DISSERTATION

Analysis of linear-quadratic optimization problems for semimartingales and application in optimal trade execution

submitted

by

Julia Ackermann

Mathematisches Institut
Justus-Liebig-Universität Gießen

Supervisor: Prof. Dr. Thomas Kruse

November 2022

Abstract

We analyze linear-quadratic (LQ) stochastic control problems that arise in optimal trade execution in models of Obizhaeva-Wang type. Extending previous literature, order book depth and resilience are both allowed to be stochastic processes. Moreover, the target position can be a random variable, and we can include a risk term with stochastic target process.

In discrete time, we find via the dynamic programming principle that the optimal trade sizes and the minimal costs are characterized by a process Y, which is defined by backward recursion, and by, for general targets, a further process ψ. We moreover investigate properties of our model such as savings in the long-time horizon, existence of profitable round trips, and premature closure of the position.

In continuous time, we go beyond the usual finite-variation strategies, and present two approaches. In the first one, we set up and solve a relevant control problem where we consider càdlàg semimartingales as execution strategies, while in the second one, we start from a typical formulation for finite-variation strategies, extend this continuously to progressively measurable strategies, and solve the extended problem via reduction to a standard LQ stochastic control problem and subsequent application of relevant literature. The counterpart of the process Y from discrete time now is the solution of a quadratic backward stochastic differential equation (BSDE), and ψ becomes the solution of a linear BSDE. It turns out that optimal strategies indeed can have infinite variation.

Zusammenfassung

Wir analysieren linear-quadratische (LQ) stochastische Kontrollprobleme, die in Modellen vom Obizhaeva-Wang Typ in der optimalen Handelsausführung auftreten. In Erweiterung zu bisheriger Literatur werden Orderbuchtiefe und Resilienz beide durch stochastische Prozesse beschrieben. Außerdem darf die Zielposition eine Zufallsvariable sein, und wir können einen Risikoterm mit stochastischem Zielprozess einbeziehen.

In diskreter Zeit erhalten wir mittels des Prinzips der dynamischen Porgammierung, dass die optimalen Handelsvolumina und die minimalen Kosten durch einen Prozess Y, der über Rückwärtsrekursion definiert ist, und, im Fall allgemeiner Zielgrößen, durch einen weiteren Prozess ψ charakterisiert sind. Wir untersuchen außerdem Eigenschaften unseres Modells wie langfristige Einsparungen, Existenz von profitablen Rundfahrten und vorzeitiges Schließen der Position.

In stetiger Zeit gehen wir über die üblichen Strategien endlicher Variation hinaus und präsentieren zwei Vorgehensweisen. Bei der ersten formulieren und lösen wir ein relevantes Kontrollproblem, bei dem wir càdlàg Semimartingale als Handelsstrategien zulassen, während wir bei der zweiten von einer typischen Formulierung für Strategien endlicher Variation starten, diese Formulierung stetig zu progressiv messbaren Strategien erweitern, und das erweiterte Problem per Reduktion zu einem standard LQ stochastischen Kontrollproblem und anschließender Anwendung von geeigneter Literatur lösen. Das Gegenstück zu dem Prozess Y aus diskreter Zeit ist nun die Lösung einer quadratischen rückwärts stochastischen Differentialgleichung (BSDE), und ψ entspricht nun der Lösung einer linearen BSDE. Es stellt sich heraus, dass optimale Strategien tatsächlich unendliche Variation haben können.

Acknowledgments

First, I would like to thank Thomas Kruse for the great supervision. He listened carefully and provided helpful advice. I enjoyed working with him and am grateful for his support during the whole time, including the periods when we had to move online.

I would further like to thank Ludger Overbeck. In particular, he took me on as a teaching and research assistant and initiated the supervision by Thomas. Moreover, I thank Ludger for agreeing to be the second referee of this thesis.

Many thanks go to my coauthors Thomas, Ludger, and Mikhail Urusov. This thesis benefited greatly from the collaboration with Thomas and Mikhail. Among others, I appreciate the time they took for discussions, the ideas and notes they shared, the friendly work atmosphere, and their feedback. I am also glad I had the opportunity to work with Thomas and Ludger on a project on affine Volterra processes.

For supporting me in attending conferences, I would like to thank Thomas, Ludger, and Mikhail. Moreover, I thank Mikhail for hospitality during a short research visit.

Furthermore, I would like to thank all people I had the pleasure to work with as part of my employment at the Mathematical Institute of the University of Gießen. Special thanks go to my former office mate Marius Scheld for encouragement and for interesting discussions on various topics.

Finally, I thank my parents and my brother for their general support, and Jonas for being there for me.

Contents

1 Introduction

In a stochastic control problem one seeks to steer, by choosing from a set of controls, a randomly evolving state in such a way that a performance criterion is optimized. Stochastic control problems arise in various applications, in particular, but not limited to, in finance (see, e.g., the examples in [Pha09, Chapter 2]). In this thesis, we analyze some linear-quadratic (LQ) stochastic control problems coming from optimal trade execution.

1.1 Optimal trade execution

Institutional investors regularly face the task to sell or buy a large amount of shares. Typically, it is not advisable to complete the whole task at once, since trading large volumes can have a substantial adverse impact on the price due to illiquidity in the market. One often can do better, i.e., reduce execution costs, by splitting up a large order into several smaller ones that are executed one after another. The issue thus consists in finding a good timing and appropriate sizes of these orders. When splitting up a large order, one has to keep in mind that there usually is a fixed time by which the original task needs to be finished, with a typical time scale ranging from some hours to a few days. Even if there is no fixed terminal time, taking longer to finish the task may bear greater uncertainty. Trading too slowly can therefore be problematic, e.g., by enforcing a costly trade at the terminal time, whereas trading too fast may accumulate avoidable costs beforehand. The optimization of such trading schedules is called optimal trade execution or optimal liquidation problem.

To treat optimal trade execution mathematically, the typical procedure is to model the impact of the large agent on the price, formulate a control problem based on this, and solve the control problem (analytically or numerically).

Admissible strategies

One needs to decide between a discrete-time and a continuous-time formulation of the model, and what trading strategies to allow for. Trading strategies X in the literature usually have the interpretation that at time s, the value of X_s indicates the (sometimes relative to the goal, and possibly negative) position of the agent in this asset. The task to sell or buy a certain amount of shares over the given trading period then translates to the requirement that a specific position has to be reached at terminal time, starting from a given initial position. Optimal trade execution problems are also sometimes formulated as the problem to close an initial position $x \in \mathbb{R}$ up to the terminal time, where a negative value $x < 0$ means a buy objective and a positive value $x > 0$ a sell objective. In this case, the constraint on trading strategies consists in starting in x and being 0 at terminal time.

Still, one often imposes further (application-motivated and/or technical) conditions on trading strategies. In some literature (e.g., [OW13], [BF14], [PSS11]), only pure buy or pure sell strategies are considered. In mathematical terms, such works only admit monotone functions. Others (e.g., [FSU14], [GZ15]) choose strategies that are composed of a pure buy and a pure sell strategy. In [Alm12], strategies are assumed to be absolutely continuous and therefore are fully described by their derivative, called trading rate. Also in, for example, [GH17] and [HX19], strategies are given via a trading rate. In contrast, the strategies in [OW13, Section 6] have an absolutely continuous component and a jump component, where jumps of the trading strategy are called block trades.

Observe that most assumptions found in the literature restrict strategies to be, in particular, of finite variation. Rarely, more general strategies are taken into account. For instance, in [LS13] semimartingales are considered as strategies in a model that extends [OW13] to an underlying semimartingale price process. An extension of proceeds of a large investor from continuous finite-variation strategies to more general classes is investigated in [BBF19]. Moreover, strategies of infinite variation show up in [HK21] when an instantaneous price impact factor tends to zero.

Cost criterion

Before searching for optimal strategies among the respective class of admissible strategies, one needs to specify a criterion for optimality. In view of the context, this should certainly involve the (expected) execution costs, where the expression for the execution costs is tied to other modeling choices such as the set of admissible strategies and the impact of trading on the price development. Besides that, the optimality criterion can also contain further aspects. In addition to the strict requirements on admissible strategies as discussed above, it is possible at this place to incorporate some preferences on the strategies.

This in particular allows to model risk-aversion of the agent. For example, [AC01] and [Alm12] use a mean-variance criterion. Although the main part of [OW13] deals with expected overall execution costs, [OW13, Section 8.3] also contains a result for mean-variance minimization. The works [SST10] and [SS09] show how to perform expected utility optimization in a model of the type of [AC01].

Moreover, there are articles such as [AK15], [GH17], [HX19], and [HK21] that include a quadratic risk term into the formulation of the cost criterion. Additionally, a target process can be followed in, e.g., [BSV17] and [BV18]. A risk term with p-th power, $p > 1$, of the position is considered in, e.g., [AJK14] and [GHS18].

Price impact

To set up an expression for the execution costs, one needs to describe how trading according to a strategy affects the price. A common assumption to start with is that the actual price is the sum of an unaffected price one would observe in absence of the agent and a price component which contains the impact of the agent's trading on the price (see, e.g., [OW13], [AC01], [Alm12], [LS13]). This second component is then often called (price) deviation. To avoid the possibility of negative prices, some works (e.g., [BL98, Section 3], [GZ15], [BBF18a]) assume that the price component describing the impact contributes multiplicatively, instead of additively, to the actual price.

Further, the literature distinguishes permanent, instantaneous (also called temporary), and transient price impact (see, e.g., [GS13, Sections 22.3 and 22.4]). Permanent price impact means that each trade induces a lasting and unchanging impact on the price. Especially, the impact of a trade affects this trade and all future trades equally. This kind of price impact can be found, for example, in [BL98, Section 2]. In contrast, if the impact applies only to the trade that provoked it, this is called instantaneous price impact. Instantaneous price impact is for instance considered in [BL98, Section 3]. A popular model type that combines a permanent price impact component and an instantaneous one goes back to Almgren and Chriss (see [AC01]).

Meanwhile, models of Obizhaeva-Wang type (initiated by [OW13]) use transient price impact. As for permanent price impact, the impact of a trade here affects future trades. However, the transient impact induced by a particular trade develops over time. In the model of [OW13], the transient impact of a trade on the price decays exponentially and thus has a stronger influence on trades closely thereafter than it has on trades much later in time. This transient impact is modeled by two components called price impact (coefficient) and resilience (coefficient). Price impact in this sense is the inverse of the order book depth in an underlying order book model, and resilience describes the change of the price deviation after a trade, a phenomenon that has been observed empirically in, e.g., [BHS95], [BGPW04], [Lar07], [LH15], and [May06, Chapter 4].

Note that [OW13] provide a motivation of their model via a simplified limit order book model. In Figure 1.1 we depict a limit order book and explain how to derive a simplification as used in [OW13]. In Figure 1.2 we illustrate the effect of a trade in such a stylized order book. More details are provided in the paragraph "A static order book model à la Obizhaeva-Wang" after next.

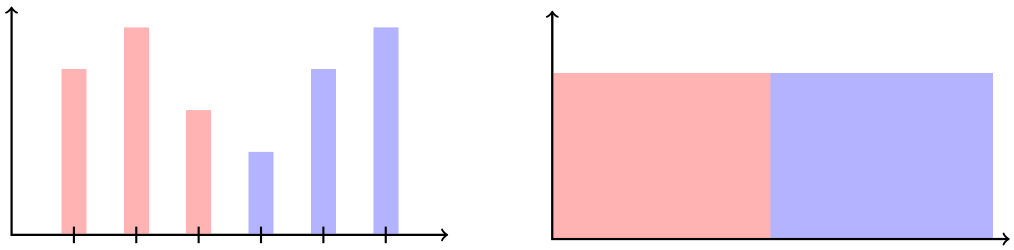


Figure 1.1: From a limit order book (left) to a simplified block-shaped order book model (right). The bid side of the order book is colored in red, whereas the ask side is colored in blue. In the limit order book (left), the height of the bar at each tick represents the amount of shares that is currently available in the order book on the respective price level. To obtain a simplification (right) as in [OW13], we assume that the bars have all the same (positive, real-valued) height and that the spacing between price levels becomes infinitesimally small.

Limit order books

Limit order books are used to store the limit orders of all market participants for a particular asset in an electronic market. There are different types of orders that a market participant can place, with limit orders and market orders being the most important ones.

A limit order is an order to buy (or sell) a certain amount of shares at a chosen price level. Here, the amount and the price level are specified by the agent, but the time when the order is executed depends on availability in the market. Until a limit order can be matched with orders of other market participants, it stays in the limit order book.

Market orders, on the other hand, are executed immediately (on a first come first serve basis) against the best limit orders available in the order book. This means that, at the expense of having to accept the current price in the market, the agent does not need to wait for their order to be executed. The agent in most literature on optimal trade execution can only use such market orders (exceptions are, e.g., [GLFT12, BL14, CJ15]).

To introduce further terminology, the bid side of the order book contains the stored buy orders, whereas the ask side comprises the stored sell orders. The best bid price is the highest price for which one can find a buy order stored in the order book. Similarly, the best ask price is the lowest price at which one can buy from the order book. The distance between the best ask and the best bid price is referred to as bid-ask spread. An order book model is called symmetric if the ask side resembles a reflection of the bid side. It is said to be block-shaped if the amount of shares available is the same for all price levels on the respective side (more general shapes are considered in, e.g., [AFS10, AS10, PSS11, AA14]). The height of the blocks in a symmetric block-shaped order book is called order book depth (or market depth).

A static order book model à la Obizhaeva-Wang

As a specific example and as a preparation for the control problems considered in this thesis, we now explain a variant of the order book model by Obizhaeva and Wang; this is not exactly the same formulation as in [OW13], but it illustrates the basics for models of Obizhaeva-Wang type. To this end, we fix some terminal time $T > 0$, and we let the order book depth q and the resilience coefficient ρ be strictly positive deterministic constants as in [OW13]. In [OW13], $\frac{1}{q}$ is split up in a permanent and a transient price impact coefficient, both assumed to be nonnegative deterministic constants. We consider only a transient price impact coefficient, which then is given by $\gamma = \frac{1}{q}$ (see also Remark 2.1.5 or Remark 5.1.2 for inclusion of a constant deterministic permanent price impact coefficient), and we will often call γ just price impact. Together, the order book parameters γ and ρ will describe the transient price impact of trading.

Let $x \in \mathbb{R}$ be the initial position with the meaning that $|x|$ is the amount of shares to be liquidated (if $x > 0$) or to be acquired (if $x < 0$) over the trading period $[0, T]$. In the following, we consider strategies $X = (X_s)_{s \in [0-, T]}$ that are càdlàg and of finite variation, equipped with the initial position $X_{0-} = x$, and required to meet $X_T = 0$. The interpretation is that for each time $s \in [0, T]$, the quantity $|X_{s-}|$ describes the amount of shares that the agent would have to sell (if $X_{s-} > 0$) or buy (if $X_{s-} < 0$) at time s in order to close the position. A jump of the strategy X at time $s \in [0, T]$ is interpreted as a block trade and denoted by $\Delta X_s = X_s - X_{s-}$.

As a meta-model for the impact of trading on the price, we assume that the actual price of a share is the sum of an unaffected price S^0 that is a càdlàg martingale and a deviation D^X, so that $S^0_t + D^X_{t-}$ is the price immediately prior to trading at time $t \in [0, T]$ and $S^0_t + D^X_t$ is the price immediately after trading at time t (a block trade ΔX_t becomes effective only immediately after a possible jump of S^0 at time t, which is economically reasonable, see also [LS13, Remark 2.1]). For simplicity, let S^0 be equivalent to 0 in what follows; this reduction is essentially without loss of generality (see also, e.g., Remark 2.1.4 or Remark 5.1.1).

We want to derive a control problem from trading in the simplified block-shaped order book model of Figure 1.1 with the constant deterministic order book depth $q > 0$. Suppose that the agent performs a block trade $\Delta X_t > 0$, i.e., a buy trade (the case of a sell trade works analogously), at time $t \in [0, T]$. This market order is matched with the best limit sell orders stored in the order book, taking them away. As visualized in Figure 1.2, this leads to an increase in the deviation from D^X_{t-} immediately prior to the trade to D^X_t afterwards. To obtain the shift of the deviation, we consider the volume removed from the order book

$$\Delta X_t = \int_{D^X_{t-}}^{D^X_t} q \, dy = (D^X_t - D^X_{t-}) q,$$

which gives $\Delta D_t^X = \gamma \Delta X_t$. If the agent does not trade between time t and time $s > t$, the deviation is assumed to decay exponentially at the constant deterministic rate $\rho > 0$:

$$D_s^X = D_t^X e^{-\rho(s-t)}.$$

In the case where trading is only allowed at given times $t_0 < t_1 < \ldots < t_N$ for some $N \in \mathbb{N}$ and $[t_0, t_N] \subseteq [0, T]$, i.e., if the strategy consists only of block trades at $t_0 < t_1 < \ldots < t_N$, then the deviation $D^X = (D_s^X)_{s \in [0-,T]}$ can be expressed as

$$D_s^X = e^{-\rho s} d + \sum_{t_j \leq s} e^{-\rho(s-t_j)} \gamma \Delta X_{t_j}, \quad s \in [0-, T], \tag{1.1}$$

where $d \in \mathbb{R}$ is the initial deviation (typically $d = 0$) with which the agent enters the trading period $[0, T]$. For trading according to a càdlàg finite-variation strategy X, this construction naturally leads to the formulation

$$dD_s^X = -\rho D_s^X ds + \gamma dX_s, \quad s \in [0, T], \quad D_{0-}^X = d. \tag{1.2}$$

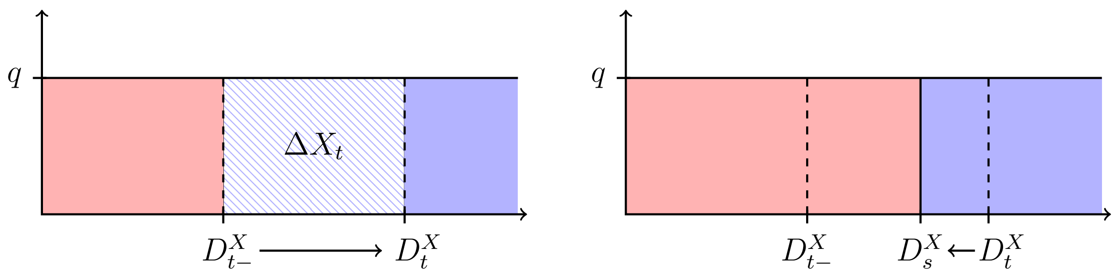


Figure 1.2: Visualization of trading in a (stylized) symmetric block-shaped limit order book model at time t (and then waiting until time $s > t$). The price is on the horizontal axis. The order book depth is depicted on the vertical axis. The red block on the left-hand side stands for the limit buy orders, the blue block on the right-hand side for the limit sell orders stored in the order book. Left: Observe that the buy trade ΔX_t takes away the left-most part of the sell-order block and shifts the price from D_{t-}^X to D_t^X. Right: After having waited until time $s > t$, some of the ask side has been replenished by new limit sell orders (and the bid side has closed the remaining gap, since we assume a model with zero bid-ask spread).

The costs to be paid for the trade ΔX_t correspond to

$$\int_0^{\Delta D_t^X} (D_{t-}^X + y) q \, dy = D_{t-}^X \Delta X_t + \frac{1}{2q} (\Delta X_t)^2 = \left(D_{t-}^X + \frac{\gamma}{2} \Delta X_t \right) \Delta X_t,$$

which is the same as if the agent would buy all ΔX_t shares at the mid-price

$$D_{t-}^X + \frac{\gamma}{2} \Delta X_t$$

between D^X_{t-} and D^X_t. To obtain the execution costs, we have to consider the costs accumulated by all trading activities of the agent during the whole trading interval $[0,T]$. In the setting where the agent can trade only with block trades at given times $t_0 < t_1 < \ldots < t_N$, this leads to the execution costs

$$\sum_{j=0}^{N} \left(D^X_{t_j-} + \frac{\gamma}{2}\Delta X_{t_j} \right) \Delta X_{t_j}. \tag{1.3}$$

When the agent more generally can use a càdlàg finite-variation strategy X, then the execution costs amount to

$$\int_{[0,T]} \left(D^X_{s-} + \frac{\gamma}{2}\Delta X_s \right) dX_s. \tag{1.4}$$

We can now set up the following continuous-time (deterministic) control problem: Let the set of functions $X = (X_s)_{s\in[0-,T]}$ that are càdlàg and of finite variation with $X_{0-} = x$ and $X_T = 0$ form the set of admissible strategies. Consider then minimization of the execution costs (1.4) subject to the deviation dynamics (1.2) over all admissible strategies.

Note that in stochastic settings where, e.g., γ, ρ, or admissible strategies are stochastic quantities, one would consider the expected value (or conditional expected value at initial time) in (1.4), as we do in the body of this thesis.

By restricting the set of admissible strategies to functions $X = (X_s)_{s\in[0-,T]}$ that are càdlàg, that satisfy $X_{0-} = x$ and $X_T = 0$, and that are constant except for jumps at the times $t_0 < t_1 < \ldots < t_N$, we get a discrete-time problem that is embedded in the continuous-time problem. It is worth noting that this in general leads to a different optimization problem whose solution does not simply follow from the one of the continuous-time problem.

Control problems from optimal trade execution

To summarize the introduction so far, we usually have the following set-up in control problems coming from optimal trade execution. The price (or, more common, related quantities such as the deviation) is taken as the state. Trading strategies (the position or related quantities such as trading rate or trade sizes) act as the control for the state. The aim is to minimize a cost functional, which contains execution costs (and possibly further ingredients such as a risk term), over all admissible trading strategies (which typically transform a given initial position into a specific terminal position). Less commonly, the problem is formulated (equivalently) as a maximization problem.

One often uses a dynamic formulation where the initial time, the initial position, and the initial deviation are regarded as variables. The cost functional then is a function of the strategy, the initial time, the initial position, the initial deviation, and, if one considers conditional expectations, the sample space Ω of the underlying filtered probability space. In the example above, we would have the cost functional

$$J_t(x,d,X) = \int_{[t,T]} \left(D^X_{s-} + \frac{\gamma}{2}\Delta X_s\right) dX_s, \quad t \in [0,T], x,d \in \mathbb{R}, X \in \mathcal{A}_t(x,d),$$

where we denote, for $t \in [0,T]$ and $x,d \in \mathbb{R}$, by $\mathcal{A}_t(x,d)$ the class of admissible strategies, i.e., of càdlàg finite-variation functions $X = (X_s)_{s\in[t-,T]}$ with $X_{t-} = x$ and $X_T = 0$, and $D^X = (D^X_s)_{s\in[t-,T]}$ is given by $dD^X_s = -\rho D^X_s ds + \gamma dX_s$, $s \in [t,T]$, $D^X_{t-} = d$. For each initial time, initial position, and initial deviation, the (possibly also dependent on Ω) value function provides us with the minimal costs; it is defined as the (essential) infimum of the cost functional (for these initial values) over all admissible strategies. In the example above, the value function would be

$$V_t(x,d) = \inf_{X\in\mathcal{A}_t(x,d)} J_t(x,d,X), \quad t \in [0,T], x,d \in \mathbb{R}.$$

If, given an initial time, initial position, and initial deviation, there exists an admissible strategy for which the cost functional attains its (essential) infimum (within the set of all admissible strategies), this strategy is called an optimal strategy. To stay with our example, given $t \in [0,T]$ and $x,d \in \mathbb{R}$, an optimal strategy (if existent) would be an element $X^* \in \mathcal{A}_t(x,d)$ such that

$$J_t(x,d,X^*) = \inf_{X\in\mathcal{A}_t(x,d)} J_t(x,d,X).$$

We remark that a particular difficulty in many control problems from optimal trade execution arises from the requirement that a given position has to be reached at terminal time, which creates a nontrivial restriction on the set of admissible controls.

Some optimal trade execution models, in particular models of Obizhaeva-Wang type for a block-shaped limit order book, lead to control problems of a linear-quadratic kind: e.g., observe that the deviation in (1.1) is linear in the trade sizes, and that the costs in (1.3) are quadratic in the pair of deviation and trade sizes. Similar observations hold for the variant (1.4) with (1.2) of this problem. Note that there, the strategy comes in via its jump process and furthermore as integrator both in the state dynamics and in the cost functional, and thus its influence is comparable to the one of the trade sizes in (1.3) with (1.1). These observations also apply to, e.g., the related to (1.1)&(1.3), respectively related to (1.2)&(1.4), stochastic control problems of Section 2.1, respectively of Section 7.1, in the body of this thesis.

The problem in Section 7.1 is not a standard stochastic control problem, but we will show in Chapter 8 how to derive a related standard LQ stochastic control problem, where now the control – and not the state, as before – is (a scaled version of) the deviation. We in this thesis use the term "standard stochastic control problem" for control problems where the state is driven by a controlled stochastic differential equation (SDE) (with a drift and a diffusion term) and the control is a progressively measurable process (not restricted to meet a terminal goal) that acts as one of the arguments in that SDE and as one of the arguments in the integrand of the target functional. Problems in this standard form are rather typical for the literature on stochastic optimal control. For less familiar readers, we suggest, e.g., [YZ99] or [Pha09] (and similar resources) for background on stochastic control theory. In particular, let us mention that in the linear-quadratic case there is a tight connection between standard LQ stochastic control problems and Riccati-type backward stochastic differential equations, dating back at least to the works [Bis76] and [Bis78] by Bismut.

1.2 From constant via time-varying towards stochastic order book parameters

Early versions of optimal trade execution models (e.g., [BL98], [AC01], [OW13]) assume the parameters describing the impact of a trade on the price to be deterministic and constant in time. However, it is established (see also, e.g., [CRS01], [ABC01], [LO09], [Alm12]) that liquidity varies over time, exhibits among others intra-day patterns, and can be stochastic. To reproduce market activity more realistically, an active direction of research on optimal trade execution thus is to incorporate randomly evolving liquidity features. In the sequel, we review the development from constant via time-varying towards stochastic parameters for the model of [OW13] in greater depth. Works on optimal trade execution in other models with stochastic parameters include, but are not limited to, [Alm12, AJK14, AK15, BV18, CS14, GHQ15, GH17, GHS18, HQZ16, HX19, KP16b, PZ19, Sch13, HK21, BBF18b].

For the model of [OW13] (note that an earlier version of the work [OW13] appeared in SSRN already in 2005), an extension of the resilience coefficient from a strictly positive constant to a deterministic, time-varying, strictly positive function is analyzed in [AFS08]. As in [OW13], Alfonsi, Fruth, and Schied in [AFS08] assume a symmetric block-shaped limit order book model (with possible bid-ask spread) and consider additive price impact with a fraction of the price impact being permanent and the other part being transient with exponential resilience. The price impact coefficients are taken to be constants in [AFS08], just as in [OW13]. The unaffected price in [AFS08] is assumed to be a martingale and can have jumps, which is more general than the Bachelier model in [OW13]. Both works assume a fixed, finite time horizon. In [AFS08], trading is allowed at finitely many given time points, which do not need to be equally spaced. A trading strategy thus is described by the collection of trade sizes for each of these time points. Without loss of generality, a buy objective is assumed. Trading is allowed in both directions and with random, adapted sizes (however, optimal strategies turn out to be deterministic pure buy strategies). Trade sizes need to be bounded from below and strict liquidation is required of admissible strategies. The aim in [AFS08] is to minimize expected overall trading costs. Alfonsi, Fruth, and Schied show that cost minimization in their model reduces to the minimization of a certain quadratic form, which makes it possible to include additional linear constraints on admissible strategies. This problem is then treated using the Kuhn-Tucker theorem.

Alfonsi and Schied moreover investigate a similar, in certain aspects more general, model in [AS10], with a view towards existence of price manipulation. The price impact coefficient is constant in time and deterministic, and the resilience coefficient is a deterministic, time-varying, strictly positive function, both as in [AFS08]. The two main differences are that the order book model in [AS10] is not restricted to be block-shaped, and that an admissible strategy in [AS10] consists of a sequence of a fixed number of nondecreasing stopping times within a fixed, finite time horizon, and corresponding trade sizes (satisfying assumptions as in [AFS08]). Optimal strategies (for a buy objective) are found to be deterministic pure buy strategies with trades at homogeneously (with respect to the averaged resilience rate between consecutive trades) spaced time points.

Alfonsi and Acevedo in [AA14] extend [AS10] to time-dependent price impact. More precisely, they assume exponential resilience with a deterministic, time-varying, strictly positive, continuously differentiable resilience coefficient, and a price impact that is a deterministic, twice continuously differentiable, strictly positive function of time (multiplied by a deterministic, constant in time shape function in case of a non-block-shaped order book model). For discrete-time trading, admissible strategies are the same as in [AFS08]. The solution approach to the discrete-time problem is in the spirit of [AFS08] and [AS10]. Furthermore, Alfonsi and Acevedo consider a continuous-time version of the problem (in the same article [AA14]). Their admissible strategies in continuous time also require strict liquidation, and are adapted, left-continuous, and have finite variation. The result for the continuous-time problem is obtained from a guess based on the discrete-time solution and subsequent verification. Optimal strategies in both cases are deterministic, and Alfonsi and Acevedo provide conditions under which they are monotone. Furthermore, optimal strategies in continuous time in general have block trades at the beginning and at the end of the trading period, but not in between (this is as in [OW13]).

Another work that, too, in an Obizhaeva-Wang type model treats deterministic, time-varying price impact and resilience coefficients is [BF14]. In comparison to [AA14], Bank and Fruth in [BF14] impose stronger assumptions on admissible strategies, but less strict assumptions on resilience and price impact functions to obtain their results. They study a continuous-time problem which is based on a one-sided block-shaped order book model with (only) transient price impact and exponential resilience, where only buy trades are allowed. Thus, admissible strategies need to be nondecreasing. Furthermore, admissible strategies are assumed to be deterministic, right-continuous, and such that the associated overall execution costs, which are to be minimized, are finite. Completion of the buy task in general is only required at infinity, but a fixed time horizon with a given position at this finite terminal time can be enforced by setting the market depth (which corresponds to the inverse of the price impact coefficient) to zero from the desired terminal time on. An unaffected price process is left out of the set-up and only the deviation is introduced, where a nonzero initial deviation is possible. Bank and Fruth first reduce their problem to a convex optimization problem and then obtain the minimal costs, a characterization of existence of optimal strategies, and, in this case, a formula for optimal strategies, by using methods from convex analysis. It is worth noting that their optimal strategies can have block trades inside the trading period.

A two-sided symmetric block-shaped order book model with exponential resilience and with the resilience and the transient price impact coefficients being deterministic time-varying is investigated in [FSU14]. The resilience coefficient is assumed to be a deterministic, strictly positive, Lebesgue-integrable function of time, and the transient price impact coefficient is supposed to be a deterministic, strictly positive, bounded function of time (with more assumptions needed for most continuous-time results). There is also a permanent price impact component, but this is described by a deterministic constant, and the pertaining costs are the same for all strategies. To include a trading-dependent bid-ask spread, Fruth, Schöneborn, and Urusov explicitly model each of the deviations of the unaffected best ask, respectively bid, price, where nonzero initial deviation is possible and the unaffected prices are assumed to be càdlàg martingales. The cost criterion is to minimize expected overall execution costs for a fixed, finite time horizon. Both, a discrete-time and a continuous-time problem, are considered. In continuous time, an admissible strategy initially is a pair of two nondecreasing, adapted, bounded, càglàd processes starting in zero such that the difference of the buy and the sell component reaches a prescribed deterministic terminal value. For discrete time, the set of admissible strategies is restricted to strategies that only trade at a finite number of given times. Fruth, Schöneborn, and Urusov show that mixing buy and sell trades in their model can not be optimal, which reduces the problem. Furthermore, it suffices to consider deterministic strategies. A further reduction concerns the dimension of the arguments of the value function due to linearity of optimal strategies in the initial state. Fruth, Schöneborn, and Urusov provide a characterization of the solution via a wait and a trade region and state a formula for the unique optimal strategy. In discrete time, the proof is based on dynamic programming. The continuous-time results are derived by an approximation from discrete time and need stronger assumptions on the price impact coefficient and the resilience coefficient.

An extension of the Obizhaeva-Wang model to stochastic parameters is analyzed in [Fru11] and [FSU19], though neither of them yet takes price impact coefficient and resilience coefficient both to be stochastic within the same model. The model in [FSU19] exhibits stochastic price impact, but deterministic resilience. Conversely, in the last chapter of her PhD thesis [Fru11, Chapter 4], Fruth discusses the inclusion of stochastic resilience in a simplified model with three trading instances when the price impact coefficient is constant. We also mention [CKW18, Section 3] which numerically deals with optimal trade execution in a setting where the order book depth in a block-shaped order book with deterministic exponential resilience is given by a discrete-time Markov chain.

Fruth, Schöneborn, and Urusov in [FSU19] extend the model of their previous article [FSU14] to a price impact coefficient given by a strictly positive, possibly time-inhomogeneous, Markov process with finite first moments. Resilience, as in [FSU14], is exponential with a deterministic, time-varying, strictly positive resilience coefficient. Note that in contrast to [FSU14], it is assumed in [FSU19] already from the beginning on that there is no permanent price impact component and that admissible strategies (except for the ones in [FSU19, Section 8]) are pure buy strategies. As in [FSU14], a discrete-time and a continuous-time variant of the problem are analyzed. In line with the result in the deterministic case [FSU14], the authors find that in a subsetting where the price impact coefficient has a special diffusion structure (which also comprises conditions in relation with the resilience coefficient), the solution is given in terms of a wait and a trade region. However, contrasting [FSU14], they provide examples that for more general, necessarily nondeterministic, specifications of the price impact coefficient, there can arise situations where this is no longer true; e.g., there can be multiple wait regions. Moreover, note that optimal strategies in [FSU19] can be nondeterministic. Several intermediate results in [FSU19] are cognate with the ones in [FSU14], e.g., that the dimension of the value function can be reduced by one, that the cost functional admits a helpful alternative representation, and that the continuous-time results can be approximated from discrete time. On the other hand, the proof techniques in [FSU19] differ from those in [FSU14] due to the stochastic setting in [FSU19].

1.3 Overview and contribution of this thesis

Our work continues the above stream of literature on Obizhaeva-Wang models by taking price impact and resilience both to be stochastic processes (in a symmetric block-shaped order book model with zero bid-ask spread and fixed, finite terminal time). We first study a discrete-time model in Chapter 2 and then devote the remainder of the thesis to the continuous-time case, where we consider semimartingale strategies in Chapter 5 and Chapter 6, and progressively measurable strategies in Chapter 7 and Chapter 8. For the continuous-time case, we moreover analyze a certain Riccati-type backward stochastic differential equation (BSDE) in Chapter 4.

The work presented in this thesis is based on the publications [AKU21b, AKU21a, AKU22b] and the preprint [AKU22a], all of which are joint work with Thomas Kruse and Mikhail Urusov.

Discrete time with stochastic price impact and resilience

In Chapter 2, we study optimal trade execution in an order book model in the sense of [OW13] in discrete time where price impact $(\gamma_k)_{k\in\mathbb{Z}}$ and resilience $(\beta_k)_{k\in\mathbb{Z}}$ are positive, adapted, sufficiently integrable processes (see Section 2.1). For interpretation of the resilience process β, note that exponential resilience with resilience coefficient ρ as in the example of Section 1.1 corresponds to the special case $\beta_k = e^{-\rho}$, $k \in \mathbb{Z}$. The stochastic control problem we look at is a linear-quadratic one with value function V defined in (2.4):

$$V_n(x,d) = \operatorname*{ess\,inf}_{X\in\mathcal{A}_n^{\text{disc}}(x,d)} E_n\left[\sum_{j=n}^{N}\left(D_{j-}+\frac{\gamma_j}{2}\xi_j\right)\xi_j + \sum_{j=n}^{N}\gamma_j\lambda_j(X_j-\zeta_j)^2\right],$$
$$n\in\mathbb{Z}\cap(-\infty,N], x\in\mathbb{R}, d\in\mathbb{R},$$

subject to the deviation evolution defined in (2.1):

$$D_{n-} = d \quad \text{and} \quad D_{k-} = (D_{(k-1)-}+\gamma_{k-1}\xi_{k-1})\beta_k, \quad k\in\{n+1,\dots,N\}.$$

$N \in \mathbb{N}$ denotes the (fixed) terminal time. The second sum in the value function is a risk term (with appropriate stochastic processes λ and ζ) that we will discuss later in this introduction and that can be ignored at the moment. Optimization happens over the set $\mathcal{A}_n^{\text{disc}}(x,d)$ of real-valued adapted stochastic processes $X = (X_k)_{k\in\{n-1,n,\dots,N\}}$ with $X_k \in L^{2+}(\mathcal{F}_k)$ for all $k \in \{n,\dots,N\}$ that are equipped with initial position $X_{n-1} = x$ and satisfy $X_N = \hat{\xi}$, where $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$ is the terminal position to be achieved through trading. $\xi_j = X_j - X_{j-1}$, $j \in \{n,\dots,N\}$, denote the trade sizes that correspond to such a strategy $X \in \mathcal{A}_n^{\text{disc}}(x,d)$. Note that trading is allowed in both directions, but is only possible at a fixed set of finitely many time points.

