observe that

$$S_0 = U_\omega S S_2^{-1} S \in \mathrm{PSL}_2(\mathsf{E}).$$

Therefore, we obtain $\langle S_0, S_1, S_2 \rangle \leqslant \mathrm{PSL}_2(\mathsf{E})$.

Conversely, from $S_0 = U_\omega S S_2^{-1} S$, which has been shown above, we get

$$U_\omega = S_0 S_1 S_2 S_1.$$

Meanwhile, we have

$$\begin{bmatrix} 0 & -\bar{\omega} \\ \omega & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\omega \\ \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \bar{\omega} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\omega-\bar{\omega} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \bar{\omega} \\ 0 & 1 \end{bmatrix}^{-1}.$$

Thus,

$$T = U_{\omega}^{-1} S_1 S_0^{-1} S_1 S_2^{-1}.$$

Therefore, we see that $\mathrm{PSL}_2(\mathsf{E}) \leqslant \langle S_0, S_1, S_2 \rangle$, which leads to

$$\mathrm{PSL}_2(\mathsf{E}) = \langle S_0, S_1, S_\omega \rangle \cong W^+(A_2^{++}).$$

□

10.3.2 Type C_2

The C_2 root system is not simply laced; it has simple roots

$$\alpha_1 = \frac{1}{\sqrt{2}}, \text{ and } \alpha_2 = \frac{-1+\mathbf{i}}{\sqrt{2}}.$$

Note that $\|\alpha_1\| = \dfrac{1}{\sqrt{2}}$ while $\|\alpha_2\| = 1$. The highest root in the root lattice Λ_{C_2} is $\theta = \dfrac{1+\mathbf{i}}{\sqrt{2}}$. As a result of Equation 10.2, the associated hyperbolic over-extension C_2^{++} has simple roots

$$A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_0 = \begin{pmatrix} 1 & \frac{-1-\mathbf{i}}{\sqrt{2}} \\ \frac{-1+\mathbf{i}}{\sqrt{2}} & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & \frac{-1+\mathbf{i}}{\sqrt{2}} \\ \frac{-1-\mathbf{i}}{\sqrt{2}} & 0 \end{pmatrix},$$

which are labeled as follows

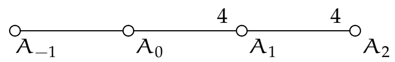


Figure 10.3: Coxeter diagram of C_2^{++}

As a consequence of Theorem 10.1.1, $W^+(C_2^{++})$, the direct subgroup of $W(C_2^{++})$, is isomorphic to a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ generated by

$$S_0 = \begin{bmatrix} 1 & \frac{1-\mathbf{i}}{\sqrt{2}} \\ \frac{-1-\mathbf{i}}{\sqrt{2}} & 0 \end{bmatrix}, \; S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{ and } S_2 = \begin{bmatrix} 0 & \frac{1+\mathbf{i}}{\sqrt{2}} \\ \frac{-1+\mathbf{i}}{\sqrt{2}} & 0 \end{bmatrix}.$$

The following proposition reveals the relation between $W^+(C_2^{++})$ and the Gaussian modular group.

Proposition 10.3.2 (Prop. 6;[FKN09]).

$$W^+(C_2^{++}) \cong \mathrm{PSL}_2(G) \rtimes 2.$$

Proof. Let $P = \begin{bmatrix} \bar{\theta}^{1/2} & 0 \\ 0 & \theta^{1/2} \end{bmatrix}$ whose inverse is $P^{-1} = \begin{bmatrix} \theta^{1/2} & 0 \\ 0 & \bar{\theta}^{1/2} \end{bmatrix}$, where $\theta = \frac{1+\mathbf{i}}{\sqrt{2}}$ is the highest root of C_2. Set $\tilde{S}_i = PS_iP^{-1}$, $i = 0,1,2$. Then

$$\begin{aligned}
\tilde{S}_0 &= \begin{bmatrix} \bar{\theta}^{1/2} & 0 \\ 0 & \theta^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \bar{\theta} \\ -\theta & 0 \end{bmatrix} \begin{bmatrix} \theta^{1/2} & 0 \\ 0 & \bar{\theta}^{1/2} \end{bmatrix} = \begin{bmatrix} 1 & \bar{\theta}^2 \\ -\theta^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}, \\
\tilde{S}_1 &= \begin{bmatrix} \bar{\theta}^{1/2} & 0 \\ 0 & \theta^{1/2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \theta^{1/2} & 0 \\ 0 & \bar{\theta}^{1/2} \end{bmatrix} = \begin{bmatrix} 0 & -\bar{\theta} \\ \theta & 0 \end{bmatrix}, \\
\tilde{S}_2 &= \begin{bmatrix} \bar{\theta}^{1/2} & 0 \\ 0 & \theta^{1/2} \end{bmatrix} \begin{bmatrix} 0 & \theta \\ -\bar{\theta} & 0 \end{bmatrix} \begin{bmatrix} \theta^{1/2} & 0 \\ 0 & \bar{\theta}^{1/2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\end{aligned}$$

Obviously, $W^+(C_2^{++}) \simeq \langle \tilde{S}_0, \tilde{S}_1, \tilde{S}_2 \rangle$. On the other hand, recall that

$$\mathrm{PSL}_2(G) = \langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, U_{\mathbf{i}} = \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix} \rangle.$$

Let $D = \begin{bmatrix} \theta & 0 \\ 0 & \bar{\theta} \end{bmatrix}$. We have

$$\begin{cases} \tilde{S}_0 = D^{-1}T^{-1}D^{-1}S,\ \tilde{S}_1 = SD,\ \tilde{S}_2 = S, \\ S = \tilde{S}_2,\ D = \tilde{S}_2\tilde{S}_1,\ T = \tilde{S}_1^{-1}\tilde{S}_0^{-1}\tilde{S}_1^{-1}\tilde{S}_2,\ U_{\mathbf{i}} = \tilde{S}_2\tilde{S}_1\tilde{S}_0\tilde{S}_1\tilde{S}_2\tilde{S}_0^{-1}\tilde{S}_1^{-1}\tilde{S}_2\tilde{S}_1^{-1}\tilde{S}_2, \end{cases}$$

which imply

$$\langle \tilde{S}_0, \tilde{S}_1, \tilde{S}_2 \rangle = \langle S, T, U_{\mathbf{i}}, D \rangle.$$

We claim that $\langle S, T, U_{\mathbf{i}}, D \rangle$ is an index 2 extension of $\langle S, T, U_{\mathbf{i}} \rangle$, which would indicate that $W^+(C_2^{++}) \cong \mathrm{PSL}_2(G) \rtimes 2$.

