Automorphisms of Buildings

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AUTOMORPHISMS
of
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The energy of the mind is the essence of life.

Aristotle

Automorphisms of Buildings

Knowledge comes but wisdom lingers and I linger on the shore.
And the individual withers, and the world is more and more.

Alfred, Lord Tennyson

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DEUTSCHE ZUSAMMENFASSUNG

Ziel dieser Arbeit ist es, die Struktur von Gebäude-Automorphismen besser zu verstehen. Dazu wird insbesondere für einen Automorphismus θ auf einem Gebäude $\mathcal{B}$ mit Weylgruppe W und Weylmetrik δ die Menge W_θ untersucht. Dies ist die Menge aller Elemente der zugrundeliegenden Weylgruppe, welche Abstand von einer Kammer zu ihrem Bild sind. Wir bezeichnen die Elemente in W_θ als Verschiebungsabstand (für θ). Es wird zuerst gezeigt, dass für Gebäude mit unendlicher irreduzibler Weylgruppe und typerhaltendem Automorphismus θ die Menge W_θ nicht identisch mit W ist. Weiter wird auch gezeigt, dass $W_\theta \neq W$ gilt, falls θ ein Automorphismus eines affinen Gebäudes ist. Im darauffolgenden Teil wird mit der CAT(0)-Struktur von Gebäuden gearbeitet.
Sei $\mathrm{M}_C(\theta)$ die Menge der Verschiebungsabstände von Kammern, deren geometrische Realisierung einen Punkt enthält, der minimal verschoben wird. Wir zeigen, dass für jeden Automorphismus θ einer Coxetergruppe W die Weylverschiebungen genau die θ-Konjugate der Worte in $\mathrm{M}_C(\theta)$ sind. Weiter wird eine Bedingung für Automorphismen von Gebäuden angegeben, unter welcher eine analoge Aussage für diese Automorphismen richtig ist.
Im Anschluss werden Graphen definiert, welche eine Baumstruktur für ein Gebäude beschreiben. Wenn solch ein Graph (V, E) für ein Gebäude $\mathcal{B}$ existiert und ein Automorphismus θ von $\mathcal{B}$ auf diesen Baum operiert, so sei M die Menge der Kammern, die in Knoten von V liegen, die minimalen Abstand zu ihrem Bild haben. Dann entspricht die Menge W_θ den θ-Konjugaten von Verschiebungsabständen von Kammern in M. Wir zeigen, dass für alle nicht-zwei-sphärischen Gebäude so ein Baum existiert. Ein Spezialfall von diesen Bäumen sind die Residuenbäume, für welche alle Knoten Residuen des Gebäudes sind und die ungerichteten Kanten den Inklusionen entsprechen. Die Existenz eines Residuenbaumes für ein Coxetersystem (W, S) impliziert bereits die Existenz eines Residuenbaumes für jedes Gebäude vom Typ (W, S).

Im letzten Abschnitt der Arbeit wird die Struktur von affinen Gebäuden bzgl. der Gruppe $\mathrm{SL}_n(K)$ für diskrete Bewertungskörper K beschrieben. Für solch ein Gebäude $\mathcal{B}$ wird die Wirkung von $\mathrm{GL}_n(K)$ auf $\mathcal{B}$ analysiert. Wir beschreiben einen Algorithmus, welcher es ermöglicht, den Weylabstand von zwei Kammern zu bestimmen, wenn diese Kammern als Bilder der fundamentalen Kammer für zwei Matrizen in $\mathrm{GL}_n(K)$ gegeben sind. Dieses Resultat ist die Grundlage für das im Anhang beschriebene Programm für Sage, mit dem Weylabstände von Kammern in $\mathcal{B}$ berechnet werden können.

PART I

Introduction

If someone was about to ask me: *What is the most essential part in modern mathematical research?* My answer would probably be: *The interaction of different fields enriching each other, providing new tools and a new point of view.*
To gain access to the knowledge of a different field, a transition of the concepts and structures has to be found. The theory of buildings can be seen as a framework for such a transition. For example, the theory of buildings provides a metric space for several algebraic structures such as semisimple algebraic groups. This is one of the reasons I became so fascinated by this theory.

Some History

Invented by Jacques Tits in 1950's and 1960's to understand finite semisimple complex Lie groups, the theory of buildings applies to a far wider class of objects than those groups. At first buildings were seen as simplicial structures arising from Weyl groups which may be understood as groups of reflections on a tiling of some space. The maximal simplices are called chambers and a building is covered by apartments which are subsets isomorphic to a simplicial realization of the corresponding Weyl group. These buildings are called simplicial buildings. In the 1980's came a different approach towards buildings. A building admits a metric, called Weyl metric, measuring distances between chambers as elements of the corresponding Weyl group W. One can also define a W-metric building as a set of chambers together with a metric into W satisfying certain conditions. It turns out that both concepts are equivalent and a building admits a realization as a chamber complex and a simplicial complex. After Davis and Moussong showed that every building admits a CAT(0)-structure (see [Dav08, Dav98]), a third realization for buildings was found which allows a very geometric analysis and gives new tools to work with. An example of such a very important tool is Bruhat-Tits' fixed point theorem, see 3.6.9.
Bruhat and Tits developed the concept of affine buildings based on their analysis of affine BN-pairs in [BT66]. These buildings correspond to semisimple algebraic groups over fields with discrete valuation (see also part V of this thesis). As a generalization of spherical buildings which are the buildings whose corresponding Weyl group is finite, the concept of twin buildings was invented. The idea behind this is a twinning of two buildings given by an opposition relation. Twin buildings correspond to Kac-Moody groups which are infinite, finitely generated, but possibly not finitely represented groups. These groups can be seen as an infinite dimensional analogue of the initially studied objects.

This Thesis

In the following let $\mathcal{B}$ be a building with Weyl group W and Weyl metric δ. The Weyl metric δ induces a metric d on the set of chambers of $\mathcal{B}$ and an automorphism of $\mathcal{B}$ is a map $\theta\colon \mathcal{B} \to \mathcal{B}$ mapping chambers to chambers, preserving the metric d. During the study of buildings there arises a natural and often researched question, which is the central question of this thesis:

What can one say about $\boldsymbol{\theta}$?

Is it possible to "classify" all automorphisms of $\mathcal{B}$? Can we say something about properties / orbits / fixed points of a given class of automorphisms or a specific automorphism?

This thesis represents our own little contribution to this question. In particular, the reader should keep the following question in mind while reading this thesis, which drove much of the research in it:

What can we say about the set $\boldsymbol{W_\theta}$ of (Weyl) displacements of $\boldsymbol{\theta}$?

Here, the set W_θ is the set of all elements in W which are the distance of (at least one) chamber $C \in \mathcal{B}$ to its image $\theta(C)$, i.e.

$$W_\theta := \{w \in W \mid \exists\ C \in \mathcal{B} : \delta(C, \theta(C)) = w\}.$$

This set might consist of exactly one element, it might be infinite, and it might be anything else in between. Therefore it is natural to ask, how does this set look like for a specific automorphism θ and on the other hand given a subset $X \subset W$, is there an automorphism with $W_\theta = X$?

A general concept of this thesis is to find a small subset $\mathcal{C}$ of the chambers of $\mathcal{B}$ such that the Weyl displacements in W_θ can be attained from $\mathcal{C}$. [See theorems 11.5.1, 12.1.32, 13.1.9].

As to why we consider the set W_θ to be interesting: One motivation for this comes from Deligne-Lusztig theory:

For a split connected reductive group G over a finite field, let B be a fixed Borel subgroup, T a maximal split torus, and W the corresponding Weyl group. In 1976 Deligne and Lusztig constructed a family of algebraic varieties given the Frobenius automorphism σ:

$$X_w := \{g \in G/B \mid g^{-1}\sigma(g) \in BwB\}.$$

The structure of G/B is a building corresponding to the Weyl group W and the set X_w is the set of all chambers which have displacement w. For a fixed w this classical Deligne-Lusztig variety is smooth and equidimensional of dimension $l(w)$. Such varieties are used to define Deligne-Lusztig characters as in [Car93, section 7.2]. The varieties used there are given in the form $G^\sigma := L^{-1}(1)$ for the Lang map $L(g) := g^{-1}\sigma(g)$. At the moment there is some interest in generalizing this setup to the theory of affine root systems, based on their relation to the reduction

modulo p of Shimura varieties [1] (see [Bea12, GHKR10, GH10, He14, Rap05]).
In the affine case, the group G is defined over a field with discrete valuation. In particular: Let $\overline{\mathbb{F}}_q$ be an algebraic closure of a finite field with q elements. Let $\mathcal{O}$ be the valuation ring of $\overline{\mathbb{F}}_q$ corresponding to a uniformizing parameter. Then G is defined over the fraction field of $\mathcal{O}$. The affine building (Bruhat-Tits building) corresponding to this group is G/I, where I is the standard Iwahori-subgroup G (see part V). The affine Deligne-Lusztig varieties for $w \in W$ and $b \in G$ are defined as:

$$X_w(b) := \{g \in G/\mathrm{I} \mid g^{-1}b\sigma(g) \in \mathrm{I}\,w\,\mathrm{I}\}.$$

The main problem in this setup is to know when $X_w(b)$ is empty.

Structure and scope

The first part of this thesis is a summary of the basic objects needed for this work. It contains definitions of graphs, Cayley graphs, (free) amalgamated products, graph products, CAT(0) spaces, the gate property, simplicial complexes, and chamber systems. Everything is only given in the most essential way to allow us to work with them. Afterward, in the second part, the main objects, Coxeter systems and buildings are introduced. As we work mainly on buildings, the section about Coxeter systems is relatively short compared to its important role in the theory of buildings. Nevertheless it covers everything we need to work with them. Part I and part II might serve as a reminder for those familiar with these topics. The reader not familiar with the subjects should find everything needed, but it is recommended to take a look at more detailed works. A very good reference for those areas is [AB08]. At the beginning of the third part we will give some examples to indicate that it is not easy to obtain general answers to the above questions. In particular it will turn out that affine buildings have a very special behavior concerning automorphisms. One might understand the problem in the following way: It does not matter how far we zoom out, the structure (of the building and of any isomorphism on it) will always look the same. Whereas for example the structure of $\mathrm{PGL}(2,\mathbb{Z})$ looks from far away like a tree and all isomorphisms behave like isomorphisms on trees. This observation is the foundation of the tie tree approach in chapter 12.

This work follows two different concepts to answer the above mentioned questions and presents an algorithmic approach for certain affine buildings, namely the Bruhat-Tits buildings of SL_n over the Laurent series of finite fields.

A geometric approach

The first approach (chapter 11) uses the complete CAT(0) structure of buildings. The existence of a CAT(0) realization $\mathcal{X}$ shows that a building only admits auto-

[1] Shimura varieties are an infinite dimensional analogue of modular curves related to a quotient of Hermitian symmetric spaces by an congruence subgroup of a reductive group defined over $\mathbb{Q}$. They are used in several areas of number theory and play an important role in the Langlands program.

morphisms which induce hyperbolic or elliptic isometries on $\mathcal{X}$. The first means that we find geodesics in the given realization, on which θ acts as a translation and the elements on these geodesics are exactly the elements with minimal distance to their image. An isometry is elliptic if it has a fixed point. The set $\mathrm{Min}(\theta)$ is defined as the set of all points with minimal distance to their image. A geodesic ray on which θ acts as a translation is called translation axis.

Let $\mathrm{M}_C(\theta)$ be the set of all chambers D of $\mathcal{B}$ whose geometric realization $|D|$ intersects $\mathrm{Min}(\theta)$ non-trivially. We show that given any automorphism of a building $\mathcal{B}$ for every chamber $C \in \mathcal{B}$ there exists a minimal gallery $(C, \ldots, D, \ldots, \theta(D))$ for some chamber $D \in \mathrm{M}_C(\theta)$. In other words: We can always construct a minimal gallery coming as close as possible to the set $\mathrm{Min}(\theta)$. If for every chamber $C \in \mathcal{B}$ there exists an apartment Σ_C containing the chamber $\theta(C)$ and a gallery $(C, \ldots, D, \ldots, \theta(D))$ for some $D \in \mathrm{M}_C(\theta)$, then the elements of W_θ are the θ-conjugates of the displacements of elements in $\mathrm{M}_C(\theta)$. One conclusion of this is that the Weyl displacements for an automorphism θ of a Coxeter system are exactly the θ-conjugates of the displacements of chambers in $\mathrm{M}_C(\theta)$. A crucial aspect of this geometric approach is the existence of an apartment containing a given chamber and a subray of a translation axis. This might be understood in the following way: The further we go along a translation axis, the smaller become the angles of the remaining ray and geodesics issuing from the given chamber going through the remaining ray. Thus after some point there cannot be any wall separating a proper subray of the remaining ray from the given chamber.

An approach using tree-like structures

The second approach (chapter 12) uses tree-like structures called tie trees. Tie trees relate to a coarser structure on a building (as a chamber system) identifying vertices with gated sets, called knots and ties whose edges correspond to the containment relation. These trees are a reasonable structure for us, as we can ensure that minimal galleries in the building relate to minimal paths in the tree. Given a tie tree, we can simplify our analysis of W_θ using the tree-like structure. The set $\mathrm{Min}(\theta)$ for the induced action on the tree does not have to contain any vertex, but we might take the support $\mathrm{supp}(\mathrm{Min}(\theta))$ which has the property that every path from a vertex to its image has to pass through it. Therefore we can easily calculate all Weyl displacements, once the displacements inside $\mathrm{supp}(\mathrm{Min}(\theta))$ are known. We will see that such a structure can be obtained (if it exists) for a building of type (W, S) directly from a tie tree structure of (W, S) (if it exists). Examples for such buildings are all non 2-spherical buildings.

Affine buildings and an implementation

A rather important class of buildings are the affine buildings. Sadly, most of the given results can not be adapted to affine buildings. A first result that can be applied to some class of automorphisms of affine buildings is a structure theorem for automorphisms stabilizing a connected subset $\mathcal{C}$ which separates every chamber outside of $\mathcal{C}$ from its image. In part V we will introduce the affine buildings $\mathcal{B}$ corresponding to $\mathrm{SL}_n(K)$ for a field K with discrete valuation. For this group together with the group $\mathrm{GL}_n(K)$ acting on $\mathcal{B}$ we describe an algorithm which computes the distance $w = \delta(C, D)$ of two chambers C, D in $\mathcal{B}$. The first amounts to finding a monomial matrix representing the same Iwahori-double coset as w. This procedure can be understood as the retraction of D onto the fundamental apartment based at the chamber C. The second step is an algorithm computing an expression v for the word w from the monomial matrix of the first procedure. These results led to the implementation of a program for computating Weyl distances in $\mathcal{B}$ (see Appendix A, B, C).

Results overview

At the end of this introduction we want to mention the most relevant results of this thesis.
Discussing some introductory examples in section 10 we show

Theorem 10.1.15. *For every automorphism θ of an affine building $\mathcal{B}$ with Coxeter system (W, S) one has $W \neq W_\theta$.*

and

Corollary 10.1.11. *By 10.1.10 every infinite Coxeter systems contains straight elements. In particular, for any type-preserving automorphism θ of a building of type (W, S) with infinite Coxeter group W, one has $W \neq W_\theta$.*

It is known that for any geodesic ray inside (the Davis realization of) a building there exists a geometric apartment containing this ray (see [CH09, 6.3]). In section 11.3 we show

Proposition 11.3.11. *Let θ be an hyperbolic action on a building $\mathcal{B}$. Let C be a chamber of $\mathcal{B}$ and let γ be a translation axis of θ. There exists a geometric apartment $|\Sigma'|$ containing $|C|$ and $\gamma((z, \infty))$ for some $z \in \mathbb{R}$.*

Working with the geometric structure of buildings we obtain our main result of chapter 11:

Theorem 11.5.1. *If an automorphism θ on a building $\mathcal{B}$ satisfies (MW), then any displacement $w \in W_\theta$ is a θ-conjugate of some displacement $w' \in W_{\mathrm{Min}(\theta)}$, i.e. $w = w_1 \cdot w_2 \cdot \theta(w_1)^{-1}$ for some $w_2 \in W_{\mathrm{Min}(\theta)}$.*

The (MW) condition ensures that for any chamber $C \in \mathcal{B}$ there exists a chamber $D \in \mathcal{B}$ whose geometric realization contains a point with minimal displacement such that D lies on a minimal gallery from C to $\theta(D)$. Examples for this are all automorphisms of Coxeter systems, as well as all autormorphisms of buildings whose Coxeter group is universal, and elliptic actions on affine building fixing exactly one (geometric) wall.

The main result on tie trees is the following:

Theorem 12.1.32. *If an automorphism θ of a building $\mathcal{B}$ admits a tie tree $\mathcal{T}$, then the displacements of θ on $\mathcal{B}$ are exactly the θ-conjugates $v \cdot w \cdot \theta(v^{-1})$ of the displacements w of chambers in $\mathrm{SM}(\theta)$ such that $l(vw\theta(v^{-1})) = 2l(v) + l(w)$.*

We obtain several examples for those buildings from section 12.2, where we take a closer look at a specialization of tie trees. It's worth to mention here that all non-2-spherical buildings admit a tie tree structure.

This slightly weaker result in section 13 gives us some information about automorphisms of affine buildings stabilizing exactly one apartment (see 13.2.6) and automorphisms preserving the wall tree of an affine building (see 13.3.4):

Theorem 13.1.9. *Let θ be an automorphism of a building $\mathcal{B}$. If there exists a θ-invariant connected subset Y of $\mathcal{B}$ such that for every chamber $C \in \mathcal{B}$ a minimal gallery from C to $\theta(C)$ has to contain an element of Y, then every displacement of θ is a reduced word of the form $w_1 w_0 \hat{w}_1$, where w_0 is an element of $W_{\mathrm{SM}(\theta)}$ and w_1 is a Weyl distance of a chamber to $\mathrm{proj}_Y(C)$.*

In part V let K be a field with discrete valuation and let $\mathcal{B}$ be the affine building associated to $\mathrm{SL}_n(K)$. We obtain a formula how to compute the Weyl displacements of chambers in $\mathcal{B}$ under the action of elements in $\mathrm{GL}_n(K)$:

Theorem 14.5.29. *Let $g \in \mathrm{GL}_n(K)$ and let $M_d = \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix}$ be a diagonal matrix with $M := M_d \cdot M_{\hat{w}} \in \mathrm{I} \cdot g \cdot \mathrm{I}$ for some word $\hat{w}$ over $\{s_1 \ldots, s_n\}$. For $k \in \{0, \ldots, n\}$, let $L_k := \sum_{i=1}^{k} l_i$ and set $L_0 := 1$. Then for every chamber $C \in \mathcal{B}$:*

$$\delta(\overline{C}, g(\overline{C})) = \prod_{i=1}^{n} \big((\sigma^{L_{i-1}}(w_{i-1}^{-1})) \cdot \sigma^{L_{i-1}}(w_{l \cdot (n-1)}) \cdot \sigma^{L_i}(w_{i-1}) \big) \cdot \sigma^{L_n}(\hat{w}).$$

Using this very last result, we developed some software tools to compute displacements in certain affine buildings. The code of the main tools can be found in Appendix A, B, C. Some explanations are given in section 15.

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At this point I want to take the time to say some words in a thankful manner. During the process of research and writing this thesis I have worked with a lot of wonderful people and I want to give my gratitude to all of them and mention some directly.

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A list of theorems and results

Overview of examples and explicitly discussed cases

Standalone results

p. 53 Let (W, S) be a Coxeter system with a straight element $w \in W$, $\mathcal{B}$ a building of type (W, S) and θ a type preserving automorphism of $\mathcal{B}$. Then $W_\theta \neq W$.
By 10.1.11 this holds in particular for every building with infinite irreducible Coxeter system.

p. 56 For any affine building $\mathcal{B}$ which is not of type $\tilde{A}_n$ (for any n) and any non-type-preserving automorphism θ of $\mathcal{B}$: $W_\theta \neq W$.

p. 56, 57 For any building $\mathcal{B}$ of type $\tilde{A}_n$ and any automorphism θ of $\mathcal{B}$: $W_\theta \neq W$.

p. 59 For the affine building corresponding to $\mathrm{SL}_4(\mathbb{F}_q((t)))$, the action of $\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ has different displacements for different values of q.

Conclusions from the geometric approach

11.6.1 A displacement result for arbitrary Coxeter systems.

11.8 A displacement result for affine buildings with an elliptic automorphism fixing exactly one wall.

Results and examples for tie trees and residue trees

12.2.13 Every non-2-spherical building admits a residue tree. The exact result is that the non-2-spherical Coxeter systems are exactly the ones admitting a non-trivial special tree of groups decomposition.

12.2.10 A building admits a residue tree, if its Coxeter systems admits a non-trivial special tree of groups decomposition (even with infinite set S).

12.3.4 Description of a tie tree for a building whose Coxeter systems admits a right-angled attached generator (some $s \in S$ with $m_{s,t} \in \{2, \infty\}$ for all $t \neq s \in S$ and $m_{s,t} = \infty$ for at least one $t \in S$).

12.3.7 Short description for a residue tree for a building whose Coxeter system corresponds to $\mathrm{PGL}(2, \mathbb{Z})$.

12.3.9 Description of a residue tree for a buildings of type $\tilde{A}_1 \times \tilde{A}_1$, i.e. its Coxeter group is the direct product of two copies of groups of type $\tilde{A}_1$.

Additional results for certain residue trees

12.1.33 How to construct a tie tree from a Coxeter group which splits as a free product.

12.2.11 Given a non-trivial special tree of groups decomposition $\mathcal{G}$ for a Coxeter system (W, S), there exists a residue tree for any building of type (W, S) whose vertices correspond exactly to the residues whose type sets correspond to the vertices and edges of $\mathcal{G}$.

12.2.15 Every automorphism of a building whose Coxeter system is virtually free preserves the residue tree corresponding to a given tree of groups decomposition.

12.2.18 The existence of a residual tie tree structure for some automorphism θ implies a non-trivial special tree of groups decomposition for the corresponding Coxeter system.

12.3.6 A more detailed description of displacements on buildings whose Coxeter system corresponds to an universal Coxeter group.

12.3.6 A very detailed description of displacements on buildings of type $\tilde{A}_1$.

12.3.9 There is no choice for a residue tree for a Coxeter system which is the product of two groups of type $\tilde{A}_1$. In particular, a residue tree can generally not be chosen to have 2-spherical residues as vertices.

Examples for the theorem about stabilized connected subsets.

These are cases in which we can apply theorem 13.1.9.

13.2.6 A hyperbolic action on a thick building which stabilizes exactly one (geometric) apartment.

13.3.4 An action on an affine building preserving a wall tree.

PART II

Basic Objects and Notation

CHAPTER

ONE

PAIRS, GRAPHS AND GRAPHS OF GROUPS

1.1 Pairs and Graphs

This section is based on [AB08, Kra08, Dav08, Ser03]

Definition 1.1.1. Let V be a set. An **undirected** or **unordered pair of V** (or **2-element subset**) is a set $\{v_1, v_2\}$ of two (not necessarily distinguished) elements v_1, v_2 of V. An **ordered** or **directed pair of V** is a set $\{\{v_1\}, \{v_1, v_2\}\}$ of (not necessarily distinguished) elements v_1, v_2 of V. A **pair** (v_1, v_2) of V is either a directed pair ($(v_1, v_2) = \{v_1, v_2\}$) or a undirected pair ($\{\{v_1\}, \{v_1, v_2\}\}$).

Definition 1.1.2. A **graph** is an ordered pair (V, E), where V is a set, and E is a set of pairs of V. The elements of V are called **vertices**. The elements of E are called **edges**. A graph is called **undirected** if the elements of E are undirected. It is called **directed** if the edges are directed.

Definition 1.1.3. Let (V, E) be a graph. A **subgraph** of (V, E) is a graph (V', E') with $V' \subseteq V$ and $E' \subseteq E$.

Definition 1.1.4. A graph (V, E) is called **simple** if it is an undirected graph without loops and with unique edges. This means that E does not contain any edges of the form (v, v) for $v \in V$ and given any edge $e = (v, v')$ then e is the only edge with vertices v and v'.

Definition 1.1.5. The set of edges E of an undirected graph (V, E) induces a relation $\sim$ on V, by defining

$$v_1 \sim v_2 \Leftrightarrow (v_1, v_2) \in E.$$

In this case we say that v_1 and v_2 are **adjacent**. This symmetric relation is called **adjacency** relation.

Definition 1.1.6. Let (V, E) be a graph. A **path** Γ in (V, E) is a finite sequence of vertices $v_0, \ldots, v_n$ such that $(v_i, v_{i+1}) \in E$ for $i \in \{0, \ldots, n-1\}$. The **length** $\boldsymbol{l(\Gamma)}$ of Γ is defined to be n. We say that two vertices v_1, v_2 are **connected by a path** if there exists a path in (V, E) from v_1 to v_2. A graph is called **connected** if any two vertices can be connected by a path.

Definition 1.1.7. A **cycle** in a graph is a closed path without any interiour repetitions, this means that it is a path $v_0, \ldots, v_n$ issuing and ending with the same vertex $v_0 = v_n$ such that $v_i \neq v_j$ for all $i \neq j \in \{0, \ldots, n-1\}$.

Definition 1.1.8. A **tree** is a connected simple graph (V, E) where the path connecting two vertices is unique. This means that a tree is a connected simple graph without cycles.

Definition 1.1.9. Let (V, E) be an undirected graph. A **spanning tree** for (V, E) is a subgraph (V', E') of (V, E) with $V' = V$ which is a tree.

1.2 Cayley Graphs

Let G be a group and S a symmetric set of generators of G, i.e. $S = S^{-1}$ which does not contain the identity.

Definition 1.2.1. The **Cayley graph** of (G, S) is the (undirected) graph whose vertices are the elements of G, and whose edges are the (unordered) pairs (g, gs), for $s \in S$ and $g \in G$. Let $s \in S$. Two elements g_1, g_2 in G are called **s-adjacent** $g_1 \sim_s g_2$ if $g_1 = g_2 \cdot s$.

Definition 1.2.2. Two elements g_1, g_2 of G are adjacent with respect to S if they are s-adjacent for some $s \in S$.

Remark 1.2.3. Two elements of G are adjacent with respect to S if and only if the corresponding elements in the Cayley graph of (G, S) are adjacent.

Definition 1.2.4. Let $\Gamma = (g_0, \ldots, g_n)$ be path in the Cayley graph of (G, S). The **type $\boldsymbol{\tau}(\boldsymbol{\Gamma})$** of Γ is the word $w = s_1 \ldots s_n$, where $g_i \sim_{s_i} g_{i-1}$, for $i \in \{1, \ldots, n\}$.

Definition 1.2.5. Let $g \in G$. We call an expression $s_1 \cdots s_n$ a **decomposition** for g if there exists a path from 1_G to G in the Cayley graph of (G, S) of type $s_1 \cdots s_n$. The length of such a decomposition is n.

Definition 1.2.6. The minimal length $l_S(g)$ of an element $g \in G$ in (G, S) is the length of a minimal path from 1_G to g in the Cayley graph of (G, S).

Definition 1.2.7. Let $g \in G$. A decomposition $s_1, \ldots, s_n$ for g in (W, S) is called **reduced** if $n = l_S(g)$.

1.3 Ends of Groups

The notion of Ends is used in proposition 1.3.3 which will be used in 5.2.21. We do not use this concept any further in this thesis, thus we will only give the definition and the used proposition. One might think of the ends of groups being the number of connected components at infinity.

Definition 1.3.1. Let G be a finitely generated group with a finite generating set S. Let Ω be its Cayley graph and let $\mathcal{C}$ be the posets of subgraphs ordered by inclusion. The **ends** of Ω is the inverse limit of the path components over the system $\{\Omega \setminus C\}_{C\in\mathcal{C}}$

$$\mathrm{Ends}(G) := \mathrm{Ends}(\Omega) := \varprojlim \pi_0(\Omega \setminus C).$$

Remark 1.3.2 (see [Dav08, G.1])**.** The set $\mathcal{C}' := \{\Omega \setminus C\}_{C\in\mathcal{C}}$ carries a poset structure (with relation $\leq$) with respect to the containment relation which yields an inverse system given the natural embeddings $\iota^{C_1}_{C_2}\colon C_1 \hookrightarrow C_2$ for all pairs $C_1 \leq C_2$, i.e. for every $C \in \mathcal{C}'$: $\iota^C_C = id_C$, and for all $C_1 \leq C_2 \leq C_3$: $\iota^{C_2}_{C_3} \circ \iota^{C_1}_{C_2} = \iota^{C_1}_{C_3}$. The inverse limit $\varprojlim \mathcal{C}'$ is the subset of the direct product $\prod\limits_{C\in\mathcal{C}'} C$ consisting of all tupels $(a_C)_{C\in\mathcal{C}'}$ such that $\iota^{C_1}_{C_2}(a_{C_1}) = a_{C_2}$ for all $C_1 \leq C_2$.
The inverse limit exists and is unique up to canonical isomorphism.

Proposition 1.3.3 ([Dav08, 8.6.1] originally [Hop44, Hauptsatz, Satz 1])**.** *Suppose G is a finitely generated group. Then G has either $0, 1, 2$ or infinitely many ends.*

(i) *G is 0-ended if and only if G is finite.*

(ii) *G is 2-ended if and only if G is virtually infinitely cyclic, i.e. G has an infinite cyclic subgroup of finite index.*

(iii) *If G has infinitely many ends then the number of ends is uncountable. Moreover, each point of $\mathrm{Ends}(G)$ is an accumulation point.*

CHAPTER

TWO

AMALGAMATED PRODUCTS

For further references and detailed proofs, one may look at [Rob96] and [Ser03].

2.1 Free Group

Definition 2.1.1. A free group $\mathrm{F}(S)$ over a set S is defined by the following universal property: For any group G and any map $\phi\colon S \to G$, there exists a unique group homomorphism $\phi' : \mathrm{F}(S) \to G$ whose restriction to S equals ϕ:

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, hook] & G \\ \mathrm{F}(S) \arrow[ur, dashed, "\exists! \phi'"'] & \end{tikzcd}$$

Definition 2.1.2. A **presentation** $\langle S \mid R \rangle$ for a group G is a set of generators S and a set of relations $R \subset \mathrm{F}(S)$ such that G is the quotient $\mathrm{F}(S)/\langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the smallest normal subgroup of $\mathrm{F}(S)$ containing R, called the normal closure of R in $\mathrm{F}(S)$. A group G is called **finitely generated** if there exists a presentation for G with a finite generating set. A group G is called **finitely presented** if there exists a presentation $\langle S, R \rangle$ for G with finite sets S and R.

2.2 Free Product

Let $\{G_i\}_{i \in I}$ be a collection of groups.
The idea of the free product of $\{G_i\}$ is to construct a group whose set of generators is the union of the generators of the G_i as disjoint sets, and having the relations given by the G_i.

Definition 2.2.1. The **free product** $\coprod_{i \in I} G_i$ is a group G and a collection of homomorphisms $\iota_i : G_i \to G$ with the following property. Given a set of homomorphisms $\phi_i : G_i \to H$ into a group H, then there exists a unique homomorphism

$G \to H$ such that $\iota_i \circ \phi = \phi_i$. This means that the diagram below commutes for all $i \in I$.

$$\begin{tikzcd} G_i \arrow[r, "\phi_i"] \arrow[d, "\iota_i"'] & H \\ G \arrow[ur, dashed, "\exists!\phi"'] & \end{tikzcd}$$

Remark 2.2.2. The free product of groups equals the coproduct in the category of groups.

Notation 2.2.3. We will denote the free product of two groups G_1 and G_2 by $G_1 * G_2$.

Proposition 2.2.4. *The free product of groups exists and is unique up to isomorphism.*

Remark 2.2.5. The existence of the free product can be shown, by a direct construction. One takes the union U of the groups G_i assuming the given groups are pairwise disjoint. The multiplicative structure on the set of all words $F(U)$ over U and defines an equivalence relation $\sim$ on those words in the following way: Let $g, f \in F(U)$, then $g \sim f$ if one can pass from g to f by applying a finite sequence of the following operations:

(i) Inserting the element 1_{G_i} for an $i \in I$.

(ii) Deleting the element 1_{G_i} for an $i \in I$.

(iii) Replacing consecutive elements $g_1 g_2$ which belong to the same group by their product $g' = g_1 g_2$. (*Contraction*)

(iv) Replacing an element g' of G_i by two elements $g_1, g_2 \in G_i$, where $g' = g_1 g_2$. (*Expansion*)

The group of these equivalence classes is a free product of the groups G_i together with the natural embeddings $G_i \to G$, where $x \in G_i$ is mapped to the equivalence class containing the word x.

2.3 Amalgamated Products

The concept of amalgamated free products generalizes the concept of free products.

Definition 2.3.1. Let $\{G_i\}_{i \in I}$ be a non-empty set of groups. Let H be a group together with monomorphisms $\varphi_i \colon H \to G_i$ for each $i \in I$. Let $F = \coprod_{i \in I} G_i$ and let $\langle\langle N \rangle\rangle$ the normal closure of $N := \{\varphi_i(h) \cdot \varphi_j(h)^{-1} \mid i, j \in I,\, h \in H\}$, i.e. the smallest normal subgroup of F containing N. Then the **amalgamated (free) product** of $\{G_i\}_{i \in I}$ along H (with respect to $\{\phi_i\}_{i \in I}$) is defined as $F / \langle\langle N \rangle\rangle$.

Notation 2.3.2. Let $\phi_1 : A \hookrightarrow G_1$ and $\phi_2 : A \hookrightarrow G_2$ be two monomorphisms of groups A, G_1, G_2. We will denote the amalgamated (free) product of G_1 and G_2 along A (with respect to φ_1 and φ_2) by $G_1 *_{A,\{\varphi_1,\varphi_2\}} G_2$. In case φ_1 and φ_2 are known, we omit them and write $G_1 *_A G_2$. In the same way, we will denote for some index set I the amalgamated product of $\{G_i\}_{i\in I}$ along A by $*_{A,\{\varphi_i\}}\{G_i\}$.

Remark 2.3.3. The idea of the amalgamated product is to find a group G which is generated by the G_i such that the images of the H_i are identified inside G.

Remark 2.3.4. The free amalgamated product $G := G_1 *_{A,\{\varphi_1,\varphi_2\}} G_2$ of groups G_1, G_2, A satisfies the following universal property:
Let H be a group and let $\psi_1 \colon G_1 \to H, \psi_2 \colon G_2 \to H$ be homomorphisms such that

$$\begin{tikzcd}
& G_1 \arrow[dr, "\psi_1"] & \\
A \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] \arrow[rr, phantom, "\circlearrowleft"] & & H \\
& G_2 \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

commutes, then the following diagram commutes everywhere.

$$\begin{tikzcd}
& G_1 \arrow[dr, "\iota_1"] \arrow[drr, "\psi_1"] & & \\
A \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & G \arrow[r, dashed, "\exists!\phi"'] & H \\
& G_2 \arrow[ur, "\iota_2"] \arrow[urr, "\psi_2"'] & &
\end{tikzcd}$$

Definition 2.3.5. Let $\{G_i\}_{i\in I}$ be a family of groups and let $F_{i,j}$ be a set of homomorphisms from G_i to G_j. The direct limit $\varinjlim G_i$ is a group G and a family of homomorphisms $\iota_i : G_i \to G$ such that $\iota_j \circ f = \iota_i$ for all $f \in F_{i,j}$, satisfying the following universal property:
Let H be a group and let $h_i \colon G_i \to H$ be a family of homomorphism satisfying: $h_j \circ f = h_i$ for each $f \in F_{i,j}$, then there exists exactly one homomorphism $h \colon G \to H$ such that $h_i = h \circ \iota_i$.
This means: If the diagram

$$\begin{tikzcd}
G_i \arrow[dd, "f"'] \arrow[dr, "h_i"] & \\
\arrow[r, phantom, "\circlearrowleft"] & H \\
G_j \arrow[ur, "h_j"'] &
\end{tikzcd}$$

commutes for all $i, j \in I$, and all $f \in F_{i,j}$, then there exists a unique homomorphism $h : G \to H$ such that

$$\begin{tikzcd}
G_i \arrow[dd, "f"] \arrow[dr, "\iota_1"] \arrow[drr, "h_i"] & & \\
& G \arrow[r, dashed, "\exists! h"'] & H \\
G_j \arrow[ur, "\iota_2"] \arrow[urr, "h_j"'] & &
\end{tikzcd}$$

commutes everywhere for all $i, j \in I$, and all $f \in F_{i,j}$.

Remark 2.3.6. The amalgamated (free) product of the groups G_i along the group A equals the direct limit $\varinjlim\{A\} \cup \{G_i\}_{i \in I}$, where the homomorphisms used are just the embeddings of A into the given groups.

2.4 Reduced Words

We follow the chapter 1.2 in [Ser03]. Let G be the amalgamated product $*_{A,\{\phi_i\}}\{G_i\}$ and let A denote its image in each of the G_i. For all $i \in I$, let S_i denote a set of right-coset representatives for G_i/A and assume $1 \in S_i$. The map $(a, s) \mapsto as$ is a bijection of $A \times S_i$ onto G_i mapping $A \times (S_i \setminus A)$ onto $G_i \setminus A$.
Let $i = (i_1, \ldots, i_n)$ be a sequence of elements of I (for $n \geq 0$) satisfying:

$$i_m \neq i_{m-1} \quad \text{for } 1 \leq m \leq n-1. \tag{$*$}$$

Definition 2.4.1. A **reduced word** of type i is any family $m = (a, s_1, \ldots, s_n)$ where $a \in A, s_1 \in S_{i_1}, \ldots s_n \in S_{i_n}$ and $s_j \neq 1$ fro all j.

Let f denote the canonical homomorphism of A into G and f_i the canonical homomorphism of G_i into G.

Theorem 2.4.2 ([Ser03, Theorem 1])**.** *For all $g \in G$, there is a sequence i satisfying $(*)$ and a reduced word $m = (a, s_1 \ldots, s_n)$ of type i such that*

$$g = f(a) f_{i_1}(s_1) \ldots f_{i_n}(s_n).$$

Furthermore, i and m are unique.

2.5 Word Problem

Let G be a finitely presented group, say $G = \langle S \mid R \rangle$, where S is a finite set, R is a finite subset of the free group $\mathrm{F}(S)$ on S. Let $\pi : \mathrm{F}(S) \to G$ be the natural map.

Definition 2.5.1 (**Word Problem**)**.** A finitely presented group $G = \langle S \mid R \rangle$ has a solvable word problem if there exists an algorithm which decides for any $w \in \mathrm{F}(S)$ whether or not $\pi(w) = 1$.

2.6 Graph of Groups

The concept of graphs of groups is based on the Bass-Serre theory. They will appear later in one of the main results 5.2.21. We will also use them to show that buildings whose Coxeter group is virtually admit a tie tree structure.
Let (V, E) be a graph.

Definition 2.6.1. A **graph of groups** $\mathcal{G}$ over (V, E) is an assignment of groups structures to (V, E) as follows: For each vertex $v \in V$, let $\mathcal{G}(v)$ be a group and for each edge $e \in E$ let $\mathcal{G}(e)$ be a group. Further let $\mathcal{G}(e, 0)\colon \mathcal{G}(e) \to \mathcal{G}(v_0)$ and $\mathcal{G}(e, 1)\colon \mathcal{G}(e) \to \mathcal{G}(v_1)$ be monomorphisms for each edge $e \in E$ with initial vertex v_0 and end vertex v_1.

Definition 2.6.2. Let $\mathcal{G}$ be a graph of groups over a graph (V, E) and let (V', E') be a spanning tree of (V, E). For each edge $e \in E$, let y_e denote a symbol. The **fundamental group** $G_{\mathcal{G}}$ is the quotient of $\coprod_{v \in V} G_v * \mathrm{F}(\{y_e \mid e \in E\})$ by the normal subgroup generated by the relations:

(i) $y_{\overline{e}} = y_e^{-1}$ for any edge e if $\overline{e}$ is the edge e with reversed orientation,

(ii) $y_e \mathcal{G}(e, 0)(a) y_e^{-1} = \mathcal{G}(e, 1)(a)$ for all $a \in \mathcal{G}(e)$,

(iii) $y_e = 1$ if $e \in E'$.

Notation 2.6.3. Once we fix a graph of groups $\mathcal{G}$ over (V, E), we use the following notation for the group $G = G_{\mathcal{G}}$:

- For every vertex $v \in V$, the image of the vertex group $\mathcal{G}(v)$ under the natural embedding in G will be denoted by G_v.
- For every edge $e \in E$, the image of the edge group $\mathcal{G}(e)$ under the natural embedding in G will be denoted by G_e.
- For every edge $e = (v_0, v_1) \in E$, the monomorphisms $\mathcal{G}(e, 0)$ and $\mathcal{G}(e, 1)$ induce monomorphisms from G_e into G_{v_1} and G_{v_2} which will be denoted by $\psi_{e,0}$ and $\psi_{e,1}$.

Remark 2.6.4. The fundamental group of a graph of groups is independent of the choice of the spanning tree. (See also [Bas93, Theorem 1.17, Remark 1.18, Section 2] for a second version of a definition for the fundamental group using a base vertex.)

Definition 2.6.5. We call a graph of groups **non-trivial** if the underlying graph consists of more than one vertex and none of its monomorphisms $\mathcal{G}(e, 0)$ or $\mathcal{G}(e, 1)$ is the identity.

Definition 2.6.6. A group G is said to **decompose as a graph of groups** if there exists a non-trivial graph of groups $\mathcal{G}$ over a graph (V, E) with $G_{\mathcal{G}} = G$. If the graph (V, E) is a tree, the group G decomposes as a **tree of groups**.

CHAPTER

THREE

CAT(0) SPACES

CAT(0) *spaces are a generalization of non-positively curved manifolds, describing metric spaces sharing essential attributes with those manifolds. They are uniquely geodesic spaces, the distance function is convex and they are contractible. Examples of such spaces are Euclidean spaces, hyperbolic spaces, and symmetric spaces without a compact factor. The important condition for* CAT(0) *spaces can be interpretet as: Every geodesic triangle is thinner than a comparison triangle in the Euclidean plane.*
The definitions and notations are taken from [BH99].

3.1 Metric Spaces

Definition 3.1.1. Let X be a set. A **pseudometric** on X is a real-valued function $d : X \times X \to \mathbb{R}$ satisfying the following properties, for all $x, y, z \in X$:

Positivity: $d(x, y) \geq 0$ and $d(x, x) = 0$.

Symmetry: $d(x, y) = d(y, x)$.

Triangle Inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

A pseudometric is called a **metric** if it is **positive definite**, i.e

$$d(x, y) > 0 \text{ if } x \neq y.$$

Notation 3.1.2. We will call $d(x, y)$ the **distance** of x and y.

Definition 3.1.3. A **metric space** is a pair (X, d), where X is a set and d is a metric on X. A metric space is said to be **complete** if every Cauchy sequence in (X, d) converges. If Y is a subset of X for a metric space (X, d), then the restriction of d to $Y \times Y$ is the **induced metric** on Y. If not stated otherwise, we will assume a subset to carry the induced metric.

Definition 3.1.4. A map $f : (X, d_X) \to (Y, d_Y)$ from a metric space (X, d_X) to a metric space (Y, d_Y) is called isometric if $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. An **isometry** is a isometric bijection $f : X \to Y$. If such a map exists, (X, d_X) and (Y, d_Y) are said to be **isometric**.

Notation 3.1.5. If no ambiguity may arise, a metric space X refers to a metric space (X, d).

3.2 Geodesics

Definition 3.2.1. Let (X, d) be a metric space. A **geodesic path** joining $x \in X$ to $y \in X$ (or **geodesic** from x to y) is an isometric map γ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(l) = y$. If $\gamma(0) = x$, then we say that γ issues from x. The image of γ is called **geodesic segment** with endpoints x and y. It will be denoted by $[x, y]$ or by $|\gamma]$.
A **geodesic ray** in a metric space (X, d) is an isometric map $\gamma : [0, \infty) \to X$. Its image $[\gamma]$ will also called geodesic ray.
A **geodesic line** in a metric space (X, d) is a isometric map $\gamma : \mathbb{R} \to X$. Its image $[\gamma]$ will also be called geodesic line.

Definition 3.2.2. A metric space (X, d) is called a **geodesic metric space** (or **geodesic space**) if every two points of X can be joined by a geodesic. It is called **unique geodesic space** if every two points can be joined by exactly one geodesic.

Definition 3.2.3. A subset of a metric space C is called **convex** if every pair of points $x, y \in C$ can be joined by a geodesic in X and if every such geodesic is contained in C.

3.3 Gate Property

The standard reference to gated sets is the work [DS87] by Dress and Scharlau. Their motivation was to generalize the known gate property of lower-dimensional stars in buildings using the projection maps, which were introduced by Tits in [Tit74] (as a product of simplices). The concept of gated sets was already known (see [GW70, Isb80, Hed83]).

Definition 3.3.1. A subset Y of a metric set (X, d) is called **gated** (in (X,d)) if the following holds:

Gate Property: For every $x \in X$, there exists a $y_x \in Y$ such that

$$d(x, y) = d(x, y_x) + d(y_x, y) \quad \text{for all } y \in Y.$$

The element y_x is called the **projection** (or **gate**) of x onto Y. It will be denoted by $\mathrm{proj}_Y(x)$.

Remark 3.3.2. The gate $\mathrm{proj}_Y(x)$ is uniquely determined by x.

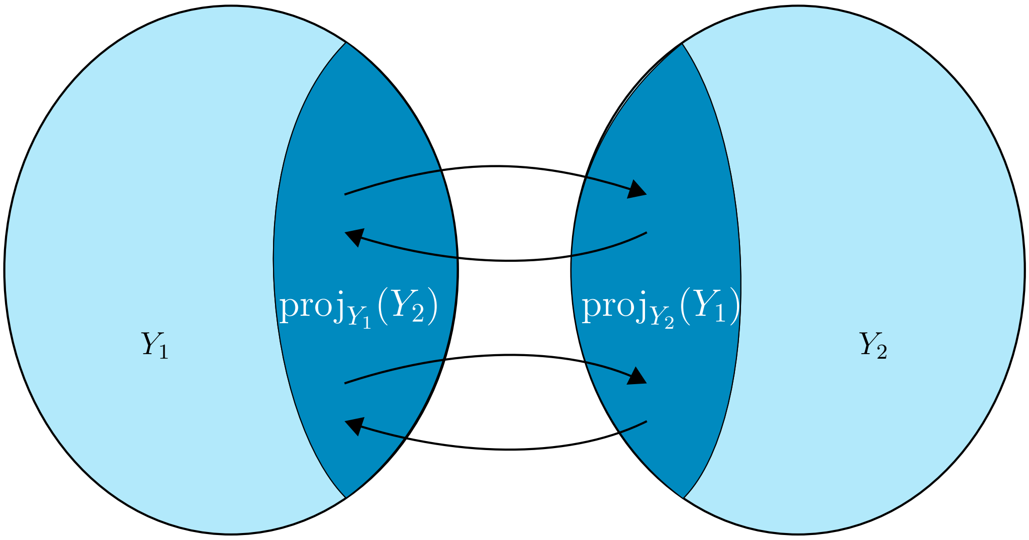


Figure 3.1: Projection maps. (*The idea of the image is taken from [DS87]*)

Remark 3.3.3. For unique geodesic spaces, it is possible to define a geodesic segment using the gate condition. Let x, y be elements of a unique geodesic space (X, d), then

$$[x, y] := \{z \in X \mid d(x, y) = d(x, z) + d(z, y)\}.$$

Remark 3.3.4 ([DS87, Proposition 1])**.** Every gated subset is convex.

Definition 3.3.5. Let A, B be two subsets of a metric space (X, d). We define their **distance** by:

$$d(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\}.$$

Remark 3.3.6. Let Y be a gated subset of a metric space (X, d) and let $x \in X$. The projection $\mathrm{proj}_Y(x)$ is the unique element of Y such that $d(x, \mathrm{proj}_Y(x)) = d(x, Y)$.

Remark 3.3.7. Let $Y \subset (X, d), x \in X$. The existence of a unique element in Y closest to x does not imply the gate property.
One may look at the closed disk of radius 1 in $\mathbb{R}^2$.

Proposition 3.3.8 ([DS87, Proposition 2], also [Hed83, Theorem 1.9], [Isb80, 1.8])**.** *Let $Z \subset Y$ be gated sets of a metric space (X, d). Then Z is gated in Y and $\mathrm{proj}_Z = \mathrm{proj}_Z^Y \circ \mathrm{proj}_Y$, where proj_Z^Y denotes the projection onto Z inside Y.*

Lemma 3.3.9. *If Y is a gated subset of a metric space (X, d) and $x \in X$, then:*

$$\textit{for all} \quad z \in [x, \mathrm{proj}_Y(x)] : \quad \mathrm{proj}_Y(z) = \mathrm{proj}_Y(x).$$

Theorem 3.3.10 ([DS87, Theorem])**.** *Let Y_1, Y_2 be two gated sets in a metric space (X, d). Let $Z_1 := \mathrm{proj}_{Y_1}(Y_2)$ and $Z_2 := \mathrm{proj}_{Y_2}(Y_1)$. Then*

(i) *The projections* proj_{Y_1} *and* proj_{Y_2} *induce isometries between* Z_1 *and* Z_2 *which are inverse to each other.*

(ii) *For* $x_1 \in Y_1, x_2 \in Y_2$*, the following two statements are equivalent:*

  (a) $d(x_1, x_2) = d(Y_1, Y_2)$.

  (b) $x_1 = \mathrm{proj}_{Y_1}(x_2)$ *and* $x_2 = \mathrm{proj}_{Y_2}(x_1)$.

(iii) *The sets* Z_1 *and* Z_2 *are gated. The projection* proj_{Y_1} *equals* $\mathrm{proj}_{Y_1} \circ \mathrm{proj}_{Y_2}$ *and* $\mathrm{proj}_{Y_2} = \mathrm{proj}_{Y_2} \circ \mathrm{proj}_{Y_1}$.

3.4 The CAT (0) Inequality

For the general definition of $\mathrm{CAT}(\kappa)$ *spaces, see [BH99, chapter II.1]. The present work only deals with* $\mathrm{CAT}(0)$ *spaces, thus only the* $\mathrm{CAT}(0)$ *inequality will be given.* Throughout this section let X be a metric space.

Definition 3.4.1. A **geodesic triangle** in X consists of three points $x, y, z \in X$, called vertices, and a choice of three geodesic segments $[x, y], [y, z], [z, x]$, called the **sides**. It will be denoted by $\Delta([x, y], [y, z], [z, x])$. For a unique geodesic space X, the choices of the geodesic segments are unique and we will write $\Delta(x, y, z)$. An element $p \in X$ is said to be in $\Delta = \Delta([x, y], [y, z], [z, x])$ if p is an element of the union of $[x, y], [y, z]$, and $[z, x]$. In this case we write $x \in \Delta$.

Definition 3.4.2. Let $\Delta = \Delta([x, y], [y, z], [z, x])$ be a geodesic triangle in X. A **comparison triangle** (in $(\mathbb{R}^2, d_{\mathbb{R}^2})$ for Δ is a geodesic triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{R}^2$ with $d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}), d(y, z) = d_{\mathbb{R}^2}(\bar{y}, \bar{z}), d(z, x) = d_{\mathbb{R}^2}(\bar{z}, \bar{x})$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called **comparison point** for $p \in [x, y]$ if $d_{\mathbb{R}^2}(\bar{x}, \bar{p}) = d(x, p)$. Comparison points for elements on $[y, z]$ and $[z, x]$ are defined in the same way.

Remark 3.4.3. A comparison triangle (in the above sense) is unique up to isometry.

Definition 3.4.4. A geodesic triangle Δ in X is said to satisfy the $\mathrm{CAT}(0)$ inequality if for a comparison triangle $\bar{\Delta}$, all $p, q \in \Delta$, and all comparison points $(\bar{p}, \bar{q})$ of $\bar{\Delta}$,

$$d(p, q) \leq d_{\mathbb{R}^2}(\bar{p}, \bar{q}).$$

Definition 3.4.5. A **CAT (0) space** is a metric space X whose geodesic triangles satisfy the $\mathrm{CAT}(0)$ inequality.

3.5 The Alexandrov Angle

Definition 3.5.1. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a comparison triangle for points $x, y, z \in X$. The interior angle at $\bar{x}$ in $\Delta(\bar{x}, \bar{y}, \bar{z})$ is called the **comparison angle** between y and z at x. It will be denoted by $\overline{\measuredangle_x}(y, z)$.