We are able to solve this optimization problem in that, under appropriate conditions (see Theorem 2.2.1), we obtain existence of a unique optimal strategy and a characterization of the optimal strategy and the value function in terms of a process $Y = (Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ and a process $\psi = (\psi_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ that are defined by backward recursion (see (2.7), respectively (2.8)) with terminal value $Y_N = \frac{1}{2}$, respectively $-\frac{1}{2}\sqrt{\gamma_N}\hat{\xi}$. The proof of this result is based on dynamic programming and the quadratic structure of the problem. An ansatz for the value function V_n at time n as a (bivariate) quadratic function of the pair of initial position and initial deviation $(x,d)\in\mathbb{R}^2$, and an application of the dynamic programming principle, lead to recursive descriptions for the coefficients in the value function, and to a characterization of the optimal trade size $\xi_n^*(x,d)$ at time n in the pair of initial position and initial deviation $(x,d)\in\mathbb{R}^2$ as the minimizer of a (univariate) quadratic function.

Long-time horizon

It furthermore turns out that Y_n (in the basic setting of [AKU21b] where $\hat{\xi} = 0$ and $\lambda \equiv 0$) can be interpreted as the (divided by 2) ratio between, in the denominator, the costs for selling $x = 1$ unit immediately at initial time n with initial deviation $d = 0$ and, in the numerator, the minimal costs for the same task; i.e., $Y_n = \frac{1}{2}\frac{V_n(1,0)}{\gamma_n/2}$. To determine how much better than immediate execution our optimal strategies perform in the long run, we then investigate the long-time limit $\lim_{n\to-\infty} Y_n$. We observe that this limit does not always exist, see Lemma 2.4.3 for such a situation. Existence is guaranteed if the price impact process $(\gamma_k)_{k\in\mathbb{Z}\cap(-\infty,N]}$ up to terminal time N is a supermartingale, i.e., when the liquidity in the model increases in time on average, see Proposition 2.4.1.

In the "time-homogeneous expectations"-setting of Proposition 2.4.2, Y is deterministic, the limit also exists, and we compute the limit explicitly. We find that there are three different subcases depending on the relation to 1 of the average resilience and of the average multiplicative increments of the price impact process. In particular, if the resilience is 1 in expectation throughout the trading period, which means that the impact of trades on the price is expected to be permanent, then $Y_n = \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$ (see also Corollary 2.3.2), and thus selling the unit immediately is optimal. If the price impact process is nonincreasing on average, which due to a structural assumption in our model (cf. Theorem 2.2.1) entails that the resilience is smaller than 1 on average, then our minimal execution costs vanish asymptotically in the sense that $\lim_{n\to-\infty} Y_n = 0$.

Round trips

We moreover investigate if, in our model, trading can be beneficial although one has no open position. Formulated differently, this is the question on existence of profitable round trips, or yet the existence of price manipulation. The notion of price manipulation in optimal trade execution models was coined in [HS04] and further studied in, e.g., [Gat10] and [AS10].

We have as a direct consequence of Theorem 2.2.1 that our model does not exhibit price manipulation whenever the initial deviation is 0 (cf. (2.53)). This is in line with the findings in [AS10, Corollary 2.8 and Remark 3.2], where it is established that price manipulation is not possible in a block-shaped Obizhaeva-Wang type model with zero bid-ask spread, constant price impact, and time-varying (possibly stochastic) exponential resilience. Our result extends this to stochastic price impact and more general forms of resilience.

However, if prior to the trading period, the agent has already induced some price deviation, then round trips can become profitable under some market conditions (also for constant price impact and exponential resilience); see Section 2.5 for details. Similar conclusions in related models were obtained in [FSU14, Section 8] and [FSU19, Section 8].

To decide whether there exist profitable round trips at initial time n for initial deviation $d \neq 0$ in our model, it suffices to study the event $\{Y_n = \frac{1}{2}\}$, as we explain at the beginning of Section 2.5 (cf. (2.53)). We thus characterize this event in Proposition 2.5.2

and discuss consequences in subsequent results and examples. E.g., in the "processes with independent multiplicative increments"-setting where for all $k \in \mathbb{Z} \cap (-\infty, N]$, the resilience β_k and the multiplicative increment of the price impact, $\frac{\gamma_k}{\gamma_{k-1}}$, are independent of the sigma-algebra $\mathcal{F}_{k-1}$ (see Section 2.3), existence of profitable round trips for nonzero initial deviation can be decided based on the resilience process alone (see Corollary 2.5.5).

Closing the position in one go

We also look at the question under which conditions one should close the position in one go, i.e., when it is optimal to execute the outstanding order at once. To this end, we consider the event $\{\xi_n^*(x,d) = -x \,\forall x, d \in \mathbb{R}\}$. Again, the process Y plays a prominent role in the description of this event (see Proposition 2.6.2). Besides, Proposition 2.6.2 (note also (2.61)) yields a connection between the existence of profitable round trips for initial deviation $d \neq 0$ and optimality of closing the position in one go.

On the event $\{Y_n = \frac{1}{2}\}$, where round trips for initial deviation $d \neq 0$ can not be profitable, we have that immediate closure is always optimal. However, we show in Example 2.6.7 that it can be optimal to close any position in one go although there exist profitable round trips for $d \neq 0$. In general, optimality of closing the position in one go does not necessarily mean to stop trading entirely after the closure. For instance, in the situation of Example 2.6.7, it is optimal to build up a new position at the next time point.

From Lemma 2.6.1, which is a direct consequence of Theorem 2.2.1, we derive in Proposition 2.6.4 that $\{\xi_n^*(x,d) = -x \,\forall x, d \in \mathbb{R}\}$ is either Ω or $\emptyset$ in the "processes with independent multiplicative increments"-setting (but the optimal trade sizes can still be random). In this setting we also provide equivalent statements to closing in one go.

Continuous time with stochastic price impact and resilience

The base setting for continuous time (see Section 3.1) includes a price impact process $(\gamma_s)_{s\in[0,T]}$ and a resilience process $(R_s)_{s\in[0,T]}$ ($T \in (0,\infty)$ is the fixed terminal time). These stochastic processes are assumed to possess a certain structure (cf. (3.2) and (3.1)):

$$d\gamma_s = \gamma_s \mu_s d[M^{(1)}]_s + \gamma_s \sigma_s dM_s^{(1)}, \quad s \in [0,T], \quad \gamma_0 \in (0,\infty),$$

and

$$dR_s = \rho_s d[M^R]_s + \eta_s dM_s^R, \quad s \in [0,T], \quad R_0 = 0,$$

where $M^{(1)}, M^R$ are continuous local martingales (Brownian motions in Chapter 6–Chapter 8) with $[-1,1]$-valued correlation $\overline{r}$, and $\mu, \sigma, \rho, \eta, \overline{r}$ are progressively measurable, sufficiently integrable (often bounded) processes. While the price impact processes in discrete and in continuous time have the same interpretation, we point out that the resilience process R has a slightly different meaning than β from discrete time: the multiplicative increments of the stochastic exponential of $-R$ are comparable to β.

When the choice of the set of admissible strategies in discrete time was rather straightforward, this becomes an interesting aspect in continuous time. As in discrete time, we do not want to restrict trading to one direction. Furthermore, we expect that optimal strategies should respond quickly to fluctuations in the market conditions. As the price impact process (and also the resilience process) in our model can have infinite variation, we therefore aim to include strategies that can have infinite variation. Our decision is backed up by empirical evidence and mathematical motivation in favor of infinite-variation strategies/inventories in similar situations – we refer to [CW19], [CL21] and to [LS13], [GP16], [BBF19], [HK21], [FHX22a].

However, a vast part of the literature on optimal trade execution considers strategies to be, in particular, of finite variation, and it is not obvious how to formulate (and later, solve) an appropriate control problem for strategies of infinite variation. We take two approaches.

Càdlàg semimartingale strategies

In the first approach, since the conventional, finite-variation formulation (5.7)&(5.8) (compare also with (1.2)&(1.4)) of the control problem contains the strategy in the integrator, we assume our strategies to be càdlàg semimartingales. Further, we demand of admissible strategies $X = (X_s)_{s\in[t-,T]} \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$ (where $t \in [0,T]$, $x, d \in \mathbb{R}$) the initial position $X_{t-} = x$, strict liquidation $X_T = 0$, and that certain integrability conditions are satisfied (see Section 5.1.1).

We show in Example 5.1.4 and Example 5.1.6 that, for strategies with infinite variation, the conventional, finite-variation formulation (5.7)&(5.8) of the control problem can result in an ill-posed optimization problem. With the modified deviation dynamics (5.1):

$$dD_s^X = -D_s^X dR_s + \gamma_s dX_s + d[\gamma, X]_s, \quad s \in [t,T], \quad D_{t-}^X = d,$$

and the modified cost functional (5.2):

$$\begin{aligned} J_t^{\mathrm{sem}}(x,d,X) = E_t\left[\int_{[t,T]} D_{s-}^X dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s - \int_t^T D_s^X d[X,R]_s\right] \\ + E_t\left[\int_t^T \gamma_s \lambda_s X_s^2 d[M^{(1)}]_s\right], \quad t \in [0,T], x, d \in \mathbb{R}, X \in \mathcal{A}_t^{\mathrm{sem}}(x,d), \end{aligned}$$

we provide an appropriate formulation of the control problem for càdlàg semimartingale strategies, which is motivated by a heuristic limit from our discrete-time model (see Section 3.2). The last term in the cost functional J^{sem} is a risk term to be discussed later, with an appropriate stochastic process λ.

We solve this control problem by purely probabilistic means, see Theorem 5.2.6 for the main result. Under appropriate assumptions, the value function $V_t^{\mathrm{sem}}(x,d) = \operatorname{ess\,inf}_{X\in\mathcal{A}_t^{\mathrm{sem}}(x,d)} J_t^{\mathrm{sem}}(x,d,X)$, $t\in[0,T]$, $x,d\in\mathbb{R}$, has a representation very similar to the representation of the value function in the discrete-time problem, with a process $Y=(Y_s)_{s\in[0,T]}$ that is the first solution component of the quadratic BSDE (4.1) (in analogy to the discrete-time process defined by backward recursion in Theorem 2.2.1, see also Section 3.3).

To obtain the representation for the value function V^{sem}, we first introduce Y into the cost functional J^{sem}, which in Theorem 5.2.1 leads to the representation (5.23)

$$\begin{aligned}
J_t^{\mathrm{sem}}(x,d,X) &= \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}\\
&\quad + E_t\bigg[\int_t^T \frac{1}{\gamma_s}\Big(\widetilde{\vartheta}_s(\gamma_s X_s - D_s^X) + D_s^X\Big)^2 \big((\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)Y_s+\kappa_s+\lambda_s\big)\, d[M^{(1)}]_s\bigg],\\
&t\in[0,T], x,d\in\mathbb{R}, X\in\mathcal{A}_t^{\mathrm{sem}}(x,d),
\end{aligned}$$

of the cost functional J^{sem} as the sum of the (later to be identified) minimal costs and a second, nonnegative term. Therein, $\kappa=\frac{1}{2}(2\rho+\mu-\sigma^2-\eta^2-2\sigma\eta\overline{r})$ (see (3.6)), and

$$\widetilde{\vartheta} = \frac{(\rho+\mu)Y+(\sigma+\eta\overline{r})Z^{(1)}+\eta\sqrt{1-\overline{r}^2}Z^{(2)}+\lambda}{(\sigma^2+\eta^2+2\sigma\eta\overline{r})Y+\kappa+\lambda}$$

(see (5.22), compare also with the driver (4.2) of BSDE (4.1)), where $(Z^{(1)},Z^{(2)})^\top$ is the second solution component of BSDE (4.1). In particular, this representation for J^{sem} implies a lower bound for the value function V^{sem}. We subsequently argue that there is equality by approximating the second term in the representation of the cost functional J^{sem}. More precisely, we show in Lemma 5.2.9 that the auxiliary process $\widetilde{\vartheta}$ can be approximated by a sequence of càdlàg semimartingales $(\vartheta^n)_{n\in\mathbb{N}}$. Based on this sequence $(\vartheta^n)_{n\in\mathbb{N}}$, we further define a sequence of strategies $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}_t^{\mathrm{sem}}(x,d)$ for which we, in Lemma 5.2.10, establish certain helpful properties for the convergence of the second term in the representation of the cost functional J^{sem} to zero.

For the characterization of existence of a minimizer and for the formula of optimal strategies, we make use of the representation of the cost functional J^{sem} in combination with the representation of the value function V^{sem}. It turns out that (under the overall assumptions of Theorem 5.2.6) there exists an optimal strategy if and only if $\widetilde{\vartheta}$ is equal $\mathcal{D}_{M^{(1)}}$-a.e. to a càdlàg semimartingale ϑ. In this case, the ($\mathcal{D}_{M^{(1)}}$-a.e. unique) optimal strategy is given by (5.36). This is a product of three factors, two of which are of particular interest when we examine properties of optimal strategies in Section 5.3, Section 5.4, and Chapter 6.

In particular, we find that in several situations we really obtain optimal strategies of infinite variation. This does not only concern the setting of Example 5.3.1, where price impact and/or resilience have infinite variation, but also certain situations with smooth price impact and resilience (see, e.g., Example 5.3.3 and Example 5.3.4). On the other hand, it is interesting to observe that in the specific setting of Section 5.4.2, infinite variation in the price impact and in the resilience can cancel out such that the optimal strategy has finite variation.

We moreover show how to produce block trades of the optimal strategy inside the trading interval (see Section 5.4.3, but also the examples in Chapter 6). Recall that for models similar to [AC01], jumps of the optimal strategy can not occur at all since admissible strategies are absolutely continuous, and that for models of Obizhaeva-Wang type, it is typical to obtain optimal strategies with jumps only at the beginning and at the end of the trading period.

Furthermore, we observe that for constant deterministic price impact γ (i.e., $\mu \equiv 0 \equiv \sigma$) and for a constant deterministic resilience coefficient $\rho > 0$ (while $\eta \equiv 0$), which is the setting of [OW13], our (in some sense more general) optimization problem results in the same optimal strategy as in [OW13, Proposition 3] (see the relevant subcase in Section 5.4.2).

We also provide an example where, although the value function V^{sem} is finite, an optimal strategy in $\mathcal{A}_t^{\text{sem}}(x,d)$ does not exist (see Section 5.4.1) because there is no càdlàg semimartingale ϑ that $\widetilde{\vartheta}$ is equal to $\mathcal{D}_{M^{(1)}}$-a.e. (cf. Theorem 5.2.6). This example motivates to try to go beyond semimartingales in the formulation of the control problem.

Extension to progressively measurable strategies

In the second approach, we begin with the conventional, finite-variation formulation of the deviation dynamics and costs in Obizhaeva-Wang type models and establish in Theorem 7.5.2 a continuous extension to progressively measurable strategies. The precise formulation of our stochastic control problem for finite-variation strategies is stated in Section 7.1 (compare also with (5.7)&(5.8), and with (1.2)&(1.4)). We here repeat the definition of the deviation (7.3):

$$dD_s^X = -D_s^X dR_s + \gamma_s dX_s, \quad s \in [t,T], \quad D_{t-}^X = d,$$

and the definition of the cost functional (7.4):

$$J_t^{\text{fv}}(x,d,X) = E_t\left[\int_{[t,T]} \left(D_{s-}^X + \frac{\gamma_s}{2}\Delta X_s\right) dX_s\right] + E_t\left[\int_t^T \gamma_s \lambda_s \left(X_s - \zeta_s\right)^2 ds\right],$$
$$t \in [0,T], x,d \in \mathbb{R}, X \in \mathcal{A}_t^{\text{fv}}(x,d).$$

Again, the last term in the cost functional is a risk term that will be discussed later, and λ and ζ are appropriate stochastic processes. The set of admissible strategies $\mathcal{A}_t^{\text{fv}}(x,d)$ (for $t \in [0,T]$, $x,d \in \mathbb{R}$) comprises all càdlàg finite-variation processes $X = (X_s)_{s\in[t-,T]}$ that satisfy suitable integrability conditions and possess initial position $X_{t-} = x$ and terminal position $X_T = \hat{\xi}$ for a suitable, fixed, $\mathcal{F}_T$-measurable $\hat{\xi}$.

The extension of this stochastic control problem for finite-variation strategies to progressively measurable strategies is made possible in the first place by the alternative representations for the deviation and the costs in Proposition 7.2.1, since these remove the strategy from the integrator. Now, these alternative expressions are also well-defined for progressively measurable strategies, which allows us to introduce the extended problem of Section 7.3, with the deviation (7.14):

$$D_s^X = \gamma_s X_s + \nu_s^{-1}\left(d - \gamma_t x - \int_t^s X_r d(\nu_r\gamma_r)\right), \quad s \in [t,T], \quad D_{t-}^X = d,$$

where ν^{-1} is the stochastic exponential of $-R$, and with the cost functional (7.16):

$$J_t^{\mathrm{pm}}(x,d,X) = E_t\left[\frac{1}{2}\gamma_T^{-1}(D_T^X)^2 + \int_t^T \gamma_s^{-1}(D_s^X)^2\kappa_s\,ds + \int_t^T \gamma_s\lambda_s\left(X_s - \zeta_s\right)^2 ds\right] - \frac{d^2}{2\gamma_t},$$
$t \in [0,T], x, d \in \mathbb{R}, X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$,

where $\kappa = \frac{1}{2}(2\rho + \mu - \sigma^2 - \eta^2 - 2\sigma\eta\overline{r})$ (see (3.6)). The superset $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ of $\mathcal{A}_t^{\mathrm{fv}}(x,d)$ consists of all progressively measurable processes $X = (X_s)_{s\in[t-,T]}$ that are equipped with an initial position $X_{t-} = x$, end in $X_T = \hat{\xi}$, satisfy $\int_t^T X_s^2 ds < \infty$ a.s., and whose associated deviation D^X meets $E[\int_t^T \gamma_s^{-1}(D_s^X)^2 ds] < \infty$.

We then in (7.30) define by $\mathbf{d}(X,\widetilde{X}) = (E[\int_t^T (D_s^X - D_s^{\widetilde{X}})^2\gamma_s^{-1}ds])^{\frac{1}{2}}$ a metric $\mathbf{d}$ on the set of progressively measurable strategies $\mathcal{A}_t^{\mathrm{pm}}(x,d)$, where the distance between two progressively measurable strategies $X, \widetilde{X} \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ is measured by some kind of weighted $\mathcal{L}_t^2$-distance between their associated deviation processes $D^X, D^{\widetilde{X}}$. With respect to this metric $\mathbf{d}$, the cost functional J^{pm} is continuous in the strategy, the set of finite-variation strategies $\mathcal{A}_t^{\mathrm{fv}}(x,d)$ is dense in the set of progressively measurable strategies $\mathcal{A}_t^{\mathrm{pm}}(x,d)$, and the set of progressively measurable strategies $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ is complete. This result, Theorem 7.5.2, provides a strong justification for our extended problem, and in particular shows that this problem and the finite-variation problem are equivalent in the sense that their value functions coincide (see Corollary 7.5.3).

To show that the cost functional J^{pm} is continuous in the strategy, we use the convergence of the deviation processes and the convergence in Lemma 7.4.3 of the "scaled hidden deviation processes". For the claim that any progressively measurable strategy $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ can be approximated, in our metric $\mathbf{d}$, by a sequence of finite-variation strategies $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}_t^{\mathrm{fv}}(x,d)$, we, in a sense, approximate the deviation process D^X of the progressively measurable strategy X with the help of Lemma 7.5.4 by a sequence that ends up to consist of deviation processes D^n, $n \in \mathbb{N}$, for the desired finite-variation strategies X^n, $n \in \mathbb{N}$. That the set of progressively measurable strategies $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ with respect to $\mathbf{d}$ is complete essentially comes from completeness of the $\mathcal{L}_t^2$-space of square-integrable, progressively measurable processes.

The previously mentioned scaled hidden deviation process is the continuous process $\overline{H}^X = \gamma^{-\frac{1}{2}}D^X - \gamma^{\frac{1}{2}}X$ that we associate to a strategy $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ and its deviation D^X (see Section 7.4). It is not only relevant for the proof of Theorem 5.2.6, but plays also an important role in the next step.

We observe in Proposition 7.4.2 that the scaled hidden deviation process $\overline{H}^X$ satisfies an SDE that is linear in $(\overline{H}^X, \gamma^{-\frac{1}{2}} D^X)$, and that the cost functional J^{pm} of the extended problem depends in a quadratic way on $(\overline{H}^X, \gamma^{-\frac{1}{2}} D^X)$. Thus, we in Section 8.1.1 reinterpret the process $\gamma^{-\frac{1}{2}} D^X$ as a control process $u \in \mathcal{L}_t^2$ and $\overline{H}^X$ as the associated state process. This leads to a standard LQ stochastic control problem which is equivalent to the extended (and thus also to the finite-variation) problem, see Corollary 8.1.3. Importantly, there is a one-to-one correspondence between square-integrable controls $u \in \mathcal{L}_t^2$ for this standard LQ stochastic control problem and progressively measurable execution strategies $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ for our extended problem. In particular, it is possible to recover the minimizer of the extended problem from the minimizer of the LQ problem (see Corollary 8.1.4).

We then apply stochastic control literature to solve the LQ problem. More exactly, we first transform the LQ problem of Section 8.1.1 with cross-terms to one without cross-terms in Section 8.1.2, and subsequently apply results of Kohlmann and Tang [KT02] (see Section 8.2). Under the assumptions of Theorem 8.2.3, there always exists a ($\mathcal{D}_{W^{(1)}}$-a.e. unique) optimal control, and the optimal control and its associated costs can be described by the BSDE of Riccati-type [KT02, (9)], which in our case corresponds to BSDE (4.1), together with the linear BSDE (8.7). We in Corollary 8.2.4 trace everything back and obtain a ($\mathcal{D}_{W^{(1)}}$-a.e. unique) optimal execution strategy in the class of progressively measurable strategies, given by the formula (8.14).

Further features

Our control problems feature some further details that we now want to highlight.

Negative and diffusive resilience

The discrete-time problem as well as the continuous-time problems exhibit more general types of resilience than the frequently used exponential resilience described by a strictly positive resilience coefficient.

In discrete time, we multiply the deviation $D_{(k-1)-} + \gamma_{k-1}\xi_{k-1}$ directly after a trade ξ_{k-1} at time $k-1$ by β_k to get the deviation D_{k-} immediately prior to the next trade at time k. The case of exponential resilience corresponds to $\beta_k = e^{-\int_{k-1}^{k} \rho_s ds}$ for some resilience coefficient ρ.

Note that we assume β only to be strictly positive (aside from some integrability and from the joint structural assumption with the price impact, see Theorem 2.2.1). In particular, we can have values greater than 1 for the resilience β, which reinforces the price deviation. Also the case $\beta_k = 1$ is allowed, which means that there is no change of the price deviation between the trade at time $k-1$ and the next trade at time k.

In the case of $(0,1)$-valued resilience β, we find that there exist profitable round trips for nonzero initial deviation (cf. Corollary 2.5.4(ii)). This is in accordance with the results in [FSU14] and [FSU19] who consider only $(0,1)$-valued resilience, more precisely, exponential resilience with strictly positive resilience coefficient. In contrast, for $(0,\infty)$-valued resilience β, it can happen that there do not exist profitable round trips for any initial deviation $d \in \mathbb{R}$ (see, e.g., Corollary 2.5.5). A necessary (but not sufficient, cf. Example 2.5.6) condition for nonexistence of profitable round trips for $d \neq 0$ is that the agent expects the resilience to be 1 (cf. Corollary 2.5.4(ii)).

Moreover, we observe that, if in the "processes with independent multiplicative increments"-setting closing in one go is optimal, then the resilience right after this trade is expected to be greater than or equal to 1 (see Corollary 2.6.5). In particular, closing in one go can not be optimal in the conventional setting with $(0,1)$-valued, deterministic resilience β and deterministic price impact γ; for stochastic β, γ the situation can be different, even with $(0,1)$-valued β, see Example 2.6.6 and the preceding discussion. But also in a deterministic situation with now $(0,\infty)$-valued β we can produce closing in one go (see Example 2.6.7).

In continuous time, we describe resilience by the resilience process $dR_s = \rho_s d[M^{(R)}]_s + \eta_s dM^R_s$, $s \in [0,T]$, $R_0 = 0$, which enters the deviation process via the stochastic exponential of $-R$. If $\eta \equiv 0$, we are in the case of exponential resilience with resilience coefficient ρ. Otherwise, the resilience still has an exponential structure (see, e.g., Section 3.2 and Section 5.1.1), but our resilience process R contains an additional diffusion part.

Note that [AKU22a] allows a diffusive resilience, whereas [AKU21a] originally does not. We in this thesis extend the semimartingale setting of [AKU21a] to also include a diffusion part in the resilience. This makes it necessary to adjust the cost functional (5.9) from [AKU21a] to (5.2), i.e., to J^{sem} (see Section 3.2 and Example 5.1.5). Also, in comparison to [AKU21a], we need to consider a more general BSDE (see Chapter 4, motivated by Section 3.3). The respective results and proofs in this thesis are extensions of those in [AKU21a].

With diffusive resilience, we observe two effects (see also [AKU22a, Section 4]). In Example 5.3.1, we see that not only infinite variation in the price impact process, but also diffusive resilience can lead to optimal strategies of infinite variation. In a rather specific setting (see Section 5.4.2), we find that diffusive resilience can override infinite variation from the price impact so that the optimal strategy has finite variation.

Furthermore, we point out that R can take negative values – due to the diffusion part, but also due to the resilience coefficient which we do not restrict to be positive. When R is negative, this means an enhanced price impact. Therefore, a resilience process that is negative during some time can be used to model self-excitement effects where the trading activities of the large investor animate other market participants to trade in the same direction. For more details, see the introduction of Chapter 6.

In Chapter 6 (cf. [AKU22b]), we focus on the resilience coefficient, and investigate the effects of a negative resilience coefficient in a subsetting of the semimartingale problem. To this end, we first obtain existence and structure of the optimal strategy via Theorem 5.2.6, where existence of a solution to the BSDE is guaranteed by Section 4.4. Then, we examine what we call (see Definition 6.1.1) “overjumping zero” and “premature closure” of the optimal strategy, which are defined in terms of the process ϑ from Theorem 5.2.6. Roughly speaking, we show that a necessary condition for overjumping zero or premature closure is to have a negative resilience coefficient at least for some time (see Proposition 6.1.4), while a sufficient condition for that is to have a negative resilience coefficient for some time close to the time horizon T (see Proposition 6.1.6). In a setting with piecewise constant resilience coefficient ρ and a simple price impact process, we further discuss properties of optimal strategies with respect to positive and negative values of ρ (see Section 6.2). In particular, in Proposition 6.2.1, the optimal strategy for initial position $x > 0$ and initial deviation $d = 0$ is strictly increasing (respectively, strictly decreasing) on the regime where ρ is strictly negative (respectively, strictly positive). Moreover, we are able to construct a setting – necessarily with a resilience coefficient that is not everywhere strictly positive – where it is optimal to close the position prematurely, and, after a while, reopen again (see Section 6.3).

For completeness, we mention that without resilience (i.e., $\rho \equiv 0 \equiv \eta$) it is optimal to close the position immediately and quit trading (see Proposition 5.2.3).

Risk term and stochastic targets

To incorporate the possibility that the target position is not known at the beginning of trading but only revealed at terminal time, we allow the prescribed terminal position $\hat{\xi}$ to be a random variable (measurable at terminal time, and with suitable integrability). Situations with random target positions may arise for instance when an airline company buys kerosene on forward markets, not knowing their precise demand beforehand because it depends on factors in the future such as ticket sales and weather conditions. Random variables as terminal targets have also been considered in, e.g., [AK15, Section 3.2], [BSV17], and [BV18].

As in the models of [BSV17] and [BV18], we furthermore include a risk term of the form $E_n[\sum_{j=n}^N \gamma_j \lambda_j (X_j - \zeta_j)^2]$, respectively $E_t[\int_t^T \gamma_s \lambda_s (X_s - \zeta_s)^2 ds]$, into the cost functional; deviations of the position X from the target process ζ are penalized via the risk coefficient process λ (the scaling by the price impact process γ is for convenience). The risk term can be used to model some kind of risk aversion of the large agent. We point out that in this thesis the notion “risk aversion” is used for the setting with nonvanishing λ in the cost functional and does not mean risk aversion in the sense of utility theory.

The target process ζ in the risk term allows to, e.g., take client preferences or regulations into account, or to improve, but closely follow, popular trading strategies. Moreover, a possible and natural choice would be $\zeta_k = E_k[\hat{\xi}]$, $k \in \{n, \dots, N\}$, respectively $\zeta_s = E_s[\hat{\xi}]$, $s \in [t, T]$, so that the risk term ensures that any optimal strategy X^* does not deviate too much from the (expected) target position $\hat{\xi}$ during the course of the trading period.

In the discrete-time model of Chapter 2, we generalize [AKU21b] to $\mathcal{F}_N$-measurable terminal targets $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$ and adapted risk coefficient, respectively target, processes $\lambda = (\lambda_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$ and $\zeta = (\zeta_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$ that satisfy $\lambda_k \geq 0$, $\lambda_k \in L^{\infty-}(\mathcal{F}_k)$, and $\zeta_k \in L^{2+}(\mathcal{F}_k)$ for all $k \in \mathbb{Z} \cap (-\infty, N]$. Note that in [AKU21b] there is no risk term, i.e., $\lambda \equiv 0$, and positions are required to be closed, i.e., $\hat{\xi} = 0$. It is shown in [AKU21b, Theorem 2.1] that the value function and the optimal strategy in this subsetting are characterized by the process Y of (2.34). This result now becomes a corollary (see Corollary 2.2.4) of Theorem 2.2.1 of the present thesis.

When we include λ, we modify the definition of Y from (2.34) to (2.7), and λ appears also in the optimal trade sizes (2.10). More interestingly, if we have a nonzero target position $\hat{\xi}$ or nonzero λ and ζ, then we need a second process ψ (see (2.8) or Remark 2.2.3) in addition to Y in order to describe the value function and the optimal strategy. For the proof of Theorem 2.2.1, we proceed similarly to the proof of [AKU21b, Theorem 2.1] with the main difference that we now have to consider a more general, but still quadratic, structure of the value function, leading to the two recursively defined processes Y and ψ. A further discussion on the influence of $\lambda, \zeta, \hat{\xi}$ on Y, ψ, the value function, and optimal strategies is contained in Section 2.2.2.

It is natural to treat the question on existence of profitable round trips only in a risk-neutral setting and for deterministic terminal targets. However, we comment in Remark 2.5.10 that most results of Section 2.5 continue to hold for general λ. For closing in one go (Section 2.6), we consider $\hat{\xi} = 0 \equiv \zeta$ and provide a somewhat counterintuitive Example 2.6.8 where it is optimal for a risk-neutral agent, but not for a risk-averse agent, to close the whole position at time $N - 2$.

In the semimartingale problem of Chapter 5, we introduce a bounded, progressively measurable process $\lambda = (\lambda_s)_{s \in [0,T]}$ (typically nonnegative) into the setting of [AKU21a]. That is, we require to close the position and we incorporate a quadratic risk term $E_t[\int_t^T \gamma_s \lambda_s X_s^2 d[M^{(1)}]_s]$ with zero moving target; in other models, such risk terms have been considered in, e.g., [AK15], [GH17], [HX19], and [HK21]. In comparison to [AKU21a], λ now is part of the driver of the BSDE and of the auxiliary process $\widetilde{\vartheta}$ of (5.22). Moreover, λ enters the optimal strategies, but only via ϑ, and the value function, but only via Y. This is shown in the main result Theorem 5.2.6. The proof of the alternative representation of the cost functional J^{sem} (i.e., Theorem 5.2.1) and the proof of Theorem 5.2.6 are generalizations of those in [AKU21a].

In the finite-variation problem and its continuous extension (Chapter 7–Chapter 8), we allow for all of $\hat{\xi}$, λ, ζ. The terminal target $\hat{\xi}$ is an $\mathcal{F}_T$-measurable random variable with $E[\gamma_T \hat{\xi}^2] < \infty$, the risk coefficient process $\lambda = (\lambda_s)_{s\in[0,T]}$ is a bounded, progressively measurable process (typically nonnegative), and the moving target $\zeta = (\zeta_s)_{s\in[0,T]}$ is a progressively measurable process with $E[\int_0^T \gamma_s \zeta_s^2 ds] < \infty$.

There are no major difficulties due to the terminal target $\hat{\xi}$ or due to the risk term $E_t[\int_t^T \gamma_s \lambda_s (X_s - \zeta_s)^2 ds]$ with moving target ζ in establishing the continuous extension of the cost functional from J^{fv} to J^{pm} and the reduction to a standard LQ stochastic control problem. A minor inconvenience comes from the possible presence of the risk term. For nonzero λ, the standard LQ problem after the first reduction in Section 8.1.1 contains cross-terms and we have to perform a second transformation in Section 8.1.2 to match the formulation of [KT02]. Since the setting in [KT02] (in contrast to, e.g., [SXY21]) allows for inhomogeneities in the cost functional and in the state process such as those produced by nonzero $\hat{\xi}$ or ζ, we then obtain the solution via [KT02] without further additional work also in the general case.

As a result, we get in Corollary 8.2.4 that the optimal strategies and minimal costs of the extended problem are characterized by the process Y from BSDE (4.1) (cf. [KT02, (9)]) and the process ψ from the linear BSDE (8.7) (cf. [KT02, (85)]). Similar to what we observe in discrete time, Y includes λ, but none of $\hat{\xi}$ and ζ, whereas ψ contains λ and ζ (and also $\widetilde{\vartheta}$ of (5.22)) in the driver and has terminal value $\psi_T = -\frac{1}{2}\sqrt{\gamma_T}\hat{\xi}$. If $\hat{\xi} = 0$ and at least one of λ, ζ vanishes, then $\psi \equiv 0$, and formulas simplify (see also the discussion at the end of Section 8.2).

The fact that we were able to incorporate an $\mathcal{F}_T$-measurable random variable $\hat{\xi}$ and a progressively measurable process ζ, satisfying suitable integrability conditions (see (7.1) and (7.2)), into our analysis, allows us to consider in Section 8.3 the Obizhaeva-Wang model with random targets. In particular, in the subsetting with only a random terminal target $\hat{\xi}$, we find that we have to include updates about this random terminal target in form of a zero-mean stochastic integral into the deterministic optimal strategy of [OW13, Proposition 3].

2

Optimal trade execution in a discrete-time model

We first consider an optimal trade execution problem in discrete time. Trading is allowed at a given finite number of time points and in both directions. This means that a strategy can comprise both buy and sell trades. A strategy is determined by the size and direction of a trade at each time point. A trade that equals 0 means no trading at this time, a negative value of a trade corresponds to selling, and a positive value of a trade stands for a buy order. Note that we only consider market orders. Admissible strategies need to reach a prescribed position at terminal time, which may be stochastic.

In contrast to [AKU21b], we interpret strategies as the development of the position in time and not as the progression of trades. Both concepts are equivalent (see also Remark 2.1.1). The interpretation in the sense of positions is in line with the notion of a strategy in the continuous-time models in later chapters (see also [AKU21a] and [AKU22a]).

We work in a stylized symmetric order book model similar to and extending the one of [OW13, Section 3]; see also the basic example of Obizhaeva-Wang type models in Section 1.1. Recall that in that example and in [OW13] the order book parameters, i.e., price impact and resilience coefficient, are deterministic constants[1], and that the only source of randomness is the underlying unaffected price. However, price impact and resilience reflect the trading activity of other market participants and are therefore described more realistically by stochastic processes. For an overview of the development of Obizhaeva-Wang models towards this direction we refer to Section 1.2. In our model, we now allow price impact and resilience both to be described by stochastic processes. Additionally, resilience does not need to be exponential.

Furthermore, we incorporate the possibility to prescribe a stochastic target position. This extends the typical setting where one wants to get from an initial position $x \in \mathbb{R}$ to terminal position 0 (or, more generally, from a deterministic initial position to a deterministic terminal position). Moreover, we include a risk term in our cost

[1] see also [OW13, Section 8.1] for a comment on extension to time-varying deterministic resilience

functional so that the agent can nudge the strategy to follow a target process during the course of the trading period. Note that such a risk term and a possibly stochastic target position expand the setting of [AKU21b].

In Section 2.1, we provide the mathematical formulation of our control problem and more detail on its financial interpretation. Section 2.1 contains also some relevant remarks on the model. We subsequently solve the control problem in Section 2.2. The main result is Theorem 2.2.1. We therein characterize the optimal strategy and the minimal costs by two processes Y and ψ that are defined by backward recursion. The proof is given in Section 2.2.1. In Section 2.2.2 we comment on the main theorem. Among others, we provide another representation of the process ψ, and we obtain the main result from [AKU21b] as a special case of our main theorem. Subsequently, we in Section 2.3 consider a subsetting within our general model that also serves as a framework for some results and examples in further sections. In Section 2.4 we explain that the process Y (in the setting of [AKU21b]) has an economic interpretation as a savings factor and investigate its long-time limit. We study in Section 2.5 the existence of profitable round trips, and in Section 2.6 optimality of closing the position in one go.

This chapter is based on and uses material from the publication [AKU21b] (joint work with Thomas Kruse and Mikhail Urusov).

2.1 The discrete-time model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k\in\mathbb{Z}}, P)$ be a filtered probability space. For all $p \in (0,\infty)$ denote by $L^p = L^p(\Omega, \mathcal{F}, P)$ the set of random variables Z on $(\Omega, \mathcal{F}, P)$ such that $E[|Z|^p] < \infty$. Denote $L^{\infty-} = \bigcap_{p\in[1,\infty)} L^p(\Omega, \mathcal{F}, P)$ and $L^{2+} = \bigcup_{\varepsilon>0} L^{2+\varepsilon}(\Omega, \mathcal{F}, P)$.