In fact, note that $\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix} \in \mathrm{PSL}_2(G) = \langle S, T, U_{\mathbf{i}} \rangle$, which implies $\begin{bmatrix} \mathbf{i} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \in \langle S, T, U_{\mathbf{i}} \rangle$. Thus, we obtain

$$D^2 = \begin{bmatrix} \theta^2 & 0 \\ 0 & \bar{\theta}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \in \langle S, T, U_{\mathbf{i}} \rangle.$$

□

We know that the B_2 root system is the same as the C_2 root system. However, the over-extension constructed from $B_2 \simeq C_2$ is C_2^{++} rather than B_2^{++}. In fact, we can over-extend the B_2 root system and obtain the twisted affine root system $D_2^{(2)+}$, as explained in Appendix A.1 in [FKN09]. In particular, we have

$$W^+(D_2^{(2)+}) \cong W^+(B_2^{++}) \cong \mathrm{PSL}_2(\mathsf{G}) \rtimes 2.$$

10.3.3 Type G_2

The G_2 root system is not simply laced; it has simple roots:

$$\alpha_1 = 1, \quad \alpha_2 = \frac{-\sqrt{3} + \mathbf{i}}{2\sqrt{3}}.$$

The highest root in Λ_{G_2} is

$$\theta = \frac{1 + \sqrt{3}\mathbf{i}}{2} = -\bar{\omega}.$$

By over-extending G_2 we obtain the hyperbolic root system of G_2^{++}.

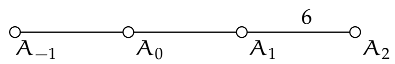


Figure 10.4: Coxeter diagram of G_2^{++} with numbering of nodes

G_2^{++} has the following simple roots in the root space $\mathfrak{h}_2(\mathbb{C})$:

$$A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_0 = \begin{pmatrix} 1 & \bar{\omega} \\ \omega & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & -\mathbf{i}\bar{\omega} \\ \mathbf{i}\omega & 0 \end{pmatrix}.$$

Moreover, the direct subgroup

$$W^+(G_2^{++}) \cong \langle S_0 = \begin{bmatrix} 1 & -\omega \\ \bar{\omega} & 0 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & \mathbf{i}\omega \\ \mathbf{i}\bar{\omega} & 0 \end{bmatrix} \rangle.$$

Proposition 10.3.3.

$$W^+(G_2^{++}) \cong \mathrm{PSL}_2(\mathsf{E}) \rtimes 2.$$

Proof. It is easy to see that $(S_1S_2)^2S_1 = \begin{bmatrix} 0 & -\omega \\ \bar{\omega} & 0 \end{bmatrix}$, which, together with Equation 10.5, implies that

$$W^+(A_2^{++}) \cong \langle S_0, S_1, (S_1S_2)^2S_1 \rangle.$$

In addition, since we have proved that $W^+(A_2^{++}) \cong \mathrm{PSL}_2(\mathsf{E})$, we obtain

$$\mathrm{PSL}_2(\mathsf{E}) = \langle S_0, S_1, (S_1S_2)^2S_1\rangle.$$

Since S_2 is a reflection obeying $S_2^2 = \mathbf{1}$, in order to prove that $W^+(G_2^{++}) \cong \mathrm{PSL}_2(\mathsf{E}) \rtimes 2$, it suffices to show that

$$\langle S_0, S_1, S_2\rangle = \langle S_0, S_1, (S_1S_2)^2S_1\rangle \rtimes \langle S_2\rangle.$$

This is true because of the exchange condition of Coxeter groups and the fact that the conjugation action of S_2 on $\langle S_0, S_1, (S_1S_2)^2S_1\rangle$ is a group automorphism. □

Furthermore, under the action given in Equation 6.2, the group $\mathrm{PSL}_2(\mathsf{E})$ can be identified as a discrete subgroup of $\mathtt{Isom}\big(\mathcal{H}(\mathbb{C})\big)$. Then following from the isomorphism in Proposition 10.3.3, the Weyl group $W(G_2^{++})$ can also be viewed as a subgroup of $\mathtt{Isom}\big(\mathcal{H}(\mathbb{C})\big)$. In particular, as illustrated in [Main Theorem; [Hil07]], $\mathcal{H}(\mathbb{C})/W(G_2^{++})$ is a cusped complete hyperbolic 3-orbifold whose volume is minimal among all such cusped hyperbolic 3-orbifolds.

10.4 $\mathbb{K} = \mathbb{H}$

Within the normed division algebra of quaternions one can find the root systems of types A_4, B_4, C_4, D_4 and F_4. However, since the hyperbolic Weyl group of A_4^{++} cannot be characterized by any of the previously defined modular groups, we will only focus on the other four finite root systems. The quaternionic case is more subtle than the commutative cases because the criterion for selecting the matrix group to which the even Weyl group belongs cannot be so easily done via a determinant.