Definition 3.5.2. Let γ_1, γ_2 be two geodesics in a CAT(0) space X, issuing from the same point p. The **Alexandrov angle** $\measuredangle(\gamma_1, \gamma_2)$ between γ_1 and γ_2 is defined by

$$\measuredangle(\gamma_1, \gamma_2) := \limsup_{t,t' \to 0} \overline{\measuredangle_p}(\gamma_1(t), \gamma_2(t')).$$

Proposition 3.5.3 ([BH99, Proposition 3.1]). *One can express the angle in a* CAT(0) *space in the following way:*

$$\cos(\measuredangle(\gamma_1, \gamma_2)) = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d(\gamma_1(t), \gamma_2(t)).$$

3.6 Properties of CAT (0) Spaces

Let X be a CAT(0) space.

Lemma 3.6.1 ([BH99, II.1.4]). *X is a unique geodesic space.*

Remark 3.6.2 ([BH99, II.2.2]). The metric on a CAT(0) space is convex, i.e. any two geodesics $\gamma_1 : [0,1] \to X, \gamma_2 : [0,1] \to X$ satisfy for all $t \in [0,1]$:

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t) d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1)).$$

Theorem 3.6.3 (The Flat Strip Theorem, [BH99, II.2.13]). *Let γ_1, γ_2 be two geodesics lines in X. If γ_1 and γ_2 are **asymptotic**, i.e. there exists a constant K such that $d(\gamma_1(t), \gamma_2(t)) \leq K$ for all $t \in \mathbb{R}$, then the convex hull of $\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})$ is isometric to a flat strip $\mathbb{R} \times [0, D] \subset \mathbb{R}^2$.*

Remark 3.6.4. In a CAT(0) space the terms asymptotic and **parallel** are used synonymously.

Proposition 3.6.5 ([BH99, I.2.4]). *Let X be a* CAT(0) *space, and let C be a convex subset which is complete in the induced metric. Then,*

(i) *for every $x \in X$, there exists a unique point $\mathrm{proj}_C(x) \in X$ such that $d(x, \mathrm{proj}_C(x)) = d(x, C) := \inf_{y \in C} d(x,y)$;*

(ii) *if x' belongs to the geodesic segment $[x, \mathrm{proj}_C(x)]$, then $\mathrm{proj}_C(x') = \mathrm{proj}_C(x)$;*

(iii) *given $x \notin C$ and $y \in C$ if $y \neq \mathrm{proj}_C(x)$ then $\angle_{\mathrm{proj}_{C(x)}}(x, y) \geq \pi/2$;*

(iv) *the map $x \mapsto \mathrm{proj}_C(x)$ is a retraction of X onto C which does not increase distances; the map $H : X \times [0,1] \to X$ associating to (x,t) the point at distance $t \cdot d(x, \mathrm{proj}_C(x))$ from x on the geodesic segment $[x, \mathrm{proj}_C(x)]$ is a continuous homotopy from the identity map of X to proj_C.*

Definition 3.6.6. Let B be a non-empty bounded set of X. The midpoint of the closed ball containing B of minimal radius is called the **circumcenter** of B.

Theorem 3.6.7 ([AB08, 11.26])**.** *Let X be a complete* CAT(0) *space, let A be a nonempty bounded subset. Then A admits exactly one circumcenter.*

Theorem 3.6.8 ([AB08, 11.27])**.** *Let X be a complete* CAT(0) *space, let A be a nonempty bounded subset, and let Y be the smallest closed convex subset of X that contains A. Then the circumcenter of A is contained in Y.*

Theorem 3.6.9 (**Bruhat-Tits Fixed-Point Theorem**, see [AB08, 11.23])**.** *Let X be a complete* CAT(0) *space and let B be a bounded subset of X. If a group G of isometries of X stabilizes B, then G fixes the circumcenter of B.*

Notation 3.6.10. We will also call the circumcenter of a bounded set its **barycenter** if X is a complete CAT(0) space.

3.7 Isometries of CAT (0) Spaces

Definition 3.7.1. Let X be a metric space and let θ be an isometry of X. The **displacement function** of θ is the function $d_\theta : X \to \mathbb{R}_{\geq 0}$, defined by $d_\theta(x) = d(x, \theta(x))$. The **translation length** of θ is the number $|\theta| := \inf\{d_\theta(x) \mid x \in X\}$. The set of points with minimal displacement $\{x \in X \mid d_\theta(x) = |\theta|\}$ will be denoted by $\mathrm{Min}(\theta)$. An isometry is called

semi-simple if $\mathrm{Min}(\theta) \neq \emptyset$,

elliptic if θ has a fixed point, i.e. $\mathrm{Min}(\theta) \neq \emptyset$ and $|\theta| = 0$,

hyperbolic if $\mathrm{Min}(\theta) \neq \emptyset$ and $|\theta| > 0$,

parabolic if $\mathrm{Min}(\theta) = \emptyset$.

Every isometry is either elliptic, hyperbolic or parabolic.

Proposition 3.7.2 ([BH99, II.6.2])**.** *Let X be a metric space with an isometry θ.*

(i) *The set $\mathrm{Min}(\theta)$ is θ-invariant.*

(ii) *If X is a* CAT(0) *space, then the displacement function is convex, i.e. given any geodesic $\gamma : I \to X$, for all $t, t' \in I$ and all $s \in [0, 1]$ the following inequality holds:*

$$d_\theta(\gamma(\,(1-s)t + st'\,)\,) \leq (1-s)d_\theta(\gamma(t)) + s\,d_\theta(\gamma(t')).$$

Hence $\mathrm{Min}(\theta)$ is a closed convex set.

Proposition 3.7.3 ([BH99, II.6.8])**.** *Let X be a* CAT(0) *space.*

(i) *An isometry θ of X is hyperbolic if and only if there exists a geodesic line $c : \mathbb{R} \to X$ which is translated non-trivially by θ, namely $\theta(c(t)) = c(t + a)$, for some $a > 0$ and all $t \in \mathbb{R}$. The set $c(\mathbb{R})$ is called axis of θ. For any such axis, the number a is actually equal to $|\theta|$.*

(ii) *If X is complete and θ^m is hyperbolic for some integer $m \neq 0$, then θ is hyperbolic.*

Let θ be a hyperbolic isometry of X.

(iii) *The axes of θ are parallel to each other and their union is* $\mathrm{Min}(\theta)$.

(iv) $\mathrm{Min}(\theta)$ *is isometric to a product $Y \times \mathbb{R}$, and the restriction of θ to* $\mathrm{Min}(\theta)$ *is of the form $(y, t) \mapsto (y, t + |\theta|)$, where $y \in Y$ and $t \in \mathbb{R}$.*

(v) *Every isometry α that commutes with θ leaves* $\mathrm{Min}(\theta) = Y \times \mathbb{R}$ *invariant, and its restriction to $Y \times \mathbb{R}$ is of the form (α', α''), where α' is an isometry of Y and α'' a translation of $\mathbb{R}$.*

Corollary 3.7.4. *For any metric space $\mathcal{X}$ and any isometry θ on $\mathcal{X}$, the following holds:*

(i) $\mathrm{Min}(\theta) = \mathrm{Min}(\theta^{-1})$.

(ii) $x \in \mathrm{Min}(\theta) \Leftrightarrow \theta(x) \in \mathrm{Min}(\theta)$.

Proof. (i): $\mathrm{Min}(\theta) = \{x \in \mathcal{X} \mid d(x, \theta(x)) = d\} = \{y \in \mathcal{X} \mid d(\theta^{-1}(y), y) = d\} = \mathrm{Min}(\theta^{-1})$.
(ii): By [BH99][II.6.2] the set $\mathrm{Min}(\theta)$ is θ-invariant. Therefore $x \in \mathrm{Min}(\theta)$ yields $\theta(x) \in \mathrm{Min}(\theta)$. Now $\theta(x) \in \mathrm{Min}(\theta)$ implies $\theta(x) \in \mathrm{Min}(\theta^{-1})$ and $x = \theta^{-1}(\theta(x)) \in \mathrm{Min}(\theta^{-1}) = \mathrm{Min}(\theta)$. □

Lemma 3.7.5. *For any isometry θ of a metric space $\mathcal{X}$,* proj_θ *and θ commute. I.e. for any element $x \in \mathcal{X}$, we have*

$$\theta(\mathrm{proj}_{\mathrm{Min}(\theta)}(x)) = \mathrm{proj}_{\mathrm{Min}(\theta)}(\theta(x)).$$

Proof. Let $d := d(x, \mathrm{proj}_{\mathrm{Min}(\theta)}(x)) = d(\mathrm{proj}_{\mathrm{Min}(\theta)}(\theta(x)), \theta(x))$ and let $z \in \mathrm{Min}(\theta)$ with $d(z, \theta(x)) \leq d$. Then $\theta^{-1}(z)$ is an element of $\mathrm{Min}(\theta)$ and $d(\theta^{-1}(z), x) \leq d$ which shows $\theta^{-1}(z) = \mathrm{proj}_{\mathrm{Min}(\theta)}(x)$. Thus $z = \theta(\mathrm{proj}_{\mathrm{Min}(\theta)}(x))$ is the unique element of $\mathrm{Min}(\theta)$ with minimal distance to $\theta(x)$ and thus $\theta(\mathrm{proj}_{\mathrm{Min}(\theta)}(x)) = \mathrm{proj}_{\mathrm{Min}(\theta)}(\theta(x))$. □

Corollary 3.7.6 ([BH99][II.2.8])**.** *If $\mathcal{X}$ is a complete* CAT(0) *space, and if Γ is a group of isometries of $\mathcal{X}$ with bounded orbit, then the fixed-point set of Γ is a non-empty convex subspace of $\mathcal{X}$.*

Proposition 3.7.7 (Flat Triangle Lemma, see [BH99][I.2.9])**.** *Let Δ be a geodesic triangle in a* CAT(0) *space $\mathcal{X}$. If one of the vertex angles of Δ is equal to the corresponding vertex angle in a comparison triangle $\bar{\Delta} \subset \mathbb{E}^2$ for Δ, then Δ is flat, i.e. the convex hull of Δ in $\mathcal{X}$ is isometric to the convex hull of $\bar{\Delta}$ in $\mathbb{E}^2$.*

CHAPTER

FOUR

SIMPLICIAL STRUCTURES

This section is taken from [AB08, Appendix A].

4.1 Simplicial Complexes

Definition 4.1.1. A **simplicial complex** with a set $\mathcal{V}$ of vertices is a collection Σ of finite subsets of $\mathcal{V}$ (called **simplices**) such that every singleton $\{v\}$ is a simplex and every subset of a simplex A is a simplex (called **face** of A).

Definition 4.1.2. The **rank** of a simplex A is its cardinality, and its **dimension** is defined to be its rank -1.

Remark 4.1.3. In this work, the empty set is considered to be a simplex. It has rank 0 and dimension -1.

Definition 4.1.4. A **subcomplex** of a simplicial complex Σ is a subset Σ' of Σ containing every face for each simplex its contains. Thus a subcomplex is again a simplicial complex.

Remark 4.1.5. The relation $A \leq B$ if A is a face of B turns a simplicial complex into a poset. Therefore:

(a) Any two elements $A, B \in \Sigma$ have a a greatest lower bound $A \cap B$.

(b) For any $A \in \Sigma$, the poset $\Sigma_{\leq A}$ of faces of A is isomorphic to the poset of subsets of $\{1, \dots, r\}$ for some $r \geq 0$.

Remark 4.1.6. A non-empty poset Σ satisfying (a) and (b) is a simplicial complex. The elements of Σ are the simplices and the elements of rank -1 are its vertices.

Definition 4.1.7. Two simplices A, B of a simplicial complex Σ are called **joinable** if they have an upper bound, i.e. there exists a simplex $C \in \Sigma$, with A and B being faces of C. In particular, if A and B are joinable the least upper bound $A \cup B$ is the simplex whose vertex set is the union of the vertices of A and B.

Definition 4.1.8. Let Σ be a simplicial complex. The **star** $\mathrm{st}_\Sigma(A)$ (or just $\mathrm{st}(A)$) of a simplex A in Σ is the set of all simplices $B \in \Sigma$ having a face in A. The **link** $\mathrm{lk}_\Sigma(A)$ (or just $\mathrm{lk}(A)$) of a simplex A in Σ is the subcomplex of Σ consisting of all simplices in Σ which are disjoint, but joinable with A.

Remark 4.1.9. We can use the definition of the star of a simplex A to define its link $\mathrm{lk}(A)$ by $\mathrm{lk}(A) = \mathrm{st}(A) \setminus A$.

Remark 4.1.10. We can turn $\mathrm{lk}(A)$ into a poset by the identification of an element $C \in \mathrm{lk}(A)$ with its union $C \cup A$ with A. In particular, the maximal simplices in $\mathrm{lk}(A)$ are in one-to-one correspondence with the maximal simplices of Σ containing A.

4.2 Flag Complexes

Definition 4.2.1. Let P be a set. A binary relation is called **incidence relation** if it is reflexive and symmetric.

Definition 4.2.2. A **flag** of a set P with an incidence relation $\sim$ is a set of pairwise incident element of P.

Definition 4.2.3. A **flag complex** $\mathcal{F}(P)$ associated to a set P with an incidence relation $\sim$ is the simplicial complex $\Sigma(P)$, where P is the set of vertices and the simplices are the sets of finite flags.

Definition 4.2.4. A flag complex of dimension 2 is called **incidence graph**.

4.3 Chamber Complexes

Definition 4.3.1. A **gallery** in a simplicial complex is a sequence of maximal simplices such that two consecutive elements are distinct and share a common maximal proper face.

Definition 4.3.2. Let Σ be a finite-dimensional simplicial complex. We call Σ a (connected) **chamber complex** if it satisfies:

(i) All maximal simplices have the same dimension.

(ii) Every two maximal simplices can be connected by a gallery.

Definition 4.3.3. A **chamber** of a chamber complex is a maximal simplex. A **panel** is a codimension-1 face of a chamber.

Definition 4.3.4. Let $d\colon \mathrm{Cham}(\Sigma) \times \mathrm{Cham}(\Sigma) \to \mathbb{N}$ be the well-defined distance function on $\mathrm{Cham}(\Sigma)$ given by the minimal length of the galleries joining two chambers. The diameter $\mathrm{diam}(\Sigma)$ is the diameter of the metric space $(\mathrm{Cham}(\Sigma), d)$

Remark 4.3.5. The metric in 4.3.4 is the standard metric on the chamber graph of Σ.

Definition 4.3.6. Let Σ be a chamber complex of rank n and let I be a set with n elements. A **type function** on Σ is a function τ on Σ with values in I that assigns to each vertex v an element $\tau(v) \in I$ such that for every maximal simplex Δ the vertices of Δ are mapped bijectively to I. For a simplex A, the image $\tau(A)$ is called the **type** of A. The **cotype** of a simplex A is the set $I \setminus \tau(A)$.

Definition 4.3.7. A chamber complex is called **colorable**, if it admits a type function.

Definition 4.3.8. A **chamber subcomplex** of a chamber complex Σ is a subcomplex of Σ which is also a chamber complex of the same dimension as Σ. The chambers of a chamber subcomplex Σ' are chambers of Σ which can be connected via a gallery inside Σ'.

Definition 4.3.9. If Σ and Σ' are chamber complexes of the same dimension, then a simplicial map $\theta\colon \Sigma \to \Sigma'$ is called a chamber map if it maps chambers to chambers.

Remark 4.3.10. One may note that a chamber map maps adjacent chambers to acjacent chambers. The image of a chamber map is always a chamber subcomplex.

4.4 Chamber Systems

Definition 4.4.1. A **chamber system** over a non-empty set I is a set $\mathcal{C}$ with an equivalence relation $\sim_i$ on $\mathcal{C}$ for each $i \in I$. The elements in $\mathcal{C}$ are called **chambers** and two chambers C, D are called ***i*-adjacent** if $C \sim_i D$. We call two chambers C, D adjacent if they are adjacent for some $i \in I$ and write $C \sim D$. The equivalence classes with respect to the i-adjacent relation are called ***i*-panels**. A panel is an i-panel for some $i \in I$.

Notation 4.4.2. If X is a structure carrying the structure of a chamber system, then the set of chambers of X will be denoted by $\mathrm{Cham}(X)$.

Definition 4.4.3. The **rank** of a chamber system over I equals the cardinality of I.

Definition 4.4.4. A **gallery** in a chamber system is a finite sequence $(C_0, \dots, C_n)$ of elements in $\mathcal{C}$ with $C_i \sim C_{i-1}$ for $i \in \{1, \dots, n\}$. A gallery is of **type** $i_1 \dots i_n$ (as a word in the free monoid over I) if $C_j \sim_{i_j} C_{j-1}$ for $i \in \{1, \dots, n\}$. A gallery $(C_0, \dots, C_n)$ is called **stuttering** if $C_i = C_{i-1}$ for some $i \in \{1, \dots, n\}$.

Notation 4.4.5. In this work a gallery is always a non-stuttering gallery, unless stated otherwise.

Definition 4.4.6. A gallery $\Gamma = (C_0, \dots, C_n)$ is called a ***J*-gallery** for some $J \subseteq I$ if Γ is of type $i_1 \dots i_n$ and $i_j \in J$ for $j \in \{1, \dots, n\}$.

Remark 4.4.7. The type of a gallery in a chamber system does not need to be unique. But in the theory of buildings, two adjacent chambers will be i-adjacent for exactly one $i \in I$.

Definition 4.4.8. A chamber system $\mathcal{C}$ is called **connected** or (J-connected for some $J \subseteq I$) if any two chambers of $\mathcal{C}$ can be joined by a gallery (by a J-gallery).

Definition 4.4.9. Let $J \subseteq I$. A ***J*-residue** (or residue of type J) of a chamber system $\mathcal{C}$, is a J-connected component of $\mathcal{C}$. A residue, is a J-residue of $\mathcal{C}$ for some $J \subseteq I$.

Remark 4.4.10. Every J-residue is a connected chamber system and the i-panels are the residues of type $\{i\}$. The rank 0-residues of a chamber system are exactly its chambers.

Definition 4.4.11. Let $\mathcal{C}$ be a chamber system over a set I and let $\mathcal{D}$ be a chamber system over a set J.
A **morphism** $\varphi : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is a map $\varphi : \mathcal{C} \to \mathcal{D}$ which preserves adjacency, i.e. if $C \sim D$, then $\varphi(C) \sim \varphi(D)$.
If I equals J, then a morhpism $\varphi : \mathcal{C} \to \mathcal{D}$ is called **type-preserving morphism** if it preserves the i-adjacency relation for all $i \in I$, i.e. if $C \sim_i D$, then $\varphi(C) \sim_i \varphi(D)$.
An isomorphism of chamber systems is a morphism of chamber systems admitting a two-sided inverse morphism. An automorphism of chamber systems is a isomorphism from a chamber system onto itself.

Proposition 4.4.12 ([AB08, A.20])**.** *Let Σ be a colorable chamber complex. Assume that the link of every simplex is again a chamber complex and that every panel is a face of at least two chambers. Then Σ is determined (up to isomorphism) by its chamber system* $\mathrm{Cham}(\Sigma)$. *More precisely:*

(i) *For every simplex A, the set $\mathcal{C}_{\geq A}$ of chambers having A as a face is a J-residue, where J is the cotype of A.*

(ii) *Every residue has the form $\mathcal{C}_{\geq A}$ for some simplex A.*

(iii) *For any simplex A, we can recover A from $\mathcal{C}_{\geq A}$ by*

$$A = \bigcap_{C \geq A} C$$

(iv) *The chamber complex Σ is isomoprhic (as a poset) to the set of residues in* $\mathrm{Cham}(\Sigma)$ *ordered by reverse inclusion.*

PART III

Introducing The Main Objects

CHAPTER

FIVE

COXETER SYSTEMS

Let W be a group with a symmetric set $1 \notin S$ of involutory generators, i.e. elements of order 2. Let $l = l_S$ be its length function. We want to have a closer look at the following conditions:

(A) The Action Condition:
Let T be the set of conjugates of elements of S. There is an action of W on $T \times \{\pm 1\}$ such that a generator $s \in S$ acts as the involution ρ_s given by

$$\rho_s(t,\epsilon) = \begin{cases} (sts,\epsilon) & \text{if } s \neq t, \\ (s,-\epsilon) & \text{if } s = t. \end{cases}$$

(C) The Coxeter Condition:
W admits the presentation

$$\left\langle S \mid (st)^{m(s,t)} = 1 \right\rangle,$$

where $m(s,t)$ is the order of st and there is one relation for each pair s,t with $m(s,t) < \infty$.

(D) The Deletion Condition: If $w = s_1 \cdots s_m$ with $m > l(w)$, then there are indices $i < j$ such that

$$w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_m,$$

where $\widehat{s}$ indicates an deleted element.

(E) The Exchange Condition:
Given $w \in W$, $s \in S$, and any reduced decomposition $w = s_1 \cdots s_d$ of w, either $l(sw) = d+1$ or else there is an index i such that

$$w = ss_1 \cdots \widehat{s_i} \cdots s_d,$$

where $\widehat{s}$ indicates an deleted element.

(F) The Folding Condition:
Given $w \in W$ and $s,t \in S$ such that $l(sw) = l(w)+1$ and $l(wt) = l(w)+1$, either

$$l(swt) = l(w) + 2 \quad \text{or else} \quad swt = w.$$

5.1 Conditions on (W, S)

Coxeter systems play a very important role in the theory of buildings. One might at this point think of a building being a set of isomorphic Coxeter systems glued together in a nice way. Whenever we look at a path from one element of a building to another one, then this path is a path inside one of those Coxeter systems. We see that understanding Coxeter systems is a crucial part of understanding buildings. The idea of Coxeter systems is an abstraction of reflection groups which are discrete groups generated by reflection of a finite dimensional Euclidean space ([Cox34]). Despite their significance for buildings, we will only give a short overview about Coxeter systems, listing the things of major importance needed in this thesis. It may be taken as a reminder for the reader familiar with Coxeter systems. For the interested reader, not familiar with this topic, we suggest to take a closer look at [AB08, chapter 1-4], or [Hum90].

Theorem 5.1.1 ([AB08, 2.49])**.** *The conditions (A), (C), (D), (E), and (F) are equivalent.*

Definition 5.1.2. A pair (W, S) of a group W and a set S of generators of order 2 for W is called a **Coxeter system** if the equivalent conditions in 5.1.1 are satisfied. A group W for which a generating set S exists such that (W, S) is a Coxeter system is called **Coxeter group**. The matrix $(m(s, t))$ will be called **Coxeter matrix** of (W, S), and the cardinality of S will be called the **rank of** (W, S).

Definition 5.1.3. Let (W, S) be a Coxeter system and let (V, E) be a graph whose vertex set is S and where the edges are given by:

- If s and t commute (i.e. $m_{st} = 2$) then there is no edge between the corresponding vertices.
- If $m_{st} = 3$ for $s, t \in S$, then the corresponding vertices are connected by an edge.
- If $m_{st} > 3$ for $s, t \in S$, then the corresponding vertices are connected by an edge which is labeled by m_{st}.

The graph Γ is called the **Dynkin diagram** (or **Coxeter diagram**) for (W, S).

Definition 5.1.4. For a subset $T \subset S$, we define $W_T := \langle T \rangle$. These sets are called **standard subgroups** (**or standard parabolic subgroups**) (**or special subgroups**) of W.

Definition 5.1.5. A subset $T \subset S$ is **spherical** if W_T is a finite subgroup of W. In this case W_T is called spherical subgroup of W.

Definition 5.1.6. A coset of W of the form wW_T, where $w \in W, T \subset S$ is called **standard coset** of W.

Remark 5.1.7. Following the concept of elementary M-operations on W by Jacques Tits [Tit69], one can show that the word problem for Coxeter groups is solvable. The elementary M-operations are two kinds of operations on expressions over S:

(i) Deleting a subword of the form ss for some $s \in S$.

(ii) Replacing an alternating subword of the form $sts\cdots$ of length m_{st} by the word $tst\cdots$ of length m_{st}.

5.2 Coxeter Complex

Definition 5.2.1. The **Coxeter complex** of a Coxeter system (W,S) is the poset $\Sigma(W,S)$ of standard cosets in W ordered by reverse inclusion is a simplicial complex. The maximal simplices are the singletons $\{w\}$ and can be identified with the elements of W. The simplices of the form $w\langle s\rangle = \{w, ws\}$ are the panels. The chamber 1_W is called the **fundamental chamber**.

Remark 5.2.2. The relation $B \leq A$ holds in $\Sigma(W,S)$ if and only if $A \subseteq B$ in W.

Theorem 5.2.3 ([AB08, Theorem 3.5]). *The Coxeter complex $\Sigma(W,S)$ is a simplicial complex. It is a thin chamber complex of rank $|S|$. It is colorable and the action of W on $\Sigma(W,S)$ is type-preserving.*

Definition 5.2.4. A simplicial complex Σ is called a Coxeter complex if it is isomorphic to the Coxeter complex $\Sigma(W,S)$ of a Coxeter system (W,S). It is called spherical if it is finite.

Remark 5.2.5. A Coxeter complex has no specific chamber with the property of being fundamental (a chamber in $\Sigma(W,S)$ which corresponds to $1_W \in W$). This allows us to choose a chamber in Σ as a fundamental chamber.

Definition 5.2.6. Let Σ be a thin chamber complex. A **root** (or **half-apartment**) is a subcomplex α whose set of chambers is of the form

$$\mathrm{Cham}(\alpha) = \{D \in \mathrm{Cham}(\Sigma) \mid d(D,C) < d(D,C')\},$$

where C and C' are two adjacent chambers. The root $-\alpha$ is defined by

$$\mathrm{Cham}(-\alpha) = \{D \in \mathrm{Cham}(\Sigma) \mid d(D,C) > d(D,C')\},.$$

It is called the root opposite to α.

Definition 5.2.7. A **wall** is the intersection of a root α and its opposite root $-\alpha$. If it is given by a root α, then it will be denoted by $\partial\alpha$ or $\overline{\alpha}$.

Remark 5.2.8 ([AB08, Section 3.4]). A wall $\partial\alpha$ determines an automorphism s_α of Σ which has the properties:

(i) The automorphism s_α is the unique non-trivial automorphism of Σ which fixes the wall $\partial\alpha$ pointwise.

(ii) The roots α and $-\alpha$ are interchanged by s_α.

Remark 5.2.9. The roots α and $-\alpha$ are given by the wall $\partial\alpha$ and any panel in $\partial\alpha$ determines the same pair of roots, i.e. if P is a panel in $\partial\alpha$, then the two chambers C, C' of P determine α and $-\alpha$.

Theorem 5.2.10 ([AB08, 3.65])**.** *A thin chamber complex Σ is a Coxeter complex if and only if every pair of adjacent chambers is separated by a wall.*

Theorem 5.2.11 ([AB08, 3,72])**.** *Let Σ be a Coxeter complex and let C be an arbitrary chamber in Σ called the fundamental chamber. Let S be the set of reflections of Σ interchanging C with an adjacent chamber. Let $W \leq \mathrm{Aut}(\Sigma)$ be the subgroup generated by S. The pair (W, S) is a Coxeter system.*

To show this theorem, the following results were used:

Lemma 5.2.12 ([AB08, 3.66])**.** *The group W acts transitively on Σ.*

Theorem 5.2.13 ([AB08, 3.67])**.** *A Coxeter system is colorable.*

Lemma 5.2.14 ([AB08, 3.69])**.** *If $\Gamma = (C_0, \ldots, C_n)$ is a minimal gallery, then the walls crossed by Γ are distinct and are precisely the walls separating C_0 from C_n. Hence the distance of two chambers is the number of walls separating them.*

Theorem 5.2.15 ([AB08, 3.68, 3.71])**.** *The action of W on Σ is type-preserving and W acts simply transitive on the chambers of Σ.*

Theorem 5.2.16 ([AB08, 3.85])**.** *Let τ be a type function for a Coxeter system Σ with values in a set S. The Coxeter matrix defined by $(m_{s,t})_{s,t\in S}$ with $m_{s,t} = \mathrm{diam}(\mathrm{lk}(A))$ determines a Coxeter system (W_M, S). The Weyl group of Σ is defined as W_M. There exists a type-preserving isomorphism $\Sigma \cong \Sigma(W_M, S)$.*

Definition 5.2.17. Let (W, S) be a Coxeter system with Coxeter matrix M. A Coxeter complex Σ is said to be of **type** (W, S) (or of type M) if Σ admits a type function with values in S such that the corresponding Coxeter matrix is M. This is equivalent to the existence of a type-preserving isomorphism $\Sigma \cong \Sigma(W, S)$.

Remark 5.2.18. Let C, D be two chambers in the chamber system of a Coxeter complex Σ. By $\Sigma \cong \Sigma(W, S)$ for some Coxeter system (W, S), we have a canonical type function. Let $\delta(C, D)$ be the type of a minimal gallery from C to D. The chambers in $\Sigma(W, S)$ correspond to the elements in W and thus a gallery of type $s_1 \ldots s_d$ with $s_1, \ldots, s_d \in S$ from w_1 to w_2 has the form $w_1, w_1s_1, \ldots, w_1s_1 \cdots s_d = w_2$ and we get $\delta(C, D) = w_1^{-1}w_2$ which is independent of the choice of the gallery. We see that after choosing a fundamental chamber $C \in \Sigma$ we can define a distance function δ on the chamber system of Σ by $\delta(w_1C, w_2C) := w_1^{-1}w_2$

Definition 5.2.19. Let Σ be a Coxeter complex of type (W, S), and let A and B be arbitrary simplices. Then there is an element $\delta(A, B)$ in W with $\delta(A, B) = \delta(C_0, C_l)$ for a minimal gallery $C_0, \ldots, C_l$ from A to B. In particular

$$d(A, B) = l(\delta(A, B)).$$

Proposition 5.2.20 ([AB08, Corollary 3.17])**.** *The Coxeter complex* $\Sigma = \Sigma(W, S)$ *is completely determined by its underlying chamber system. More precisely, the simplices of* Σ *are in* $1 - 1$ *correspondence with the residues in* $\mathrm{Cham}(\Sigma)$*, ordered by reverse inclusion.*
A simplex Δ *corresponds to the residue* $\mathrm{Cham}(\Sigma)_{\geq \Delta}$*, consisting of all chambers containing* Δ *as a face.*

Proposition 5.2.21 ([Dav08, 8.8.2])**.** *Any Coxeter system decomposes as a tree of groups, where each vertex group is a* 0*- or* 1*-ended special subgroup and each edge group is a finite special subgroup.*

Proposition 5.2.22 ([Dav08, 8.8.2])**.** *A Coxeter group* W *is two-ended if and only if* (W, S) *decomposes as* $(W_1, S_1) \times (W_2, S_2)$*, where* W_1 *is finite and* W_2 *is the infinite dihedral group.*

Corollary 5.2.23 ([Dav08, 8.8.5])**.** *A Coxeter group is virtually free if and only if it has a tree of groups decomposition.*

Remark 5.2.24. A Coxeter group is virtually free if it can be written as an iterated amalgamated product of finite special subgroups along finite special subgroups.

CHAPTER

SIX

BUILDINGS

Buildings can be seen as a rich tool for studying semisimple algebraic groups over arbitrary fields. They provide a simplicial structure, a CAT(0) *space, and a combinatorial structure. Hence they play an important role as trees do in the theory of free groups. We will introduce buildings as they evolved. Starting with the simplicial approach we will go over to the combinatorial approach and show that both concepts coincide (as long as we assume the generating set S of the underlying Coxeter group to be finite). According to our study aim we then discuss some aspects of groups acting on buildings before we present the Davis realization of buildings. A typical example for a building is the coset space $\mathrm{SL}_n(k)/B$, where B is the subgroup of upper triangular matrices in $\mathrm{SL}_n(k)$.*

6.1 Buildings as Simplicial Complexes

The definitions are taken from [AB08, Chapter 4].

Definition 6.1.1. A **building** is a simplicial complex $\mathcal{B}$ that can be expressed as the union of subcomplexes Σ (called **apartments**) satisfying the following axioms:

(B0): Each apartment Σ is a Coxeter complex.

(B1): For any two simplices $A, B \in \mathcal{B}$, there exists an apartment Σ containing both of them.

(B3): If Σ and Σ' are two apartments containing two simplices A, B, then there exists an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Remark 6.1.2. By taking the simplices A, B in $(B3)$ to be the empty simplex, one sees that all apartments are isomorphic. Further $\mathcal{B}$ is finite dimensional and its dimension is the common dimension of its apartments.

Remark 6.1.3. A building $\mathcal{B}$ is a chamber complex. Its chambers are the maximal simplices. For any two maximal simplices, there exists an apartment containing them. Thus they have the same dimension and are connected by a gallery.

Definition 6.1.4. Let $\mathcal{B}$ be a building. Then every set of subcomplexes satisfying the axioms $(B1), (B2)$, and $(B3)$ is called a **system of apartments** for $\mathcal{B}$.

Definition 6.1.5. A building is called **thick** if every panel is a face of at least three chambers. It is called **thin** if every panel is a face of exactly two chambers and **weak** if every panel is a face of at least two chambers. We further say that a building is **locally finite** if every panel is a face of only finitely many chambers.

Proposition 6.1.6 ([AB08, Remark 4.3, Remark 4.4])**.** *The axiom $(B2)$ can be replaced by one of the following axioms:*

(B2'): *Let Σ and Σ' be two apartments containing a simplex A and a chamber C (i.e. a maximal simplex C of Σ). Then there exists an isomorphism $\Sigma \to \Sigma'$ fixing A and C pointwise.*

(B2"): *Let Σ and Σ' be two apartments containing a chamber C. Then there exists an isomorphism $\Sigma \to \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.*

Remark 6.1.7. A Coxeter complex is a thin building with exactly one apartment.

Remark 6.1.8 ([AB08, Proposition 4.6])**.** A building is colorable and the isomorphisms $\Sigma \to \Sigma'$ in axiom $(B2)$ can be taken to be type-preserving.

Remark 6.1.9. All apartments of a building have the same Coxeter matrix. Thus we can define the Coxeter matrix of a building to be the Coxeter matrix of any of its apartments. Further we have a Coxeter system of a building. One may note that these properties are independent from the given apartment system.

Remark 6.1.10. Let (W, S) be the Coxeter system associated to the Coxeter matrix of a building $\mathcal{B}$. For any apartment Σ of $\mathcal{B}$, there exists a type-preserving isomorphism from Σ to the Coxeter complex $\Sigma(W, S)$.

Proposition 6.1.11 ([AB08, Corollary 4.11])**.** *A building $\mathcal{B}$ is completely determined by its underlying chamber system. More precisely, the simplices of $\mathcal{B}$ are in $1-1$ correspondence with the residues of its chamber systems* $\mathrm{Cham}(\mathcal{B})$*, ordered by reverse inclusion. A simplex A corresponds to the residue $\mathcal{C}_{\geq A}$ of all chambers having A as a face.*

Remark 6.1.12. Let (W, S) be a Coxeter system with Coxeter matrix M. A building $\mathcal{B}$ is said to be of type (W, S) if it admits a type function with values in S such that the Coxeter matrix of $\mathcal{B}$ is M.

Proposition 6.1.13 ([AB08, 4.9])**.** *If $\mathcal{B}$ is a building and A is a simplex in $\mathcal{B}$, then* $\mathrm{lk}(A)$ *is a building. In particular* $\mathrm{lk}(A)$ *is a chamber complex.*

The Complete System of Apartments

Theorem 6.1.14 ([AB08, 4.54])**.** *If $\mathcal{B}$ is a building, then the union of any family of apartment systems is again an apartment system. Consequently, $\mathcal{B}$ admits a largest system of apartments.*

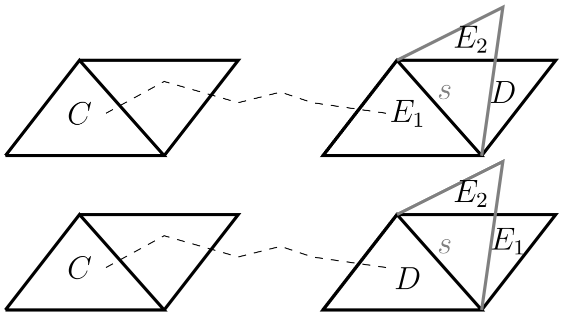


Figure 6.1: $\delta(C,D) = w, \delta(D,E_1) = \delta(D,E_2) = s$

Definition 6.1.15. For a building $\mathcal{B}$, the maximal apartment system will be called the **complete apartment system** or **complete system of apartments**. We denote this apartment system by $\mathcal{A}(\mathcal{B})$.

Proposition 6.1.16 ([AB08, 4.59])**.** *If Σ is a subcomplex of a building $\mathcal{B}$ of type M which is isomorphic to Σ_M, then Σ is an apartment in the complete system of apartments.*

Definition 6.1.17. A **root** in a building $\mathcal{B}$ is a subcomplex which is contained in an apartment Σ of $\mathcal{B}$ and which is a root inside Σ.

6.2 Buildings as W-Metric Sets

Definition 6.2.1. A **building** of type (W,S) is a pair $(\mathcal{C},\delta)$ with a non-empty set $\mathcal{C}$ and a map $\delta : \mathcal{C} \times \mathcal{C} \to W$, satisfying the following properties:

(i) $\delta(C,C') = 1$ if and only if $C = C'$.

(ii) If $\delta(C,D) = w \in W$ and $\delta(C,C') = s \in S$ then $\delta(C',D) \in \{w, sw\}$. Furthermore, if $l(w) < l(sw)$, then $\delta(C',D) = sw$.

(iii) If $w = \delta(C,D)$ for $C,D \in \mathcal{C}$ and $s \in S$, then there exists a chamber $C' \in \mathcal{C}$ with $\delta(C',D) = sw$.

The elements of $\mathcal{C}$ are called **chambers** and the map δ is called the **Weyl distance function** on $\mathcal{C}$.

Definition 6.2.2. Let $s \in S$. Two chambers C,D are called **s-adjacent** if $\delta(C,D) \in \{1,s\}$ and we write $C \sim_s D$. We call two chambers adjacent if they are s-adjacent for some $s \in S$ and we write $C \sim D$.

Lemma 6.2.3 ([AB08, Lemma 5.3])**.** *Let C,D,E be chambers of a building $(\mathcal{C},\delta)$. If $\delta(C,D) = s$ then $\delta(D,C) = s$. If $\delta(C,D) = \delta(D,E) = s \in S$, then $\delta(C,E) \in \{1,s\}$.*

With the previous lemma we can define panels:

Definition 6.2.4. Let $s \in S$. A **panel** (or s-panel) is an equivalence class under the adjacency relation (or s-adjacency relation).

Definition 6.2.5. A sequence $\Gamma = (C_0, \dots, C_n)$ of of chambers with $C_i \sim_{s_i} C_{i+1} \neq C_i$ for $0 \leq i < n-1$ is called a **gallery** of length n. The word $s_0 \cdots s_{n-1}$ is called the **type of Γ**. If there exists no gallery of length $< n$ from C_0 to C_n, we say that C_0 and C_n have distance n. If n is the distance of C_0 and C_n, we say that Γ is a **minimal gallery**.

Lemma 6.2.6 ([AB08, 5.17])**.** *For any two chamber $C, D \in \mathcal{B}$, we have*

$$\delta(C, D) = \delta(D, C)^{-1}.$$

Remark 6.2.7 ([AB08, Proposition 4.41])**.** Let $\Gamma = (C_0, \dots, C_n)$ be a gallery of type $w = s_1 \cdots s_n$. Then Γ is a minimal gallery if and only if w is a reduced word.

Definition 6.2.8. The Weyl distance function δ of a building $(\mathcal{C}, \delta)$ induces a metric $d \colon \mathcal{C} \times \mathcal{C} \to \mathbb{N}$ on $(\mathcal{C}, \delta)$ by

$$d(C, D) := l(\delta(C, D)).$$

Definition 6.2.9. Let $J \subset S$. A J-residue R of a building $\mathcal{B}$ is defined as a J-connected component. It is a set of chambers such that

$$R = \{D \in \mathrm{Cham}(\mathcal{B}) \mid \delta(D, C) \in W_J\}$$

for some chamber C in $\mathcal{B}$. A **residue** is a J-residue for some $J \subset S$.

Proposition 6.2.10 ([AB08, 5.30])**.** *A J-residue R together with the Weyl metric δ restricted to R is a building of type* (W_J, J)

Definition 6.2.11. A subset $\mathcal{C} \subset \mathrm{Cham}(\mathcal{B})$ of a building $\mathcal{B}$ is called **convex** if for any two chambers $C, D \in \mathcal{C}$ every minimal gallery from C to D is also contained in $\mathcal{C}$. The **convex hull** $\mathrm{conv}(\mathcal{C})$ of a subset $\mathcal{C} \subset \mathrm{Cham}(\mathcal{B})$ is the smallest convex subset of $\mathrm{Cham}(\mathcal{B})$ containing $\mathcal{C}$.

Remark 6.2.12. Residues are convex subsets.

Definition 6.2.13. A building of type (W, S) is a connected chamber system (see 4.4). Thus we can define a **building morphism** $\phi : (\mathcal{C}, \delta) \to (\mathcal{C}', \delta')$ of buildings to be a morphism $\phi : (\mathcal{C}, \delta) \to (\mathcal{C}', \delta')$ of chamber systems. The definition of a building isomorphism and building automorphism follow directly.

Definition 6.2.14. Let θ be an automorphism of a building $(\mathcal{C}, \delta)$. The set

$$\boldsymbol{W_\theta} := \{w \in W \mid \exists\, C \in (C) : \ \delta(C, \theta(C)) = w\}$$

is called the **Weyl displacement set**. The elements in W_θ are called **Weyl displacements**.

6.3 Simplicial Complexes vs. W-Metric

We will follow [AB08, section 5.6] to show that the simplicial approach and the Weyl metric approach for buildings yield the same objects. To do so, we will construct a simplicial complex corresponding to a building of type (W, S) as a chamber system with Weyl metric δ and we will obtain a Weyl metric δ on the set of chambers for a building as a simplicial complex.

Remark 6.3.1. We have to restrict the type of the building to be of finite rank, i.e. the generating set S for the pair (W, S) is finite. This is due to the structure of simplicial complexes.

We have seen that a building (as a simplicial complex) has a unique Coxeter matrix which yields a Coxeter group W. Furthermore for the coloring in 6.1.8, the set of colors can be taken to be the set S for a Coxeter system (W, S).
By 6.1.11 we get a chamber system $\mathrm{Cham}(\mathcal{B})$ for $\mathcal{B}$. Apartments are convex, i.e. every minimal gallery of $\mathrm{Cham}(\mathcal{B})$ is contained in an apartment and every apartment Σ carries a well-defined Weyl distance function $\delta_\Sigma\colon \Sigma \times \Sigma \to W$. If C, D are chambers of $\mathcal{B}$, then by definition there exists an apartment Σ containing both chambers and we can define $\delta(C, D) := \delta_\Sigma(C, D)$. By 6.1.8 this definition is independent from the choice of Σ. From this we get a function $\delta\colon \mathcal{B} \times \mathcal{B} \to W$ which we will call the **Weyl distance function** of $\mathcal{B}$.

Proposition 6.3.2 ([AB08, 4.84])**.** *The Weyl distance function on a building $\mathcal{B}$ as a simplicial complex satisfies the conditions for $(\mathrm{Cham}(\mathcal{B}), \delta)$ to be a building as a W-metric set.*

Definition 6.3.3. Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S) (of finite rank). We define a poset

$$\Delta(\mathcal{B}) := \{R \mid R \text{ is a residue of } \mathcal{B}\},$$

with partial order

$$R \leq R' \Leftrightarrow R \supseteq R'.$$

We define a map

$$\tau(R) = S \setminus J,$$

where J is the type of R

Remark 6.3.4. Following section 5.6 in [AB08], the poset $\Delta(\mathcal{B})$ is a colorable chamber complex whose chambers are exactly the chambers of $\mathcal{B}$ and τ is a type-function. In the simplicial complex $\Delta(\mathcal{B})$ one might understand the residue R as the simplex whose link corresponds to the chambers in R (seen as a residue in $\mathcal{B}$).

Theorem 6.3.5 ([AB08, 5.93])**.**

(i) *Let $\mathcal{B}$ be a building (as a simplicial complex) of type (W, S), and let $(\mathrm{Cham}(\mathcal{B}), \delta)$ be the W-metric building associated to $\mathcal{B}$ (see 6.3.2). Then the chamber complex $\Delta(\mathrm{Cham}(\mathcal{B}))$ is canonically isormorphic to $\mathcal{B}$.*

(ii) *Let* $(\mathcal{B}, \delta)$ *be a building (as a* W*-metric set) of type* (W, S)*, and let* $\Delta(\mathcal{B})$ *be the corresponding simplicial building of type* (W, S)*. Then the* W*-metric building associated to* $\Delta(\mathcal{B})$ *is equal to the original building* $(\mathcal{B}, \delta)$*.*

CHAPTER

SEVEN

BUILDINGS AND GROUPS

7.1 Weyl Transitive Action

Let $\mathcal{B}$ be a building of type (W, S). Let G be a group acting on $\mathcal{B}$ type-preservingly.

Definition 7.1.1. An action of G on $\mathcal{B}$ is **chamber transitive** if it acts transitively on the set $\mathrm{Cham}(\mathcal{B})$. It is **Weyl transitive** if for each $w \in W$, the action is transitive on the set of ordered pairs (C, D) of chambers with $\delta(C, D) = w$.

Proposition 7.1.2 ([AB08, 6.11])**.** *Let G be a group acting chamber transitively on $\mathcal{B}$. Let C be a chamber and Σ a apartment containing C (in the complete apartment system of $\mathcal{B}$). Let B be the stabilizer of C in G. Then the action of G on $\mathcal{B}$ is Weyl transitive if and only if*

$$\mathcal{B} = \bigcup_{b \in B} b\Sigma.$$

Remark 7.1.3. Assume G acts Weyl transitively on $\mathcal{B}$. Let C be a chamber and let B be its stabilizer in G. We can identify the set $\mathrm{Cham}(\mathcal{B})$ of chambers with G/B of left cosets gB via $gC \leftrightarrow gB$ for $g \in G$. By the Weyl transitive action, the B-orbits in $\mathrm{Cham}(\mathcal{B})$ are in $1-1$ correspondence with the elements of W, with the orbit of a chamber D corresponding to $w = \delta(C, D)$. But the B-orbits in in $\mathrm{Cham}(\mathcal{B})$ correspond to the B-orbits in G/B and hence to double cosets BgB.

Theorem 7.1.4 ([AB08, 6.17])**.** *Assume that the action of G on $\mathcal{B}$ is Weyl transitive, and let B be the stabilizer of a chamber C. Then there is a bijection $B\backslash G/B \to W$ given by $BgB \mapsto \delta(C, gC)$. Hence $G = \coprod_{w \in W} C(w)$, where $w \mapsto C(w)$ is the inverse bijection.*

7.2 Bruhat Decomposition

Let G be a group, $B \leq G$ a subgroup, (W, S) a Coxeter system.

Definition 7.2.1. If there exists a bijection $C : W \to B\backslash G/B$ satisfying:

(*B*): For all $s \in S, w \in W$:

$$C(sw) \subset C(s)C(w) \subset C(sw) \cup C(w),$$

and if $l(sw) = l(w) + 1$, then $C(s)C(w) = C(sw)$,

then C is said to provide a **Bruhat decomposition** of type (W, S).

Definition 7.2.2. Given a Bruhat decomposition C for G, B, let $T \subset S$ and let A be a face of the fundamental chamber of cotype T. The stabilizer of A in G is

$$P_T := \bigcup_{w \in W_T} C(w).$$

By [AB08, 6.27] the sets P_T are groups. We call these groups standard parabolic subgroups of W and their left cosets standard parabolic cosets. We denote by $\mathcal{B}(G, B)$ the poset of standard parabolic cosets, ordered by reverse inclusion.

Proposition 7.2.3 ([AB08, Proposition 6.34])**.** *Given a Bruhat decomposition for (G, B), the poset $\mathcal{B}(G, B)$ is a building, and the natural action of G on $\mathcal{B}$ by left translation is Weyl transitive and has B as the stabilizer of a fundamental chamber. Conversely, if a group G admits a Weyl transitive action on a building $\mathcal{B}$ and B is the stabilizer of a fundamental chamber, then (G, B) admits a Bruhat decomposition and $\mathcal{B}$ is canonically isomorphic to $\mathcal{B}(G, B)$.*

Axioms 7.2.4. Let G be a group, B a subgroup, (W, S) a Coxeter system, and $C\colon W \to B\backslash G/B$ a function. Consider the following axioms:

(Bru 1) $C(w) = B$ if and only if $w = 1$.

(Bru 2) $C\colon W \to B\backslash G/B$ is surjective, i.e.

$$G = \bigcup_{w \in W} C(w).$$

(Bru 3) For any $s \in S$ and $w \in W$:

$$C(sw) \subset C(s)C(w) \subset C(sw) \cup C(w).$$

(Bru 3′) For any $s \in S$ and $w \in W$:

$$C(ws) \subset C(w)C(s) \subset C(ws) \cup C(w).$$

Proposition 7.2.5 ([AB08, Proposition 6.36])**.** *Let G be a group and B a subgroup. Suppose we are given a group W, a generating set S consisting of elements of order 2, and a function $C\colon W \to B\backslash G/B$ satisfying (Bru 1), (Bru 2), and (Bru 3). Then the six conditions below are satisfied.*
In particular, C provides a Bruhat decomposition for (G, B) if (W, S) is a Coxeter system.

(i) *C is a bijection, i.e.*

$$G = \coprod_{w \in W} C(w).$$

(ii) *$C(w)^{-1} = C(w^{-1})$ for all $w \in W$. Consequently, (Bru 3′) holds.*

(iii) *If $l(sw) \geq l(w)$ with $s \in S$ and $w \in W$, then $C(s)C(w) = C(sw)$.*

(iv) *Given a reduced decomposition $w = s_1 \cdots s_l$ of an element $w \in W$, we have $C(w) = C(s_1) \cdots C(s_l)$.*

(v) *If $l(sw) \leq l(w)$ with $s \in S$ and $w \in W$, and if $[C(s) : B] \geq 2$, then $C(s)C(w) = C(sw) \cup C(w)$.*

(vi) *Let $J \subset S$ be an arbitrary subset. Then $P_J := \bigcup_{w \in W_J} C(w)$ is a subgroup of G. It is generated by the cosets $C(s)$ with $s \in J$.*

7.3 BN-Pairs

Definition 7.3.1. A pair of subgroups B, N of a group G is a ***BN*-pair** if B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S satisfying:

***BN*1:** For all $s \in S, w \in W$:

$$sBw \subset BswB \cup BwB.$$

***BN*2:** For all $s \in S$:

$$sBs^{-1} \not\subseteq B.$$

Theorem 7.3.2 ([AB08, Theorem 6.56])**.**

(i) *Given a BN-pair (B, N) in G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. Define $\Delta(B, N)$ as the set of B-cosets with Weyl metric $\delta(gB, hB) := w \Leftrightarrow Bg^{-1}hB = BwB$. Then $\Delta(B, N)$ is a thick building that admits a strongly transitive G-action such that B is the stabilizer of a fundamental chamber and N stabilizes a fundamental apartment and is transitive on its chambers.*

(ii) *Conversely, suppose a group G acts strongly transitively on a thick building $\mathcal{B}$ with fundamental apartment Σ and fundamental chamber C. Let B be the stabilizer of C and let N be a subgroup of G that stabilizes Σ and is transitive on the chambers of Σ. Then (B, N) is a BN-pair in G, and $\mathcal{B}$ is canonically isomorphic to $\Delta(B, N)$.*

7.4 Gate Property of Residues

Proposition 7.4.1 ([AB08] 5.34)**.** *Let R be a residue and D a chamber of a building. Then there exists a unique chamber C_1 of R such that $d(C_1, D) = d(R, D)$. The chamber C_1 has the following properties:*

(i) $\delta(D_1, D) = \min(\delta(R, D))$.

(ii) $\delta(C, D) = \delta(C, C_1)\delta(C_1, D)$ *for all* $C \in R$.

(iii) $d(C, D) = d(C, C_1) + d(c_1, D)$ *for all* $C \in R$.

Corollary 7.4.2. *Residues are gated sets.*

7.5 Isometries

Definition 7.5.1. An **isomorphism** of Coxeter systems $\sigma\colon (W, S) \to (W', S')$ is a group isomorphism $\sigma\colon W \to W'$ such that $\sigma(S) = S'$.

Remark 7.5.2. An isomorphism $\sigma\colon (W, S) \to (W', S')$ of Coxeter systems can be seen as a relabeling of the generator set S. Thus it can be identified with an isomorphism of the Coxeter diagrams.