Observe that $L^{\infty-} \subseteq L^{2+} \subseteq L^2$, and that the following hold (see also [AKU21b, Appendix B]): If $Z_1, Z_2 \in L^{\infty-}$, then $Z_1 Z_2 \in L^{\infty-}$. If $Z_1 \in L^{\infty-}$ and $Z_2 \in L^{2+}$, then $Z_1 Z_2 \in L^{2+}$.

Furthermore, for $k \in \mathbb{Z}$, write $L^{\infty-}(\mathcal{F}_k)$ (resp. $L^{2+}(\mathcal{F}_k)$) for the set of $\mathcal{F}_k$-measurable random variables in $L^{\infty-}$ (resp. L^{2+}). In the sequel, we will use the convention for sums and products that $\sum_{j=n}^{k} := 0$ and $\prod_{j=n}^{k} := 1$ if $n, k \in \mathbb{Z}$ with $n > k$.

Let $\beta = (\beta_k)_{k\in\mathbb{Z}}$ and $\gamma = (\gamma_k)_{k\in\mathbb{Z}}$ be strictly positive adapted stochastic processes, called the *resilience (process)* and the *price impact (process)*, respectively. Assume that $\beta_k, \gamma_k, \frac{1}{\gamma_k} \in L^{\infty-}$ for all $k \in \mathbb{Z}$. It turns out to be convenient to denote the multiplicative increments of γ by $\Gamma_k = \frac{\gamma_k}{\gamma_{k-1}}$, $k \in \mathbb{Z}$.

Let $N \in \mathbb{N}$. We introduce a random variable $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$, an adapted stochastic process $\zeta = (\zeta_k)_{k\in\mathbb{Z}\cap(-\infty,N]}$, and a nonnegative adapted stochastic process $\lambda = (\lambda_k)_{k\in\mathbb{Z}\cap(-\infty,N]}$. Assume that $\zeta_k \in L^{2+}$ and $\lambda_k \in L^{\infty-}$ for all $k \in \mathbb{Z}\cap(-\infty, N]$. For $n \in \mathbb{Z}\cap(-\infty, N]$ and $x, d \in \mathbb{R}$ we denote by $\mathcal{A}_n^{\mathrm{disc}}(x, d)$ the set of real-valued adapted stochastic processes $X = (X_k)_{k\in\{n-1,n,\dots,N\}}$ with $X_k \in L^{2+}$ for all $k \in \{n, \dots, N\}$

that are equipped with initial position[2] $X_{n-1} = x$ and satisfy $X_N = \hat{\xi}$. Elements of $\mathcal{A}_n^{\text{disc}}(x,d)$ are called *execution strategies*. For such an execution strategy $X = (X_k)_{k\in\{n-1,n,\dots,N\}}$ we furthermore introduce its associated *trade process* $\xi = (\xi_k)_{k\in\{n,\dots,N\}}$ defined by $\xi_k = X_k - X_{k-1}$, $k \in \{n,\dots,N\}$.

Remark 2.1.1. Note that for $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, and an execution strategy $X = (X_k)_{k\in\{n-1,n,\dots,N\}} \in \mathcal{A}_n^{\text{disc}}(x,d)$, its trade process ξ is a real-valued adapted stochastic process with $\xi_k \in L^{2+}$ for all $k \in \{n,\dots,N\}$ and $x + \sum_{j=n}^N \xi_j = \hat{\xi}$. Observe that it is equivalently possible to start from a real-valued adapted stochastic process $\xi = (\xi_k)_{k\in\{n,\dots,N\}}$ satisfying $x + \sum_{j=n}^N \xi_j = \hat{\xi}$ and $\xi_k \in L^{2+}$ for all $k \in \{n,\dots,N\}$, and to define an execution strategy $X = (X_k)_{k\in\{n-1,n,\dots,N\}} \in \mathcal{A}_n^{\text{disc}}(x,d)$ via $X_{n-1} = x$, $X_k = x + \sum_{j=n}^k \xi_j$, $k \in \{n,\dots,N\}$. This execution strategy then has ξ as its trade process. Thus, (for fixed $n \in \mathbb{Z} \cap (-\infty, N]$, $x \in \mathbb{R}$, and $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$) there is a one-to-one correspondence between execution strategies and real-valued adapted stochastic processes (trade processes) $\xi = (\xi_k)_{k\in\{n,\dots,N\}}$ satisfying $x + \sum_{j=n}^N \xi_j = \hat{\xi}$ and $\xi_k \in L^{2+}$ for all $k \in \{n,\dots,N\}$.

For $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, and $X \in \mathcal{A}_n^{\text{disc}}(x,d)$ we define the *deviation process* $D = (D_{k-})_{k\in\{n,\dots,N\}}$ associated to the execution strategy X recursively by

$$D_{n-} = d \quad \text{and} \quad D_{k-} = (D_{(k-1)-} + \gamma_{k-1}\xi_{k-1})\beta_k, \quad k \in \{n+1,\dots,N\}, \tag{2.1}$$

where ξ is the trade process for X. Note that the process $D = (D_{k-})_{k\in\{n,\dots,N\}}$ is adapted. In addition to the recursive definition of the deviation process, we have the following explicit representation.

Remark 2.1.2. For $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, and $X \in \mathcal{A}_n^{\text{disc}}(x,d)$, the deviation process $D = (D_{k-})_{k\in\{n,\dots,N\}}$ associated to X is given explicitly by

$$D_{k-} = d \prod_{l=n+1}^{k} \beta_l + \sum_{i=n+1}^{k} \gamma_{i-1}\xi_{i-1} \prod_{l=i}^{k} \beta_l, \quad k \in \{n,\dots,N\}, \tag{2.2}$$

where ξ is the trade process for X. This can be established by induction on $k \in \{n,\dots,N\}$. Observe furthermore that we see from (2.2) and the assumptions $\beta_k, \gamma_k \in L^{\infty-}$ for all $k \in \mathbb{Z}$, $\xi_k \in L^{2+}$ for all $k \in \{n,\dots,N\}$, that $D_{k-} \in L^{2+}$ for all $k \in \{n,\dots,N\}$.

For $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, we want to minimize over $X \in \mathcal{A}_n^{\text{disc}}(x,d)$ the expected costs

$$E_n\left[\sum_{j=n}^{N} \left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j\right] + E_n\left[\sum_{j=n}^{N} \gamma_j\lambda_j(X_j - \zeta_j)^2\right], \tag{2.3}$$

[2] d will be the initial value for the state process associated to X, see also (2.1).

where d is the starting point of the process D in (2.1), ξ is the trade process associated to X, and $E_n[\cdot]$ is a shorthand notation for $E[\cdot|\mathcal{F}_n]$.

Remark 2.1.3. Note that the expected costs (2.3) are finite. To show this, we verify that for all $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, $X \in \mathcal{A}_n^{\text{disc}}(x, d)$ with trade process ξ, each summand $\left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j$, $j \in \{n, \dots, N\}$, and $\gamma_j \lambda_j (X_j - \zeta_j)^2$, $j \in \{n, \dots, N\}$, is integrable. Since $\gamma_j \in L^{\infty-}$ and $\xi_j \in L^{2+}$, the product $\gamma_j \xi_j$ is in L^{2+}. By Remark 2.1.2, $D_{j-} \in L^{2+}$ as well. Hence, $D_{j-} + \frac{\gamma_j}{2}\xi_j \in L^{2+}$. The Cauchy-Schwarz inequality thus yields the integrability of $\left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j$. For $\gamma_j \lambda_j (X_j - \zeta_j)^2$, note that $X_j - \zeta_j \in L^{2+}$. Since $\gamma_j, \lambda_j \in L^{\infty-}$, it follows that $\gamma_j \lambda_j (X_j - \zeta_j) \in L^{2+}$. Again by the Cauchy-Schwarz inequality we have that $\gamma_j \lambda_j (X_j - \zeta_j)^2$ is integrable.

We then define the value function $V \colon \Omega \times (\mathbb{Z} \cap (-\infty, N]) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$V_n(x, d) = \underset{X \in \mathcal{A}_n^{\text{disc}}(x,d)}{\operatorname{ess\,inf}} E_n \left[\sum_{j=n}^{N} \left(D_{j-} + \frac{\gamma_j}{2}\xi_j \right) \xi_j + \sum_{j=n}^{N} \gamma_j \lambda_j (X_j - \zeta_j)^2 \right], \tag{2.4}$$
$$n \in \mathbb{Z} \cap (-\infty, N], x \in \mathbb{R}, d \in \mathbb{R}.$$

Let us now explain the financial interpretation of the model. The numbers $N \in \mathbb{N}$ and $n \in \mathbb{Z} \cap (-\infty, N]$ specify the end and the beginning of the trading period, respectively. The possible trading times are given by the set $\{n, \dots, N\}$. The number $x \in \mathbb{R}$ represents the initial position of the agent, while the random variable $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$ prescribes the target position at terminal time. Since $\hat{\xi}$ is assumed to be $\mathcal{F}_N$-measurable, it is possible to model a situation where the value of the target position is only revealed at terminal time. The condition to close the position corresponds to the choice $\hat{\xi} = 0$. An execution strategy $X \in \mathcal{A}_n^{\text{disc}}(x, d)$ tracks the agent's position with the given constraints that the initial position is fixed at x and the terminal position at $\hat{\xi}$. For a time point $k \in \{n, \dots, N-1\}$, the value of X_k reflects the position after the trade at time k and prior to the trade at time $k+1$. The difference $\xi_k = X_k - X_{k-1}$ corresponds to the number of shares traded by the agent at time $k \in \{n, \dots, N\}$ and will therefore sometimes be called trade size. If $\xi_k > 0$, the agent buys shares, whereas a negative value $\xi_k < 0$ means selling. With the last trade ξ_N, the target position $\hat{\xi}$ needs to be reached, i.e., $\xi_N = \hat{\xi} - X_{N-1} = \hat{\xi} - x - \sum_{j=n}^{N-1} \xi_j$.

The process D describes the deviation of the price of a share from the unaffected price caused by the past trades of the agent. Typically, the initial deviation $d \in \mathbb{R}$ immediately prior to the considered trading period $\{n, \dots, N\}$ is 0. Given a deviation of size $D_{(k-1)-}$ directly prior to the trade at time $k - 1 \in \{n, \dots, N\}$, the deviation directly after a trade of size ξ_{k-1} equals $D_{(k-1)-} + \gamma_{k-1}\xi_{k-1}$. In particular, the change of the deviation is proportional to the size of the trade, and the proportionality factor is given by the price impact process γ. In the language of the literature on optimal trade execution problems, our model thus includes a linear price impact. This corresponds to a block-shaped symmetric limit order book, i.e., limit orders are uniformly distributed to the left and to the right of the mid-market price. Note that in our idealized model the bid-ask spread is always assumed to be 0. The height of the order book at time $k \in \mathbb{Z}$ is given by $\frac{1}{\gamma_k}$. In particular, our model allows the height of the limit order book to evolve randomly in time and thereby captures stochastic market liquidity. Note that since γ is positive, a purchase $\xi_k > 0$ at time $k \in \{n, \dots, N\}$ increases the deviation, whereas a sale $\xi_k < 0$ decreases it.

In the period after the trade at time $k-1$ and before the trade at time k, the deviation changes from $D_{(k-1)-} + \gamma_{k-1}\xi_{k-1}$ to $D_{k-} = (D_{(k-1)-} + \gamma_{k-1}\xi_{k-1})\beta_k$ due to resilience effects in the market. In the literature on optimal execution the resilience process β is often assumed to take values in $(0, 1)$ and describes the speed with which the deviation tends back to zero between two trades, where values of β close to zero signify a faster reversion to zero. In this case, i.e., for $(0, 1)$-valued β, the price impact is usually called transient (cf., e.g., [ASS12]). The case $\beta \equiv 1$ corresponds to permanent impact. In our work we assume β only to be positive. If $D_{(k-1)-} + \gamma_{k-1}\xi_{k-1}$ has the same sign as the trade ξ_{k-1} at time $k-1$, which typically is the case, then a value $\beta_k > 1$ describes the effect when the deviation continues to move in the direction of the trade for some time after the trade. In any case, $\beta > 1$ reinforces the deviation. Note that not only γ, but also β evolves randomly in time.

At each time $k \in \{n, \dots, N\}$ the illiquidity costs incurred by a trade ξ_k amount to $(D_{k-} + \frac{\gamma_k}{2}\xi_k)\xi_k$. This means that the price per share that the agent has to pay in addition to the unaffected price equals the mean of the deviation before the trade D_{k-} and the deviation after the trade $D_{k-} + \gamma_k\xi_k$. The overall illiquidity costs during the trading period $\{n, \dots, N\}$ are given by $\sum_{k=n}^{N}(D_{k-} + \frac{\gamma_k}{2}\xi_k)\xi_k$.

To these illiquidity costs, we add some costs due to risk preferences $\sum_{k=n}^{N} \gamma_k\lambda_k(X_k - \zeta_k)^2$. These additional costs should be viewed as penalization or steering of strategies and are not necessarily of a financial nature. The value ζ_k describes the agent's preferred position at time $k \in \{n, \dots, N\}$. In most cases, one might want to choose $\zeta_N = \hat{\xi}$ at terminal time since any admissible strategy X needs to satisfy $X_N = \hat{\xi}$ anyway. Furthermore, a typical choice for the process ζ is $\zeta_k = E_k[\hat{\xi}]$ for all $k \in \{n, \dots, N\}$. This means that at each time $k \in \{n, \dots, N\}$ during the trading period, the agent aims for a position that is not too far from the best current prediction $E_k[\hat{\xi}]$ of the target $\hat{\xi}$. The coefficient $\gamma_k\lambda_k$ describes how strict discrepancies of the position X_k from the target position ζ_k at time $k \in \{n, \dots, N\}$ are penalized. Note that we use the parametrization $\gamma_k\lambda_k$ instead of simply $\widetilde{\lambda}_k$ to match the notation in [AKU22a] and also for more convenience in Section 2.2.

To sum up, control problem (2.4) corresponds to minimizing the expected costs (including risk preferences) of transferring an initial position of size $x \in \mathbb{R}$ within the trading period $\{n, \dots, N\}$ to position $\hat{\xi}$ at time N given initial deviation $d \in \mathbb{R}$, where the minimization is performed in an extension of symmetric block-shaped limit order book models to the case of randomly evolving order book depth and resilience.

We remark that the above model can be extended to explicitly include an unaffected price as long as the unaffected price is a square-integrable martingale. This is a fairly standard assumption in the literature on optimal trade execution (see, e.g., [AFS08, AFS10, AS10, OW13, PSS11], and [GS13, Section 22.2]). For an example where the dependence of optimal strategies on a possible drift in the underlying unaffected price process is analyzed (in a continuous-time model of Obizhaeva-Wang type), we mention [LS13].

Remark 2.1.4. We can also include an unaffected price process in the model. Indeed, if the unaffected price process is given by a square-integrable martingale $S^0 = (S^0_k)_{k\in\mathbb{Z}\cap(-\infty,N]}$, then, for all $n \in \mathbb{Z}\cap(-\infty,N]$, $x, d \in \mathbb{R}$, and $X \in \mathcal{A}^{\text{disc}}_n(x,d)$, we get

$$\begin{aligned}
\sum_{j=n}^{N} S^0_j \xi_j &= \sum_{j=n}^{N} S^0_j (X_j - X_{j-1}) \\
&= \sum_{j=n}^{N} S^0_{j-1}(X_j - X_{j-1}) + \sum_{j=n}^{N} (S^0_j - S^0_{j-1}) X_j - \sum_{j=n}^{N} (S^0_j - S^0_{j-1}) X_{j-1} \\
&= S^0_N X_N - S^0_{n-1} X_{n-1} - \sum_{j=n}^{N-1} (S^0_{j+1} - S^0_j) X_j - (S^0_n - S^0_{n-1}) X_{n-1} \\
&= \hat{\xi} S^0_N - x S^0_n - \sum_{j=n}^{N-1} (S^0_{j+1} - S^0_j) X_j,
\end{aligned}$$

and thus

$$E_n\left[\sum_{j=n}^{N} S^0_j \xi_j\right] = E_n\left[\hat{\xi} S^0_N - x S^0_n - \sum_{j=n}^{N-1} X_j (S^0_{j+1} - S^0_j)\right] = E_n[\hat{\xi} S^0_N] - x S^0_n.$$

It follows that for all $n \in \mathbb{Z}\cap(-\infty,N]$, $x, d \in \mathbb{R}$, the expected costs generated by an execution strategy $X \in \mathcal{A}^{\text{disc}}_n(x,d)$ with trade process ξ and deviation process $(D_{k-})_{k\in\{n,\dots,N\}}$ of (2.1) satisfy

$$\begin{aligned}
&E_n\left[\sum_{j=n}^{N} \left(S^0_j + D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j + \sum_{j=n}^{N} \gamma_j \lambda_j (X_j - \zeta_j)^2\right] \\
&= E_n[\hat{\xi} S^0_N] - x S^0_n + E_n\left[\sum_{j=n}^{N} \left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j + \sum_{j=n}^{N} \gamma_j \lambda_j (X_j - \zeta_j)^2\right].
\end{aligned} \tag{2.5}$$

Hence, minimizing $E_n[\sum_{j=n}^{N}(S^0_j + D_{j-} + \frac{\gamma_j}{2}\xi_j)\xi_j + \sum_{j=n}^{N}\gamma_j\lambda_j(X_j - \zeta_j)^2]$ is equivalent to (2.4).

As an extension to our model, we could additionally include a constant permanent price impact coefficient without changing the analysis (see also, e.g., [AFS08] or [FSU14]).

Remark 2.1.5. To set up a variant of our model with transient and permanent price impact, let $q = (q_k)_{k\in\mathbb{Z}}$ be a strictly positive adapted process such that $\hat{c} = \frac{1}{q_k} - \gamma_k$ is a strictly positive constant for all $k \in \mathbb{Z}$. The setting usually considered in this chapter corresponds to the choice $q_k = \frac{1}{\gamma_k}$, $k \in \mathbb{Z}$. In general, we interpret q as the order book depth, γ as the transient price impact coefficient, and $\hat{c}$ as the permanent price impact coefficient. For $n \in \mathbb{Z}\cap(-\infty, N]$, $x, d \in \mathbb{R}$, and $X \in \mathcal{A}_n^{\text{disc}}(x,d)$ (with trade process ξ), we now consider the deviation process $\hat{D} = (\hat{D}_{k-})_{k\in\{n,\dots,N\}}$ given by

$$\begin{aligned}\hat{D}_{k-} &= d\prod_{l=n+1}^{k}\beta_l + \sum_{i=n+1}^{k}\gamma_{i-1}\xi_{i-1}\prod_{l=i}^{k}\beta_l + \sum_{i=n+1}^{k}\hat{c}\,\xi_{i-1}\\ &= D_{k-} + (X_{k-1}-x)\hat{c}, \quad k\in\{n,\dots,N\},\end{aligned}$$

and the expected costs[3]

$$E_n\left[\sum_{j=n}^{N}\left(\hat{D}_{j-} + \frac{1}{2q_j}\xi_j\right)\xi_j\right] + E_n\left[\sum_{j=n}^{N}\gamma_j\lambda_j(X_j-\zeta_j)^2\right]. \tag{2.6}$$

It holds for all $n \in \mathbb{Z}\cap(-\infty, N]$, $x, d \in \mathbb{R}$, and $X \in \mathcal{A}_n^{\text{disc}}(x,d)$ that

$$\sum_{j=n}^{N}\left(\hat{D}_{j-} + \frac{1}{2q_j}\xi_j\right)\xi_j = \sum_{j=n}^{N}\left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j + \hat{c}\sum_{j=n}^{N}\left(X_{j-1} - x + \frac{1}{2}\xi_j\right)\xi_j,$$

and for the second term we further obtain that

$$\begin{aligned}\sum_{j=n}^{N}\left(X_{j-1} - x + \frac{1}{2}\xi_j\right)\xi_j &= \sum_{j=n}^{N}\left(\frac{1}{2}X_{j-1} - x + \frac{1}{2}X_j\right)(X_j - X_{j-1})\\ &= \frac{1}{2}\sum_{j=n}^{N}\left(X_j^2 - X_{j-1}^2\right) - x(X_N - x)\\ &= \frac{1}{2}(\hat{\xi}^2 - x^2) - x(\hat{\xi} - x).\end{aligned}$$

It follows that the additional costs due to the permanent price impact coefficient do not depend on the choice of the strategy. This shows that minimizing (2.6) is equivalent to (2.4).

[3] Recall that the scaling of λ by γ in the risk term is only for convenience.

Furthermore, let us mention that the optimization problem for initial position $x+b$, where $x, b \in \mathbb{R}$, and terminal position b is not different from the problem to close the initial position x (the assumption that the terminal positions are deterministic is important). This is the content of the next remark.

Remark 2.1.6. Let $n \in \mathbb{Z} \cap (-\infty, N]$, and $x, b, d \in \mathbb{R}$. We denote by $\mathcal{A}_n^0(x, d)$ (resp. $\mathcal{A}_n^b(x+b, d)$) the set of execution strategies $X = (X_k)_{k \in \{n-1,\dots,N\}}$ with initial value $X_{n-1} = x$ (resp. $X_{n-1} = x+b$) and terminal value $X_N = 0$ (resp. $X_N = b$). Suppose first that $X \in \mathcal{A}_n^0(x, d)$ with associated trade process ξ. Then, the definition $X_k^{(b)} = X_k + b$ for all $k \in \{n-1, \dots, N\}$ yields an execution strategy $X^{(b)} = (X_k^{(b)})_{k \in \{n-1,\dots,N\}} \in \mathcal{A}_n^b(x+b, d)$, and the associated trade process $\xi^{(b)} = (\xi_k^{(b)})_{k \in \{n,\dots,N\}}$ is given by $\xi_k^{(b)} = X_k^{(b)} - X_{k-1}^{(b)} = X_k - X_{k-1} = \xi_k$, $k \in \{n, \dots, N\}$. Conversely, starting from $X^{(b)} \in \mathcal{A}_n^b(x+b, d)$, we can recover $X \in \mathcal{A}_n^0(x, d)$ via $X_k = X_k^{(b)} - b$, $k \in \{n-1, \dots, N\}$. In fact, we obtain in this way a one-to-one correspondence between strategies in $\mathcal{A}_n^0(x, d)$ and strategies in $\mathcal{A}_n^b(x+b, d)$, and their trade processes coincide. Since the deviation process and the illiquidity costs only depend on the initial deviation d and the trade process, but not on the initial or terminal position, it holds that the illiquidity costs associated to $X \in \mathcal{A}_n^0(x, d)$ and the illiquidity costs associated to $X^{(b)} = (X_k + b)_{k \in \{n-1,\dots,N\}} \in \mathcal{A}_n^b(x+b, d)$ are the same. It follows that

$$\underset{X \in \mathcal{A}_n^0(x,d)}{\operatorname{ess\,inf}} E_n \left[\sum_{j=n}^{N} \left(D_{j-} + \frac{\gamma_j}{2} \xi_j \right) \xi_j \right] = \underset{X^{(b)} \in \mathcal{A}_n^b(x+b,d)}{\operatorname{ess\,inf}} E_n \left[\sum_{j=n}^{N} \left(D_{j-} + \frac{\gamma_j}{2} \xi_j \right) \xi_j \right].$$

Therefore, in the risk-neutral case, the minimization problems are the same, and if there exists an optimal strategy of one problem, it is a shifted version of the optimal strategy of the other problem, and both have the same trade process and the same costs. We can also use a risk preference term as in (2.4) provided that both problems use the same λ and that the target processes ζ, $\zeta^{(b)}$ are linked via $\zeta_k^{(b)} = \zeta_k + b$, $k \in \{n, \dots, N\}$. Moreover, we can include for both problems the same square-integrable martingale S^0 as an unaffected price (see Remark 2.1.4).

2.2 Optimal strategies and minimal costs

The following main result Theorem 2.2.1 provides a solution to the stochastic control problem (2.4). It shows that the value function and the optimal strategy in (2.4) are characterized by two processes Y and ψ. The process Y is defined via the backward recursion (2.7) and involves only the resilience β, the multiplicative increments Γ of the price impact, and λ (recall the risk term $\sum_{j=n}^{N} \gamma_j \lambda_j (X_j - \zeta_j)^2$ in (2.4)). In case of a nonzero target $\hat{\xi}$ or nonzero λ and ζ, the process ψ (defined by (2.8); see also Remark 2.2.3) enters the representation of the value function and the optimal strategy.

We provide a proof of Theorem 2.2.1 in Section 2.2.1, and subsequently discuss Theorem 2.2.1 in Section 2.2.2.

Theorem 2.2.1. *Recall the assumptions that* $\beta_n, \gamma_n, \frac{1}{\gamma_n}, \lambda_n \in L^{\infty-}$, $\zeta_n \in L^{2+}$, *and* $\beta_n, \gamma_n > 0$, $\lambda_n \geq 0$ *for all* $n \in \mathbb{Z} \cap (-\infty, N]$ *and that* $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$. *Suppose moreover that for all* $n \in \mathbb{Z} \cap (-\infty, N-1]$ *it holds that* $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ *a.s. and that* $(1 - E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}])^{-1} \in L^{\infty-}$. *Let* $Y = (Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ *be the process that is recursively defined by* $Y_N = \frac{1}{2}$ *and, for* $n \in \mathbb{Z} \cap (-\infty, N-1]$,

$$Y_n = E_n[\Gamma_{n+1} Y_{n+1}] + \lambda_n - \frac{\left(E_n\left[Y_{n+1}\left(\beta_{n+1} - \Gamma_{n+1}\right)\right] - \lambda_n\right)^2}{E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1} - \Gamma_{n+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + \lambda_n\right]}. \tag{2.7}$$

Furthermore, let $\psi = (\psi_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ *be the process that is recursively defined by* $\psi_N = -\frac{1}{2}\sqrt{\gamma_N}\hat{\xi}$ *and, for* $n \in \mathbb{Z} \cap (-\infty, N-1]$,

$$\begin{aligned} \psi_n = {} & E_n\left[\sqrt{\Gamma_{n+1}}\psi_{n+1}\right] - \sqrt{\gamma_n}\lambda_n\zeta_n + E_n\left[\sqrt{\Gamma_{n+1}}\psi_{n+1}\left(1 - \frac{\beta_{n+1}}{\Gamma_{n+1}}\right) - \sqrt{\gamma_n}\lambda_n\zeta_n\right] \\ & \cdot \frac{E_n\left[Y_{n+1}\left(\beta_{n+1} - \Gamma_{n+1}\right)\right] - \lambda_n}{E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1} - \Gamma_{n+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + \lambda_n\right]}. \end{aligned} \tag{2.8}$$

(i) It holds for all $n \in \mathbb{Z} \cap (-\infty, N]$ *that* $0 < Y_n \leq \frac{1}{2}$ *and* $\psi_n \in L^{2+}(\mathcal{F}_n)$.

(ii) It holds for all $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, *that*

$$\begin{aligned} V_n(x,d) = {} & \frac{Y_n}{\gamma_n}\left(d - \gamma_n x\right)^2 - \frac{d^2}{2\gamma_n} - 2\frac{\psi_n}{\sqrt{\gamma_n}}(d - \gamma_n x) \\ & + E_n\left[\frac{\gamma_N}{2}\hat{\xi}^2 + \gamma_N\lambda_N\left(\hat{\xi} - \zeta_N\right)^2 + \sum_{j=n}^{N-1}\gamma_j\lambda_j\zeta_j^2\right] \\ & - \sum_{j=n}^{N-1} E_n\left[\frac{\left(E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1 - \frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right] - \sqrt{\gamma_j}\lambda_j\zeta_j\right)^2}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1} - \Gamma_{j+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right) + \lambda_j\right]}\right]. \end{aligned} \tag{2.9}$$

(iii) For all $x, d \in \mathbb{R}$ *the (up to* P*-null sets) unique optimal trade size is given by*

$$\begin{aligned} \xi_n^*(x,d) = {} & \frac{E_n\left[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n}{E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1} - \Gamma_{n+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + \lambda_n\right]}\left(x - \frac{d}{\gamma_n}\right) - \frac{d}{\gamma_n} \\ & - \frac{E_n\left[\sqrt{\Gamma_{n+1}}\psi_{n+1}\left(1 - \frac{\beta_{n+1}}{\Gamma_{n+1}}\right)\right] - \sqrt{\gamma_n}\lambda_n\zeta_n}{\sqrt{\gamma_n}E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1} - \Gamma_{n+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + \lambda_n\right]} \end{aligned} \tag{2.10}$$

for all $n \in \mathbb{Z} \cap (-\infty, N-1]$, *and* $\xi_N^*(x,d) = \hat{\xi} - x$. *It holds that* $\xi_n^*(x,d) \in L^{2+}(\mathcal{F}_n)$ *for all* $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$.

(iv) In particular, for all $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, *the process* $X^* = (X_k^*)_{k \in \{n-1,\dots,N\}}$ *recursively defined by* $X_{n-1}^* = x$, $D_{n-}^* = d$,

$$\begin{aligned} X_k^* &= X_{k-1}^* + \xi_k^*(X_{k-1}^*, D_{k-}^*), \\ D_{(k+1)-}^* &= \left(D_{k-}^* + \gamma_k \xi_k^*(X_{k-1}^*, D_{k-}^*)\right) \beta_{k+1}, \quad k \in \{n, \dots, N\}, \end{aligned} \tag{2.11}$$

is a unique optimal strategy in $\mathcal{A}_n^{disc}(x,d)$ *for* (2.4) *with associated trade process* $(\xi_k^*(X_{k-1}^*, D_{k-}^*))_{k \in \{n,\dots,N\}}$ *and associated deviation process* $(D_{k-}^*)_{k \in \{n,\dots,N\}}$.

2.2.1 Proof of the main theorem

In this subsection, we prove Theorem 2.2.1 by using the same techniques as in the proof of [AKU21b, Theorem 2.1]. In particular, we rely on the dynamic programming principle and the quadratic nature of the value function.

The main difference is that the value function in [AKU21b, Theorem 2.1] has the structure $\widetilde{V}_n(x,d) = \widetilde{v}_{1,n} d^2 + \widetilde{v}_{2,n} x^2 + \widetilde{v}_{3,n} xd$, $x, d \in \mathbb{R}$, $n \in \mathbb{Z} \cap (-\infty, N]$, for some $\mathcal{F}_n$-measurable coefficients $\widetilde{v}_{j,n}$, $j \in \{1,2,3\}$, $n \in \mathbb{Z} \cap (-\infty, N]$, whereas we here need to consider $V_n(x,d) = v_{1,n} d^2 + v_{2,n} x^2 + v_{3,n} xd + v_{4,n} d + v_{5,n} x + v_{6,n}$, $x, d \in \mathbb{R}$, $n \in \mathbb{Z} \cap (-\infty, N]$, for some $\mathcal{F}_n$-measurable coefficients $v_{j,n}$, $j \in \{1,2,3,4,5,6\}$, $n \in \mathbb{Z} \cap (-\infty, N]$. It turns out that, in contrast to [AKU21b, Theorem 2.1], a single process Y is in general not sufficient to describe the value function $V_n(x,d)$, $x, d \in \mathbb{R}$, $n \in \mathbb{Z} \cap (-\infty, N]$, that has possibly nonvanishing coefficients $v_{4,n}$ of d and $v_{5,n}$ of x and shift $v_{6,n}$, $n \in \mathbb{Z} \cap (-\infty, N]$.

We first show (i)–(iii) of Theorem 2.2.1 simultaneously by backward induction, and subsequently (iv) by forward induction.

Proof of (i)–(iii)

For the base case $n = N$ we have $Y_N = \frac{1}{2} \in (0, \frac{1}{2}]$. Since $\gamma_N \in L^{\infty-}(\mathcal{F}_N)$, it holds by Jensen's inequality that also $\sqrt{\gamma_N} \in L^{\infty-}(\mathcal{F}_N)$. Together with $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$, we then have that $\psi_N = -\frac{1}{2}\sqrt{\gamma_N}\hat{\xi} \in L^{2+}(\mathcal{F}_N)$. Observe that, for all $x, d \in \mathbb{R}$, the admissible set $\mathcal{A}_n^{\mathrm{disc}}(x,d)$ consists exactly of the process $X = (X_k)_{k \in \{N-1,N\}}$ given by $X_{N-1} = x$ and $X_N = \hat{\xi}$. The associated trade process is given by the single trade $\xi_N = \hat{\xi} - x$. This implies for all $x, d \in \mathbb{R}$ that $V_N(x,d) = (d + \frac{\gamma_N}{2}(\hat{\xi} - x))(\hat{\xi} - x) + \gamma_N \lambda_N (\hat{\xi} - \zeta_N)^2$, and that $\xi_N^*(x,d) = \hat{\xi} - x \in L^{2+}(\mathcal{F}_N)$ is the unique optimal trade size. Since, for all

$x, d \in \mathbb{R}$,

$$\begin{aligned}\left(d + \frac{\gamma_N}{2}(\hat{\xi} - x)\right)(\hat{\xi} - x) &= (\hat{\xi} - x)d + \frac{\gamma_N}{2}(x^2 - 2\hat{\xi}x + \hat{\xi}^2) \\ &= -xd + \frac{1}{2}\gamma_N x^2 + \hat{\xi}(d - \gamma_N x) + \frac{\gamma_N}{2}\hat{\xi}^2 \\ &= \frac{Y_N}{\gamma_N}(d - \gamma_N x)^2 - \frac{d^2}{2\gamma_N} - \frac{2\psi_N}{\sqrt{\gamma_N}}(d - \gamma_N x) + E_N\left[\frac{\gamma_N}{2}\hat{\xi}^2\right],\end{aligned}$$

we conclude that (2.9) holds for $n = N$.

Consider now the induction step $\mathbb{Z} \cap (-\infty, N] \ni n+1 \to n \in \mathbb{Z} \cap (-\infty, N-1]$. For all $x, d \in \mathbb{R}$ let

$$a_n = \gamma_n E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}(\beta_{n+1} - \Gamma_{n+1})^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + \lambda_n\right],$$

$$\begin{aligned}b_n(x,d) = {} & E_n\left[d\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + 2Y_{n+1}\left(\frac{\beta_{n+1}}{\Gamma_{n+1}} - 1\right)(\beta_{n+1}d - \gamma_{n+1}x)\right] \\ & + E_n\left[2\sqrt{\gamma_{n+1}}\psi_{n+1}\left(1 - \frac{\beta_{n+1}}{\Gamma_{n+1}}\right)\right] + 2\gamma_n\lambda_n(x - \zeta_n),\end{aligned}$$

$$\begin{aligned}c_n(x,d) = {} & E_n\left[\frac{Y_{n+1}}{\gamma_{n+1}}(\beta_{n+1}d - \gamma_{n+1}x)^2 - \frac{d^2\beta_{n+1}^2}{2\gamma_{n+1}}\right] + E_n\left[2\sqrt{\gamma_{n+1}}\psi_{n+1}\left(x - \frac{\beta_{n+1}}{\gamma_{n+1}}d\right)\right] \\ & + \gamma_n\lambda_n(x - \zeta_n)^2 + E_n\left[\frac{\gamma_N}{2}\hat{\xi}^2 + \gamma_N\lambda_N\left(\hat{\xi} - \zeta_N\right)^2 + \sum_{j=n+1}^{N-1}\gamma_j\lambda_j\zeta_j^2\right] \\ & - \sum_{j=n+1}^{N-1} E_n\left[\frac{\left(E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1 - \frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right] - \sqrt{\gamma_j}\lambda_j\zeta_j\right)^2}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}(\beta_{j+1} - \Gamma_{j+1})^2 + \frac{1}{2}\left(1 - \frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right) + \lambda_j\right]}\right].\end{aligned} \tag{2.12}$$

The random variables a_n, $b_n(x,d), c_n(x,d)$ are well-defined and it holds that $a_n \in L^{\infty-}$, $b_n(x,d) \in L^{2+}$, and $c_n(x,d) \in L^1$ for all $x, d \in \mathbb{R}$. This relies on the assumptions that $\beta_k, \gamma_k, \frac{1}{\gamma_k}, \lambda_k \in L^{\infty-}$ and $\zeta_k, \hat{\xi} \in L^{2+}$ for all $k \in \mathbb{Z} \cap (-\infty, N]$, as well as on the induction hypothesis $0 < Y_j \le \frac{1}{2}$ and $\psi_j \in L^{2+}(\mathcal{F}_j)$ for all $j \in \{n+1, \ldots, N\}$. For the last term in the definition of $c_n(x,d)$, $x, d \in \mathbb{R}$, we also use the assumptions $E_k[\frac{\beta_{k+1}^2}{\Gamma_{k+1}}] < 1$ and $(1 - E_k[\frac{\beta_{k+1}^2}{\Gamma_{k+1}}])^{-1} \in L^{\infty-}$ for all $k \in \mathbb{Z} \cap (-\infty, N-1]$. Let us treat this last term in more detail.