10.4.1 Type D_4

The D_4 root system is simply-laced and has simple roots [FKN09]

$$\begin{cases} \alpha_1 = 1, \quad \alpha_2 = \frac{1}{2}(-1+\mathbf{i}-\mathbf{j}-\mathbf{k}) = \mathbf{i}-\mathbf{h}, \\ \alpha_3 = \frac{1}{2}(-1-\mathbf{i}+\mathbf{j}-\mathbf{k}) = \mathbf{j}-\mathbf{h},\ \alpha_4 = \frac{1}{2}(-1-\mathbf{i}-\mathbf{j}+\mathbf{k}) = \mathbf{k}-\mathbf{h}, \end{cases}$$

where $\mathbf{h} = \frac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k})$. The highest root in the root lattice Λ_{D_4} is

$$\theta = 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1 - \mathbf{h} = \bar{\mathbf{h}}.$$

The over-extension of D_4 is D_4^{++}, which is hyperbolic and has Coxeter diagram

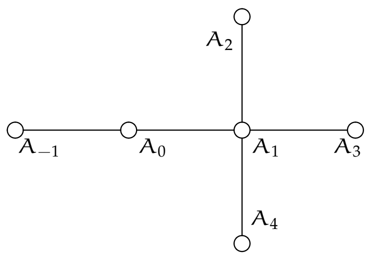


Figure 10.5: Coxeter diagram of D_4 with numbering of nodes

The simple roots of D_4^{++} are given by

$$A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ A_0 = \begin{pmatrix} 1 & -\bar{\mathbf{h}} \\ -\mathbf{h} & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & \mathbf{i}-\mathbf{h} \\ -\mathbf{i}-\bar{\mathbf{h}} & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & \mathbf{j}-\mathbf{h} \\ -\mathbf{j}-\bar{\mathbf{h}} & 0 \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & \mathbf{k}-\mathbf{h} \\ -\mathbf{k}-\bar{\mathbf{h}} & 0 \end{pmatrix}.$$

The even hyperbolic Weyl group $W^+(D_4^{++})$ is isomorphic to a discrete subgroup of $\mathrm{PSL}_2(\mathrm{H})$ generated by the following matrices

$$\left\{ \begin{array}{l} S_0 = \begin{bmatrix} 1 & \mathbf{h} \\ -\bar{\mathbf{h}} & 0 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ S_2 = \begin{bmatrix} 0 & \mathbf{i}+\bar{\mathbf{h}} \\ \mathbf{i}-\mathbf{h} & 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 & \mathbf{j}+\bar{\mathbf{h}} \\ \mathbf{j}-\mathbf{h} & 0 \end{bmatrix}, S_4 = \begin{bmatrix} 0 & \mathbf{k}+\bar{\mathbf{h}} \\ \mathbf{k}-\mathbf{h} & 0 \end{bmatrix}. \end{array} \right. \tag{10.6}$$

Proposition 10.4.1.

$$W^+(D_4^{++}) \cong \mathrm{PSL}_2^*(\mathrm{H}).$$

Proof. Recall from Equation 5.4 that the Hurwitzian modular group $\mathrm{PSL}_2^*(\mathrm{H})$ is generated by

$$\left\{ \begin{array}{l} S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ U_{\mathbf{i}} = \begin{bmatrix} 1 & \mathbf{i} \\ 0 & 1 \end{bmatrix},\ U_{\mathbf{j}} = \begin{bmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{bmatrix},\ U_{\mathbf{k}} = \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix}. \end{array} \right.$$

Applying Equation 8.8 to $\mathbf{h}$ gives rise to

$$\begin{bmatrix} \mathbf{h} & 0 \\ 0 & \bar{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{h} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which implies

$$\begin{bmatrix} \mathbf{h} & 0 \\ 0 & \bar{\mathbf{h}} \end{bmatrix} \in \mathrm{PSL}_2^*(\mathsf{H}).$$

Thus,

$$\begin{aligned} S_0 = \begin{bmatrix} 1 & \mathbf{h} \\ -\bar{\mathbf{h}} & 0 \end{bmatrix} &= \begin{bmatrix} 1 & -\mathbf{h} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{h}} & 0 \\ 0 & \mathbf{h} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{h} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{h} & 0 \\ 0 & \bar{\mathbf{h}} \end{bmatrix}^{-1} \in \mathrm{PSL}_2^*(\mathsf{H}). \end{aligned}$$

Similarly, by applying Equation 8.8 to $\mathbf{i}-\mathbf{h}$ we obtain $\begin{bmatrix} \mathbf{i}-\mathbf{h} & 0 \\ 0 & -\mathbf{i}-\bar{\mathbf{h}} \end{bmatrix} \in \mathrm{PSL}_2^*(\mathsf{H})$.
Then

$$S_2 = \begin{bmatrix} 0 & \mathbf{i}+\bar{\mathbf{h}} \\ \mathbf{i}-\mathbf{h} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i}-\mathbf{h} & 0 \\ 0 & -\mathbf{i}-\bar{\mathbf{h}} \end{bmatrix} \in \mathrm{PSL}_2^*(\mathsf{H}).$$

Analogously, we can show that $S_3, S_4 \in \mathrm{PSL}_2^*(\mathsf{H})$. Consequently, we get

$$\langle S_I \rangle \triangleq \langle S_i \mid i = 0, 1, \ldots, 4 \rangle \leqslant \mathrm{PSL}_2^*(\mathsf{H}).$$

Conversely, one may explicitly calculate that

$$S_2S_3 = \begin{bmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, \; S_3S_4 = \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{j} \end{bmatrix}, \; S_4S_2 = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{k} \end{bmatrix},$$

which indicates that

$$D_{\mathbf{i}} := \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{bmatrix} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix} = (S_4S_2)(S_1S_2S_3S_1) \in \langle S_I \rangle.$$

Similarly, we have $D_{\mathbf{j}} := \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{j} \end{bmatrix}, D_{\mathbf{k}} := \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{k} \end{bmatrix} \in \langle S_I \rangle$.

Moreover, from

$$S_0S_2 = \begin{bmatrix} 1 & \mathbf{h} \\ -\bar{\mathbf{h}} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{i}+\bar{\mathbf{h}} \\ \mathbf{i}-\mathbf{h} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{h}\mathbf{i}-\mathbf{h}^2 & \mathbf{i}+\bar{\mathbf{h}} \\ 0 & -\bar{\mathbf{h}}\mathbf{i}-\bar{\mathbf{h}}^2 \end{bmatrix} = \begin{bmatrix} -\mathbf{k} & \mathbf{i}+\bar{\mathbf{h}} \\ 0 & \mathbf{j} \end{bmatrix}$$

we get

$$\begin{bmatrix} 1 & \mathbf{i}+\bar{\mathbf{h}} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\mathbf{j} \end{bmatrix} S_0S_2 \begin{bmatrix} \mathbf{k} & 0 \\ 0 & 1 \end{bmatrix} = D_{\mathbf{j}}^{-1} S_0S_2S_1D_{\mathbf{k}}S_1 \in \langle S_I \rangle.$$