Definition 7.5.3. Let $(\mathcal{B}, \delta)$ be a building of type (W, S) and $(\mathcal{B}', \delta')$ be a building of type (W', S') and let σ be an isomorphism of (W, S) to (W', S'). A σ-**isometry** from $\mathcal{B}$ to $\mathcal{B}'$ is a map $\phi\colon \mathcal{B} \to \mathcal{B}'$ satisfying

$$\delta'(\phi(C), \phi(D)) = \sigma(\delta(C, D)),$$

for all $C, D \in \mathcal{B}$. If $(W, S) = (W', S')$ and σ is the identity, we call ϕ an isometry.

Lemma 7.5.4 ([AB08, Lemma 5.61])**.** *Let $(\mathcal{B}, \delta)$ be a building of type (W, S) and let $(\mathcal{B}', \delta')$ be a building of type (W', S'). Let σ be an isomorphism from (W, S) to (W', S'). Then a map $\phi : \mathcal{B} \to \mathcal{B}'$ is a σ-isometry if and only if it takes s-adjacent chambers to $\sigma(s)$-adjacent chambers for all $s \in S$.*

Definition 7.5.5. A **simple root** corresponding to $s \in S$ of a Coxeter system (W, S) is a set of the form

$$\alpha_s := \{w \in W \mid l(sw) > l(w)\}.$$

A **root** of a building $\mathcal{B}$ of type (W, S) is a subset $\alpha \subset \mathcal{B}$ if it is isometric to a simple root $\alpha_s \subset W$ for some $s \in S$.

Proposition 7.5.6 ([AB08, Proposition 5.82])**.** *Let α be a root of a building $\mathcal{B}$. Then*

(i) *α is a convex subset of $\mathcal{B}$.*

(ii) *α is contained in an apartment of $\mathcal{B}$.*

(iii) *If $\mathcal{B}$ is a thin building of type (W, S), then $\alpha = w\alpha_s$ for some $s \in S$ and $w \in W$.*

Proposition 7.5.7 ([AB08] 5.73)**.** *Let (X, δ) be a building of type (W, S), and let $V \subset W$ be an arbitrary subset. Then any isometry $\theta : V \to X$ can be extended to an isometry $\bar{\theta} : W \to X$. Consequently, any subset of X that is isometric to a subset of W is contained in an apartment.*

CHAPTER

EIGHT

CAT(0) REALIZATION

Davis showed in [Dav98] that buildings carry a CAT(0) *metric. This result is based on the work by Moussong presenting a* CAT(0) *realization for Coxeter systems.*

8.1 The Geometric Realization of a Simplicial Complex

Definition 8.1.1. Let Δ be a simplex with vertex set $\mathcal{V}$. Let V be a vector space with basis $\mathcal{V}$. The **geometric realization** of a simplex $A \in \Delta$ is the convex hull of the element of V associated to its vertices, i.e.

$$|A| := \sum_{v\in A} \lambda_v v \quad \text{with} \quad \lambda_v \geq 0 \quad \text{and} \quad \sum_{v\in A} \lambda_v = 1.$$

Let Σ be a simplicial complex with vertex set $\mathcal{V}$. For each simplex A in Σ let $\mathcal{V}(A)$ denoted the vertex set of A. Let $\tilde{V}$ be a vector space with $\mathcal{V}$ as a basis. The **geometric realization** of Σ (over $\tilde{V}$) is defined as

$$|\Sigma| := \bigcup_{\Delta\in\Sigma} |\Delta|,$$

where $|A|$ is the geometric realization of $|A|$ over the subspace of $\tilde{V}$ with basis $\mathcal{V}(A)$.

8.2 The Davis Realization of a Building

We will sketch a geometric realization for buildings which is CAT(0)*. This realization will be called the Davis realization.*

Let (W, S) be an arbitrary Coxeter system, and let $\mathcal{B}$ be a building of type (W, S).

Notation 8.2.1. Let $\mathcal{S}$ be the poset of spherical subsets of S, ordered by inclusion. Let Z be the geometric realization $|\mathcal{F}(\mathcal{S})|$ of the flag complex $\mathcal{F}(\mathcal{S})$. For $s \in S$, let $\mathcal{S}_s$ be the elements of $\mathcal{S}$ containing s and let $Z_s := |\mathcal{F}(\mathcal{S}_s)|$. For $z \in Z$, let $S_z := \{s \in S \mid z \in Z_s\}$ and define $W_z := \langle S_z \rangle$.
On the product $\mathrm{Cham}(\mathcal{B}) \times Z$ we define $\sim$ to be the equivalence relation given by

$$(C, z) \sim (D, z') \quad \Leftrightarrow \quad z = z' \quad \text{and} \quad \delta(C, D) \in W_z. \qquad (*)$$

Definition 8.2.2. The **Z-realization** $Z(\mathcal{B})$ of a building $\mathcal{B}$ is the quotient of $\mathrm{Cham}(\mathcal{B}) \times Z$ by the equivalence given in $(*)$. The equivalence class of (C, z) will be denoted by $[C, z]$ and we define $Z(C) := \{[C, z] \mid z \in Z\}$ for any chamber C of $\mathcal{B}$.

Definition 8.2.3. The **type function** on $Z(\mathcal{B})$ is the map $\tau\colon Z(\mathcal{B}) \to Z$ defined by $\tau([C, z]) := z$. For any $x \in Z(\mathcal{B})$, the image $\tau(x)$ is called the **type** of x.

Remark 8.2.4. The type function induces a bijection from every $Z(C)$ to Z.

Definition 8.2.5. The **dual Coxeter complex** $\Sigma_d(W, S)$ is a regular cell complex whose cells are the cells of the form

$$e_A := |\mathcal{F}(\Sigma(W, S))_{\geq A}|.$$

Its (nonempty) cells correspond to the finite standard cosets in W, ordered by inclusion. The stabilizers are the finite parabolic subgroups of W, i.e. conjugates of the finite standard parabolic subgroups.

Remark 8.2.6 ([AB08, section 12.3.3])**.** If $\Sigma(W, S)$ is finite, then $|\Sigma_d(W, S)|$ is topologically the cone over $\Sigma(W, S)$.

Proposition 8.2.7 ([AB08, Proposition 12.55])**.** *As a set, $|\Sigma_d(W, S)|$ is canonically in $1-1$ correspondence with $Z(W, S)$.*

Definition 8.2.8. Let (W, S) be a finite Coxeter system. Let x be a point in the interior of the fundamental chamber in the canonical linear representation of W, i.e. on $\mathbb{R}^S$, with basis $(e_s)_{s \in S}$ together with a bilinear form $B(e_s, e_t) = -\cos(\frac{\pi}{m_{st}})$. The Coxeter polytope associated (W, S) is the convex polytope C_W defined as the convex hull of $W.x$ (a generic W-orbit). The action of W is given by $s.v := v - 2\frac{B(e_s, v)}{B(e_s, e_s)} e_s$.

Definition 8.2.9. The **nerve** $L(W, S)$ of the Coxeter system (W, S) is the abstract simplicial complex $\mathcal{S}_{>\emptyset}$ of all nonempty spherical subsets.

Proposition 8.2.10 ([Dav08] 7.3.4)**.** *There is a natural cell structure on $\Sigma_d(W, S)$ so that its vertex set is W, its 1-skeleton is the Cayley graph (for the generating set S), and its 2-skeleton is the Cayley 2-complex, i.e. the Cayley graph with a 2-cell attached for each relation in R for $W = \langle S \mid R \rangle$.*

Theorem 8.2.11 ([AB08, Theorem 12.58])**.** *The space $|\Sigma_d(W, S)|$ with its piecewise Euclidean metric is a* CAT(0) *space.*

Remark 8.2.12. There exists a distance function on $Z(\mathcal{B})$ which is given as follows:
A finite sequence of point $\gamma := x_1, \ldots, x_n$ in $Z(\mathcal{B})$ is called a chain if there exists a chamber C_i with $x_i, x_{i+1} \in Z(C_i)$ for each $i \in \{1, \ldots, n-1\}$. The length of a chain is defined by

$$l(\gamma) := \sum_{i=1}^{n-1} d_Z(\tau(x_i), \tau(x_{i+1})),$$

where $d_Z\colon Z(\mathcal{B}) \to \mathbb{R}$ is the metric on Z coming from the Euclidean metric on Σ_d.

Proposition 8.2.13 ([AB08, Proposition 12.10])**.** *The distance function on $Z(\mathcal{B})$ given by $d(x,y) := \inf_{\gamma} l(\gamma)$, where γ ranges over all chains from x to y, is a metric.*

Definition 8.2.14. The **Davis realization** for a building $\mathcal{B}$ is the geometric realization $Z(\mathcal{B})$ together with the metric in 8.2.13 and will be denoted by $\mathcal{X}(\mathcal{B})$, or just $\mathcal{X}$ if $\mathcal{B}$ is known.

Theorem 8.2.15 ([AB08, 12.66])**.** *For any building $\mathcal{B}$, its Davis realization is a complete* CAT(0) *space.*

Proposition 8.2.16 ([AB08, 12.3.4])**.** *Let $\mathcal{X}$ be the Davis realization of a Coxeter system (W,S). The group W operates on $\mathcal{X}$, and the cell stabilizers are the finite parabolic subgroups of W.*

Proposition 8.2.17 ([AB08, 12.16])**.** *Given a type-preserving chamber map $\phi : \mathcal{B} \to \mathcal{B}'$, the induced map $\phi : \mathcal{X} \to \mathcal{X}', \quad [c,z] \mapsto [\phi(c), z]$ is distance-decreasing, i.e.*

$$d(\phi(x), \phi(y)) \leq d(x,y)$$

for all $x, y \in \mathcal{X}$.

Corollary 8.2.18. *For any type-preserving building automorphism $\phi : \mathcal{B} \to \mathcal{B}$, the induced map $\phi : \mathcal{X} \to \mathcal{X}$ is an isometry.*

Proof. We use 8.2.17 for ϕ and ϕ^{-1} and get

$$d(x,y) \leq d(\phi(x), \phi(y)) \leq d(\phi^{-1}(\phi(x)), \phi^{-1}(\phi(y))) = d(x,y),$$

for all $x, y \in \mathcal{X}$. □

8.3 Geometric Counterparts

Let $\mathcal{X}$ be the Davis realization of a building $\mathcal{B}$.

Definition 8.3.1. For any element $z \in \mathcal{X}$, we define a spherical residue

$$\boldsymbol{R(x)} := \{C \in \mathrm{Cham}(\mathcal{B}) \mid x \in |C|\}.$$

Definition 8.3.2. Let $[\gamma]$ be a geodesic segment, ray, or line in $\mathcal{X}$. We define a relation on the elements in $[\gamma]$ by:

$$\text{For all } x, y \in [\gamma] : x \leq y \text{ if and only if } \gamma^{-1}(x) \leq \gamma^{-1}(y).$$

As γ is an isometry, one sees that this relation is independent of the choice of γ describing the segment $[\gamma]$.
Let a be the infimum and b be the supremum of the domain of γ. For any $z \in [\gamma]$, we define:

$$\boldsymbol{\gamma_z^+} := \begin{cases} \gamma([t,b]) & \text{if the domain of } \gamma \text{ has a maximum} \\ \gamma([t,b)) & \text{else} \end{cases}$$

and

$$\boldsymbol{\gamma_z^-} := \begin{cases} \gamma([t,a]) & \text{if the domain of } \gamma \text{ has a minimum} \\ \gamma([t,a)) & \text{else,} \end{cases}$$

where $t := \gamma^{-1}(z) \in \mathbb{R}$. We say γ_z^+ is the positive subgeodesic of γ starting at z and γ_z^- is the negative subgeodesic of γ starting at z.

Definition 8.3.3. For any convex subset $\mathcal{C}$ of $\mathcal{X}$, we denote by $\boldsymbol{R(\mathcal{C})}$ the union $\bigcup\limits_{x \in \mathcal{C}} R(x)$ and by $\boldsymbol{\mathcal{R}(\mathcal{C})}$ the set $\{R(x) \mid x \in \mathcal{C}\}$.

Definition 8.3.4. For any subset $\mathfrak{C}$ of $\mathcal{B}$, we define $\boldsymbol{|\mathfrak{C}| := \bigcup\limits_{C \in \mathrm{Cham}(\mathfrak{C})} |C|}$.
A subset $E \subseteq \mathcal{X}$ is called **geometric apartment** if there exists an apartment Σ such that $|\Sigma| = E$.

Notation 8.3.5. The set of all roots inside a given apartment Σ in $\mathcal{B}$ will be denoted by $\boldsymbol{\Phi(\Sigma)}$.

Notation 8.3.6. For any root α, the associated geometric root is the convex hull of $\{|C| \mid C \in \alpha\}$. It will also be denoted by α.
Let $\mathcal{X}$ be the Davis realization of a Coxeter system (W, S), and let α be a root inside (W, S). The reflection s_α mapping α to $-\alpha$ induces a reflection $\tilde{s}_\alpha$ in $\mathcal{X}$. We call the fixed point set of $\tilde{s}_\alpha$ the **wall corresponding to $\boldsymbol{\alpha}$**. A wall in $\mathcal{X}$ is a wall corresponding to a root α in (W, S).
If a root α is determined by the chambers corresponding to 1_W and s, we call s_α a **simple reflection**.

Definition 8.3.7. A **(geometric) $\boldsymbol{n}$-flat** is a closed convex subset of $\mathcal{X}$ which is isometric to $\mathbb{E}^n$.

Lemma 8.3.8 ([CH09, 6.2])**.** *Let Σ be an apartment of a building and let $\mathcal{C}$ be a set of chambers in Σ. Suppose there exists a residue R and a chamber $C \in \mathcal{C}$ such that $C \in R$ and $\mathrm{proj}_R(D) = C$ for all chambers $D \in \mathcal{C}$. Then, for any chamber $E \in R \setminus \{C\}$, there exists an apartment Σ' containing $\mathcal{C} \cup \{E\}$.*

Theorem 8.3.9 ([CH09, 6.3]). *Let (W, S) be a Coxeter system and $\mathcal{B}$ be a building of type (W, S). Let $F \subset \mathcal{X}$ be a geometric n-flat and let c_0 be a chamber such that* $\dim(F \cap |c_0|) = n$.
Define

$$C(F, c_0) := \{\mathrm{proj}_{R(x)}(c_0) \mid x \in F\}.$$

Then there exists a geometric apartment $|\Sigma|$ such that $C(F, c_0) \subset \Sigma$. In particular, we have $F \subset |\Sigma|$.

CHAPTER

NINE

AFFINE BUILDING

As stated in the introduction, affine buildings behave very special concerning our guiding questions. A very important part of their structure are the wall trees. The cases where an action preserves such a wall tree provide examples for some of the presented results. We will not use any further properties of affine buildings and thus give a very short definition based on their classification (see[Tit86]).
A Coxeter system (W, S) is called affine if its Coxeter diagram is one of the ones in figure 9.1 or if it is a direct product of such groups. A building is called affine if its Coxeter system is affine.
These objects are called affine, as the corresponding Coxeter complex carries a Euclidean metric and one might think of W inducing a tiling of an Euclidean space by (compact) polytopes.

9.1 Wall Trees in Affine Buildings

This section is based on [Wei09].

Definition 9.1.1 ([Wei09, 29.32])**.** Let $\mathcal{B}$ be a building and let α be a root in some apartment of $\mathcal{B}$. Then $\mu(\alpha)$ denotes the set of all panels P of $\mathcal{B}$ such that $|P \cap \alpha| = 1$. The set $\mu(\alpha)$ is called **wall** of the root α.

Definition 9.1.2. A wall M is said to be contained in an apartment Σ if there exists a root α contained in Σ such that $M = \mu(\alpha)$. A wall M is said to be contained in a root β if there exists a root α properly contained in β such that $M = \mu(\alpha)$.

Reminder 9.1.3. Recall that in this thesis we always consider buildings equipped with their complete system of apartment.

Definition 9.1.4. Two walls M and M' are said to be adjacent if there exists an apartment Σ containing roots α and α' such that $M = \mu(\alpha), M' = \mu(\alpha')$ and α' is contained maximally in α. If two walls M, M' are adjacent, we write $M \sim M'$.

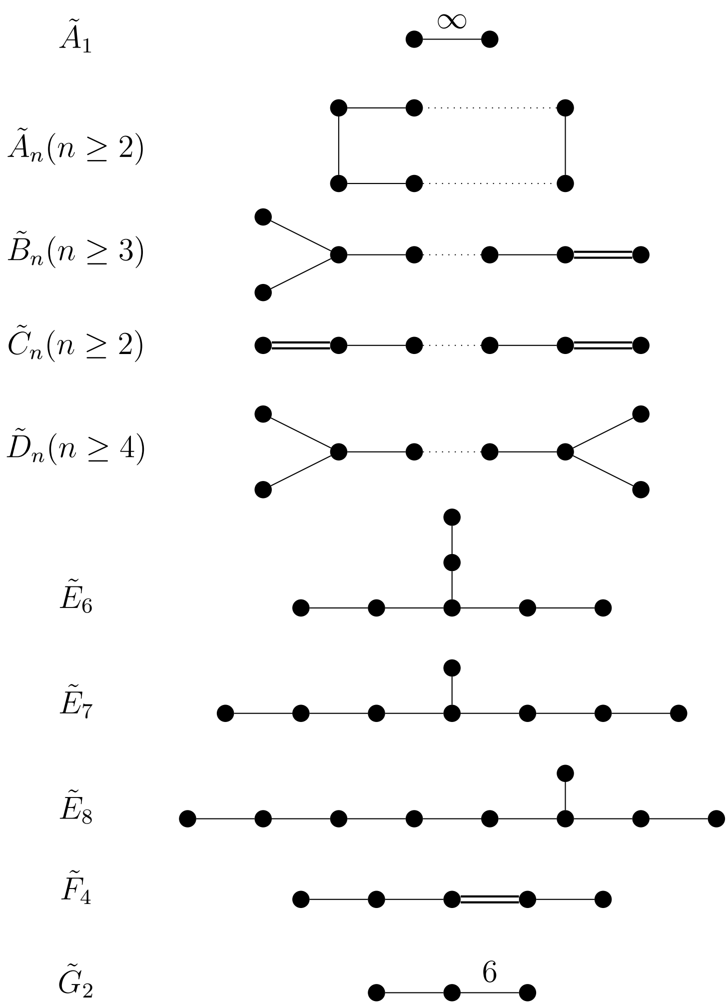


Figure 9.1: The irreducible affine Coxeter diagrams

Remark 9.1.5. If α' is maximally contained in α, then $-\alpha$ is maximally contained in $-\alpha'$. Adjacency of walls is a symmetric relation. Furthermore by [Wei09, 1.46] adjacent walls are parallel.

Definition 9.1.6. Let M and M' be adjacent walls and let Σ and α be as in 9.1.4. We set

$$[M, M'] := -\alpha' \cap \alpha.$$

Lemma 9.1.7 ([Wei09, 10.10])**.** *The complete apartment system for $\mathcal{B}$ is full, i.e. if α is a root in some apartment Σ, P is a panel determined by the wall $\bar{\alpha}$ and if C is a chamber in $P \setminus \alpha$, then the set of all apartments containing $\alpha \cup C$ is non-empty.*

Proposition 9.1.8 ([Wei09, 10.14])**.** *Let m be a parallel class of walls. Define T_m to be the graph whose vertices are the walls in m and where two vertices M, M' are connected if and only if $M \sim M'$. Then T_m is a tree.*

PART IV

Displacements In Buildings

CHAPTER
TEN

INTRODUCTORY EXAMPLES

The main goal of this thesis is to understand automorphisms of buildings in terms of Weyl displacements. To give some idea about this problem, this part starts with some examples which will motivate the two questions mentioned in the introduction:

What can we say about a given automorphism of a building?

and

What can we say about W_θ?

At the beginning we show that it is generally not possible to have an action on a building such that every element of the corresponding Weyl group is a displacement for this action. In particular, for every automorphism θ of an affine building $\mathcal{B}$:

$$W_\theta := \{w \in W \mid \exists\, C \in \mathcal{B} :\ \delta(C, \theta(C)) = w\} \neq W.$$

10.1 Some Preliminaries

Definition 10.1.1. Let (W, S) be a Coxeter system. A **Coxeter element** is an element of W, for which every expression contains each generator in S exactly once. An element $v = s_1 \cdots s_l \in W$ with $s_1, \ldots, s_l \in S$ such that $l(v^k) = l^k$ for all $k \in \mathbb{N}_{>0}$ is called **logarithmic** or **straight** (see [Mar14, BBE+12, Kra08]). This means that any power of the expression $s_1 \ldots s_l$ is a reduced expression.

Lemma 10.1.2. *Let $\mathcal{B}$ be a building and θ a type-preserving automorphism of $\mathcal{B}$. If a chamber D has a straight displacement, then the orbit of D under the action of $\langle\theta\rangle$ is unbounded.*

Proof. Let D be a chamber with a straight displacement v. Then $\delta(D, \theta^k(D)) = v^k$ is a reduced word of length $k \cdot l(v)$. Thus $d(D, \theta^k(D)) = k \cdot l(v)$ and hence the orbit of D is unbounded. □

Lemma 10.1.3. *Let $\mathcal{B}$ be a building and θ a type-preserving automorphism of $\mathcal{B}$, then the following statements are equivalent:*

(i) *θ stabilizes a spherical residue.*

(ii) *W_θ contains a spherical element.*

(iii) *For all $C \in \mathcal{B}$, the θ-orbit of C is bounded.*

Proof. $(i) \Leftrightarrow (ii)$ follows directly as θ is type-preserving. Let D be an arbitrary chamber in $\mathcal{B}$ and assume (i) holds, then there exists a chamber $E \in \mathcal{B}$ whose orbit lies inside a spherical residue R. Let d_R be the diameter of R. For $l \in \mathbb{Z}, k \in \mathbb{N}_{>0}$, we compute:

$$\begin{aligned} d(\theta^l(D), \theta^{l+k}(D)) &= d(D, \theta^k D) \\ &\leq d(D, E) + d(E, \theta^k(D)) \\ &\leq d(D, E) + d(\theta^k(E), \theta^k(D)) + d_R \\ &= d(D, E) + d(E, D) + d_R = 2 \cdot d(D, E) + d_R. \end{aligned}$$

Hence the orbit of D is bounded and thus $(i) \Rightarrow (iii)$ holds. Now assume (iii) holds and let $E \in \mathcal{B}$. Then the orbit $\langle\theta\rangle$ of the barycenter of $|E|$ in the Davis realization of $\mathcal{B}$ is bounded. By Bruhat-Tits fixed-point theorem (3.6.9) the action of $\langle\theta\rangle$ has a fixed point, say y. We conclude that $\langle\theta\rangle$ stabilizes $R(y)$, the spherical residue consisting of all chambers $D \in \mathcal{B}$ with $y \in |D|$, hence (iii) implies (i). □

An immediate consequence of the proof is:

Corollary 10.1.4. *Let $\mathcal{B}$ be a building and θ an automorphism of $\mathcal{B}$, then the following statements are equivalent:*

(i) *θ stabilizes a spherical residue.*

(ii) *For all $C \in \mathcal{B}$, the θ-orbit of C is bounded.*

Remark 10.1.5. The statements in 10.1.3 are not equivalent given a non type-preserving automorphism θ. Let (W, S) be the Coxeter system $\langle s, t \mid s^2 = t^2 = 1\rangle$ of type $\tilde{A}_1$. The displacements of the (building) automorphism θ given by

$$w \mapsto \begin{cases} sw & \ell(w) \quad \text{even} \\ tw & \ell(w) \quad \text{odd} \end{cases}$$

are either t or s and thus spherical. But $\theta^{2k}(w)$ is either $(st)^k w$ or $(ts)^k w$ and both have distance of length $2k$ to w. Thus $\langle\theta\rangle$ has no spherical orbits.

Corollary 10.1.6. *Let $\mathcal{B}$ be a building and θ a type-preserving automorphism of $\mathcal{B}$. If W_θ contains a straight element, then W_θ does not contain spherical elements. And if W_θ contains a spherical element, then W_θ does not contain straight elements.*

Lemma 10.1.7. *Let $\mathcal{B}$ be a building and let θ be an automorphism of $\mathcal{B}$. If the orbits of θ^k are unbounded, so are the orbits of θ.*

Proof. This follows directly as every orbit of θ^k is contained in an orbit of θ. □

Corollary 10.1.8. *Let $\mathcal{B}$ be a building and θ an automorphism of $\mathcal{B}$. If there exists some $k \geq 1$ such that θ^k is type-preserving and W_{θ^k} contains a straight element, then W_θ does not contain spherical elements.*

Corollary 10.1.9. *Let $\mathcal{B}$ be a building of type (W, S) and θ an automorphism of $\mathcal{B}$. If for some $k > 0$ the automorphism θ^k is type-preserving and W_{θ^k} contains a straight element, then $W \neq W_\theta$.*

In the first example we will use use the following theorem for Coxeter elements: We will use the following theorem for Coxeter elements in most examples:

Theorem 10.1.10 ([Spe09, Theorem 1] see also [BBE+12, Theorem 3.1])**.** *Let W be an infinite, irreducible Coxeter group and let $(s_1, \dots, s_n)$ be any ordering of the simple generators. Then the word $s_1 \dots s_n \dots s_1 \dots s_n$ is reduced for any number of repetitions of $s_1 \dots s_n$, i.e. $s_1 \dots s_n$ is straight.*

Corollary 10.1.11. *By 10.1.10 every infinite Coxeter systems contains straight elements. In particular, for any type-preserving automorphism θ of a building of type (W, S) with infinite Coxeter group W, one has $W \neq W_\theta$.*

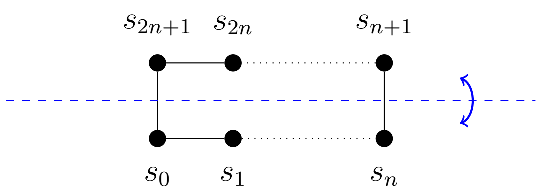
Remark 10.1.12. The next example was first done for locally finite buildings. In a discussion, Timothée Marquis pointed out that one can avoid a this condition.

Example 10.1.13. Let (W, S) be a Coxeter system with a straight element $v \in W$. Let $\mathcal{B}$ be a building of type (W, S) and let θ be a type-preserving automorphism of $\mathcal{B}$. If $v \in W_\theta$, then $1 \notin W$ by 10.1.9, hence $W \neq W_\theta$.

Let θ be non-type-preserving. We want to look at all irreducible affine cases:

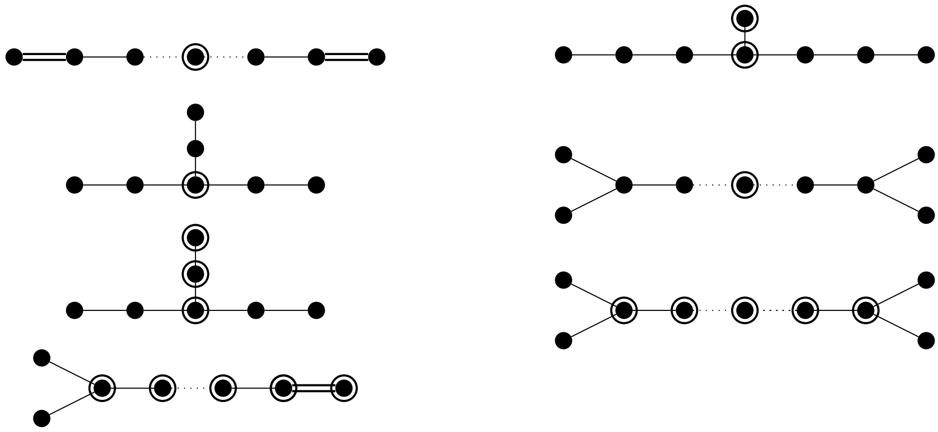
(i) Assume there exists a straight element of the form $s_1 \dots s_n$ for some $n > 1$, where $s_i \neq s_j$ for $i \neq j$ and the induced action of θ on S has the property: $\theta(s_i) = s_{i+k \pmod n}$ for some fixed $k > 0$, where $s_0 := s_n$. If a chamber D of $\mathcal{B}$ has displacement $s_1 \cdots s_k$, then θ^n is type-preserving and $(s_1 \cdots s_n)^k \in W_{\theta^n}$ is a power of a straight element and thus straight itself. Hence $1_W \notin W_\theta$ by 10.1.9. We conclude $W_\theta \neq W$.

(ii) Let $\mathcal{B}$ be a building of type $\tilde{A}_{2n+1}$ and let θ induce a reflection on the corresponding Dynkin diagram interchanging s_0 and s_{2n+1} as well as s_n and s_{n+1}.



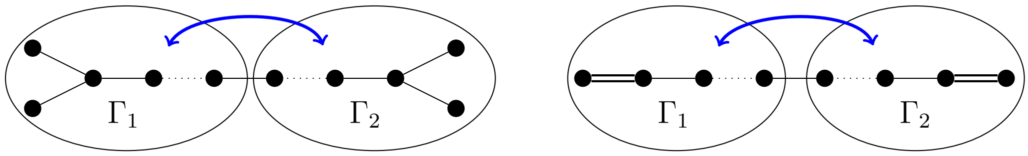
Suppose there is a chamber D with displacement $s_0 \cdots s_n$. Then $\theta(D)$ has displacement $s_{2n+1} \dots s_{n+1}$. The word $w = s_0 \dots s_n s_{2n+1} \dots s_{n+1}$ is a Coxeter element. By 10.1.10 the word w is straight. Hence W_{θ^2} contains the straight element w. The automorphism θ^2 is type-preserving and by 10.1.9 $W_\theta \neq W$.

(iii) Let $\mathcal{B}$ be an affine building with an automorphism θ corresponding to one of the following diagrams, where the marked vertices are the ones fixed by θ:



Let $V_1, \ldots, V_l$ be the set of orbits of θ on the generating set S. All generators inside an orbit commute and hence given an orbit V_i, every element of W, expressed by a word containing each generator of V_i exactly once, is invariant under the action of θ. For each orbit V_i, let v_i be such a word and let w be the product $v_1 \cdots v_l$. The element w is invariant under θ and it is a Coxeter element. Hence by theorem 10.1.10 w is straight. If $w \in W_\theta$, then $w^l \in W_{\theta^l}$ for $l \in \mathbb{N}$ with θ^l type-preserving and by 10.1.9 $W \neq W_\theta$. This implies $W_\theta \neq W$.

(iv) Let $\mathcal{B}$ be a building of type $\tilde{C}_{2n+1}$ or $\tilde{D}_{2n+1}$[1] with an automorphism θ without a fixed point on the corresponding diagram. Let Γ_1, Γ_2 be the two maximal subdiagrams which are interchanged by θ.



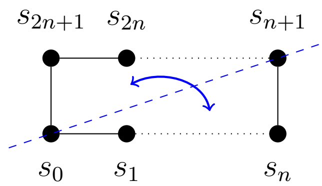
The word $w = w_1 \cdot \theta(w_1)$, where w_1 is a Coxeter element for the Weyl group corresponding to Γ_1, is a Coxeter element for W which is straight by 10.1.10. If $w \in W_{\theta^2}$, then by 10.1.9 $W_\theta \neq W$. We conclude $W_\theta \neq W$.

For the next cases we will use the structure of W as monomial matrices acting on lattice classes, see part V of this thesis.[2] *As before, we will show the existence of a straight element $w' = w \cdot \theta(w)$ which implies the unboundedness of some orbit.*

(v) Let $\mathcal{B}$ be a building of type $\tilde{A}_{2n+1}$ and let θ induce a reflection on the corre-

[1]The rank of those systems is even, i.e. they have an even number of generators.
[2]The expressions used in these cases were found using the program given in appendix VI.

sponding Dynkin diagram fixing s_0 and s_{n+1} and interchanging s_{2n+1} and s_1.



Let D be a chamber with displacement $s_0 s_1 \cdots s_{n+1}$. Then $\theta(D)$ has displacement $s_0 s_{2n+1} \dots s_{n+1}$. The word $w = s_0 s_1 \dots s_{n+1} s_0 s_{2n} \dots s_{n+1}$ corresponds to a monomial matrix of the form:

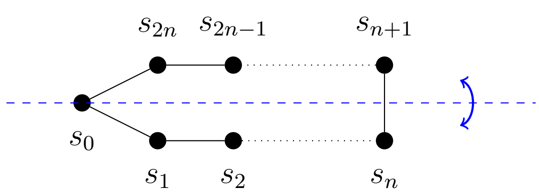
$$M_w := \begin{pmatrix} -1 & 0 & & & & & & & \\ 0 & 0 & & -\pi^{-1} & & & & & \\ 0 & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & & -1 & & & & \\ & & & & & -1 & & & \\ & & & & & 0 & \ddots & & \\ & & & & & & & -1 & \\ & & & & -\pi & & & & \end{pmatrix},$$

where the entry $-\pi^{-1}$ is in the column $(n+1)$ and the entry $-\pi$ in column $(n+3)$. $M_w)^{2n}$ is of the form

$$(M_w)^{2n} = \begin{pmatrix} 1 & & & & & & & \\ & \pi^{-2} & & & & & & \\ & & \ddots & & & & & \\ & & & \pi^{-2} & & & & \\ & & & & 1 & & & \\ & & & & & \pi^2 & & \\ & & & & & & \ddots & \\ & & & & & & & \pi^2 \end{pmatrix}.$$

As the matrix M_w^{2n} shows, w^2 acts on W as a translation, where every element of W lies on a translation axis. This means that $w^{2n} \in W_{\theta^{2n}}$ is straight and by 10.1.9 $W \neq W_\theta$. We conclude $W \neq W_\theta$. [3]

(vi) Let $\mathcal{B}$ be a building of type $\tilde{A}_{2n}$ and let θ induce a reflection on the corresponding Dynkin diagram fixing s_0 and interchanging s_1 and s_{2n}.



[3]The element w itself is already logarithmic. Thinking of the generators acting on the set of rows or columns of a matrix, one can show that erasing any two of the generators of the expression w^{2n} yields a matrix which acts differently from $(M_w)^{2n}$. The resulting matrix is either not a diagonal matrix or its entries have different valuations. Thus we cannot apply the deletion condition which implies that this expression is reduced.

Let D be a chamber with displacement $s_0 s_1 \cdots s_n$. Then $\theta(D)$ has displacement $s_0 s_{2n} \ldots s_{n+1}$. The word $w = s_0 s_1 \ldots s_n s_0 s_{2n} \ldots s_{n+1}$ corresponds to a monomial matrix of the form:

$$\begin{pmatrix}
-1 & & & & & & & & \\
0 & 0 & & & -\pi^{-1} & & & & \\
0 & 1 & & & & & & & \\
& & \ddots & & & & & & \\
& & & 1 & & & & & \\
& & & 0 & & & -1 & & \\
& & & & & & 0 & \ddots & \\
& & & & & & & & -1 \\
& & & & & -\pi & & &
\end{pmatrix},$$

where π is a uniformizing parameter for a field with discrete valuation. The entry $-\pi^{-1}$ is in column $(n+1)$ and the entry $-\pi$ in column $(n+2)$. Its $(2n)$th power has the form:

$$\begin{pmatrix}
1 & & & & & & \\
& \pi^{-2} & & & & & \\
& & \ddots & & & & \\
& & & \pi^{-2} & & & \\
& & & & \pi^{2} & & \\
& & & & & \ddots & \\
& & & & & & \pi^{2}
\end{pmatrix}.$$

As in the previous case, this matrix shows that w^{2n} acts on W as a translation, where every element of W lies on a translation axis. Therefore w^{2n} is straight and the same argument as before shows $W_\theta \neq W$.

Remark 10.1.14. Every non-type-preserving action induces a graph automorphism on the corresponding Coxeter diagram. In the case of affine buildings, the only types allowing such automorphisms are those discussed in the previous examples: $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{E}_6$ and $\tilde{E}_7$.

We come to the following conclusion:

Theorem 10.1.15. *For every automorphism θ of an affine building $\mathcal{B}$ with Coxeter system (W, S) one has $W \neq W_\theta$.*

Remark 10.1.16. One might ask whether it is possible to apply [BBE$^+$12] to the last two cases in order to obtain straight displacments. The answer is: No. The words used in those examples are equivalent to expressions starting with $s_0 s_1 s_0$ which allows the use of a non-short Braid relation by replacing $s_0 s_1 s_0$ with $s_1 s_0 s_1$. But the results in [BBE$^+$12] are for CFC elements which are elements that do not allow non-short Braid relations, in particular given a CFC element w, for any pair $s, t \in S$ with $m_{s,t} > 2$, no expression for w contains an alternating subword of the form $sts..$ of length $m_{s,t}$.

Reminder 10.1.17. The projective line $P^1(\mathbb{F}_q)$ is the set of all one-dimensional subspaces in the vector space $\mathbb{F}_q^2$. [4]

[4]The projective line can be seen as a line consisting of elements in $\mathbb{F}_q$ extended by a point at infinity. Inside a building, this reflects the concept of the root groups (one-parameter subgroups of G_q over $\mathbb{F}_q$) fixing a chamber D of a panel P and acting transitively on $P \setminus \{D\}$.

Example 10.1.18. [5] Let $\mathcal{B}_q$ be the affine building corresponding to $G_q := \mathrm{SL}_4(\mathbb{F}_q((t)))$ (see part V). This means that $\mathcal{B}_q$ is of type $\tilde{A}_3$ and the panels carry the structure of $P^1(\mathbb{F}_q)$.

Let $M_{s_1} := \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in G_q$ and consider the action of G_q on $\mathcal{B}$ (see 14.4 and 14.5 for more information about the action of $\mathrm{GL}_4(\mathbb{F}_q((t))$ or $\mathrm{SL}_4(\mathbb{F}_q((t)))$ on $\mathcal{B}$). We are going to compare the action of M_{s_1} for different values of q.[6] We check for every reduced word of length ≤ 2 in W if this word appears as a Weyl displacement for a chamber in $\mathcal{B}_q(C, 4) := \{C \in \mathcal{B} \mid d(C, D) \leq 4\}$.

w	$q \in \{2, 4\}$	$q \in \{3, 7\}$	$q \in \{5, 9\}$
1_W	✓	✗	✓
s_0	✓	✓	✓
s_1	✓	✓	✓
s_2	✓	✓	✓
s_3	✓	✓	✓
$s_0 s_1$	✗	✓	✓
$s_0 s_3$	✗	✓	✓
$s_1 s_0$	✗	✓	✓
$s_1 s_2$	✗	✓	✓
$s_2 s_0$	✗	✗	✓
$s_2 s_1$	✗	✓	✓
$s_2 s_3$	✗	✓	✓
$s_3 s_0$	✗	✓	✓
$s_3 s_1$	✗	✗	✓
$s_3 s_2$	✗	✓	✓

Explanation

We can define the matrix M_{s_1} for each of the G_q. It represents a simple reflection inside the fundamental apartment Σ of the building (its stabilizer is the set of monomial matrices). The differences in the given cases relate to the following observations:

The existence of fixed chambers: Let C be the fundamental chamber and let P be the s_1-panel containing C. Then M_{s_1} fixes a chamber in P if and only if $\mathbb{F}_q$ contains an element which squares to -1 (see 14.8). In the case of $\mathbb{F}_4$ there are exactly two chambers of P outside of Σ which are not fixed. In the case of $\mathbb{F}_5, \mathbb{F}_9$ we get exactly two fixed chambers.

Now we want to explain the existence/non-existence for Weyl displacements of length 2. Let $D \notin \Sigma$ be s-adjacent to C with $m_{ss_1} = 3$,[7] let P_1 be the s_1-panel containing D and let P_2 be its image. In the case of characteristic 2 there exists exactly one chamber E inside P_1 such that C, E and the image of E lie in a common apartment. Thus E has displacement s and M_{s_1} interchanges the two projections

[5] The calculations for this example were done using the program in the appendix.

[6] The characteristic polynomial for $\mathbb{F}_9$ used by the program is $x^2 - x - 1$.

[7] If $m_{ss_1} = 2$, then these generators commute and the corresponding Weyl displacements are either s or 1_W.

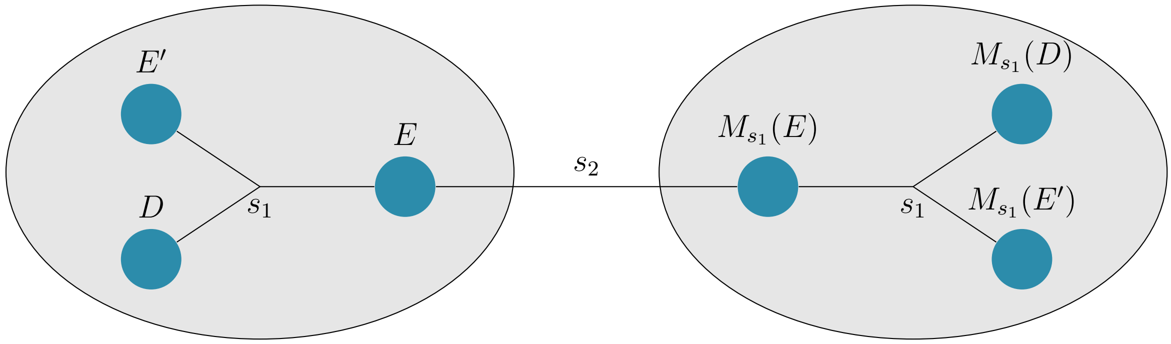


Figure 10.1: Behavior of M_{s_1} in characteristic 2

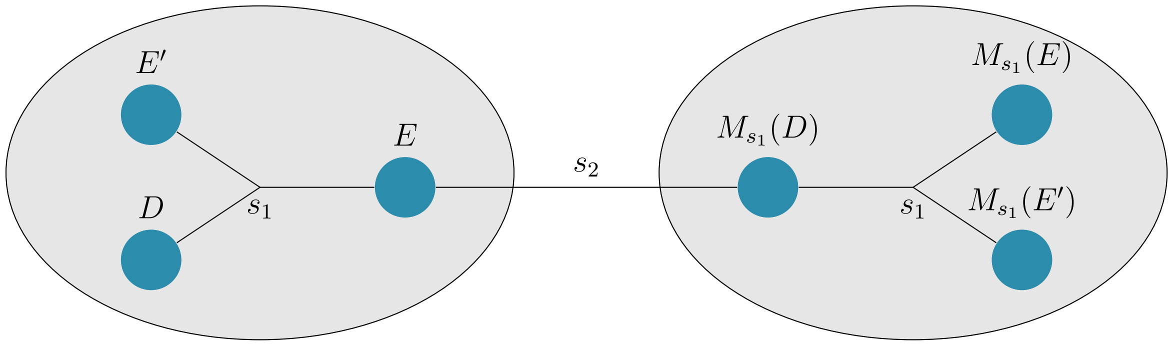


Figure 10.2: Behavior of M_{s_1} in characteristic $\neq 2$

$\mathrm{proj}_{P_2}(P_1)$ *and* $\mathrm{proj}_{P_1}(P_2) = E$. *If the characteristic of the given field is not* 2 *then the action of* M_{s_1} *looks like a translation restricted to* P_1*: The panels are not parallel and* M_{s_1} *does not interchange the projections* $\mathrm{proj}_{P_2}(P_1)$ *and* $\mathrm{proj}_{P_1}(P_2)$.

In terms of matrices we end up calculating the Weyl elements similar to the following case: Let D *be the chamber which we obtain by using the root group element corresponding to the parameter* 1 *(see 14.7.3 and 14.7.4). Then* D *is* s_2 *adjacent to* C. *The Weyl displacements for the elements in* P_1 *correspond to double cosets represented by matrices of the form:* $\begin{pmatrix} 1 & & & \\ & 0 & x & \\ 2 & 1 & & \\ & & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ 2 & 0 & x & \\ 0 & 1 & & \\ & & & 1 \end{pmatrix}$ *and* $\begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}$. *Here* $x \in \mathbb{F}_q$ *is some element* $\neq 0$. *The latter matrix corresponds to the word* $s_1 s_2 s_1$, *but for the first two matrices, depending on the characteristic of the given field, the result is either* s_2 *for both or* $s_2 s_1$ *and* $s_1 s_2$ *respectively.*

CHAPTER

ELEVEN

A GEOMETRIC APPROACH

We will use the CAT(0) *structure of buildings to show that given any automorphism θ on a building $\mathcal{B}$, for every chamber C in $\mathcal{B}$ there exists a minimal gallery of the form $(C, \ldots, D, \ldots \theta(D))$ where D is a chamber whose realization contains an element of* $\mathrm{Min}(\theta)$ *(an elements with minimal displacement). Among other things, this will yield a structure theorem for Weyl displacements in Coxeter systems (11.6.1).*

11.1 A Minimal Gallery along a CAT (0) Geodesic

In this section we will show how to construct a minimal gallery from a chamber C to a chamber D along the geodesic γ between their barycenters. Furthermore this gallery will lie entirely inside $R(\gamma)$, the set of the (spherical) residues which are given as the support of an element of γ.

Let $\mathcal{X}$ be the Davis realization of a building $\mathcal{B}$ of type (W, S).

Reminder 11.1.1 (see 8.2.12 and 8.2.14)**.** A chain in $\mathcal{X}$ is a finite sequence of points $\gamma := x_1, \ldots, x_n$ in $\mathcal{X}$, where two consecutive elements x_i, x_{i+1} are contained in a common cell. The length of a chain $l(\gamma)$ is the sum $\sum_{i=1}^{n-1} d(x_i, x_{i+1})$ and the distance of two elements x, y is the infimum $\inf\limits_{\gamma} \ l(\gamma)$ where γ ranges over the set of chains from x to y.

Lemma 11.1.2. *Let $\mathcal{X}$ be the Davis realization of a building $\mathcal{B}$ with an automorphism θ. The induced map θ' on $\mathcal{X}$ is an isometry.*

Proof. The image of $\mathcal{X}$ is the Davis realization of the building $\theta(\mathcal{B})$ and the induced map $\theta' \colon \mathcal{X} \to \theta'(\mathcal{X})$ gives a one-to-one correspondence from the geodesic segments inside a cell $|C|$ to the geodesic segments inside the cell $\theta'(|C|)$. Let $x, y \in \mathcal{X}$ and let $x_1 = x, \ldots, x_n = y$ be a chain. As θ is a chamber map, the image of $(x_1, \ldots, x_n)$ is a sequence of points $\theta'(x_i)$ such that two consecutive points $\theta'(x_i)$ and $\theta'(x_{i+1})$ are contained in a common (geometric) chamber. Thus for $i \in \{1, \ldots, n-1\}$ the geodesic segment $[x_i, x_{i+1}]$ is mapped to the geodesic segment $[\theta'(x_i), \theta'(x_{i+1})]$ and hence $\theta'(x_1), \ldots, \theta'(x_n)$ is a chain in $\theta'(\mathcal{B})$ with the same length as the chain

$x_1, \ldots, x_n$. As θ^{-1} is again a chamber map, we conclude that we have a length-preserving one-to-one correspondence from the chains from x to y to the chains from $\theta'(x)$ to $\theta'(y)$. Hence $d(\theta'(x), \theta'(y)) = d(x, y)$ by the definition of $d(x, y)$ and $d(\theta'(x), \theta'(y))$ as the infimum of those lengths. □

Reminder 11.1.3. An isometry is called semi-simple if it is elliptic (has a fixed point) or if it is hyperbolic (the set of displacements has an minimum $\neq 0$).

Proposition 11.1.4 ([Bri99, Theorem A]). *Let K be a connected M_κ-polyhedral complex (in our case a Euclidean cell complex). If the set of isometry classes of cells is finite, then every cellular isometry of K is semi-simple.*

Lemma 11.1.5. *Let $\mathcal{B}$ be a building and let $\mathcal{X}$ be its Davis realization. For every automorphism θ of $\mathcal{B}$, the induced isometry on $\mathcal{X}$ is semi-simple.*

Proof. By the definition of the Davis realization 8.2.14 (see also the definition of a Z-realization 8.2.2) the geometric realization of all chambers in $\mathcal{X}$ are isometric. The induced map is an isometry 11.1.2 (for type-preserving automorphisms also by 8.2.17) and preserves the cellular (piecewise Euclidean) structure of $\mathcal{X}$. By 11.1.4 this action is semi-simple. □

Reminder 11.1.6. A geodesic segment/ray/line $[\gamma]$ is a convex set and the set $R(\gamma)$ is defined by $R(\gamma) := \bigcup_{x \in [\gamma]} R(x)$, where $R(x)$ is the support of x, i.e. the set of all chambers whose geometric realization contains x.

Lemma 11.1.7. *Let x, y be two arbitrary elements of $\mathcal{X}$. The set of (spherical) residues in $\mathcal{R}([x, y])$ has a natural order along $[x, y]$:*

$$\mathcal{R}([x, y]) = (R_i)_{i \in \{1, \ldots, n\}}.$$

Proof. Let $[x, y] \subset \mathcal{X}$ be the geodesic joining two elements x, y. Let $(R_\lambda)_{\lambda \in \Lambda}$ be the spherical residues in $\mathcal{R}([x, y])$. We can order the family $(R_\lambda)_{\lambda \in \Lambda}$, by $R_i \leq R_j$ if and only if for any $z_1, z_2 \in [x, y]$ with $R_i = R(z_1), R_j = R(z_2)$, we have $z_1 \leq z_2$ and this is well-defined. As the set of generators S is finite, any (spherical) residue R_λ contains only finitely many residues of $\mathcal{R}([x, y])$. With the distance of x and y being finite, the set Λ is a finite set.

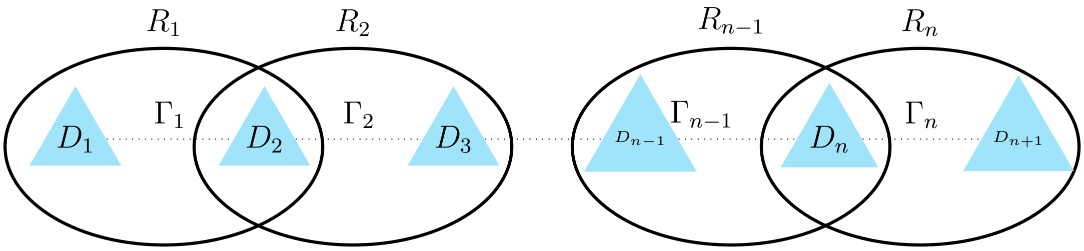
□

Lemma 11.1.8. *Let x, y be two arbitrary points in $\mathcal{X}$. The intersection of two consecutive residues $R_i, R_{i+1} \in \mathcal{R}([x, y])$ is either R_i or R_{i+1}.*

Proof. Let R_i, R_{i+1} be two consecutive residues in $\mathcal{R}([x, y])$. Let $x' \in [x, y] \cap |R_i|$ and $y' \in [x, y] \cap |R_{i+1}|$. For any element $z \in [x', y']$, the residue $R(z)$ equals either R_i or R_{i+1}. Furthermore, as $R_i \cap R_{i+1}$ cannot be empty, there exists a $z \in [x', y']$ which lies in this intersection. But $R(z)$ is a proper subresidue of R_i or R_{i+1}. Thus either $R_i \subsetneq R_{i+1}$ or $R_{i+1} \subsetneq R_i$. □

Notation 11.1.9. For two galleries $\Gamma_1 = (C_0, \ldots, C_n)$ and $\Gamma_2 = (D_0, \ldots, D_m)$ with $C_n = D_0$, we define the product $\Gamma_1 \cdot \Gamma_2$ to be the gallery $(C_0, \ldots, C_n = D_0, \ldots D_m)$.