First, the assumption $E_k[\frac{\beta_{k+1}^2}{\Gamma_{k+1}}] < 1$ for all $k \in \mathbb{Z} \cap (-\infty, N-1]$, the induction hypothesis $Y_{j+1} > 0$, $j \in \{n, \ldots, N-1\}$, and $\lambda_k \ge 0$, $\gamma_k > 0$ for all $k \in \mathbb{Z} \cap (-\infty, N]$

imply for all $j \in \{n, \dots, N-1\}$ that

$$\begin{aligned} & E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right)+\lambda_j\right] \\ & \geq \frac{1}{2}\left(1-E_j\left[\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right]\right)>0. \end{aligned} \tag{2.13}$$

For all $j \in \{n, \dots, N-1\}$ it then follows from the assumption $(1 - E_j[\frac{\beta_{j+1}^2}{\Gamma_{j+1}}])^{-1} \in L^{\infty-}$ that

$$\frac{1}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right)+\lambda_j\right]} \in L^{\infty-}. \tag{2.14}$$

Second, by the induction hypothesis, we have $\psi_{j+1} \in L^{2+}$ for all $j \in \{n, \dots, N-1\}$. Furthermore, we have the assumptions that $\beta_k, \gamma_k, \frac{1}{\gamma_k} \in L^{\infty-}$ for all $k \in \mathbb{Z} \cap (-\infty, N]$, and therefore it holds for all $j \in \{n, \dots, N-1\}$ that $\sqrt{\Gamma_{j+1}}\psi_{j+1}(1 - \frac{\beta_{j+1}}{\Gamma_{j+1}}) \in L^{2+}$. Moreover, we have for all $k \in \mathbb{Z} \cap (-\infty, N]$ that $\sqrt{\gamma_k}, \lambda_k \in L^{\infty-}$ and $\zeta_k \in L^{2+}$, and hence $\sqrt{\gamma_k}\lambda_k\zeta_k \in L^{2+}$. Therefore, it holds for all $j \in \{n, \dots, N-1\}$ that

$$E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1-\frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right]-\sqrt{\gamma_j}\lambda_j\zeta_j \in L^{2+}. \tag{2.15}$$

Next, we combine (2.15) and (2.14) to obtain for all $j \in \{n, \dots, N-1\}$ that

$$\frac{E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1-\frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right]-\sqrt{\gamma_j}\lambda_j\zeta_j}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right)+\lambda_j\right]} \in L^{2+}. \tag{2.16}$$

Further, the Cauchy-Schwarz inequality then proves for all $j \in \{n, \dots, N-1\}$ that

$$\frac{\left(E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1-\frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right]-\sqrt{\gamma_j}\lambda_j\zeta_j\right)^2}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right)+\lambda_j\right]} \in L^1.$$

We conclude that the last term in the definition of $c_n(x,d)$, $x, d \in \mathbb{R}$, is well-defined and in L^1.

We furthermore remark that $\frac{a_n}{\gamma_n} > 0$ and $a_n > 0$ due to (2.13) and $\gamma_n > 0$. Also, (2.14) and $\frac{1}{\gamma_n} \in L^{\infty-}$ show that $\frac{\gamma_n}{a_n}, \frac{1}{a_n} \in L^{\infty-}$.

Besides, note that the assumptions that for all $k \in \mathbb{Z} \cap (-\infty, N]$ it holds that $\beta_k, \gamma_k, \frac{1}{\gamma_k}, \lambda_k \in L^{\infty-}$ and $\zeta_k, \hat{\xi} \in L^{2+}$, and the fact that Y_{n+1} is bounded and $\psi_{n+1} \in L^{2+}$, ensure that all conditional expectations in the sequel are well-defined and that we can conduct all our calculations.

The remainder of the induction step is subdivided into the four paragraphs *Optimal trade size* $\xi_n^*(x,d)$, *Representation of the value function* $V_n(x,d)$, *Bounds for* Y_n, and *Integrability property for* ψ_n.

Optimal trade size $\xi_n^*(x,d)$. We now prove existence and formula (2.10) for the optimal trade size, and that $\xi_n^*(x,d) \in L^{2+}(\mathcal{F}_n)$. It holds by the dynamic programming principle and the induction hypothesis on the value function that for all $x, d \in \mathbb{R}$

$$\begin{aligned}
V_n(x,d) &= \operatorname*{ess\,inf}_{X \in \mathcal{A}_n^{\mathrm{disc}}(x,d)} \left[\left(D_{n-} + \frac{\gamma_n}{2}\xi_n \right) \xi_n + \gamma_n \lambda_n (X_n - \zeta_n)^2 + E_n \left[V_{n+1}(X_n, D_{(n+1)-}) \right] \right] \\
&= \operatorname*{ess\,inf}_{\xi \in L^{2+}(\mathcal{F}_n)} \left[\left(d + \frac{\gamma_n}{2}\xi \right) \xi + \gamma_n \lambda_n (x + \xi - \zeta_n)^2 + E_n \left[V_{n+1}\left(x+\xi, (d+\gamma_n \xi)\beta_{n+1} \right) \right] \right] \\
&= \operatorname*{ess\,inf}_{\xi \in L^{2+}(\mathcal{F}_n)} \Bigg[\left(d + \frac{\gamma_n}{2}\xi \right) \xi + \gamma_n \lambda_n (x + \xi - \zeta_n)^2 \\
&\qquad + E_n \Bigg[\frac{Y_{n+1}}{\gamma_{n+1}} \left((d + \gamma_n \xi)\beta_{n+1} - \gamma_{n+1}(x+\xi) \right)^2 - \frac{(d+\gamma_n\xi)^2 \beta_{n+1}^2}{2\gamma_{n+1}} \\
&\qquad\qquad - 2 \frac{\psi_{n+1}}{\sqrt{\gamma_{n+1}}} \left((d+\gamma_n \xi)\beta_{n+1} - \gamma_{n+1}(x+\xi) \right) + V_{n+1}(0,0) \Bigg] \Bigg] \\
&= \operatorname*{ess\,inf}_{\xi \in L^{2+}(\mathcal{F}_n)} E_n \Bigg[d\xi + \frac{\gamma_n}{2}\xi^2 + \gamma_n \lambda_n (x - \zeta_n)^2 + 2\gamma_n \lambda_n (x - \zeta_n)\xi + \gamma_n \lambda_n \xi^2 \\
&\qquad + \gamma_{n+1} Y_{n+1} \left(\frac{d\beta_{n+1}}{\gamma_{n+1}} - x + \left(\frac{\beta_{n+1}}{\Gamma_{n+1}} - 1 \right) \xi \right)^2 - \frac{d^2 \beta_{n+1}^2}{2\gamma_{n+1}} \\
&\qquad - \frac{d\beta_{n+1}^2}{\Gamma_{n+1}}\xi - \frac{\beta_{n+1}^2}{2\Gamma_{n+1}}\gamma_n \xi^2 - 2\sqrt{\gamma_{n+1}}\psi_{n+1} \left(\frac{d\beta_{n+1}}{\gamma_{n+1}} - x + \left(\frac{\beta_{n+1}}{\Gamma_{n+1}} - 1 \right)\xi \right) \\
&\qquad + V_{n+1}(0,0) \Bigg] \\
&= \operatorname*{ess\,inf}_{\xi \in L^{2+}(\mathcal{F}_n)} \Bigg[\gamma_n E_n \left[\frac{1}{2} + \lambda_n + \Gamma_{n+1} Y_{n+1} \left(\frac{\beta_{n+1}}{\Gamma_{n+1}} - 1 \right)^2 - \frac{\beta_{n+1}^2}{2\Gamma_{n+1}} \right] \xi^2 \\
&\qquad + E_n \Bigg[d + 2\gamma_n \lambda_n (x - \zeta_n) + 2\gamma_{n+1} Y_{n+1} \left(\frac{\beta_{n+1}}{\Gamma_{n+1}} - 1 \right) \left(\frac{d\beta_{n+1}}{\gamma_{n+1}} - x \right) \\
&\qquad\qquad - \frac{d\beta_{n+1}^2}{\Gamma_{n+1}} - 2\sqrt{\gamma_{n+1}}\psi_{n+1} \left(\frac{\beta_{n+1}}{\Gamma_{n+1}} - 1 \right) \Bigg] \xi \\
&\qquad + E_n \Bigg[\gamma_n \lambda_n (x - \zeta_n)^2 + \gamma_{n+1} Y_{n+1} \left(\frac{d\beta_{n+1}}{\gamma_{n+1}} - x \right)^2 - \frac{d^2 \beta_{n+1}^2}{2\gamma_{n+1}} \\
&\qquad\qquad - 2\sqrt{\gamma_{n+1}}\psi_{n+1} \left(\frac{d\beta_{n+1}}{\gamma_{n+1}} - x \right) + V_{n+1}(0,0) \Bigg] \Bigg].
\end{aligned}$$

We thus obtain the representation

$$V_n(x,d) = \operatorname*{ess\,inf}_{\xi \in L^{2+}(\mathcal{F}_n)} \left[a_n \xi^2 + b_n(x,d)\xi + c_n(x,d) \right], \quad x, d \in \mathbb{R}. \tag{2.17}$$

For all $x, d \in \mathbb{R}$ we find $\xi_n^*(x,d) = -\frac{b_n(x,d)}{2a_n}$ to be the unique minimizer of $\xi \mapsto a_n\xi^2 + b_n(x,d)\xi + c_n(x,d)$. Note that, for all $x, d \in \mathbb{R}$, the facts that $b_n(x,d) \in L^{2+}(\mathcal{F}_n)$ and $\frac{1}{a_n} \in L^{\infty-}(\mathcal{F}_n)$ imply that $\xi_n^*(x,d) \in L^{2+}(\mathcal{F}_n)$. Observe further that for all $x, d \in \mathbb{R}$ it holds that

$$\begin{aligned}
&-\frac{b_n(x,d)}{2a_n} \\
&= -\frac{E_n\left[\frac{d}{\gamma_n}\frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + Y_{n+1}\left(\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\frac{d}{\gamma_n} - \beta_{n+1}\frac{d}{\gamma_n} - \beta_{n+1}x + \Gamma_{n+1}x\right)\right]}{a_n/\gamma_n} \\
&\quad - \frac{E_n\left[\sqrt{\gamma_{n+1}}\psi_{n+1}\left(1 - \frac{\beta_{n+1}}{\Gamma_{n+1}}\right)\right] + \gamma_n\lambda_n(x - \zeta_n)}{a_n} \\
&= -\frac{d}{\gamma_n}\frac{E_n\left[\frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + \frac{Y_{n+1}}{\Gamma_{n+1}}(\beta_{n+1} - \Gamma_{n+1})^2 + \lambda_n + Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n}{a_n/\gamma_n} \\
&\quad - x\frac{E_n\left[Y_{n+1}(\Gamma_{n+1} - \beta_{n+1})\right] + \lambda_n}{a_n/\gamma_n} - \frac{E_n\left[\sqrt{\gamma_{n+1}}\psi_{n+1}\left(1 - \frac{\beta_{n+1}}{\Gamma_{n+1}}\right)\right] - \gamma_n\lambda_n\zeta_n}{a_n} \\
&= -\frac{d}{\gamma_n} + \left(x - \frac{d}{\gamma_n}\right)\frac{E_n\left[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n}{a_n/\gamma_n} \\
&\quad - \frac{E_n\left[\sqrt{\Gamma_{n+1}}\psi_{n+1}\left(1 - \frac{\beta_{n+1}}{\Gamma_{n+1}}\right)\right] - \sqrt{\gamma_n}\lambda_n\zeta_n}{a_n/\sqrt{\gamma_n}},
\end{aligned}$$

which yields the representation of $\xi_n^*(x,d)$ in (2.10).

Representation of the value function $V_n(x,d)$. We next establish representation (2.9) of the value function $V_n(x,d)$.

By inserting the optimal trade size $\xi_n^*(x,d) = -\frac{b_n(x,d)}{2a_n}$ into (2.17), we obtain for all $x, d \in \mathbb{R}$ that

$$V_n(x,d) = -\frac{b_n(x,d)^2}{4a_n} + c_n(x,d). \tag{2.18}$$

Note that by (2.18) and (2.12) it holds that for almost all $\omega \in \Omega$, V_n is a quadratic function in $(x,d) \in \mathbb{R}^2$. We thus have for all $x, d \in \mathbb{R}$ that

$$\begin{aligned}
V_n(x,d) = {} & \frac{(\partial_{dd}^2 V_n)(0,0)}{2}d^2 + \frac{(\partial_{xx}^2 V_n)(0,0)}{2}x^2 + [(\partial_{dx}^2 V_n)(0,0)]xd \\
& + [(\partial_d V_n)(0,0)]d + [(\partial_x V_n)(0,0)]x + V_n(0,0).
\end{aligned} \tag{2.19}$$

The dynamic programming principle ensures for all $x, d, h \in \mathbb{R}$ that

$$\begin{aligned}
V_n(x,d) - \left(d + \frac{\gamma_n}{2}h\right) h &= \operatorname*{ess\,inf}_{X \in \mathcal{A}_n^{\mathrm{disc}}(x,d)} \Bigg[\left(d + \frac{\gamma_n}{2}\xi_n\right) \xi_n + \gamma_n \lambda_n (X_n - \zeta_n)^2 - \left(d + \frac{\gamma_n}{2}h\right) h \\
&\qquad + E_n \left[V_{n+1}(x + \xi_n, (d + \gamma_n \xi_n)\beta_{n+1})\right] \Bigg] \\
&= \operatorname*{ess\,inf}_{\xi \in L^{2+}(\mathcal{F}_n)} \Bigg[\left(d + \frac{\gamma_n}{2}(\xi + h)\right) (\xi - h) + \gamma_n \lambda_n (x + \xi - \zeta_n)^2 \\
&\qquad + E_n \left[V_{n+1}(x + \xi, (d + \gamma_n \xi)\beta_{n+1})\right] \Bigg] \\
&= \operatorname*{ess\,inf}_{\tilde{\xi} \in L^{2+}(\mathcal{F}_n)} \Bigg[\left(d + \gamma_n h + \frac{\gamma_n}{2}\tilde{\xi}\right) \tilde{\xi} + \gamma_n \lambda_n (x + h + \tilde{\xi} - \zeta_n)^2 \\
&\qquad + E_n \left[V_{n+1}(x + h + \tilde{\xi}, (d + \gamma_n (h + \tilde{\xi}))\beta_{n+1})\right] \Bigg] \\
&= V_n(x + h, d + \gamma_n h).
\end{aligned}$$

It follows for all $x, d \in \mathbb{R}$ that

$$(\partial_x V_n)(x,d) + \gamma_n (\partial_d V_n)(x,d) \leftarrow \frac{V_n(x+h, d+\gamma_n h) - V_n(x,d)}{h} = -\left(d + \frac{\gamma_n}{2}h\right) \to -d \tag{2.20}$$

as $h \to 0$. Consequently, we obtain that

$$\begin{aligned}
(\partial_x V_n)(0,0) + \gamma_n (\partial_d V_n)(0,0) &= 0, \\
(\partial^2_{xx} V_n)(0,0) + \gamma_n (\partial^2_{dx} V_n)(0,0) &= 0, \\
(\partial^2_{xd} V_n)(0,0) + \gamma_n (\partial^2_{dd} V_n)(0,0) &= -1.
\end{aligned}$$

This implies that

$$\begin{aligned}
(\partial_d V_n)(0,0) &= -\frac{1}{\gamma_n} (\partial_x V_n)(0,0), \\
(\partial^2_{dx} V_n)(0,0) &= -\frac{1}{\gamma_n} (\partial^2_{xx} V_n)(0,0), \\
(\partial^2_{dd} V_n)(0,0) &= \frac{1}{\gamma_n} \big(-(\partial^2_{dx} V_n)(0,0) - 1\big) = \frac{1}{\gamma_n^2} (\partial^2_{xx} V_n)(0,0) - \frac{1}{\gamma_n}.
\end{aligned}$$

Inserting this into (2.19) proves for all $x, d \in \mathbb{R}$ that

$$V_n(x,d) = \frac{(\partial^2_{xx} V_n)(0,0)}{2} \left(\frac{d}{\gamma_n} - x\right)^2 - \frac{d^2}{2\gamma_n} + (\partial_x V_n)(0,0) \left(x - \frac{d}{\gamma_n}\right) + V_n(0,0). \tag{2.21}$$

We obtain $\frac{(\partial_{xx}^2 V_n)(0,0)}{2}$ (resp. $(\partial_x V_n)(0,0)$) by identifying the coefficient of x^2 (resp. x) in (2.18) using (2.12). We therefore consider for all $x \in \mathbb{R}$

$$
\begin{aligned}
V_n(x,0) &= -\frac{b_n(x,0)^2}{4a_n} + c_n(x,0) \\
&= -\frac{\left(E_n\left[\sqrt{\gamma_{n+1}}\psi_{n+1}\left(1-\frac{\beta_{n+1}}{\Gamma_{n+1}}\right) - \gamma_n\lambda_n\zeta_n\right] + E_n\left[Y_{n+1}\gamma_{n+1}\left(1-\frac{\beta_{n+1}}{\Gamma_{n+1}}\right) + \gamma_n\lambda_n\right]x\right)^2}{a_n} \\
&\quad + E_n\left[Y_{n+1}\gamma_{n+1}\right]x^2 + E_n\left[2\sqrt{\gamma_{n+1}}\psi_{n+1}\right]x + \gamma_n\lambda_n(x-\zeta_n)^2 + E_n\left[V_{n+1}(0,0)\right] \\
&= \left(E_n\left[\gamma_{n+1}Y_{n+1}\right] + \gamma_n\lambda_n - \frac{\gamma_n\left(E_n\left[Y_{n+1}\left(\Gamma_{n+1}-\beta_{n+1}\right)+\lambda_n\right]\right)^2}{a_n/\gamma_n}\right)x^2 \\
&\quad + \left(E_n\left[2\sqrt{\gamma_{n+1}}\psi_{n+1}\right] - 2\gamma_n\lambda_n\zeta_n\right)x \\
&\quad - \frac{2E_n\left[\sqrt{\gamma_{n+1}}\psi_{n+1}\left(1-\frac{\beta_{n+1}}{\Gamma_{n+1}}\right) - \gamma_n\lambda_n\zeta_n\right]E_n\left[Y_{n+1}\left(\Gamma_{n+1}-\beta_{n+1}\right)+\lambda_n\right]}{a_n/\gamma_n}x \\
&\quad + \gamma_n\lambda_n\zeta_n^2 - \frac{\left(E_n\left[\sqrt{\gamma_{n+1}}\psi_{n+1}\left(1-\frac{\beta_{n+1}}{\Gamma_{n+1}}\right) - \gamma_n\lambda_n\zeta_n\right]\right)^2}{a_n} \\
&\quad + E_n\left[\frac{\gamma_N}{2}\hat{\xi}^2 + \gamma_N\lambda_N\left(\hat{\xi}-\zeta_N\right)^2 + \sum_{j=n+1}^{N-1}\gamma_j\lambda_j\zeta_j^2\right] \\
&\quad - \sum_{j=n+1}^{N-1}E_n\left[\frac{\left(E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1-\frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right] - \sqrt{\gamma_j}\lambda_j\zeta_j\right)^2}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2 + \frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right) + \lambda_j\right]}\right] \\
&= \gamma_n Y_n x^2 + 2\sqrt{\gamma_n}\psi_n x + E_n\left[\frac{\gamma_N}{2}\hat{\xi}^2 + \gamma_N\lambda_N\left(\hat{\xi}-\zeta_N\right)^2 + \sum_{j=n}^{N-1}\gamma_j\lambda_j\zeta_j^2\right] \\
&\quad - \sum_{j=n}^{N-1}E_n\left[\frac{\left(E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1-\frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right] - \sqrt{\gamma_j}\lambda_j\zeta_j\right)^2}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2 + \frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right) + \lambda_j\right]}\right].
\end{aligned}
$$

It then follows from (2.21) for all $x, d \in \mathbb{R}$ that

$$
\begin{aligned}
V_n(x,d) = \gamma_n Y_n\left(\frac{d}{\gamma_n} - x\right)^2 - \frac{d^2}{2\gamma_n} + 2\sqrt{\gamma_n}\psi_n\left(x - \frac{d}{\gamma_n}\right) \\
+ E_n\left[\frac{\gamma_N}{2}\hat{\xi}^2 + \gamma_N\lambda_N\left(\hat{\xi}-\zeta_N\right)^2 + \sum_{j=n}^{N-1}\gamma_j\lambda_j\zeta_j^2\right] \\
- \sum_{j=n}^{N-1}E_n\left[\frac{\left(E_j\left[\sqrt{\Gamma_{j+1}}\psi_{j+1}\left(1-\frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\right] - \sqrt{\gamma_j}\lambda_j\zeta_j\right)^2}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2 + \frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right) + \lambda_j\right]}\right],
\end{aligned}
$$

which equals (2.9).

Bounds for Y_n. To show that $Y_n > 0$, observe that

$$\begin{aligned}
\frac{\gamma_n}{a_n} Y_n &= E_n\left[\Gamma_{n+1} Y_{n+1} + \lambda_n\right] \frac{a_n}{\gamma_n} - \left(E_n\left[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n\right)^2 \\
&= E_n\left[\Gamma_{n+1} Y_{n+1} + \lambda_n\right] E_n\left[Y_{n+1}\left(\frac{\beta_{n+1}^2}{\Gamma_{n+1}} - 2\beta_{n+1} + \Gamma_{n+1}\right) + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right) + \lambda_n\right] \\
&\quad - \left(E_n\left[\Gamma_{n+1} Y_{n+1} + \lambda_n\right] - E_n\left[\beta_{n+1} Y_{n+1}\right]\right)^2 \\
&= E_n\left[\Gamma_{n+1} Y_{n+1} + \lambda_n\right] E_n\left[Y_{n+1} \frac{\beta_{n+1}^2}{\Gamma_{n+1}} + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right)\right] - \left(E_n\left[\beta_{n+1} Y_{n+1}\right]\right)^2 .
\end{aligned} \tag{2.22}$$

Since $Y_{n+1} > 0$ by the induction hypothesis and $\gamma > 0$ and $\lambda \geq 0$, we have that

$$E_n\left[\Gamma_{n+1} Y_{n+1} + \lambda_n\right] \geq E_n\left[\Gamma_{n+1} Y_{n+1}\right] > 0. \tag{2.23}$$

Similarly, it holds that

$$E_n\left[Y_{n+1} \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] \geq 0. \tag{2.24}$$

By the assumption $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$, we moreover have that

$$E_n\left[\frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right)\right] = \frac{1}{2}\left(1 - E_n\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right]\right) > 0. \tag{2.25}$$

It now follows from (2.22)–(2.25) that

$$\frac{\gamma_n}{a_n} Y_n > E_n\left[\Gamma_{n+1} Y_{n+1}\right] E_n\left[Y_{n+1} \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - \left(E_n\left[\beta_{n+1} Y_{n+1}\right]\right)^2 . \tag{2.26}$$

The Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\left(E_n\left[\beta_{n+1} Y_{n+1}\right]\right)^2 &= \left(E_n\left[\frac{\beta_{n+1}}{\sqrt{\Gamma_{n+1}}} \sqrt{Y_{n+1}} \sqrt{\Gamma_{n+1}} \sqrt{Y_{n+1}}\right]\right)^2 \\
&\leq E_n\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}} Y_{n+1}\right] E_n\left[\Gamma_{n+1} Y_{n+1}\right] .
\end{aligned} \tag{2.27}$$

We thus have from (2.26) and (2.27) that $\frac{\gamma_n}{a_n} Y_n > 0$. Since $\gamma_n, a_n > 0$, we conclude that $Y_n > 0$.

For the upper bound, note that

$$
\begin{aligned}
Y_n &= E_n\left[\Gamma_{n+1} Y_{n+1}\right] + \lambda_n - \frac{\left(E_n\left[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n\right)^2}{a_n/\gamma_n} \\
&\le E_n\left[\Gamma_{n+1} Y_{n+1}\right] + \lambda_n - \frac{\left(E_n\left[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n\right)^2}{a_n/\gamma_n} \\
&\quad + \frac{a_n}{\gamma_n}\left(\frac{E_n\left[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n}{a_n/\gamma_n} + 1\right)^2 \\
&= E_n\left[\Gamma_{n+1} Y_{n+1}\right] + \lambda_n + 2\left(E_n\left[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})\right] - \lambda_n\right) + \frac{a_n}{\gamma_n} \\
&= E_n\left[-\Gamma_{n+1} Y_{n+1} + 2 Y_{n+1}\beta_{n+1} + \frac{Y_{n+1}}{\Gamma_{n+1}}(\beta_{n+1} - \Gamma_{n+1})^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right)\right] \\
&\quad + \lambda_n - 2\lambda_n + \lambda_n \\
&= \frac{1}{2} + E_n\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\left(Y_{n+1} - \frac{1}{2}\right)\right].
\end{aligned}
\tag{2.28}
$$

From the induction hypothesis $Y_{n+1} \le \frac{1}{2}$ we then obtain that $Y_n \le \frac{1}{2}$.

Integrability property for ψ_n. Clearly, ψ_n from (2.8) is $\mathcal{F}_n$-measurable. By the integrability assumptions that $\beta_k, \gamma_k, \frac{1}{\gamma_k}, \lambda_k \in L^{\infty-}$ for all $k \in \mathbb{Z} \cap (-\infty, N]$ and boundedness of Y_{n+1}, we obtain that $E_n[Y_{n+1}(\beta_{n+1} - \Gamma_{n+1})] - \lambda_n \in L^{\infty-}$. This together with $\frac{\gamma_n}{a_n} \in L^{\infty-}$ shows that

$$
\frac{E_n\left[Y_{n+1}\left(\beta_{n+1} - \Gamma_{n+1}\right)\right] - \lambda_n}{a_n/\gamma_n} \in L^{\infty-}. \tag{2.29}
$$

It then follows from (2.15) that the last term in (2.8) is in L^{2+}. Moreover, the induction hypothesis $\psi_{n+1} \in L^{2+}$ and the assumptions $\gamma_k, \frac{1}{\gamma_k} \in L^{\infty-}$ for all $k \in \mathbb{Z}$ yield that $E_n[\sqrt{\Gamma_{n+1}}\psi_{n+1}] \in L^{2+}$. Since $\gamma_n, \lambda_n \in L^{\infty-}$ and $\zeta_n \in L^{2+}$, we also have that $\sqrt{\gamma_n}\lambda_n\zeta_n \in L^{2+}$. Hence, $\psi_n \in L^{2+}(\mathcal{F}_n)$.

This completes the induction step.

Proof of (iv)

In the remainder of the proof of Theorem 2.2.1 we show for all $n \in \mathbb{Z} \cap (-\infty, N-1]$, $x, d \in \mathbb{R}$, that the process $X^* = (X_k^*)_{k \in \{n-1,\dots,N\}}$ recursively defined by (2.11) is in $\mathcal{A}_n^{\text{disc}}(x,d)$. It is obvious that then, $\xi^* = (\xi_k^*(X_{k-1}^*, D_{k-}^*))_{k \in \{n,\dots,N\}}$ and $D^* = (D_{k-}^*)_{k \in \{n,\dots,N\}}$ are the associated trade process and deviation process, respectively, and uniqueness is a consequence of part (iii).

To this end, we let $n \in \mathbb{Z} \cap (-\infty, N-1]$, $x, d \in \mathbb{R}$, define $X^* = (X_k^*)_{k \in \{n-1,\dots,N\}}$ by (2.11), and show by (forward) induction on $k \in \{n, \dots, N\}$ that X_k^* (and $\xi_k^*(X_{k-1}^*, D_{k-}^*)$) are in $L^{2+}(\mathcal{F}_k)$ for all $k \in \{n, \dots, N\}$.

For the base case $k = n$ we have $\xi_n^*(X_{n-1}^*, D_{n-}^*) = \xi_n^*(x,d)$, which by part (iii) is already known to be in $L^{2+}(\mathcal{F}_n)$. This further implies that $X_n^* = x + \xi_n^*(x,d) \in L^{2+}(\mathcal{F}_n)$.

We continue with the induction step $\{n,\dots,N-2\} \ni k-1 \to k \in \{n+1,\dots,N-1\}$. By the induction hypothesis, it holds that $X_j \in L^{2+}(\mathcal{F}_j)$ and $\xi_j^*(X_{j-1}^*, D_{j-}^*) \in L^{2+}(\mathcal{F}_j)$ for all $j \in \{n,\dots,k-1\}$. As in Remark 2.1.2, we can therefore obtain that $D_{k-}^* \in L^{2+}(\mathcal{F}_k)$. Now, consider

$$\begin{aligned}\xi_k^*(X_{k-1}^*, D_{k-}^*) = {} & \frac{E_k\left[Y_{k+1}(\beta_{k+1}-\Gamma_{k+1})\right]-\lambda_k}{E_k\left[\frac{Y_{k+1}}{\Gamma_{k+1}}\left(\beta_{k+1}-\Gamma_{k+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{k+1}^2}{\Gamma_{k+1}}\right)+\lambda_k\right]}\left(X_{k-1}^*-\frac{D_{k-}^*}{\gamma_k}\right) \\ & -\frac{D_{k-}^*}{\gamma_k}-\frac{E_k\left[\sqrt{\Gamma_{k+1}}\psi_{k+1}\left(1-\frac{\beta_{k+1}}{\Gamma_{k+1}}\right)\right]-\sqrt{\gamma_k}\lambda_k\zeta_k}{\sqrt{\gamma_k}E_k\left[\frac{Y_{k+1}}{\Gamma_{k+1}}\left(\beta_{k+1}-\Gamma_{k+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{k+1}^2}{\Gamma_{k+1}}\right)+\lambda_k\right]}.\end{aligned} \tag{2.30}$$

Clearly, this is $\mathcal{F}_k$-measurable. To prove that $\xi_k^*(X_{k-1}^*, D_{k-}^*) \in L^{2+}$, note that by Minkowski's inequality, it suffices to show that each summand is in L^{2+}. To begin with, it holds that $\frac{D_{k-}^*}{\gamma_k} \in L^{2+}$ due to $D_{k-}^* \in L^{2+}$, $\frac{1}{\gamma_k} \in L^{\infty-}$. Since $X_{k-1}^* \in L^{2+}$, we moreover have that $X_{k-1}^* - \frac{D_{k-}^*}{\gamma_k} \in L^{2+}$. It further follows with (2.29) that

$$\frac{E_k\left[Y_{k+1}\left(\beta_{k+1}-\Gamma_{k+1}\right)\right]-\lambda_k}{E_k\left[\frac{Y_{k+1}}{\Gamma_{k+1}}\left(\beta_{k+1}-\Gamma_{k+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{k+1}^2}{\Gamma_{k+1}}\right)+\lambda_k\right]}\left(X_{k-1}^*-\frac{D_{k-}^*}{\gamma_k}\right) \in L^{2+}. \tag{2.31}$$

From $\frac{1}{\gamma_k} \in L^{\infty-}$, which implies $\frac{1}{\sqrt{\gamma_k}} \in L^{\infty-}$, and (2.16) we have that the last term in (2.30) is in L^{2+} as well. Therefore, $\xi_k^*(X_{k-1}^*, D_{k-}^*) \in L^{2+}(\mathcal{F}_k)$, which together with $X_{k-1}^* \in L^{2+}(\mathcal{F}_{k-1})$ from the induction hypothesis implies that $X_k^* = X_{k-1}^* + \xi_k^*(X_{k-1}^*, D_{k-}^*) \in L^{2+}(\mathcal{F}_k)$. This finishes the induction step $\{n,\dots,N-2\} \ni k-1 \to k \in \{n+1,\dots,N-1\}$.

Finally, it also holds true that $X_N^* = X_{N-1}^* + \xi_N^*(X_{N-1}^*, D_{N-}^*) = X_{N-1}^* + \hat{\xi} - X_{N-1}^* = \hat{\xi} \in L^{2+}(\mathcal{F}_N)$. As a result, $X^* \in \mathcal{A}_n^{\text{disc}}(x,d)$.

This completes the proof of Theorem 2.2.1.

2.2.2 Comments on the main theorem

We first have the following supplement to Theorem 2.2.1.

Remark 2.2.2. Suppose that the assumptions of Theorem 2.2.1 are satisfied and that $\hat{\xi} \in L^{\infty-}$ and $\zeta_k \in L^{\infty-}$ for all $k \in \mathbb{Z} \cap (-\infty, N]$. Then, by straightforward modifications of the integrability arguments in the proof of Theorem 2.2.1, we see that ψ_n, $\xi_n^*(x,d)$, and X_k^* are in $L^{\infty-}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$, $k \in \{n-1,\dots,N\}$, $x, d \in \mathbb{R}$.

Next, observe that we have the following representation in Remark 2.2.3 below for the process ψ from Theorem 2.2.1. The recursion for ψ itself is removed, although not the recursion entering ψ indirectly via Y.

Remark 2.2.3. Under the assumptions and with the notations of Theorem 2.2.1, it holds for all $n \in \mathbb{Z} \cap (-\infty, N]$ that

$$\begin{aligned}\psi_n = &-E_n\left[\frac{\gamma_N \hat{\xi}}{2\sqrt{\gamma_n}} \prod_{j=n}^{N-1}\left(1+\left(1-\frac{\beta_{j+1}}{\Gamma_{j+1}}\right)\frac{E_j\left[Y_{j+1}(\beta_{j+1}-\Gamma_{j+1})\right]-\lambda_j}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right)\right]+\lambda_j}\right)\right]\\ &-\sum_{j=n}^{N-1} E_n\left[\frac{\gamma_j\lambda_j\zeta_j}{\sqrt{\gamma_n}}\left(1+\frac{E_j\left[Y_{j+1}(\beta_{j+1}-\Gamma_{j+1})\right]-\lambda_j}{E_j\left[\frac{Y_{j+1}}{\Gamma_{j+1}}\left(\beta_{j+1}-\Gamma_{j+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{j+1}^2}{\Gamma_{j+1}}\right)\right]+\lambda_j}\right)\right.\\ &\qquad\left.\cdot\prod_{k=n}^{j-1}\left(1+\left(1-\frac{\beta_{k+1}}{\Gamma_{k+1}}\right)\frac{E_k\left[Y_{k+1}(\beta_{k+1}-\Gamma_{k+1})\right]-\lambda_k}{E_k\left[\frac{Y_{k+1}}{\Gamma_{k+1}}\left(\beta_{k+1}-\Gamma_{k+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{k+1}^2}{\Gamma_{k+1}}\right)\right]+\lambda_k}\right)\right].\end{aligned}$$

This can be shown by backward induction.

In particular, it is evident from Remark 2.2.3 that if $\hat{\xi} \in L^{2+}(\mathcal{F}_n)$ is known at initial time $n \in \mathbb{Z} \cap (-\infty, N]$, then ψ_k for all $k \in \{n, \dots, N\}$ and almost all $\omega \in \Omega$ is an affine-linear function (depending on k) of the target position $\hat{\xi}$. A similar observation holds for the process ζ provided that ζ is known at initial time $n \in \mathbb{Z} \cap (-\infty, N]$. In contrast, the involvement of the process λ is more complicated, as it enters ψ directly at several places and also indirectly via Y. We can however observe that for $\lambda \equiv 0$, the second of the two parts in the representation for ψ in Remark 2.2.3 vanishes. This is also the case if $\zeta \equiv 0$. For $\hat{\xi} = 0$, the first part of the representation for ψ vanishes.

We can thus summarize that, if $\hat{\xi} = 0$ and at least one of ζ, λ vanishes, then $\psi \equiv 0$. Furthermore, in this case, the minimal costs (2.9) in Theorem 2.2.1 for $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, simplify to

$$V_n(x,d) = \frac{Y_n}{\gamma_n}\left(d - \gamma_n x\right)^2 - \frac{d^2}{2\gamma_n}, \tag{2.32}$$

and the optimal trade size for $n \in \mathbb{Z} \cap (-\infty, N-1]$, $x, d \in \mathbb{R}$, becomes

$$\xi_n^*(x,d) = \frac{E_n\left[Y_{n+1}(\beta_{n+1}-\Gamma_{n+1})\right]-\lambda_n}{E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1}-\Gamma_{n+1}\right)^2+\frac{1}{2}\left(1-\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right)\right]+\lambda_n}\left(x-\frac{d}{\gamma_n}\right)-\frac{d}{\gamma_n}. \tag{2.33}$$

In Corollary 2.2.4 below we state that in the important subsetting where $\hat{\xi} = 0$, $\zeta \equiv 0$, and $\lambda \equiv 0$, i.e., when one considers a risk-neutral agent who needs to close a position, Theorem 2.2.1 reduces to [AKU21b, Theorem 2.1]. Observe that the assumptions in [AKU21b, Theorem 2.1] and Theorem 2.2.1 aside from the newly introduced $\hat{\xi}$, ζ, and λ are the same.

Corollary 2.2.4. *Suppose that $\hat{\xi} = 0$, $\zeta \equiv 0$, and $\lambda \equiv 0$. Assume that for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ it holds that $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ a.s. and that $(1 - E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}])^{-1} \in L^{\infty-}$. Let $(Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ be the process that is recursively defined by $Y_N = \frac{1}{2}$ and*

$$Y_n = E_n[\Gamma_{n+1} Y_{n+1}] - \frac{\left(E_n\left[Y_{n+1}\left(\beta_{n+1} - \Gamma_{n+1}\right)\right]\right)^2}{E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1} - \Gamma_{n+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right)\right]}, n \in \mathbb{Z} \cap (-\infty, N-1]. \tag{2.34}$$

Then it holds for all $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, that

$$V_n(x,d) = \frac{Y_n}{\gamma_n}\left(d - \gamma_n x\right)^2 - \frac{d^2}{2\gamma_n} \quad \text{and} \quad 0 < Y_n \leq \frac{1}{2}.$$

Moreover, for all $x, d \in \mathbb{R}$ the (up to a P-null set) unique optimal trade size is given by

$$\xi_n^*(x,d) = \frac{E_n[Y_{n+1}\left(\beta_{n+1} - \Gamma_{n+1}\right)]}{E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1} - \Gamma_{n+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right)\right]}\left(x - \frac{d}{\gamma_n}\right) - \frac{d}{\gamma_n}, n \in \mathbb{Z} \cap (-\infty, N-1], \tag{2.35}$$

and $\xi_N^*(x,d) = -x$, and we have $\xi_n^*(x,d) \in L^{\infty-}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$ and $x, d \in \mathbb{R}$.