Note that $(\mathbf{i}+\bar{\mathbf{h}})^3 = -1$. One may explicitly compute that

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = S_2 S_1 \begin{bmatrix} 1 & -(\mathbf{i}+\bar{\mathbf{h}}) \\ 0 & 1 \end{bmatrix} S_1 S_2 \in \langle S_I \rangle.$$

Additionally, $U_{\mathbf{i}} = D_{\mathbf{i}}^{-1} T D_{\mathbf{i}}$, $U_{\mathbf{j}} = D_{\mathbf{j}}^{-1} T D_{\mathbf{j}}$, and $U_{\mathbf{k}} = D_{\mathbf{k}}^{-1} T D_{\mathbf{k}}$ all belong to $\langle S_I \rangle$. Meanwhile, we have $U_{\mathbf{h}} = \begin{bmatrix} 1 & \mathbf{i}+\bar{\mathbf{h}} \\ 0 & 1 \end{bmatrix} U_{\mathbf{j}} U_{\mathbf{k}} \in \langle S_I \rangle$. Thus,

$$\mathrm{PSL}_2^*(\mathsf{H}) \leqslant \langle S_I \rangle,$$

which completes the proof. □

It is worth mentioning that the even Weyl group $W^+(D_4^{++})$ is isomorphic to $\mathrm{PSL}_2^*(\mathsf{H})$ rather than $\mathrm{PSL}_2(\mathsf{H})$. Since the even cyclic permutation of (e_1, e_2, e_3) is realized by conjugation by the Hurwitzian unit θ, the associated diagonal matrix $\begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \in \mathrm{PSL}_2(\mathsf{H})$ solves the necessary symmetry requirements, but itself is not part of the Weyl group, and therefore has to be removed from $\mathrm{PSL}_2(\mathsf{H})$ [FKN09].

10.4.2 Type B_4

The B_4 root lattice is isomorphic to L, the hypercubic lattice of the Lipschitzian integers. The simple roots can be chosen as follows:

$$\begin{cases} \alpha_1 = 1, \quad \alpha_2 = \frac{1}{2}(-1+\mathbf{i}-\mathbf{j}-\mathbf{k}) = \mathbf{i}-\mathbf{h}, \\ \alpha_3 = \frac{1}{2}(-1-\mathbf{i}+\mathbf{j}-\mathbf{k}) = \mathbf{j}-\mathbf{h}, \quad \alpha_4 = \frac{-\mathbf{j}+\mathbf{k}}{2}. \end{cases}$$

Note that $\|\alpha_4\| = \frac{1}{\sqrt{2}}$. The highest root of B_4 is

$$\theta = 2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = \frac{1}{2}(1-\mathbf{i}-\mathbf{j}-\mathbf{k}) = \bar{\mathbf{h}}.$$

The hyperbolic over-extension B_4^{++} has the following simple roots:

$$\begin{cases} A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & -\bar{\mathbf{h}} \\ -\mathbf{h} & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ A_2 = \begin{pmatrix} 0 & \mathbf{i}-\mathbf{h} \\ -\mathbf{i}-\bar{\mathbf{h}} & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & \mathbf{j}-\mathbf{h} \\ -\mathbf{j}-\bar{\mathbf{h}} & 0 \end{pmatrix}, A_4 = \begin{bmatrix} 0 & \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} \\ \frac{\mathbf{j}-\mathbf{k}}{\sqrt{2}} & 0 \end{bmatrix}, \end{cases}$$

which are labeled as in the following graph.

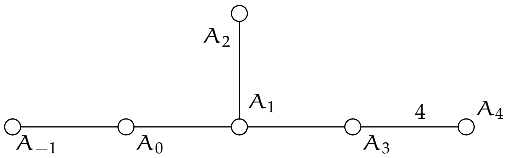


Figure 10.6: Coxeter diagram of B_4^{++}

The even Weyl group $W^+(B_4^{++})$ is isomorphic to the discrete subgroup of $\mathrm{PSL}_2(\mathbb{H})$ generated by

$$\begin{cases} S_0 = \begin{bmatrix} 1 & \mathbf{h} \\ -\bar{\mathbf{h}} & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ S_2 = \begin{bmatrix} 0 & \mathbf{i}+\bar{\mathbf{h}} \\ \mathbf{i}-\mathbf{h} & 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 & \mathbf{j}+\bar{\mathbf{h}} \\ \mathbf{j}-\mathbf{h} & 0 \end{bmatrix}, S_4 = \begin{bmatrix} 0 & \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} \\ \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} & 0 \end{bmatrix}. \end{cases}$$

Proposition 10.4.2 (Proposition 10;[FKN09]).

$$W^+(B_4^{++}) \cong \mathrm{PSL}_2^*(\mathsf{H}) \rtimes 2.$$

10.4.3 Type C_4

Even though the Weyl groups of B_4 and C_4 are isomorphic, the hyperbolic Weyl groups of B_4^{++} and C_4^{++} are nevertheless different. As explained in [FKN09], this difference is reflected only in the difference between the highest roots; for C_4 we have an octahedral unit of order four whereas for B_4 the highest root is a Hurwitz number of order six. Explicitly, the simple roots of C_4 are

$$\alpha_1 = \frac{1}{\sqrt{2}},\ \alpha_2 = \frac{-1+\mathbf{i}-\mathbf{j}-\mathbf{k}}{2\sqrt{2}},\ \alpha_3 = \frac{-1-\mathbf{i}+\mathbf{j}-\mathbf{k}}{2\sqrt{2}},\ \alpha_4 = \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}}.$$

Note that $\|\alpha_1\| = \|\alpha_2\| = \|\alpha_3\| = \frac{1}{\sqrt{2}}$ while $\|\alpha_4\| = 1$. The highest root in Λ_{C_4} is given as

$$\theta = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = \frac{-\mathbf{j}-\mathbf{k}}{\sqrt{2}}.$$

The over-extension C_4^{++} is hyperbolic and has the following Coxeter diagram.