Lemma 11.1.10. *Let $R_1, \dots, R_n$ be a sequence of spherical residues and for $i \in \{1, \dots, n\}$ let $D_i \in R_i$ such that $\operatorname{proj}_{R_i}(D_1) = D_i$ and let $D_{n+1} \in R_n$. If $D_{i+1} \in R_i$ for all $i \in \{1, \dots, n-1\}$, then there exists a minimal gallery Γ from D_1 to D_{n+1} of the form $\Gamma = \Gamma_1 \cdot \Gamma_2 \cdots \Gamma_n$, where Γ_i is a any minimal gallery inside R_i from D_i to D_{i+1}.*



Proof. The statement holds for $n = 1$, because $D_2 \in R_1$. Let $n > 1$. For $i \in \{1, \dots, n\}$, let Γ_i be a minimal gallery from D_i to D_{i+1}. If $\Gamma_1 \cdots \Gamma_{n-1}$ is a minimal gallery from D_1 to $D_n = \operatorname{proj}_{R_n}(D_1)$, then this gallery extends to a minimal gallery from D_1 to any chamber inside R_n (see 7.4.1). By assumption $D_{n+1} \in R_n$ and thus $\Gamma_1 \cdots \Gamma_{n-1} \cdot \Gamma_n$ is a minimal gallery. □

Lemma 11.1.11. *Let x, y be two arbitrary elements of $\mathcal{X}$ and let $(R_i)_{i\in\{1,\dots,n\}}$ be the ordered set of residues in $\mathcal{R}([x, y])$. Then for all $i \in \{1, \dots, n\}$ and any chamber C in $R(x)$:*

$$\operatorname{proj}_{R_{i+1}}(C) \in R_i.$$

Proof. Let $(R_i)_{i\in\{1\dots n\}} = \mathcal{R}([x, y])$. As the R_i are (spherical) residues, there exists a unique projection C_i of C onto each R_i (see 7.4.1).
As C is contained in $R(x)$, the projection of C onto R_1 is C. Using induction over the index $i > 1$, we will show $C_{i+1} \in R_i$.
If R_{i+1} is contained in R_i then $C_{i+1} \in R_{i+1} \subset R_i$, and $C_{i+1} \in R_i$.
If $R_i \subset R_{i+1}$, assume $C_{i+1} \notin R_i$. By 11.1.10 there exists an apartment Σ, containing C and $\{\operatorname{proj}_{R_j}(C) \mid 1 \le j \le i\}$. As $R_i \subset R_{i+1}$, there exists a minimal gallery from C to C_i containing C_{i+1} by the gate property of residues. Therefore also C_{i+1} is contained in Σ.
As $C_i \notin R_{i+1}$ we get $C_i \neq C_{i+1}$, which shows that there exists a chamber D in $R_{i+1} \cap \Sigma$ which is adjacent to C_i and satisfies $d(C, C_i) + 1 = d(C, D)$, and therefore $D \in R_{i+1} \setminus R_i$.
Let $\alpha \in \Phi(\Sigma)$ be a root containing C and D, but not C_i. It follows from $C \in \alpha$ that $x \in \alpha$. Let $y_D \in |D| \cap [x, y]$ and $y_i \in |C_i| \cap [x, y]$ such that $R(y_D) = R_{i+1}$ and $R(y_i) = R_i$. By the convexity of α we have $d(y_D, x) < d(y_i, x)$ which contradicts the order of R_{i+1} and R_i in $R([x, y])$. □

Proposition 11.1.12. *For two arbitrary elements $x, y \in \mathcal{X}$ and any chamber $C \in R(x)$, there exists a minimal gallery from x along $R([x, y])$ to any chamber D in $R(y)$, i.e. there exists a minimal gallery from C to D which is entirely contained in $R([x, y])$.*
In particular, for two arbitrary chambers $C, D \in \mathcal{B}$ there exists a minimal gallery

from C to D along $R([x,y])$, where x and y are arbitrary elements of $|C|,|D|$ respectively.

Proof. From 11.1.11 and 11.1.10, we get a minimal gallery from C to $\mathrm{proj}_{R(y)}(C)$ which lies entirely in $R([x,y])$. This gallery can be extended to a minimal gallery from C to any chamber inside $R(y)$ by 7.4.1. $\square$

11.2 Definitions and the Elliptic case

Let θ be an automorphism of a building $\mathcal{B}$ and let $\mathcal{X}$ be the Davis realization of $\mathcal{B}$. Let θ also denote the induced isometry of $\mathcal{X}$. For every chamber C of $\mathcal{B}$, we define:

$$\begin{aligned} b_C &:= \text{ the barycenter of } |C|, \\ \mathrm{Min}(\theta)_C &:= \mathrm{conv}(\{\theta^z(\mathrm{proj}_{\mathrm{Min}(\theta)}(b_C)) \mid z \in \mathbb{Z}\}), \\ \mathrm{M}_C(\theta) &:= R(\mathrm{Min}(\theta)). \end{aligned}$$

Remark 11.2.1. If θ is elliptic, $\mathrm{Min}(\theta)_C$ equals $\mathrm{proj}_{\mathrm{Min}(\theta)}(b_C)$.
If θ is hyperbolic, $\mathrm{Min}(\theta)$ is the translation axis of θ containing $\mathrm{proj}_{\mathrm{Min}(\theta)}(b_C)$.

Proposition 11.2.2. *Let θ be an elliptic automorphism of a building. For any chamber $C \in \mathcal{B}$, there exists a chamber $D \in \mathrm{M}_C(\theta)$ and a minimal gallery from C to $\theta(D)$ containing D.*

Proof. If $C \in \mathrm{M}_C(\theta)$ the statement follows directly. Let $C \notin \mathrm{M}_C(\theta)$. For any $z \in \mathrm{Min}(\theta)$, the residue $R(z)$ will be stabilized by θ. Thus the statement follows for $D = \mathrm{proj}_{R(z)}(C)$, because $\theta(D) \in R(z)$ and any minimal gallery from C to D can be extended to a minimal gallery from C to any chamber inside $R(z)$. $\square$

To give a similar result in the hyperbolic case, we need the main result of the following section:

11.3 The Translation-Cone of a Chamber

The aim of this section is to show that for any given chamber C in a building $\mathcal{B}$ and a translation axis of an action $\gamma : \mathbb{R} \to \mathcal{X} = \mathcal{X}(\mathcal{B})$, there exists a geometric apartment containing $|C|$ and γ_z^+ for some $z \in \mathbb{R}$.

Let θ be an hyperbolic action on a building $\mathcal{B}$. Let γ be a translation axis of θ, let b_C be the barycenter of $|C|$, and let p be the projection of b_C onto γ. Let Σ be an apartment containing γ (see 8.3.9).

Reminder 11.3.1. For a geodesic ray γ and an element $z = \gamma(t) \in [\gamma]$, the geodesic ray γ_z^- is defined by $\gamma_z^- := \gamma([t, -\infty))$.

Definition 11.3.2. Let $\gamma : \mathbb{R} \to X$ be a geodesic line in a CAT(0) space X and let $\mathcal{C}$ be a convex subset of X. If $[\gamma] \cap \mathcal{C} \neq \emptyset$, we can define the **angle** $\measuredangle(\gamma, \mathcal{C})$ in the following way: Let $x \in [\gamma]$ such that $[\gamma_x^-] \cap \overline{\mathcal{C}} = \{x\}$: Then

$$\measuredangle(\gamma, \mathcal{C}) := \inf_{y \in \mathcal{C} \setminus \{x\}} \measuredangle_x(\gamma_x^-, [x, y]).$$

Definition 11.3.3. An element $x \in \gamma$ is called **chamber cut** of γ if there exists a chamber $C \in \mathcal{B}$ such that x lies in the interior of a maximal facet $\mathcal{F}$ of $|C|$ with $\gamma_x^- \cap \mathcal{F} = \{x\}$.

Lemma 11.3.4. *Let x be a chamber cut of γ. Then $\theta(x)$ is a chamber cut of γ and there are only finitely many chamber cuts on $[x, \theta(x)]$.*

Proof. Let x be a chamber cut for γ, and let $|C|, \mathcal{F}$ be a corresponding chamber and maximal facet of $|C|$ such that $\gamma_x^- \cap \mathcal{F} = x$. As θ is a cellular isometry on $\mathcal{X}$, $\theta(x)$ lies in the maximal facet $\theta(\mathcal{F})$ of $\theta(|C|)$. Assume $\theta(\mathcal{F})$ contains another element of $\gamma_{\theta(x)}^-$, say y. Then $\theta^{-1}(y)$ lies in $\theta^{-1}(\mathcal{F}) = \mathcal{F}$. As γ is a translation axis for θ, the element $\theta^{-1}(y)$ lies in γ_x^- and therefore $\theta(x) = \gamma_{\theta(x)}^- \cap \theta(\mathcal{F})$. The geodesic segment $[x, \theta(x)]$ intersects finitely many maximal facets of chambers non-trivially, as all chambers are isometric. Hence the statement. $\square$

Lemma 11.3.5. *Let x be a chamber cut of γ and let $\mathcal{F}$ be a maximal facet of a chamber intersecting γ in x. Then the angle $\measuredangle(\gamma, \mathcal{F})$ equals the angle $\measuredangle(\gamma, \theta(\mathcal{F}))$.*

Proof. The action of θ induces an isometry of $\mathcal{F}$ to $\theta(\mathcal{F})$. Thus the geodesics of the form $[x, y]$ with $y \in \mathcal{F}$ are mapped to the geodesics of the form $[\theta(x), y']$ with $y' \in \theta(\mathcal{F}')$. Further every geodesic of the form $[\theta(x), y']$ with $y' \in \theta(\mathcal{F}')$ is mapped to the geodesic $[x, y]$ with $y \in \mathcal{F}$ by θ^{-1}. As the angles in a CAT(0) space are given by distances (see 3.5.3), any isometry preserves the angles of geodesics and hence the statement follows. $\square$

Definition 11.3.6. Let x be a chamber cut of γ. A chamber cut angle (of γ) at a chamber cut x is an angle $\measuredangle(\gamma, \mathcal{F})$ for a maximal facet $\mathcal{F}$ corresponding to the chamber cut x.

Lemma 11.3.7. *Let x, y be a chamber cuts of γ and let $\mathcal{F}, \mathcal{F}'$ be corresponding facets. The angle of γ with $\mathcal{F}'$ in y equals a chamber cut angle for γ at some point in $[x, \theta(x)]$.*

Proof. The geodesic segment $[x, \theta(x)]$ is a weak fundamental domain for the action of the group spanned by θ on γ. Thus $x' := \gamma^k(y) \in [x, \theta(x)]$ for some $k \in \mathbb{Z}$ and x' is a chamber cut of γ. Now the previous lemma shows that γ^k preserves the chamber cut angle and the statement follows. $\square$

Lemma 11.3.8. *Let x be a chamber cut for γ. The number of facets $\mathcal{F}$ with $\mathcal{F} \cap \gamma \ni x$ is finite and thus the number of different angles of facets with γ is finite.*

Proof. The number of facets containing a common point is bounded by the rank of the building. All chamber cut angles of γ equal an chamber cut angle in $[x, \theta(x)]$ for any chamber cut x. But the number of chamber cuts in $[x, \theta(x)]$ is finite and thus there are only finitely many chamber cut angles for γ. □

Lemma 11.3.9. *A chamber cut angle is always non-zero.*

Proof. If two geodesics have angle zero, then they coincide on a certain interval. Thus if a chamber cut angle $\measuredangle(\mathcal{F}, \gamma)$ at a point x was zero, then there exists an $y \in \mathcal{F}$ such that $\measuredangle_x([x, y], \gamma_x^-) = 0$.
One may note that this statement uses that facets are complete and convex.
But then there exists an element $y' \in [x, y]$ with $y' \in \gamma_x^-$ which contradicts the conditions for a chamber cut. □

Proposition 11.3.10. *Let $\Delta(x, y, z)$ be a geodesic triangle with $\alpha := \measuredangle_{\bar{z}}(\bar{x}, \bar{y})$ for a comparison triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{R}^2$. Then for any point u in the convex hull of x, y, z we have $\measuredangle_z(x, u) \leq \alpha$.*

Proof. This follows as a comparison point $\bar{u}$ lies inside the triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$ and thus $\measuredangle_{\bar{z}}(\bar{x}, \bar{u}) \leq \alpha$ and hence $\measuredangle_z(x, u) \leq \alpha$. □

Proposition 11.3.11. *Let θ be an hyperbolic action on a building $\mathcal{B}$. Let C be a chamber of $\mathcal{B}$ and let γ be a translation axis of θ. There exists a geometric apartment $|\Sigma'|$ containing $|C|$ and γ_z^+ for some $z \in \mathbb{R}$.*

Proof. Let b_C be the barycenter of $|C|$, and let p be the projection of b_C onto γ. Let Σ be an apartment containing γ (see 8.3.9). Let $\measuredangle_1$ be the minimum of all chamber cut angles $\neq 0$ of γ. Let $x \in [\gamma]$ such that $\measuredangle_{\bar{x}}(\bar{p}, \bar{b_C}) \leq \measuredangle_1$ for a comparison triangle $\Delta(\bar{x}, \bar{p}, \bar{b_C})$ in $\mathbb{R}^2$. Then the angle $\measuredangle_y(p, b_C)$ is less or equal $\measuredangle_1$ for all $y \in \gamma_x^+$. Let D be a chamber satisfying the following conditions:

(i) D is a chamber in Σ

(ii) D intersects γ_x^+ non-trivially

(iii) D contains a maximal facet $\mathcal{F}$ such that $\measuredangle(\gamma, \mathcal{F}) > 0$.

Let α_+ be the root of Σ determined by $\mathcal{F}$, with $\theta(\alpha_+ \cap \gamma) \subset \alpha_+$ and let $z \in \mathcal{F} \cap \gamma$. Let P be the panel determined by $\mathcal{F}$ and let D be the projection of C onto P. Let $\Gamma = (D' = D_0, D_1 = D, D_2, \dots, D_n = C)$ be a minimal gallery.
Assume there exists a root α_i containing $D_i, D_{i-1}, \dots, D_0$ and γ_z^+, for some $i > 0$. Then the projection of all chambers of α_i onto the panel containing D_i and D_{i+1} is D_i. Thus by 8.3.8 there exists an apartment containing α_i and D_{i+1}.
This shows that we obtain an apartment Σ' containing Γ, starting from the root α_+. We want to show that Σ' contains γ_z^+. By the construction of Σ', either γ_z^+ is contained in Σ' or there exists an element $z'' \in \gamma$ such that z'' lies on a wall $\partial\beta$ separating C from D and $\gamma_x^+ \cap \Sigma = \{x\}$.
As $\partial\beta$ separates D from C, this wall intersects the geodesic segment $[b_C, b_D]$ joining the barycenters of C and D in a point z'. Thus in the latter case, the angle of

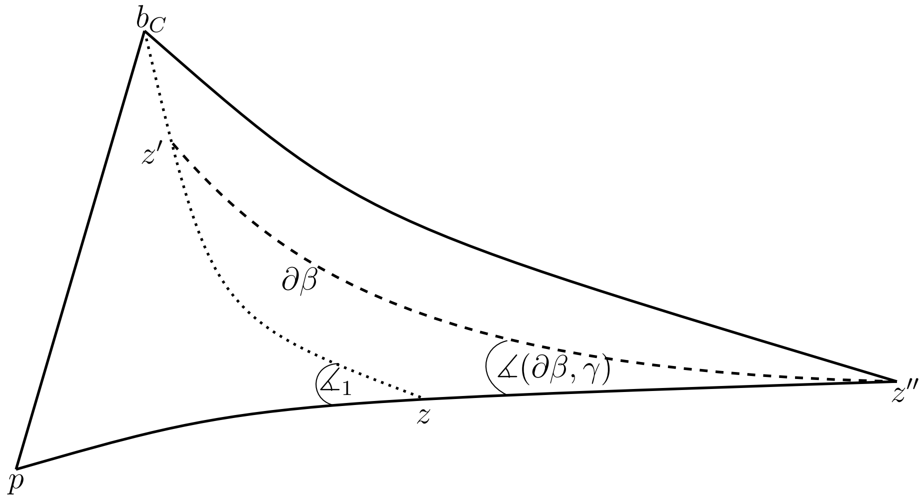


Figure 11.1: Angles within a triangle in a CAT(0) space.

$\measuredangle(\gamma, \partial\beta)$ which is a chamber cut angle, equals $\measuredangle_{z''}(z', p)$. As $z'' \in \gamma_z^+$ the angle $\measuredangle(z'', b_C, p) \leq \measuredangle(z, b_C, p)$ and as the geodesic triangle $\Delta(z'', b_C, p)$ contains the point z', we can apply 11.3.10 to see that this angle is less or equal $\measuredangle_1$. By the convexity of the wall $\partial\beta$, the angle $\measuredangle(\gamma, \partial\beta)$ is less or equal to $\measuredangle_{z''}(b_C, p)$ which contradicts that chamber cut angles along γ are greater than $\measuredangle_1$. This shows that Σ' contains γ_z^+. □

11.4 Hyperbolic Actions

This section is about the existence of a minimal gallery of the form $(C, \ldots, D, \ldots \theta(D))$ for any chamber $C \in \mathcal{B}$.

Let $\mathcal{B}$ be a building with an hyperbolic action θ on its Davis realization $\mathcal{X}$.

Proposition 11.4.1. *Let Σ be an apartment of $\mathcal{B}$ with $\theta(\Sigma \cap \mathrm{Min}(\theta)_C) \subset \Sigma$ and $\Sigma \cap \mathrm{Min}(\theta)_C \neq \emptyset$. For any root α in Σ with $\mathrm{Min}(\theta)_C \cap \alpha \nsubseteq \bar{\alpha}$, either*

(i) *$\theta(\alpha \cap \mathrm{Min}(\theta)_C) \subset \alpha$, or*

(ii) *$\theta(-\alpha \cap \mathrm{Min}(\theta)_C) \subset -\alpha$.*

Proof. Let α be a root of Σ with $\mathrm{Min}(\theta)_C \nsubseteq \bar{\alpha}$. Assume there exists a $y \in \alpha \cap \mathrm{Min}(\theta)_C$ with $\theta(y) \in -\alpha$. Let $\gamma : [0, \infty] \to \Sigma$ be the geodesic ray issuing from y containing $\theta(y)$. For $z := \min_{t \in [0,\infty)}\{\gamma(t) \in -\alpha\}$, the ray $\gamma([z, \infty))$ equals $\mathrm{Min}(\theta)_C \cap -\alpha$, and the statement follows. □

Definition 11.4.2. Let $C \in \mathcal{B}$. We define Σ_C to be the set of all apartments Σ of $\mathcal{B}$, containing C with the following properties:

(i) $|\Sigma| \cap \mathrm{Min}(\theta)_C \neq \emptyset$, and

(ii) $\theta(|\Sigma| \cap \mathrm{Min}(\theta)_C) \subset |\Sigma|$.

For each $\Sigma \in \Sigma_C$, define $\alpha(\Sigma, C)$ to be the set of all (geometric) roots $\alpha \in \Sigma$ containing C, with

(i) $\mathrm{Min}(\theta)_C \cap \alpha \not\subseteq \bar{\alpha}$,

(ii) $\theta(\alpha \cap \mathrm{Min}(\theta)_C) \subset \alpha$, and

(iii) $\alpha \cap \mathrm{Min}(\theta)_C \neq \emptyset$.

Further define $S^{\Sigma}(C) := \bigcap\{\alpha \in \alpha(\Sigma, C)\}$.

By 11.4.1 and 11.3.11 such roots exist. As C is contained in each of these roots, this set is not empty. Now we can define $\mathrm{SM}(\theta)(C, \Sigma) := \mathrm{M}_C(\theta) \cap S^{\Sigma}(C)$.

Lemma 11.4.3. *If an apartment Σ contains a chamber C and a point x, then Σ also contains $\mathrm{proj}_{R(x)}(C)$.*

Proof. If a point x is contained in Σ, then Σ has to contain at least one chamber D of $R(x)$. The support $R(x)$ is a spherical residue, thus $\mathrm{proj}_{R(x)}(C)$ is contained in the chamberwise convex hull of C and D which is contained in Σ. □

Lemma 11.4.4. *For every $\Sigma \in \Sigma_C$, there exists a chamber $D \in \mathrm{SM}(\theta)(C, \Sigma)$ with $D = \mathrm{proj}_{R(y)}(C)$ for some suitable $y \in |\mathcal{B}|$.*

Proof. Let $y \in \mathrm{Min}(\theta)_C \cap S^{\Sigma}(C)$. Assume that there is a root $\alpha \in \alpha(C, \Sigma)$ which does not contain D.
If α contains any other chamber of $R(y)$ then it has to contain D, because α is chamberwise convex and D is the projection of C onto $R(y)$. So the root α does not contain any other chamber of $R(y)$. Thus α does not contain any point inside $R(y)$ which contradicts $y \in \alpha$. □

Remark 11.4.5. By 11.3.11 the set $\mathrm{SM}(\theta)(C, \Sigma) \cap \mathrm{Min}(\theta)_C$ is not empty.

Assume $D \in \mathrm{SM}(\theta)(C, \Sigma)$ and $D \in \Sigma'$ for some other apartment $\Sigma' \in \Sigma_C$. If $D \notin \mathrm{SM}(\theta)(C, \Sigma')$ then there is a root $\beta' \in \Sigma'$ which contains C and a subray of $\mathrm{Min}(\theta)_C$, but not D. Let y' be a point of $\mathrm{Min}(\theta)_C$ which is contained in Σ and Σ' such that $D \notin R(y')$ and $E := \mathrm{proj}_{R(y')}(D) \in \beta'$.
As both apartments contain C, D and y', there is an isomorphism from Σ' to Σ fixing $\mathrm{conv}(C, D, E)$. The root β' is mapped to a root β which contains C and E but not D. Furthermore $\theta(\beta \cap \mathrm{Min}(\theta))$ has to be a subset of β which contradicts $D \in \mathrm{SM}(\theta)(C, \Sigma)$. We conclude

Proposition 11.4.6. *If a chamber D is contained in $\mathrm{SM}(\theta)(C, \Sigma)$ for some apartment $\Sigma \in \Sigma_C$, then $D \in \mathrm{SM}(\theta)(C, \Sigma')$ for every $\Sigma' \in \Sigma_C$ containing D.*

Let $D \in \mathrm{SM}(\theta)(C, \Sigma)$ with $D := \mathrm{proj}_{R(y)}(C)$ for some $y \in S^{\Sigma}(C) \cap \mathrm{Min}(\theta)_C$.

Proposition 11.4.7. *There exists a minimal gallery from C to $\theta(D)$ containing D.*

Proof. For $y \in S^{\Sigma}(C) \cap \mathrm{Min}(\theta)_C$, the geodesic segment $[y, \theta(y)]$ is contained in Σ. Let Γ be a minimal gallery from D to $\theta(D)$ along $[y, \theta(y)]$, let $(R_i)_{i\in\{1,\dots,n\}} = \mathcal{R}([y, \theta(y)], \Sigma)$ as in 11.1.12, and let Γ' be a minimal gallery from C to D. We want to show that $\Gamma'.\Gamma$ is a minimal gallery. The we have to show that the projections $\mathrm{proj}_{R_i}(C)$ and $\mathrm{proj}_{R_i}(D)$ coincide for every $i \in \{1, \dots, n\}$. Using 11.4.3 we see that both projections lie in Σ.
By the condition $D := \mathrm{proj}_{R(y)}(C)$, we get $\mathrm{proj}_{R_1}(C) = \mathrm{proj}_{R(y)}(C)$.
Assume the statement holds for $j < i$, but $C_i := \mathrm{proj}_{R_i}(C) \neq \mathrm{proj}_{R_i}(D) =: D_i$.
Then there exists a root α in Σ containing D_i and D but not C_i and C. Thus $-\alpha$ contains C_i and C. As $\bar{\alpha}$ separates C and D, and as D is the projection onto the support of a point in $\mathrm{Min}(\theta)$ we have $\mathrm{Min}(\theta)_C \cap \alpha \not\subset \bar{\alpha}$. Thus by 11.4.1 we get $\theta(-\alpha \cap \mathrm{Min}(\theta)) \subset \alpha$, as $y \in |\alpha|$ and $\theta(y) \in |-\alpha|$. This means $D \notin \mathrm{SM}(\theta)(C, \Sigma)$ which contradicts the conditions. □

Proposition 11.4.8 (see also [AB09])**.** *Let w be a θ-displacement in $\mathcal{B}$. If there is an $s \in S$ with $l(sw\theta(s)) = l(w) + 2$, then the word $sw\theta(s)$ is also a θ-displacement in $\mathcal{B}$.*

Proof. Let $C \in \mathcal{B}$ with displacement w. Let C' be a chamber in the s-panel P containing C. Then $\theta(C')$ and $\theta(C)$ lie in a common $\theta(s)$-panel $P' = \theta(P)$. As $l(w\theta(s)) > l(w)$, the chamber $\theta(C)$ is the projection of C onto P', and as $l(sw) > l(w)$, the chamber C is the projection of $\theta(C)$ onto P and we get a minimal gallery from C' to its image, containing C and $\theta(C)$. Thus $\delta(C', \theta(C') = sw\theta(s)$. □

An immediate consequence of this proof is:

Corollary 11.4.9. *If $C \in \mathcal{B}$ is a chamber with displacement w such that $l(sw\theta(s)) = l(w) + 2$, then the displacement of any chamber D in the s-panel containing C is $sw\theta(s)$.*

Lemma 11.4.10. *Let w be a θ-displacement in a $\mathcal{B}$. If there exists $s \in S$ with $l(sw) = l(w) + 1$, $l(w\theta(s)) = l(w) - 1$, then $sw\theta(s)$ and sw are θ-displacements.*

Proof. Let $C \in \mathcal{B}$ be a chamber with displacement w. Then for any chamber $E \neq C$ in the s-panel P containing C, we have $\delta(E, \theta(C)) = sw$ and thus C is the projection of $\theta(C)$ onto P. On the other hand, $l(w\theta(s)) = l(w) - 1$ shows that $\theta(C)$ is not the projection of any chamber $E \in P$ onto $\theta(P)$. Let $D \in P$ be the preimage of $\mathrm{proj}_{\theta(P)}(C) = \mathrm{proj}_{\theta(P)}(P)$ under θ. Then $\delta(D, \theta(D)) = sw\theta(s)$. Let E be a chamber in $P \setminus \{C, D\}$, then there exists a minimal gallery from E to $\theta(E)$ passing the projection $C = \mathrm{proj}_P(\theta(E))$ and $\mathrm{proj}_{\theta(P)}(E) = \theta(D)$. This gallery is of type sw. □

An immediate consequence of this proof is:

Corollary 11.4.11. *If $C \in \mathcal{B}$ is a chamber with displacement w such that $l(sw) = l(w) + 1$ and $l(w\theta(s)) = l(w) - 1$, then there exists a unique chamber E in the s-panel P containing C with displacement $sw\theta(s)$ and every chamber in $P \setminus \{C, E\}$ has displacement sw.*

11.5 The (MW) Condition

Let θ be an action on a building $\mathcal{B}$ and let θ also denote its induced action on the Davis realization $\mathcal{X}$ of $\mathcal{B}$. We define the following sets:

- $W_\theta := \{w \in W \mid \exists\ D \in \mathcal{B} : \delta(D, \theta(D)) = w\}$, the set of displacements of θ on $\mathcal{B}$, and
- $W_{\mathrm{Min}(\theta)} := \{w \in W \mid \exists\ C \in \mathcal{B}, D \in \mathrm{M}_C(\theta) : \delta(D, \theta(D)) = w\}$, the set of displacements of θ on chambers in $\bigcup\limits_{C \in \mathcal{B}} \mathrm{M}_C(\theta)$.
- For all $C \in \mathcal{B}$ we define:
  $\mathrm{Cham}(C, \theta) := \{D \in \mathrm{M}_C(\theta) \mid D$ lies on a minimal gallery from C to $\theta(D)\}$.

We have already shown that $\mathrm{Cham}(C, \theta)$ is non-empty for any chamber C in $\mathcal{B}$. For the main statement, we need the following condition on θ:

(MW) For every chamber $C \in \mathcal{B}$, there exists a chamber $D \in \mathrm{Cham}(C, \theta)$ and an apartment Σ such that Σ contains $C, \theta(D)$, and $\theta(C)$.

Theorem 11.5.1. *If an automorphism θ on a building $\mathcal{B}$ satisfies (MW), then any displacement $w \in W_\theta$ is a θ-conjugate of some displacement $w' \in W_{\mathrm{Min}(\theta)}$, i.e. $w = w_1 \cdot w_2 \cdot \theta(w_1)^{-1}$ for some $w_2 \in W_{\mathrm{Min}(\theta)}$.*

Proof. Let Σ be an apartment containing $C, \theta(C)$, and $\theta(D)$. As D is an element of $\mathrm{Cham}(C, \theta)$, also $D \in \Sigma$. From $\theta(\delta(C, D)) = \delta(\theta(C), \theta(D))$, we get $\delta(C, \theta(C)) = \delta(C, D) \cdot \delta(D, \theta(D)) \cdot \theta(\delta(D, C))$ which proves the statement. □

11.6 Displacements in Coxeter Systems

Let (W, S) be a Coxeter system. We can view (W, S) as a thin building $\mathcal{B}$ and (MW) is satisfied.

Corollary 11.6.1. *For any Coxeter system (W, S) and any isomorphism θ on (W, S), the displacements in W_θ are exactly the words $w \cdot w' \cdot \theta(w^{-1})$, where $w' \in W_{\mathrm{Min}(\theta)}$ and $l(w'w) = l(w) + l(w')$.*

Proof. Let $w' \in W_{\mathrm{Min}(\theta)}$ and $w \in W$ with $l(ww') = l(w) + l(w')$. From $w' \in W_{\mathrm{Min}(\theta)}$ it follows that there exists an element $D \in W$ with displacement w'. Furthermore from $l(ww') = l(w)l(w')$ there exists a chamber C with $\delta(C, D) = w$ and $\delta(C, \theta(D)$ = ww'. Now $\delta(\theta(D), \theta(C)) = \theta(w)$ and the result follows. □

An Open Question:

Find non-trivial necessary and / or sufficient conditions on the building $\mathcal{B}$ which ensure that (MW) is satisfied for every (type-preserving) automorphism.

Here, by a trivial condition we mean something like "the building satisfies (MW) for every automorphism". Of course conditions which are both necessary and sufficient would be most interesting, but also partial results (i.e conditions which are just necessary, or just sufficient) are of interest. An example for such conditions can be derived from section 11.7: *If the underlying Coxeter group of a building $\mathcal{B}$ is universal, the condition (MW) is satisfied for every automorphism of $\mathcal{B}$.*

A closely related open question to the given one is following:
Given a building, find a non-trivially characterization of all automorphisms satisfying (MW).

One conclusion of the next chapter is that for a class of buildings (the ones admitting a tie tree) a similar result holds. The set used to obtain all displacements as θ-conjugates will be a lot bigger than $R(\mathrm{Min}(\theta))$, but for these cases an analogue of the (MW)-condition (the existence of an apartment containing $C, D, \theta(D)$, and $\theta(C)$) will always be satisfied.

11.7 Buildings with Universal Coxeter Group

A **universal Coxeter group** W is a (Coxeter) group of the form

$$W = \langle S \mid s^2 = 1 \text{ for all } s \in S \rangle.$$

Let $\mathcal{B}$ be a building with an universal Coxeter group. Let (V, E) be the adjacency graph whose vertices are the chambers of $\mathcal{B}$ and whose edges correspond to adjacent chambers. Then (V, E) is a tree. Indeed, the type of any sequence of adjacent chambers $(C_1, C_2, \ldots, C_l)$ in this graph with $\delta(C_{i-1}, C_i) \neq \delta(C_i, C_{i+1})$ for $1 < i < l$ is a reduced word and hence the sequence is a minimal gallery and cannot be a cycle. Any automorphism θ of $\mathcal{B}$ has to preserve the adjacency relation and thus induces a graph automorphism $\tilde{\theta}$ on (V, E). Let C be an arbitrary chamber of $\mathcal{B}$. The tree structure of (V, E) implies that $\mathrm{M}_C(\theta)$ is gated and the unique gallery from C to $D := \mathrm{proj}_{\mathrm{M}_C(\theta)}(C)$ extends uniquely to a minimal gallery from C to $\theta(D)$. With $M_C(\theta^{-1}) = \mathrm{M}_C(\theta)$, we obtain a unique minimal gallery from $\theta(C)$ to D containing $\theta(D)$. Thus there exists a (unique) gallery from C to $\theta(C)$ which contains $\theta(D)$. Therefore there exists an apartment contains those chambers and hence θ satisfies (MW).
We conclude:

Lemma 11.7.1. *Let $\mathcal{B}$ be a building with universal Coxeter group. Then (MW) is satisfied for every automorphism of $\mathcal{B}$.*

11.8 Fixing Exactly One Wall

Let θ be an elliptic action on an affine building $\mathcal{B}$, with $\mathrm{Min}(\theta) = M$ for a wall M. Let C be a chamber in $\mathcal{B}$ and let $p := \mathrm{proj}_M(b_C)$. Then p determines a spherical residue R_C and as θ stabilizes R_C, for $D := \mathrm{proj}\, R_C(C)$ we get a minimal gallery from C to $\theta(D)$ containing D. This means $D \in M(\theta)$. Furthermore we see, using 9.1.7, that there exists an apartment Σ containing C and $\theta(C)$ with $|M| \subset |\Sigma|$. This implies that also D and $\theta(D)$ are contained in Σ and we conclude:

Lemma 11.8.1. *Let θ be an elliptic action on an affine building $\mathcal{B}$ fixing exactly one wall. Then θ satisfies the (MW) condition.*

If θ is a hyperbolic action of an affine building fixing exactly one wall M, then θ does not generally satisfy (MW). We will see later (13.3) that we can adjust this idea, so we don't need D to lie on a minimal gallery from C to $\theta(D)$, but we will use another chamber D' which will be a projection of $\theta(C)$ onto M such that there exists a minimal gallery from C to D' containing D. This will also mean that the displacement is not of the form $ww_2\theta(w^{-1})$ as before, as the gallery from $\theta(C)$ to D' is in general not the image of the gallery from C to D.

We will also explain in 14.8, given the affine building $\mathcal{B}$ corresponding to $\mathrm{GL}_n(K)$ for a field K with discrete valuation and finite residue field k, when the canonical reflection $\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ acting on $\mathcal{B}$ fixes exactly one wall.

CHAPTER

TWELVE

TREE-LIKE STRUCTURES

The concept of tie trees was developed to obtain a framework containing every building admitting a meaningful tree-like structure. It is based on the following observation on $\mathrm{PGL}(2,\mathbb{Z})$*: The graph whose vertices are the maximal spherical residues and their intersections and whose (undirected) edges correspond to the containment relation is a tree. Tie trees have the crucial property that minimal galleries in the building relate to minimal paths in the tree. Furthermore the vertices are gated sets, so that we can use a projection of a chamber onto such a vertex. We will see that we can obtain a tie tree structure for a building of type* (W,S) *if* (W,S) *itself admits such a structure. Theorem 12.1.32 is a structure theorem for Weyl displacements of automorphisms of such buildings. A specialization of tie trees are residue trees. For those trees, all vertices are residues.*

12.1 Tie Trees

Let $\mathcal{B}$ be a building of type (W,S).

Definition 12.1.1. A **tie** of a building $\mathcal{B}$ is a proper subset of $\mathrm{Cham}(\mathcal{B})$ (i.e. it is not empty and not the whole building). We call a tie a **knot** if it contains exactly one chamber. The set of knots will be denoted by $\boldsymbol{\mathcal{K}}$. We say that an intersection of two ties is **knotted** if their intersection is a knot.

Definition 12.1.2. A **tie graph** (of $\mathcal{B}$) is a graph (V,E), satisfying the following properties:

(TG1) The vertices of (V,E) are non-trivial pairwise different ties of $\mathcal{B}$.

(TG2) V is closed under non-trivial, non-knotted intersections, i.e. if $v,w\in V$ and $v\cap w\notin\{\emptyset\}\cup\mathcal{K}$ (i.e. $(|v\cap w|\geq 2)$ then $v\cap w\in V$.

(TG3) For $(v,w)\in E$: $v\subset w$ or $w\subset v$.

(TG4) If $v,w\in V$ with $v\subset w$[1] then $v=w_1\cap w_2$ for some $w_1,w_2\in V$ and $(w_1,v),(v,w_2)\in E$. If $v\notin\mathcal{K}$, then w_1 can be chosen to be w.

[1]The notation $v\subset w$ means a proper inclusion.

(TG5) If $v_0 \subset v_1 \subset v_2$ then $(v_0, v_1) \notin E$.

Lemma 12.1.3. *A tie graph satisfies:*

(TG6) *For $v, w \in V \setminus \mathcal{K}$: $(v, w) \in E \Leftrightarrow (v \subset w$ or $w \subset v)$.*

Proof. Let $v, w \in V \setminus \mathcal{K}$. If $(v, w) \in E$, then by (TG3) we get $v \subset w$ or $w \subset v$. If $v \subset w$, then by ((TG4)) $v = w \cap w_2$ for some $w_2 \in V$ and $(v, w) \in E$. $\square$

Definition 12.1.4. Let θ be a building automorphism of a building $\mathcal{B}$. A tie graph (V, E) (of $\mathcal{B}$) is called **tie tree** for θ if it satisfies the following properties:

(TT1) For every panel P of $\mathcal{B}$, there exists a tie $v \in V$ with $\mathrm{Cham}(P) \subset v$.

(TT2) For any edge $(v, w) \in E$, the intersection $v \cap w$ satisfies the gate property (and thus is chamberwise convex).

(TT3) The graph (V, E) is a tree.

(TT4) The action θ on $\mathcal{B}$ induces a graph automorphism on (V, E).

From now on consider $\mathcal{T} = (V, E)$ to be a tie tree for an autormophism θ on a building $\mathcal{B}$.

Example 12.1.5. Let $\mathcal{B}$ be a building of type $\underset{s_1}{\circ}\overset{3}{\text{—}}\underset{s_2}{\circ}\overset{\infty}{\text{—}}\underset{s_3}{\circ}$. The corresponding Coxeter group is isomorphic to $\mathrm{PGL}_2(\mathbb{Z})$. The graph whose vertices are the maximal spherical residues (the residues of type $\{s_1, s_2\}$, $\{s_1, s_3\}$) and their intersections (the residues of type $\{s_1\}$) form a tree if we consider the edges are given by the containment relation. (We will see later in 12.3.7 that the graph is a tree.) Any action on $\mathcal{B}$ preserves this tree structure (as it has to be type-preserving) and any a building of type $\mathrm{PGL}(2, \mathbb{Z})$ admits a tie tree for every automorphism.

Explanation (The definition of tie trees)
The aim of the definition of tie trees is to obtain minimal galleries in the building using minimal paths in the tree. For two ties v, w with minimal path $v = v_1, \ldots, v_n = w$ and chambers $C \in v, D \in w$, the gallery determined by minimal galleries from $\mathrm{proj}_{v_i}(C)$ to $\mathrm{proj}_{v_{i+1}}(C)$ for $i \in \{1, \ldots, n-1\}$ and a minimal gallery $\mathrm{proj}_w(C)$ to D is a minimal gallery from C to D and further every minimal gallery from C to D has to contain a chamber of each of the v_i. Hence for every chamber C we obtain a minimal gallery from C to $\theta(C)$ using a tie v containing C and the minimal path in $\mathcal{T}$ from v to $\theta(v)$.

We give some examples for graphs satisfying all, but one of the given conditions:

(TG2) *Consider the Coxeter system of type $\underset{s_1}{\bullet}\overset{\infty}{\text{—}}\underset{s_2}{\bullet}\ \ \underset{s_3}{\bullet}\overset{\infty}{\text{—}}\underset{s_4}{\bullet}$. Chose a chamber C and let $\mathcal{P}$ be the set of panels parallel to the s_1 panel containing C. Let V be the set of the $\{s_1, s_2, s_3\}$- and the $\{s_1, s_2, s_4\}$- residues together with the panels in $\mathcal{P}$. And let the edges correspond to the inclusion. For every automorphism preserving $\mathcal{P}$, this graph satisfies all conditions to be a tie tree but (TG2).*

(TG3) *Consider a tie tree (V, E) for some automorphism θ of a building $\mathcal{B}$ fixing a tie v which is contained in two ties w_1 and w_2 and which is not connected to any other tie of V. Add a new tie $w_1 \cup w_2$ and replace all edges of the form (v', w_1) or (v', w_2) by (v', v). Remove the ties w_1 and w_2 and add the edge $(w_1 \cup w_2, v)$. The automorphism θ induces a graph automorphism on the resulting tree which does not satisfy condition (TG3).*

(TG4) *Let (V, E) be a tie tree for an automorphism θ on a building $\mathcal{B}$ fixing a vertex v with $v \subset w$ for some $w \in V$. Let t be the union of all ties connected to v. We construct a new graph (V', E') as follows: We take V and E and replace every edge of the form (v_1, v_2) by (v_1, t) if v_2 is connected to v in E. We remove every edge with a vertex v and add the edge (t, v). Then θ induces a graph automorphism on the tree (V', E'), but the tie v is not the intersection of two ties.*

(TG5) *Let (W, S) be a Coxeter system of type $\mathrm{PGL}(2, \mathbb{Z})$ with generators chosen as in example 12.1.5. Let C be the chamber corresponding to 1_W and let R be the s_1-panel containing C. Let $\mathcal{C}$ be a set of chamber containing exactly one chamber for every s_1-panel in (W, S) such that the reflection r_{s_3} stabilizing the s_3-panel containing C preserves $\mathcal{C}$. Let V be the set of all maximal spherical residues together with their intersections and all chambers of $\mathcal{C}$. We connect every chamber in $\mathcal{C}$ to all residues containing it. The resulting graph is a tree and r_{s_3} induces a graph automorphism on it. But the graph does not satisfy (TG5).*

(TT1) *Consider the Coxeter system (W, S) of type $\underset{s_1}{\bullet}\overset{\infty}{\text{——}}\underset{s_2}{\bullet}\;\;\underset{s_3}{\bullet}\overset{\infty}{\text{——}}\underset{s_4}{\bullet}$. Chose a chamber C and let R be the $\{s_3, s_4\}$-residue containing C. Let V be the set of all $\{s_1, s_2\}$- residues together with R and all chambers in R. Let the edges correspond to the containment relation. For every automorphism θ of (W, S) preserving R, the resulting graph satisfies all conditions to be a tie tree for θ on $\mathcal{B}$, but (TT1).*

(TT2) *Consider a Coxeter system (W, S) of type $\tilde{A}_2$. For every wall β, let $\mathcal{C}_\beta$ be set of all chambers which lie in a panel determined by β. Let α be a wall and let V be the set of all $\mathcal{C}_\beta$ for all β parallel to α and their intersections. Let (V, E) be a graph, where the edges correspond to the inclusion relation. For every automorphism of (W, S) preserving the parallel class $[\alpha]$, the resulting graph satisfies all conditions to be a tie tree, but (TT2).*

(TT3) *Consider a building $\mathcal{B}$ of type $\tilde{A}_2$ and let V consists of all maximal spherical residues and all panels. The graph (V, E), with edges corresponding to the inclusion relation is not a tree but satisfies all conditions but (TT3)to be a tie tree for any automorphism of $\mathcal{B}$.*

(TT4) *Consider the Coxeter system $(W, S)$of type $\underset{s_1}{\bullet}\overset{\infty}{\text{——}}\underset{s_2}{\bullet}\;\;\underset{s_3}{\bullet}\overset{\infty}{\text{——}}\underset{s_4}{\bullet}$. Let $\mathcal{B}$ be a building of type (W, S). The graph consisting of all $\{s_1, s_4, s_2\}$-and*

$\{s_1, s_4, s_3\}$-residues and their intersections, where the edges correspond to the inclusion relation is a tie tree for every type preserving automorphism, but not for a non-type-reserving automorphism.

Definition 12.1.7. A tie is called **maximal** if it is not contained in any other tie. It is called **gated/convex** if it is gated/convex as a subset of $\mathrm{Cham}(\mathcal{B})$.

Definition 12.1.8. Let $C \in \mathcal{B}$, v a tie. If v is a gated, we denote the gate for C onto v by $\mathrm{proj}_v(C)$. This gate will also be called the projection of C onto v.

Lemma 12.1.9. *For $(v, w) \in E$, either v or w is maximal.*

Proof. From $(v, w) \in E$ it follows $v \subset w$ or $w \subset v$. We may assume $v \subset w$. If w was not maximal, then there exists a tie v' containing w. As v is a proper subset of w, the tie w is not a knot, and thus $(v', w) \in E$ by (TG6). But then v is also a subset of v' and by (TG5) we get $(w, v) \notin E$ which contradicts the choice of v and w. Thus w has to be maximal. □

Lemma 12.1.10. *For $(v, w) \in E$, the tie $v \cap w$ equals either v or w.*

Proof. This follows directly from the condition (TG3). □

Lemma 12.1.11. *For $(v, w) \in E$, at least one of v and w is gated and convex.*

Proof. This follows from (TG4) as either $v = v \cap w$ or $w = v \cap w$. □

Lemma 12.1.12. *If a tie t contains two adjacent chambers C, D, then t contains the whole panel containing C and D.*

Proof. Let P be a panel containing two chambers C and D and let t be a tie containing C and D. If t is gated, then every chamber of the panel P needs to have a unique gate onto t which implies $P \subseteq t$. Let t be a non-gated tie containing C and D and let t' be a tie containing the panel P which exists by (TT1). If $t' \subseteq t$ we are done by (TG6) and (TT2). The intersection $t'' := t \cap t'$ is a gated tie containing C and D. Hence, by the above, $P \subset t''$. □

Proposition 12.1.13. *Let $v_0, \ldots, v_n$ be an arbitrary path in $\mathcal{T}$. Then for all $i \in \{0, \ldots, n-2\}$:*

(i) *If $v_i \subset v_{i+1}$ then $v_{i+1} \supset v_{i+2}$.*

(ii) *If $v_i \supset v_{i+1}$ then $v_{i+1} \subset v_{i+2}$.*

In particular, we get an alternating relation of containment along any path in $\mathcal{T}$.

Proof. From (TG5) in the definition of a tie graph, we cannot have a sequence of the form $v_1 \subset v_2 \subset v_3$. Thus by (TG3) the relations along the path need to be alternating. □

Definition 12.1.14. A minimal gallery Γ is said to be contained in a minimal path γ of $\mathcal{T}$, if for every two consecutive chambers of Γ, there exists a tie in γ containing those chambers.

Definition 12.1.15. For two paths $\gamma_1 = (v_{1,1} \dots, v_{1,n_1}), \gamma_2 = (v_{2,1}, \dots, v_{2,n_2})$ in $\mathcal{T}$ with $v_{1,n_1} = v_{2,1}$, we define $\gamma_1.\gamma_2 := (v_{1,1}, \dots, v_{1,n_1} = v_{2,1} \dots, v_{2,n_2})$.

Lemma 12.1.16. *For every minimal gallery Γ in $\mathcal{B}$, there exists a path in $\mathcal{T}$ containing Γ.*

Proof. Let $\Gamma = (C_0, C_1, \dots, C_n)$ be a minimal gallery. For any two consecutive chambers C_i, C_{i+1} of Γ, there exists a tie v_i containing the panel P_i with $C_i, C_{i+1} \in P_i$, by (TT1). Let γ_i be the minimal path from v_i to v_{i+1}, then the path $\gamma = \gamma_1.\gamma_2 \dots \gamma_{n-1}$ contains the gallery Γ. $\square$

Proposition 12.1.17. *For any minimal gallery in $\mathcal{B}$, there exists a (uniquely determined) minimal path in $\mathcal{T}$ containing this gallery.*

Proof. Let $\Gamma = (C_0, \dots, C_n)$ be a minimal gallery and let v, w be ties with $C \in v$ and $D \in w$. There exists a unique minimal path $\gamma := v_0, \dots, v_n, n > 0$ from v to w in the tie tree $\mathcal{T}$. Let $v = u_0, \dots, u_l = w$, $l > 0$ in $\mathcal{T}$ be the path containing the minimal gallery Γ, as constructed in 12.1.16. If γ equals $u_0, \dots, u_l$, we are done. Assume the two paths do not coincide. As there are no cycles inside a tree, the path $v_0, \dots, v_n (= u_l), \dots u_0$ cannot contain any cycle. Therefore $u_i = v_{j_{i,1}} = v_{j_{i,2}}$ for some $j_{i,2} \geq j_{i,1}$ and $j_{i,2} + 1 = j_{i+1,1}$ for $0 < i < n - 1$. If u_i is convex, all chambers in Γ which are contained in the ties $v_{j_{i,1}}, \dots, v_{j_{i,2}}$ are contained in u_i. If $u_i = v_{j_{i,1}}$ is not convex, then $v_{j_{i,1}+1}$ is a convex tie by 12.1.11. Furthermore using (TT2), we see that $v_{j_{i,1}+1}$ is a tie contained in $v_{j_{i,1}} = u_i$. Thus all chambers of Γ which are contained in $v_{j_{i,1}}, \dots, v_{j_{i,2}}$ are contained in u_i and furthermore by 12.1.12 each panel determined by consecutive chambers of Γ inside $v_{j_{i,1}}, \dots, v_{j_{i,2}}$ is contained in u_i. We see that every panel containing two consecutive chambers of Γ is contained in a tie of γ and thus γ contains Γ. $\square$

Observation 12.1.18. *For any two chambers C, D and two ties v, w with $C \in v$ and $D \in w$, the unique path from v to w contains every minimal gallery from C to D.*

Proposition 12.1.19. *Ties of a tie tree are convex.*

Proof. Let v be a tie of a tie tree $\mathcal{T}$ and let C, D be two arbitrary chambers inside v. By 12.1.18 the path (v) contains every minimal gallery from C to D and hence the tie v is convex. $\square$

Definition 12.1.20. If (V, E) is tie tree for each automorphism of $\mathcal{B}$, then it is called **tie tree** (for $\mathcal{B}$).

Definition 12.1.21. Let $\Gamma = (C_0, \dots, C_n)$ be a gallery in $\mathcal{B}$. We say Γ is a **minimal gallery** from v to v' for $v, v' \in \mathcal{T}$ if Γ is minimal and $C_0 \in v, C_n \in v'$.

Lemma 12.1.22. *Let v, v' be ties containing a common chamber C. Then any tie inside the minimal path from v to v' contains C.*

Proof. Let v, v' be two ties containing a common chamber C and let $v = v_0, \dots, v_n = v'$ be the minimal path between them.If there exists a tie inside this path, not containing C, we may assume, by shortening the path if necessary, that v_0 and v_n are the only ties in this sequence containing C.
Now n needs to be larger than 1 and we see that $v_1 \subset v_0$ and $v_{n-1} \subset v_n$. As $v_{n-1} \subset v_n$, the tie v_{n-1} is gated by (TT2) and there exists a minimal gallery $\Gamma = (C_0, \dots, C_l)$ from C to $\mathrm{proj}_{v_{n-1}}(C)$ which lies entirely inside the convex set v_n, see 12.1.19. By 12.1.18, this gallery has to lie inside the path $v_0, \dots v_{n-1}$, i.e. every pair of consecutive chambers in this gallery is contained in one of the ties $v_0, \dots, v_{n-1}$. Thus one of the ties $v_0, \dots, v_{n-1}$ has to contain the chambers C_0, C_1. By assumption, the only tie in this path containing C is v_0. Hence $v_0 \cap v_n$ contains C_0 and C_1. By (TG2) $v_0 \cap v_1$ is an element of V with edges $(v_0 \cap v_1, v_0)$ and $(v_0 \cap v_1, v_1)$ by (TG6). Now $v_0, v_0 \cap v_n, v_n$ is the minimal path from v_0 to v_n and v_{n-1} contains C. This contradicts our assumption that only v_0 and v_n contain C. □

Corollary 12.1.23. *For any chamber C, the set of all ties containing C spans a connected subtree of $\mathcal{T}$. In particular, for any tie v and any chamber C there exists a unique tie w containing C which is closest to v.*

Proof. By 12.1.22 the set $V'(C)$ of all ties containing a common chamber C is connected and as $V'(C)$ spans a connected subgraph of a tree, it spans a subtree. For any subtree $\mathcal{T}_0$ of $\mathcal{T}$, (or a tree in general) and any tie v in $\mathcal{T}$, there exists a unique tie in $\mathcal{T}_0$ closest to v. This implies that for any tie v in $\mathcal{T}$ there exists a unique tie w containing a given chamber C which is closest to v. □

Lemma 12.1.24. *Let $v_1, \dots, v_n$ be a minimal path in $\mathcal{T}$ with $n > 1$. Then any minimal gallery from v_1 to v_n has to contain a chamber of v_{n-1}.*

Proof. The statement is always true if $v_n \subset v_{n-1}$ or $n \leq 3$. Now let $n > 3$. Let $\Gamma = (C_0, \dots, C_l)$ be a minimal gallery from v_1 to v_n. By 12.1.18 the chambers of Γ are contained in the path $v_1, \dots, v_n$. Let C_i be the first chamber of Γ which lies inside v_n. If C_i lies inside v_{n-1} we are done. Now assume $C_i \notin v_{n-1}$. The panel containing C_{i-1} and C_i is contained in at least one of the ties in the given path, see 12.1.18. Let v' be the last of such ties. By 12.1.22 every tie on the minimal path from v' to v_n contains C_i and thus v_{n-1} contains C_i. □

Proposition 12.1.25. *Let v, v' be two arbitrary ties in $\mathcal{T}$. For any $C \in v$, $D \in v'$ and any tie $\tilde{v}$ on the minimal path from v to v', every minimal gallery from C to D has to contain a chamber of $\tilde{v}$.*

Proof. Let Γ be a minimal gallery from $C \in v$ to $D \in v'$ and let $v_1 = v, \dots, v_n = v'$ be a minimal path. By 12.1.24 the tie v_{n-1} contains a chamber D_{n-1} of Γ. Now we can apply the same argument to the gallery we get from Γ by ending this gallery at D_{n-1}. We see that every tie on the minimal path from v to v' has to contain an element of Γ and the statement holds. □

Lemma 12.1.26. *For any tie v in $\mathcal{T}$ and any chamber C in $\mathcal{B}$, there exists a unique projection of C onto v. In particular, each tie is gated.*
We say $\mathcal{T}$ satisfies the ***gate property****.*

Proof. Let C be a chamber in $\mathcal{B}$ and let v be a tie of $\mathcal{T}$. In the case, where C is contained in v we are fine. Assume v is not gated. Let v' be the (unique) tie closest to v containing C (see 12.1.23). By 12.1.18 there exists a unique minimal path $(v = v_0, \ldots, v_n = v')$ from v to v' containing every minimal gallery from C to any chamber in v'. As v is not gated, the tie v_1 is gated and contained in v_0 by (TG3) and (TT2). Let D be an arbitrary chamber in v. By 12.1.24 any minimal gallery from C to D has to meet v_1. As v_1 is gated, we may adjust any such gallery to meet $\mathrm{proj}_{v_1}(C)$. This means that for any chamber D in v, there exists a minimal gallery from C to D containing $\mathrm{proj}_{v_1}(C)$ which shows that $\mathrm{proj}_{v_1}(C)$ is a gate for C onto v. □

An immediate consequence of this proof is:

Lemma 12.1.27. *Let $v \neq v' \in \mathcal{T}$ and let $C \in v$. Let $(v = v_0, \ldots, v_n = v')$ be the unique minimal path from v to v'. Then* $\mathrm{proj}_{v'}(C) \in v_{n-1}$.