Proof. Since $\lambda \equiv 0$, the process defined by (2.7) and the process defined by (2.34) coincide. The assumptions $\hat{\xi} = 0$, $\zeta \equiv 0$, and $\lambda \equiv 0$ imply that $\psi \equiv 0$ (cf. the representation in Remark 2.2.3). The claims thus follow from Theorem 2.2.1 (note also Remark 2.2.2). □

The requirement $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ a.s. for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ in Theorem 2.2.1 and [AKU21b, Theorem 2.1] is a structural assumption. It ensures that the minimization problem (2.4) preserves its structure with increasing number of time steps. More precisely, under this assumption the coefficients a_n in front of ξ^2 in (2.17) and the random variables Y_n in (2.7) stay positive at all times $n \in \mathbb{Z} \cap (-\infty, N-1]$. To further discuss the condition $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ a.s. for all $n \in \mathbb{Z} \cap (-\infty, N-1]$, we consider a two-period version of the problem.

Since $Y_N = \frac{1}{2}$, we can show for time $N-1$ that

$$a_{N-1} = \frac{\gamma_{N-1}}{2} E_{N-1}\left[\Gamma_N + 1 - 2\beta_N + 2\lambda_{N-1}\right]$$

and

$$Y_{N-1} = \frac{E_{N-1}\left[\Gamma_N\right] - \left(E_{N-1}\left[\beta_N\right]\right)^2 + 2\lambda_{N-1}}{2E_{N-1}\left[\Gamma_N + 1 - 2\beta_N + 2\lambda_{N-1}\right]}.$$

We already see that for Y_{N-1} to be well-defined, we need to require $a_{N-1} \neq 0$. Furthermore, note that by (2.17) the value function has the structure

$$V_{N-1}(x,d) = \operatorname*{ess\,inf}_{\xi \in L^{2+}(\mathcal{F}_{N-1})} \left[a_{N-1}\xi^2 + b_{N-1}(x,d)\xi + c_{N-1}(x,d)\right] \tag{2.36}$$

for all $x, d \in \mathbb{R}$. The quadratic function $\xi \mapsto a_{N-1}\xi^2 + b_{N-1}(x,d)\xi + c_{N-1}(x,d)$ for all $x, d \in \mathbb{R}$ is strictly convex (resp. strictly concave) if and only if $a_{N-1} > 0$ (resp. $a_{N-1} < 0$). Therefore, in the case $a_{N-1} < 0$, the minimization problem in (2.36) is ill-posed in the sense that one can generate infinite gains (in the limit) by choosing strategies with $|\xi| \to \infty$. We thus demand that $a_{N-1} > 0$. This guarantees that there exists a (unique) minimizer in (2.36).

$a_{N-1} > 0$ is however not sufficient to ensure that also $Y_{N-1} > 0$: Consider, e.g., $\beta \equiv \frac{1}{2}$, $\Gamma \equiv \frac{1}{8}$, $\lambda \equiv 0$. Then, $a_{N-1} = \frac{\gamma_{N-1}}{16} > 0$, but $Y_{N-1} = -\frac{1}{2} < 0$ and further $a_{N-2} = -\frac{17}{16}\gamma_{N-2} < 0$. This example furthermore shows that for $Y_{N-1} < 0$, a_{N-2} can become negative, which leads to an ill-posed minimization problem at time $N-2$. As a consequence, we need to impose further conditions on β_{N-1}, Γ_{N-1}, and λ_{N-1}. More precisely, given $a_{N-1} > 0$, it holds that $Y_{N-1} > 0$ if and only if $E_{N-1}[\Gamma_N] - (E_{N-1}[\beta_N])^2 + 2\lambda_{N-1} > 0$.

Note that the Cauchy-Schwarz inequality implies that

$$(E_{N-1}[\beta_N])^2 \leq E_{N-1}\left[\frac{\beta_N^2}{\Gamma_N}\right] E_{N-1}[\Gamma_N],$$

and hence it holds that

$$\frac{2E_{N-1}[\beta_N] - 1}{E_{N-1}[\Gamma_N]} \leq \frac{(E_{N-1}[\beta_N])^2}{E_{N-1}[\Gamma_N]} \leq E_{N-1}\left[\frac{\beta_N^2}{\Gamma_N}\right].$$

It thus follows that on the event $\{E_{N-1}[\frac{\beta_N^2}{\Gamma_N}] < 1\}$, we have $E_{N-1}[\Gamma_N + 1 - 2\beta_N] > 0$ and $E_{N-1}[\Gamma_N] - (E_{N-1}[\beta_N])^2 > 0$, which imply that a_{N-1} and Y_{N-1} are positive. We remark that the same still holds true on the larger event $\{\frac{(E_{N-1}[\beta_N])^2}{E_{N-1}[\Gamma_N]} < 1\}$. However, replacing the assumption $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ a.s. with the weaker one $\frac{(E_n[\beta_{n+1}])^2}{E_n[\Gamma_{n+1}]} < 1$ a.s. for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ does not in general allow to perform the backward induction, as the structure of the problem can be lost already on the step $N-1 \to N-2$. Namely, Y_{N-1} can be strictly less than $\frac{1}{2}$ (in contrast to $Y_N = \frac{1}{2}$), while $E_{N-2}[\frac{\beta_{N-1}^2}{\Gamma_{N-1}}]$ can be strictly greater than 1 (even assuming $\frac{(E_{N-2}[\beta_{N-1}])^2}{E_{N-2}[\Gamma_{N-1}]} < 1$ a.s.), and we do not necessarily get positivity of a_{N-2} (see (2.12)).

To see that the assumptions of Theorem 2.2.1, in particular the structural assumption discussed above, are satisfied for a reasonably large class of models, consider the following Example 2.2.5. There will be further examples in subsequent sections.

Example 2.2.5. Let $(r_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ and $(\beta_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ be deterministic strictly positive sequences such that

$$\frac{\beta_{n+1}^2 r_n}{r_{n+1}} < 1 \quad \text{for all } n \in \mathbb{Z} \cap (-\infty, N-1],$$

e.g., take $\beta_n \equiv \beta \in (0,1)$ for all $n \in \mathbb{Z} \cap (-\infty, N]$ and $(r_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ a nondecreasing sequence in $(0,\infty)$. Let $(\gamma_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ be given by the formula $\gamma_n = \frac{r_n}{Z_n}$,

$n \in \mathbb{Z} \cap (-\infty, N]$, where $(Z_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is a strictly positive supermartingale such that $Z_n, \frac{1}{Z_n} \in L^{\infty-}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$. Furthermore, take some $\hat{\xi} \in L^{2+}(\mathcal{F}_N)$ and adapted processes ζ, λ with $\zeta_k \in L^{2+}$, $\lambda_k \in L^{\infty-}$, $\lambda_k \geq 0$, $k \in \mathbb{Z} \cap (-\infty, N]$. It is straightforward to see that all assumptions of Theorem 2.2.1 are satisfied.

Finally, we point out that the process Y defined in (2.7) plays a major role in the analysis of the trade execution problem as it is a main ingredient to describe the optimal strategy and the optimal costs, see Theorem 2.2.1. Notice that Y involves λ, but neither $\hat{\xi}$ nor ζ, which enter the solution of the trade execution problem via the process ψ. In the case $\hat{\xi} = 0 \equiv \zeta$, the solution is solely described by the process Y, as ψ vanishes (see (2.32) and (2.33)). On the other hand, if $\hat{\xi}$ is general, but $\lambda \equiv 0$ (which implies that all terms containing ζ vanish as well), then definition (2.7) coincides with definition (2.34) from [AKU21b, Theorem 2.1].

In the subsetting where $\hat{\xi} = 0$ and $\zeta \equiv 0$, the basic observation that, under the assumptions of Theorem 2.2.1, it holds $\gamma_n Y_n = V_n(1, 0)$ for all $n \in \mathbb{Z} \cap (-\infty, N]$ (see also (2.32)), leads to the following improved upper bound for Y.

Remark 2.2.6. Let $n \in \mathbb{Z} \cap (-\infty, N]$, $\hat{\xi} = 0$, and $\zeta \equiv 0$, and suppose that the assumptions of Theorem 2.2.1 are satisfied. Note that for an initial position of size $x = 1$, a possible execution strategy is to sell the whole unit at some time $k \in \{n, \dots, N\}$. For $k \in \{n, \dots, N\}$, such a strategy $X^{(k)} = (X_j^{(k)})_{j\in\{n-1,n,\dots,N\}}$ is given by $X_j^{(k)} = 1$ for all $j \in \{n - 1, \dots, k - 1\}$ and $X_j^{(k)} = 0$ for all $j \in \{k, \dots, N\}$, with associated trade process $\xi^{(k)} = (\xi_j^{(k)})_{j\in\{n,\dots,N\}}$ that satisfies $\xi_k^{(k)} = -1$ and $\xi_j^{(k)} = 0$, $j \in \{n, \dots, N\} \setminus \{k\}$. If there is no initial deviation, i.e., $d = 0$, it follows for all $k \in \{n, \dots, N\}$ that $D_{k-}^{(k)} = 0$ (cf. (2.2)) and that the expected costs of $X^{(k)}$ amount to

$$\begin{aligned}
&E_n\left[\sum_{j=n}^{N}\left(D_{j-}^{(k)} + \frac{\gamma_j}{2}\xi_j^{(k)}\right)\xi_j^{(k)}\right] + E_n\left[\sum_{j=n}^{N}\gamma_j\lambda_j(X_j^{(k)})^2\right] \\
&= E_n\left[\left(D_{k-}^{(k)} + \frac{\gamma_k}{2}\xi_k^{(k)}\right)\xi_k^{(k)}\right] + E_n\left[\sum_{j=n}^{k-1}\gamma_j\lambda_j(X_j^{(k)})^2\right] \\
&= E_n\left[\frac{\gamma_k}{2}\right] + \sum_{j=n}^{k-1} E_n\left[\gamma_j\lambda_j\right].
\end{aligned} \tag{2.37}$$

From Theorem 2.2.1 with $\hat{\xi} = 0$ and $\zeta \equiv 0$, we have that $\gamma_n Y_n = V_n(1, 0)$. Since the expected costs in (2.37) are at least as large as the optimal costs $V_n(1, 0)$, this implies that

$$Y_n \leq \frac{\min_{k\in\{n,\dots,N\}}\left(E_n\left[\gamma_k\right] + 2\sum_{j=n}^{k-1} E_n\left[\gamma_j\lambda_j\right]\right)}{2\gamma_n}. \tag{2.38}$$

Note that

$$\frac{\min_{k\in\{n,\dots,N\}}\left(E_n\left[\gamma_k\right]+2\sum_{j=n}^{k-1}E_n\left[\gamma_j\lambda_j\right]\right)}{2\gamma_n}\leq\frac{E_n\left[\gamma_n\right]}{2\gamma_n}=\frac{1}{2}.$$

Therefore, (2.38) improves the bound $Y_n \leq \frac{1}{2}$ provided by Theorem 2.2.1.

2.3 Processes with independent multiplicative increments

In this section we introduce a subsetting within our general model where the resilience and price impact processes and λ satisfy

(PIMI) for all $k \in \mathbb{Z} \cap (-\infty, N]$ the random variables Γ_k and β_k are independent of $\mathcal{F}_{k-1}$, and λ_k is deterministic.

It turns out that in this case the process Y from Theorem 2.2.1 is deterministic.

Lemma 2.3.1. *Assume* ***(PIMI)*** *and that for all $n \in \mathbb{Z}\cap(-\infty,N]$ it holds $E[\frac{\beta_n^2}{\Gamma_n}] < 1$. Let $Y = (Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ be the process from Theorem 2.2.1 that is recursively defined by $Y_N = \frac{1}{2}$ and* (2.7)*. Then Y is deterministic, $(0, \frac{1}{2}]$-valued, and satisfies the recursion*

$$Y_n = E[\Gamma_{n+1}]Y_{n+1} + \lambda_n - \frac{\left(Y_{n+1}\left(E\left[\beta_{n+1}\right]-E\left[\Gamma_{n+1}\right]\right)-\lambda_n\right)^2}{Y_{n+1}E\left[\frac{(\beta_{n+1}-\Gamma_{n+1})^2}{\Gamma_{n+1}}\right]+\frac{1}{2}\left(1-E\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right]\right)+\lambda_n}, \quad n \in \mathbb{Z}\cap(-\infty,N-1]. \tag{2.39}$$

If furthermore $\hat{\xi} = 0$ and at least one of λ, ζ is equivalent to zero, then formula (2.10) *for optimal trade sizes in the state $(x, d) \in \mathbb{R}^2$ takes the form*

$$\xi_n^*(x,d) = \frac{Y_{n+1}\left(E\left[\beta_{n+1}\right]-E\left[\Gamma_{n+1}\right]\right)-\lambda_n}{Y_{n+1}E\left[\frac{(\beta_{n+1}-\Gamma_{n+1})^2}{\Gamma_{n+1}}\right]+\frac{1}{2}\left(1-E\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right]\right)+\lambda_n}\left(x-\frac{d}{\gamma_n}\right)-\frac{d}{\gamma_n}, \quad n \in \mathbb{Z}\cap(-\infty,N-1], \tag{2.40}$$

and $\xi_N^*(x, d) = -x$.

Proof. Since $Y_N = \frac{1}{2}$ is deterministic and we assume **(PIMI)**, recursion (2.39) follows from (2.7) by a straightforward induction argument. Formula (2.40) is an immediate consequence of (2.33), the assumption **(PIMI)**, and the fact that Y is deterministic. □

We next show that if the resilience moreover at any time has expectation 1, then the process Y stays at $\frac{1}{2}$.

Corollary 2.3.2. *Suppose that the assumptions of Lemma 2.3.1 hold true, and that moreover* $E[\beta_n] = 1$ *for all* $n \in \mathbb{Z} \cap (-\infty, N]$*. It then holds that* $Y_n = \frac{1}{2}$ *for all* $n \in \mathbb{Z} \cap (-\infty, N]$*.*

Proof. Since $E[\beta_n] = 1$ for all $n \in \mathbb{Z} \cap (-\infty, N]$, we obtain from (2.39) for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ that

$$Y_n = \frac{\left(E[\Gamma_{n+1}]Y_{n+1} + \lambda_n\right)\left(Y_{n+1} - \frac{1}{2}\right)\left(E\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - 1\right) + Y_{n+1}\left(Y_{n+1}\left(E[\Gamma_{n+1}] - 1\right) + \lambda_n\right)}{\left(Y_{n+1} - \frac{1}{2}\right)\left(E\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - 1\right) + Y_{n+1}\left(E[\Gamma_{n+1}] - 1\right) + \lambda_n},$$

which in case of $Y_{n+1} = \frac{1}{2}$ equals $\frac{1}{2}$. Due to $Y_N = \frac{1}{2}$, it follows inductively that $Y_n = \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$. □

The situation where $(\Gamma_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$ and $(\beta_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$ are deterministic[4] sequences and $\lambda \equiv 0$ constitutes a particular case of **(PIMI)**. We provide a closed-form expression for recursion (2.39) in this case. In fact, in this case, it is more convenient to work with the quantities

$$Z_k = \frac{1}{2Y_k}, \quad k \in \mathbb{Z} \cap (-\infty, N], \tag{2.41}$$

in place of Y_k, $k \in \mathbb{Z} \cap (-\infty, N]$.

Corollary 2.3.3. *Let* $\lambda \equiv 0$*. Assume that, for all* $k \in \mathbb{Z} \cap (-\infty, N]$*,* Γ_k *and* β_k *are deterministic and satisfy* $\beta_k^2 < \Gamma_k$*. Let the (deterministic) sequence* $Z = (Z_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$ *be defined by* (2.41)*, where the sequence* $Y = (Y_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$ *is recursively defined by* $Y_N = \frac{1}{2}$ *and* (2.39)*. Then* Z *is* $[1, \infty)$*-valued and it holds that*

$$Z_k = \left(\prod_{i=k+1}^{N} \frac{1}{\Gamma_i}\right) + \sum_{j=k+1}^{N} \left(\prod_{i=k+1}^{j} \frac{1}{\Gamma_i}\right) \frac{(\Gamma_j - \beta_j)^2}{\Gamma_j - \beta_j^2}, \quad k \in \mathbb{Z} \cap (-\infty, N]. \tag{2.42}$$

If furthermore $\hat{\xi} = 0$*, formula* (2.40) *for optimal trade sizes in the state* $(x, d) \in \mathbb{R}^2$ *takes the form*

$$\xi_k^*(x, d) = \frac{\beta_{k+1} - \Gamma_{k+1}}{\frac{(\Gamma_{k+1} - \beta_{k+1})^2}{\Gamma_{k+1}} + Z_{k+1}\left(1 - \frac{\beta_{k+1}^2}{\Gamma_{k+1}}\right)}\left(x - \frac{d}{\gamma_k}\right) - \frac{d}{\gamma_k}, \quad k \in \mathbb{Z} \cap (-\infty, N-1], \tag{2.43}$$

and $\xi_N^*(x, d) = -x$*.*

Proof. In the current setting, recursion (2.39) simplifies to $Y_N = \frac{1}{2}$ and

$$Y_k = \frac{\frac{1}{2}\left(1 - \frac{\beta_{k+1}^2}{\Gamma_{k+1}}\right)\Gamma_{k+1}Y_{k+1}}{Y_{k+1}\frac{(\Gamma_{k+1} - \beta_{k+1})^2}{\Gamma_{k+1}} + \frac{1}{2}\left(1 - \frac{\beta_{k+1}^2}{\Gamma_{k+1}}\right)}, \quad k \in \mathbb{Z} \cap (-\infty, N-1],$$

[4] It is worth noting that $(\gamma_k)_{k \in \mathbb{Z} \cap (-\infty, N]}$ can be random.

which, for the sequence Z, yields $Z_N = 1$ and

$$Z_k = \frac{(\Gamma_{k+1} - \beta_{k+1})^2}{\Gamma_{k+1}^2 - \Gamma_{k+1}\beta_{k+1}^2} + \frac{1}{\Gamma_{k+1}} Z_{k+1}, \quad k \in \mathbb{Z} \cap (-\infty, N-1]. \tag{2.44}$$

By backward induction, we obtain (2.42). The fact that Z is $[1,\infty)$-valued follows from the fact that Y is $(0,\frac{1}{2}]$-valued and (2.41). The statement on the optimal trade sizes follows by a straightforward transformation in (2.40). □

The formulas simplify even further when we additionally assume a constant order book depth.

Corollary 2.3.4. *Let $\lambda \equiv 0$. Assume that, for all $k \in \mathbb{Z} \cap (-\infty, N]$, $\gamma_k = \hat{\gamma}$ a.s. with some strictly positive $\bigcap_{k\in\mathbb{Z}} \mathcal{F}_k$-measurable random variable $\hat{\gamma}$ satisfying $\hat{\gamma}, \frac{1}{\hat{\gamma}} \in L^{\infty-}$. In particular, $\Gamma_k = 1$ a.s. for all $k \in \mathbb{Z} \cap (-\infty, N]$. Further, assume that the sequence $(\beta_k)_{k\in\mathbb{Z}\cap(-\infty,N]}$ is deterministic and $(0,1)$-valued. Then we are in the situation of Corollary 2.3.3, formula (2.42) simplifies to*

$$Z_k = 1 + \sum_{j=k+1}^{N} \frac{1-\beta_j}{1+\beta_j}, \quad k \in \mathbb{Z} \cap (-\infty, N],$$

and, if $\hat{\xi} = 0$, formula (2.43) for optimal trade sizes in the state $(x,d) \in \mathbb{R}^2$ takes the form

$$\begin{aligned} \xi_k^*(x,d) &= \frac{1}{1-\beta_{k+1} + (1+\beta_{k+1})Z_{k+1}} \left(\frac{d}{\hat{\gamma}} - x\right) - \frac{d}{\hat{\gamma}} \\ &= \frac{1}{2 + (1+\beta_{k+1})\sum_{j=k+2}^{N} \frac{1-\beta_j}{1+\beta_j}} \left(\frac{d}{\hat{\gamma}} - x\right) - \frac{d}{\hat{\gamma}}, \quad k \in \mathbb{Z} \cap (-\infty, N-1], \end{aligned}$$

and $\xi_N^*(x,d) = -x$.

Proof. Since $\Gamma_k = 1$ for all $k \in \mathbb{Z} \cap (-\infty, N]$, the result follows from Corollary 2.3.3 via straightforward calculations. □

2.4 Long-time horizon

Let the assumptions of Corollary 2.2.4 be satisfied. In particular, we consider $\hat{\xi} = 0$ and $\zeta \equiv 0 \equiv \lambda$. In this situation, we have the following economic interpretation of Y as a savings factor. Suppose that at time $n \in \mathbb{Z} \cap (-\infty, N]$ the task is to sell $x = 1$ share given an initial deviation of $d = 0$. Then immediate execution of the share, which corresponds to the execution strategy $X = (X_k)_{k\in\{n-1,n,\dots,N\}}$ defined by $X_{n-1} = 1$, $X_k = 0$, $k \in \{n, \dots, N\}$, generates the expected costs $\frac{\gamma_n}{2}$ (see also (2.37)). The optimal execution strategy incurs the expected costs $V_n(1,0) = \gamma_n Y_n$ (cf. Corollary 2.2.4). So, the random variable $2Y_n \colon \Omega \to [0,1]$ describes to which percentage the costs of selling the unit immediately can be reduced by executing the position optimally.

This means that if we want to study the improvement in the costs due to optimal trading, we can have a look at the process Y. A relevant question is how much better in comparison to the immediate closure we can do in the long run. To analyze this, there are basically two starting points, both based on the process Y.

One is to adopt the perspective that trading starts at a fixed point in time, e.g., at $n = 0$, and that the terminal date N when the position has to be closed is shifted further and further into the future. This corresponds to studying the limit of the sequence of random variables $(Y_0^N)_{N\in\mathbb{N}}$ as $N \to \infty$, where Y^N is the process defined as in (2.34) pertaining to the terminal time N.[5] Recall that $Y_0^N = V_0^N(1,0)/\gamma_0$, where V^N is the value function belonging to the terminal time N. Since Y_0^N is nonnegative and $V_0^N(1,0)/\gamma_0$ is nonincreasing[6] in N, it follows that $\lim_{N\to\infty} Y_0^N$ always exists (under the assumptions of Corollary 2.2.4).

Another perspective consists in fixing the terminal time N and asking what would have been if one had started trading earlier. This corresponds to investigating the limit $\lim_{n\to-\infty} Y_n$. In some settings (e.g., in a time-homogeneous deterministic framework or, more generally, in the setting of Proposition 2.4.2) one can see that both perspectives coincide by simply relabeling time instances appropriately. In contrast to $\lim_{N\to\infty} Y_0^N$, the limit $\lim_{n\to-\infty} Y_n$ does not always exist (cf. Lemma 2.4.3). In Proposition 2.4.1 we study the existence of the long-time limit $\lim_{n\to-\infty} Y_n$.

We furthermore remark that the question of the long-time limit is different from considering the continuous-time limit of the control problem, which corresponds to fixing $N \in \mathbb{N}$ and $n \in \mathbb{Z} \cap (-\infty, N]$ and letting the number of available trading times in $[n, N]$ go to infinity. A continuous-time variant of the control problem and the relation to the discrete-time results will be discussed in Chapter 3, Chapter 5, and Chapter 7–Chapter 8. In particular, the counterpart of the discrete-time process Y turns out to be a quadratic BSDE.

Proposition 2.4.1. *Let the assumptions of Corollary 2.2.4 be satisfied, and let* $(Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ *be the process that is recursively defined by* $Y_N = \frac{1}{2}$ *and* (2.34). *Fix any* $p \in [1,\infty)$.

(i) The sequence $(\gamma_n Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ *converges a.s. and in* L^p *as* $n \to -\infty$ *to a finite nonnegative random variable.*

(ii) If $(\gamma_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ *is a supermartingale, then the sequence* $(Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ *converges a.s. and in* L^p *as* $n \to -\infty$ *to a finite nonnegative random variable.*

[5]Note that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k\in\mathbb{Z}}, P)$ and the processes $(\gamma_k)_{k\in\mathbb{Z}}$, $(\beta_k)_{k\in\mathbb{Z}}$ do not depend on N. Furthermore, we currently consider the subsetting where $\hat{\xi} = 0$ and $\zeta \equiv 0 \equiv \lambda$.

[6]For $N+1$, a possible strategy is to first trade according to the execution strategy that is optimal for N, and to not trade at terminal time $N+1$.

Proof. (i) It follows from (2.34) that for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ it holds $Y_n \leq E_n[\Gamma_{n+1} Y_{n+1}] = \frac{1}{\gamma_n} E_n[\gamma_{n+1} Y_{n+1}]$. Thus, $(\gamma_n Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is a submartingale. Therefore, the backward convergence theorem implies that $(\gamma_n Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ converges a.s. as $n \to -\infty$. Moreover, $(\gamma_n Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is a positive sequence in $L^{\infty-}$. Hence, its limit is nonnegative, and, by the submartingale property and Jensen's inequality, $(\gamma_n Y_n)^p \leq (E_n[\gamma_N Y_N])^p \leq E_n[(\gamma_N Y_N)^p]$, $n \in \mathbb{Z} \cap (-\infty, N]$. It follows that the sequence $((\gamma_n Y_n)^p)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is uniformly integrable, which implies the convergence in L^p towards a finite nonnegative random variable.

(ii) If $(\gamma_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is a supermartingale, then it converges a.s. as $n \to -\infty$ to an $\mathbb{R}\cup\{+\infty\}$-valued random variable, denoted by $\gamma_{-\infty}$, due to the backward convergence theorem. As γ_n is positive for all $n \in \mathbb{Z} \cap (-\infty, N]$, the random variable $\gamma_{-\infty}$ is, in fact, $[0, +\infty]$-valued. Furthermore, it holds[7]

$$0 = E\left[\gamma_{-\infty} 1_{\{\gamma_{-\infty}=0\}}\right] \geq E\left[\gamma_N 1_{\{\gamma_{-\infty}=0\}}\right] \geq 0.$$

Together with the fact that $\gamma_N > 0$ a.s., this implies $\gamma_{-\infty} > 0$ a.s. Therefore, we have that $(\frac{1}{\gamma_n})_{n\in\mathbb{Z}\cap(-\infty,N]}$ converges a.s. as $n \to -\infty$ to the finite nonnegative random variable $\frac{1}{\gamma_{-\infty}}$. It now follows from (i) that $(Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ converges a.s. as $n \to \infty$ to a finite nonnegative random variable. As the sequence $(Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is bounded (being $(0, \frac{1}{2}]$-valued), it also converges in L^p. □

The assumption that $(\gamma_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is a supermartingale in Proposition 2.4.1(ii) means that the liquidity in the model increases in time (in average). In Lemma 2.4.3 below we have that $(\gamma_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is a submartingale and $(Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ does not converge. This shows that the claim in Proposition 2.4.1(ii) does not in general hold when the liquidity in the model decreases in time.

We are further interested in specific examples for the long-time limit $\lim_{n\to-\infty} Y_n$. In the next Proposition 2.4.2 we compute this limit assuming **(PIMI)** (see Section 2.3) and a sort of time-homogeneity for expectations.

Proposition 2.4.2. *Suppose that the assumptions of Lemma 2.3.1 hold true, that $\lambda \equiv 0$, and that $\bar{\beta} = E[\beta_{n+1}]$, $\bar{\eta} = E[\Gamma_{n+1}]$, and $\bar{\alpha} = E[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}]$ do not depend on $n \in \mathbb{Z} \cap (-\infty, N-1]$.*

(i) If $\bar{\beta} = 1$, we have $\bar{\eta} > 1$, and it holds for all $n \in \mathbb{Z} \cap (-\infty, N]$ that $Y_n = \frac{1}{2}$.

(ii) If $\bar{\eta} \leq 1$, we have $\bar{\beta} < 1$, and the sequence $Y = (Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ converges monotonically to 0 as $n \to -\infty$.

(iii) If $\bar{\beta} \neq 1$ and $\bar{\eta} > 1$, the sequence $Y = (Y_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$, as $n \to -\infty$, converges monotonically to

$$\frac{\frac{1}{2}(1-\bar{\alpha})(\bar{\eta}-1)}{(1-\bar{\alpha})(\bar{\eta}-1) + (\bar{\beta}-1)^2} \in \Big(0, \frac{1}{2}\Big). \tag{2.45}$$

[7]Here we use the convention $\infty \cdot 0 = 0$.

Proof. From (2.39) we have that

$$Y_n = \bar{\eta} Y_{n+1} - \frac{Y_{n+1}^2 \left(\bar{\beta} - \bar{\eta}\right)^2}{Y_{n+1}\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + \frac{1}{2}\left(1 - \bar{\alpha}\right)}, \quad n \in \mathbb{Z} \cap (-\infty, N-1]. \tag{2.46}$$

Define $g\colon [0,\infty) \to \mathbb{R}$,

$$g(y) = \bar{\eta} y - \frac{y^2 \left(\bar{\beta} - \bar{\eta}\right)^2}{y\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + \frac{1}{2}\left(1 - \bar{\alpha}\right)}, \qquad y \in [0,\infty). \tag{2.47}$$

Note that $\bar{\alpha} < 1$ by assumption and that $\bar{\alpha} - 2\bar{\beta} + \bar{\eta} \geq \frac{\left(\bar{\beta}-\bar{\eta}\right)^2}{\bar{\eta}} \geq 0$ because $\frac{\bar{\beta}^2}{\bar{\eta}} \leq \bar{\alpha}$ by the Cauchy-Schwarz inequality. We first show that g is strictly increasing on $[0,\infty)$. To this end, let $y \geq 0$. We compute that

$$\begin{aligned} g'(y) &= \bar{\eta} - \left(\bar{\beta} - \bar{\eta}\right)^2 \frac{2y\left(y\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + \frac{1}{2}\left(1-\bar{\alpha}\right)\right) - y^2\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right)}{\left(y\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + \frac{1}{2}\left(1-\bar{\alpha}\right)\right)^2} \\ &= \bar{\eta} - \left(\bar{\beta} - \bar{\eta}\right)^2 \frac{y^2\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + y\left(1-\bar{\alpha}\right)}{\left(y\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + \frac{1}{2}\left(1-\bar{\alpha}\right)\right)^2}. \end{aligned}$$

Hence, $g'(y) > 0$ is equivalent to

$$\bar{\eta}\left(y\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + \frac{1}{2}\left(1-\bar{\alpha}\right)\right)^2 > \left(\bar{\beta} - \bar{\eta}\right)^2 \left(y^2\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) + y\left(1-\bar{\alpha}\right)\right).$$

Divide by $\bar{\eta} > 0$ and note that $\frac{\left(\bar{\beta}-\bar{\eta}\right)^2}{\bar{\eta}} = \frac{\bar{\beta}^2}{\bar{\eta}} - 2\bar{\beta} + \bar{\eta}$. This yields the equivalent statement

$$\begin{aligned} 0 &< y^2\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right)^2 + y\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right)\left(1-\bar{\alpha}\right) + \frac{\left(1-\bar{\alpha}\right)^2}{4} - \frac{\left(\bar{\beta}-\bar{\eta}\right)^2}{\bar{\eta}} y^2\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) \\ &\quad - \frac{\left(\bar{\beta}-\bar{\eta}\right)^2}{\bar{\eta}} y\left(1-\bar{\alpha}\right) \\ &= y^2\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right)\left(\bar{\alpha} - \frac{\bar{\beta}^2}{\bar{\eta}}\right) + y\left(\bar{\alpha} - \frac{\bar{\beta}^2}{\bar{\eta}}\right)\left(1-\bar{\alpha}\right) + \frac{\left(1-\bar{\alpha}\right)^2}{4} \\ &= \left(y\left(\bar{\alpha} - \frac{\bar{\beta}^2}{\bar{\eta}}\right) + \frac{1-\bar{\alpha}}{2}\right)^2 + y^2\left(\bar{\alpha} - \frac{\bar{\beta}^2}{\bar{\eta}}\right)\frac{\left(\bar{\beta}-\bar{\eta}\right)^2}{\bar{\eta}}. \end{aligned}$$

Since $\bar{\alpha} < 1$ and $\frac{\bar{\beta}^2}{\bar{\eta}} \leq \bar{\alpha}$, this always holds true for $y \geq 0$. It follows that g is strictly increasing on $[0,\infty)$.

Recall that $0 < Y_n \leq \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ and $Y_N = \frac{1}{2}$. In particular, $Y_{N-1} \leq Y_N$. The recursion $Y_n = g(Y_{n+1})$, $n \in \mathbb{Z} \cap (-\infty, N-1]$ (cf. (2.46) and (2.47)), and the fact that g is increasing therefore imply by induction that the sequence Y is nondecreasing, i.e., $Y_{n-1} \le Y_n$ for all $n \in \mathbb{Z} \cap (-\infty, N]$. Hence, the limit $\lim_{n\to-\infty} Y_n$ exists and belongs to $[0, \frac{1}{2}]$. Moreover, it is the largest fixed point of g in $[0, \frac{1}{2}]$. Indeed, since g is increasing, for the largest fixed point $\bar{y}$ of g in $[0, \frac{1}{2}]$, we have that $y \ge \bar{y}$ implies $g(y) \ge g(\bar{y}) = \bar{y}$. Hence, $\bar{y}$ is a lower bound of Y. We obtain that $\lim_{n\to-\infty} Y_n \ge \bar{y}$ and is a fixed point of g, which means that $\lim_{n\to-\infty} Y_n = \bar{y}$.

(i) Suppose that $\bar{\beta} = 1$. The claim that $\bar{\eta} > 1$ follows from $\frac{\bar{\beta}^2}{\bar{\eta}} \le \bar{\alpha} < 1$. By Corollary 2.3.2 it holds that $Y_n = \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$.

(ii) Suppose that $\bar{\eta} \le 1$. First notice that $\bar{\beta}^2 \le \bar{\eta}\bar{\alpha} < \bar{\eta} \le 1$ and hence $\bar{\beta} < 1$. Now it follows from (2.47) that for all $y > 0$ we have $g(y) < y$. This yields that 0 is the only fixed point of g on $[0, \infty)$ and hence $\lim_{n\to-\infty} Y_n = 0$.

(iii) Suppose that $\bar{\beta} \ne 1$ and $\bar{\eta} > 1$. In this case (2.45) is a fixed point of g and the only one in $(0, \infty)$. Indeed, for $y \in (0, \infty)$ the condition $g(y) = y$ is equivalent to

$$y\left(\left(\bar{\beta} - \bar{\eta}\right)^2 - (\bar{\eta} - 1)\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right)\right) = \frac{1}{2}(1 - \bar{\alpha})(\bar{\eta} - 1).$$

From the fact that

$$\left(\bar{\beta} - \bar{\eta}\right)^2 - (\bar{\eta} - 1)\left(\bar{\alpha} - 2\bar{\beta} + \bar{\eta}\right) = (1 - \bar{\alpha})(\bar{\eta} - 1) + \left(\bar{\beta} - 1\right)^2 > (1 - \bar{\alpha})(\bar{\eta} - 1) > 0$$

we deduce (2.45). □

To discuss Proposition 2.4.2, recall that in the setting of Corollary 2.2.4, $2Y_n = \frac{V_n(1,0)}{\gamma_n/2}$ compares the costs $\frac{\gamma_n}{2}$ of selling one unit immediately at time $n \in \mathbb{Z} \cap (-\infty, N]$ given initial deviation 0 to the corresponding optimal costs $V_n(1, 0)$. In general, dividing a large order into many small orders and executing them at consecutive time points can be profitable compared to the immediate execution because of the following reasons:

- the price impact process γ penalizes trades at different times in a different way whenever γ is nonconstant,
- the resilience process β changes the deviation process D between the trades whenever β is not identically 1.

From this viewpoint the claims of Proposition 2.4.2 are naturally interpreted as follows.

If the resilience is in expectation 1 ($\bar{\beta} = 1$), then the price impact process γ is increasing in average (as $\bar{\eta} > 1$), and neither of the above reasons suggests dividing a large order into many small orders.

We can asymptotically get rid of the execution costs in the case of nonincreasing price impact (in the sense $\bar{\eta} \le 1$). Notice that, in this case, the price impact is allowed to be constant, but we anyway profit from the resilience, which, in expectation, drives the deviation back to zero between two trades ($\bar{\beta} < 1$).

In the remaining case of a nontrivial resilience and a geometrically increasing price impact (in the sense $\bar{\beta} \neq 1$ and $\bar{\eta} > 1$) we can not fully get rid of the execution costs regardless of how large our time horizon for execution is.