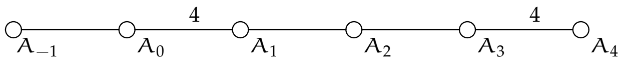


Figure 10.7: Coxeter diagram of C_4^{++}

where the simple roots are given by

$$\begin{cases} A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}} \\ \frac{-\mathbf{j}-\mathbf{k}}{\sqrt{2}} & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ A_2 = \begin{pmatrix} 0 & \mathbf{i}-\mathbf{h} \\ -\mathbf{i}-\bar{\mathbf{h}} & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & \mathbf{j}-\mathbf{h} \\ -\mathbf{j}-\bar{\mathbf{h}} & 0 \end{pmatrix}, A_4 = \begin{bmatrix} 0 & \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} \\ \frac{\mathbf{j}-\mathbf{k}}{\sqrt{2}} & 0 \end{bmatrix}. \end{cases}$$

The even Weyl group $W^+(C_4^{++})$ is then isomorphic to the discrete subgroup of $\mathrm{PSL}_2(\mathbb{H})$ generated by

$$\begin{cases} S_0 = \begin{bmatrix} 1 & \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}} \\ \frac{-\mathbf{j}-\mathbf{k}}{\sqrt{2}} & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ S_2 = \begin{bmatrix} 0 & \mathbf{i}+\bar{\mathbf{h}} \\ \mathbf{i}-\mathbf{h} & 0 \end{bmatrix}, S_3 = \begin{bmatrix} 0 & \mathbf{j}+\bar{\mathbf{h}} \\ \mathbf{j}-\mathbf{h} & 0 \end{bmatrix}, S_4 = \begin{bmatrix} 0 & \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} \\ \frac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} & 0 \end{bmatrix}. \end{cases}$$

Obviously, the generators S_I, $I = 1, \ldots, 4$, are identical to those of B_4^{++}. In fact, we have

Proposition 10.4.3 (Proposition 11; [FKN09]).

$$W^+(C_4^{++}) \cong \widetilde{\mathrm{PSL}}_2^*(\mathsf{H}) \rtimes 2.$$

where

$$\widetilde{\mathrm{PSL}}_2^*(\mathsf{H}) = \{ \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} X \begin{bmatrix} \bar{\theta} & 0 \\ 0 & 1 \end{bmatrix} \mid X \in \mathrm{PSL}_2^*(\mathsf{H}) \}$$

is the unitary transformation of $\mathrm{PSL}_2^*(\mathsf{H})$ *determined by the highest root* θ *of* C_4.

10.4.4 Type F_4

The root system of F_4 consists of two copies of the D_4, one of which is rescaled. It has the following simple roots:

$$\alpha_1 = 1,\ \alpha_2 = \frac{1}{2}(-1+\mathbf{i}-\mathbf{j}-\mathbf{k}),\ \alpha_3 = \frac{-\mathbf{i}+\mathbf{j}}{2},\ \alpha_4 = \frac{-\mathbf{j}+\mathbf{k}}{2},$$

where $\|\alpha_1\| = \|\alpha_2\| = 1$ while $\|\alpha_3\| = \|\alpha_4\| = \dfrac{1}{\sqrt{2}}$. The highest root of F_4 is

$$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}) = \bar{\mathbf{h}}.$$

The Coxeter diagram for the over-extension F_4^{++} is

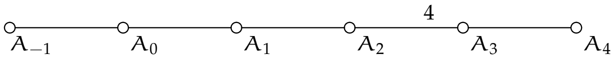


Figure 10.8: Coxeter diagram of F_4^{++}

Following from Theorem 10.1.1, the even Weyl group $W^+(F_4^{++}) \cong \langle S_I \mid I = 0, 1, \ldots, 4\rangle$ with

$$\begin{cases} S_0 = \begin{bmatrix} 1 & \mathbf{h} \\ -\bar{\mathbf{h}} & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ S_2 = \begin{bmatrix} 0 & \mathbf{i} + \bar{\mathbf{h}} \\ \mathbf{i} - \mathbf{h} & 0 \end{bmatrix}, \; S_3 = \begin{bmatrix} 0 & \dfrac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}} \\ \dfrac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}} & 0 \end{bmatrix}, \; S_4 = \begin{bmatrix} 0 & \dfrac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} \\ \dfrac{-\mathbf{j}+\mathbf{k}}{\sqrt{2}} & 0 \end{bmatrix}. \end{cases}$$

Proposition 10.4.4 (Prop. 12;[FKN09]).

$$W^+(F_4^{++}) \cong PSL_2^*(H) \rtimes \mathcal{S}_3 \cong PSL_2(H) \rtimes 2,$$

where $\mathcal{S}_3$ is the symmetric group on three letters.

Proof. First, we explicitly compute that

$$S_3S_2S_3 = \begin{bmatrix} 0 & \mathbf{j} + \bar{\mathbf{h}} \\ \mathbf{j} - \mathbf{h} & 0 \end{bmatrix}, \; S_4S_3S_2S_3S_4 = \begin{bmatrix} 0 & \mathbf{k} + \bar{\mathbf{h}} \\ \mathbf{k} - \mathbf{h} & 0 \end{bmatrix}.$$

Obviously, it results from Equation 10.6 that

$$W^+(D_4^{++}) \cong \langle S_0, S_1, S_2, S_3S_2S_3, S_4S_3S_2S_3S_4\rangle.$$

Recall that $W^+(D_4^{++}) \cong PSL_2^*(H)$, thus,

$$PSL_2^*(H) \cong \langle S_0, S_1, S_2, S_3S_2S_3, S_4S_3S_2S_3S_4\rangle.$$

On the other hand, it is clear that

$$W^+(F_4^{++}) \cong \langle S_0, S_1, S_2, S_3S_2S_3, S_4S_3S_2S_3S_4, S_3, S_4\rangle.$$

Notice that $\langle S_3, S_4\rangle \cong \mathcal{S}_3$. Thus, it suffices to prove that

$$\langle S_0, S_1, S_2, S_3S_2S_3, S_4S_3S_2S_3S_4, S_3, S_4\rangle \cong \langle S_0, S_1, S_2, S_3S_2S_3, S_4S_3S_2S_3S_4\rangle \rtimes \langle S_3, S_4\rangle,$$

which can be verified directly. □

Furthermore, considering the action given in Equation 6.2 we may view $W(\mathsf{F}_4^{++})$ as a discrete subgroup of $\mathtt{Isom}\big(\mathcal{H}(\mathbb{H})\big)$. Especially, the quotient $\mathcal{H}(\mathbb{H})/W(\mathsf{F}_4^{++})$ is a cusped complete hyperbolic 5-orbifold whose volume is minimal among all such cusped hyperbolic 5-orbifolds [Hil07].