Proposition 12.1.28. *Let v, v' be two distinct ties in $\mathcal{T}$ and let $\bar{v}$ be a tie lying on a minimal path between v and v' in $\mathcal{T}$.*
Then for each chamber $C \in \mathrm{Cham}(v)$ and each chamber $D \in \mathrm{Cham}(v')$ there exists a minimal gallery from C to D that contains $\mathrm{proj}_{\bar{v}}(C)$ and $\mathrm{proj}_{\bar{v}}(D)$.

Proof. Let $\Gamma = (C = C_0, \ldots, C_n = D)$ be a minimal gallery from C to D. We can assume that $C_k = \mathrm{proj}_{\bar{v}}(C)$ for some $k \in \{1, \ldots, n\}$, as Γ has to meet every tie on the minimal path from v to v' by 12.1.25 and as all ties are gated by 12.1.26. From the gate property of $\bar{v}$ there also exists a minimal gallery Γ' from E_k to D containing $\mathrm{proj}_{\bar{v}}(D)$. Thus $(C = C_0, \ldots, C_k) \cdot \Gamma'$ is a minimal gallery from C to D containing $\mathrm{proj}_{\bar{v}}(C)$ and $\mathrm{proj}_{\bar{v}}(D)$. □

Definition 12.1.29. Let θ be an automorphism of a building $\mathcal{B}$. If $\mathcal{T}$ is a tie tree for θ on $\mathcal{B}$, then it admits a CAT(0) structure such that θ induces an isometry on $\mathcal{T}$. We will make no difference in the notation for ties and edges and their corresponding realizations. Therefore we have the non-empty, convex set $\mathrm{Min}(\theta)(\mathcal{T})$ of all points with minimal displacement by 3.7.2 (ii) and 11.1.4.
If the set of ties inside $\mathrm{Min}(\theta)(\mathcal{T})$ is empty, then θ stabilizes an edge $(v, w) \in \mathcal{T}$, but does not fix it. From the convexity we get that the midpoint of the realization of this edge is the only fixed point for θ and thus the only element of $\mathrm{Min}(\theta)(\mathcal{T})$. Therefore we can define the following:
The **support $\mathbf{SM}\,(\boldsymbol{\theta})(\boldsymbol{\mathcal{T}})$** of $\mathrm{Min}(\theta)(\mathcal{T})$ is either

(i) the set $\{v \in \mathcal{T} \mid v \in \mathrm{Min}(\theta)(\mathcal{T})\}$, or

(ii) it is the set $\{v, w\}$ for an edge (v, w) which is stabilized but not fixed by θ.

For a chamber C in $\mathrm{Cham}(\mathrm{SM}(\theta))$, we also write $C \in \mathrm{SM}(\theta)$.

Remark 12.1.30. As $\mathrm{Min}(\theta)(\mathcal{T})$ is a convex set, we see that $\mathrm{SM}(\theta)(\mathcal{T})$ is a convex set, i.e. if $v, w \in \mathrm{SM}(\theta)(\mathcal{T})$, then any tie u which lies on the minimal path from v to w is also contained in $\mathrm{SM}(\theta)(\mathcal{T})$.
It follows from 3.7.2 (i) that θ stabilizes $\mathrm{SM}(\theta)(\mathcal{T})$.

Furthermore, for any tie $v \in \mathcal{T}$, there exists a unique tie in $\mathrm{SM}(\theta)(\mathcal{T})$ which is closest to v. This follows directly from the tree structure of $\mathcal{T}$ and the convexity of $\mathrm{SM}(\theta)(\mathcal{T})$.

Proposition 12.1.31. *Let $C \notin \mathrm{SM}(\theta)(\mathcal{T})$ be a chamber of a building $\mathcal{B}$ with an automorphism θ and a tie tree $\mathcal{T}$ for θ. Let v be a tie containing C and let u be the unique tie of $\mathrm{SM}(\theta)(\mathcal{T})$ closest to v. Then*

$$\theta(\mathrm{proj}_u(C)) = \mathrm{proj}_{\theta(u)}(\theta(C))$$

Proof. If $\mathrm{proj}_{\theta(u)}(\theta(C))$ has a shorter distance to $\theta(C)$ as $\theta(\mathrm{proj}_u(C))$, then its image under θ^{-1} is an element of u which has a shorter distance to C as $\mathrm{proj}_u(C)$. But this contradicts the definition of the projection. □

Theorem 12.1.32. *If an automorphism θ of a building $\mathcal{B}$ admits a tie tree $\mathcal{T}$, then the displacements of θ on $\mathcal{B}$ are exactly the θ-conjugates $v \cdot w \cdot \theta(v^{-1})$ of the displacements w of chambers in $\mathrm{SM}(\theta)$ such that $l(vw\theta(v^{-1})) = 2l(v) + l(w)$.*

Proof. Let C be a chamber of $\mathcal{B}$. Assume $C \notin \mathrm{SM}(\theta)$ and let v be a tie containing C. As $\mathcal{T}$ is a tree, there exists a unique tie u of $\mathrm{SM}(\theta)$ which is closest to v (see 12.1.30). Let $D := \mathrm{proj}_u(C)$ and let $u' := \theta(u)$. From 12.1.31 we get $\theta(D) = \mathrm{proj}_{u'}(\theta(C))$.

By the convexity of $\mathrm{SM}(\theta)$ there exists a minimal path in $\mathcal{T}$ from v to u' containing u and by 12.1.28 there exists a minimal gallery from C to $\theta(D)$ containing D. Now we have two cases:

$u' \neq u$: The tree structure of $\mathcal{T}$ shows that there exists a minimal path in $\mathcal{T}$ from v to $\theta(v)$ containing u and u'. By 12.1.28 we obtain a minimal gallery from C to $\theta(C)$ containing D and $\theta(D)$.

$u' = u$: For this case, let $(u = v_0, \dots, v_n = v)$ be the minimal path from u to v and let $(u = w_0, \dots, w_n = \theta(v))$ be the minimal path from u' to $\theta(v)$.

- **$v_1 \neq w_1$:** As u connects the two paths $(v_1 \dots, v_n)$ and $(w_1, \dots, w_n)$, the tree structure yields that $v_n, \dots, v_1, u, w_1, \dots, w_n$ is a minimal path in $\mathcal{T}$ and by 12.1.28 we get a minimal gallery from C to $\theta(C)$ containing D and $\theta(D)$.
- **$v_1 = w_1$:** In this case v_1 is a fixed point of the action of θ on $\mathcal{T}$ and thus v_1 is an element of $\mathrm{SM}(\theta)$ closer to v than u, which contradicts the choice of u. So this case does not happen.

This shows that we get a minimal gallery from C to $\theta(C)$ containing D and $\theta(D)$. The type of this gallery is a word of the form $vw\theta(v^{-1})$ with $l(vw\theta(v^{-1})) = 2l(v) + l(w)$. Thus every displacement is a reduced word of such a form.
Now let w be a displacement for a chamber $C \in \mathrm{SM}(\theta)(\mathcal{T})$ and let v be a word with $l(vw\theta(v^{-1})) = 2l(v) + l(w)$. This means that a gallery of type $vw\theta(v^{-1})$ is minimal as its type is a reduced word. Let Γ_1 be a minimal gallery from a chamber E to C of type v and let Γ_2 be a minimal gallery from C to $\theta(C)$ of type w. The image $\theta(\Gamma_1^{-1})$ is a minimal gallery of type $\theta(v^{-1})$ and the concatenation $\Gamma_1 \cdot \Gamma_2 \cdot \theta(\Gamma_1^{-1})$ is a gallery of type $vw\theta(v^{-1})$ and thus a minimal gallery.
This shows that every word with the desired conditions is a displacement for θ. □

Example 12.1.33 (Free Products)**.** *In this section, we will have a closer look at Coxeter systems which split as a free product of Coxeter systems. Buildings of these types are examples for which a tie tree will have knots. An attempt to construct a tie tree by taking the maximal spherical residues and their intersections, as we did for* $\mathrm{PGL}(2,\mathbb{Z})$*, will not work. Those intersections can be knots and connecting the knots to the ties containing them will generally yield cycles in the graph. This is where we need the condition (TG5): If* $v_0 \subset v_1 \subset v_2$ *then* $(v_0, v_1) \notin E$*.* The Coxeter system (W, S) with

$$\begin{aligned} W := \langle s_1, s_2, s_3, s_4, s_4, s_5 \mid \\ & s_1^2 = s_2^2 = s_3^2 = s_4^2 = s_5^2 = s_6^2 = 1 \\ & = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_4 s_5)^3 = (s_5 s_6)^3 \rangle \end{aligned}$$

can be decomposed as a tree of groups in the following way:

$$\begin{tikzcd} W_{\{s_1,s_2\}} \arrow[r, dash, "W_{\{s_2\}}"] & W_{\{s_2,s_3\}} \arrow[r, dash, dashed, "W_\emptyset"] & W_{\{s_4,s_5\}} \arrow[r, dash, "W_{\{s_5\}}"] & W_{\{s_5,s_6\}} \end{tikzcd}$$

We see that this group decomposes as the free product of $W_{\{s_1,s_2,s_3\}} * W_{\{s_4,s_5,s_6\}}$. Removing the edge corresponding to the trivial group yields a decomposition of the tree into two separate trees:

$$\begin{tikzcd} W_{\{s_1,s_2\}} \arrow[r, dash, "W_{\{s_2\}}"] & W_{\{s_2,s_3\}} & W_{\{s_4,s_5\}} \arrow[r, dash, "W_{\{s_5\}}"] & W_{\{s_5,s_6\}} \end{tikzcd}$$

There are several ways to connect the two components, for example:

$$\begin{tikzcd} W_{\{s_1,s_2\}} \arrow[r, dash, "W_{\{s_2\}}"] & W_{\{s_2,s_3\}} \arrow[rr, dash, dashed, bend left, "W_\emptyset"] & W_{\{s_4,s_5\}} \arrow[r, dash, "W_{\{s_5\}}"] & W_{\{s_5,s_6\}} \end{tikzcd}$$

Using the special subgroups corresponding to the edge and vertex groups in the tree of groups decomposition and connecting them corresponding to the given tree, we derive a tree structures as follows:

$$\begin{array}{c|c}
W_{\{s_1,s_2\}} = R_{12}(1_W) & W_{\{s_1,s_2\}} = R_{12}(1_W) \\
W_{\{s_2\}} = R_2(1_W) & W_{\{s_2\}} = R_2(1_W) \\
W_{\{s_2,s_3\}} = R_{23}(1_W) & W_{\{s_2,s_3\}} = R_{23}(1_W) \\
\{1_W\} & \{1_W\} \\
W_{\{s_4,s_5\}} = R_{45}(1_W) & W_{\{s_5,s_6\}} = R_{56}(1_W) \\
W_{\{s_5\}} = R_5(1_W) & W_{\{s_5\}} = R_5(1_W) \\
W_{\{s_5,s_6\}} = R_{56}(1_W) & W_{\{s_4,s_5\}} = R_{45}(1_W) \\
 & \\
\text{Graph in the } 1^{\text{st}} \text{ case} & \text{Graph in the } 2^{\text{nd}} \text{ case}
\end{array}$$

Here we denote a residue corresponding to a special subgroup of type $s_{i_1}, \dots s_{i_l}$, with $i_1 \dots, i_l \in \{1, \dots, n\}$ by $R_{i_1,\dots,i_n}$. This structure yields the same graph as the one given in the tree of groups decomposition.
We extend this tree to a tree structure on the whole Coxeter group by translating these residues along the whole Coxeter group. This means, we multiply the given residues which each element of the Coxeter group. By identifying two vertices in the resulting graph if and only if they describe the same residue, we yield the connectedness of the graph. (see figure 12.1).

Remark 12.1.34. If a Coxeter system decomposes as a graph of groups, some edge groups might be trivial groups. Examples are the universal the Coxeter systems which are of the form $G := \langle s_1, \dots, s_n \mid s_1^2 = s_2^2 = \cdots = s_n^2 = 1 \rangle$. One can describe such groups as a graph of groups with n vertices, one for each special subgroup corresponding to a generator $s_i, i = 1, \dots, n$, and no edges.
It is also possible to decompose this group as a tree of groups, with the same set of vertex groups, but adding trivial edge groups such that the resulting graph is a tree. It is clear that adding trivial edge groups has no effect on the resulting group.

If a group decomposes as a tree of groups, we might remove all edges which correspond to trivial edge groups. This yields a set of connected components of the graph which are again trees. If there is more than one connected component, the given group splits as a free product of the groups corresponding to the connected components.

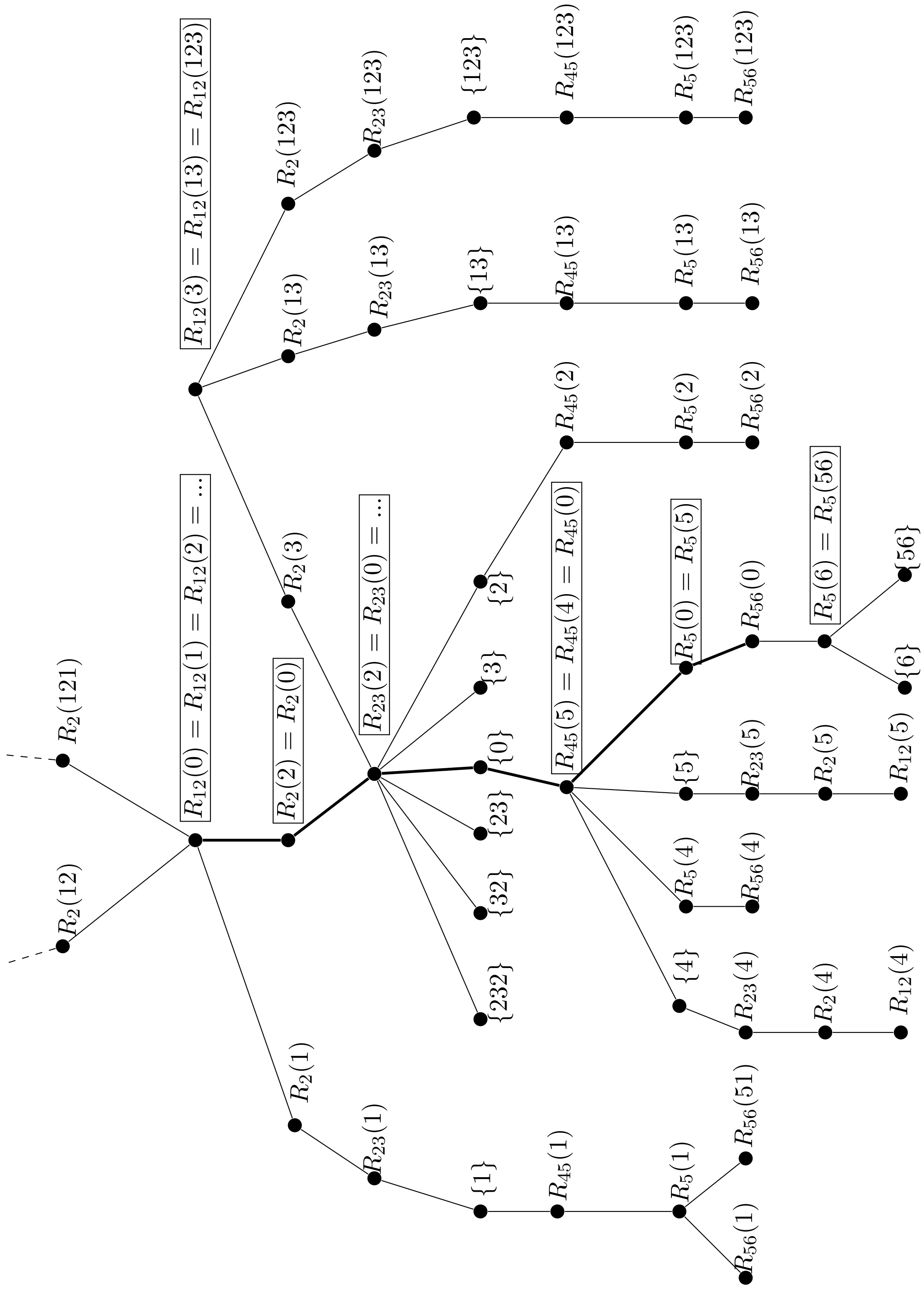


Figure 12.1: Excerpt of a residue tree for Example 12.1.33. The thick lines come from the tree of groups decomposition. For readability, a different notation as usual is used in this graph. The number 45 in $R_{45}(13)$ represents the set $\{s_4, s_5\}$ and the number 13 in $R_{45}(13)$ represents the word $s_1 s_3 \in W$, where 0 represents its neutral element 1_W.

12.2 Residue Trees

Let's have a closer look at a specialization of tie trees. Instead of defining them as tie trees with additional conditions, we present a definition which will make it easier to work with them later on.

Definition 12.2.1. Let (W, S) be a Coxeter system. A **residue graph** for a building $\mathcal{B}$ of type (W, S) is a simple graph (V, E) with the following conditions:

(RG1) The vertices are distinguished residues $\emptyset \neq R \neq \mathcal{B}$ of $\mathcal{B}$.

(RG2) The set V is closed under intersections of rank ≥ 1, i.e. for all $v, w \in V$ with $\operatorname{rank}(v \cap w) \geq 1$: $v \cap w \in V$.

(RG3) If a vertex is a residue of type $J \subset S$, then every J-residue is a vertex of (V, E).

(RG4) If $(v, w) \in E$, then either $v \subset w$ or $w \subset v$.

(RG5) Every panel is contained in at least one vertex $v \in V$ (and thus every chamber is contained in at least one vertex).

(RG6) If $v, w \in V$ with $v \subset w$ then $v = w_1 \cap w_2$ for some $w_1, w_2 \in V$ and $(w_1, v), (v, w_2) \in E$. If $\operatorname{rank} v \geq 1$, then w_1 can be chosen to be w.

(RG7) If $v_0 \subset v_1 \subset v_2$, then $(v_0, v_1) \notin E$.

A Residue graph which is a tree is called **residue tree**.

Lemma 12.2.2. *A residue graph satisfies*

(RG8) *If* $v, w \in V$ *with* $v \subset w$ *and* $\operatorname{rank}(v) \geq 1$, *then* $(v, w) \in E$.

Proof. Let $v, w \in V$ with $v \subset w$ and $\operatorname{rank}(v) \geq 1$. Then by (RG6) $v = w \cap w_2$ for some $w_2 \in V$ and $(v, w) \in E$. $\square$

Remark 12.2.3. A residue graph (V, E) of a Coxeter system (W, S) determines a unique residue graph (V', E') for any building of type (W, S) by choosing the vertices of (V', E') to be the set of residues whose type is the same as a vertex of V and the edges in E' connecting incident residues if the vertices of V with the same types are connected in E.

Lemma 12.2.4. *A residue tree* (V, E) *for a building* $\mathcal{B}$ *is a tie tree for any type-preserving action on* $\mathcal{B}$.

Proof. There are several direct equalities: (TG2) and (RG2), (TG3) and (RG4), (TG4) and (RG6), (TG5) and (RG7), (TT1) and (RG5), as well as (TT3) and the condition that a tie tree is a tree. There are only a few remaining conditions to check:

(TG1) Every residue is a proper subset of $\mathcal{B}$ and thus a tie.

(TT2) Every residue is gated.

(TT4) Assume θ is a type-preserving automorphism of $\mathcal{B}$. This means that θ preserves the set of J-residues for any $J \subset S$. Therefore θ preserves the residue tree structure. $\square$

Definition 12.2.5. Let $\mathcal{G}$ be a tree of groups decomposition of a Coxeter system (W, S). If $\mathcal{G}$ contains more than one vertex, it is said to be non-trivial.
If every vertex of $\mathcal{G}$ is a special subgroup of (W, S), and if no vertex group of $\mathcal{G}$ embeds into any other vertex group of $\mathcal{G}$, then $\mathcal{G}$ is called a **special tree of groups decomposition** for (W, S).

Definition 12.2.6. Let $\mathcal{G}$ be a non-trivial special tree of groups decomposition for a Coxeter system (W, S), and let $\mathcal{B}$ be a building of type (W, S).
A **residue graph** of $\mathcal{B}$ associated to $\mathcal{G}$ is a residue graph (V, E) for $\mathcal{B}$, where the vertices are the residues of $\mathcal{B}$ whose type is the type of some vertex or edge group of $\mathcal{G}$.

Lemma 12.2.7. *Let $\mathcal{G}$ be a non-trivial special tree of groups decomposition of a Coxeter system (W, S). Let $\mathcal{B}$ be a building of type (W, S). If no edge of $\mathcal{G}$ corresponds to the trivial group, then the residue graph of $\mathcal{B}$ associated to $\mathcal{G}$ is unique.*

Proof. We need to show that the set of edges E is uniquely determined.
As there are no trivial edge groups, every vertex is a residue of rank ≥ 1. Thus the edges are uniquely given by the relation $(v, w) \in E \Leftrightarrow v \subset w$ or $w \subset v$. $\square$

Lemma 12.2.8. *Let (V, E) be a residue tree for a Coxeter system (W, S) corresponding to a tree of groups decomposition. For any building $\mathcal{B}$ of type (W, S), the residue graph (V', E') associated to (V, E) is a tree.*

Proof. Let $v_1, \ldots, v_n$ be an arbitrary path in (V', E') without any repetitions and $v_2 \subset v_1$. Let n' be the maximal even number in $\{1, \ldots, n\}$, let C_1 be a chamber in v_1, C_{n+1} a chamber in v_n and let C_i be the projection of C_1 onto v_i for the even values in $i \in \{2, \ldots, n'\}$. For each even number $i \in \{2, \ldots, n' - 2\}$, let Γ'_i be a minimal gallery from C_i to C_{i+2} and let C_{i+1} be the first chamber of Γ'_i contained in $v_{i+1} \backslash v_i$. Let $\Gamma'_n + 1$ be a minimal from $C_{n'}$ to C_{n+1}. Now for $i \in \{1, \ldots, n'\}$ let Γ_i be a minimal gallery from C_i to C_{i+1}, and let w_i be their types. Further let Γ be the gallery $\Gamma_1 \cdots \Gamma_{n'+1}$. The type of this gallery is the expression $w_1 w_2 \ldots w_{n'} \cdot w_{n'+1}$ which is an element of the free amalgamated product $W_0 *_{W_1} W_2 *_{W_3} \ldots$, where W_i is the special subgroup whose type is the type of the residue v_i. But this is a reduced word[2] which means that it describes a minimal gallery and thus $C \neq D$ which shows that $v_1, \ldots, v_n$ cannot be a cycle. Thus (V', E') does not contain any cycles. It remains to show that (V', E') is connected, but this follows directly from the covering of $\mathcal{B}$ by apartments. $\square$

[2] A (free) amalgamated product $W_1 *_{W_2} W_3$ has the property, that a word which is a product of reduced words $w_1 w_2 w_3 \cdots$ is reduced, if $w_1, w_4, \cdots \in W_1 \backslash W_2$, $\quad w_2, w_5, \cdots \in W_2$ and $w_3, w_6, \cdots \in W_3$, see 2.4.

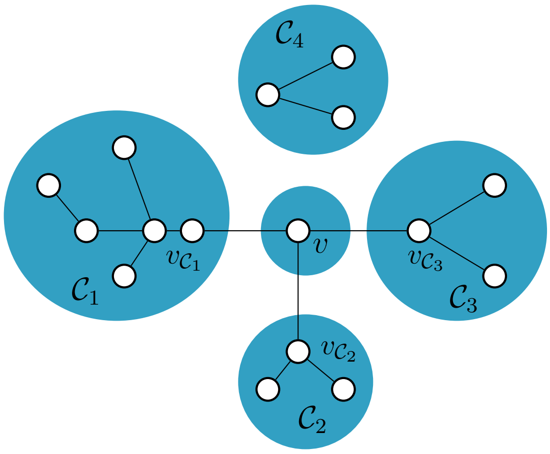


Figure 12.2: An example, where $\Gamma \setminus \{v\}$ splits into 4 connected components.

Theorem 12.2.9. *Let $\mathcal{G}$ be a non-trivial special tree of groups decomposition for a Coxeter system (W, S). Let v be a vertex of $\mathcal{G}$. Then $V(\mathcal{G}) \setminus \{v\}$ with*

$$E(\mathcal{G}) \setminus \{(v_1, v_2) \in E \mid v_1 = v \text{ or } v_2 = v \text{ or } G_{(v_1,v_2)} = \{1_W\}\}$$

yields a graph of groups $\mathcal{G}'$ whose connected components are trees of groups. The group corresponding to $\mathcal{G}'$ is the free product of those collection of tree of groups which are again special subgroups of W. If every building of type of one of those special subgroups admits a residue tree associated to the corresponding tree, then $\mathcal{B}$ admits a residue tree associated to $\mathcal{G}$.

Proof. The idea of the proof is the following: We use the graph Γ of the non-trivial special tree of groups decomposition for W and construct an new graph Γ' whose vertices are the connected components of $\Gamma \setminus \{v\}$ and $\{v\}$. The graph Γ' will be extended to a graph covering the whole building, where every vertex of the extended graph has the same type set as a vertex of Γ'. In the last two steps we will show that the extended graph is a tree and that it satisfies the conditions to be a residue tree.

A graph (V', E') whose vertices are the connected components and v:
Let $\mathfrak{C}$ be the family of connected components of $\mathcal{G}'$. For each connected component $\mathcal{C}$, the vertex v is connected to at most one vertex $v_{\mathcal{C}}$ of $\mathcal{C}$. If no vertex of $\mathcal{C}$ is connected to v, then $\mathcal{C}$ is a factor for W as a free product. In this case we chose an arbitrary vertex of $\mathcal{C}$ for $v_{\mathcal{C}}$. In the other case, v and $v_{\mathcal{C}}$ intersect non-trivially, but the intersection of v with any other vertex of $\mathcal{C}$ is 1_W.

Let J be the type set of v, $K_{\mathcal{C}}$ be the type set of the special subgroup corresponding to $\mathcal{C}$ and let $J_{\mathcal{C}} := J \cap K_{\mathcal{C}}$. Let (V', E') be the graph whose vertices are the residues whose type set is an element of

$$\{J\} \cup \{K_{\mathcal{C}} \mid C \in \mathfrak{C}\} \cup \{J_{\mathcal{C}} \mid C \in \mathfrak{C}\},$$

where $(w, w') \in E'$ if and only if one of the following holds:

(i) $w \subset w'$ and w' is of type J.

(ii) $w' \subset w$ and w is of type J.

(iii) $w \subset w'$, w' is not of type J and w is maximal in w'.

(iv) $w' \subset w$, w is not of type J and w' is maximal in w.

Here maximal means the following: The vertex w is maximal in w' if for any vertex $w'' \in V'$ with $w \subset w'' \subseteq w'$, we have $w = w''$.
The condition on the maximality is needed as a vertex corresponding to a trivial group should not be connected to a connected component which has non-trivial intersection with v.

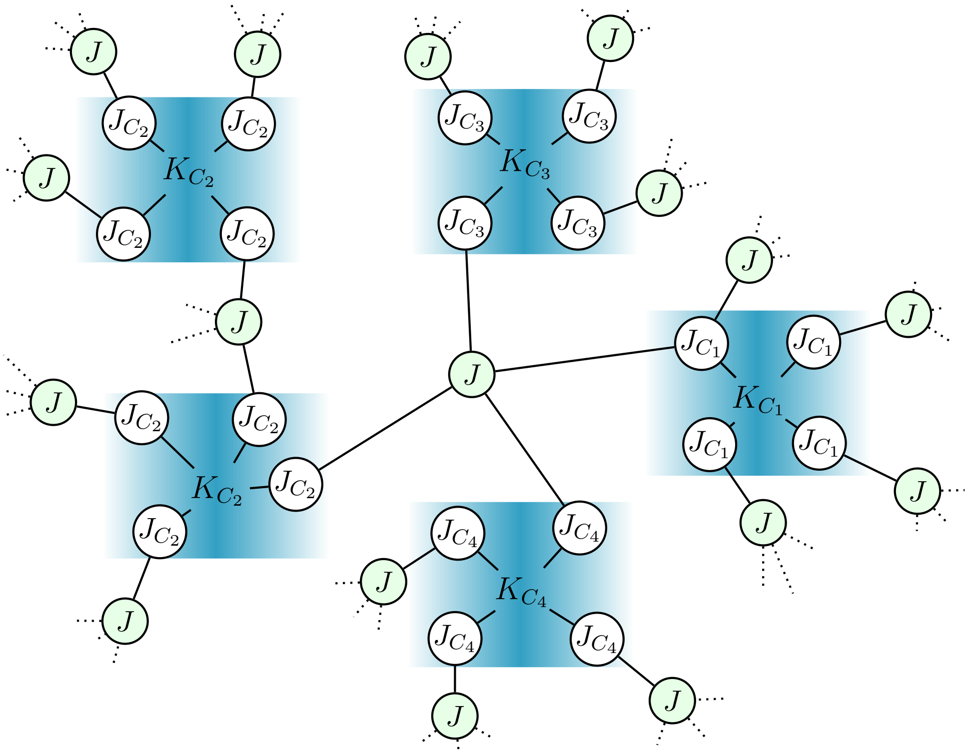


Figure 12.3: Excerpt of (V', E') corresponding to figure 12.2

A finer structure on (V', E'):
We substitute a vertex v' corresponding to a connected component $\mathcal{C}$, i.e. v' is of type $K_{\mathcal{C}}$ for some $\mathcal{C} \in \mathfrak{C}$, with a copy $T(v')$ of the residue tree corresponding to the residue associated to $\mathcal{C}$.

Each residue of type $\tau(v_{\mathcal{C}})$ inside $T(v')$ contains a residue of type $J_{\mathcal{C}}$ which is a vertex of the ambient tree. The tree $T(v')$ itself does not contain any vertex of type $J_{\mathcal{C}}$. The only possibility for such a vertex would be a residue of type $\emptyset$, i.e. a single chamber. But as we erased all edges which correspond to the trivial group, there are no such vertices in any connected component. As $T(v')$ does not contain such a vertex, we can connect the residue corresponding to $v_{\mathcal{C}}$ with each of its subresidues of type $J_{\mathcal{C}}$. If we do this procedure for any vertex corresponding to a connected component of $\mathcal{G}$, we get a connected graph. If (V', E') is a tree, extending vertices by trees and connecting them without constructing cycles yields again a tree. There we need to show that there are no cycles inside (V', E').

The tree structure on (V', E'):
Let $v_1, v_2, \ldots, v_n$ be a path in (V', E') such that $(v_i, v_j) \in E \Leftrightarrow j = (i \pm 1)$ for $i = 2, \ldots, n-1$ and $v_2 \subset v_1$. Let $J_1 \ldots, J_n$ denote the type set of $v_1, \ldots, v_n$ respectively.
As $v_2 \subset v_1$ it follows also that $v_2 \subset v_3$. If v_2 is a single chamber, say C, then moving from any chamber in $v_1 \setminus \{C\}$, to any chamber in $v_3 \setminus \{C\}$ yields a Weyl distance of the form ww', where every generator in w has no relation with any generator in w'.

If $J_2 \neq \emptyset$, then the Weyl distance from any chamber in $v_1 \setminus v_2$ to any chamber in $v_3 \setminus v_2$ is a word of the form $ws\tilde{w}tw'$, where $\tilde{w}$ is a word in $J_2 = J_1 \cap J_3$ and $s \in J_1$, $t \in J_3$. The elements s, t have no relation in W.

We are looking at the amalgamated product $W = W_k *_{W_0} W'$, where W' is the free product of the groups corresponding to the elements in $\mathfrak{C}$ and W_0 is the free product of the groups of the form $W_{J_\mathcal{C}}$, where $\mathcal{C}$ runs over the set $\mathfrak{C}$. For the given path $v_1, \ldots, v_n$, we can find a gallery in the following way: We take a chamber C_1 inside v_1 and a minimal gallery from C_1 to its projection P_1 onto v_2. Then we extend this gallery with a minimal gallery from P_1 to its projection P_1' onto v_4. Let D_1 be the first chamber inside this gallery, lying inside $v_3 \setminus v_2$. Its projection onto v_4 is also P_1'. Now we can take a minimal gallery from P_1' to its projection P_2 onto v_6. Here we take C_2 as the first chamber of this gallery, which lies inside $v_5 \setminus v_4$. Now we can iterate this procedure and get a sequence of chambers $(C_1, P_1, D_1, P_1', \ldots, C_l, \hat{P}_l, \hat{D}_l, \hat{P}_l')$, with $1 < l < n$, where $\hat{}$ denotes that this chamber possibly does not exists.

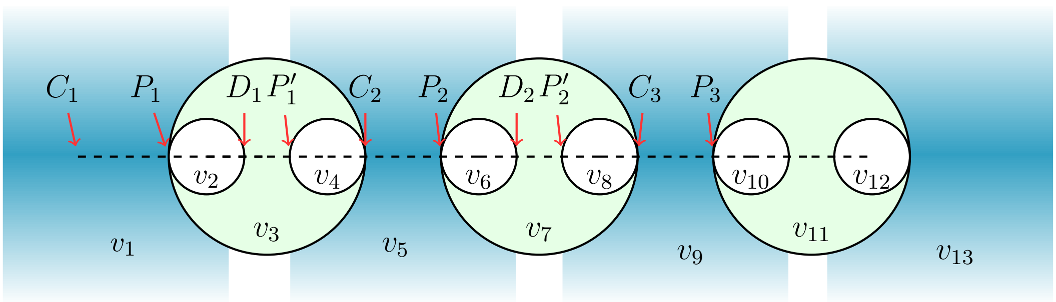


Figure 12.4: Example for the construction of the gallery

The given gallery is of type

$$(\hat{u}_1\hat{k}_1\hat{w}_1 j_1)\ (u_2 k_2 w_2 j_2) \cdots (u_l \hat{k}_l \hat{w}_l \hat{k}_l),$$

where $u_i, w_i \in W_0$, $k_i \in W_1 \setminus W_0$ and $j_i \in W' \setminus W_0$, here the $\hat{}$ denotes that these elements possibly do not appear.
Furthermore we see that $u_i k_i w_i$ and $w_i j_i u_{i+1}$ are reduced words. Now the structure

of amalgamated products shows that this expression is a reduced word. (One may take $u_i k_i$ as a representative for a right-coset of W_0 in K and $w_i j_i$ as a representative of W_0 in W'.) Thus it is not possible to reach any element of v_i again, by moving away from v_i in (V', E') and this means that (V', E') contains no cycles.

Conditions for a residue graph:

(RG1) By construction, all vertices are distinguished residues.

(RG2) To see that the set of vertices is closed under intersections of residues of rank ≥ 2, we note that the intersections of rank ≥ 2 correspond exactly to the non-trivial edges of $\mathcal{G}$. By assumption, this condition is satisfied for the connected components. Thus we only need to check the non-trivial edges in the tree of groups decomposition where one of the vertices is v. But by definition, the special subgroups corresponding to those edges determine some $J_{\mathcal{C}}$ which are vertices of the tree.

(RG3) Also, by construction: If a vertex is a residue of type $J \subset S$, then every J-residue is a vertex of (V, E).

(RG8) If $v', w' \in V$ with $v' \subset w'$ and $\operatorname{rank}(v') \geq 1$ then $(v', w') \in E'$ if $w' \neq v$ and $v' \not\subseteq v$. If $w' = v$, then v' is the intersection of v with an connected component and thus v' is an vertex of the graph (V', E') connected to v. If $v' \subset v$ and w' is not v, then v' is the intersection of v and w which is added in (V', E') and connected to w and v'.

(RG4) If $(v, w) \in E$, then by construction either $v \subset w$ or $w \subset v$.

(RG5) From the tree of groups decomposition $\mathcal{G}$ we see that every generator $s \in S$ is contained in at least one vertex of $\mathcal{G}$ and thus every panel is contained in at least one residue of the given tree.

(RG6) The vertices of the resulting graph correspond to the vertices and edges of $\mathcal{G}$. Therefore, every vertex contained in some other vertex is the intersection of two ambient vertices.

(RG7) The last thing to check is: Given $v_0 \subset v_1 \subset v_2$, then $(v_0, v_1) \notin E$.
A sequence of residues $v_0 \subset v_1 \subset v_2$ in the tree shows that there exist vertex and edge groups $G_0 \subset G_1 \subset G_2$ of $\mathcal{G}$. We may assume G_2 is a vertex group by extending to some ambient vertex group if necessary. As we have a non-trivial tree of groups decomposition, the groups G_0 and G_1 are edge groups and thus $G_0 = G_2 \cap G'$ and $G_1 = G_2 \cap G''$ for two vertex groups G' and G''. If G_0 is not of rank 0 we get a cycle $G_2 \supset G_1 \subset G' \supset G' \cap G'' \subseteq G_0 \subset G_2$ which does not exist.
Now we assume that one of the groups G_2, G', G'' corresponds to the vertex v and that G_0 is the trivial group. As seen before, the trees corresponding to connected components do not contain any copies of the trivial group. Thus G_0

corresponds to the edge of v to some connected component $\mathcal{C}$. This implies that also G_1 corresponds to the edge of v to some connected component $\mathcal{C}' \neq \mathcal{C}$. By our construction, there is no edge connecting $\mathcal{C}$ and $\mathcal{C}'$ and thus such a sequence cannot appear. □

Corollary 12.2.10. *There exists a residue tree for every non-trivial special tree of groups decomposition (even with infinite set S).*

Proof. Let $\mathcal{G}$ be an arbitrary non-trivial special tree of groups decomposition. Let v be any vertex of $\mathcal{G}$. Then the graph (V', E') in the proof of 12.2.9 is a tree. Furthermore the last part of that proof shows that (V', E') is a residue graph. □

Corollary 12.2.11. *Let $\mathcal{B}$ be a building of type (W, S). We get a tie tree $\mathcal{T}$ for any non-trivial special tree of groups decomposition $\mathcal{G}$, where the vertices of $\mathcal{T}$ are exactly the residues of $\mathcal{B}$ whose type sets are the typesets of some vertex or edge group of $\mathcal{G}$.*

Proof. As for a building the set S is finite, we can apply the previous proposition iterating over the set S and get a tree structure whose residues correspond exactly to the vertices of the non-trivial tree of groups decomposition. □

Theorem 12.2.12. *Let (W, S) be a Coxeter system which splits as a free product of special subgroups $(W_1, S_1) * \cdots * (W_n, S_n)$ such that none of the given factors splits as a free product of special subgroups. Let $\mathcal{B}$ be a building of type (W, S). The graph (V, E), where V is the set of all $S_1, \ldots, S_n$-residues and all chambers of $\mathcal{B}$ and where the (undirected) edges correspond to the inclusion relation, is a tie tree for any action θ on $\mathcal{B}$.*

Proof. We can adjust the proof of 12.2.9 by adding an additional vertex v corresponding to the trivial group to the non-trivial special tree of groups decomposition. The same construction of the graph (V', E') as in the proof gives a graph (V', E') by using v and the factors of W. The proof shows, that (V', E') is a tree. But furthermore, in this case (V', E') equals (V, E) and this is a residue graph for $\mathcal{B}$. Now any action on $\mathcal{B}$ has to preserve the given decomposition as free products, and hence (V, E) is a tie tree for any action on $\mathcal{B}$. □

Corollary 12.2.13. *A Coxeter system (W, S) admits a non-trivial special tree of groups decomposition if and only if its diagram is not 2-spherical.*

Proof. Assume the diagram is not 2-spherical, then there are generators $s, t \in S$ without any relation. Let W_1, W_2, W_0 be the special subgroups of W generated by $S \setminus \{s\}$, $S \setminus \{t\}$ and $S \setminus \{s, t\}$ respectively. Then W is the amalgamated product $W_1 *_{W_0} W_2$ which gives us a special tree of groups decomposition for W.
On the other hand if we have given a non-trivial special tree of groups decomposition for W, let $W_1 \neq W_2$ be the special subgroups corresponding to two arbitrary vertices of the given decomposition. As the decomposition is non-trivial, there exists $s \in W_1 \setminus W_2$ and $t \in W_2 \setminus W_1$ and thus s and t have no relation which means that their corresponding vertices in the Coxeter diagram are connected by an infinity. □

Lemma 12.2.14. *Any action on a building preserves the set of maximal spherical residues.*

Proof. Let R be a maximal spherical residue. Then $\theta(R)$ is again a residue and it needs to be spherical. If $\theta(R)$ is not maximal spherical then there exists a spherical residue R' containing $\theta(R)$ and $\theta^{-1}(R')$ is a spherical residue containing R properly which does not exist. □

Lemma 12.2.15. *Every building of type (W, S) with W virtually free admits a tie tree for any action.*

Proof. A virtually free Coxeter system admits a non-trivial (special) tree of groups decomposition $\mathcal{G}$ whose vertex groups are all spherical. All of these vertices correspond to maximal spherical subgroups. Indeed, if $v \in \mathcal{G}$ is not a maximal spherical special subgroup, let J be its type set and let $J \subset J'$ be a maximal spherical type set. For $s \in J' \setminus J$, there exists a vertex $v' \in \mathcal{G}$ containing s. As we do not allow any embeddings, the two vertices v, v' of $\mathcal{G}$ show that the order of st needs to be infinite which contradicts the sphericity of J'.
Now by 12.2.14 any action on $\mathcal{B}$ preserves the structure of maximal spherical residues and thus it preserves the unique residue tree associated to $\mathcal{G}$ for $\mathcal{B}$. □

Definition 12.2.16. A tie tree for some action θ on a building $\mathcal{B}$ is called residual if its set of vertices is the set of all $J_1, \dots, J_n$-residues, for some $J_1, \dots, J_n \subset S$.

Lemma 12.2.17. *Let $\mathcal{T}$ be a residual tie tree for some action θ on a building $\mathcal{B}$. Every maximal spherical residue of $\mathcal{B}$ is contained in at least one tie of $\mathcal{T}$*

Proof. Assume $J \subset S$ is maximal spherical and J is not contained in any typeset of the residues of $\mathcal{T}$. Let $s, t \in J$ such that $s \in J_1 \setminus J_2$ and $t \in J_2 \setminus J_1$ for two typesets J_1, J_2 of residues of $\mathcal{T}$. Let C be a chamber in $\mathcal{B}$ and let v be a residue of type J_1 containing C. As $o(ts) = n$ for some $n > 0$, we can construct a gallery of type $(ts)^n$ from C to itself. This gallery gives a sequence of J_1 and J_2-residues of length $2n$ from v to v which is a cycle. As cycles do not exist in $\mathcal{T}$, it is not possible to find such s and t which proofs the statement. □

Lemma 12.2.18. *Let $\mathcal{T}$ be a residual tie tree for some action θ on a building $\mathcal{B}$ of type (W, S). Then (W, S) admits a non-trivial special tree of groups decomposition.*

Proof. Let J_1, J_2 be two different types of residues of $\mathcal{T}$. Let $s \in J_1 \setminus J_2, t \in J_2 \setminus J_1$. Let C be a chamber in $\mathcal{B}$. Let v a residue of $\mathcal{T}$ containing C. Then any gallery of type $(st)^n$ describes a path in $\mathcal{T}$ along J_1 and J_2 residues issuing at v. As there are no cycles inside $\mathcal{T}$, this path cannot end in v again. And thus the element st has order infinity.
Thus there exists a non-trivial special tree of groups decomposition for (W, S). □

12.3 Examples

Example 12.3.1 (Right-Angle Attached Generators)**.** *This example shows the existence of non-residual tie trees for some action.*

Definition 12.3.2. Let (W, S) be a Coxeter system. An element s of S is said to be **right-angle attached** to (W, S) if the order $\mathrm{o}(st) \in \{2, \infty\}$ for all $t \in S \setminus \{s\}$, and if there exists an $t \in S$ with $\mathrm{o}(st) = \infty$.

Let (W, S) be Coxeter system with some right-angle attached generator $s \in S$. For every s-panel P in $\mathcal{B}$, we define R_P to be the chamberwise union of all $S \setminus \{s\}$-residues intersecting P non-trivially. Let (V, E) be the graph where the vertices are the R_P for all s-panels P in $\mathcal{B}$ together with all $S \setminus \{s\}$-residues of $\mathcal{B}$, and where the (undirected) edges correspond to the inclusion relation.

We show that (V, E) is a tie tree for $\mathcal{B}$ for any action which preserves the type $\{s\}$.

Lemma 12.3.3. *If two distinct $S \setminus \{s\}$-residues of some R_P intersect an s-panel P' non-trivially, then $R_P = R_{P'}$.*

Proof. Let v, v' be two distinct $S \setminus \{s\}$-residues of R_P which both intersect an s-residue P' non-trivially. The intersection of v and v' is empty and they intersect each panel P and P' in exactly one chamber. Let $C := v \cap P, C' := v \cap P'$ and let $E := v' \cap P, E' := v' \cap P'$. Then there exists a gallery from C to C' over E and E' of type $\delta(C, E) = sws$ and s has to commute with w. Thus P and P' are parallel with a Weyl distance in $W_{S \setminus \{s\}}$ and the $S \setminus \{s\}$ residues intersecting P non-trivially are exactly the $S \setminus \{s\}$ residues intersecting P' non-trivially. $\square$

Theorem 12.3.4. *The graph (V, E) is a tie tree for any action on $\mathcal{B}$ preserving the element $s \in S$.*

Proof. The conditions (TG1), (TG3), (TG4), (TG5) follow directly.

(TG2) V is closed under non-trivial and non-knotted intersections:

As residues of the same type are either equal or intersect trivially, we only have to check that for two s-panels $P \neq P'$ the intersection $R_P \cap R_{P'}$ is either empty or a $S \setminus \{s\}$-panel.

Assume $R_P \cap R_{P'}$ contains a chamber C, then the unique $S \setminus \{s\}$-residue R containing C is contained in $R_P \cap R_{P'}$. Assume there exists a chamber $D \in R_P \cap R_{P'}$ with $D \notin R$. From the definition of R_P, the chamber D lies in a $S \setminus \{s\}$-residue R' such that R and R' intersect an s-panel P'' non-trivially. This shows that $R_P = R_{P''} = R_{P'}$.

Connectedness of (V, E) Let $v, v' \in V$. Let $C \in v$, $D \in v'$ be arbitrary chambers. Let Γ be a minimal gallery from C to D. Let $w = \delta(C, D)$ and let $w_1, \ldots, w_n$ be the subwords of w such that $w = w_1 s w_2 s \cdots s w_n$. Let C_i be the unique chamber in Γ with $\delta(C, C_i) = w_1 s w_2 s \cdots w_{i-1} s$. We can construct a path in (V, E) from v to v' in the following way: Let $v_0 = v$. Let $v_i := R_{P_i}$ where P_i is the unique s-panel containing C_i. Then the v_i intersects v_{i-1} non-trivially for $0 < i < n$. Now v_0 and v_1 are either equal, v_0 is contained in v_1 or they intersect non-trivially. Same holds for v_n and v'. Therefore we get a path from v to v' along the v_i.

(TT1) Every panel is contained in a tie:
If P is a panel of type s, then R_P contains P.
If P is a panel of type $s' \neq s$, then it is contained in a $S \setminus \{s\}$-residue.

(TT2) For any edge (v, w), the intersection $v \cap w$ is convex and gated:
As seen before, $v \cap w$ is a residue and therefore convex and gated.

(TT3) (V, E) is a tree: Let (V', E') be the residue tree which is associated to the decomposition of (W, S) as an amalgamated product $\langle S \setminus \{s\}\rangle *_{\langle S' \rangle} \langle \{s\} \cup S' \rangle$, where $S' := \{t \in S \mid st = ts\}$. As every element of S' commutes with s, the $\{s\} \cup S'$-panels are two S'-panels along a set of parallel s-panels.
Let P be an s-panel and let $v'_P \in V'$ be the unique vertex in the residue tree containing P. The set R_P equals the set $V(P)$ of all $S \setminus \{s\}$-residues which are connected to v'_P in (V', E'). If we identify for each s-panel P the set R_P with $V(P)$ and connect any two elements $V(P)$ and $V(P')$ if and only if they intersect in an $S \setminus \{s\}$-residue, we get a tree (V'', E'').
Now, for each edge $(v_1, v_2) \in E''$ we add the element $v_1 \cap v_2$ to V'' and replace any edge (v_1, v_2) of E'' with edges $(v_1, v_1 \cap v_2)$ and $(v_1 \cap v_2, v_2)$.
The resulting graph equals (V, E), and as it is again a tree, (V, E) is a tree.

(TT4) Let θ be any action on $\mathcal{B}$ preserving the element $s \in S$. Then θ has to stabilize the set $S \setminus \{s\}$ and therefore it preserves the structure (V, E). $\square$

Remark 12.3.5. If the vertex corresponding to the generator s is not right-angle attached, i.e. there exists an element $t \in S$ with $\mathrm{o}(st) = n \notin \{2, \infty\}$, then the graph (V, E) is not a tree. One may take a chamber C and an $S \setminus \{s\}$ residue R containing C. Let Σ be an apartment containing C, and let $(C, C_1, \ldots, C_{2n})$ be a gallery inside Σ of type $(st)^n$. This gallery gives a sequence of $S \setminus \{s\}$-residues along some s-residues. But $C_{2n} = C$ and thus this sequence is a cycle.

Example 12.3.6 (Universal Coxeter Groups)**.** A universal Coxeter group W of rank n is generated by n reflections $s_1, \ldots, s_n$ which have pairwise no relation. The Cayley graph of such a Coxeter group corresponding to the generator set $\{s_1, \ldots, s_n\}$ is a n-valent tree. Let $\mathcal{B}$ be a thick building of type $(W, \{s_1, \ldots, s_n\})$. Let (V, E) be the graph, where the vertices are the panels in $\mathcal{B}$, and their pairwise non-empty intersections, and where the (undirected) edges correspond to the inclusion relation. As W is the free product of the groups $\langle s_1 \rangle * \cdots * \langle s_n \rangle$, we can apply theorem 12.2.9 to see that (V, E) is a residue tree for $\mathcal{B}$. As any action on $\mathcal{B}$ preserves the set of panels and the set of chambers, this is a tie tree for any action θ on $\mathcal{B}$.