With Lemma 2.4.3, we now provide an example within the **(PIMI)** setting where, in contrast to Proposition 2.4.2, the process $Y = (Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ does not converge as $n \to -\infty$. The idea behind this construction is to alternate between setting (i) and (iii) in Proposition 2.4.2 and thereby create two subsequences that converge towards different values.

Lemma 2.4.3. *Suppose that the assumptions of Lemma 2.3.1 hold true and that* $\lambda \equiv 0$. *Let* $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\eta}_1$, $\bar{\eta}_2 \in (0, \infty)$, *and* $\bar{\alpha}_1$, $\bar{\alpha}_2 \in (0,1)$, *such that for all* $k \in \mathbb{N}_0$ *it holds* $\bar{\beta}_1 = E[\beta_{N-2k-1}] = 1$, $\bar{\beta}_2 = E[\beta_{N-2k}] \neq 1$, $\bar{\eta}_1 = E[\Gamma_{N-2k-1}]$, $\bar{\eta}_2 = E[\Gamma_{N-2k}] > 1$, $\bar{\alpha}_1 = E[\frac{\beta^2_{N-2k-1}}{\Gamma_{N-2k-1}}]$, *and* $\bar{\alpha}_2 = E[\frac{\beta^2_{N-2k}}{\Gamma_{N-2k}}]$.

Then, γ *is a submartingale and* $Y = (Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ *does not converge as* $n \to -\infty$. *In particular, the sequence* Y *is not monotone.*

Proof. Note first that $\bar{\beta}_1 = 1$ and $\bar{\alpha}_1 < 1$ imply that $\bar{\eta}_1 > 1$ by the Cauchy-Schwarz inequality. It follows from

$$\begin{aligned} 1 < \bar{\eta}_1 = E\left[\Gamma_{N-2k-1}\right] = E_{N-2k-2}\left[\Gamma_{N-2k-1}\right] = E_{N-2k-2}\left[\frac{\gamma_{N-2k-1}}{\gamma_{N-2k-2}}\right] \\ = \frac{1}{\gamma_{N-2k-2}} E_{N-2k-2}\left[\gamma_{N-2k-1}\right] \end{aligned}$$

and

$$1 < \bar{\eta}_2 = \frac{1}{\gamma_{N-2k-1}} E_{N-2k-1}\left[\gamma_{N-2k}\right]$$

for all $k \in \mathbb{N}_0$ that γ is a submartingale.

For $j \in \{1,2\}$, denote by g_j the function defined by (2.47) with $\bar{\beta} = \bar{\beta}_j$, $\bar{\eta} = \bar{\eta}_j$, and $\bar{\alpha} = \bar{\alpha}_j$. Recall that g_1, g_2 are strictly increasing, and note that for $k \in \mathbb{N}_0$, we have $Y_{N-2k-2} = g_1(Y_{N-2k-1})$ and $Y_{N-2k-1} = g_2(Y_{N-2k})$. Furthermore, the equations $g_j(y) = y$, $j \in \{1,2\}$, are quadratic ones, and neither g_1 nor g_2 is the identity function. Hence, each of the functions g_1 and g_2 has at most two fixed points. Clearly, 0 is a fixed point. In view of the proof of Proposition 2.4.2, we conclude that the only fixed points of g_1 are 0 and $\frac{1}{2}$, and the only fixed points of g_2 are given by 0 and $\bar{y} \in (0, \frac{1}{2})$ from (2.45). We also notice that $g_1(y) > y$ for $y \in (0, \frac{1}{2})$. Indeed, since $\bar{\beta}_1 = 1$ and $\bar{\eta}_1 > 1$, we compute for all $y \in (0, \frac{1}{2})$ that

$$g_1(y) - y = \frac{(\bar{\eta}_1 - 1)(1 - \bar{\alpha}_1) y \left(\frac{1}{2} - y\right)}{y(\bar{\alpha}_1 - 2\bar{\beta}_1 + \Gamma_1) + \frac{1}{2}(1 - \bar{\alpha}_1)} > 0.$$

We now prove by induction that $Y_{N-m} > \bar{y}$ for all $m \in \mathbb{N}_0$. The case $m = 0$ is clear. For the induction step $\mathbb{N}_0 \ni m \to m+1 \in \mathbb{N}$, if m is even, we have $Y_{N-m-1} = g_2\left(Y_{N-m}\right) > g_2\left(\bar{y}\right) = \bar{y}$. If m is odd, it holds $Y_{N-m-1} = g_1\left(Y_{N-m}\right) > g_1\left(\bar{y}\right) > \bar{y}$.

We next show inductively that $Y_{N-m} \geq Y_{N-m-2}$ for all $m \in \mathbb{N}_0$. For $m = 0$, this follows from $Y_{N-2} \leq \frac{1}{2} = Y_N$. Consider then the induction step $\mathbb{N}_0 \ni m \to m+1 \in \mathbb{N}$. If m is even, we have $Y_{N-m-3} = g_2(Y_{N-m-2}) \leq g_2(Y_{N-m}) = Y_{N-m-1}$. If m is odd, we have $Y_{N-m-3} = g_1(Y_{N-m-2}) \leq g_1(Y_{N-m}) = Y_{N-m-1}$.

Therefore, the subsequences $(Y_{N-2k})_{k\in\mathbb{N}_0}$ and $(Y_{N-2k-1})_{k\in\mathbb{N}_0}$ of Y are nonincreasing in $k \in \mathbb{N}_0$ and bounded from below by $\bar{y}$, which implies that the limits $\bar{Y}^{(e)} = \lim_{k\to\infty} Y_{N-2k} \geq \bar{y}$ and $\bar{Y}^{(o)} = \lim_{k\to\infty} Y_{N-2k-1} \geq \bar{y}$ exist. Taking limits on both sides of $Y_{N-2k-1} = g_2(Y_{N-2k})$, we obtain $\bar{Y}^{(o)} = g_2(\bar{Y}^{(e)})$ by continuity of g_2. Similarly, it holds that $\bar{Y}^{(e)} = g_1(\bar{Y}^{(o)})$. Now, if $\bar{Y}^{(e)}$ and $\bar{Y}^{(o)}$ were equal, then $\bar{Y}^{(e)} = \bar{Y}^{(o)}$ would be a common fixed point of g_1 and g_2 and hence 0, which is a contradiction to $\bar{Y}^{(e)} \geq \bar{y} > 0$. We thus conclude that Y does not converge. □

We next present examples that fall outside the **(PIMI)** framework.

Example 2.4.4. A simple observation is that under the assumptions of Corollary 2.2.4 we have $\lim_{n\to-\infty} Y_n = 0$ a.s. whenever $(\gamma_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ satisfies $\lim_{n\to-\infty}\gamma_n = +\infty$ a.s. This follows from statement (i) of Proposition 2.4.1.

Example 2.4.5. Suppose, in addition to the assumptions of Corollary 2.2.4, that

$$\Gamma_n = \beta_n \quad \text{for all } n \in \mathbb{Z}\cap(-\infty, N]. \tag{2.48}$$

It is worth noting that, in this setting, the optimal strategy given initial deviation $d = 0$ is to wait until the terminal time N and to close the position at time N. In contrast, if $d \neq 0$, the optimal strategy in general consists of nontrivial trades at all time points. For the sake of discussing the long-time limit $\lim_{n\to-\infty} Y_n$ in this setting we observe that the requirement $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ a.s. from Corollary 2.2.4 under (2.48) becomes

$$E_n[\Gamma_{n+1}] < 1 \text{ a.s. for all } n \in \mathbb{Z}\cap(-\infty, N-1]. \tag{2.49}$$

Hence, $(\gamma_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is a supermartingale. By statement (ii) of Proposition 2.4.1, $\lim_{n\to-\infty} Y_n$ always exists in this setting. Moreover, we have $Y_n = E_n[\Gamma_{n+1}Y_{n+1}]$ for all $n \in \mathbb{Z}\cap(-\infty, N-1]$, and hence by induction

$$Y_n = \frac{1}{2}E_n\left[\prod_{j=n+1}^{N}\Gamma_j\right] = \frac{1}{2}\frac{E_n[\gamma_N]}{\gamma_n} \tag{2.50}$$

for all $n \in \mathbb{Z}\cap(-\infty, N]$. In general, we still can have different values for the long-time limit. Therefore, we now discuss several more specific examples.

(i) Assume there exists $c \in (0,1)$ such that $E_n[\Gamma_{n+1}] \leq c$ a.s. (cf. (2.49)) for all $n \in \mathbb{Z}\cap(-\infty, N-1]$. By intermediate conditioning, it follows from (2.50) that $Y_n \leq \frac{1}{2}c^{N-n}$ a.s. for all $n \in \mathbb{Z}\cap(-\infty, N]$, hence $\lim_{n\to-\infty} Y_n = 0$ a.s.

(ii) On the other hand, it is clear from (2.50) that, even with suitable deterministic sequences $(\Gamma_n)_{n\in\mathbb{Z}}$, we can achieve for the long-time limit $\lim_{n\to-\infty} Y_n$ any deterministic value in $(0, \frac{1}{2})$.

(iii) In order to present an explicit and, possibly, nondeterministic long-time limit, we finally consider the following construction. Let $(r_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ be a strictly decreasing sequence of nonnegative real numbers. Let Z_n, $n \in \mathbb{Z}\cap(-\infty, N]$, and K be random variables such that $(Z_n)_{n\in\mathbb{Z}\cap(-\infty,N]}$ is an i.i.d. sequence independent of K, and such that $Z_N, K \geq 0$ and $Z_N, K \in L^{\infty-}$ (and thus $Z_n \geq 0$, $Z_n \in L^{\infty-}$ for all $n \in \mathbb{Z}\cap(-\infty, N]$). We also require at least one of the conditions (a) $r_N > 0$ or (b) $\frac{1}{Z_N}, \frac{1}{K} \in L^{\infty-}$. We now define

$$U_n = \sum_{j=n}^{N} Z_j, \quad \mathcal{F}_n = \sigma(K, U_j; j \in \mathbb{Z}\cap(-\infty, n]), \quad n \in \mathbb{Z}\cap(-\infty, N],$$

and set

$$\gamma_n = r_n + \frac{1}{N-n+1} U_n K, \quad n \in \mathbb{Z}\cap(-\infty, N].$$

Note that $\gamma_n > 0$ for all $n \in \mathbb{Z}\cap(-\infty, N]$, and define $\beta_n = \Gamma_n = \frac{\gamma_n}{\gamma_{n-1}}$, $n \in \mathbb{Z}\cap(-\infty, N]$. Thus, we are in setting (2.48), and we now verify that the assumptions of Corollary 2.2.4 are satisfied.

Since r_n is deterministic and $Z_n, K \in L^{\infty-}$ for all $n \in \mathbb{Z}\cap(-\infty, N]$, it holds that $\gamma_n \in L^{\infty-}$ for all $n \in \mathbb{Z}\cap(-\infty, N]$. To see that also $\frac{1}{\gamma_n} \in L^{\infty-}$ for all $n \in \mathbb{Z}\cap(-\infty, N]$, note that $\gamma_n > r_N$ and $\gamma_n \geq \frac{1}{N-n+1} Z_N K$ for all $n \in \mathbb{Z}\cap(-\infty, N]$, and use condition (a) or (b). Clearly, we then also have that $\beta_n \in L^{\infty-}$ for all $n \in \mathbb{Z}\cap(-\infty, N]$. Further, for $n \in \mathbb{Z}\cap(-\infty, N]$ and $j \in \{n, \dots, N\}$, it holds that $E_n[Z_j] = \frac{1}{N-n+1} U_n$. Hence, for all $n \in \mathbb{Z}\cap(-\infty, N]$,

$$\begin{aligned} E_n[\gamma_{n+1}] &= r_{n+1} + \frac{1}{N-n} \sum_{j=n+1}^{N} E_n[Z_j] K = r_{n+1} + \frac{1}{N-n+1} U_n K \\ &< r_n + \frac{1}{N-n+1} U_n K = \gamma_n \text{ a.s.}, \end{aligned}$$

i.e., requirement (2.49) holds true (which is $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ a.s.). Furthermore, we have that

$$\left(1 - E_n\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right]\right)^{-1} = \left(\frac{\gamma_n - E_n[\gamma_{n+1}]}{\gamma_n}\right)^{-1} = \frac{\gamma_n}{r_n - r_{n+1}} \in L^{\infty-}$$

for all $n \in \mathbb{Z}\cap(-\infty, N-1]$.

By the strong law of large numbers it holds that $\frac{1}{N-n+1} U_n \to E[Z_N]$ a.s., as $n \to -\infty$. Setting $r_{-\infty} = \lim_{n\to-\infty} r_n$ $(\in (0, \infty])$, we obtain $\lim_{n\to-\infty} \gamma_n = r_{-\infty} + E[Z_N]K$ a.s.

Furthermore, $\lim_{n\to-\infty} E_n[\gamma_N] = r_N + E[Z_N]K$ a.s., and hence

$$\lim_{n\to-\infty} Y_n = \frac{1}{2}\lim_{n\to-\infty}\frac{E_n[\gamma_N]}{\gamma_n} = \frac{1}{2}\frac{r_N + E[Z_N]K}{r_{-\infty} + E[Z_N]K} \text{ a.s.,}$$

which is, in general, nondeterministic.

2.5 Round trips

We now turn to the question if an agent who has no initial position in the asset and also requires position 0 at terminal time nevertheless can expect to benefit from trading[8].

To this end, let $x = 0$, $\hat{\xi} = 0$, and $\lambda \equiv 0 \equiv \zeta$ throughout this section (except for Remark 2.5.10). In particular, if we assume in addition that for all $k \in \mathbb{Z}\cap(-\infty, N-1]$ it holds that $E_k[\frac{\beta_{k+1}^2}{\Gamma_{k+1}}] < 1$ a.s. and that $(1 - E_k[\frac{\beta_{k+1}^2}{\Gamma_{k+1}}])^{-1} \in L^{\infty-}$, we are in the setting of Corollary 2.2.4.

Definition 2.5.1. Let $\hat{\xi} = 0$ and $\lambda \equiv 0 \equiv \zeta$. For any $d \in \mathbb{R}$, we call an execution strategy $X \in \mathcal{A}_n^{\text{disc}}(0, d)$ a *round trip* at time $n \in \mathbb{Z}\cap(-\infty, N-1]$. A round trip $X \in \mathcal{A}_n^{\text{disc}}(0, d)$ at time $n \in \mathbb{Z}\cap(-\infty, N-1]$ is said to be *profitable* for initial deviation $d \in \mathbb{R}$ if for the associated costs it holds

$$\begin{aligned} &P\left(E_n\left[\sum_{j=n}^{N}\left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j\right] \le 0\right) = 1 \text{ and} \\ &P\left(E_n\left[\sum_{j=n}^{N}\left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j\right] < 0\right) > 0, \end{aligned} \tag{2.51}$$

where ξ is the associated trade process and D is the associated deviation process with $D_{n-} = d$.

We formalize our previous question and ask whether there exist profitable round trips (at time $n \in \mathbb{Z}\cap(-\infty, N-1]$, for given initial deviation $d \in \mathbb{R}$). The existence of profitable round trips is sometimes also referred to as price manipulation (see, e.g., [AS10], [Gat10], or [HS04]).

Note that, for $n \in \mathbb{Z}\cap(-\infty, N-1]$, $d \in \mathbb{R}$, and under the assumptions of Corollary 2.2.4, there exist profitable round trips at time n for initial deviation[9] d if and only if

$$P(V_n(0, d) < 0) > 0. \tag{2.52}$$

[8] We could pose a similar question for initial position x and terminal position x, given $x \in \mathbb{R}$. However, this is equivalent to the problem treated here, see Remark 2.1.6 and (2.51).

[9] Note that for existence of profitable round trips we in fact only have to distinguish between $d = 0$ and $d \neq 0$, see the subsequent discussion.

In this case, the optimal strategy from Corollary 2.2.4 is such a profitable round trip.

To see that this indeed holds true, fix, for this paragraph, $n \in \mathbb{Z} \cap (-\infty, N-1]$, $d \in \mathbb{R}$, and let the assumptions of Corollary 2.2.4 be in force. Observe that from Corollary 2.2.4 we have the existence of an optimal strategy $X^* \in \mathcal{A}_n^{\text{disc}}(0,d)$ and that

$$V_n(0,d) = \frac{d^2}{\gamma_n}\left(Y_n - \frac{1}{2}\right) \tag{2.53}$$

with $(0, \frac{1}{2}]$-valued Y_n. Suppose first that there exists a profitable round trip $X \in \mathcal{A}_n^{\text{disc}}(0,d)$. It then follows that a.s.

$$E_n\left[\sum_{j=n}^{N}\left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j\right] \geq E_n\left[\sum_{j=n}^{N}\left(D_{j-}^* + \frac{\gamma_j}{2}\xi_j^*\right)\xi_j^*\right] = V_n(0,d).$$

The fact that X is profitable implies that

$$P\left(V_n(0,d) < 0\right) \geq P\left(E_n\left[\sum_{j=n}^{N}\left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j\right] < 0\right) > 0.$$

Suppose now that $P(V_n(0,d) < 0) > 0$. Since γ_n is positive and Y_n is $(0, \frac{1}{2}]$-valued, it follows from (2.53) that furthermore $P(V_n(0,d) \leq 0) = 1$. The optimal strategy X^* thus satisfies (2.51), i.e., X^* is a profitable round trip.

Observe that (2.53) implies in particular that $V_n(0,0) = 0$ for all $n \in \mathbb{Z} \cap (-\infty, N-1]$. Thus, by (2.52), there are no profitable round trips whenever $d = 0$. This means that without initial deviation of the price process the agent can not make profits in expectation. Moreover, this shows that, if there is no initial deviation of the price process, our model does not admit price manipulation.

Note also that (2.52) and (2.53) imply that if, for given $n \in \mathbb{Z} \cap (-\infty, N-1]$, $d_0 \in \mathbb{R}$, there exists a profitable round trip at time n for initial deviation d_0, then there exist profitable round trips at time n for any initial deviation $d \neq 0$.

In the sequel, we study existence of profitable round trips when the price of a share deviates from the unaffected price, i.e., when it holds $d \neq 0$. For $n \in \mathbb{Z} \cap (-\infty, N-1]$ and $d \neq 0$, (2.52) and (2.53) imply the following classification:

- If $P(Y_n < \frac{1}{2}) > 0$, there exist profitable round trips,
- if $P(Y_n = \frac{1}{2}) = 1$, there are no profitable round trips.

Thus, the question reduces to finding a tractable description of the event $\{Y_n = \frac{1}{2}\}$, $n \in \mathbb{Z} \cap (-\infty, N-1]$. We characterize this event in the next proposition and then discuss several consequences of this characterization.

Proposition 2.5.2. *Let the assumptions of Corollary 2.2.4 be satisfied. Then we have*

$$\left\{Y_n = \frac{1}{2}\right\} = \left\{E_n\left[Y_{n+1}\right] = \frac{1}{2}, E_n\left[\beta_{n+1}\right] = 1\right\}, \quad n \in \mathbb{Z} \cap (-\infty, N-1],$$

where here and below we understand equalities or inclusions for events up to P-null sets.

Proof. Throughout the proof we fix $n \in \mathbb{Z} \cap (-\infty, N-1]$. With the notation

$$\nu_{n+1} = \frac{1}{2} - \left(\frac{1}{2} - Y_{n+1}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}},$$

and with

$$a_n = \gamma_n E_n\left[\frac{Y_{n+1}}{\Gamma_{n+1}}\left(\beta_{n+1} - \Gamma_{n+1}\right)^2 + \frac{1}{2}\left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right)\right]$$

from (2.12), we obtain from (2.34) that

$$\begin{aligned}
Y_n &= E_n[\Gamma_{n+1}Y_{n+1}] - \frac{\left(E_n[Y_{n+1}\beta_{n+1}]\right)^2 - 2E_n[Y_{n+1}\beta_{n+1}]\,E_n[Y_{n+1}\Gamma_{n+1}] + \left(E_n[Y_{n+1}\Gamma_{n+1}]\right)^2}{E_n\left[\nu_{n+1} - 2Y_{n+1}\beta_{n+1} + Y_{n+1}\Gamma_{n+1}\right]} \\
&= \frac{E_n\left[\Gamma_{n+1}Y_{n+1}\right]E_n\left[\nu_{n+1}\right] - \left(E_n\left[Y_{n+1}\beta_{n+1}\right]\right)^2}{E_n\left[\nu_{n+1} - 2Y_{n+1}\beta_{n+1} + Y_{n+1}\Gamma_{n+1}\right]} \\
&= \frac{E_n\left[\nu_{n+1}\right]E_n\left[\nu_{n+1} - 2Y_{n+1}\beta_{n+1} + Y_{n+1}\Gamma_{n+1}\right] - \left(E_n\left[\nu_{n+1} - Y_{n+1}\beta_{n+1}\right]\right)^2}{E_n\left[\nu_{n+1} - 2Y_{n+1}\beta_{n+1} + Y_{n+1}\Gamma_{n+1}\right]} \\
&= \frac{1}{2} - E_n\left[\left(\frac{1}{2} - Y_{n+1}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - \frac{\gamma_n}{a_n}\left(\frac{1}{2} - E_n\left[\left(\frac{1}{2} - Y_{n+1}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - E_n[Y_{n+1}\beta_{n+1}]\right)^2.
\end{aligned}$$

Since $\Gamma_{n+1}, \gamma_n, a_n > 0$ and $Y_{n+1} \le \frac{1}{2}$ a.s., it now follows that

$$\left\{Y_n = \frac{1}{2}\right\} = \left\{E_n\left[\left(\frac{1}{2} - Y_{n+1}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] = 0, E_n\left[Y_{n+1}\beta_{n+1}\right] = \frac{1}{2}\right\}. \tag{2.54}$$

Let

$$C_n = \left\{E_n\left[\left(\frac{1}{2} - Y_{n+1}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] = 0\right\}$$

and

$$B_n = \left\{E_n[Y_{n+1}] = \frac{1}{2}\right\}.$$

We show that $C_n = B_n$. For the inclusion $C_n \supseteq B_n$ note first that, due to $\{E_n[Y_{n+1}] = \frac{1}{2}\} \in \mathcal{F}_n$, it holds

$$\int_{\left\{E_n[Y_{n+1}]=\frac{1}{2}\right\}} Y_{n+1}\,dP = \int_{\left\{E_n[Y_{n+1}]=\frac{1}{2}\right\}} E_n\left[Y_{n+1}\right]\,dP = \int_{\left\{E_n[Y_{n+1}]=\frac{1}{2}\right\}} \frac{1}{2}\,dP, \tag{2.55}$$

which yields that $Y_{n+1} = \frac{1}{2}$ on B_n. This together with the fact that $B_n \in \mathcal{F}_n$ implies

$$1_{B_n} E_n \left[\left(\frac{1}{2} - Y_{n+1} \right) \frac{\beta_{n+1}^2}{\Gamma_{n+1}} \right] = E_n \left[1_{B_n} \left(\frac{1}{2} - Y_{n+1} \right) \frac{\beta_{n+1}^2}{\Gamma_{n+1}} \right] = 0.$$

To prove $C_n \subseteq B_n$, observe that $C_n \in \mathcal{F}_n$, and that

$$C_n \subseteq \left\{ \left(\frac{1}{2} - Y_{n+1} \right) \frac{\beta_{n+1}^2}{\Gamma_{n+1}} = 0 \right\} = \left\{ Y_{n+1} = \frac{1}{2} \right\}$$

(by an argument similar to (2.55)) since $\beta_{n+1}, \Gamma_{n+1} > 0$ and $Y_{n+1} \leq \frac{1}{2}$ a.s. It thus holds that

$$1_{C_n} E_n [Y_{n+1}] = E_n [1_{C_n} Y_{n+1}] = 1_{C_n} \frac{1}{2}.$$

From $C_n = B_n$ together with (2.54) we obtain

$$\left\{ Y_n = \frac{1}{2} \right\} = \left\{ E_n[Y_{n+1}] = \frac{1}{2}, E_n [Y_{n+1} \beta_{n+1}] = \frac{1}{2} \right\}.$$

Furthermore, we have

$$1_{B_n} E_n [Y_{n+1} \beta_{n+1}] = E_n [1_{B_n} Y_{n+1} \beta_{n+1}] = 1_{B_n} \frac{1}{2} E_n [\beta_{n+1}],$$

and hence

$$\left\{ Y_n = \frac{1}{2} \right\} = \left\{ E_n[Y_{n+1}] = \frac{1}{2}, E_n [\beta_{n+1}] = 1 \right\}.$$

□

Corollary 2.5.3. *Under the assumptions of Corollary 2.2.4 it holds that*

$$\left\{ Y_{N-1} = \frac{1}{2} \right\} = \{ E_{N-1} [\beta_N] = 1 \}.$$

Proof. The result is immediate from Proposition 2.5.2 because $Y_N = \frac{1}{2}$. □

Corollary 2.5.4. *Under the assumptions of Corollary 2.2.4 we have the following inclusions for $n \in \mathbb{Z} \cap (-\infty, N-1]$.*

(i) It holds that

$$\left\{ Y_n = \frac{1}{2} \right\} \subseteq \left\{ Y_{n+1} = \frac{1}{2} \right\}$$

(equivalently, $\{Y_{n+1} < \frac{1}{2}\} \subseteq \{Y_n < \frac{1}{2}\}$).

(ii) It holds that

$$\left\{ Y_n = \frac{1}{2} \right\} \subseteq \{ E_n [\beta_{n+1}] = 1 \} \subseteq \{ E_n [\beta_{n+1}] \geq 1 \} \subseteq \{ E_n [\Gamma_{n+1}] > 1 \}$$

(equivalently, $\{E_n[\Gamma_{n+1}] \leq 1\} \subseteq \{E_n[\beta_{n+1}] < 1\} \subseteq \{E_n[\beta_{n+1}] \neq 1\} \subseteq \{Y_n < \frac{1}{2}\}$).

Proof. We fix $n \in \mathbb{Z} \cap (-\infty, N-1]$.

(i) The claim follows from

$$\left\{Y_n = \frac{1}{2}\right\} \subseteq \left\{E_n\left[Y_{n+1}\right] = \frac{1}{2}\right\} \subseteq \left\{Y_{n+1} = \frac{1}{2}\right\},$$

where the first inclusion is immediate from Proposition 2.5.2, and the second one follows from the facts that $Y_{n+1} \leq \frac{1}{2}$ a.s. and (2.55).

(ii) Due to Proposition 2.5.2 only the inclusion $\{E_n[\beta_{n+1}] \geq 1\} \subseteq \{E_n[\Gamma_{n+1}] > 1\}$ needs to be proved. By the Cauchy-Schwarz inequality and the assumption $E_n[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$ a.s. we get

$$\left(E_n\left[\beta_{n+1}\right]\right)^2 \leq E_n\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] E_n\left[\Gamma_{n+1}\right] < E_n\left[\Gamma_{n+1}\right] \quad \text{a.s.},$$

which implies the claim. □

In case of a $(0,1)$-valued resilience process β, Corollary 2.5.4(ii) and the discussion preceding Proposition 2.5.2 imply that for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ and $d \neq 0$ we have profitable round trips. We also mention the discussions on existence of profitable round trips for nonzero initial deviation in similar models with $(0,1)$-valued resilience in [FSU14, Remark 8.2] and in [FSU19] (after Model 8.3). In particular, they observe that for a conventional symmetric block-shaped order book model with zero bid-ask spread, constant price impact, and nonzero initial deviation, the knowledge that the deviation will be driven towards zero due to the ($(0,1)$-valued) resilience allows to construct profitable round trips. E.g., even without using Corollary 2.2.4, we can directly compute in a setting[10] where for the trading period $\{n, \ldots, N\}$ for fixed $n \in \mathbb{Z} \cap (-\infty, N-1]$, $\gamma_n = \gamma_{n+1}$, β_{n+1} is $(0,1)$-valued, and the initial deviation $d \in \mathbb{R}$ is nonzero, that the strategy $X \in \mathcal{A}_n^{\text{disc}}(0,d)$ with trades $\xi_n = -\frac{d}{2\gamma_n} = -\xi_{n+1}$ and $\xi_j = 0$, $j \in \{n+2, \ldots, N\}$, leads to a.s. negative expected costs $-\frac{d^2}{4\gamma_n}(1 - E_n[\beta_{n+1}])$ and thus is a profitable round trip (cf. [FSU14, Remark 8.2]).

The assumption of $(0,1)$-valued resilience is typical in the literature on optimal trade execution. We more generally assume that the resilience takes values in $(0,\infty)$. It follows from Corollary 2.5.4(ii) and the discussion preceding Proposition 2.5.2 that also if $P(E_n[\beta_{n+1}] \neq 1) > 0$ there are profitable round trips for $n \in \mathbb{Z} \cap (-\infty, N-1]$, $d \neq 0$. That means, to have existence of profitable round trips, it is enough to expect the resilience to go in some direction.

A new qualitative effect in our setting is that the situation of nonexistence of profitable round trips is possible not only for $d = 0$, but also for $d \neq 0$ (see also Corollary 2.5.5). The previous discussion and Corollary 2.5.4(ii) explain that $P(E_n[\beta_{n+1}] =$

[10] Of course, we also assume that $(\gamma_k)_{k\in\mathbb{Z}}$, $(\beta_k)_{k\in\mathbb{Z}}$ are adapted, positive, and satisfy $\gamma_k, \frac{1}{\gamma_k}, \beta_k \in L^{\infty-}$ for all $k \in \mathbb{Z}$. Furthermore, recall that in this section we have set $\lambda \equiv 0 \equiv \zeta$ and $\hat{\xi} = 0$.

$1) = 1$ is necessary for the nonexistence of profitable round trips for $n \in \mathbb{Z} \cap (-\infty, N-1]$, $d \neq 0$.

A somewhat unexpected effect is that the inclusion $\{Y_n = \frac{1}{2}\} \subseteq \{E_n[\beta_{n+1}] = 1\}$ can be strict, and hence there might exist profitable round trips even though $P(E_n[\beta_{n+1}] = 1) = 1$ for $n \in \mathbb{Z} \cap (-\infty, N-1]$, $d \neq 0$ (see Example 2.5.6). In particular, for $n \in \mathbb{Z} \cap (-\infty, N-2]$, we can not distinguish $Y_n = \frac{1}{2}$ from $Y_n < \frac{1}{2}$ on the basis of $E_n[\beta_{n+1}]$ alone, and, indeed, the exact characterization of the event $\{Y_n = \frac{1}{2}\}$ also includes $E_n[Y_{n+1}]$ (see Proposition 2.5.2).

A special case where we obtain an explicit criterion to distinguish between $Y_n = \frac{1}{2}$ and $Y_n < \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ only in terms of the process β is the case of processes with independent multiplicative increments **(PIMI)** as in Section 2.3. We treat this in the next Corollary 2.5.5.

Furthermore, we mention that if $P(E_{N-1}[\beta_N] \neq 1) > 0$, then also in the general setting, by Corollary 2.5.3 it holds that $P(Y_{N-1} < \frac{1}{2}) > 0$, and it further follows from Corollary 2.5.4(i) that $P(Y_{N-2} < \frac{1}{2}) \geq P(Y_{N-1} < \frac{1}{2}) > 0$. Inductively, we obtain from Corollary 2.5.4(i) that in this case there exist profitable round trips at any time $n \in \mathbb{Z} \cap (-\infty, N-1]$ for $d \neq 0$. More generally, i.e., without assuming $P(E_{N-1}[\beta_N] \neq 1) > 0$, Corollary 2.5.4(i) implies that if there exist profitable round trips at some time $j \in \mathbb{Z} \cap (-\infty, N-1]$ for $d \neq 0$, then there also exist profitable round trips at all earlier times for nonzero initial deviation. An intuitive explanation is the following. Suppose that for a fixed time $j \in \mathbb{Z} \cap (-\infty, N-1]$ there exist profitable round trips at time j for all nonzero deviations. Then, if our trading period is $\{n, \dots, N\}$ for some $n \in \mathbb{Z} \cap (-\infty, j-1]$ with initial deviation $d \neq 0$, we can wait until time j and then make a profitable round trip (since $D_{j-} = d \prod_{l=n+1}^{k} \beta_l \neq 0$). Hence, we have constructed a profitable round trip at time $n < j$.

Similar to the last paragraph, we obtain from Corollary 2.5.4(i) that nonexistence of profitable round trips at some time $j \in \mathbb{Z} \cap (-\infty, N-1]$ for $d \neq 0$ implies nonexistence at all later times $k \in \{j+1, \dots, N-1\}$.

The next corollary contains the announced result on round trips in the setting of **(PIMI)**.

Corollary 2.5.5. *Let the assumptions of Lemma 2.3.1 and Corollary 2.2.4 be in force. Suppose that* $E[\beta_j] \neq 1$ *for some* $j \in \mathbb{Z} \cap (-\infty, N]$ *and define*[11]

$$n_0 = N \wedge \inf\{n \in \mathbb{Z} \cap (-\infty, N-1] : E[\beta_k] = 1 \textit{ for all } k \in \mathbb{Z} \cap [n+1, N]\}.$$

Then, for the (deterministic) process Y*, we have* $Y_n < \frac{1}{2}$ *for* $n \in \mathbb{Z} \cap (-\infty, n_0)$ *and* $Y_n = \frac{1}{2}$ *for* $n \in \mathbb{Z} \cap [n_0, N]$.

Proof. The result follows from Proposition 2.5.2 and the fact that, by Lemma 2.3.1, the process Y is deterministic. □

[11] We use the convention that $\inf \emptyset = \infty$.

In addition, note that if $E[\beta_n] = 1$ for all $n \in \mathbb{Z} \cap (-\infty, N]$ in the setting of Lemma 2.3.1, it holds that $Y_n = \frac{1}{2}$ for all $n \in \mathbb{Z} \cap (-\infty, N]$ (cf. Corollary 2.3.2).

We now discuss the inclusion $\{Y_n = \frac{1}{2}\} \subseteq \{E_n[\beta_{n+1}] = 1\}$ (see Corollary 2.5.4(ii)) in more detail. First, we present a simple example where for $n = N-2$ this inclusion is strict.

Example 2.5.6. We take any deterministic sequences β and γ with $\beta_N \neq 1$ and $\beta_{N-1} = 1$ that satisfy the assumptions of Corollary 2.2.4. Then the process Y is deterministic. Corollary 2.5.3 implies that $Y_{N-1} < \frac{1}{2}$. Hence, by Corollary 2.5.4(i), it holds that $Y_{N-2} < \frac{1}{2}$. We thus have

$$\left\{Y_{N-2} = \frac{1}{2}\right\} = \emptyset \subsetneq \Omega = \{E_{N-2}[\beta_{N-1}] = 1\}.$$

In other words, we have profitable round trips at time $N-2$ for $d \neq 0$ despite $E_{N-2}[\beta_{N-1}] = 1$.

This is not surprising in this example: First, we see that profitable round trips are already present when we start at time $N-1$ due to $Y_{N-1} < \frac{1}{2}$, which is caused by $\beta_N \neq 1$. Second, since $\beta_{N-1} = 1$, the deviation will not change from time $N-2$ to $N-1$ if we do not trade at time $N-2$.

One might, therefore, intuitively expect that here all profitable round trips do not contain a trade at time $N-2$, but this is not the case. If $d \neq 0$, then we have for the (here, deterministic) optimal trade size $\xi^*_{N-2}(0,d)$ of (2.35) that $\xi^*_{N-2}(0,d) \neq 0$.

To see this, observe that for $d \neq 0$ and due to the facts that β, Γ, Y are deterministic, (2.35) implies that $\xi^*_{N-2}(0,d) \neq 0$ is equivalent to

$$1 = \frac{Y_{N-1}(\Gamma_{N-1} - 1)}{Y_{N-1}\frac{(1-\Gamma_{N-1})^2}{\Gamma_{N-1}} + \frac{1}{2}\left(1 - \frac{1}{\Gamma_{N-1}}\right)}.$$

This in turn holds true if and only if

$$\left(\frac{1}{2} - Y_{N-1}\right)\left(1 - \frac{1}{\Gamma_{N-1}}\right) = 0,$$

which in this example is not satisfied because $Y_{N-1} < \frac{1}{2}$ and $\frac{1}{\Gamma_{N-1}} = \frac{\beta^2_{N-1}}{\Gamma_{N-1}} < 1$ (recall the assumptions of Corollary 2.2.4).

To summarize, in Example 2.5.6, we have existence of profitable round trips at time $N-2$ for $d \neq 0$ in a (deterministic) setting where $P(E_{N-1}[\beta_N] = 1) = 0$ and $P(E_{N-2}[\beta_{N-1}] = 1) > 0$. We next study if profitable round trips at time $n \in \mathbb{Z} \cap (-\infty, N-2]$ for initial deviation $d \neq 0$ can also occur if $P(\bigcap_{k=n}^{N-1}\{E_k[\beta_{k+1}] = 1\}) > 0$. Corollary 2.5.5 implies that this is impossible in the framework of **(PIMI)** (let alone with deterministic β and γ). But, in general, such a phenomenon is possible, and we present a specific example after the following lemma.

Lemma 2.5.7. *Let the assumptions of Corollary 2.2.4 be in force.*

(i) It holds for all $n \in \mathbb{Z} \cap (-\infty, N-1]$ that

$$\left\{Y_n = \frac{1}{2}\right\} \subseteq \bigcap_{k=n}^{N-1} \{E_k[\beta_{k+1}] = 1\}. \tag{2.56}$$

(ii) Let $n \in \mathbb{Z} \cap (-\infty, N-1]$. The inclusion in (2.56) *is strict (in the sense that the set difference has positive P-probability) if and only if*

$$\bigcap_{k=n}^{N-1} \{E_k[\beta_{k+1}] = 1\} \notin \overline{\mathcal{F}}_n, \tag{2.57}$$

where $\overline{\mathcal{F}}_n = \sigma(\mathcal{F}_n \cup \mathcal{N})$ with $\mathcal{N} = \{A \in \mathcal{F} : P(A) = 0\}$.