10.5 $\mathbb{K} = \mathbb{O}$

10.5.1 Type E_8

The root system of type E_8 is simply-laced, whose root lattice is the only non-trivial positive-definite, even, unimodular lattice of rank 8. Specifically, Λ_{E_8} consists of point in $\mathbb{R}^8$ that satisfies

- the coordinates are either all integers or all half-integers;
- the sum of the eight coordinates is an even integer.

Direct calculations show that Λ_{E_8} is isometric to O, the integral lattice of octaves. Recall that $\mathsf{O} = \bigoplus_{i=1}^{8} \mathbb{Z}\epsilon_i$ with

$$\begin{aligned}
\epsilon_1 &= \frac{1}{2}(1 - e_1 - e_5 - e_6), & \epsilon_2 &= e_1,\\
\epsilon_3 &= \frac{1}{2}(-e_1 - e_2 + e_6 + e_7), & \epsilon_4 &= e_2,\\
\epsilon_5 &= \frac{1}{2}(-e_2 - e_3 - e_4 - e_7), & \epsilon_6 &= e_3,\\
\epsilon_7 &= \frac{1}{2}(-e_3 + e_5 - e_6 + e_7), & \epsilon_8 &= e_4.
\end{aligned}$$

Hence, the vectors $\{\epsilon_i\}_{i=1}^{8}$ can be viewed as simple roots of E_8, which are labeled below in the same way as in [KMW].

With respect to the root basis $\{\epsilon_i\}_{i=1}^{8}$, the highest root is then equal to

$$\theta = 2\epsilon_1 + 3\epsilon_2 + 4\epsilon_3 + 5\epsilon_4 + 6\epsilon_5 + 4\epsilon_6 + 2\epsilon_7 + 3\epsilon_8 = 1.$$

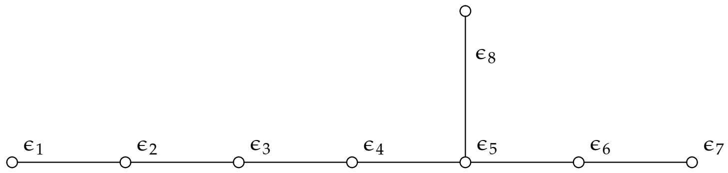


Figure 10.9: Coxeter diagram of E_8 with numbering of nodes

The over-extension of E_8 root system is the hyperbolic root system E_{10}, whose root lattice is also even and unimodular. E_{10} has the following root basis:

$$A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ A_0 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix},\ A_i = \begin{pmatrix} 0 & \epsilon_i \\ \bar{\epsilon}_i & 0 \end{pmatrix},\ i = 1, \dots, 8,$$

which are labeled as in the graph below.

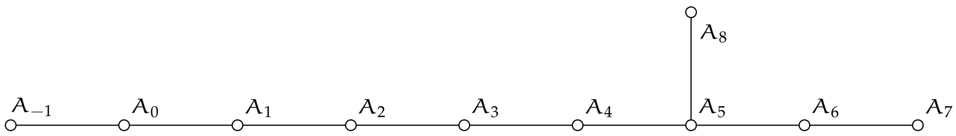


Figure 10.10: Coxeter diagram of E_{10} with numbering of nodes

Moreover, from Theorem 10.1.1 we obtain

$$W^+(E_{10}) \cong \langle S_I : I = 0, 1, \dots, 8 \rangle,$$

where

$$S_0 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},\ S_I = \begin{bmatrix} 0 & -\bar{\epsilon}_I \\ \epsilon_I & 0 \end{bmatrix} \text{ for } I = 1, \dots, 8.$$

Regarding the even Weyl group of E_{10}, we have the following theorem.

Theorem 10.5.1.

$$W^+(E_{10}) \cong \mathrm{PSL}_2^*(\mathbb{O})$$

Proof. Write $K := \langle S_I : I = 0, 1, \dots, 8 \rangle \leqslant \mathrm{PSL}_2(\mathbb{O})$. Then it suffices to prove $K \cong \mathrm{PSL}_2^*(\mathbb{O})$. We have already shown in Equation 5.6 that the Octavian modular group

$$\mathrm{PSL}_2^*(\mathbb{O}) = \langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, U_{\epsilon_I} = \begin{bmatrix} 1 & \epsilon_I \\ 0 & 1 \end{bmatrix},\ I = 1, \dots, 8 \rangle.$$

First, the fact $\theta = 2\epsilon_1 + 3\epsilon_2 + 4\epsilon_3 + 5\epsilon_4 + 6\epsilon_5 + 4\epsilon_6 + 2\epsilon_7 + 3\epsilon_8 = 1$ implies

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix} = U_{\epsilon_1}^2 U_{\epsilon_2}^3 U_{\epsilon_3}^4 U_{\epsilon_4}^5 U_{\epsilon_5}^6 U_{\epsilon_6}^4 U_{\epsilon_7}^2 U_{\epsilon_8}^2 \in \mathrm{PSL}_2^*(\mathbb{O}).$$

Then

$$\begin{cases} \begin{bmatrix} 1 & \bar{\epsilon}_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1-\epsilon_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon_1 \\ 0 & 1 \end{bmatrix}^{-1} \in \mathrm{PSL}_2^*(O) \\ \begin{bmatrix} 1 & \bar{\epsilon}_I \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon_I \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} S1 & \epsilon_I \\ 0 & 1 \end{bmatrix}^{-1} \in \mathrm{PSL}_2^*(O) \text{ for } I \geqslant 2. \end{cases}$$

Thus, applying Equation 8.8 to ϵ_I shows that for all $I = 1, \dots, 8$,

$$\begin{bmatrix} 0 & -\epsilon_I \\ \bar{\epsilon}_I & 0 \end{bmatrix} = \begin{bmatrix} 1 & \epsilon_I \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \bar{\epsilon}_I \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \epsilon_I \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}_2^*(O).$$

Hence, we obtain

$$\begin{cases} S_0 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{PSL}_2^*(O), \\ S_I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\epsilon_I \\ \bar{\epsilon}_I & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{PSL}_2^*(O), \end{cases}$$

and thus, $K \leqslant \mathrm{PSL}_2^*(O)$.