Let θ be an action on a thick building $\mathcal{B}$ of type of a universal Coxeter system (W, S). Let (V, E) be the residue graph of $\mathcal{B}$ whose vertices are all panels and chambers, with (undirected) edges corresponding to the inclusion relation.
If θ acts on (V, E) as a translation, then there exists a unique axis γ for this action in (V, E). Let C be a chamber of $\mathcal{B}$ with minimal Weyl displacement and define $d := l(\delta(C, \theta(C)))$. Let

$$W_d := \{w \in W \mid l(w) = d\}$$

θ is hyperbolic and		θ is elliptic and	
type-preserving	not type-preserving	type-preserving	not type-pres.
		1_W (fixed chamber)	1_W
$(ts)^l t(st)^n(st)^l$	$(st)^l st(st)^n(st)^l$	$(ts)^l t(st)^l,\ t \in S'$	$(ts)^l t(st)^l$
$s(ts)^l t(st)^n(st)^l s$	$t(st)^l st(st)^n(st)^l s$	$s(ts)^l t(st)^l s,\ t \in S'$	$s(ts)^l t(st)^l t$
$(st)^l s(ts)^n(ts)^l$	$(ts)^l ts(ts)^n(ts)^l$	$(st)^l s(ts)^l,\ s \in S'$	$(st)^l s(st)^l$
$t(st)^l s(ts)^n(ts)^l t$	$s(ts)^l ts(ts)^n(ts)^l t$	$t(st)^l s(ts)^l t,\ s \in S'$	$t(st)^l s(ts)t^l$

Table 12.1: Weyl displacements in the case of $\tilde{A}_1$ (l is arbitrary ≥ 0).

and

$$W'_\theta := \{w \in W \mid \exists v, v' \in V \cap \gamma \colon v, v' \in \mathrm{Cham}(\mathcal{B}), \delta(v, v') = w, l(\delta(\theta(v), v')) < l(w)\}.$$

The minimal displacements of θ are the elements in $W_d \cap W'_\theta$. And the displacements of θ are the words of the form

$$w_1 sw\theta(w_1^{-1}) \quad \text{with} \quad l(w_1 sw) = l(w_1) + 1 + l(w), \quad w \in W_d \cap W'_\theta.$$

Indeed, if C is any chamber of $\mathcal{B}$ which is not a vertex of γ, then let D be its projection onto the vertex of γ which is closest to C. Then D is contained in some s-panel which is a vertex of γ, and its displacement is $s \cdot w$, where $w \in W_d \cap W'_\theta$. Applying now theorem 12.1.32 shows the statement.

If θ has a fixed point on (V, E), then

(i) either no vertex is fixed and θ stabilizes a s-panel for some $s \in \{s_1, \ldots, s_n\}$. In this case s is the only minimal Weyl displacement, where s is the type of the stabilized panel. The displacements of θ are the words of the form wsw^{-1} where $l(ws) = l(w) + 1$.

(ii) or θ fixes a chamber and 1_W is the only minimal Weyl displacement. If the action is not type-preserving, there is exactly one fixed chamber and the displacements of θ are the words of the form $1_W, ws\theta((ws)^{-1})$, where $l(ws) = l(w) + 1$. If the action is type-preserving, then the displacements of θ are the words of the form 1_W, wsw^{-1}, where $l(ws) = l(w) + 1$, and where s is the type of a panel which is stabilized, but not fixed by θ.

If $\mathcal{B}$ is of type $\tilde{A}_1$, i.e its Coxeter group is generated by two elements s, t without any relation, then we can write down all Weyl displacements more explicitly. In the case, where θ is hyperbolic, the words in $W_d \cap W'_\theta$ are of the form $(st)^n, (ts)^n$ if θ is type-preserving, and $(st)^n s, (ts)^n t$ if θ swaps s and t.
In the case, where θ is elliptic and not type-preserving the Weyl displacements are 1_W and all reduced words of the form $ws\theta(w)$ and $wt\theta(w)^{-1}$. If θ is elliptic and type-preserving, then the Weyl displacements are exactly the reduced words of the form $ws'\theta(w)^{-1}$ where s' ranges over the types S' of stabilized but not fixed panels and 1_W if there exists a fixed chamber. The element s' can be just s, just t or both.

Example 12.3.7 (PGL$(2,\mathbb{Z})$)**.** Let $\mathcal{B}$ be a building of type $\underset{s_1}{\circ}\overset{3}{\text{—}}\underset{s_2}{\circ}\overset{\infty}{\text{—}}\underset{s_3}{\circ}$. Here s_3 is right-angle attached generator. The Coxeter system admits a non-trivial special tree of groups decomposition of the following form: $\underset{\langle s_1,s_2\rangle}{\circ}\overset{\langle s_1\rangle}{\text{———}}\underset{\langle s_1,s_3\rangle}{\circ}$. Thus we get a residue graph whose vertices are the residues of type $J_1 := \{s_1,s_2\}$, $J_2 := \{s_1,s_3\}$ and $J_0 := \{s_1\}$. The residues of type J_1 and J_2 are the maximal spherical residues in $\mathcal{B}$. By 12.2.14, any action on $\mathcal{B}$ preserves this structure. Thus a building of type PGL$(2,\mathbb{Z})$ admits a tie tree. Its vertices are exactly the maximal spherical residues and their intersections.

Example 12.3.8 (Right-Angled Buildings)**.** Let $\mathcal{B}$ be a right-angled non-spherical building. As the building is not spherical, there exist generators s,t whose product has infinite order. Thus s is a right-angle attached generator of W and we get a tie tree for any action which preserves the type s.
Furthermore, we get a residue graph containing all $S\setminus\{s\}$-,$S\setminus\{t\}$- and $S\setminus\{s,t\}$-residues which yields a tie tree for any action preserving the set $\{s,t\}$.

Example 12.3.9 (The direct product of two $\tilde{A}_1$-Coxeter systems)**.** *This example will show that it is generally not possible to construct a residue tree whose vertices are* 2*-spherical.*

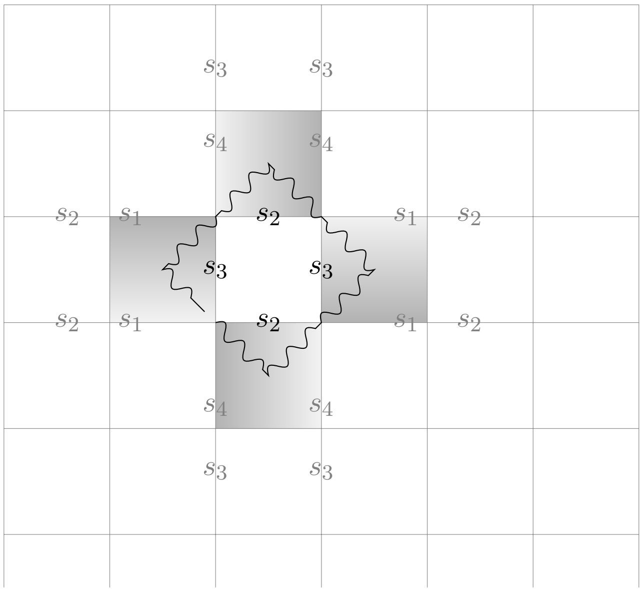
Let (W,S) be a Coxeter system of type $\underset{s_1}{\bullet}\overset{\infty}{\text{——}}\underset{s_2}{\bullet}\quad\underset{s_3}{\bullet}\overset{\infty}{\text{——}}\underset{s_4}{\bullet}$. The group W is the amalgamated product $W_{\{s_1,s_2,s_3\}} *_{W_{\{s_1,s_2\}}} W_{\{s_1,s_2,s_4\}}$ and thus we get a residue tree, by taking the residues of type $\{s_1,s_2,s_3\}$, $\{s_1,s_2,s_4\}$ and $\{s_1,s_2\}$. In figure 12.3.9 the 1-skeletons of the cubical complexes are shown inside an excerpt of the Cayley graph of W with respect to S. Of course we might also take the type sets $\{s_3,s_4,s_1\}$, $\{s_3,s_4,s_2\}$ and $\{s_3,s_4\}$.

The 2-spherical type sets are exactly the spherical type sets:

$$\{s_1,s_3\},\{s_2,s_3\},\{s_1,s_4\},\{s_2,s_4\},\{s_1\},\{s_2\},\{s_3\},\{s_4\}.$$

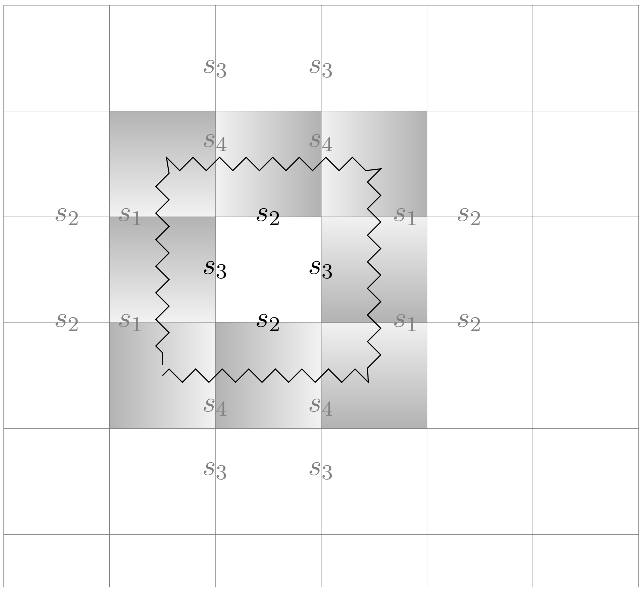
If we could get a residue tree with 2-spherical residues, we will have to take at least 2 residues of rank 2.

2 rank 2-types given: If $\{s_1, s_3\}$ and $\{s_2, s_4\}$ are two of the given type sets, then any path in W of type $s_3s_2s_3s_2$, yields a cycle in the given residue graph.



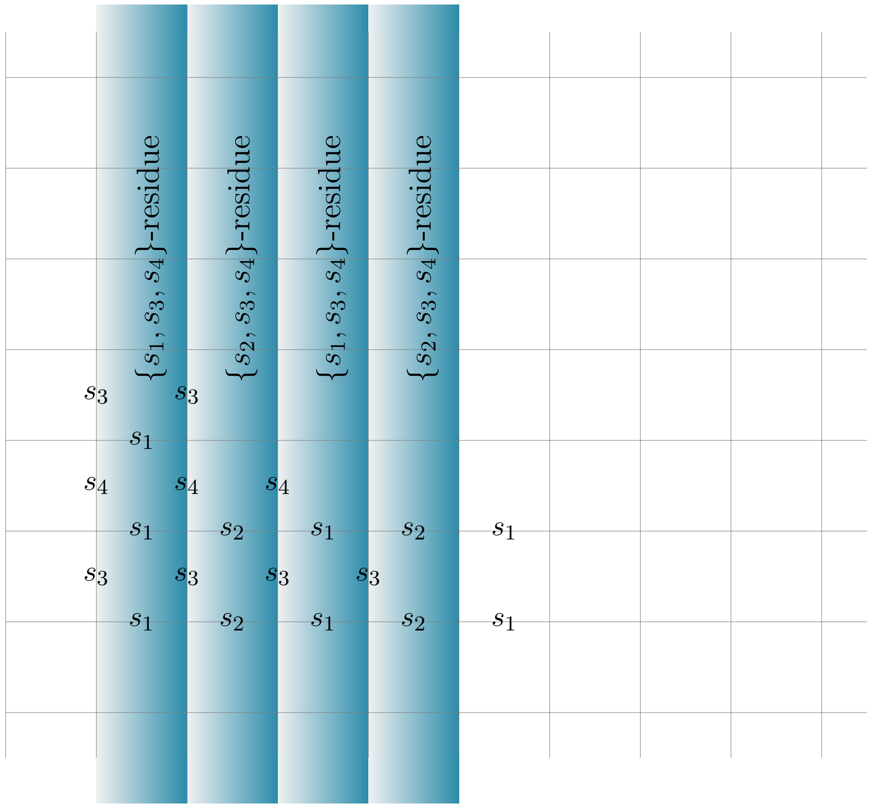
All rank 2 types given: If all rank 2-residues are vertices of the residue graph, then the set of all vertices sharing a common chamber describe a cycle.

3 rank 2 types given: If $\{s_1, s_3\}, \{s_1, s_4\}$ and $\{s_2, s_4\}$ are the type sets for the residue graph, then any word of type s_2, s_3, s_2, s_3 yields a cycle in the residue graph.



We conclude the following lemma:

Lemma 12.3.10. *There are buildings admitting a residue tree, but not admitting a residue tree whose residues are all* 2*-spherical.*



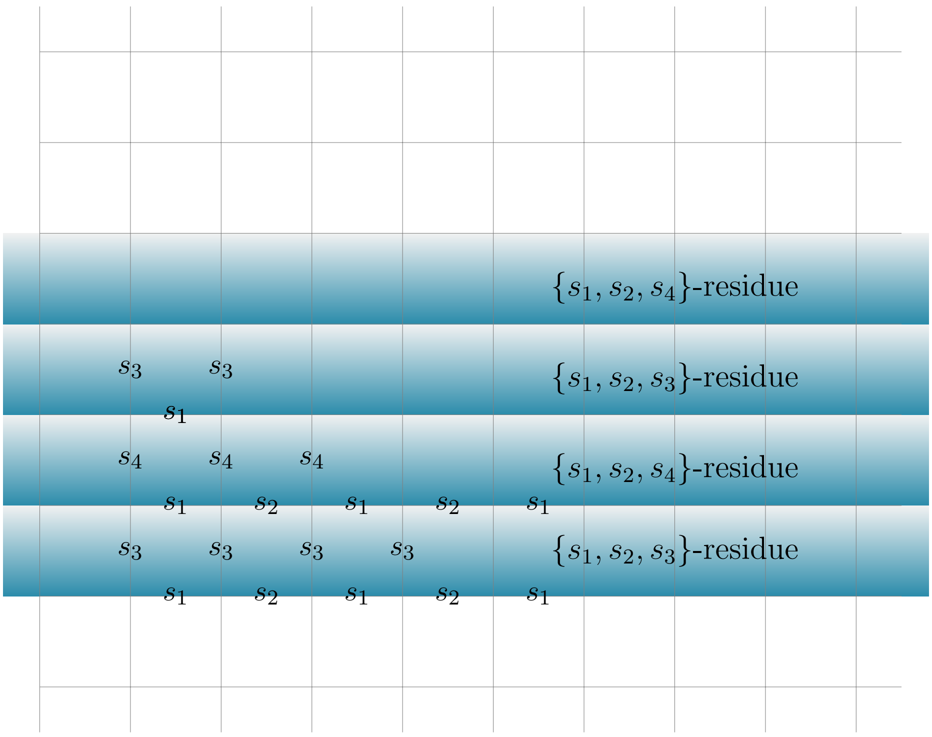


Figure 12.5: Two possibilities for a residue tree on the direct product $\tilde{A}_1 \times \tilde{A}_1$. The small cubes are the maximal spherical residues and the corners of the cubes are the chambers of the building.

CHAPTER

THIRTEEN

STABILIZED CONNECTED SUBSETS

In this chapter we will give a structure result on Weyl displacements under the following condition: Given an action θ on a building $\mathcal{B}$ there exists a proper subset $\mathcal{C}$ such that for every chamber C every minimal gallery from C to its image has to pass through $\mathcal{C}$. One may think of $\mathcal{C}$ as a set separating every chamber outside of $\mathcal{C}$ from its image. The result we obtain is a lot weaker than the previous results as we will not obtain the Weyl displacements as θ-conjugates of other displacements.

13.1 Basics

Let $\mathcal{B}$ be a building.

Definition 13.1.1. A subset Y of $\mathcal{B}$ is called **connected** if for two elements $C, D \in Y$ there exists a minimal gallery from C to D entirely contained in Y.

Definition 13.1.2. Let Y be a connected subset of $\mathcal{B}$ and let C be a chamber in $\mathcal{B}$. We define the projection of C onto Y by

$$\mathrm{proj}_Y(C) := \{D \in Y \mid \text{ for every minimal gallery } \Gamma = (C, \dots, D)\colon \Gamma \cap Y = \{D\}\}.$$

We call the elements in $\mathrm{proj}_Y(C)$ the pre-gates for C onto Y.

Remark 13.1.3. If the set Y is gated, then $\mathrm{proj}_Y(C)$ is the gate for C onto Y.

Lemma 13.1.4. *Let Y be a connected subset of $\mathcal{B}$ and let $C \in \mathcal{B}$. For every chamber $E \in Y$, there exists a minimal gallery from C to E containing an element of* $\mathrm{proj}_Y(C)$.

Proof. Let $E \in Y$ and $C \in \mathcal{B}$. Let Γ be a minimal gallery from C to E and let D_1 be the first chamber in Γ which lies inside Y. If there exists a minimal gallery from C to D_1 which contains another element D_2 of Y, than there exists a minimal gallery from C to E containing D_2 and D_1. As the distance of C to E is finite, we can iterate this process until we find a chamber $D_l \in Y$ which lies on a minimal gallery from C to E, containing D_i for $i \in \{1, \dots, l-1\}$ such that for any minimal gallery Γ' from C to D_l the only chamber inside Γ' intersecting Y is D_l. Thus D_l is a projection of C onto Y. □

Lemma 13.1.5. *In the situation of 13.1.4, there exists a minimal gallery from C to E containing an element of $\mathrm{proj}_Y(C)$ with minimal distance to E.*

Proof. The statement holds for all chambers in $\mathrm{proj}_Y(C)$. We use induction over the minimal distance of a chamber in Y to C. Assume the statement holds for all chambers in Y of distance l to C. Let $E \in Y \setminus \mathrm{proj}_Y(C)$ with $d(C, E) = l + 1$. Let E' be a chamber in Y adjacent to E with $d(C, E') = l$ and let D be an element of $\mathrm{proj}_Y(C)$ with minimal distance to E'. Then E' is the projection of C onto the panel containing E and E' and it is the projection of D onto this panel. Thus D lies on a minimal gallery from C to E'. □

Lemma 13.1.6. *Let Y be a connected subset of $\mathcal{B}$, $C \in \mathcal{B}$. The pre-gates for C in Y do not need to have the same distance to C.*

Proof. Let R be a spherical residue inside a Coxeter group and let D, D' be two opposite chambers in R. Let Γ be a minimal gallery from D to D'. And let C be a chamber in R adjacent to D but not contained in Γ. The gallery Γ is a connected set but the pre-gate D does not lie on a minimal gallery from C to D' which means that D is not a pre-gate for D' and thus the pre-gate for D' has distance greater than $d(C, D)$. □

Lemma 13.1.7. *Let θ be an automorphism of a building $\mathcal{B}$ and let Y be θ-invariant connected subset of $\mathcal{B}$. Let $C \in \mathcal{B}$ and $D \in \mathrm{proj}_Y(C)$, then $\theta(D) \in \mathrm{proj}_Y(\theta(C))$.*

Proof. Assume there exists a minimal gallery Γ from $\theta(C)$ to $\theta(D)$ which contains some element E of Y. Then $\theta^{-1}(\Gamma)$ is a minimal gallery from C to D containing $\theta^{-1}(E) \in Y$. But then $\theta^{-1}(E)$ equals D and thus $E = \theta(D)$. □

Definition 13.1.8. Let θ be an automorphism of $\mathcal{B}$ with an θ-invariant connected subset Y. For every $C \in \mathcal{B}$, we define

$\boldsymbol{W^C_{\mathrm{SM}(\theta)}} :=$ $\{w \in W \mid w = \delta(D, E), D \in \mathrm{proj}_Y(C), E \in \mathrm{proj}_Y(\theta(C))\}$,

$\boldsymbol{W_{\mathrm{SM}(\theta),C}} :=$ $\{w \in W^C_{\mathrm{SM}(\theta)} \mid$ for all $w' \in W^C_{\mathrm{SM}(\theta)} \colon l(w) \leq l(w')\}$, and

$\boldsymbol{W_{\mathrm{SM}(\theta)}} :=$ $\bigcup_{C \in \mathcal{B}} W_{\mathrm{SM}(\theta),C}$.

Let $C \in \mathcal{B}$, $D \in \mathrm{proj}_Y(C), E \in \mathrm{proj}_Y(\theta(C))$ with $\delta(D, E) \in W_{\mathrm{SM}(\theta),C}$. If $w = \delta(C, D)$, we define $\hat{w} := \delta(\theta(C), E)$.

Theorem 13.1.9. *Let θ be an automorphism of a building $\mathcal{B}$. If there exists a θ-invariant connected subset Y of $\mathcal{B}$ such that for every chamber $C \in \mathcal{B}$ a minimal gallery from C to $\theta(C)$ has to contain an element of Y, then every displacement of θ is a reduced word of the form $w_1 w_0 \hat{w}_1$, where w_0 is an element of $W_{\mathrm{SM}(\theta)}$ and w_1 is a Weyl distance of a chamber to $\mathrm{proj}_Y(C)$.*

Proof. If $C \in Y$ the statement follows directly as C is its own (pre-)gate and $\theta(C)$ is its image.
Let $C \notin Y$ and let $D \in \text{proj}_Y(C), E \in \text{proj}_Y(\theta(C))$ such that $\delta(D, E) \in W_{\text{SM}(\theta),C}$. Then by 13.1.6 D lies on a minimal gallery from C to E and E lies on a minimal gallery from $\theta(C)$ to D. Thus we can construct a minimal gallery from C to E containing D and can extend to a gallery from C to $\theta(C)$. By the condition that every minimal gallery from C to $\theta(C)$ has to pass through Y, we see that this gallery cannot be shortened and thus is a minimal gallery. □

Remark 13.1.10. The condition that for every chamber C a minimal gallery from C to $\theta(C)$ has to contain an element of Y is similar to the (MW)-condition of theorem 11.5.1.

An example for this are automorphisms of affine buildings which preserve a wall tree.

13.2 Examples

Example 13.2.1 (Stabilizing Exactly One Apartment)**.** Let θ be a hyperbolic action on a thick building $\mathcal{B}$ with the following properties:

(i) There exists an apartment Σ which is covered by translation axes of θ, i.e. $|\Sigma| \subseteq \text{Min}(\theta)$.

(ii) No wall of Σ is stabilized.

Remark 13.2.2. The condition to avoid stabilized walls is needed to avoid any kind of parallelity of residues. This will enable us to ensure that $|\Sigma|$ equals $\text{Min}(\theta)$. For example: If there is a stabilized wall it might happen that for a panel determined by this wall, there exists a chamber outside of Σ which is contained in $\text{Min}(\theta)$.

Lemma 13.2.3. *We have* $|\Sigma| = \text{Min}(\theta)$.

Proof. Let D be a chamber of $\mathcal{B}$ with $|D| \cap \text{Min}(\theta) \neq \emptyset$ and $|D| \not\subset \text{Min}(\theta)$. The support of $|D| \cap \text{Min}(\theta)$ is a spherical residue R whose intersection with Σ is a residue of the same type. Let y be an element of $|D| \setminus |\Sigma|$ and let $D' \in R \cap \Sigma$. There exists an element $x \in |D'| \setminus |D|$ such that the geodesic from x to y has to pass a chamber adjacent to D'. If $y \in \text{Min}(\theta)$ then $\gamma \subset \text{Min}(\theta)$. Thus it suffices to show:
For any chamber $E \notin \Sigma$ which lies inside a panel of R containing a chamber of Σ, we have

$$\text{for all} \;\; y \in |E|: \quad y \in \text{Min}(\theta) \Leftrightarrow y \in |\Sigma|.$$

Let E_1, E_2 be two adjacent chambers of Σ and let $E_1 \sim D \sim E_2$. As θ does not preserve any wall of Σ, the panel P containing these three chambers cannot be parallel to its image. We can assume E_2 to be the projection $\text{proj}_P(\theta(P))$. Then $\theta(E_1)$ is the projection $\text{proj}_{\theta(P)}(P)$ as action is hyperbolic and stabilizes $|\Sigma|$. If

$y \in |D \setminus |\Sigma|$ lies in $\mathrm{Min}(\theta)$ then it has minimal distance to its image and thus $\theta(y)$ has to lie in $\mathrm{proj}_{\theta(P)}(P)$. But then $\theta(D) = \theta(E_1)$ which is not possible. Thus $y \notin \mathrm{Min}(\theta)$. □

Corollary 13.2.4. *The apartment Σ is uniquely determined by θ.*

Remark 13.2.5. By 3.7.3 the set $\mathrm{Min}(\theta)$ is isometric to the product $\mathbb{R} \times Y$. We can conclude that the apartments of $\mathcal{B}$ are isometric to $\mathbb{R} \times Y$.

Lemma 13.2.6. *Let $\mathcal{B}$ be an affine building and θ as in 13.2.1. Then for every chamber $C \in \mathcal{B}$ every minimal gallery from C to $\theta(C)$ has to contain a chamber of Σ. In particular, we can apply 13.1.9 to θ and* $\mathrm{Min}(\theta) = \Sigma$.

Proof. The statement holds for any chamber in Σ. Let C be an arbitrary chamber of $\mathcal{B} \setminus \Sigma$ and let b_C be its barycenter. The projection $\mathrm{proj}_{\mathrm{Min}(\theta)}(b_C)$ lies in Σ and its support is a spherical residue R of rank > 1. Let $D := \mathrm{proj}_R(C)$. As a first observation, we see that $\theta(D)$ is the projection $\mathrm{proj}_{R'}(\theta(C))$, where $R' = \theta(R)$ is the support of $\mathrm{proj}_{\mathrm{Min}(\theta)}(b_{\theta(C)})$, where $b_{\theta(C)}$ is the barycenter of $\theta(C)$. As Σ is covered with translation axes, the walls of Σ determined by R are parallel to the walls of Σ determined by R'. Let P be a panel in R containing two chambers in Σ. Let α be a root of Σ determined by P. As θ does not preserve any wall, we know that either $\theta(\alpha) \subset \alpha$ or $\theta(-\alpha) \subset -\alpha$. Thus we may assume $\theta(\alpha) \subset \alpha$. Let $E' := P \cap \alpha$ and let E be the projection of C onto P. Then E lies on a minimal gallery from D to E' and $E \notin \Sigma$. By 9.1.7 we can find an apartment Σ_1 containing α and E. Let β be the root of Σ_1 containing α minimally, then $D \in \beta$, as at most one wall of a parallel class can separate a spherical residue. We can iterate this procedure along the walls parallel to $\bar{\alpha}$ separating C from E' to obtain an apartment Σ_l containing a root β' which contains C and β. Now we can use the same procedure starting at $\theta(P)$ and the root $\theta(-\alpha')$ of Σ_l to obtain an apartment Σ'_l containing $C, D, \theta(D)$, and $\theta(C)$.
By the convexity of apartments, every minimal gallery from C to $\theta(C)$ has to lie in Σ' and thus every minimal gallery Γ from C to $\theta(C)$ has to contain an element of Σ'_l or more specific: Γ intersects $\alpha \cap \theta(-\alpha)$ non-trivially. □

13.3 Tree Structures from Connected Subsets of Wall Trees

Let $\mathcal{B}$ be an affine building and let $T = T_m$ be the thick wall tree corresponding to a parallel class m of walls.

Definition 13.3.1. For every wall $M \in m$, we denote the set $\bigcup_{M' \sim M} [M, M']$ by v_M.

Remark 13.3.2. Let $M \neq M' \in m$. Then $v_M \cap v_{M'}$ is either empty or it is $[M, M']$. In particular, if $v_M \cap v_{M'}$ is non-empty, then it is connected.

Proof. Let $M = M_0, \dots, M_n = M'$ be the unique path in T_m from M to M'. By [Wei09][10.11] there exists an apartment Σ containing roots $\alpha_0, \dots, \alpha_n$ such that $M_i = \mu(\alpha_i)$ for all $i \in \{0, \ldots, n\}$ with α_i containing α_{i-1} minimally for all $i \in \{1, \ldots, n\}$. Therefore $[M_i, M_{i+1}] \cap [M_j, M_{j+1}] \neq \emptyset$ if and only if $i = j$. If $v_M \cap v_{M'} \neq \emptyset$, then there exist walls $\tilde{M} \sim M$ and $\tilde{M}' \sim M'$ such that $[M, \tilde{M}]$ and $[M', \tilde{M}']$ intersect non-trivially. Now we get a minimal path in T_m containing these walls. But from $M \neq M'$ we see that $M = \tilde{M}'$ and $M' = \tilde{M}$. Thus the statement holds. $\square$

Observation 13.3.3. *Two strips $[M_0, M_1]$ and $[M_1, M_2]$ intersect non-trivially if and only if they are equal.*

Corollary 13.3.4. *Let θ be an automorphism preserving T_m, then it preserves the set of connected subsets v_M. As θ induces a graph automorphism on T_m, we have set of vertices with minimal displacement in T_m. This θ-invariant set is convex and thus it determines a θ-invariant connected subset $\mathrm{SM}(\theta)$ of $\mathcal{B}$. Then we can apply 13.1.9 to θ and $\mathrm{SM}(\theta)$.*

Proof. Wee need to show that for any chamber $C \in \mathcal{B}$, every minimal gallery from C to $\theta(C)$ has to contain an element of $\mathrm{SM}(\theta)$. We have seen in the proof of 13.2.6 that we can reach every chamber C by extending a root α corresponding to a wall $M \in m$, with $\theta(\alpha \operatorname{Min}(\theta)) \subset \alpha$. The existence of such a root follows by 9.1.7 as θ preserves the tree T_m. Let v be a vertex of T_m containing C and let v' be the projection of v onto $\operatorname{Min}(\theta)(T_m)$. Then $\theta(v')$ is the projection onto $\operatorname{Min}(\theta)(T_m)$ of $\theta(v)$. By 9.1.7 there exists an apartment containing C and $\theta(C)$ and every minimal gallery from C to $\theta(C)$ has to contain a chamber corresponding to convex hull $\operatorname{conv}(v, v')$. $\square$

PART V

An Algorithmic Approach

CHAPTER

FOURTEEN

THE BRUHAT-TITS BUILDING FOR $\mathrm{GL_n(K)}$

For a better understanding of actions on affine buildings, a program modeling the Bruhat-Tits building of $\mathrm{SL}_n(K)$ *over a discrete valuation ring* K *was implemented. This chapter will give the basic definitions of the related objects together with some observations which make it possible to develop such a program.*
The mathematical background for this concept is mainly taken from chapter 6 in [AB08].

14.1 Discrete Valuations

Let K be a field.

Definition 14.1.1. A **discrete valuation** on K is a surjective homomorphism $v : K^* \twoheadrightarrow \mathbb{Z}$ from the multiplicative group K^* of K into $\mathbb{Z}$ which satisfies the following inequality:

$$v(x + y) \geq \min\{v(x), v(y)\}$$

for all $x, y \in K^*$ with $x + y \neq 0$.

Notation 14.1.2. We define $v(0) := +\infty$ in order to extend a discrete valuation to K.

Definition 14.1.3. The ring $A_v := \{x \in K \mid v(x) \geq 0\}$ is called the **valuation ring** associated to K. Any ring that arises from a discrete valuation in this way is called **(discrete) valuation ring**. For every (discrete) valuation ring A let K be the corresponding field and v the corresponding valuation.

Remark 14.1.4. The group A^* of units of a valuation ring A is precisely the kernel $v^{-1}(0)$. Let π be an element with $v(\pi) = 1$, then every $x \in K^*$ is uniquely expressible in the form $x = u \cdot \pi^k$ with $u \in A^*$ and $k := v(x) \in \mathbb{Z}$.

Definition 14.1.5. Let A be a valuation ring. An element of valuation 1 in A is called **uniformizing parameter**. A uniformizing parameter π generates the unique maximal ideal $\pi A = \{x \in K \mid v(x) > 0\}$ of A. The field $k := A/\pi A$ is called the **residue field** of K associated to the valuation v.

Definition 14.1.6. A discrete valuation v on K induces a **real valued absolute value** on K defined by

$$|x| = e^{-v(x)}$$

which has the following property:

$$|xy| = |x| \cdot |y| \quad \text{and} \quad |x+y| \leq |x| + |y|\,.$$

Remark 14.1.7. The absolute value $|\cdot| : K \to \mathbb{R}$ induces a metric d on K defined by $d(x,y) := |x-y|$. This gives the possibility to define completeness in the sense of converging Cauchy sequences. The completion of K with respect to this metric is obtained by adding the limits of all Cauchy sequences to K. It will be denoted by $\hat{K}$. All field operations and the valuation v extend onto $\hat{K}$ and $\hat{K}$ is a field with discrete valuation. Its valuation ring is $\hat{A}$, the closure of A in $\hat{K}$. The residue field of $\hat{K}$ is again k.

Example 14.1.8. *k(t)*: Let $k(t)$ be the function field over a field k.
We want to have a look at two discrete valuations v_0 and v_∞ on $k(t)$, the **order of vanishing at 0** and the **order of vanishing at ∞**.

Two discrete valuations: Let $f \in k(t)$ and let $g_f, h_f \in k[t]$ such that $f = t^n \frac{g_1}{g_2}$ and t does not divide g_1 or g_2. We define $v_0(f) := n$. For $g_1, g_2 \in k[t]$ with $f = \frac{g_1(t)}{g_2(t)}$ we define $v_\infty(f) := \deg(g_2) - \deg(g_1)$.
One can show that $v_\infty(f(t)) = v_0(f(t^{-1}))$.

The completion: The completion of $k(t)$ corresponding to v_0 can be identified with $k((t))$, the ring or formal Laurent series $\sum_{i\in\mathbb{Z}} a_i t^i$ with $a_i \in k$ and $a_n = 0$ for $n \ll 0$. Similarly, the completion of $k(t)$ corresponding to v_∞ is $k((t^{-1}))$.

14.2 The Affine Building of $\mathrm{SL_n(K)}$

Let K be a field with a discrete valuation v and let A be its valuation ring, π an uniformizing parameter, and k its residue field. We get the following diagram of matrix groups:

$$\begin{tikzcd} \mathrm{SL}_n(A) \arrow[r, hook, "\iota"] \arrow[d, "\rho"'] & \mathrm{SL}_n(K) \\ \mathrm{SL}_n(k) & \end{tikzcd}$$

Definition 14.2.1. Let B be the upper triangular subgroup of $\mathrm{SL}_n(k)$ which is called the **standard Borel subgroup** of $\mathrm{SL}_n(k)$. The standard **Iwahori subgroup** I of $\mathrm{SL}_n(K)$ is defined as

$$\mathrm{I} := \iota \circ \rho^{-1}(B).$$

Remark 14.2.2. The standard Borel subgroup of $\mathrm{SL}_n(k)$ is of the form

$$B = \begin{pmatrix} v=0 & & & & \\ & v=0 & & v \geq 0 & \\ & & \ddots & & \\ & 0 & & v=0 & \\ & & & & v=0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \middle| \, v(a_{i,i}) = 0, \ v(a_{i,j}) \geq 0 \text{ for } j > i, \ a_{i,j} = 0 \text{ for } j > i \right\}.$$

The Iwahori subgroup I is of the following form:

$$\begin{aligned} \mathrm{I} &= \iota \circ \rho^{-1}(B) \\ &= \begin{pmatrix} v=0 & & & & \\ & v=0 & & v \geq 0 & \\ & & \ddots & & \\ & v \geq 1 & & v=0 & \\ & & & & v=0 \end{pmatrix}, \end{aligned}$$

which is the set

$$\left\{ \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \middle| \, v(a_{i,i}) = 0, \ v(a_{i,j}) \geq 0 \text{ for } j > i, \ v(a_{i,j}) \geq 1 \text{ for } j > i \right\}.$$

Proposition 14.2.3 ([AB08, 6.9.2])**.** *Let* I *be the standard Iwahori subgroup of* $\mathrm{SL}_n(K)$ *and let* N *be the set of monomial matrices in* $\mathrm{SL}_n(K)$*. Then* (I, N) *is a* BN*-pair for* $\mathrm{SL}_n(K)$*.*

Reminder 14.2.4. There exists a building corresponding to the BN-pair (I, N), see 7.3.2.

Definition 14.2.5. The building $\Delta(\mathrm{I}, N)$ from the BN-pair in 14.2.3 will be called the **affine building** (or **Bruhat-Tits building**) associated to $\mathrm{SL}_n(K)$.

14.3 The Affine Weyl Group

Let $\mathcal{B}$ be the affine building associated to $\mathrm{SL}_n(K)$. This section will introduce a factorization for the Weyl group of $\Delta(\mathrm{I}, N)$. Later on we will give a rough idea of how one can see this factorization on the lattice classes and its geometric realization.

Notation 14.3.1. Let $G := \mathrm{SL}_n(K)$ and let

- I be the standard Iwohori subgroup,

- B be the standard Borel subgroup,
- N be subgroup of monomial matrices,
- $T(K)$ be the group of diagonal matrices,
- $T(A)$ be the group of diagonal matrices of $\mathrm{SL}_n(A)$,
- $W := N/T(A)$ be the Weyl group of $\Delta(I,N)$,
- $\overline{W} := N/T(K)$ be the symmetric group on n letters (the Weyl group of $\Delta(B,N)$ of type A_{n-1}), and
- $F := T(K)/T(A) \cong \mathbb{Z}^{n-1}$.

Remark 14.3.2. The matrices in $T(A)$ are of the form

$$\begin{pmatrix} v=0 & & & & \\ & v=0 & & 0 & \\ & & v=0 & & \\ & 0 & & v=0 & \\ & & & & v=0 \end{pmatrix}.$$

Notation 14.3.3. For $M \in N$ we denote its image in $W = N/T(A)$ by $[M]$. If a monomial matrix is given in the form $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$, its image will be denoted by $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$.

Notation 14.3.4 (cite[6.9.3]AB08)**.** As in [AB08, 6.9], we choose the following set of generators for W :

$$s_i = [M_{s_i}] \text{ for } i \in \{0, \ldots, n\}$$

where

$$M_{s_0} := \begin{pmatrix} 0 & & & -\pi^{-1} \\ & 1 & & \\ & & \ddots & \\ & & 1 & \\ \pi & & & 0 \end{pmatrix}, M_{s_1} := \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \ \ldots, \ M_{s_{n-1}} := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

14.4 Lattice Classes

This section introduces the concept of lattices and lattice classes in order to understand the structure of the affine building corresponding to $\mathrm{SL}_n(K)$*.*
Let K be a field with discrete valuation v, A its valuation ring and π a uniformizing parameter. Let $V := K^n$ with standard basis $e_1, \ldots, e_n$.

Definition 14.4.1. A **lattice** (or A-lattice) of V is an A-submodule $L < V$ of the form $Ab_1 \oplus \cdots \oplus Ab_n$ for some basis $b_1, \ldots, b_n$ for V. In particular, L is a free A-module of rank n. The lattice $A^n = Ae_1 \oplus \cdots \oplus Ae_n$ is called the **standard lattice** of V.

Lemma 14.4.2. *Let L, L' be two lattices in V. There exists a basis $b_1, \ldots, b_n$ for V with $L = Ab_1 \oplus \cdots \oplus Ab_n$ such that $L' = A(a_1 b_1) \oplus \cdots \oplus A(a_n b_n)$ for suitable $a_1, \ldots, a_n \in K^*$.*

Proof. We take two bases $\mathcal{B} = b_1, \ldots, b_n$ and $\mathcal{B}' = b'_1, \ldots, b'_n$ for V with $L = Ab_1 \oplus \cdots \oplus Ab_n$ and $L' = Ab'_1 \oplus \cdots \oplus Ab'_n$. Expressing every basis element in $\mathcal{B}$ as a linear combination of the elements in $\mathcal{B}'$, we obtain a matrix ${}_{\mathcal{B}}M_{\mathcal{B}'}$ in $\mathrm{GL}_n(K)$. We can transform ${}_{\mathcal{B}}M_{\mathcal{B}'}$ to a monomial matrix by multiplying elementary matrices of $\mathrm{SL}_n(A)$ from left and right. (See the proof of 14.5.12.)
The row and column operations correspond to base changes for L and L' respectively. This means that turning ${}_{\mathcal{B}}M_{\mathcal{B}'}$ into a monomial matrix corresponds to replacing the two bases for L and L' such that the new basis elements of L' are scalar multiples of the new basis elements of L. □

Definition 14.4.3. Two lattices L, L' are called **equivalent** if $L = a \cdot L'$ for some $a \in K^*$. The equivalence class $[L]$ of a lattice L will be called a **lattice class**. If a lattice L is given as $Ab_1 \oplus \cdots \oplus Ab_n$ for some basis $b_1, \ldots, b_n$ of K^n then $[L]$ will also be denoted by $[[b_1, \ldots, b_n]]$.

Remark 14.4.4. Note that the scalar a in definition 14.4.3 can always be chosen to be a power of the uniformizing parameter π.

Remark 14.4.5. The canonical action of $\mathrm{GL}_n(K)$ on the set of lattice classes of V is transitive. The stabilizer of the lattice class of A^n is $K^* \cdot \mathrm{GL}_n(A)$ and the determinants of the elements in this subgroup have valuation $0 \mod (n)$.

Definition 14.4.6. Let $\Lambda = [[f_1, \ldots, f_n]]$ be a lattice class and let $g_\Lambda \in \mathrm{GL}_n(K)$ be the matrix whose columns are the f_i. The type of a lattice class Λ is defined as $\mathrm{type}(\Lambda) := v(\det(g_\Lambda)) + n\mathbb{Z}$.

Remark 14.4.7. Note that the element g_Λ in 14.4.6 has the property

$$g_\Lambda.[A^n] = \Lambda.$$

Definition 14.4.8. Two lattice classes $\Lambda_1 \neq \Lambda_2$ are said to be **incident** if there exist $L_1 \in \Lambda_1, L_2 \in \Lambda_2$ with

$$\pi L_1 < L_2 < L_1.$$

It follows that $\pi L_2 < \pi L_1 < L_2$, thus the incidence relation is symmetric.

Remark 14.4.9. The flag-complex $\mathcal{F}$ arising from the incidence structure on the set of lattice classes is a simplicial complex where the vertices are the lattice classes and the simplices are the sets of pairwise incident classes, see 4.2. The flag complex $\mathcal{F}$ is a chamber complex and the action of $GL_n(K)$ (and thus of $\mathrm{SL}_n(K)$) on the lattice classes induces a chamber map on $\mathcal{F}$.

Proposition 14.4.10 ([AB08, Section 6.9])**.** *The flag complex arising from the incidence structure on the set of lattice classes is isomorphic to the building $\Delta(I, N)$*

of 14.2.3. The fundamental chamber C of $\Delta(\mathrm{I}, N)$ corresponds to the simplex determined by the vertices

$$[[e_1, \ldots, e_i, \pi e_{i+1}, \ldots, \pi e_n]], \quad i = 1, \ldots, n$$

where $(e_1, \ldots, e_n)$ is the standard basis in V. The stabilizer of C in $\mathrm{SL}_n(K)$ is the intersection of the stabilizer of all of its vertices and equals the standard Iwahori subgroup I.

Remark 14.4.11. The vertices of the fundamental apartment are exactly the lattice classes of the form $[[\pi^{d_1}e_1, \ldots, \pi^{d_n}e_n]]$ where $d_1, \ldots, d_n \in \mathbb{Z}$. The stabilizer of the fundamental apartment is the set of monomial matrices and every apartment of the above building is determined by a basis $b_1, \ldots, b_n$.

Remark 14.4.12. The stabilizer of $[A^n]$ is the subgroup $\mathrm{SL}_n(K) \cap K^* \,\mathrm{GL}_n(A) = \mathrm{SL}_n(A)$. The stabilizer of the lattice $g.[A^n]$ is the subgroup of the form

$$\mathrm{Stab}_{\mathrm{GL}_n(K)}(g.[A^n]) = g.\,\mathrm{Stab}_{\mathrm{GL}_n(K)}([A^n]).g^{-1} = g.\,\mathrm{SL}_n(A).g^{-1}.$$

Remark 14.4.13. The (affine) Weylgroup W corresponding to $\mathcal{B}$ splits as a semi-direct product $F \rtimes \overline{W}$ where F is a free abelian group of rank $n-1$ and $\overline{W}$ is the span of the generators $\{s_1, \ldots, s_{n-1}\}$ of W. We can identify the elements of the factor F by coordinates of the form $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ with $\sum\limits_{i=1}^{n} = 0$. We obtain an equivalence relation on F.

Notation 14.4.14. We denote the $\{s_1, \ldots, s_{n-1}\}$-residue containing the element corresponding to the coordinates $(x_1, \ldots, x_n)$ by $R([x_1, \ldots, x_n])$.

Explanation (A geometric description for $F \rtimes \overline{W}$)
Let $(D, w) \in F \rtimes \overline{W}$. The element w determines a chamber C in $R([0, \ldots, 0])$. The action of D on the vertices (as lattices classes) of C determines a unique chamber in $R(D)$ (see also figure 14.1). This can be seen geometrically as shifting C along the direction of D.
Consider the product $(1, w).(D, 1) = (D^w, w)$ in the case where $w = s_i$ for some $i \in \{1, \ldots, n\}$. For D corresponding to $(x_1, x_2, \ldots, x_n)$, the conjugation D^{s_i} interchanges the coordinates x_i and x_{i+1}. Thus for every $s \in \{s_1, \ldots, s_n\}$, the conjugation D^s reflects the barycenter of $R(D)$ along the hyperplane determined by s. It follows that (D^s, s) is the element of W which we obtain by reflecting the element 1_W by s, shifting the residue containing 1_W by D and then reflecting this residue along the hyperplane corresponding to s. This means that (D^s, s) is the reflection of the element D along the hyperplane corresponding to s.

Example 14.4.16. To understand the factor F in the Weyl group W of type $\tilde{A}_2$, look at figure 14.1: The coordinates of the form $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right]$ correspond to the lattice classes $[\pi^{a_1}b_1, \pi^{a_2}b_2, b_3]$ for some basis b_1, b_2, b_3. The filled chambers are images of a chosen base chamber $\overline{C}$ under the action of F and the thickened hexagons bound the residues of type $\{s_1, s_2\}$.

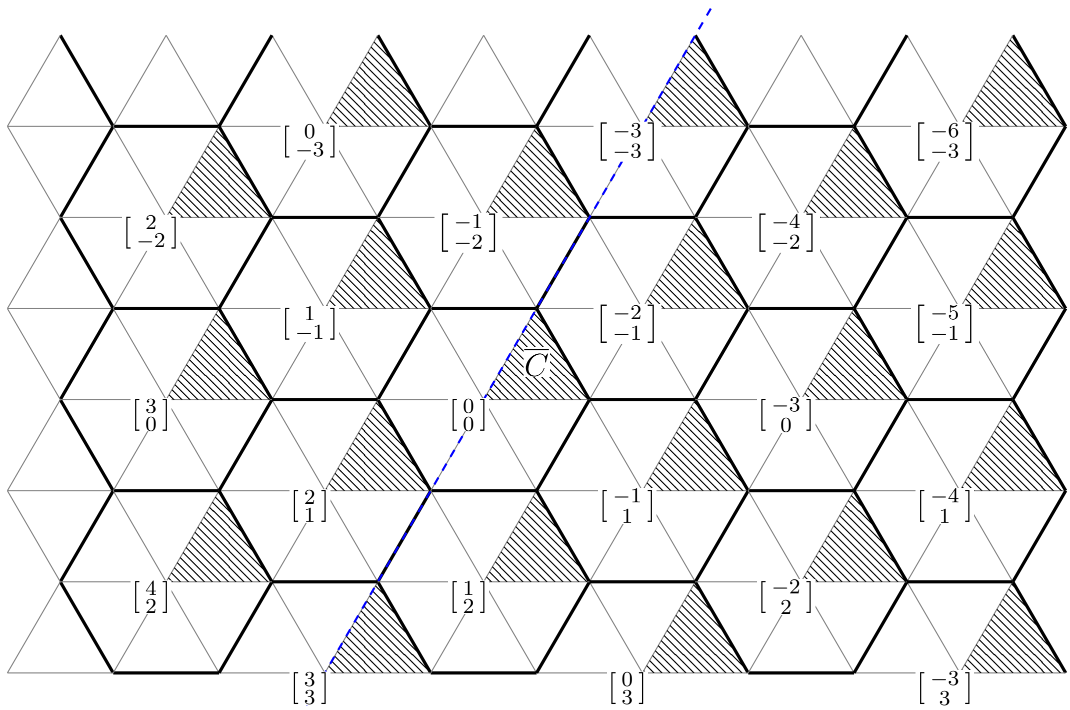


Figure 14.1: The action of F on W.

14.5 The Action of $\mathbf{GL_n(K)}$

Let $\overline{C}$ be the fundamental chamber of the building $\mathcal{B} = \Delta(\mathrm{I}, N)$ with Weyl metric δ. The canonical action of $\mathrm{GL}_n(K)$ on the lattice classes of K^n induces an action on the affine building $\mathcal{B}$ in the following way:

Definition 14.5.1. For any chamber C in $\mathcal{B}$ let $\boldsymbol{\Lambda(C)}$ be the set of lattice classes which correspond to the vertices of C. For every set Λ of lattice classes with $\Lambda = \Lambda(C)$ for some chamber C in $\mathcal{B}$, we define $[\Lambda] := C$.

Definition 14.5.2. Let $g \in \mathrm{GL}_n(K)$ and $C \in \mathrm{Cham}(\mathcal{B})$. We define an action of $\mathrm{GL}_n(K)$ on $\mathrm{Cham}(\mathcal{B})$ given by:

$$\mathrm{GL}_n(K) \times \mathrm{Cham}(\mathcal{B}) \to \mathrm{Cham}(\mathcal{B}) \quad (g, C) \mapsto g(C) = [g.\Lambda(C)].$$

Remark 14.5.3. This action is well defined as the action of $\mathrm{GL}_n(K)$ on the lattice classes induced from the action of $\mathrm{GL}_n(K)$ on K^n is well defined.

Lemma 14.5.4. *The action defined in 14.5.2 extends the action of* $\mathrm{SL}_n(K)$ *on* $\mathcal{B}$.

Proof. Let $g \in \mathrm{SL}_n(K)$, $C \in \mathrm{Cham}(\mathcal{B})$, then $g(C) = [g.\Lambda(C)] = [\Lambda(g.C)] = g.C$ $\square$

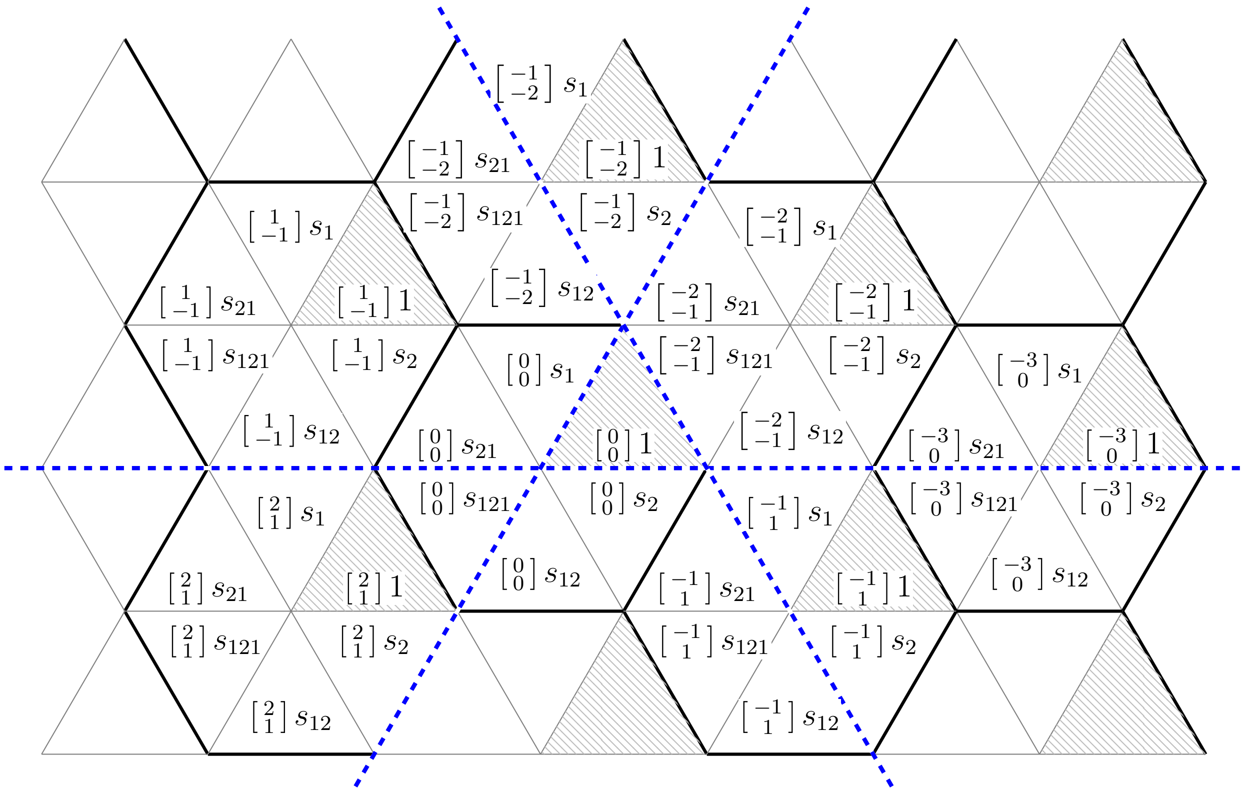


Figure 14.2: Elements of W in the form $f.w$ with $f \in F$, $w \in \overline{W}$.

Definition 14.5.5. We define

$$M_\sigma := \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\pi & & & 0 & 0 \end{pmatrix} \in \mathrm{GL}_n(K), \text{ thus } M_\sigma^{-1} = \begin{pmatrix} 0 & & & & -\pi^{-1} \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

This matrix is called the **shift matrix**.