Proof. (i) We proceed by backward induction. Corollary 2.5.3 shows that the claim holds true for $n = N-1$. Consider then the induction step $\mathbb{Z} \cap (-\infty, N-1] \ni n+1 \to n \in \mathbb{Z} \cap (-\infty, N-2]$. It follows from Corollary 2.5.4(i) and the induction hypothesis that $\{Y_n = \frac{1}{2}\} \subseteq \{Y_{n+1} = \frac{1}{2}\} \subseteq \bigcap_{k=n+1}^{N-1} \{E_k[\beta_{k+1}] = 1\}$. Furthermore, we have from Corollary 2.5.4(ii) that $\{Y_n = \frac{1}{2}\} \subseteq \{E_n[\beta_{n+1}] = 1\}$. This yields (2.56).

(ii) Let $n \in \mathbb{Z} \cap (-\infty, N-1]$. Under (2.57) it holds that the inclusion in (2.56) is strict (note that $\{Y_n = \frac{1}{2}\} \in \mathcal{F}_n$). It remains to prove that, if there is $A_n \in \mathcal{F}_n$ which is (up to a P-null set) equal to $\bigcap_{k=n}^{N-1} \{E_k[\beta_{k+1}] = 1\}$, then $Y_n = \frac{1}{2}$ a.s. on A_n.

To show this, we first establish by backward induction that $E_j[Y_{j+1}] = \frac{1}{2}$ a.s. on A_n for all $j \in \{n, \dots, N-1\}$. For the base case $j = N-1$, we have that $E_{N-1}[Y_N] = \frac{1}{2}$ due to $Y_N = \frac{1}{2}$. If $n = N-1$, we are done; otherwise, consider the induction step $\{n+1, \dots, N-1\} \ni j+1 \to j \in \{n, \dots, N-2\}$. By the induction hypothesis, it holds that $E_{j+1}[Y_{j+2}] = \frac{1}{2}$ a.s. on A_n. Since $E_{j+1}[\beta_{j+2}] = 1$ a.s. on A_n, Proposition 2.5.2 thus implies that $Y_{j+1} = \frac{1}{2}$ a.s on A_n. As $A_n \in \mathcal{F}_n \subseteq \mathcal{F}_j$, we then obtain that $E_j[Y_{j+1}] = \frac{1}{2}$ a.s. on A_n. This completes the induction.

In particular, we now have that $E_n[Y_{n+1}] = \frac{1}{2}$ a.s. on A_n. Again, the argument that $E_n[\beta_{n+1}] = 1$ a.s. on A_n and Proposition 2.5.2 then imply that $Y_n = \frac{1}{2}$ a.s. on A_n. □

We next present a specific example where for $n = N-2$ the inclusion in (2.56) is strict, or, in other words, $P(Y_{N-2} < \frac{1}{2}, E_{N-2}[\beta_{N-1}] = E_{N-1}[\beta_N] = 1) > 0$. In particular, in this example there exist profitable round trips at time $N-2$ for all $d \neq 0$, and, at the same time, the event $\{E_{N-2}[\beta_{N-1}] = 1\} \cap \{E_{N-1}[\beta_N] = 1\}$ has positive probability.

Example 2.5.8. Take arbitrary $a, p \in (0,1)$. Let $\mathcal{F}_n = \{\emptyset, \Omega\}$ for $n \in \mathbb{Z} \cap (-\infty, N-2]$, $\mathcal{F}_{N-1} = \mathcal{F}_N = \sigma(\beta_{N-1})$ with β_{N-1} being distributed according to $P(\beta_{N-1} = 1) = 1-p$ and $P(\beta_{N-1} = 1 \pm a) = p/2$. We set $\beta_N = \beta_{N-1}$ and choose any process γ (and β_k for the remaining $k \in \mathbb{Z} \setminus \{N-1, N\}$) satisfying the assumptions of Corollary 2.2.4 (e.g., one can easily take deterministic γ). It then holds that $E_{N-2}[\beta_{N-1}] = E[\beta_{N-1}] = 1$, and hence $\{E_{N-2}[\beta_{N-1}] = 1\} \cap \{E_{N-1}[\beta_N] = 1\} = \{E_{N-1}[\beta_N] = 1\} = \{\beta_N = 1\}$, which is an event of probability $1 - p \in (0,1)$. We thus obtain (2.57) for $n = N-2$. By Lemma 2.5.7, the inclusion in (2.56) for $n = N-2$ is strict. As a result, we get $P(Y_{N-2} < \frac{1}{2}, E_{N-2}[\beta_{N-1}] = E_{N-1}[\beta_N] = 1) > 0$.

Our discussion of existence of profitable round trips has mainly focused on the resilience β (and Y). In Lemma 2.5.9 we provide a different sufficient condition for existence of profitable round trips based on the price impact γ (and Y).

Lemma 2.5.9. *Under the assumptions of Corollary 2.2.4 it holds for all* $n \in \mathbb{Z} \cap (-\infty, N-1]$ *that*

$$\left\{Y_n = \frac{1}{2}\right\} \subseteq \left\{\min_{k \in \{n+1,\dots,N\}} E_n[\gamma_k] \geq \gamma_n\right\}$$

(equivalently, $\{\min_{k\in\{n+1,\dots,N\}} E_n[\gamma_k] < \gamma_n\} \subseteq \{Y_n < \frac{1}{2}\}$*).*

Proof. While the result can be again inferred from the characterization of the event $\{Y_n = \frac{1}{2}\}$ in Proposition 2.5.2, it is shorter to observe that $Y_n < \frac{1}{2}$ on the event $\{\min_{k\in\{n+1,\dots,N\}} E_n[\gamma_k] < \gamma_n\}$ due to Remark 2.2.6 and $\lambda \equiv 0$. □

In this section, we consider a risk-neutral setting only. One might wonder about existence of profitable round trips from the view point of a risk-averse agent. Or one might ask if the results for the set $\{Y_n = \frac{1}{2}\}$, $n \in \mathbb{Z} \cap (-\infty, N-1]$, still hold in the general setting. We comment on this in the next remark.

Remark 2.5.10. When we consider a risk-averse agent and want to discuss round trips, we first have to think about what we want to understand by a profitable round trip in this case.

One possibility is to use exactly the same definition as in the risk-neutral case, with the rationale that the risk term often just implements a penalization and does not represent actual financial costs. Furthermore, in the context of price manipulation, one can argue that whether price manipulation is possible or not should be a property of the model irrespective of the risk-preferences of the particular agent (see also [GS13, Section 22.2]).

Another option is to replace the term $E_n[\sum_{j=n}^N (D_{j-} + \frac{\gamma_j}{2}\xi_j)\xi_j]$ in Definition 2.5.1 by (2.3). The interpretation of a profitable round trip then is to make sure that the agent has nonpositive financial costs and does not deviate too much from their preferred strategy.

We now examine this definition and the mathematical results for general λ, ζ further. To this end, let the assumptions of Theorem 2.2.1 be satisfied.

Observe that the risk-neutral costs associated to a strategy are always smaller than or equal to the risk-averse costs associated to this strategy. Therefore, if there exists a profitable round trip for the risk-averse agent, this is also a profitable round trip for the risk-neutral agent. In particular, this implies that also in the risk-averse case, for any $n \in \mathbb{Z} \cap (-\infty, N-1]$, there do not exist profitable round trips at time n for $d = 0$.

Suppose for this paragraph that $\zeta \equiv 0$. Since moreover $\hat{\xi} = 0$, we then have (2.53) and that, for $n \in \mathbb{Z} \cap (-\infty, N-1]$, $d \in \mathbb{R}$, there exist profitable round trips at time n for initial deviation d if and only if (2.52) is satisfied. We again obtain the classification of existence of profitable round trips at time $n \in \mathbb{Z} \cap (-\infty, N-1]$ for $d \neq 0$ via the set $\{Y_n = \frac{1}{2}\}$.

The results in Proposition 2.5.2, Corollary 2.5.3, and Corollary 2.5.4 continue to hold for general λ (and even general $\hat{\xi}$, ζ, since the focus is on Y). The crucial point is to observe that for $n \in \mathbb{Z} \cap (-\infty, N-1]$ we have that

$$Y_n = \frac{1}{2} - E_n\left[\left(\frac{1}{2} - Y_{n+1}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - \frac{\gamma_n}{a_n}\left(\frac{1}{2} - E_n\left[\left(\frac{1}{2} - Y_{n+1}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - E_n[Y_{n+1}\beta_{n+1}]\right)^2,$$

with a_n from (2.12). The only difference to the proof of Proposition 2.5.2 is that now, a_n contains λ, which is however not relevant for the further arguments in Proposition 2.5.2, Corollary 2.5.3, and Corollary 2.5.4.

We next show that[12], if there exists $n \in \mathbb{Z} \cap (-\infty, N-1]$ such that $\{Y_n = \frac{1}{2}\} = \Omega$, then also $\{Y_n^0 = \frac{1}{2}\} = \Omega$, where we denote by Y^0 the process defined by (2.34). To this end, let $n \in \mathbb{Z} \cap (-\infty, N-1]$ such that $\{Y_n = \frac{1}{2}\} = \Omega$. It then follows from Corollary 2.5.4(i) that $\{Y_k = \frac{1}{2}\} = \Omega$ for all $k \in \{n, \ldots, N-1\}$. Corollary 2.5.4(ii) then implies that $\{E_k[\beta_{k+1}] = 1\} = \Omega$ for all $k \in \{n, \ldots, N-1\}$. From this and Proposition 2.5.2 we obtain that $\{Y_k^0 = \frac{1}{2}\} = \{E_k[Y_{k+1}^0] = \frac{1}{2}\}$ for all $k \in \{n, \ldots, N-1\}$. Using this equality, we can show by backward induction that $\{Y_n^0 = \frac{1}{2}\} = \Omega$.

If $\zeta = 0$, this means that nonexistence of profitable round trips at time $n \in \mathbb{Z} \cap (-\infty, N-1]$ for $d \neq 0$ for the risk-averse agent implies nonexistence of profitable round trips at time n for $d \neq 0$ for the risk-neutral agent.

To conclude, if $\zeta = 0$, then existence of profitable round trips for a risk-averse agent does not differ from existence of profitable round trips for a risk-neutral agent. This completes the current remark.

We finally remark that we could also have defined profitable round trips at time $n \in \mathbb{Z} \cap (-\infty, N-1]$ for initial deviation $d \in \mathbb{R}$ to be execution strategies $X \in \mathcal{A}_n^{\text{disc}}(0, d)$ such that $E_n[\sum_{j=n}^{N} \left(D_{j-} + \frac{\gamma_j}{2}\xi_j\right)\xi_j] < 0$ a.s. (instead of (2.51)), for which existence translates to $V_n(0, d) < 0$ a.s. (instead of (2.52)). Note that the class of, in this sense, profitable round trips is a subset of our class used in this section, and they coincide in a deterministic setting. Furthermore, since our mathematical analysis is based on the description of the event $\{Y_n = \frac{1}{2}\}$ (or, equivalently, $\{Y_n < \frac{1}{2}\}$), the results and proofs are the same for both notions of profitable round trips, and only the discussions would need some slight modifications.

[12]Recall that we understand equalities of events only up to P-null sets.

2.6 Closing the position in one go

A main motivation to consider optimal trade execution problems is the observation that splitting up a large order into several smaller orders can be advantageous over the naive strategy to immediately complete the whole task. There are, however, situations when it is in fact optimal to execute the order at once (for instance in Example 2.6.6 below). To examine under what conditions it is optimal to close any position in one go[13] is the topic of this section.

Let the assumptions of Theorem 2.2.1 be in force. Assume that $\hat{\xi} = 0$ and that at least one of λ, ζ is equivalent to zero. Let $n \in \mathbb{Z} \cap (-\infty, N-1]$. We are interested in the situation where $\xi_n^*(x,d) = -x$ for all $x, d \in \mathbb{R}$. Recall that, for each $x, d \in \mathbb{R}$, a version of the optimal trade size $\xi_n^*(x,d)$ (which is defined up to a P-null set) is given by the right-hand side of (2.33). We choose the versions in such a way that the random field $(x,d) \mapsto \xi_n^*(x,d)$ is continuous (the most natural choice in view of (2.33)). Then we have that

$$\{\xi_n^*(x,d) = -x \; \forall x, d \in \mathbb{R}\} = \{\xi_n^*(x,d) = -x \; \forall x, d \in \mathbb{Q}\} = \bigcap_{x,d \in \mathbb{Q}} \{\xi_n^*(x,d) = -x\}\,;$$

hence, $\{\xi_n^*(x,d) = -x \; \forall x, d \in \mathbb{R}\}$ is an $\mathcal{F}_n$-measurable event (as a countable intersection of such events). We have the following description of this event.

Lemma 2.6.1. *Let the assumptions of Theorem 2.2.1 be in force. Assume that $\hat{\xi} = 0$ and that at least one of λ, ζ is equivalent to zero. Let $n \in \mathbb{Z} \cap (-\infty, N-1]$. It then holds that*

$$\{\xi_n^*(x,d) = -x \; \forall x, d \in \mathbb{R}\} = \left\{ E_n \left[\left(Y_{n+1} - \frac{1}{2} \right) \frac{\beta_{n+1}^2}{\Gamma_{n+1}} - Y_{n+1}\beta_{n+1} + \frac{1}{2} \right] = 0 \right\}, \tag{2.58}$$

where here and below we understand equalities or inclusions for events up to P-null sets.

Proof. We compute from (2.33) for all $x, d \in \mathbb{R}$ that

$$\xi_n^*(x,d) + x = \frac{E_n \left[\left(Y_{n+1} - \frac{1}{2} \right) \frac{\beta_{n+1}^2}{\Gamma_{n+1}} - Y_{n+1}\beta_{n+1} + \frac{1}{2} \right]}{E_n \left[\frac{Y_{n+1}}{\Gamma_{n+1}} \left(\beta_{n+1} - \Gamma_{n+1} \right)^2 + \frac{1}{2} \left(1 - \frac{\beta_{n+1}^2}{\Gamma_{n+1}} \right) + \lambda_n \right]} \left(x - \frac{d}{\gamma_n} \right), \tag{2.59}$$

from which we obtain (2.58). □

[13] We remark that in our analysis we do not exclude the possibility that afterwards the position reopens again and one appends a round trip (see also Example 2.6.7). Moreover, we mention that in view of Remark 2.1.6 and for a risk-neutral setting, closing the position $x \in \mathbb{R}$ in one go is equivalent to executing the order x at once for deterministic terminal position $b \in \mathbb{R}$ and current position $x + b$.

The next result presents a relation between the previously (see Section 2.5) studied question of nonexistence of profitable round trips at time n for initial deviation $d \neq 0$ and the currently studied question of closing the position in one go.

Proposition 2.6.2. *Let the assumptions of Theorem 2.2.1 be in force. Assume that* $\hat{\xi} = 0$ *and that at least one of* λ, ζ *is equivalent to zero. Let* $n \in \mathbb{Z} \cap (-\infty, N-1]$. *It then holds that*

$$\left\{Y_n = \frac{1}{2}\right\} = \{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\} \cap \left\{E_n[Y_{n+1}] = \frac{1}{2}\right\}.$$

Proof. We first establish the inclusion "$\subseteq$". Recall that by Proposition 2.5.2 and Corollary 2.5.4 (see also Remark 2.5.10) we have that

$$\left\{Y_n = \frac{1}{2}\right\} = \left\{E_n\left[Y_{n+1}\right] = \frac{1}{2}, E_n\left[\beta_{n+1}\right] = 1\right\} \subseteq \left\{Y_{n+1} = \frac{1}{2}\right\}.$$

In particular, on the event $\{Y_n = \frac{1}{2}\} \in \mathcal{F}_n$ it holds that $Y_{n+1} = \frac{1}{2}$ and $E_n[\beta_{n+1}] = 1$, which implies that on the event $\{Y_n = \frac{1}{2}\} \in \mathcal{F}_n$ we have that

$$E_n\left[\left(Y_{n+1} - \frac{1}{2}\right)\frac{\beta_{n+1}^2}{\Gamma_{n+1}} - Y_{n+1}\beta_{n+1} + \frac{1}{2}\right] = 0.$$

Lemma 2.6.1 now yields that

$$\left\{Y_n = \frac{1}{2}\right\} \subseteq \{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\}.$$

Since also $E_n\left[Y_{n+1}\right] = \frac{1}{2}$ on $\{Y_n = \frac{1}{2}\} \in \mathcal{F}_n$, it follows that

$$\left\{Y_n = \frac{1}{2}\right\} \subseteq \{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\} \cap \left\{E_n[Y_{n+1}] = \frac{1}{2}\right\}.$$

To prove the reverse inclusion "$\supseteq$" we first note that

$$\left\{E_n[Y_{n+1}] = \frac{1}{2}\right\} \subseteq \left\{Y_{n+1} = \frac{1}{2}\right\} \tag{2.60}$$

because $Y_{n+1} \leq \frac{1}{2}$ a.s. It follows from (2.58) and (2.60) that on the $\mathcal{F}_n$-measurable set

$$A_n := \{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\} \cap \left\{E_n[Y_{n+1}] = \frac{1}{2}\right\}$$

it holds that $\frac{1}{2}E_n[\beta_{n+1}] = E_n[Y_{n+1}\beta_{n+1}] = \frac{1}{2}$, i.e., $E_n[\beta_{n+1}] = 1$. Hence,

$$A_n \subseteq \left\{E_n\left[Y_{n+1}\right] = \frac{1}{2}, E_n\left[\beta_{n+1}\right] = 1\right\} = \left\{Y_n = \frac{1}{2}\right\},$$

where the set equality is again Proposition 2.5.2 (see also Remark 2.5.10). □

In particular, we have that

$$\left\{Y_n = \frac{1}{2}\right\} \subseteq \{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\}. \tag{2.61}$$

It is worth noting that the inclusion in (2.61) can be strict in the sense that the set difference can be nonnegligible. In Example 2.6.7 further below (we also use Proposition 2.6.4 for Example 2.6.7) we show that, indeed, there can be profitable round trips (cf. Section 2.5) at time n for initial deviation $d \neq 0$ and still it can be optimal to close the whole position at time n.

However, at time $N-1$, we always have equality in (2.61).

Corollary 2.6.3. *Let the assumptions of Theorem 2.2.1 be in force. Assume that $\hat{\xi} = 0$ and that at least one of λ, ζ is equivalent to zero. It then holds that*

$$\left\{Y_{N-1} = \frac{1}{2}\right\} = \left\{\xi_{N-1}^*(x,d) = -x \ \forall x, d \in \mathbb{R}\right\}.$$

Proof. This follows from Proposition 2.6.2 and the fact that $Y_N = \frac{1}{2}$. □

We next provide more details on closing the position in one go for the case of processes with independent multiplicative increments **(PIMI)** of Section 2.3. We recall that in this case the process Y is deterministic. Notice, however, that the optimal trade sizes $\xi_n^*(x,d)$, $x, d \in \mathbb{R}$, in general are still random because of the randomness in γ_n, see (2.40).

Proposition 2.6.4. *Let the assumptions of Lemma 2.3.1 be satisfied. Assume that $\hat{\xi} = 0$ and that at least one of λ, ζ is equivalent to zero. Let $n \in \mathbb{Z} \cap (-\infty, N-1]$. It then holds that $\{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\}$ is either Ω or $\emptyset$. Furthermore, the following statements are equivalent:*

(i) $\{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\} = \Omega$.

(ii) There exist $x, d \in \mathbb{R}$ with $P(\gamma_n x \neq d) > 0$ such that $\{\xi_n^*(x,d) = -x\} = \Omega$.

(iii) It holds that

$$E[\beta_{n+1}] = 1 + \frac{\left(1 - E\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right]\right)\left(\frac{1}{2} - Y_{n+1}\right)}{Y_{n+1}}. \tag{2.62}$$

Proof. Since Y is deterministic and Γ_{n+1} and β_{n+1} are independent of $\mathcal{F}_n$, Lemma 2.6.1 yields

$$\{\xi_n^*(x,d) = -x \ \forall x, d \in \mathbb{R}\} = \left\{\left(Y_{n+1} - \frac{1}{2}\right) E\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - Y_{n+1} E\left[\beta_{n+1}\right] + \frac{1}{2} = 0\right\}, \tag{2.63}$$

which can be either Ω or $\emptyset$.

The equivalence between (i) and (ii) follows from (2.59) and the fact that the factor in front of $(x-\frac{d}{\gamma_n})$ on the right-hand side of (2.59) is deterministic under our assumptions.

For the equivalence between (i) and (iii), note that (2.63) shows that (i) is equivalent to

$$Y_{n+1}E\left[\beta_{n+1}\right] = Y_{n+1} + \left(Y_{n+1} - \frac{1}{2}\right)\left(E\left[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}\right] - 1\right),$$

which clearly is equivalent to (2.62). □

Corollary 2.6.5. *Let the assumptions of Lemma 2.3.1 be satisfied. Assume that $\hat{\xi} = 0$ and that at least one of λ, ζ is equivalent to zero. Let $n \in \mathbb{Z} \cap (-\infty, N-1]$, and assume that $\{\xi_n^*(x,d) = -x \; \forall x, d \in \mathbb{R}\} \neq \emptyset$. It then holds that $E[\beta_{n+1}] \geq 1$, and, if $Y_{n+1} < \frac{1}{2}$, even that $E[\beta_{n+1}] > 1$.*

Proof. By Proposition 2.6.4 we have from $\{\xi_n^*(x,d) = -x \; \forall x, d \in \mathbb{R}\} \neq \emptyset$ that (i) in Proposition 2.6.4 holds, and thus also (iii). Since Y_{n+1} is $(0, \frac{1}{2}]$-valued and $E[\frac{\beta_{n+1}^2}{\Gamma_{n+1}}] < 1$, the claim follows from (2.62). □

The meaning of Corollary 2.6.5 is that in the case of **(PIMI)** (special case: deterministic processes β and γ), closing the position in one go is never optimal in the conventional framework, where the resilience process β is assumed to be $(0,1)$-valued (and $\lambda \equiv 0$).

This raises the question whether closing the position in one go can be optimal in general (that is, beyond **(PIMI)**) with the resilience process β taking values in $(0,1)$ and $\lambda \equiv 0$. In our setting the answer is affirmative (see the next Example 2.6.6). It is worth noting that in the related setting where trading is constrained only in one direction and the process β is $(0,1)$-valued (and $\lambda \equiv 0$), the answer is negative, i.e., closing the position in one go is never optimal in that setting (see [FSU19, Proposition A.3] and [FSU14, Proposition 5.6]).

Example 2.6.6. In this example we consider a version of our model with three time points for trading $N-2$, $N-1$, and N where the resilience process β is $(0,1)$-valued and $\lambda \equiv 0$ and still it is optimal at time $N-2$ to close the position in one go. To this end, assume that $\lambda \equiv 0$, that $\mathcal{F}_{N-2} = \{\emptyset, \Omega\}$, and that we can specify the positive random variables γ_{N-1}, γ_N, and the $(0,1)$-valued random variable β_N in such a way that $E_{N-1}[\frac{\beta_N^2}{\Gamma_N}] < 1$, $(1 - E_{N-1}[\frac{\beta_N^2}{\Gamma_N}])^{-1} \in L^{\infty-}$ and that Y_{N-1} and $\frac{1}{\gamma_{N-1}}$ are strictly negatively correlated, i.e.,

$$E\left[\frac{Y_{N-1}}{\gamma_{N-1}}\right] - E\left[Y_{N-1}\right]E\left[\frac{1}{\gamma_{N-1}}\right] < 0. \tag{2.64}$$

At the end of this example we present a specific choice such that these assumptions are satisfied.

To continue, by (2.64) we can choose a deterministic

$$\beta_{N-1} \in \left(\frac{E\left[\frac{Y_{N-1}}{\gamma_{N-1}}\right]}{E\left[Y_{N-1}\right] E\left[\frac{1}{\gamma_{N-1}}\right]}, 1 \right) \tag{2.65}$$

and then define

$$\gamma_{N-2} = \frac{E\left[\frac{1}{2} - Y_{N-1}\beta_{N-1}\right]}{E\left[\left(\frac{1}{2} - Y_{N-1}\right)\frac{\beta_{N-1}^2}{\gamma_{N-1}}\right]}. \tag{2.66}$$

Note that, indeed, $\beta_{N-1} \in (0,1)$ and $\gamma_{N-2} > 0$. Next, we verify that $E[\frac{\beta_{N-1}^2}{\Gamma_{N-1}}] < 1$ (recall $\mathcal{F}_{N-2} = \{\emptyset, \Omega\}$). By (2.65) it holds that

$$E\left[\beta_{N-1}Y_{N-1}\right] E\left[\frac{1}{\gamma_{N-1}}\right] > E\left[\frac{Y_{N-1}}{\gamma_{N-1}}\right].$$

This implies that

$$E\left[\frac{1}{2} - \beta_{N-1}Y_{N-1}\right] E\left[\frac{\beta_{N-1}^2}{\gamma_{N-1}}\right] < E\left[\left(\frac{1}{2} - Y_{N-1}\right)\frac{\beta_{N-1}^2}{\gamma_{N-1}}\right].$$

Due to $E[\frac{1}{2} - \beta_{N-1}Y_{N-1}] > 0$, it follows that

$$E\left[\frac{\beta_{N-1}^2}{\gamma_{N-1}}\right] < \frac{E\left[\left(\frac{1}{2} - Y_{N-1}\right)\frac{\beta_{N-1}^2}{\gamma_{N-1}}\right]}{E\left[\frac{1}{2} - \beta_{N-1}Y_{N-1}\right]} = \frac{1}{\gamma_{N-2}}.$$

Since γ_{N-2} is deterministic and $\Gamma_{N-1} = \frac{\gamma_{N-1}}{\gamma_{N-2}}$, we get $E[\frac{\beta_{N-1}^2}{\Gamma_{N-1}}] < 1$.

Having set up the model, we now examine closing the position in one go. From (2.66) we obtain that

$$E\left[\left(Y_{N-1} - \frac{1}{2}\right)\frac{\beta_{N-1}^2}{\Gamma_{N-1}} - Y_{N-1}\beta_{N-1} + \frac{1}{2}\right] = 0.$$

Therefore, it follows from Lemma 2.6.1 and $\mathcal{F}_{N-2} = \{\emptyset, \Omega\}$ that for all $x, d \in \mathbb{R}$ it holds that $\xi_{N-2}^*(x,d) = -x$, i.e., it is optimal to close the whole position at time $N-2$.

It remains to specify positive γ_{N-1}, γ_N, and $(0,1)$-valued β_N, such that $E_{N-1}[\frac{\beta_N^2}{\Gamma_N}] < 1$, $(1 - E_{N-1}[\frac{\beta_N^2}{\Gamma_N}])^{-1} \in L^{\infty-}$, and (2.64) are satisfied. To this end, let γ_{N-1} be $\{\frac{1}{2}, 1\}$-valued with $P(\gamma_{N-1} = 1) = p \in (0,1)$ and $P(\gamma_{N-1} = \frac{1}{2}) = 1 - p$. Define $\gamma_N = \gamma_{N-1}^2$ and $\beta_N = \frac{\gamma_{N-1}}{2}$.

Note that β_N is $(0,1)$-valued, that $\gamma_{N-1}, \gamma_N > 0$, and that $\Gamma_N = \gamma_{N-1}$. It further holds that $E_{N-1}[\frac{\beta_N^2}{\Gamma_N}] = \frac{\gamma_{N-1}}{4} \leq \frac{1}{4} < 1$, and then $(1 - E_{N-1}[\frac{\beta_N^2}{\Gamma_N}])^{-1} \in L^{\infty-}$.

For (2.64), we first compute from (2.34) and $Y_N = \frac{1}{2}$ that

$$Y_{N-1} = \frac{1}{2}E_{N-1}\left[\Gamma_N\right] - \frac{\frac{1}{2}\left(E_{N-1}\left[\beta_N\right] - E_{N-1}\left[\Gamma_N\right]\right)^2}{E_{N-1}\left[1 - 2\beta_N + \Gamma_N\right]} = \frac{1}{2}\frac{E_{N-1}\left[\Gamma_N\right] - \left(E_{N-1}\left[\beta_N\right]\right)^2}{1 - 2E_{N-1}\left[\beta_N\right] + E_{N-1}[\Gamma_N]}.$$

Using $E_{N-1}[\Gamma_N] = E_{N-1}[\gamma_{N-1}] = \gamma_{N-1}$ and $E_{N-1}[\beta_N] = E_{N-1}[\frac{\gamma_{N-1}}{2}] = \frac{\gamma_{N-1}}{2}$, we thus have that

$$Y_{N-1} = \frac{1}{2}\left(\gamma_{N-1} - \frac{\gamma_{N-1}^2}{4}\right).$$

Since

$$E\left[\gamma_{N-1}\right] = p + \frac{1}{2}(1-p), \quad E\left[\gamma_{N-1}^2\right] = p + \frac{1}{4}(1-p), \quad \text{and} \quad E\left[\frac{1}{\gamma_{N-1}}\right] = p + 2(1-p),$$

we obtain (2.64):

$$\begin{aligned}
&E\left[\frac{Y_{N-1}}{\gamma_{N-1}}\right] - E\left[Y_{N-1}\right] E\left[\frac{1}{\gamma_{N-1}}\right] \\
&= \frac{1}{2}\left(1 - \frac{1}{4}E\left[\gamma_{N-1}\right]\right) - \frac{1}{2}\left(E\left[\gamma_{N-1}\right] - \frac{1}{4}E\left[\gamma_{N-1}^2\right]\right) E\left[\frac{1}{\gamma_{N-1}}\right] \\
&= \frac{1}{2}\frac{5}{16}p(p-1) < 0.
\end{aligned}$$

For completeness, we mention that for $k \in \{N-2, N-1, N\}$ the assumptions $\beta_k, \gamma_k, \frac{1}{\gamma_k} \in L^{\infty-}$ are trivially satisfied.

We next provide the announced example on inclusion (2.61).

Example 2.6.7. Let $\lambda \equiv 0$. Consider a resilience process β and a price impact process γ that satisfy the assumptions of Lemma 2.3.1 (in particular, **(PIMI)**) and, moreover, $E[\beta_N] \neq 1$ and

$$E[\beta_{N-1}] = 1 + \frac{\left(1 - E\left[\frac{\beta_{N-1}^2}{\Gamma_{N-1}}\right]\right)\left(\frac{1}{2} - Y_{N-1}\right)}{Y_{N-1}}. \tag{2.67}$$

Below we present a specific choice of the parameters such that (2.67) is satisfied.

We now argue that, with these assumptions, the inclusion in (2.61) for $n = N-2$ is strict. On the one hand, notice that by Proposition 2.6.4, condition (2.67) is equivalent to

$$\left\{\xi_{N-2}^*(x,d) = -x \;\forall x, d \in \mathbb{R}\right\} = \Omega. \tag{2.68}$$

On the other hand, $E[\beta_N] \neq 1$ and Corollary 2.5.5 imply that for the deterministic process Y it holds that $Y_k < \frac{1}{2}$ for all $k \in \mathbb{Z} \cap (-\infty, N-1]$, hence

$$\left\{Y_{N-2} = \frac{1}{2}\right\} = \emptyset.$$

This does not only show that in our example the inclusion in (2.61) is strict, but also that there exist profitable round trips at time $N-2$ and at time $N-1$ for nonzero initial deviation. Concurrently, it is optimal at time $N-2$ to close the position in one go.

Moreover, we can compute from (2.40) and $Y_N = \frac{1}{2}$ that for all $d \in \mathbb{R}$

$$\xi^*_{N-1}(0,d) = \frac{E[\beta_N] - 1}{1 + E[\Gamma_N] - 2E[\beta_N]} \frac{d}{\gamma_{N-1}}. \tag{2.69}$$

Therefore, the optimal strategy in this example for the trading period $\{N-2, N-1, N\}$, any initial position $x \in \mathbb{R}$, and any initial deviation $d \in \mathbb{R}$ is to close the position at time $N-2$ (cf. (2.68)), to build up a new position at time $N-1$ (at least if $D_{(N-1)-} = (d - \gamma_{N-2}x)\beta_{N-1} \neq 0$, cf. (2.69) and $E[\beta_N] \neq 1$), and to close this position at time N.

It remains to explain how we can satisfy (2.67). A possible example where the requirements on β and γ listed above are satisfied can be constructed with deterministic sequences β and γ as follows. Choose arbitrary deterministic $\gamma_N, \gamma_{N-1} > 0$ and $\beta_N \in (0, \sqrt{\Gamma_N}) \setminus \{1\}$. Then we clearly have $\frac{\beta_N^2}{\Gamma_N} < 1$. Furthermore, these inputs yield a deterministic $Y_{N-1} \in (0, \frac{1}{2})$ (see Corollary 2.5.3). Take a sufficiently small $a > 0$ such that

$$\frac{aY_{N-1}}{\frac{1}{2} - Y_{N-1}} \in (0,1).$$

Finally, set $\beta_{N-1} = 1 + a$ and choose $\gamma_{N-2} > 0$ to satisfy

$$\frac{aY_{N-1}}{\frac{1}{2} - Y_{N-1}} = 1 - \frac{(1+a)^2}{\Gamma_{N-1}}$$

(recall that $\Gamma_{N-1} = \frac{\gamma_{N-1}}{\gamma_{N-2}}$). This choice gives us (2.67) together with $\frac{\beta_{N-1}^2}{\Gamma_{N-1}} < 1$.

We briefly discuss the difference between a risk-neutral agent and a risk-averse agent with respect to closing the position in one go (for $\zeta \equiv 0 = \hat{\xi}$). Observe that the strategy to close the position immediately and then stop trading yields the same costs for both agents (cf. (2.3)). Moreover, for any strategy, the associated costs in the risk-neutral setting are smaller than or equal to those in the risk-averse setting. Therefore, if the optimal strategy for the risk-neutral agent consists only of a single trade at the beginning, then the same holds true for the risk-averse agent. However, optimality of closing in one go for the risk-neutral agent does not imply that closing in one go is necessarily optimal for the risk-averse agent. Consider the following example where the risk-neutral agent closes in one go, but the risk-averse agent does not.

Example 2.6.8. Let the resilience process β and the price impact process γ be chosen as in Example 2.6.7. Denote by $Y = (Y_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ again the (deterministic) process from Example 2.6.7, and by $\xi^*_n(x,d)$, $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, the optimal trade sizes from Example 2.6.7. To compare the risk-neutral agent from Example 2.6.7 to a risk-averse agent in the same setting, we now include some deterministic $\lambda = (\lambda_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ satisfying $\lambda_{N-1} > 0$. Note that we are still in the setting of Lemma 2.3.1. Denote by $\widetilde{Y} = (\widetilde{Y}_n)_{n \in \mathbb{Z} \cap (-\infty, N]}$ the (deterministic) process given by $\widetilde{Y}_N = \frac{1}{2}$ and (2.39) for this λ, and denote by $\widetilde{\xi}_n^*(x,d)$, $n \in \mathbb{Z} \cap (-\infty, N]$, $x, d \in \mathbb{R}$, the optimal trade sizes (2.40) for this λ.

Observe that

$$\widetilde{Y}_{N-1} - Y_{N-1} = \frac{\lambda_{N-1} (1 - E[\beta_N])^2}{(1 - 2E[\beta_N] + E[\Gamma_N] + 2\lambda_{N-1}) (1 - 2E[\beta_N] + E[\Gamma_N])} > 0$$

due to $\lambda_{N-1} > 0$ and $E[\beta_N] \neq 1$. In particular, we have that $\widetilde{Y}_{N-1} \neq Y_{N-1}$. This implies that (2.67) does not hold with Y_{N-1} replaced by $\widetilde{Y}_{N-1}$. It follows from Proposition 2.6.4 that $\{\widetilde{\xi}_{N-2}^*(x,d) = -x \; \forall x, d \in \mathbb{R}\} = \emptyset$, and, using (2.59), that it holds for all $x, d \in \mathbb{R}$ with $P(\gamma_{N-2} x \neq d) = 1$ that

$$\left\{\widetilde{\xi}_{N-2}^*(x,d) = -x\right\} = \left\{\left(\widetilde{Y}_{N-1} - \frac{1}{2}\right) E\left[\frac{\beta_{N-1}^2}{\Gamma_{N-1}}\right] - \widetilde{Y}_{N-1} E[\beta_{N-1}] + \frac{1}{2} = 0\right\} = \emptyset.$$

To sum up, if $x, d \in \mathbb{R}$ with $\gamma_{N-2} x \neq d$, then the risk-averse agent does not close their position x at time $N-2$, whereas the risk-neutral agent closes the position at time $N-2$ to append a (nontrivial) round trip at time $N-1$ (cf. Example 2.6.7).

If $x, d \in \mathbb{R}$ such that $\gamma_{N-2} x = d$, then both agents close their position at time $N-2$. Moreover, the risk-neutral agent in this case does not open a new position at time $N-1$ (cf. (2.69)). Since only the denominator in (2.69) changes for $\lambda_{N-1} > 0$, the risk-averse agent does not open a new position either. I.e., we have the situation described prior to this example, where it is optimal for both agents to close the position immediately and quit trading.