Conversely, let $s_\theta \in W(E_8)$ denote the reflection on the hyperplane H_θ that is orthogonal to the highest root θ. It follows from Equation 10.1 that $s_\theta(\mathbf{z}) = -\bar{\mathbf{z}}$. Then in the over-extension E_{10}, as displayed in Equation 10.3, it corresponds to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Especially, the element $s_{-1}s_\theta \in W^+(E_{10})$ corresponds to the matrix

$$S_\theta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which implies that $S = S_\theta \in K$.

It is known that E_9 is the affine extension of the E_8 root system. Following from Corollary 9.1.6 we get

$$W(E_9) = \Gamma \rtimes W(E_8)$$

with

$$\Gamma = \langle t_\alpha \mid \alpha \in \Lambda_{E_8} \rangle \equiv \langle t_{\epsilon_i} \mid i = 1, \dots, 8 \rangle \cong \mathbb{Z}^8.$$

In particular, it is clear that

$$W^+(E_9) = \Gamma \rtimes W^+(E_8).$$

Note that $\langle U_{\epsilon_I} \mid I = 1, \dots, 8\rangle$ is a free abelian group of rank 8. Thus, $\langle U_{\epsilon_I}\rangle \cong \Gamma$ can be considered as a normal subgroup of $W^+(E_9)$, and hence a subgroup of $K \cong W^+(E_{10})$. As we have already shown that $S \in K$, we thus obtain

$$\mathrm{PSL}_2^*(\mathcal{O}) = \langle U_{\epsilon_I}, S\rangle \leqslant K,$$

which completes the proof. □

Consider the action of $\mathrm{PSL}_2^*(\mathcal{O})$ on $\mathcal{H}(\mathbb{O})$ given in Equation 6.2. It is clear that $\mathrm{PSL}_2^*(\mathcal{O}) \leqslant \mathtt{Isom}\big(\mathcal{H}(\mathbb{O})\big)$. Then as a result of Theorem 10.5.1, the group $W^+(E_{10})$ can also be viewed as a subgroup of $\mathtt{Isom}\big(\mathcal{H}(\mathbb{O})\big)$. Moreover, we define

$$\mathrm{PGL}_2^*(\mathcal{O}) = \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon_I \\ 0 & 1 \end{bmatrix}, \ I = 1, \dots, 8\rangle.$$

Then it is clear that $W(E_{10}) \cong \mathrm{PGL}_2^*(\mathcal{O}) \leqslant \mathtt{Isom}\big(\mathcal{H}(\mathbb{O})\big)$. In particular, it has been shown in [Main Theorem; [Hil07]] that $\mathcal{H}(\mathbb{O})/W(E_{10}) \cong \mathcal{H}^9/\mathrm{PGL}_2^*(\mathcal{O})$ is a cusped complete hyperbolic 9-orbifold whose volume is minimal among all such cusped hyperbolic 9-orbifolds.

10.5.2 Type D_8 and B_8

The simple roots of D_8 are given by

$$\begin{aligned}
\epsilon_1 &= e_3, & \epsilon_2 &= \frac{1}{2}(-e_1 - e_2 - e_3 + e_4),\\
\epsilon_3 &= e_1, & \epsilon_4 &= \frac{1}{2}(-1 - e_1 - e_4 + e_5),\\
\epsilon_5 &= 1, & \epsilon_6 &= \frac{1}{2}(-1 - e_5 - e_6 - e_7),\\
\epsilon_7 &= \frac{1}{2}(e_2 - e_3 + e_6 - e_7), & \epsilon_8 &= \frac{1}{2}(-1 + e_2 + e_4 + e_7).
\end{aligned}$$

The highest root of D_8 is

$$\theta_D = 2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 + 2\epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8 = \frac{1}{2}(e_3 + e_4 + e_5 - e_7).$$

The simple roots of B_8 are

$$\begin{aligned}
\epsilon_1 &= e_3, & \epsilon_2 &= \frac{1}{2}(-e_1 - e_2 - e_3 + e_4),\\
\epsilon_3 &= e_1, & \epsilon_4 &= \frac{1}{2}(-1 - e_1 - e_4 + e_5),
\end{aligned}$$

$$\epsilon_5 = 1, \qquad \epsilon_6 = \frac{1}{2}(-1 - e_5 - e_6 - e_7),$$
$$\epsilon_7 = \frac{1}{2}(e_2 - e_3 + e_6 - e_7), \qquad \epsilon_8 = \frac{1}{4}(e_2 + e_4 + e_5 + e_6 + 2e_7).$$

Note that ϵ_8 is no longer an octave, in agreement with the fact that B_8 is not a subalgebra of E_8. The highest root of B_8 is

$$\theta_B = 2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 + 2\epsilon_5 + 2\epsilon_6 + \epsilon_7 + 2\epsilon_8 = \frac{1}{2}(e_3 + e_4 + e_5 - e_7).$$

It follows [FKN09] that $W^+(D_8^{++})$ is an index 135 subgroup of $PSL_2(\mathsf{O})$ and $W^+(B_8^{++}) \simeq W^+(D_8^{++}) \rtimes 2$. Besides this, we know very little about the hyperbolic Weyl groups of D_8^{++} and B_8^{++}.

Bibliography

[Art11] E. Artin. *Geometric Algebra*. Wiley Classics Library. Wiley, 2011.

[Asl96] Helmer Aslaksen. Quaternionic determinants. *The Mathematical Intelligencer*, 18(3):57–65, 1996.

[Bae02] John Baez. The octonions. *Bulletin of the American Mathematical Society*, 39(2):145–205, 2002.

[Bou02] Nicolas Bourbaki. Lie groups and lie algebras. chapters 4–6., 2002.

[CFK+97] James W Cannon, William J Floyd, Richard Kenyon, Walter R Parry, et al. Hyperbolic geometry. *Flavors of geometry*, 31:59–115, 1997.