Lemma 14.5.6. *The matrix M_σ fixes $\overline{C}$.*

Proof. The fundamental chamber $\overline{C}$ is the simplex corresponding to the vertices

$$[[e_1, \ldots, e_i, \pi e_{i+1}, \ldots, \pi e_n]], \quad i = 1, \ldots, n$$

where $(e_1, \ldots, e_n)$ is the standard basis of K^n. Let $i \in \{2, \ldots, n\}$, then $M_\sigma \cdot e_i = e_{i-1}$, $M_\sigma \cdot (\pi \cdot e_i) = \pi \cdot e_{i-1}$, and $M_\sigma \cdot e_1 = \pi e_n$. Hence

$$\begin{aligned} M_\sigma.[[e_1, \ldots, e_n]] &= [[e_1, \ldots, e_{n-1}, \pi e_n]] \\ M_\sigma.[[e_1, \ldots, e_{n-1}, \pi e_n]] &= [[e_1, \ldots, e_{n-2}, \pi e_{n-1}, \pi e_n]] \\ M_\sigma.[[e_1, \ldots, e_i, \pi e_{i+1}, \ldots, \pi e_n]] &= [[e_1, \ldots, e_{i-1}, \pi e_i, \ldots \pi e_n]] \\ M_\sigma.[[e_1, \pi e_2, \ldots, \pi e_n]] &= [[\pi e_1, \ldots, \pi e_n]] = [[e_1, \ldots, e_n]]. \end{aligned}$$

The matrix M_σ permutes the lattice classes corresponding to the fundamental chamber, mapping the lattice class of type i to the one of type $i + 1$ (mod n). $\square$

Notation 14.5.7. For $i \neq j \in \{1, \ldots, n\}$, $a \in K$, let $E_{i,j}(a)$ be the elementary matrix in $\mathrm{GL}_n(K)$ whose entry in row i and column j is a.

Lemma 14.5.8. *Let $M = (b_{ij})$ be an arbitrary matrix in $\mathrm{GL}_n(K)$. For every $l \in \{1, \ldots n\}$ let $m \in \{1, \ldots, n\}$ with $b_{l,m}$ being the entry in column m which has minimal valuation in its column such that $v(b_{i,m}) > v(b_{l,m})$ for $i < l$. Then $\mathrm{I} \cdot M$ contains a matrix (c_{ij}) where $c_{l,m}$ is the only non-zero entry in column l and $c_{l,j} = b_{l,j}$ for all $j \in \{1, \ldots, n\}$.*

Proof. For $a \in K$ the left multiplication $E_{i,j}(a) \cdot M$ adds the a - multiple of the row j of M to the row i of M. The matrix $E_{i,j}(a)$ is an element of I if $i < j$ and $v(a) \geq 0$ or if $i > j$ and $v(a) \geq 1$. Let $b_{l,m}$ be the entry in column m of M which has minimal valuation in its column such that $v(b_{i,m}) > v(b_{l,m})$ for $i < l$. Then for $i \in \{1, \ldots, n\}$ the elementary matrix $E_{i,m}(-b_{i,m} \cdot b_{l,m}^{-1})$ is an element of I. Thus multiplying M with those elementary matrices from left results in a matrix whose column m contains exactly one non-zero entry. $\square$

Lemma 14.5.9. *Let $M = (b_{ij})$ be an arbitrary matrix in $\mathrm{GL}_n(K)$. The set $\mathrm{I} \cdot M \cdot \mathrm{I}$ contains a matrix (c_{ij}) such that there exist $k, l \in \{1, \ldots, n\}$ with $c_{l,m}$ being the only non-zero element in row l and column m.*

Proof. Let $b_{l,m}$ be the entry satisfying:

(i) $b_{l,m}$ has minimal valuation among the entries in column m.

(ii) $v(b_{i,m}) > v(b_{l,m})$ for $i < l$.

(iii) $b_{l,m}$ has minimal valuation among the entries in row l.

(iv) $v(b_{l,j}) > v(b_{l,m})$ for $j > m$.

By 14.5.8 there exists an element in $\mathrm{I} \cdot M$ with $c_{l,m}$ being the only non-zero entry in its row and $c_{l,j} = b_{l,j}$ for $j \in \{1, \ldots, n\}$. Thus $c_{l,m}$ has minimal valuation among the entries in row l and $v(c_{l,j}) > v(c_{l,m})$ for $j > m$. We see that for $j \in \{1, \ldots, n\}$ the matrix $E_{l,j}(-c_{l,j} \cdot c_{l,m}^{-1})$ is an element of I. The right multiplication $M \cdot E_{l,j}(-c_{l,j} \cdot c_{l,m}^{-1})$ adds the $(-c_{l,j} \cdot c_{l,m}^{-1})$ - multiple of column l of (c_{ij}) to column j of (c_{ij}). Hence applying those right multiplications results in a matrix whose entry in row l and column m is the only non-zero element in its row and column. The statement follows. $\square$

Definition 14.5.10. We denote by $\mathbf{N}_{\boldsymbol{\pi}}$ the set of monomial matrices of $\mathrm{GL}_n(K)$ whore non-zero entries are of the form π^k for some $k \in \mathbb{Z}$.

Remark 14.5.11. The set N_π is a subgroup of N and $N_\pi T(A) = N$ as every element of K can be expressed as a product of an element of A and a power of π. Hence by the second isomorphism theorem $W \cong N/T \cong N_\pi / T \cap N_\pi$.

Lemma 14.5.12. *Every* I*-double coset of* $\mathrm{GL}_n(K)$ *has representative in* N_π*.*

Proof. Let $M = (a_{ij}) \in \mathrm{GL}_n(K)$ and let $k, l \in \{1, \dots, l\}$ such that $a_{k,l}$ is not the only non-zero element in its row and its column and such that $a_{k,l}$ satisfies the conditions in 14.5.9. Following the proofs of 14.5.9 and 14.5.8, the set $\mathrm{I}\, M\, \mathrm{I}$ contains a matrix $(c_{i,j})$ with $c_{k,l}$ being the only non-zero element in its row and column and for all k', l' with $a_{k',l'}$ being the only non-zero entry in its row and column, the same holds for $c_{k',l'}$. We can iterate this process to obtain a monomial matrix M' in $\mathrm{I}\, M\, \mathrm{I}$. For every element $f \in A$ with $v(f) = z$ the element $f \cdot \pi^{-z}$ has valuation 0 and thus is lies in A. For $i \in \{1, \dots, n\}$ let f_i be the non-zero entry of M' in row i. The matrix $D_1 := \mathrm{diag}((f_1 \cdot \pi^{-v(f_1)})^{-1}, \dots, (f_n \cdot \pi^{-v(f_n)})^{-1})$ is an element of I and $D_1 \cdot M'$ is a monomial matrix in $\mathrm{I}\, M\, \mathrm{I}$ whose non-zero entries are powers of π. $\square$

Lemma 14.5.13. *Let $g \in \mathrm{GL}_n(K)$. Then for every element $g' \in \mathrm{I} \cdot g \cdot \mathrm{I}$:*

$$\delta(\overline{C}, g(\overline{C})) = \delta(\overline{C}, g'(\overline{C})).$$

Proof. Suppose $g_1, g_2 \in \mathrm{I}$. Since $g_1 \in \mathrm{SL}_n(K)$ we have

$$\delta(\overline{C}, (g_1 \cdot g \cdot g_2)(\overline{C})) = \delta(g_1^{-1}(\overline{C}), g_1^{-1} \cdot g_1 \cdot g \cdot g_2(\overline{C})) = \delta(\overline{C}, g(\overline{C})).$$

$\square$

Corollary 14.5.14. *For every $g \in \mathrm{GL}_n(K)$ there exists $M_g \in N_\pi$ such that $\delta(\overline{C}, g(\overline{C})) = \delta(C, M_g(\overline{C}))$.*

Proof. By 14.5.12 there exists $M_g \in (\mathrm{I} \cdot g \cdot \mathrm{I}) \cap N_\sigma$. By 14.5.13 the Weyl distance $\delta(\overline{C}, g(\overline{C}))$ equals $\delta(\overline{C}, M_g(\overline{C}))$. $\square$

Lemma 14.5.15. *For all $i \in \{0, \dots, n-1\}$ the following equation holds:*

$$M_\sigma M_{s_i} = M_{s_{(i-1 \pmod n)}} M_\sigma.$$

Proof. We calculate the M_σ conjugates of the matrices $M_{s_0}, \dots, M_{s_{n-1}}$.
Let $i \in \{2, \dots, n-1\}$, then

$$\begin{aligned}
&M_\sigma \cdot M_{s_i} \cdot (M_\sigma)^{-1} \\
&= \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\pi & & & 0 & 0 \end{pmatrix}
\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}
\begin{pmatrix} 0 & & & & -\pi^{-1} \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\pi & & & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & & & & & -\pi^{-1} \\ 1 & \ddots & & & & \\ & 0 & -1 & 0 & & \\ & 1 & 0 & 0 & & \\ & & & 1 & \ddots & \\ & & & & 1 & 0 \end{pmatrix} = M_{s_{i-1}}.
\end{aligned}$$

For the conjugation of M_{s_1} we obtain:

$$
\begin{aligned}
&M_\sigma \cdot M_{s_1} \cdot (M_\sigma)^{-1} \\
&= \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\pi & & & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}
\begin{pmatrix} 0 & & & & -\pi^{-1} \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\pi & & & 0 & 0 \end{pmatrix}
\begin{pmatrix} -1 & & & & 0 \\ & 0 & & & -\pi^{-1} \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix} = M_{s_0}.
\end{aligned}
$$

And in the case of M_{s_0}:

$$
\begin{aligned}
&M_\sigma \cdot M_{s_0} \cdot (M_\sigma)^{-1} \\
&= \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\pi & & & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & & & & -\pi^{-1} \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \pi & & & & 0 \end{pmatrix}
\begin{pmatrix} 0 & & & & -\pi^{-1} \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\pi & & & 0 & 0 \end{pmatrix}
\begin{pmatrix} 0 & & & & -\pi^{-1} & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & 1 & 0 & \\ & & & & & -1 \end{pmatrix} = M_{s_{n-1}}. \quad \square
\end{aligned}
$$

Definition 14.5.16. We define

- We denote the diagonal matrix with diagonal $(a_1, \ldots, a_n)$ by $\operatorname{diag}(a_1, \ldots, a_n)$.
- $D := \operatorname{diag}(1, \ldots, 1, (-1)^{n-1})$ and $M_D := D \cdot M_\sigma$.
- For $i \in \{1, \ldots, n-1\}$ let $\epsilon_i := \begin{cases} -1 & \text{for } i \in \{n-1, 0\}, \\ 1 & \text{else .} \end{cases}$
- We define $\mathcal{D} := \{\operatorname{diag}(a_1, \ldots, a_n) \mid a_i \in \{1, -1\}\}$.

Remark 14.5.17. The matrix M_D is an element of $\mathrm{SL}_n(K)$ although $M_\sigma \in \mathrm{SL}_n(K)$ if and only if n is odd.

Lemma 14.5.18. *For $i \in \{0, \ldots, n-2\}$ let $j := i - 2$. Then*

$$(D \cdot M_\sigma) M_{s_i} = (M_{s_j})^{\epsilon_j} (D \cdot M_\sigma).$$

Proof. The result follows as the conjugation by D multiplies the n-th row and the n-th column with -1. $\square$

Definition 14.5.19. Let σ be the n-cycle $(n, n-1, \dots, 1) \in \mathrm{Sym}(n)$. We let σ act on W by acting on the indices of the generators of S, i.e.

$$\sigma(w) = \sigma(s_{i_1} \cdots s_{i_l}) = s_{\sigma(i_1)} \cdots s_{\sigma(i_l)}.$$

Corollary 14.5.20. *Given a product $M_{s_{i_1}} \dots M_{s_{i_l}}$ with $i_1, \dots, i_l \in \{0, \dots, n-1\}$ there exists an element $D' \in \mathcal{D}$ such that the following equation holds:*

$$\begin{aligned} M_D \cdot M_{s_{i_1}} \dots M_{s_{i_l}} &= M_{\sigma(s_{i_1})} \dots M_{\sigma(s_{i_l})} \cdot D' \cdot M_D \\ &= M_{s_{\sigma(i_1)}} \dots M_{s_{\sigma(i_l)}} \cdot D' \cdot M_D. \end{aligned}$$

Definition 14.5.21. Let $\hat{w} = s_{i_1} \dots s_{i_l}$ be an reduced expression for an element $w \in W$. We define the matrix $M_{\hat{w}} := M_{s_{i_1}} \cdots M_{s_{i_l}}$. For $\hat{w} = 1_W$ we define $M_{\hat{w}}$ to be the identity in $\mathrm{GL}_n(K)$.

Lemma 14.5.22. *Let $\hat{w}_1, \hat{w}_2$ be two different expressions of the same element $w \in W$. Then $M_{\hat{w}_1}(\overline{C}) = M_{\hat{w}_2}(\overline{C})$.*

Proof. The two expressions $\hat{w}_1$ and $\hat{w}_2$ represent the same coset in $N/T(A)$. Hence the two matrices $M_{\hat{w}_1}$ and $M_{\hat{w}_2}$ differ by only a factor in $T(A)$. But $T(A) \subset \mathrm{I}$ and thus the elements in $T(A)$ stabilize $\overline{C}$. The statement follows. $\square$

Lemma 14.5.23. *The following equation holds:*

$$\begin{pmatrix} 0 & & & -1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} = M_{s_1} \dots M_{s_{n-1}} \cdot D.$$

Proof.

$$\begin{aligned} M_{s_1} \cdots M_{s_{n-1}} &= \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & \\ & 0 & -1 & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -1 & 0 & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & -1 & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 0 & & (-1)^{n-1} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \end{aligned}$$

Multiplying both sides with D from the right yields the statement. $\square$

Notation 14.5.24.
- For $i, j \in \mathbb{Z}$ with $j \geq i$ we define $s_i \dots s_j := \prod_{z=i}^{j} s_{z \pmod n}$.
- For $i, j \in \mathbb{Z}$ with $j \leq i$ we define $s_i \dots s_j := (s_j \dots s_i)^{-1}$

- For $z \in \mathbb{Z}$ we define $w_z := \begin{cases} 1_W & \text{if } z = 0 \\ s_1 \dots s_z & \text{if } z > 0 \\ s_0 \dots s_z & \text{if } z < 0. \end{cases}$
- For $l \in \mathbb{N}$ and a word $w = s_{i_1} \dots s_{i_l}$ over S, let
$$M_w := \prod_{j=1}^{l} M_{s_{i_j}}.$$

Proposition 14.5.25. *For all $l \in \mathbb{Z}$ and all chambers $C \in \mathrm{Cham}(\mathcal{B})$ the following equation holds:*
$$\begin{pmatrix} \pi^l & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}(C) = \left(M_{w_{l\cdot(n-1)}} \cdot D' \cdot (M_D)^l\right)(C)$$
with $D' \in \mathcal{D}$.

Proof. Let C be an arbitrary chamber of $\mathcal{B}$.

The case $l = 0$: In the case of $l = 0$ choose $D' = id = M_{1_W}$.

Positive exponents: We calculate:
$$\begin{aligned} \begin{pmatrix} \pi^l & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} &= \begin{pmatrix} \pi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}^l = \left(\begin{pmatrix} 0 & & & -1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \cdot M_\sigma\right)^l \\ &\overset{14.5.23}{=} \left(M_{s_1} \cdots M_{s_{n-1}} \cdot D \cdot M_\sigma\right)^l \\ &= \left(M_{s_1} \cdots M_{s_{n-1}} \cdot M_D\right)^l \end{aligned}$$

Hence:
$$\begin{pmatrix} \pi^l & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}(C) = \left(M_{s_1} \cdots M_{s_{n-1}} \cdot M_D\right)^l(C).$$

We apply 14.5.20 and use that for $i \in \{1, \dots, n-1\}$ the matrix $M_{s_i}^{-1}$ equals $M_{s_i} \cdot D'$ for some $D' \in \mathcal{D}$ and for every $D_1 \in \mathcal{D}$, we get $D_1 M_{s_i} = M_{s_i} D_2$ for some $D_2 \in \mathcal{D}$.
$$\begin{aligned} \begin{pmatrix} \pi^l & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}(C) &\overset{14.5.20}{=} (M_{s_1} \cdots M_{s_{n-1}}) \cdot (M_{s_0} M_{s_1} \cdots M_{s_{n-2}}) \cdot D_1 \cdot (M_D)^2 \\ &\quad \cdot \left(M_{s_1} \cdots M_{s_{n-1}} \cdot M_D\right)^{l-2}(C) \end{aligned}$$
for some $D_1 \in \mathcal{D}$ and
$$\begin{pmatrix} \pi^l & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}(C) = M_{s_1 \cdots s_{l\cdot(n-1)}} \cdot D' \cdot (M_D)^l(C) = M_{w_{l(n-1)}} \cdot D' \cdot (M_D)^l(\overline{C})$$
for some $D' \in \mathcal{D}$.

Negative exponents: We calculate

$$\begin{aligned}
\begin{pmatrix} \pi^{-l} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} &= \begin{pmatrix} \pi^{l} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}^{-1} \\
&= \left(\left(M_{s_1} \cdots M_{s_{n-1}} \cdot D \cdot M_\sigma\right)^l\right)^{-1} \\
&= \left((M_D)^{-1} \cdot (M_{s_{n-1}})^{-1} \cdots (M_{s_1})^{-1}\right)^l.
\end{aligned}$$

Hence:

$$\begin{aligned}
\begin{pmatrix} \pi^{-l} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}(C) &= \left((M_D)^{-1} \cdot (M_{s_{n-1}})^{-1} \cdots (M_{s_1})^{-1}\right)^l (C) \\
&= \left((M_D)^{-1} \cdot D_1 \cdot M_{s_{n-1}} \cdots M_{s_1}\right)^l (C) \\
&\overset{14.5.20}{=} M_{s_n} \cdots M_{s_2} \cdot D_2 \cdot M_D^{-1} \\
&\quad \cdot \left((M_D)^{-1} \cdot D_1 \cdot M_{s_{n-1}} \cdots M_{s_1}\right)^{l-1} (C)
\end{aligned}$$

for some $D_1, D_2 \in \mathcal{D}$ and

$$\begin{aligned}
\begin{pmatrix} \pi^{-l} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}(C) &= (M_{s_n} \cdots M_{s_2} M_{s_1} M_{s_n} \cdots M_{s_3}) \cdot D_3 \cdot (M_D)^{-2} \\
&\quad \cdot \left((M_D)^{-1} \cdot D_1 \cdot M_{s_{n-1}} \cdots M_{s_1}\right)^{l-2} (C) \\
&= \mathrm{M}_{s_0 \cdots s_{-l \cdot (n-1)}} \cdot D' \cdot (M_D)^{-l}(C) \\
&= M_{w_{-l(n-1)}} \cdot D' \cdot (M_D)^{-l}(\overline{C})
\end{aligned}$$

for some $D_3, D' \in \mathcal{D}$. □

Lemma 14.5.26. *Let M be a diagonal matrix in $\mathrm{GL}_n(K)$ with diagonal $(a_1, \ldots, a_n)$ and let $i \in \{1, \ldots, n-1\}$. Then the product $(M_{s_i})^{-1} \cdot M \cdot M_{s_i}$ is the diagonal matrix obtained from M by interchanging a_i and a_{i+1}. In particular, the conjugate of $(M_{s_1 \ldots s_i})^{-1} \cdot M \cdot M_{s_1 \cdots s_i}$ is the diagonal matrix*

- $\mathrm{diag}(a_2, \ldots, a_{i+1}, a_1, a_{i+2} \ldots a_n)$ *for $i < n-2$, and*
- $\mathrm{diag}(a_2, \ldots, a_n, a_1)$ *for $i = n-1$.*

Proof. The statement follows directly from the following computation:

$$\begin{aligned}
(M_{s_i})^{-1} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \cdot M_{s_i} &= \begin{pmatrix} a_1 & & & & \\ & \ddots & & & \\ & & \begin{matrix} 0 & a_{i+1} \\ -a_i & 0 \end{matrix} & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix} \cdot M_{s_i} \\
&= \begin{pmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_{i+1} & & \\ & & & a_i & \\ & & & & \ddots \\ & & & & & a_n \end{pmatrix}.
\end{aligned}$$

□

Notation 14.5.27. For $i \in \{2, \dots, n\}$ let $S_i := s_1 \cdots s_{i-1}$ and $S_1 := 1_W$.

Theorem 14.5.28. *Let $l_1, \dots, l_n$ be elements in $\mathbb{Z}$ and let M_d be a diagonal matrix* $\operatorname{diag}(\pi^{l_1}, \dots, \pi^{l_n})$. *Let $\hat{w}$ be a word over S and let M be the product $M \cdot M_{\hat{w}}$. For $k \in \{0, \dots, n\}$ let $L_k := \sum_{i=1}^{k} l_i$ and set $L_0 := 1$. Then $M(\overline{C}) = M_{\hat{v}}(\overline{C})$ where*

$$\hat{v} = \prod_{i=1}^{n} \left((\sigma^{L_{i-1}}(w_{i-1}^{-1})) \cdot \sigma^{L_{i-1}}(w_{l_i \cdot (n-1)}) \cdot \sigma^{L_i}(w_{i-1}) \right) \cdot \sigma^{L_n}(\hat{w}).$$

Proof. For $i \in \{1, \dots, n-1\}$ let M_i be the diagonal matrix in $\mathrm{GL}_n(K)$ whose diagonal entry i is π^{l_i} and all other diagonal entries are 1. The matrix M_d equals the product $\prod_{i=1}^{n} M_i$ and by 14.5.26 this equals $\prod_{i=1}^{n} (M_{w_{i-1}})^{-1} \begin{pmatrix} \pi^{l_i} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} M_{w_{i-1}}$. By 14.5.25 the action of $\begin{pmatrix} \pi^{l_i} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ on $\mathrm{Cham}(\mathcal{B})$ equals the action of $\left(M_{w_{l_i \cdot (n-1)}} \cdot D_i \cdot (M_D)^l \right)$ for some $D_i \in \mathcal{D}$. Furthermore the $M_{s_i}^{-1} = M_{s_i} D_{s_i}$ for some $D_{s_i} \in \mathcal{D}$. Then

$$\begin{aligned} M_d &= \left(\prod_{i=1}^{n} M_{w_{i-1}^{-1}} \cdot \begin{pmatrix} \pi^{l_i} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot M_{w_{i-1}} \right) \\ &= \left(\prod_{i=1}^{n} M_{w_{i-1}^{-1}} \cdot \left(M_{w_{l_i \cdot (n-1)}} \cdot D_i \cdot (M_D)^{l_i} \right) \cdot M_{w_{i-1}} \right) \\ &= \left(\prod_{i=1}^{n} M_{w_{i-1}^{-1}} \cdot \left(M_{w_{l_i \cdot (n-1)}} \right) \cdot M_{\sigma^{l_i}(w_{i-1})} \cdot D_i' \cdot (M_D)^{l_i} \right) \end{aligned}$$

for some $D_i' \in \mathcal{D}$ for $i \in \{1, \dots, l\}$. Moving the D_i' and the $(M_D)^{L_i}$ out of the product yields

$$M_d = \left(\prod_{i=1}^{n} (M_{\sigma^{L_{i-1}}(w_{i-1}^{-1})}) \cdot \left(M_{\sigma^{L_{i-1}}(w_{l_i \cdot (n-1)})} \right) \cdot (M_{\sigma^{L_i}(w_{i-1})}) \right) \cdot \left(D' \cdot (M_D)^{L_n} \right)$$

for some $D' \in \mathcal{D}$. As $D' \cdot M_D^{L_n} \cdot M_{\hat{w}} = M_{\sigma^{L_n}(\hat{w})} \cdot D'' \cdot M_D^{L_n}$ for some $D'' \in \mathcal{D}$ and as M_D and D'' fix $\overline{C}$ (note that $D'' \in T(A)$), we conclude

$$\begin{aligned} M(\overline{C}) &= (M_d \cdot M_{\hat{w}})(\overline{C}) \\ &= \left(\prod_{i=1}^{n} M_{\sigma^{L_{i-1}}(w_{i-1}^{-1})} \cdot \left(M_{\sigma^{L_{i-1}}(w_{l_i \cdot (n-1)})} \right) \cdot M_{\sigma^{L_i}(w_{i-1})} \right) \cdot M_{\sigma^{L_n}(\hat{w})}(\overline{C}). \end{aligned}$$ □

Lemma 14.5.29. *Let $g \in \mathrm{GL}_n(K)$ and let $M_d = \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix}$ be a diagonal matrix with $M := M_d \cdot M_{\hat{w}} \in \mathrm{I} \cdot g \cdot \mathrm{I}$ for some word $\hat{w}$ over $\{s_1 \dots, s_n\}$. For $k \in \{0, \dots, n\}$ let $L_k := \sum_{i=1}^{k} l_i$ and set $L_0 := 1$. Then*

$$\delta(\overline{C}, g(\overline{C})) = \prod_{i=1}^{n} \left((\sigma^{L_{i-1}}(w_{i-1}^{-1})) \cdot \sigma^{L_{i-1}}(w_{l \cdot (n-1)}) \cdot \sigma^{L_i}(w_{i-1}) \right) \cdot \sigma^{L_n}(\hat{w}).$$

Proof. By 14.5.14 we get $\delta(\overline{C}, M(\overline{C})) = \delta(\overline{C}, g(\overline{C}))$ and by 14.5.28: $M(\overline{C}) = M_{\hat{v}}(\overline{C})$, where $\hat{v} := \prod_{i=1}^{n}(\sigma^{L_{i-1}}(w_{i-1}^{-1})) \cdot \sigma^{L_{i-1}}(w_{l\cdot(n-1)}) \cdot \sigma^{L_i}(w_{i-1}) \cdot \sigma^{L_n}(\hat{w})$. The matrix $M_{\hat{v}}$ is an element of $\mathrm{SL}_n(K)$ and thus $\delta(\overline{C}, g(\overline{C})) = \delta(\mathrm{I}, M_{\hat{v}}.\mathrm{I}) = \mathrm{I}\backslash M_{\hat{v}}/\mathrm{I} = \hat{v}$. □

Lemma 14.5.30. *Let $g \in \mathrm{GL}_n(K)$ and $k := v(\det(g))$. For the automorphism of W given by $(s_i)^g := s_{i-k \pmod n}$ we get*

$$\delta(g(\overline{C}), \overline{C}) = \delta(\overline{C}, g^{-1}(\overline{C}))^g.$$

In particular, for two chamber $C, D \in \mathcal{B}$: $\delta(g(C), D) = \delta(C, g^{-1}(D))^g$.

Proof. Let $M_g := \mathrm{diag}(\det(g \cdot M_\sigma^{-k})^{-1}, 1, \dots, 1)$. The action of M_g on the lattice classes preserves every lattice classes of the form $Ae_1 \oplus \cdots \oplus Ae_n$ as $M_g(\pi^l e_i) = \pi^l e_i$ for $i \in \{2, \dots, n\}$ and $M_g(A\pi^l e_1) = A\pi^l e_1$ for every $l \in \mathbb{Z}$. Thus M_g fixes $\overline{C}$. The product $g \cdot (M_\sigma)^{-k} \cdot M_g$ is an element of $\mathrm{SL}_n(K)$, hence

$$\delta(g(\overline{C}), \overline{C}) = \delta(\left(g \cdot (M_\sigma^{-k}) \cdot M_g\right).\overline{C}, \overline{C}) = \delta(\overline{C}, \left(M_g^{-1} \cdot (M_\sigma)^k \cdot g^{-1}\right).\overline{C}).$$

The chamber $g^{-1}(\overline{C})$ equals $M_{\hat{v}}.\overline{C}$ for some word $\hat{v}$ over S. Thus

$$\begin{aligned}
(M_g^{-1} \cdot M_\sigma^k \cdot g^{-1})(\overline{C}) &= M_g^{-1}(M_\sigma)^k(g^{-1}(\overline{C})) \\
&= M_g^{-1} \cdot (M_\sigma)^k(M_{\hat{v}}.\overline{C}) \\
&= (M_g^{-1}(M_\sigma)^k \cdot M_{\hat{v}})(\overline{C}) \\
&\overset{14.5.20}{=} (M_g^{-1} \cdot M_{\sigma^k(\hat{v})} . (M_\sigma)^k)(\overline{C})) \\
&= (M_g^{-1} \cdot M_{\sigma^k(\hat{v})})(\overline{C}) \\
&= (M_{\sigma^k(\hat{v})} \cdot M')(\overline{C})
\end{aligned}$$

where M' is a diagonal matrix whose diagonal is a permutation of the diagonal of M_g. There exists some $j \in \{1, \dots, n\}$ such that for all $l \in \mathbb{Z}$, the matrix M' stabilizes $A\pi^l e_j$ and fixes e_i for $i \neq j$. Thus M' fixes the lattices classes of the form $Ae_1 \oplus \cdots \oplus Ae_n$, hence M' fixes $\overline{C}$. We conclude

$$\begin{aligned}
\delta(g(\overline{C}), \overline{C}) &= \delta(\overline{C}, M_{\sigma^k(\hat{v})}.\overline{C}) &&= \sigma^k(\hat{v}) \\
&= \sigma^{-k}\delta(\overline{C}, M_{\hat{v}} \cdot \overline{C}) &&= \sigma^k\delta(\overline{C}, g^{-1}(\overline{C})) \\
&= \delta(\overline{C}, g^{-1}(\overline{C}))^g.
\end{aligned}$$

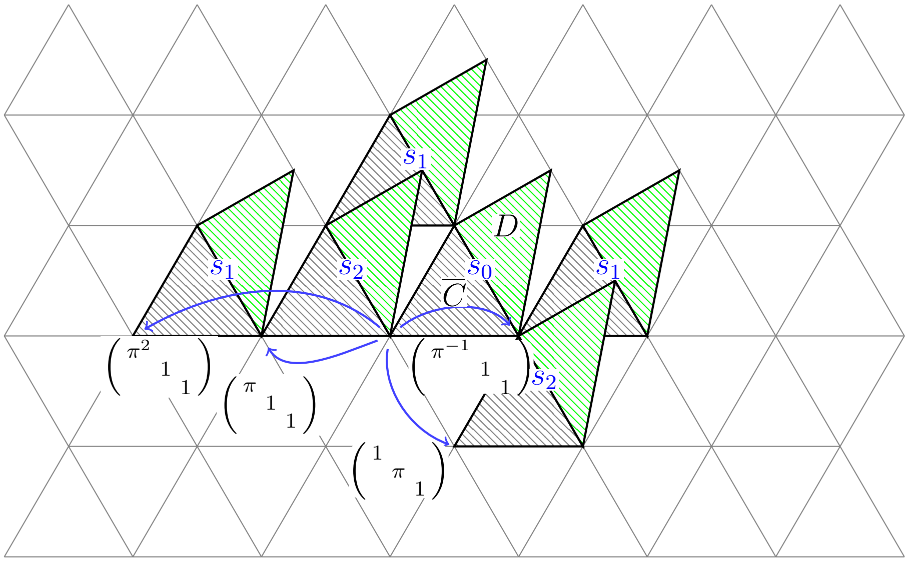
Let $C, D \in \mathrm{Cham}(\mathcal{B})$. Then $\delta(g(C), D) = \delta(g(g_1.\overline{C}), g_2.\overline{C})$ for some $g_1, g_2 \in \mathrm{SL}_n(K)$ and

$$\begin{aligned}
\delta(g(g_1.C), g_2.C) &= \delta((g_2^{-1} \cdot g \cdot g_1)(\overline{C}), \overline{C}) \\
&= \delta(\overline{C}, ((g_2^{-1} \cdot g \cdot g_1)^{-1}(\overline{C})^{(g_2^{-1} \cdot g \cdot g_1)} \\
&= \delta(\overline{C}, (g_1^{-1} \cdot g^{-1} \cdot g_2)(\overline{C}))^g \\
&= \delta(g_1.\overline{C}, g^{-1}(g_2.\overline{C}))^g = \delta(C, g^{-1}D)^g.
\end{aligned}$$

□

14.6 Examples

Example 14.6.1. Consider a thick building $\mathcal{B}$ of type $\tilde{A}_2$ with fundamental chamber $\overline{C}$.



Let $g := \begin{pmatrix} \pi^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ \pi & & 1 \end{pmatrix} \begin{pmatrix} & & -\pi^{-1} \\ & 1 & \\ \pi & & \end{pmatrix}$ We calculate $\delta(\overline{C}, g(\overline{C}))$:

$$
\begin{aligned}
\delta(\overline{C}, g(\overline{C})) \quad &\hat{=} \quad \delta\left(\overline{C}, \begin{pmatrix} \pi^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ \pi & & 1 \end{pmatrix} \begin{pmatrix} & & -\pi^{-1} \\ & 1 & \\ \pi & & \end{pmatrix} (\overline{C})\right) \\
&= \quad \delta\left(\overline{C}, \begin{pmatrix} & & -\pi^{-2} \\ & 1 & \\ \pi & 0 & -1 \end{pmatrix} (\overline{C})\right) \\
&\overset{\text{monomial in}}{\underset{\mathrm{I}\,g\,\mathrm{I}}{=}} \quad \delta\left(\overline{C}, \begin{pmatrix} & & -\pi^{-2} \\ & 1 & \\ \pi & & \end{pmatrix} (\overline{C})\right) \\
&= \quad \delta\left(\overline{C}, \begin{pmatrix} -\pi^{-2} & & \\ & 1 & \\ & & \pi \end{pmatrix} \cdot M_{s_1 s_2 s_1}(\overline{C})\right) \\
&= \quad \delta\left(\overline{C}, \begin{pmatrix} -\pi^{-3} & & \\ & \pi^{-1} & \\ & & 1 \end{pmatrix} \cdot M_{s_1 s_2 s_1}(\overline{C})\right) \\
&= \quad w_{-3\cdot 2} \cdot \sigma^{-3}(s_1) \cdot \sigma^{-3}(w_{-1\cdot 2}) \cdot \sigma^{-4}(s_1) \cdot \sigma^{-4}(s_! s_2 s_1) \\
&= \quad s_0 s_2 s_1 s_0 s_2 s_1 \cdot s_1 \cdot s_0 s_2 \cdot s_2 \cdot s_2 s_0 s_2 \\
&= \quad s_0 s_2 s_1 \cdot s_0 s_2 s_2 s_0 \\
&= \quad s_0 s_2 s_1.
\end{aligned}
$$

As a last step we calculate $\delta(gC, gD)$. The valuation of the determinant of g is -1 and we get:

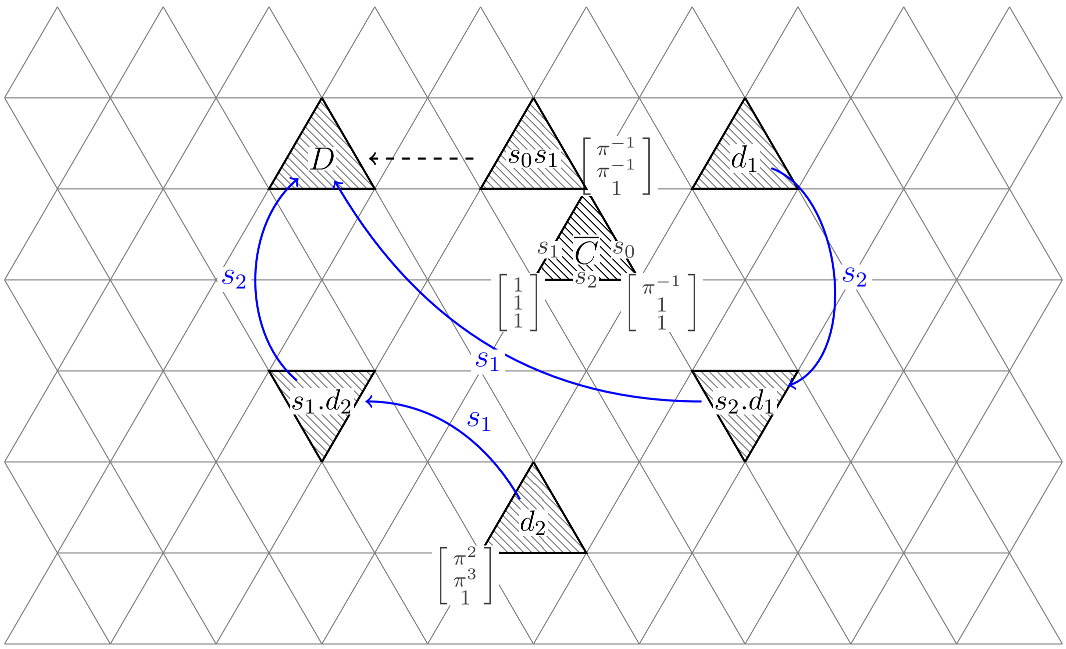
$$\begin{aligned} \delta(gC, gD) &= \delta(C, g^{-1}gD)^g = \delta(C, D)^g \\ &= (s_0)^g = s_{0+1 \pmod n} = s_1. \end{aligned}$$

Example 14.6.2. In this example we will show three ways to move to a specific chamber D in the fundamental apartment starting from $\overline{C}$. Let $D := M_{s_1 s_2 s_0 s_1 s_2 s_1}(\overline{C})$. For a reminder about the action of F (shifting a chamber) take a look at 14.4.16.

(i) First we act with $s_0 s_1$ on C and then shift this chamber by the matrix $\begin{pmatrix} \pi^2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

(ii) First we shift C by the matrix $\begin{pmatrix} \pi^{-2} & & \\ & \pi^{-1} & \\ & & 1 \end{pmatrix}$ (see d_1 in the figure), and then we act by s_2 following by the action of s_1.

(iii) First we shift C by the matrix $\begin{pmatrix} \pi^2 & & \\ & \pi^3 & \\ & & 1 \end{pmatrix}$ (see d_2 in the figure) and then we act by s_1 following by an action of s_2.



14.7 The Blueprint Construction

This section is taken from [Ron09, Chapter 7]. The blueprint construction allows us to realize every chamber of the building $\Delta(I, N)$ as a product of root group elements and reflections (acting on the fundamental chamber).

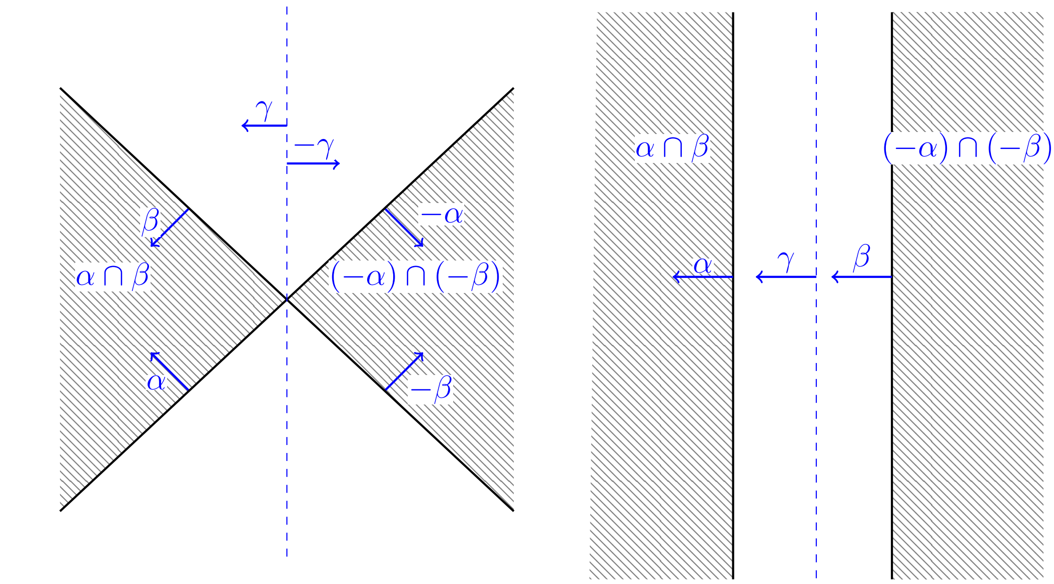


Figure 14.3: Visualizations of a prenilpotent roots α, β with a root $\gamma \in (\alpha, \beta)$

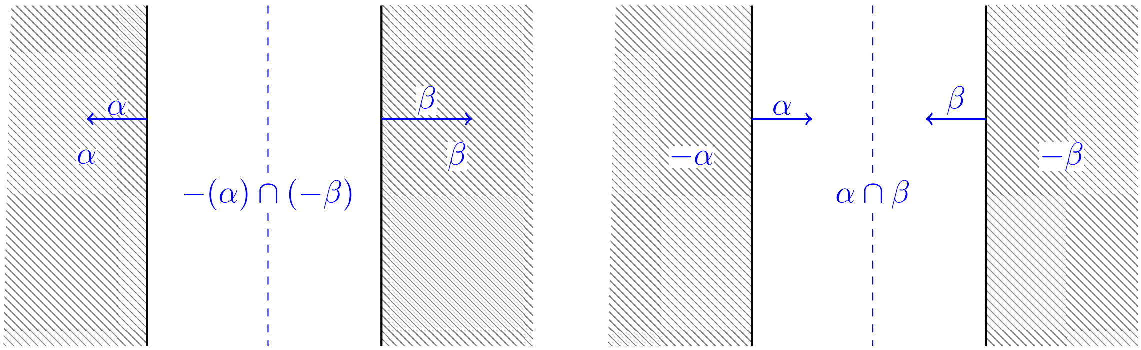


Figure 14.4: Visualizations of a two roots α, β which are not prenilpotent

Let $\mathcal{B}$ be a building, $\mathcal{I}$ its type-set (its set of adjacency relations), W its Weyl group, Σ its fundamental apartment, C its fundamental chamber and Φ the set of roots in Σ.

Definition 14.7.1. Two roots $\alpha, \beta \in \Phi$ are called **prenilpotent** if $\alpha \cap \beta \neq \emptyset$ and $(-\alpha) \cap (-\beta) \neq \emptyset$. For a prenilpotent pair α, β we define

$$[\alpha, \beta] := \{\gamma \in \Phi \mid \alpha \cap \beta \subseteq \gamma \text{ and } (-\alpha) \cap (-\beta) \subseteq -\gamma\},$$

and

$$(\alpha, \beta) = [\alpha, \beta] \setminus \{\alpha, \beta\}.$$

Remark 14.7.2. Two roots are prenilpotent if their walls intersect or one of them is contained in the other one.

Definition 14.7.3. A building $\mathcal{B}$ is called **Moufang** if there exists a set of groups $(U_\alpha)_{\alpha\in\Phi}$ satisfying:

M1 If P is a panel of $\partial\alpha$, and $D \in \mathrm{Cham}(P \cap \alpha)$, then U_α fixes all chambers of α and acts simple-transitive on $\mathrm{Cham}(P) \setminus \{D\}$.

M2 If $\{\alpha, \beta\}$ is a prenilpotent pair of distinct roots, then $[U_\alpha, U_\beta] \leq U_{(\alpha,\beta)} := \langle U_\gamma \mid \gamma \in (\alpha, \beta)\rangle$.

M3 For each $u \in U_\alpha \setminus \{1\}$ there exists $m(u) \in U_{-\alpha}uU_\alpha$ stabilizing Σ, i.e. interchanging α and $-\alpha$.

M4 If $n = m(u)$ then for any root β, $nU_\beta n^{-1} = U_{n\beta}$.

The groups U_α are called **root groups**.

Proposition 14.7.4 ([Ron09, Proposition 6.14])**.** *A root group U_α fixes every chamber having a panel in $\alpha \setminus \partial\alpha$.*

For the following definition we identify W with the automorphism group of Σ.

Definition 14.7.5. Let $w = s_{i_1} \dots s_{i_l} \in W$ be a reduced expression. If $\beta \in \Phi$ denotes the unique root of Σ containing $w_{j-1}(C)$ but not $w_j(C)$, then the β_j are precisely the roots containing C but not $w(C)$ and we define:

$$U_w := U_{\beta_1} \dots U_{\beta_l}.$$

Theorem 14.7.6 ([Ron09, Theorem 6.15])**.** *If $\mathcal{B}$ is a Moufang building, then U_w acts simple-transitively on the set of chambers D such that $\delta(C, D) = w$. In particular if (B, N) is a BN pair of $\mathcal{B}$ with B stabilizing C and N stabilizing Σ, then every such chamber can be written uniquely as a coset uwB where $u \in U_w$.*

Let P_i be the panel of type i containing C and define $\alpha_i \in \Phi$ to be the root containing C with $P_i \in \partial\alpha_i$. Let s_i denote the reflection interchanging α_i and $-\alpha_i$ and write $U_i := U_{\alpha_i}$. For each $i \in I$ select some element $e_i \in U_i \setminus \{1\}$.

Lemma 14.7.7 ([Ron09, 7.3])**.** *For $n_i := m(e_i)$ we get:*

$$n_i n_j \dots = n_j n_i \dots$$

for m_{ij} alternating terms of n_i and m_i on both sides. Further, for any $w \in W$ there exists a unique $n(w)$ stabilizing Σ, with

$$n(w) = n_{i_1} \dots n_{i_l} \quad \textit{for } w = s_{i_1} \dots s_{i_l}$$

Lemma 14.7.8 ([Ron09, Section 7.2])**.** *Each chamber $D \in \mathcal{B}$ can be written as an equivalence class $un(w)$ of elements of the form $u_1 n_{i_1} \dots u_k n_{u_k}$*

Let R be an arbitrary i-residue of $\mathcal{B}$ and let $\mathrm{proj}_R(C) = D$, $\delta(C, D) = w$. As cosets of B (or the standard Iwahori subgroup I), chambers may be written as $un(w)B$ (for D) and $un(w)vn_i$ with $u \in U_w$ and $v \in U_i$.

Remark 14.7.9. If $\mathcal{B}$ is a Moufang building with fundamental chamber C, fundamental apartment Σ, and a system $(U_\alpha)_{\alpha\in\Phi}$ of root groups. We define:

- $G := \langle U_\alpha \mid \alpha \in \Phi\rangle$,
- $N := \langle m(u) \mid u \in U_\alpha, \alpha \in \Phi\rangle$,
- $H := \{h \in N \mid h.D = D \text{ for all } D \in \mathrm{Cham}(\Sigma)\}$,
- $\Phi_+ := \{\alpha \in \Phi \mid C \in \mathrm{Cham}(\alpha)\}$,
- $B := \langle H, U_\alpha \mid \alpha \in \Phi^+\rangle$.

Then (B, N) is a BN pair for G and $B \cap N = H$.

Example 14.7.10. We consider K to be $k(\pi)$ for some transcendent element π over k. We give an example for a system of root groups for the affine building $\Delta(I, N)$ corresponding to $\mathrm{SL}_n(K)$.

We define:

$$u_1(a) := \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad u_{-1}(a) := \begin{pmatrix} 1 & & & \\ a & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$u_2(b) := \begin{pmatrix} 1 & & & & \\ & 1 & b & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad u_{-2}(b) := \begin{pmatrix} 1 & & & & \\ b & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$\vdots$$

$$u_n(c) := \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ c\pi & & & 1 \end{pmatrix} \qquad u_{-n}(c) := \begin{pmatrix} 1 & & & c\pi^{-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

where a, b, c range over the residue field k of K. And for $i \in \{1, \dots, n\}$ we define $U_i := \{u_i(a) \mid a \in k\}$. These are the fundamental roots of the building.
Every root group U_α of the fundamental apartment can be described as a translation of a root group corresponding to a wall separating the $\{s_1, \dots, s_{n-1}\}$-residue R containing C which we obtain by taking commutators of the fundamental root groups $U_1, \dots, U_{n-1}$. The tanslations of the fundamental root groups look as follows:

$$\begin{pmatrix} 1 & a\pi^z & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \pi^z & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot U_1 \cdot \begin{pmatrix} 1 & & & \\ & \pi^{-z} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ a\pi^z & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \pi^z & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot U_{-1} \cdot \begin{pmatrix} 1 & & & \\ & \pi^{-z} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & 1 & b\pi^z & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \pi^z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot U_2 \cdot \begin{pmatrix} \pi^{-z} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \pi^{-z} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot U_2 \cdot \begin{pmatrix} 1 & & & \\ & \pi^z & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ c\pi^{z+1} & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \pi^z \end{pmatrix} \cdot U_n \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \pi^{-z} \end{pmatrix}$$

$$= \begin{pmatrix} \pi^{-z} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot U_n \cdot \begin{pmatrix} \pi^z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & d\pi^{z-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \pi^z \end{pmatrix} \cdot U_{-n} \cdot \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \pi^{-z} \end{pmatrix}$$

$$= \begin{pmatrix} \pi^{-z} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot U_{-n} \cdot \begin{pmatrix} \pi^z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

where a, b, c, d range over k. The wall corresponding to the root group of U_n does not separate chambers the maximal spherical residue R. But the wall corresponding to the reflection interchanging C and its opposite chamber in P is a translate of U_{-n}:

$$\begin{pmatrix} 1 & & & k \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \pi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot U_{-n} \cdot \begin{pmatrix} \pi^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \pi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & a\pi^{-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \pi^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

where a ranges over all elements in k.

Remark 14.7.11. The last example is very close to the concept of **affine extensions** for spherical Weyl groups. For further information about this, one might have a look at [Bou68][VI, §4 , section 3].

14.8 An Example

Let $K := k(\pi)$ be an extension of k with π transcendent over k and let $\mathcal{B}$ be the affine building corresponding to $\mathrm{SL}_n(K)$.

The matrix $M_{s_1} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ represents a fundamental reflection, i.e. a reflection in the fundamental apartment along a wall $\partial\alpha_{s_1}$ determined by the fundamental chamber C. This means that the action θ of M_{s_1} on $\mathcal{B}$ interchanges $C = \mathrm{I}$ and $s_1C = M_{s_1}\,\mathrm{I}$ and it satisfies:

$$|\partial\alpha_{s_1}| \subseteq \mathrm{Min}(\theta).$$

Let P be the panel determined by $\partial\alpha_{s_1}$ and C. If there exists an element $y \in \mathrm{Min}(\theta)$ with $y \notin |\partial\alpha_{s_1}|$, then there exists an element $z \in |P| \setminus |\partial\alpha_{s_1}|$ which is fixed by θ. This is due to the fact that $\mathrm{Min}(\theta)$ is convex (see 3.7.2) and the convex hull of $|\partial\alpha_{s_1}|$ and y intersects the interior of one (geometric) chamber $|D|$ inside $|P|$ non-trivially. The action θ is type-preserving, as the determinant of M_{s_1} has valuation 0, see 14.5.30. Hence $|D|$ is fixed by θ. We conclude that θ fixes exactly $|\partial\alpha_{s_1}|$ if and only if θ does not fix any chamber of P. We will show that the existence of a fixed chamber in P is equivalent to the existence of an element $a \in k$ with $a^2 = -1$.

Let's calculate the displacements for the chambers in P. We can represent each of those chambers by an element of U_1 times M_{s_1} or by the identity (for C). The elements of U_1 are given by their parameter a (see 14.7.10). We calculate the Weyl element $\delta(D, \theta(D))$ for all chambers D in P:

(i) For the two chambers in P which lie the fundamental apartment and which are represented by the identity and M_{s_1}, we get the displacement s_1.

(ii) If $D = u_1(a) \cdot M_{s_1}$ we get

$$\begin{aligned}\delta(D, M_{s_1}.D) &= \delta(u_1(a)M_{s_1}I, M_{s_1}u_1(a)M_{s_1}I)\\ &= \delta(I, (u(a)M_{s_1})^{-1}M_{s_1}u_1(a)M_{s_1}I)\\ &= \delta(I, M_{s_1}^{-1}u(a)^{-1}M_{s_1}u_1(a)M_{s_1}I)\\ &= \delta(IM_{s_1}^{-1}u(a)^{-1}M_{s_1}u_1(a)M_{s_1}I).\end{aligned}$$

For the double coset I $M_{s_1}^{-1}u(a)^{-1}M_{s_1}u(a)M_{s_1}$ I we find a representative by:

$$M_{s_1}^{-1}u(a)^{-1}M_{s_1}u(a)M_{s_1}$$

$$= \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -a & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & & & \\ -1 & a & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} & -1 & & & \\ 1 & a & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a & & & \\ a & 1+a^2 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} a & -1 & & & \\ 1+a^2 & -a & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

If $a^2 = -1$, this matrix lies in I and thus the chamber D has Weyl displacement 1_W, i.e. $\delta(D, M_{s_1}.D) = 1_W$. Otherwise a monomial representation for this double coset has the form of M_{s_1} and thus $\delta(D, M_{s_1}.D) = s_1$.