[Cox46] H. S. M. Coxeter. Integral cayley numbers. *Duke Mathematical Journal*, 13(4):561–578, 12 1946.

[CS03] J.H. Conway and D.A. Smith. *On Quaternions and Octonions*. Ak Peters Series. Taylor & Francis, 2003.

[Die43] Jean Dieudonné. Les déterminants sur un corps non commutatif. *Bull. Soc. Math. France*, 71(171-180):95, 1943.

[DMR16] Tom De Medts and Erik Rijcken. Local moufang sets and PSL_2 over a local ring. *Journal of Group Theory*, 19(5):815–847, 2016.

[DMS09] Tom De Medts and Yoav Segev. A course on moufang sets. *Innov. Incidence Geom*, 9(79-122):2, 2009.

[DMW06] Tom De Medts and Richard M Weiss. Moufang sets and jordan division algebras. *Mathematische Annalen*, 335(2):415–433, 2006.

[dra]

[EGM13] Jürgen Elstrodt, Fritz Grunewald, and Jens Mennicke. *Groups acting on hyperbolic space: Harmonic analysis and number theory*. Springer Science & Business Media, 2013.

[Esc97] J.H. Eschenburg. Lecture notes on symmetric spaces. *preprint*, 1997.

[FF83] Alex J Feingold and Igor B Frenkel. A hyperbolic kac-moody algebra and the theory of siegel modular forms of genus 2. *Mathematische Annalen*, 263(1):87–144, 1983.

[Fin89] Benjamin Fine. Algebraic theory of the bianchi groups. 1989.

[FKN09] Alex J Feingold, Axel Kleinschmidt, and Hermann Nicolai. Hyperbolic weyl groups and the four normed division algebras. *Journal of algebra*, 322(4):1295–1339, 2009.

[For04] B Foreman. Conjugacy invariants of sl (2, h). *Linear algebra and its applications*, 381:25–35, 2004.

[GGRW05] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, and Robert Lee Wilson. Quasideterminants. *Advances in Mathematics*, 193(1):56–141, 2005.

[GP11] Skip Garibaldi and Holger P. Petersson. Wild pfister forms over henselian fields, k-theory, and conic division algebras. *Journal of Algebra*, 327(1):386 – 465, 2011.

[GY08] Xianfeng David Gu and Shing-Tung Yau. *Computational conformal geometry*. International Press Somerville, Mass, USA, 2008.

[Hil07] Thierry Hild. The cusped hyperbolic orbifolds of minimal volume in dimensions less than ten. *Journal of algebra*, 313(1):208–222, 2007.

[Hum92] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1992.

[Jac49] N. Jacobson. Derivation algebras and multiplication algebras of semi-simple jordan algebras. *Annals of Mathematics*, 50(4):866–874, 1949.

[JJ49] FD Jacobson and Nathan Jacobson. Classification and representation of semi-simple jordan algebras. *Transactions of the American Mathematical Society*, 65(2):141–169, 1949.

[JvNW93] Pascual Jordan, John von Neumann, and Eugene P Wigner. On an algebraic generalization of the quantum mechanical formalism. In *The Collected Works of Eugene Paul Wigner*, pages 298–333. Springer, 1993.

[JW99a] Norman W Johnson and Asia Ivic Weiss. Quadratic integers and coxeter groups. *Canadian Journal of Mathematics*, 51(6):1307–1336, 1999.

[JW99b] Norman W Johnson and Asia Ivić Weiss. Quaternionic modular groups. *Linear algebra and its applications*, 295(1-3):159–189, 1999.

[Kac94] V.G. Kac. *Infinite-Dimensional Lie Algebras*. Progress in mathematics. Cambridge University Press, 1994.

[KM68] DA Kazhdan and GA Margulis. Proof of selbergâĂŹs lemma. *Mat. Sb.(NS)*, 75:162–168, 1968.

[KMW] VG Kac, RV Moody, and M Wakimoto. On e_{10}, differential geometrical methods in theoretical physics (como, 1987), 109–128. *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci*, 250.

[Kna02] A.W. Knapp. *Lie Groups Beyond an Introduction*. Progress in Mathematics. Birkhäuser Boston, 2002.

[Kna13] Anthony W Knapp. *Lie groups beyond an introduction*, volume 140. Springer Science & Business Media, 2013.

[KNP$^+$12] Axel Kleinschmidt, Hermann Nicolai, Jakob Palmkvist, et al. Modular realizations of hyperbolic weyl groups. *Advances in Theoretical and Mathematical Physics*, 16(1):97–148, 2012.

[MD99] Corinne A Manogue and Tevian Dray. Octonionic möbius transformations. *Modern Physics Letters A*, 14(19):1243–1255, 1999.

[Mei13] Eckhard Meinrenken. *Clifford algebras and Lie theory*, volume 58. Springer, 2013.

[Mor15] D.W. Morris. *Introduction to Arithmetic Groups*. Deductive Press, 2015.

[Mou33] Ruth Moufang. Alternativkörper und der satz vom vollständigen vierseit (d9). In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 9, pages 207–222. Springer, 1933.

[MS93] Corinne A Manogue and Jörg Schray. Finite lorentz transformations, automorphisms, and division algebras. *Journal of Mathematical Physics*, 34(8):3746–3767, 1993.

[PS09] John R Parker and Ian Short. Conjugacy classification of quaternionic möbius transformations. *Computational methods and function theory*, 9(1):13–25, 2009.

[Sud84] A Sudbery. Division algebras, (pseudo)orthogonal groups and spinors. *Journal of Physics A: Mathematical and General*, 17(5):939, 1984.

[Swa71] Richard G Swan. Generators and relations for certain special linear groups. *Advances in Mathematics*, 6(1):1–77, 1971.

[Vei14] Jean Pierre Veiro. Octonionic presentation for the lie group $SL(2, \mathbb{O})$. *Journal of Algebra and Its Applications*, 13(06):1450017, 2014.

[VP91] Paul Van Praag. Sur la norme réduite du déterminant de dieudonné des matrices quaternioniennes. *Journal of Algebra*, 136(2):265–274, 1991.

[Zha97] Fuzhen Zhang. Quaternions and matrices of quaternions. *Linear algebra and its applications*, 251:21–57, 1997.

Index