This shows that the action of M_{s_1} fixes exactly the (geometric) wall $|\partial\alpha_{s_1}|$ if and only if the residue field k contains an element a with $a^2 = -1$.

The same result holds for the other fundamental reflections by analog computations.

CHAPTER

FIFTEEN

THE IMPLEMENTATION

This section gives an algorithmic version of the two main important steps for calculating a Weyl element corresponding to an I*-double coset. Further some explanations on the internal structures of the program, the usage and its performance are given.*

The program for the calculations is written for Sage (`http://www.sagemath.org`). The version used for the implementation is 6.3.

15.1 Algorithms

To make the first algorithm more readable, one step will be defined here as an external function:
Let **Find_Entry_With_Lowest_Valuation(**M**,**R**,**C**)** be a function which returns for a given matrix M and two subsets R, C of $\{1, \ldots, n\}$ (n being the rank of M) a pair (i, j) with the following properties:

(i) The entry a_{ij} has minimal valuation in the ith row and jth column.

(ii) If an entry a_{il} has the same valuation as a_{ij}, then $l > j$.

(iii) If an entry a_{mj} has the same valuation as a_{ij}, then $m < i$.

Algorithm 1: Transforming a matrix of $\mathrm{GL}_n(K)$ to a monomial matrix - An algorithmic version of Lemma 14.5.12

input : A matrix M of $\mathrm{GL}_n(K)$
output: A monomial matrix M' with $M' \in I \cdot M \cdot I$ whose entries are powers of π.

M' $\leftarrow$ `Change_Entries(`M`)` Remaining_Rows $\leftarrow \{1, \ldots, n\}$ Remaining_Columns $\leftarrow \{1, \ldots, n\}$
foreach counter *in* $\{0, \ldots, n\}$ **do**
  $(i, j) \leftarrow$ `Get_Entry_With_Lowest_Valuation` (M',Remaining_Rows,Remaining_Columns) `// The` ith `row and the` j`th column will have exactly one element` $\neq$ `0 after this.`
  `// Adjust the row and column index sets`
  Remaining_Rows $\leftarrow$ Remaining_Rows.Remove$(-i)$
  Remaining_Columns $\leftarrow$ Remaining_Columns.Remove$(-j)$
  `// Turn the other entries in the corresponding row and column to zero.`
  if $a_{ij} = c\pi^l$ *is not the only non-zero element in its row* **then**
    foreach k *in* Remaining_Rows **do**
      if $a_{kj} = c'\pi^{l'} \neq 0$ **then**
        Subtract the $(c)^{-1} \cdot c' \cdot \pi^{l'-l}$ multiple of the ith row from the kth row.
  else
    foreach l *in* Remaining_Columns **do**
      $a_{il} \leftarrow 0$
      `// Apart from` a_{ij} `the entries in column` j `are all zero. Thus subtracting a multiple of column` j `from any other column has only an effect on the` i`th entry.`

Algorithm 2: Calculating the Weyl element
- An algorithmic version of Theorem 14.5.29

input : The result M' of Algorithm 1 for a matrix of $\mathrm{GL}_n(K)$.
output: A word w over the alphabet S with $M \in I \cdot M_w \cdot I$.

$M_w \leftarrow$ permutation matrix, s.th. $M' \cdot M_w$ is diagonal
$w' \leftarrow$ a word over $S \setminus \{s_0\}$ describing M_w $M' \leftarrow M' \cdot M_w$
`// Store the diagonal entries of the given matrix`
foreach i *in* $\{0,\ldots,\mathrm{rank}(M)\}$ **do**
    `// A matrix is internally stored as a two-dimensional array`
    $a_i \leftarrow v(M[i][i])$
    $S_i \leftarrow \sum_{l=0}^{i} a_i$
`// Reminder:` σ `is the permutation on` W `induced by the cycle` $(s_1,\ldots,s_{n_1},s_0)$`.`
$w \leftarrow 1_W$
`// Iterate through the diagonal entries of the given matrix`
foreach i *in* $\{0,\ldots,\mathrm{rank}(M)\}$ **do**
    if $a_i \neq 0$ **then**
        $w = w \cdot \sigma^{S_i}(s_{i-1}\cdots s_1)$ **if** $(a_1 > 0)$ **then**
            $w \leftarrow w \cdot \sigma^{S_i}\big((s_1\cdots s_{n-1}s_0)^{(a_i(n-1)/n)}s_1\cdots s_{a_i\%n}\big)$
            `// in the case of` $(a_i\%n)=0$ `the last term vanishes`
        else
            $w \leftarrow w \cdot \sigma^{S_i}\big((s_0s_{n-1}\cdots s_1)^{(-a_i(n-1)/n)}s_0s_{n-1}s_{n+1-(-a_i\%n)}\big)$
            `// in the case of` $(-a_i\%n)=1$`, the last term is just` s_0 `if` $(-a_i\%n)=0$ `it vanishes.`
        $w \leftarrow w \cdot \sigma^{S_{i+1}}(s_{i-1}\cdots s_1)$
return $w \cdot \sigma_n^S(w')$

15.2 The Program

The program uses matrices defined over the rational function field $K = k(t)$ of a finite field k.

Definition 15.2.1. For $i \in \{1, \dots, n\}$ we define:

$$M_{s_1} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \dots, M_{s_n} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix},$$

$$M_{s_0} = \begin{pmatrix} 0 & & & & -\pi^{-1} \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \pi & & & & 0 \end{pmatrix}$$

(see also 14.3.4). For $i \in \{1, \dots, n\}, \alpha \in k$ we define

$$u_{s_1}(\alpha) = \begin{pmatrix} 1 & \alpha & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \dots, u_{s_n}(\alpha) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \alpha \\ & & & & 1 \end{pmatrix}$$

$$u_{s_0}(\alpha) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \alpha \cdot t & & & & 1 \end{pmatrix}.$$

Definition 15.2.2. In the program, a chamber D of $\Delta(\mathrm{I}, N)$ corresponds to the equivalence class of matrices M with $M(\overline{C}) = D$ where $\overline{C}$ denotes the fundamental chamber.

Explanation (The chamber representations)
The chambers of $\Delta(\mathrm{I}, N)$ can be represented by a sequence of products $x_i(\alpha) := R_{s_i}(\alpha) \cdot M_{s_i}$. This allows us to use an easy description for chambers in terms of lists $[i_1, a_1, \dots, i_l, a_l]$ representing the product $x_{i_1}(a_1) \cdots x_{i_l}(a_l)$. Internally chambers are stored as matrices over K which allows the user to use actions on the building in terms of matrix operations.
Given an arbitrary matrix M of $\mathrm{GL}_n(K)$, we can compute a chamber description, i.e. a list of the form $[i_1, a_1, \dots, i_l, a_l]$ for the chamber D represented by M. For this, we first calculate the Weyl distance of the fundamental chamber (the identity matrix) to M. This will give us a sequence of types of panels and we can move along those panels searching for the unique chamber on that panel with minimal distance to D - the projection of D onto the panel.

Explanation (Calculating the action on S)
If an automorphism θ on the building is not given as the left multiplication by a matrix, we cannot obtain the action of θ on S directly as in 14.5.30. But we can look at the chamber descriptions for the image of the identity matrix and for the images of the generators. From these descriptions we obtain matrices representing those images and we can compute their Weyl distances which gives the induced action on S.

Explanation (The choice of the rational function field)
In terms of buildings, one might prefer to work with the ring over (formal) Laurent series (see [GH10, GHKR10, He14, Bea12]) as it is complete (see 14.1.8). But internally (in the used version of Sage) these series are truncated. Thus the inverse of $1+t$ will be stored in the form $1+t+t^2+t^3+O(4)$, depending on the chosen precision depth. This will result in wrong calculations as $(1+t)\cdot(1+t)^{-1}$ will never return 1 in this case.

Explanation (Performance)
The goal of this implementation is to get Weyl distances for given chambers as quickly as possible. The main calculations were optimized in the following ways:

(i) *Replace matrix multiplications by list comprehension if possible. This applies for the multiplication with the generators and the root group elements.*

(ii) *Once a chamber is given as a matrix M, use the matrix inversion of Sage for the computation of Weyl distances. Here the given inversion is much faster than constructing a new object corresponding to the inverse M^{-1}.*

(iii) *Using 14.5.12 we transform the given matrix into a monomial one without any matrix multiplications.*

(iv) *Using algorithm 2 we get an expression for the desired Weyl element by just extending precalculated lists. In order to obtain a reduced word for this expression we only need to transform the resulting list at the very end into an element of the affine permutation group corresponding to W. Sage already provides a function returning a reduced expression for such a list.*

One of the purposes of this program was to be able to calculate the Weyl displacements of all chambers within a certain radius around a given chamber. In the affine building $\Delta(\mathrm{I}, N)$ of rank 4 over a field with 25 elements, a ball of radius 4 contains already more than 13.5 million chambers. Therefore going through every single chamber is far too time consuming for this approach to be a useful tool. Using the presented implementation, the test-computer, an Intel i7-4770K (3.50 GHz) 32GB RAM running Sage on an Oracle VM VirualBox (v 4.3.10) inside Windows 7 64bit) can calculate per kernel the Weyl distances for about 160.000 pairs of chambers within one hour. Although parallel computations on n kernels reduce this time nearly by the factor n, as the computations on each kernel are independent from each other, this will still be far from being fast.

Explanation (Displacment Balls)
We want to calculate the Weyl displacements of all chambers inside some ball around a given chamber C within an acceptable time range. To do so, one can reduce the amount of chambers needed to obtain the desired results. Let $s \in S$ and let D be a chamber, for which we already know its Weyl displacement, say w. When we look at 11.4.9 and 11.4.11 we see that we already have some information about the possible Weyl displacements for the s-panel P containing D. In particular:

(i) *If $l(sw\theta(s)) = l(w)+2$, then every chamber in $P \setminus \{D\}$ has Weyl displacement $sw\theta(s)$.*

(ii) *If $l(sw) = l(w) + 1$ and $l(w\theta(s)) = l(w) - 1$, then the Weyl displacements in $P \setminus \{D\}$ are $w\theta(s)$ (for exactly one chamber) and sw for all others.*

(iii) *If $l(sw) = l(w) - 1$ and $l(w\theta(s)) = l(w) + 1$ then the Weyl displacements in $P \setminus \{S\}$ are $sw\theta(s)$ (for exactly one chamber) and $w\theta(s)$ for all others.*

(iv) *If $l(sw\theta(s)) = l(w) - 2$ we need to find the projection of $\theta(P)$ onto P and for this element, we can apply one of the previous statements.*

(v) *If $sw\theta(s) = w$, we have to calculate all chambers of $P \setminus \{D\}$. The panels D and $\theta(D)$ are parallel and the given information is not enough for any kind of reduction.*

We see that depending on the amount of panels being mapped to parallel panels, we can immensely reduce the set of chambers to be considered during the calculations.

The essence of mathematics lies entirely in its freedom.

Georg Cantor

PART VI

Appendix

APPENDIX

ONE

THE MAIN PROGRAM CODE

Aff_Buildings.sage

```
###### SETTING SOME BASIC VARIABLES ####
from time import gmtime, strftime
global FundamentalMatrix
global folderstring
folderstring = "/Data/"
print("Ensure, you are using version 6.3 or later")
print("Use load_attach_path(_path_) to tell sage the directory _path_,")
print(" which contains the files of this package")
print("Type Weylrepinfo() for further information")
load ('Chamber_based_Functions.sage')

###### THE MAIN CALCULATIONS ###########
def GetRepWithDiagonalMat(matrix):
    """ matrix has to be a monomial matrix.
    Returns a pair w1, DiagonalValues, where w1 represents a permuta-
    tion matrix as a product of generators, such that matrix * w1 is
    the diagonal  matrix whose non-zero entries are the entries in
    DiagonalValues  """
    w1 = []
    diagonalValues = []
    columnPositions = []
    # compute the non-zero entries and their positions
    for i in range(globalrank):
        for j in range(globalrank):
            if matrix[i][j] != 0:
                diagonalValues.append(_myval(matrix[i][j]))
                columnPositions.append(j)
                break
    # compute the permutation to turn matrix into a diagonal matrix
    for i in range(globalrank-1):
        position = columnPositions[i]
        _list = range(i,position)
        _list.reverse()
        for j in _list:
            w1.append(j+1)
            l = columnPositions.index(j)
```

```
            m = columnPositions.index(j+1)
            columnPositions[l], columnPositions[m] = \
                columnPositions[m], columnPositions[l]
    return w1, diagonalValues

def GetWordForSingleEntry(diagval, shiftval):
    """ Returns the word corresponding to diagonal matrix
        diag(diagval,1,..,1) (see  14.5.25) """
    if (diagval >0):
        return [Modulo(1+val-shiftval)
            for val in range(diagval * (globalrank -1))];
    return [Modulo(-val-shiftval)
        for val in range(-diagval * (globalrank -1))];
def GetWeylRepresentativeForMonomialMatrix(matrix):
    """ Returns a word w in W such that the corresponding matrix M_w
        represents the same Iwahori double coset as matrix (see 14.5.29).
    """
    # turn the matrix into diagonal matrix:
    w1, diagonalValues = GetRepWithDiagonalMat(matrix)
    w=[]
    shiftlist =[0]
    shiftsum = 0;
    # calculate the Weyl word corresponding to the diagonal matrix
    for i in range(globalrank):
        shiftlist.append(i)
        if (diagonalValues[i] == 0):
            continue;
        shiftlist.reverse()
        w.extend(SigmaShift(shiftlist,-(shiftsum)))
        w.extend(GetWordForSingleEntry(diagonalValues[i], shiftsum))
        shiftsum += diagonalValues[i]
        shiftlist.reverse()
        w.extend(SigmaShift(shiftlist, -(shiftsum)));
    # extend w with the correctly shifted w1
    w1.reverse();
    w1 = SigmaShift(w1, -shiftsum);
    w.extend(w1);
    WeylRepresentant = W.from_reduced_word(w)
    return WeylRepresentant.reduced_word()

def GetMonomialRepresentative(matrix):
    """ Returns a monomial matrix representing the same Iwahori- double
        coset as matrix - see Algorithm 1 on page 132.
    """
    return EraseInList([[y for y in row ] for row in matrix.rows()])

def EraseInList(matrix, rows = [],cols = []):
    """ __ internally used function __
    this is the main function for computing a monomial representative  """
    if rows ==[]:
        rows = range(len(matrix))
        cols = rows[:]
    pair = GetLowValuation(matrix,rows,cols)
    row = pair[0]
    col = pair[1]
```

```
    cols.remove(col)
    integralval = (matrix[row][col] *\
             t^(-_myval(matrix[row][col])))^-1
    # multiply the col's column such that matrix[row][col] is a monomial
    for i in rows:
        matrix[i][col] = matrix[i][col] *integralval
    # remove the other entries in the row's row
    for j in cols:
        if matrix[row][j] == 0:
            continue
        factor = (matrix[row][col])^(-1) * matrix[row][j]
        for i in rows:
            if matrix[i][col] == 0:
                continue
            matrix[i][j] -= matrix[i][col] * factor
    rows.remove(row)
    # remove the remaining entries in the column
    for i in rows:
        matrix[i][col] = 0
    if len(rows)>1:
        return EraseInList(matrix,rows,cols)
    else:
        return matrix

###### IMPORTANT SUB-ROUTINES ##########
def DiagonalMatrix(list):
    """ Returns a diagonal matrix, whose diagonal equals list """
    n = len(list)
    list = [list[i] if i<n else 1 for i in range(globalrank)]
    return matrix(K, [[list[i] if i == j else 0 for i in range(globalrank)]
                        for j in range(globalrank)])

def MakeGeneratorList():
    """ Returns a list of the generators of W, as they are used in this
    program, where the 0th entry corresponds to the affine extension """
    global globalrank
    GeneratorList = []
    return [MakeGenerator(i) for i in xrange(globalrank)]

def MakeGenerator(x):
    """ Returns the generator (element of S) corresponding to the value x """
    m = identity_matrix(K, globalrank)
    MultWithGenFromLeft(x,m)
    return m

def MultWithGenFromRight(type,Matrix):
    """ Changes Matrix to ( Matrix *s_{type} ) """
    if type == 0:
        firstcolumn = Matrix.column(0)
        Matrix.set_column(0,t * Matrix.column(globalrank-1))
        Matrix.set_column(globalrank-1,-t^-1*firstcolumn)
    else:
        firstcolumn = Matrix.column(type-1)
        Matrix.set_column(type-1, Matrix.column(type))
        Matrix.set_column(type, -1 * firstcolumn)
```

```
def MultWithRootElementFromRight(type, val, Matrix):
    """ Changes Matrix to (Matrix * u_{type}(val)) """
    if val == 0:
        return
    if type == 0:
        Matrix.set_column(0,
              val * t * Matrix.column(globalrank -1)+\
              Matrix.column(0))
    else:
        Matrix.set_column(type,
                Matrix.column(type)+ \
                val*Matrix.column(type-1))

def MultWithGenFromLeft(type,Matrix, factor = 1):
    """ Changes Matrix to ( s_{type} * Matrix) """
    if type == 0:
        firstrow = Matrix.row(0)
        Matrix.set_row(0,factor * -t^-1 * Matrix.row(globalrank-1))
        Matrix.set_row(globalrank-1,factor * t * firstrow)
    else:
        firstrow = Matrix.row(type-1)
        Matrix.set_row(type-1, factor * -1 *  Matrix.row(type))
        Matrix.set_row(type, factor * firstrow)

def MultWithGenFromLeftInverse(type,Matrix):
    """ Changes Matrix into (s_{type] * Matrix)^-1 """
    MultWithGenFromLeft(type, Matrix, -1)

def Modulo(x):
    """ Returns x (mod n) """
    if (x <0):
        return globalrank-((-x)%globalrank)
    else:
        if (x >= globalrank):
            return (x%globalrank);
        else:
            return x

def SigmaShift(list,k):
    """ returns a the list {(x + k ) mod globalrank | x in list} """
    return [Modulo(y+k) for y in list]

def RootElement(type,value):
    """ Returns u_{type}(value)
    w.r.t the fundamental chamber, this returns the matrix
    one needs to go to the type-adjacent chambers corresponding
    to the parameter value  """
    m = identity_matrix(K, globalrank)
    MultWithRootElementFromLeft(type, value, m)
    return m

def GetNeighbour(matrix,type_of_panel,field_value):
    """ Returns a matrix representing the chamber corresponding to
    u_{type_of_panel}(fieldval)(C), where C corresponds to matrix """
```

```
    m = copy(matrix)
    MultWithRootElementFromRight(type_of_panel, field_value, m)
    MultWithGenFromRight(type_of_panel,m)
    return m

def GoToNeighbour(matrix, type_of_panel, field_value):
    """ For the chamber C represented by matrix:
    Changes matrix into the matrix representing the chamber corresponding
    to u_{type_of_panel}(fieldval)(C) """
    MultWithRootElementFromRight(type_of_panel, field_value, matrix)
    MultWithGenFromRight(type_of_panel,matrix)

def GetLowValuation(vals,rows=[],cols=[]):
    """ Returns the index pair of an entry, such that this entry
       has lowest valuation in its row and column and among those
       has lowest row index and greatest column index
       then i' > i """
    if rows == []:
        rows = range(len(vals))
        cols = rows[:]
    startvalfound = False
    pair = GetNonZeroIndex(vals,rows,cols)
    newval = True
    while True:
        new_index = LowestValuationInList(
            vals[pair[0]], cols, False)
        if new_index <> pair[1]:
            pair[1] = new_index
            continue
        pair[1] = new_index
        # check the row
        new_index = LowestValuationInList(
            [v[pair[1]] for v in vals], rows, True)
        if new_index == pair[0]:
            return pair
        pair[0] = new_index

def LowestValuationInList(vals,indices, _is_column):
    """_is_column == True: return the greatest index of an entry
                  with minimal valuation
        _is_column == False: return the lowest index of an entry
                  with minimal valuation  """
    resultindex = -1
    for index in indices:
        if vals[index] == 0:
            continue
        if resultindex == -1:
            val = _myval(vals[index])
            resultindex = index
            continue
        compval = _myval(vals[index])
        if compval < val:
            val = compval
            resultindex = index
        elif _is_column == True:
```

```
            if compval == val:
                resultindex = index
    return resultindex

def GetNonZeroIndex(matrix, rows, cols):
    for row in rows:
        for col in cols:
            if matrix[row][col] != 0:
                return [row,col]
                break
    return pair

def _myval(f):
    """ If f is an element of tha basefield, it is not interpreted as an
    element of K or L and thus doesn't have a valuation.  """
    try:
        return f.valuation(t)
    except:
        return 0

#### WHAT ARE ROOTS AND GENERATORS ####
""" Example for globalrank = 3
    The generators are:

        s_0             s_!         s_2

   / 0  0 -t^-1\   / 0 -1  0 \   / 1  0  0 \
   | 0  1  0   |   | 1  0  0 |   | 0  0  -1|
   \ t  0  1   /   \ 0  0  1 /   \ 0  1  0 /

   The roots (with value val) denoted by u_{type}(val) are:

       u_0(val)        u_1(val)      u_2(val)

    / 1    0  0 \    / 1 val 0 \    / 1  0  0 \
    | 0    1  0 |    | 0  1  0 |    | 0  1 val|
    \val*t 0  1 /    \ 0  0  1 /    \ 0  0  1 /  """
```

APPENDIX

TWO

THE CODE INTERFACE FOR THE USER

Chamber_based_Functions.sage

```
def WeylrepSettings(rank,field):
    """ Sets the basic Data. Example: MyField.<a> = FiniteField(25)
    WeylrepSettings(3,MyField,’alph’) sets SL_3(F_7(t))
    as the building of rank 3 """
    global GeneratorList, globalrank, ShiftDisplacement
    globalrank = rank
    global L,t, K, ProgrammMyField, W, inverseShiftDisplacement
    ProgrammMyField = field
    L.<t> = LaurentPolynomialRing(field)
    K.<t> = FractionField(L)
    W = AffinePermutationGroup(["A",globalrank-1,1])
    GeneratorList = MakeGeneratorList()
    inverseShiftDisplacement = range(1,globalrank)
    ShiftDisplacement = range(1,globalrank)
    ShiftDisplacement.reverse()

def GetWeylelement(matrix):
    """ return the type of a gallery from the fundamental chamber
        to the chamber represented by matrix """
    return GetWeylRepresentativeForMonomialMatrix(
          GetMonomialRepresentative(matrix))

def GetWeylelements(list_of_matrices):
    """ Returns a list of list of the following form:
    [matrix from the list, reduced word, length of the word] """
    return [[A, GetWeylelement(A)] for A in list_of_matrices]

def WeylDistance(chamber_one,chamber_two):
    """ Returns the type of a gallery from the chamber represented by
        chamber_one to the chamber represented by chamber_two """
    return GetWeylelement(chamber_one^(-1) *chamber_two)

def GetDisplacement(Chamber,action):
    """ Returns the type of a gallery from the chamber represented by
        Chamber to its image under action """
    return GetWeylelement(Chamber^(-1) *(action(Chamber)))
```

```
def ChamberOfRep(matrix):
    """ Returns an element of the building corresponding to the given matrix
     -- it may be needed if one changes the type-set,
        the fundamental chamber, or the fundamental apartment """
    return GetChamber(GetChamberRepresentation(matrix))

def CalculateImagesOfGenerators(Function):
    global TypeImages
    TypeImages = []
    M = (ChamberOfRep(
                Function(DiagonalMatrix([]))))^(-1)
    for i in range(globalrank):
        TypeImages.append(GetWeylelement(
                        M*GetChamber(
                                GetChamberRepresentation(
                                    Function(GeneratorList[i])))))
    return TypeImages

def GetProjection(panel_chamber, paneltype, chamber):
    """ Computes the projection of chamber onto the paneltype-panel
        containing panel_chamber
        Returns the projection and the corresponding field value """
    distance = len(GetWeylelement(panel_chamber^(-1)* chamber))
    return GetProjection_p(panel_chamber, paneltype, chamber, distance)

def GetProjection_p(panel_chamber, paneltype, chamber, distance):
    """ Computes the projection of chamber onto the panel of type
    paneltype containing the chamber panel_chamber
    returns the pair D,val if
    D = GetNeighbour(panel_chamber, paneltype, val)
    is closer to chamber than panelchamber
    else returns the panel_chamber and an empty list """
    for __,val in enumerate(ProgrammMyField):
        D = GetNeighbour(panel_chamber,paneltype, val)
        if (len(WeylDistance(D,chamber)) < distance):
            return D, val
    return panel_chamber, []

def GetChamberRepresentation(matrix):
    """ Returns a gallery describing the chamber corresponding to matrix
        the gallery can be used as a parameter GetChamber """
    Wdistance = GetWeylelement(matrix)
    C = DiagonalMatrix([])
    gallery=[]
    n = len(Wdistance)
    for i in range(len(Wdistance)):
        gallery.extend([Wdistance[i]])
        C, val = GetProjection_p(C, Wdistance[i], matrix, n-i)
        if val == []:
            raise Exception('could not determine a projection.\r\n'+\
        'the gallery is:' +str(gallery) +'\r\n' +\
        'the Weyldistance is: ' + str(Wdistance))
        gallery.append(val)
    if len(gallery) < 2* n:
```

```
        raise Exception ('could not determine a minimal gallery.'+\
        'the gallery is:' +str(gallery) +'\r\n' +\
        'the Weyldistance is: ' + str(Wdistance))
    return gallery

def GeneratorProduct(List):
    """ Returns the product of the generators
    s_(i_1) .... s_(i_l) if list == [i_1,...,i_2] """
    m = identity_matrix(K, globalrank)
    for x in List:
        MultWithGenFromRight(x,m)
    return m

def GetRandomGallery(pathlength, showpath = False):
    """ Returns a random list, which can be used as a chamber description """
    pathList = []
    printList = []
    list = W.random_element_of_length(pathlength).reduced_word()
    for i in range (pathlength):
        printList.append(list[i])
        pathList.append(list[i])
        pathList.append(ProgrammMyField.random_element())
    if showpath =="path":
        print(pathList)
    if showpath =="gallery":
        print(printList)
    return pathList

def GetRandomChamber(pathlength,basechamber =None, showpath = False):
    """ Returns a random chamber of the building, by constructing a gallery
    from the fundamental chamber of length $pathlength """
    if basechamber == None:
        basechamber =  DiagonalMatrix([])
    return GetChamber(GetRandomGallery(pathlength, showpath),basechamber)

def GetDisplacementForFunction(Function,A):
    """ Returns the Weyldistance of the given matrix A
        to its image under the map Function """
    return GetWeylelement(A^(-1) * Function(A))

def GetAllNeighbours(matrix,type_of_panel):
    """ Given the matrix A and i in the Typeset of W,
    we calculate all neighbours w.r.t. the given Weylgroup """
    C = []
    for j,x in enumerate(ProgrammMyField):
        C.append(GetNeighbour(matrix,type_of_panel,x))
    return C

def GetChamber(arglist, basechamber =None):
    """ Returns a matrix, which represents the chamber C, whose path
        from the baseChamber is given by the argument list[],
        i.e. list = [s_n,a_n,...,s_2,a_2, s_1,a_1] or
        [[s_n,a_n],...,[s_1,a_1]], where w = s_1...s_n is the
        Weyldistance of the chamber to the base chamber
        The parameter a_i is an element of the given field -
```

```
        it determines the parameter of the rootgroup action
        correponding to the simple root associated to s_i """
    if basechamber == None:
        basechamber =  DiagonalMatrix([])
    B = copy(basechamber)
    if len(arglist) == 0:
        return B
    for i in range(len(arglist)//2):
        B = GetNeighbour(B,arglist[i*2],arglist[i*2+1])
    return B

def GetChamberInverse(arglist, baseChamber = None):
    """ Returns the inverse of GetChamber(arglist, basechamber)
        !! If a chamber is given as a matrix M using M^-1 is in most
        cases a faster option """
    if baseChamber == None:
        baseChamber =  DiagonalMatrix([])
    B = copy(baseChamber)
    if len(arglist) == 0:
        return B
    for i in range(len(arglist)//2):
        GoToNeighbourInverse(B,arglist[i*2],arglist[i*2+1])
    return B
```

APPENDIX

THREE

DISPLACEMENT BALL VERSION ONE

This code provides a routine for calculating Weyl displacements. It reduces the amount of calculations by comparing the possible Weyl displacements with Weyl displacements which already occurred. The calculations are very fast for a small radius, but the overall time increases a lot if a panel gets mapped onto a parallel panel and one of the possible Weyl displacements has not been a Weyl displacement before. The output which will be stored directly into a file, contains information about every step the algorithm went through.

```
load('Aff_Buildings.sage')
nr_of_processes = 3, break_at_maxvals = True, quick = True, startlength = 1
maxFieldElements= 0, _initialpath =[], precalculate=True, preList=[]
use_reduction = True, _do_parallel = True, _parallel_threshold = 40
""" About some parameters:
    initialpath - will be placed in front of every gallery
    quick = True - step over computations if the list
            displacementslist contains already all possible results
    precalculate = True - works together with the quick option,
        some random chambers are calculated at the beginning of the
        main computation
    - preList is added to _kowngalleries at the beginning of the computations
    - maxFieldElements - the maximal amount of galleries per gallery
        type. If set to 0: no restriction is applied """

def GetDisplacementOfRandomChamber(radius, action,
            basechamber = None, showpath = false):
    """ Computes the displacement of a randomly chosen chamber corresponding
    to action. The choice of the chamber is limited by the radius around
    fundamental chamber or the basechamber, if given. """
    if basechamber is None:
        basechamber = DiagonalMatrix([])
    D = GetRandomChamber(radius, basechamber, showpath)
    return GetWeylelement(D^(-1)*action(D))

def MakeDisplacementList(radius, steps, action, basechamber = None):
    """ Computes the displacements of steps random chambers in the Ball of
    radius radius around the basechamber  """
    if basechamber is None:
```

```
        basechamber = DiagonalMatrix([])
    list = []
    for i in range(steps):
        D = GetRandomChamber(radius, basechamber)
        list.append(MatrixInvert(action(D))*D)
    return list

def SetFunction(_Function):
    global Function
    Function = _Function

def InitDisplacmentBall():
    """ Initializes certain parameters for the main computations """
    global filetext
    filetext = ""

    # a counter for testing purposes
    global valuesovermaxval
    valuesovermaxval = 0

    #compute the action of _Function on the set of generators
    global TypeImages
    TypeImages = CalculateImagesOfGenerators(Function)
    print("The images of the Types are:")
    for i in range(globalrank):
        print(str(i)+ "<-->" + str(TypeImages[i]))
    filetext = AddInitToFiletext(description, _displacementSet)

    # set some global variables for counting
    global _count_calculated_chambers
    _count_calculated_chambers = 0
    global _count_calculated_chambers_per_gallery
    _count_calculated_chambers_per_gallery =0

    # Compute the displacment of the basechamber
    w = GetDisplacement(basechamber, Function)
    _displacementSet.append(w)

    #initialize the dictionary for the computed galleries
    global _knowngalleries
    _knowngalleries = dict({"":[w]})
    filetext += str(w) +"-\t- [] -\t- []\r\n"

def GetDisplacementBall(_Function,
        maxpathlength, _basechamber = None,
        _description = ""):
    """ This function computes all displacements corresponding Function
    that appear inside a ball of
    radius maxpathlength around the basechamber
    The parameter: description - is used for the stored file  """
    global basechamber
    if _basechamber == None:
        basechamber = DiagonalMatrix([])
    else:
        basechamber = _basechamber
```

```
    global description
    description = _description
    SetFunction(_Function)
    #reset the _displacementSet
    global _displacementSet
    _displacementSet = []

    # apply some reduction for the computations
    if quick == True:
        # add previously computated displacements
        _displacementSet += preList
        if precalculate==True:
            # compute the displacements for a random set of chambers
            PreCalculation(globalrank**2 * maxpathlength*5,
                                maxpathlength)

    # initialize computation - get Images of the generators
    #   compute the displacement of the basechamber
    InitDisplacmentBall()

    # passing through all chambers in a Ball of diameter pathlength
    # around BaseChamber in the following way:
    # We start with the BaseChamber itself
    # We will use the method elements_of_length(n) to describe
    # the chambers around the base chamber

    # use the pathlength as a global parameter for the calculations
    global pathlength
    for pathlength in range(startlength,maxpathlength+1):
        CalculatePathlength()

    # computation is finished - save filetext to disc
    print("Done.")
    PrintDataInFile(filetext, _displacementSet)
    return

def CalculatePathlength():
    """ Computes the displacements for the chambers with distance
    pathlength from basechamber  """
    print("pathlenght: " +str(pathlength))
    global filetext
    filetext += "\r\n    pathlength: "+str(pathlength) +"\r\n=====\n"
    filetext += "Displacement \t Type of Gallery \t Gallery\r\n"
    paths = W.elements_of_length(pathlength)

    # The list fieldlementslists contains all sequences of
    #   length pathlength of elements in the given field
    global fieldelementlists
    fieldelementlists = MakeFieldelementLists(
        maxFieldElements, pathlength)

    # iterate through all paths of length pathlength
    for path in paths:
        # CalculatePath checks whether a complete computation is needed
        if CalculatePath(path) == True:
```

```
            InternalDisplacementCalculations()

def CalculatePath(path):
    """ Returns False if internal computations can be avoided """
    _count_calculated_chambers_per_gallery =0
    # tmp_dict stores the displacements, which can be computed by
    # comparing given words. If a complete computation can be avoided
    # tmp_dict will be added to _known_galleries
    global tmp_dict
    tmp_dict = dict()
    # pathword is a list describing the type of path
    global pathword
    pathword = path.reduced_word()
    global pathstring
    pathstring = ""

    # boolean variables to check whether a complete computation is needed
    check_path_is_extended =False
    check_extension_works = True

    # maxvals is the maximal possible number of displacements to be
    # obtained from a given gallery
    global maxvals
    maxvals = []
    for pos in range(pathlength-1):
        pathstring += str(pathword[pos])
    _len = len(pathstring)
    global extensionstring
    extensionstring = ""
    # filetext - for writing information on disk
    global filetext
    for key in _knowngalleries:
        if len(key) == _len:
            # _check all possible new displacements
            _CheckKey(key)
            if check_path_is_extended == True:
                break
    if check_path_is_extended ==True and \
       check_extension_works == True:
        _knowngalleries.update(tmp_dict)
        filetext += extensionstring
        if use_reduction == True:
            # no need for internal calculations
            return False
    pathstring += str(pathword[-1])
    _knowngalleries.update({pathstring:[]})
    global maxvalnumber
    maxvalnumber = len(maxvals)
    filetext += "maximal possible values: \r\n"
    for l in maxvals:
        filetext+= str(l) +"-"
    filetext += "\r\n"
    if quick == True:
        maxvals_in_displacementSet = True
        for val in maxvals:
```

```
                if val not in _displacementSet:
                    maxvals_in_displacementSet = False
            if maxvals_in_displacementSet == True:
                filetext += "All displacements are known \r\n"
                for val in maxvals:
                    _knowngalleries[pathstring].append(val)
                # no need for internal calculations
                return False
        return True

def AddInitToFiletext(description, _displacementSet):
    filetext = "globalrank: " + str(globalrank) +"\r\n"
    filetext += "fieldsize:" + str(ProgrammMyField) +"\r\n"
    filetext += "Extra description: " +description+"\r\n"
    filetext += "displacementSet at start (after PreCalculation):\r\n"
    for val in _displacementSet:
        filetext += " - " + str(val) + "\r\n"
    print("pathlenght: 0")
    filetext += "\r\n    pathlength: " +str(0) +"\r\n=============\r\n"
    filetext += "Displacement \t Type of Gallery \t Gallery\r\n"
    return filetext

def PrintDataInFile(filetext, _displacmentSet):
    filestring =load_attach_path()[-1] + "/" +\
        folderstring+getCurrentTimeString()
    file = open(filestring+".txt",'w')
    file.write(filetext)
    file.close()
    _displacementSet.sort(lambda x,y: cmp(len(x), len(y)))
    file = open(filestring+"_Results.txt",'w')
    file.write("globalrank: " + str(globalrank) +"\r\n")
    file.write("fieldsize:" + str(ProgrammMyField) +"\r\n")
    file.write("Extra description: " +description +"\r\n")
    file.write("The images of the Types are:\r\n")
    for i in range(globalrank):
        file.write(str(i)+ "<-->" + str(TypeImages[i])+"\r\n")
    file.write("Values over maxval during calculations: " +\
        str(valuesovermaxval)+"\r\nDisplacments:\r\n")
    for x in _displacementSet:
        file.write(str(x) +"\r\n")
    file.close()

def InternalDisplacementCalculations():
    global maxvalreached, len_fieldvals
    maxvalreached = False
    len_fieldvals = len(fieldelementlists)
    if _do_parallel == True and\
            len(fieldelementlists) > 2* _parallel_threshold:
        ParallelComputation()
        return
    SequentielComputation()

def SequentielComputation():
    # declaring global variables which may be altered
    global _count_calculated_chambers, filetext
```

```
    global maxvalreached, filetext, maxvals
    global valuesovermaxval
    global _knowngalleries, _displacementSet
    additionalvalues =0
    newvalues = []
    for fieldelementlist in fieldelementlists:
        # user information about the amount of done calculations
        if _count_calculated_chambers % 10000 == 9999:
            print("calculated " +str(_count_calculated_chambers)+\
                "chambers.")
            print("Current memory usage: " + str(get_memory_usage()))
            print("Current number of chambers per gallery: "+\
                str(len(fieldelementlists)))
        _count_calculated_chambers +=1

        if maxvalnumber >0 and len(newvalues) == maxvalnumber and\
            maxvalreached == False:
            # check if computations shall be finished at maxvals
            if break_at_maxvals:
                return
            filetext += "Reached maxvals\r\n"
            maxvalreached = True
        gallery = _initialpath

        for i in range(pathlength):
            gallery  = gallery+[pathword[i], fieldelementlist[i]]
        C = GetChamber(gallery, basechamber)
        w = GetDisplacement(C, Function)
        if w not in newvalues:
            newvalues.append(w)
            if maxvalreached == True:
                valuesovermaxval +=1
            if w not in _knowngalleries[pathstring]:
                _knowngalleries[pathstring].append(w)
            if (w not in _displacementSet):
                _displacementSet.append(w)
        filetext += str(w) +"-\t-" +\
            str(pathword) +"-\t-" +\
            str(gallery)+ "\r\n"
    for b in newvalues:
        if b in maxvals:
            maxvals.remove(b)
    filetext += "values that did not appear: "+\
        str(maxvals)+"\r\n"

def ParallelComputation():
    #declaring the global variables, which may be altered
    global filetext, valuesovermaxval
    global _knowngalleries, _displacementSet
    global valuesovermaxval
    maxcount = len(maxvals)
    newvalues = []
    filetext += "parallel\r\n=============\r\n"
    fullparts = len_fieldvals // _parallel_threshold
    # construct the argumentslist for parallel processing
```

```
    # each list contains a pair (x,), where
    # x = [Function, pathword, list], where the list contains at most
    #   _parallel_threshold number of entries

    _splittedlist = [([Function, pathword,
        fieldelementlists[i*_parallel_threshold:\
                        (i+1)*_parallel_threshold]],)\
                        for i in range(fullparts)]

    # split the argumentlist in nr_of_processes parts
    # to be able to break the parallel processing if maxvals is reached
    for i in range(len(_splittedlist)/nr_of_processes +1 ):
        # check if the number of maximal possible new values
        # has been reached
        if len(newvalues) >= maxvals:
            if break_at_maxvals:
                filetext += "Reached maxvals \r\n"
                return
        _biglist = list(
                GetDisplacmentsForList(
                    _splittedlist[i*nr_of_processes: \
                    (i+1)*nr_of_processes]))
        for partresult in _biglist:
            for w in partresult[1]:
                if w not in newvalues:
                    newvalues.append(w)
                    filetext += str(w) +"-\t-" +\
                    str(pathword) +"-\t-" +"\r\n"
                    if w not in _knowngalleries[pathstring]:
                        _knowngalleries[pathstring].append(w)
                    if w not in _displacementSet:
                        _displacementSet.append(w)

    valuesovermaxval += len(newvalues) - maxcount
    return

def MakeFieldelementLists(maxFieldElements, pathlength):
    """ Constructs a list containing all lists needed to describe every
    chamber (up to the maxFieldElements restriction) corresponding
    to a given gallery:
    Example:
    For pathlength = 3 and FiniteField(3) the function returns:
    [ [0,0,0], [0,0,1], [0,0,2], [0,1,0], [0,1,1], [0,1,2], ...
        [2,1,0], [2,1,1], [2,1,2],  [2,2,0], [2,2,1], [2,2,2] ] """
    if maxFieldElements == 0 or pathlength <3:
        fieldelementlists = [[]]
        for  i in range(pathlength):
            fieldelementlists =[
                    [x]+y for j,x in enumerate(ProgrammMyField)\
                          for y in fieldelementlists]
        return fieldelementlists
    # no restriction on the amount of lists
    fieldelementlists = []
    max = maxFieldElements * pathlength
    for i in range(max):
```

```
        tmplist = []
        for j in range(pathlength):
            tmplist.append(ProgrammMyField.random_element())
        fieldelementlists.append(tmplist)
    return fieldelementlists

def PreCalculation(amount, _maxpathlength):
    """ compute some displacements before the main computation """
    global _displacementSet
    for i in range(amount):
        w = GetDisplacementOfRandomChamber(
                randint(0,_maxpathlength), Function, basechamber)
        for w1 in W.from_reduced_word(w).reduced_words():
            if w1 not in _displacementSet:
                _displacementSet.append(w1)

def _CheckKey(key):
    """ Given a key, i.e. a string describing a gallerytype,
    Checks if computations are needed or can be derived by the
    previously computed displacements """
    if W.from_word([int(c) for c in key]) != \
        W.from_word(pathword[:-1]):
        return

    global filetext, keyname
    global tmp_dict, keycount
    keyname = pathstring + str(pathword[-1])
    tmp_dict.update({keyname :[]})
    keycount = len(_knowngalleries[key])
    global valcounter
    valcounter = 0
    filetext += "======\r\n"+str(keycount)+" pregalleries inside ["+\
        str(key)+ "]:\r\n"
    for val in _knowngalleries[key]:
        filetext += str(val ) + " -- "
    filetext += "\r\n"
    global pregallery
    for pregallery in _knowngalleries[key]:
        valcounter+=1
        if (CheckTrivialPath() == True):
            continue
        if (CheckWordOfLengthOne() == True):
            continue
        w = W.from_word([pathword[-1]]+\
                pregallery+\
                TypeImages[pathword[-1]]).reduced_word()
        if (CheckTwoSidedProjection(w) == True):
            continue
        w1 = W.from_word([pathword[-1]]+\
                pregallery).reduced_word()
        w2 = W.from_word(pregallery+\
                TypeImages[pathword[-1]]).reduced_word()
        if (CheckTwoSidedReduction(w,w1,w2) == True):
            continue
        if (CheckExtendingOnTheLeft(w,w1,w2) == True):
```

```
            continue
        if (CheckExtendingOnTheRight(w,w1,w2) == True):
            continue
        # the remaining case of parallel panels:
        CaseNoExtension(w,w1,w2)


def CheckTrivialPath():
    """ A special treatment for the empty list """
    if (pregallery != [] or \
            [pathword[-1]] != TypeImages[pathword[-1]]):
        return False
    global maxvals, check_extension_works, filetext
    if [] not in maxvals:
        maxvals.append([])
    if [pathword[-1]] not in maxvals:
        maxvals.append([pathword[-1]])
    check_extension_works=False
    filetext +="no extension for trivial path[type 1]:\r\n"
    return True

def CheckWordOfLengthOne():
    """ A special treatment for a list of length 1 """
    if (len(pregallery) !=1 or \
        [pathword[-1]] != pregallery or\
        pregallery != TypeImages[pathword[-1]]):
        return False
    global maxvals, check_extension_works, filetext
    if [] not in maxvals:
        maxvals.append([])
    if [pathword[-1]] not in maxvals:
        maxvals.append([pathword[-1]])
    check_extension_works=False
    filetext += "no extension for path[type 2]"+\
      "(mingallery has same type as extending generator):\r\n"
    return True


def CheckTwoSidedReduction(w,w1,w2):
    """ For non-parallel panels:
    There are two cases:
    1:
    Some chamber D in the (pathword[-1]panel P containing C is the
    projection of its  image onto P and its image D'
    is the projection of D onto the image of P.

     C--D --.......-- D'--C'  (D is a chamber in P and D' its image)
    2:
    Some chamber D in the (pathword[-1]panel P containing C is the
    projection of its  image onto P, but its image D'
    is not the projection of P onto its image.

                         D'
                        /
     C--D --.......-- E'--C'  (D is a chamber in P and D' its image)
```

```
     Without knowing the image of the projection, we cannot do a
     reduction """
    if (len(w) != len(pregallery)-2):
        return False
    global check_path_is_extended, extensionstring, filetext
    global check_extension_works
    check_path_is_extended =True
    check_extension_works = False
    AddValues([pregallery,w,w1,w2])
    # adjust text based variables
    extensionstring+= str(w) +"-\t-"+\
        str(pathword) +"-\t- extending"+\
        str(pregallery)+" ("+str(valcounter)+\
        " of "+str(keycount)+") \r\n"
    filetext += "reduction (two sided)\r\n"
    return True

def CheckTwoSidedProjection(w):
    """ The previously computed chamber, say C, is the projection of its
    image onto the (pathword[-1]panel P containing C and its image C'
    is the projection of C onto the image of P

     D--C --.......-- C'--D'  (D is a chamber in P and D' its image)  """
    if (len(w) != len(pregallery)+2):
        return False
    global check_path_is_extended, extensionstring, filetext

    check_path_is_extended =True
    AddValues([w])

    # adjust text based variables
    extensionstring+= str(w) +"-\t-"+\
        str(pathword) +"-\t- extending"+\
        str(pregallery)+" ("+str(valcounter)+\
        " of "+str(keycount)+") \r\n"
    filetext += "extension (two sided)\r\n"
    return True

def CheckExtendingOnTheLeft(w,w1,w2):
    """ The previously computed chamber, say C, is the projection of its
    image onto the (pathword[-1]panel P containing C, but its image C'
    is not the projection of C onto the image of P

     D--C --.......-- D'--C'
     (D is a chamber in P and D' its image) """
    if (len(w1) != len(w) +1 or\
            len(w2) != len(w) -1):
        return False
    global extensionstring, check_path_is_extended, filetext
    check_extension_works = False
    check_path_is_extended = True
    AddValues([w1,w])

    # adjust text based variables
```

```
    extensionstring+= str(w) +"-\t-" +str(pathword) +"-\t- extending"+\
        str(pregallery)+" ("+str(valcounter)+ " of "+str(keycount)+\
        ") - left sided [the projection preimage] \r\n"
    extensionstring+= str(w1) + "-\t-" + str(pathword) +\
        "-\t- extending"+str(pregallery)+" ("+str(valcounter)+" of "+ \
        str(keycount)+ ") - left sided [the rest]\r\n"
    filetext += "Extension (only left):\r\n"
    return True

def CheckExtendingOnTheRight(w,w1,w2):
    """ The previously computed chamber, say C, is not the projection of
    its image onto the (pathword[-1]panel P containing C, but its image
    C’ is the projection of C onto the image of P

     C--D --.......-- C’--D’
     (D is a chamber in P and D’ its image) """
    if (len(w1) != len(w) -1 or\
            len(w2) != len(w) +1):
        return False
    global extensionstring, check_path_is_extended, filetext

    check_path_is_extended = True
    AddValues([w2,w])

    # adjust text based variables
    extensionstring+= str(w) +"-\t-" +str(pathword) +"-\t- extending"+\
        str(pregallery)+" ("+str(valcounter)+ " of "+str(keycount)+\
        ") - right sided [the projection preimage] \r\n"
    extensionstring+= str(w2)+"-\t-" + str(pathword) +"-\t- extending"+\
        str(pregallery)+" ("+str(valcounter)+" of "+str(keycount)+\
        ") - right sided [the rest]\r\n"
    filetext += "Extension (only right):\r\n"
    return True

def CaseNoExtension(w,w1,w2):
    global maxvals
    check_extension_works=False
    extends = [pregallery, w1, w2,w]
    UpdateMaxVals(extends)
    global filetext
    filetext += "no extension for path[type 3]:\r\n"


def AddValues(valuelist):
    """ Adds the given words in valuelist to tmp_dict and _displacementSet
    -- adjust maxvals  """
    global tmp_dict, _displacementSet
    for word in valuelist:
        if word not in tmp_dict[keyname]:
            tmp_dict[keyname].append(word)
        if word not in _displacementSet:
            _displacementSet.append(word)
    UpdateMaxVals(valuelist)

```

```
def UpdateMaxVals(valuelist):
    global maxvals
    for word in valuelist:
        iscontained = False
        w12 = W.from_reduced_word(word)
        for val in maxvals:
            if w12 == W.from_reduced_word(val):
                iscontained =True
        if iscontained == False:
            maxvals.append(word)

@parallel
def GetDisplacmentsForList(parameterlist):
    """ The parameterlist: [Function,pathword, fieldlementlists]
    Computes the displacements for all chambers corresponding to
    the parameterlist  """
    l = parameterlist
    returnlist = []
    pathlength = len(l[2][0])
    for _fieldlist in l[2]:
        gallery = []
        for i in range(pathlength):
            gallery  = gallery+[l[1][i], _fieldlist[i]]
        C = GetChamber(gallery)
        returnlist.append(GetWeylelement(C^-1 * l[0](C)))
    return returnlist

###### FUNCTIONS FOR INTERNAL DATA #####

def ChangeFolderForStream(folder):
    folderstring = folder

def getCurrentTimeString():
    return strftime("%Y-%m-%d-%H-%M-%S",gmtime())

# About some global variables
"""
    filtetext - contains a string about the computations
                it will be saved as a file at the very end
    tmp_dict  - a temporary dictionary about displacements
    pathstring - a string describing the type of a gallery
    pathword   - a list describing a sequence of panel types
    _knowngalleries - a (string, list) dictionary
            where the keys describe gallery types and the values
            describe the displacments of chambers that can be reached
            from the basechamber along a gallery of type "key"
    maxvals - the maximal number of displacements to appear within an
            internal computation - this helps to break certain loops
    newvalues - a temporary list of displacements used to determine
            the amount of different displacements within an internal
            computation

    valuesovermaxval - parameters used for testing """
```

GLOSSARY

(V, E)	a graph,	1
$-\alpha$	opposite root of α,	27
θ	isometry / automorphism of a building,	16
α	root,	27
$\Sigma(W, S)$	Coxeter complex of (W, S),	27
ΣC	apartments containing the chamber C satisfying,	67
$\mathcal{A}(\mathcal{B})$	the complete apartmentsystem of $\mathcal{B}$,	33
$*$	free product,	6
$*_A$	(free) amalgamated product over A,	6
conv	convex hull,	34
δ	Weyl-distance function,	33
$\mathcal{X}$	CAT(0)-space / the Davis realization of a building,	43
Γ	path in a graph / gallery,	2
γ_z^+	the positive subgeodesic of γ starting at z,	45
γ_z^-	the negative subgeodesic of γ starting at z,	45
$\Lambda(C)$	lattice classes corresponding to the chamber C,	113
$\mathcal{C}$	set of chambers,	33
$\mathcal{F}$	a facet,	65
$\mathcal{G}$	graph of groups,	9
$\mathcal{G}(e)$	edge group,	9
$\mathcal{G}(v)$	vertex group,	9
$\mathcal{K}$	set of knots,	73
$\measuredangle$	Alexandrov angle,	15
Min	set of elements with minimal displacement,	16
$\mathrm{Min}(\theta)$	set of elements with minimal displacement,	16
$\mathrm{Min}(\theta)_C$	$\mathrm{conv}(\{\theta^z(\mathrm{proj}_{\mathrm{Min}(\theta)}(b_C)) \mid z \in \mathbb{Z}\})$,	64
$\mathrm{M}_C(\theta)$	$R(\mathrm{Min}(\theta)_C)$,	64
$\mu(\alpha)$	set of panels containing exactly one chamber of the wall α,	49
$\overline{\alpha}$	wall corresp. to α,	27
$\partial\alpha$	wall corresp. to α,	27
proj	projection,	13
$\psi_{e,0}$	embedding of the edge group to the vertex group of the initial vertex,	9
$\psi_{e,1}$	embedding of the edge group to the vertex group of the end vertex,	9

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SELBSTÄNDIGKEITSERKLÄRUNG

Ich erkläre:

Ich habe die vorgelegte Dissertation selbständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis“ niedergelegt sind, eingehalten.

Gießen, 24. April 2015

Markus-Ludwig Wermer

Automorphisms of Buildings

Markus-Ludwig Wermer