We further mention that in the case $x = 0$ and $d \neq 0$, both agents perform a nontrivial round trip: the risk-averse agent starts their round trip at time $N-2$, since closing in one go at time $N-2$ is not optimal for them, while the risk-neutral agent waits until time $N-1$ and then starts trading at time $N-1$ due to (2.69).

3

From discrete to continuous time: base setting and heuristics

For the remainder of the thesis, we are interested in optimal trade execution in a continuous-time version of the model from Chapter 2. That is, we want to consider a model of Obizhaeva-Wang type with stochastic resilience and stochastic order book depth where we allow trading during the whole time interval $[0, T]$ (for given terminal time $T > 0$) instead of only at the time points $\{0, 1, \dots, N\}$ (for given terminal time $N \in \mathbb{N}$).

Observe that for our continuous-time model, in contrast to Chapter 2, we assume a certain structure of the processes that describe the resilience and the order book depth. This set-up is introduced in Section 3.1. Within this set-up, we in Chapter 5 formulate and solve a continuous-time control problem where we consider càdlàg semimartingales as execution strategies. In Chapter 7 we, also within the set-up of Section 3.1, start from a typical formulation for finite-variation strategies, extend this to progressively measurable strategies, and solve the extended problem in Chapter 8 via reduction to a standard LQ stochastic control problem. We refer to, e.g., Chapter 1, Section 5.3, and Section 5.4.1 for reasons why we want to allow for more general than finite-variation strategies.

Section 3.2 and Section 3.3 are purely heuristic treatments and serve to motivate, in the risk-neutral setting, in particular the semimartingale control problem and its solution by the discrete-time problem and results from Chapter 2. We first, in Section 3.2, derive appropriate definitions of the deviation dynamics and of the costs in the continuous-time model for semimartingale strategies via a limiting procedure from the discrete-time setting. It is worth noting that for semimartingale strategies, there in general appear certain covariation terms (see also Section 5.1.2 for further discussion). Subsequently, in Section 3.3, we heuristically show that the process defined by backward recursion in (2.34) that characterizes the solution of the discrete-time problem (see Corollary 2.2.4) gives rise to a quadratic BSDE. This BSDE in fact describes the solution of the continuous-time problems (see Theorem 5.2.6, and compare also with Chapter 8). We further analyze this BSDE in Chapter 4.

Section 3.2 and Section 3.3 are extensions (to the setting of possibly diffusive resilience) of [AKU21a, Appendix A] and [AKU21a, Appendix B], respectively.

3.1 Base setting and notations for continuous time

The following mathematical set-up and notations provide a basis for the remaining chapters of the thesis, where we often specify subsettings of this framework. We here also introduce some assumptions that are frequently used at several places in the thesis.

Let $T \in (0,\infty)$ be the terminal time, and let $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_T$. Furthermore, we suppose that for some $m \in \mathbb{N}$, $m \geq 2$, the filtered probability space supports m independent continuous local martingales[1] $M^{(j)} = (M^{(j)}_s)_{s\in[0,T]}$, $j \in \{1,\dots,m\}$, such that the quadratic variation processes $[M^{(j)}]$, $j \in \{1,\dots,m\}$, are pairwise indistinguishable. In particular, $[M^{(1)}] = [M^{(2)}]$ and $[M^{(1)}, M^{(2)}] = 0$.

We now introduce some notation. For a (possibly multidimensional) Brownian motion $W = (W_s)_{s\in[0,T]}$, the augmented natural filtration of W is denoted by $(\mathcal{F}^W_s)_{s\in[0,T]}$. For $t \in [0,T]$, the Borel sigma-algebra on $[t,T]$ is written as $\mathcal{B}([t,T])$. The Lebesgue measure on $([0,T], \mathcal{B}([0,T]))$ is called Leb. We denote by $\mathcal{D}_{M^{(1)}}$ the Doléans measure associated to $M^{(1)}$ on $(\Omega\times[0,T], \mathcal{F}\otimes\mathcal{B}([0,T]))$, i.e., $\mathcal{D}_{M^{(1)}}(C) = E[\int_0^T 1_C(\cdot,s)d[M^{(1)}]_s]$ for $C \in \mathcal{F}\otimes\mathcal{B}([0,T])$. For $t \in [0,T]$ we use the notation $\mathcal{D}_{M^{(1)}}|_{[t,T]}$ for the restriction of the Doléans measure $\mathcal{D}_{M^{(1)}}$ to $(\Omega\times[t,T], \mathcal{F}\otimes\mathcal{B}([t,T]))$. For $t \in [0,T]$ conditional expectations with respect to $\mathcal{F}_t$ are denoted by $E_t[\cdot]$. For $t \in [0,T]$ and a càdlàg process $X = (X_s)_{s\in[t-,T]}$, a jump at time $s \in [t,T]$ is denoted by $\Delta X_s = X_s - X_{s-}$. We follow the convention that, for $t \in [0,T]$, $r \in [t,T]$, and a càdlàg semimartingale $L = (L_s)_{s\in[t-,T]}$, jumps of the càdlàg integrator L at time t contribute to integrals of the form $\int_{[t,r]} \dots dL_s$. In contrast, we write $\int_{(t,r]} \dots dL_s$ when we do not include jumps of L at time t into the integral. The notation $\int_t^r \dots dL_s$ is sometimes used for continuous integrators L. For a continuous semimartingale $Q = (Q_s)_{s\in[0,T]}$ we denote by $\mathcal{E}(Q) = (\mathcal{E}(Q)_s)_{s\in[0,T]}$ its stochastic exponential, i.e., $\mathcal{E}(Q)_s = \exp\left(Q_s - Q_0 - \frac{1}{2}[Q]_s\right)$, $s \in [0,T]$. For $t \leq s$ in $[0,T]$ we also use the notation $\mathcal{E}(Q)_{t,s} = \frac{\mathcal{E}(Q)_s}{\mathcal{E}(Q)_t} = \exp\left(Q_s - Q_t - \frac{1}{2}([Q]_s - [Q]_t)\right)$. A superscript $^\top$ of a matrix denotes transpose. A superscript ^{c} of a set means its complement. For $n \in \mathbb{N}$ and $y \in \mathbb{R}^n$ let $\|y\|_n = (\sum_{j=1}^n y_j^2)^{\frac{1}{2}}$. For every $t \in [0,T]$ we mean by $L^1(\Omega, \mathcal{F}_t, P)$ the space of all real-valued $\mathcal{F}_t$-measurable random variables K such that $\|K\|_{L^1} = E[|K|] < \infty$. For $t \in [0,T]$, let $\mathcal{L}^2_t = \mathcal{L}^2(\Omega\times[t,T], \mathrm{Prog}(\Omega\times[t,T]), \mathcal{D}_{M^{(1)}}|_{[t,T]})$ denote the space of all

[1] We remark that in the dynamics for the resilience and price impact process, we only make use of the local martingales $M^{(1)}$ and $M^{(2)}$. The reason why we do not just let $m = 2$ is that in Proposition 4.3.2 and Section 8.2, to apply the results from the literature on LQ stochastic control, we assume that the filtration $(\mathcal{F}_s)_{s\in[0,T]}$ is generated by an m-dimensional Brownian motion $(W^{(1)},\dots,W^{(m)})^\top$. The components $W^{(3)},\dots,W^{(m)}$ will therefore serve as further sources of randomness on which the model inputs may depend.

(equivalence classes of) real-valued progressively measurable processes $u = (u_s)_{s\in[t,T]}$ such that $\|u\|_{\mathcal{L}_t^2} = (E[\int_t^T u_s^2 d[M^{(1)}]_s])^{\frac{1}{2}} < \infty$.

Our model requires six progressively measurable processes $\mu = (\mu_s)_{s\in[0,T]}$, $\sigma = (\sigma_s)_{s\in[0,T]}$, $\rho = (\rho_s)_{s\in[0,T]}$, $\eta = (\eta_s)_{s\in[0,T]}$, $\overline{r} = (\overline{r}_s)_{s\in[0,T]}$, and $\lambda = (\lambda_s)_{s\in[0,T]}$ such that $\int_0^T (|\rho_s| + |\mu_s| + \sigma_s^2 + \eta_s^2) d[M^1]_s < \infty$ a.s., and such that λ is $\mathcal{D}_{M^{(1)}}$-a.e. bounded and $\overline{r}$ is $[-1,1]$-valued. We define the continuous local martingale $M^R = (M_s^R)_{s\in[0,T]}$ by

$$dM_s^R = \overline{r}_s dM_s^{(1)} + \sqrt{1-\overline{r}_s^2} dM_s^{(2)}, \quad s \in [0,T], \quad M_0^R = 0,$$

and refer to $\overline{r}$ as the *correlation (process)*. Observe that

$$d[M^R]_s = \overline{r}_s^2 d[M^{(1)}]_s + (1-\overline{r}_s^2) d[M^{(2)}]_s = d[M^{(1)}]_s, \quad s \in [0,T],$$

due to $[M^{(1)}] = [M^{(2)}]$ and independence of $M^{(1)}, M^{(2)}$. The process ρ, called *resilience coefficient*, and the process η together define the *resilience (process)*[2] $R = (R_s)_{s\in[0,T]}$, which is a continuous semimartingale, via

$$dR_s = \rho_s d[M^R]_s + \eta_s dM_s^R, \quad s \in [0,T], \quad R_0 = 0. \tag{3.1}$$

Based on the inputs μ and σ, the *price impact (process)* $\gamma = (\gamma_s)_{s\in[0,T]}$, a strictly positive continuous semimartingale, is modeled by

$$d\gamma_s = \gamma_s \mu_s d[M^{(1)}]_s + \gamma_s \sigma_s dM_s^{(1)}, \quad s \in [0,T], \quad \gamma_0 \in (0,\infty). \tag{3.2}$$

The price impact process has the representation

$$\gamma_s = \gamma_0 \exp\left(\int_0^s \left(\mu_r - \frac{\sigma_r^2}{2}\right) d[M^{(1)}]_r + \int_0^s \sigma_r dM_r^{(1)}\right), \quad s \in [0,T].$$

For future reference, note that by an application of Itô's formula it holds for all $s \in [0,T]$ that

$$d\gamma_s^{-1} = \gamma_s^{-1}\left(-(\mu_s - \sigma_s^2)\, d[M^{(1)}]_s - \sigma_s\, dM_s^{(1)}\right), \tag{3.3}$$

$$d\gamma_s^{\frac{1}{2}} = \gamma_s^{\frac{1}{2}}\left(\frac{1}{2}\mu_s - \frac{1}{8}\sigma_s^2\right) d[M^{(1)}]_s + \frac{1}{2}\gamma_s^{\frac{1}{2}}\sigma_s dM_s^{(1)}, \tag{3.4}$$

$$d\gamma_s^{-\frac{1}{2}} = \gamma_s^{-\frac{1}{2}}\left(-\frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2\right) d[M^{(1)}]_s - \frac{1}{2}\gamma_s^{-\frac{1}{2}}\sigma_s dM_s^{(1)}. \tag{3.5}$$

[2]Note that although we call R resilience process, it does not play the same role as the resilience process β in Chapter 2. Instead, the multiplicative increments of the stochastic exponential of $-R$ are comparable to β from Chapter 2 (cf., e.g., Section 3.2). Further, R does not have the same meaning as the resilience coefficient (also called resilience rate) in, e.g., [OW13] and [FSU19]. If η vanishes in our model, then the resilience is described by ρ only, and ρ is what most articles call resilience (coefficient/rate); see also [AKU21a] for this subsetting (however, ρ in [AKU21a] is called resilience process to emphasize that it can be stochastic).

We moreover introduce, for convenience, the process $\kappa = (\kappa_s)_{s\in[0,T]}$ defined by

$$\kappa_s = \frac{1}{2}\left(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s\eta_s\overline{r}_s\right), \quad s \in [0,T]. \tag{3.6}$$

We now formulate some assumptions that we invoke at several places in the thesis. For most of our results, we assume that:

Assumption ($\mathbf{C_{>0}}$). $\kappa + \lambda > 0$ $\mathcal{D}_{M^{(1)}}$-a.e.

This is a structural condition on the input processes which, roughly speaking, ensures that the minimization problems that we consider are convex. To see this, we refer to the representation in Theorem 5.2.1. Notice also the similar condition in [FSU19, Proposition 6.2]. Observe furthermore that condition ($\mathbf{C_{>0}}$) ensures that the denominator in the driver of the quadratic BSDE in Chapter 4 stays strictly positive. We sometimes (e.g., in Chapter 4 and Chapter 6) strengthen condition ($\mathbf{C_{>0}}$) to boundedness away from zero, i.e.:

Assumption ($\mathbf{C_{\geq\varepsilon}}$). There exists $\varepsilon \in (0,\infty)$ such that $\kappa + \lambda \geq \varepsilon$ $\mathcal{D}_{M^{(1)}}$-a.e.

This condition appears also when we consider the "regular case" in [KT02]. For the "singular case" in [KT02], we introduce the following assumption (note that it always holds that $\sigma^2 + \eta^2 + 2\sigma\eta\overline{r} \geq 0$ $\mathcal{D}_{M^{(1)}}$-a.e.).

Assumption ($\mathbf{C_s}$). There exists $\overline{\varepsilon} \in (0,\infty)$ such that $\sigma^2 + \eta^2 + 2\sigma\eta\overline{r} \geq \overline{\varepsilon}$ $\mathcal{D}_{M^{(1)}}$-a.e.

Further, we sometimes (e.g., in Section 8.2, and most of the time in Chapter 4) require that λ and κ on their own are nonnegative:

Assumption ($\mathbf{C_{nonneg}}$). $\kappa \geq 0$ $\mathcal{D}_{M^{(1)}}$-a.e. and $\lambda \geq 0$ $\mathcal{D}_{M^{(1)}}$-a.e.

Note that if $\lambda \equiv 0$, then $\kappa \geq 0$ is already implied by ($\mathbf{C_{>0}}$) (which, in turn, is an implication of ($\mathbf{C_{\geq\varepsilon}}$)). In Chapter 4, Chapter 6, and Chapter 8, as well as for almost all of the results in Chapter 7 and some of the results in Chapter 5, we assume that the input processes are bounded:

Assumption ($\mathbf{C_{bdd}}$). There exist $c_\rho, c_\mu, c_\sigma, c_\eta \in (0,\infty)$ such that $|\rho| \leq c_\rho$, $|\mu| \leq c_\mu$, $|\sigma| \leq c_\sigma$, $|\eta| \leq c_\eta$ $\mathcal{D}_{M^{(1)}}$-a.e.

In particular, we rely on this when we prove existence of the BSDE in Chapter 4. Further, ($\mathbf{C_{bdd}}$) allows to apply results from the LQ literature in Section 8.2, since the coefficient processes in the control problems of such works are typically assumed to be bounded. To obtain a standard LQ problem in Chapter 8 and for our examples (including Chapter 6), we consider $M^{(1)}$ (and hence by Lévy's characterization all $M^{(j)}$, $j \in \{1,\dots,m\}$) to be a Brownian motion. When dealing with general continuous local martingales, we often require the following condition, which, in particular, is satisfied in case of an $(\mathcal{F}_s)_{s\in[0,T]}$-Brownian motion $M^{(1)} = W^{(1)}$:

Assumption ($\mathbf{C_{[M^{(1)}]}}$). For all $c \in (0,\infty)$: $E\left[\exp(c\,[M^{(1)}]_T)\right] < \infty$.

3.2 Motivation for the deviation dynamics and the costs

We fix an initial position $x \in \mathbb{R}$ and an initial deviation $d \in \mathbb{R}$, and consider a semimartingale execution strategy X for the trading interval $[0,T]$. By this, we mean a càdlàg semimartingale $X = (X_s)_{s\in[0-,T]}$ with $X_{0-} = x$, $X_T = 0$, and suitable integrability conditions. For any (large) $N \in \mathbb{N}$, we set $h = \frac{T}{N}$ and consider discrete-time trading at points of the grid $\{kh\colon k = 0,\dots,N\}$. More precisely, the continuous-time strategy $X = (X_s)_{s\in[0-,T]}$ is approximated by the discrete-time strategy $(X_{kh})_{k\in\{-1,\dots,N\}}$ with initial value $X_{-h} = x$ and terminal value $X_{Nh} = 0$. The discrete-time strategy thus consists of trades ξ_{kh}, $k \in \{0,\dots,N\}$, at the grid points, where $\xi_{kh} = X_{kh} - X_{(k-1)h}$, $k \in \{0,\dots,N\}$. Notice that ξ_{kh} is $\mathcal{F}_{kh}$-measurable, $k \in \{0,\dots,N\}$.

Furthermore, we take as discrete-time price impact process[3] $(\gamma_{kh})_{k\in\{0,\dots,N\}}$, and as discrete-time resilience process $(\beta_{kh})_{k\in\{1,\dots,N\}}$ defined by

$$\beta_{kh} = e^{-(R_{kh}-R_{(k-1)h})-\frac{1}{2}([R]_{kh}-[R]_{(k-1)h})} = \mathcal{E}(-R)_{(k-1)h,kh}, \quad k \in \{1,\dots,N\}. \tag{3.7}$$

We remark that the arguments in the present section do not rely on the specific dynamics of R or γ. More generally, we could take R to be a continuous semimartingale with $R_0 = 0$ and γ a continuous positive semimartingale.

Recall that the discrete-time deviation process $(D^h_{(kh)-})_{k\in\{0,\dots,N\}}$ is defined by (2.1) and has the alternative representation (cf. (2.2))[4]

$$D^h_{(kh)-} = d\prod_{l=1}^{k}\beta_{lh} + \sum_{i=1}^{k}\gamma_{(i-1)h}\xi_{(i-1)h}\prod_{l=i}^{k}\beta_{lh}, \quad k \in \{0,\dots,N\},$$

where $\sum_{i=1}^{0} := 0$, $\prod_{l=1}^{0} := 1$. Substituting the definition of $(\beta_{kh})_{k\in\{1,\dots,N\}}$, we obtain that, for all $k \in \{1,\dots,N\}$,

$$\begin{aligned} D^h_{(kh)-} &= e^{-R_{kh}-\frac{1}{2}[R]_{kh}}d + \sum_{i=1}^{k}\gamma_{(i-1)h}\xi_{(i-1)h}e^{-(R_{kh}-R_{(i-1)h})-\frac{1}{2}([R]_{kh}-[R]_{(i-1)h})} \\ &= \mathcal{E}(-R)_{kh}\left(d + \sum_{i=1}^{k}\gamma_{(i-1)h}\mathcal{E}(-R)^{-1}_{(i-1)h}\xi_{(i-1)h}\right) = \mathcal{E}(-R)_{kh}L^h_{(k-1)h}, \end{aligned} \tag{3.8}$$

[3] To be more precise, the price impact and resilience process in Chapter 2 are defined for all times in $\mathbb{Z}$. We here only rely on these processes on the time points of the grid and could define them outside this set of time points in any way that is in accordance with Chapter 2, which obviously is possible.

[4] The minus in the subscript of $D^h_{(kh)-}$ is purely notational (this is a discrete-time process), the meaning of $D^h_{(kh)-}$ is that this is the deviation at time kh directly prior to the trade ξ_{kh} at time kh, see Section 2.1.

where, for $k \in \{0, \dots, N\}$, we set

$$\begin{aligned} L^h_{kh} &= d + \sum_{j=0}^{k} \gamma_{jh} \mathcal{E}(-R)^{-1}_{jh} \xi_{jh} \\ &= d + \gamma_0 (X_0 - x) + \sum_{j=1}^{k} \gamma_{jh} \mathcal{E}(-R)^{-1}_{jh} \left(X_{jh} - X_{(j-1)h} \right) \\ &= d + \gamma_0 (X_0 - x) + \sum_{j=1}^{k} \gamma_{(j-1)h} \mathcal{E}(-R)^{-1}_{(j-1)h} \left(X_{jh} - X_{(j-1)h} \right) \\ &\quad + \sum_{j=1}^{k} \left(\gamma_{jh} \mathcal{E}(-R)^{-1}_{jh} - \gamma_{(j-1)h} \mathcal{E}(-R)^{-1}_{(j-1)h} \right) \left(X_{jh} - X_{(j-1)h} \right). \end{aligned}$$

The last expression shows that the continuous-time limit of the processes $(L^h_{kh})_{k \in \{0,\dots,N\}}$, as $N \to \infty$ (and $h = \frac{T}{N} \to 0$), is the process $(L_s)_{s \in [0-,T]}$ given by $L_{0-} = d$,

$$L_s = d + \int_{[0,s]} \gamma_r \mathcal{E}(-R)^{-1}_r \, dX_r + \int_{[0,s]} d[\gamma \mathcal{E}(-R)^{-1}, X]_r, \quad s \in [0,T],$$

(apply [JS03, Proposition I.4.44 and Theorem I.4.47]). Further, note that

$$d(\gamma_s \mathcal{E}(-R)^{-1}_s) = \mathcal{E}(-R)^{-1}_s d\gamma_s + \gamma_s \mathcal{E}(-R)^{-1}_s d(R_s + [R]_s) + d[\gamma, \mathcal{E}(-R)^{-1}]_s, \quad s \in [0,T],$$

and thus

$$d[\gamma \mathcal{E}(-R)^{-1}, X]_s = \mathcal{E}(-R)^{-1}_s d[\gamma, X]_s + \gamma_s \mathcal{E}(-R)^{-1}_s d[R, X]_s, \quad s \in [0,T].$$

It hence turns out that the continuous-time limit of the processes $(D^h_{(kh)-})_{k \in \{0,\dots,N\}}$ is the process $(D_s)_{s \in [0-,T]}$ given by $D_{0-} = d$,

$$\begin{aligned} D_s = \mathcal{E}(-R)_s L_s = e^{-R_s - \frac{1}{2}[R]_s} \bigg(d &+ \int_{[0,s]} \gamma_r e^{R_r + \frac{1}{2}[R]_r} dX_r + \int_{[0,s]} e^{R_r + \frac{1}{2}[R]_r} d[\gamma, X]_r \\ &+ \int_{[0,s]} \gamma_r e^{R_r + \frac{1}{2}[R]_r} d[R, X]_r \bigg), \quad s \in [0,T]. \end{aligned} \tag{3.9}$$

Observe that by, e.g., [Pro05, Theorem V.7 and Theorem V.52] this is the unique solution of the equation

$$dD_s = -D_s dR_s + \gamma_s dX_s + d[\gamma, X]_s, \quad s \in [0,T], \quad D_{0-} = d. \tag{3.10}$$

The above discussion suggests to define the deviation process in the continuous-time model by (3.10) (or, equivalently, by (3.9)).

We now turn to the cost functional. In the discrete-time setting the costs over the whole trading period for a risk-neutral agent are (cf. Section 2.1)

$$\sum_{j=0}^{N} \left(D^h_{(jh)-} + \frac{\gamma_{jh}}{2} \xi_{jh} \right) \xi_{jh}.$$

It holds that

$$\begin{aligned}\sum_{j=0}^{N} \left(D^h_{(jh)-} + \frac{\gamma_{jh}}{2} \xi_{jh} \right) \xi_{jh} &= \sum_{j=0}^{N} D^h_{(jh)-} \left(X_{jh} - X_{(j-1)h} \right) + \sum_{j=0}^{N} \frac{\gamma_{(j-1)h}}{2} \left(X_{jh} - X_{(j-1)h} \right)^2 \\ &\quad + \sum_{j=0}^{N} \frac{1}{2} \left(\gamma_{jh} - \gamma_{(j-1)h} \right) \left(X_{jh} - X_{(j-1)h} \right)^2 .\end{aligned} \tag{3.11}$$

For the first term on the right-hand side of (3.11), we have that

$$\begin{aligned}\sum_{j=0}^{N} D^h_{(jh)-} \left(X_{jh} - X_{(j-1)h} \right) &= \sum_{j=0}^{N} \mathcal{E}(-R)_{jh} L^h_{(j-1)h} \left(X_{jh} - X_{(j-1)h} \right) \\ &= \sum_{j=0}^{N} \mathcal{E}(-R)_{(j-1)h} L^h_{(j-1)h} \left(X_{jh} - X_{(j-1)h} \right) \\ &\quad + \sum_{j=0}^{N} L^h_{(j-1)h} \left(\mathcal{E}(-R)_{jh} - \mathcal{E}(-R)_{(j-1)h} \right) \left(X_{jh} - X_{(j-1)h} \right),\end{aligned}$$

which has the continuous-time limit

$$\int_{[0,T]} \mathcal{E}(-R)_s L_{s-} dX_s + \int_{[0,T]} L_{s-} d[\mathcal{E}(-R), X]_s = \int_{[0,T]} D_{s-} dX_s - \int_{[0,T]} D_{s-} d[R, X]_s,$$

as $\mathcal{E}(-R)$ is a continuous process with $d\mathcal{E}(-R)_s = -\mathcal{E}(-R)_s dR_s$, $s \in [0,T]$. Further, the second term on the right-hand side of (3.11) tends to $\int_{[0,T]} \frac{\gamma_s}{2} d[X]_s$ and the third term to $\frac{1}{2}[\gamma, [X]]_T = 0$ because γ is continuous. As the continuous-time limit of the discrete-time costs we thus obtain

$$\int_{[0,T]} D_{s-} dX_s - \int_{[0,T]} D_{s-} d[R, X]_s + \int_{[0,T]} \frac{\gamma_s}{2} d[X]_s. \tag{3.12}$$

3.3 Motivation for the BSDE

Now that we have suggested an appropriate problem formulation for semimartingale strategies based on the discrete-time model, we want to draw inspiration from the discrete-time results also for its solution. Recall that in discrete time, the minimal costs and optimal strategies for zero terminal position in the risk-neutral setting are characterized by a process defined via backward recursion (see Corollary 2.2.4). We therefore guess that a continuous-time counterpart of this process might also play a crucial role for the solution in continuous time.

Let us consider a discrete-time version of the stochastic control problem to minimize the expected costs in (3.12) over the set of semimartingale execution strategies (càdlàg semimartingales with given initial position, terminal position 0, and some integrability properties) with deviation (3.10). For $h > 0$ such that $h = \frac{T}{N}$ for some $N \in \mathbb{N}$, $t \in [0,T]$, and $x, d \in \mathbb{R}$, let $V_t^h(x,d)$ denote the value function of the problem to minimize only over the subset of all semimartingale execution strategies $X = (X_s)_{s\in[t-,T]}$ of the form $X_s = \sum_{k=0}^N X_{(kh)\vee t} 1_{[kh,(k+1)h)}(s)$, $s \in [t,T]$, (and $X_{t-} = x$). Then it follows from Corollary 2.2.4 that for each $h > 0$ with $h = \frac{T}{N}$ for some $N \in \mathbb{N}$ there exists a process $Y^h = (Y_t^h)_{t\in\{0,h,\dots,T\}}$ such that $V_t^h(x,d) = \frac{Y_t^h}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$, $x, d \in \mathbb{R}$, $t \in \{0,h,\dots,T\}$. The discrete-time process $Y^h = (Y_t^h)_{t\in\{0,h,\dots,T\}}$ is given by the backward recursion $Y_T^h = \frac{1}{2}$ and, for $t \in \{0,h,\dots,T-h\}$,

$$\begin{aligned} Y_t^h = E_t\left[\frac{\gamma_{t+h}}{\gamma_t} Y_{t+h}^h\right] - \left(E_t\left[Y_{t+h}^h\left(\mathcal{E}(-R)_{t,t+h} - \frac{\gamma_{t+h}}{\gamma_t}\right)\right]\right)^2 \\ \cdot \left(E_t\left[Y_{t+h}^h \frac{\gamma_t}{\gamma_{t+h}}\left(\mathcal{E}(-R)_{t,t+h} - \frac{\gamma_{t+h}}{\gamma_t}\right)^2 + \frac{1}{2}\left(1 - \frac{\gamma_t}{\gamma_{t+h}}\mathcal{E}(-R)_{t,t+h}^2\right)\right]\right)^{-1}. \end{aligned} \tag{3.13}$$

It seems plausible to expect that also the value function of the continuous-time problem will turn out to have the form $\frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$, $x, d \in \mathbb{R}$, $t \in [0,T]$, for some continuous-time variant Y of Y^h. The aim is thus to derive heuristically the limit of Y^h when the distance h between the time points for trading tends to 0. We hence make the ansatz that there is a continuous-time limit $Y = (Y_t)_{t\in[0,T]}$ of Y^h as $h \to 0$, and that Y can be decomposed as

$$dY_t = a_t d[M^{(1)}]_t + Z_t^{(1)} dM_t^{(1)} + Z_t^{(2)} dM_t^{(2)} + dM_t^{\perp}, \quad t \in [0,T], \tag{3.14}$$

where $(a_t)_{t\in[0,T]}$, $(Z_t^{(j)})_{t\in[0,T]}$, $j \in \{1,2\}$, are progressively measurable processes (the process $(a_t)_{t\in[0,T]}$ is to be determined) and $M^{\perp} = (M_t^{\perp})_{t\in[0,T]}$ is a local martingale orthogonal to $M^{(1)}$ and $M^{(2)}$.

From (3.14) we deduce that $(a_t)_{t\in[0,T]}$ should be identified as the limit

$$a_t = \lim_{h\to 0} \frac{E_t[Y_{t+h}] - Y_t}{E_t\left[[M^{(1)}]_{t+h}\right] - [M^{(1)}]_t}, \quad t \in [0,T].$$

Assume that replacing Y^h with Y of (3.13) introduces an error only of the magnitude $o\left(E_t\left[[M^{(1)}]_{t+h}\right] - [M^{(1)}]_t\right)$. Then we can get the expression for a_t by evaluating the limit

$$a_t = \lim_{h\to 0} \frac{1}{E_t\left[[M^{(1)}]_{t+h}\right] - [M^{(1)}]_t} \left(E_t\left[Y_{t+h}\right] - E_t\left[\frac{\gamma_{t+h}}{\gamma_t} Y_{t+h}\right] \right.$$
$$\left. + \frac{\left(E_t\left[Y_{t+h}\left(\mathcal{E}(-R)_{t,t+h} - \frac{\gamma_{t+h}}{\gamma_t}\right)\right]\right)^2}{E_t\left[Y_{t+h}\frac{\gamma_t}{\gamma_{t+h}}\left(\mathcal{E}(-R)_{t,t+h} - \frac{\gamma_{t+h}}{\gamma_t}\right)^2 + \frac{1}{2}\left(1 - \frac{\gamma_t}{\gamma_{t+h}}\mathcal{E}(-R)^2_{t,t+h}\right)\right]} \right), \; t\in[0,T]. \tag{3.15}$$

For the remainder of this section we fix $t \in [0,T]$ and assume that all stochastic integrals with respect to $dM^{(1)}$, $dM^{(2)}$, and $dM^{\perp}$ that appear are true martingales. We define the process $\Gamma = (\Gamma_s)_{s\in[t,T]}$ by $\Gamma_s = \frac{\gamma_s}{\gamma_t}$ for $s \in [t,T]$.

Since

$$\begin{aligned} d(\Gamma_s Y_s) &= \Gamma_s dY_s + Y_s d\Gamma_s + d[\Gamma, Y]_s \\ &= \Gamma_s a_s d[M^{(1)}]_s + \Gamma_s Z^{(1)}_s dM^{(1)}_s + \Gamma_s Z^{(2)}_s dM^{(2)}_s + \Gamma_s dM^{\perp}_s \\ &\quad + Y_s\Gamma_s\mu_s d[M^{(1)}]_s + Y_s\Gamma_s\sigma_s dM^{(1)}_s + \Gamma_s\sigma_s Z^{(1)}_s d[M^{(1)}]_s \\ &= (\Gamma_s a_s + Y_s\Gamma_s\mu_s + \Gamma_s\sigma_s Z^{(1)}_s) d[M^{(1)}]_s + (\Gamma_s Z^{(1)}_s + Y_s\Gamma_s\sigma_s) dM^{(1)}_s \\ &\quad + \Gamma_s Z^{(2)}_s dM^{(2)}_s + \Gamma_s dM^{\perp}_s, \quad s \in [t,T], \end{aligned}$$

it holds for all $h \in (0, T-t)$ that

$$E_t\left[\Gamma_{t+h}Y_{t+h}\right] = Y_t + E_t\left[\int_t^{t+h} \left(\Gamma_s a_s + Y_s\Gamma_s\mu_s + \Gamma_s\sigma_s Z^{(1)}_s\right) d[M^{(1)}]_s\right]. \tag{3.16}$$

Together with

$$E_t\left[Y_{t+h}\right] = Y_t + E_t\left[\int_t^{t+h} a_s d[M^{(1)}]_s\right], \quad h \in (0, T-t),$$

we obtain heuristically that

$$\begin{aligned} \frac{E_t\left[Y_{t+h}\right] - E_t\left[\Gamma_{t+h}Y_{t+h}\right]}{E_t\left[[M^{(1)}]_{t+h}\right] - [M^{(1)}]_t} &= \frac{E_t\left[\int_t^{t+h}\left(a_s(1-\Gamma_s) - Y_s\Gamma_s\mu_s - \Gamma_s\sigma_s Z^{(1)}_s\right) d[M^{(1)}]_s\right]}{E_t\left[\int_t^{t+h} d[M^{(1)}]_s\right]} \\ &\xrightarrow[h\to 0]{} -Y_t\mu_t - \sigma_t Z^{(1)}_t. \end{aligned} \tag{3.17}$$

Furthermore, it holds that

$$
\begin{aligned}
d(Y_s\mathcal{E}(-R)_{t,s}) &= -Y_s\mathcal{E}(-R)_{t,s}dR_s + \mathcal{E}(-R)_{t,s}a_s d[M^{(1)}]_s + \mathcal{E}(-R)_{t,s}Z_s^{(1)}dM_s^{(1)} \\
&\quad + \mathcal{E}(-R)_{t,s}Z_s^{(2)}dM_s^{(2)} + \mathcal{E}(-R)_{t,s}dM_s^{\perp} - \mathcal{E}(-R)_{t,s}d[Y,R]_s \\
&= -Y_s\mathcal{E}(-R)_{t,s}\rho_s d[M^{(1)}]_s - Y_s\mathcal{E}(-R)_{t,s}\eta_s\overline{r}_s dM_s^{(1)} \\
&\quad - Y_s\mathcal{E}(-R)_{t,s}\eta_s\sqrt{1-\overline{r}_s^2}dM_s^{(2)} + \mathcal{E}(-R)_{t,s}a_s d[M^{(1)}]_s \\
&\quad + \mathcal{E}(-R)_{t,s}Z_s^{(1)}dM_s^{(1)} + \mathcal{E}(-R)_{t,s}Z_s^{(2)}dM_s^{(2)} + \mathcal{E}(-R)_{t,s}dM_s^{\perp} \\
&\quad - \mathcal{E}(-R)_{t,s}Z_s^{(1)}\eta_s\overline{r}_s d[M^{(1)}]_s - \mathcal{E}(-R)_{t,s}Z_s^{(2)}\eta_s\sqrt{1-\overline{r}_s^2}d[M^{(2)}]_s \\
&= \mathcal{E}(-R)_{t,s}\left(-Y_s\rho_s + a_s - Z_s^{(1)}\eta_s\overline{r}_s - Z_s^{(2)}\eta_s\sqrt{1-\overline{r}_s^2}\right)d[M^{(1)}]_s \\
&\quad + \mathcal{E}(-R)_{t,s}\left(Z_s^{(1)} - Y_s\eta_s\overline{r}_s\right)dM_s^{(1)} + \mathcal{E}(-R)_{t,s}dM_s^{\perp} \\
&\quad + \mathcal{E}(-R)_{t,s}\left(Z_s^{(2)} - Y_s\eta_s\sqrt{1-\overline{r}_s^2}\right)dM_s^{(2)}, \quad s\in[t,T].
\end{aligned}
$$

We then have for all $h \in (0, T-t)$ that

$$
\begin{aligned}
&E_t\left[Y_{t+h}\mathcal{E}(-R)_{t,t+h}\right] \\
&= Y_t + E_t\left[\int_t^{t+h}\mathcal{E}(-R)_{t,s}\left(-Y_s\rho_s + a_s - Z_s^{(1)}\eta_s\overline{r}_s - Z_s^{(2)}\eta_s\sqrt{1-\overline{r}_s^2}\right)d[M^{(1)}]_s\right].
\end{aligned}
\tag{3.18}
$$

From (3.16) and (3.18) we derive heuristically that

$$
\begin{aligned}
&\frac{E_t\left[Y_{t+h}\left(\mathcal{E}(-R)_{t,t+h} - \Gamma_{t+h}\right)\right]}{E_t\left[[M^{(1)}]_{t+h}\right] - [M^{(1)}]_t} \\
&= E_t\Bigg[\int_t^{t+h}\Bigg(\mathcal{E}(-R)_{t,s}\left(-Y_s\rho_s + a_s - Z_s^{(1)}\eta_s\overline{r}_s - Z_s^{(2)}\eta_s\sqrt{1-\overline{r}_s^2}\right) \\
&\qquad - \Gamma_s\left(a_s + Y_s\mu_s + \sigma_s Z_s^{(1)}\right)\Bigg)d[M^{(1)}]_s\Bigg]\left(E_t\left[\int_t^{t+h}d[M^{(1)}]_s\right]\right)^{-1} \\
&\xrightarrow[h\to 0]{} -Y_t\rho_t - Z_t^{(1)}\eta_t\overline{r}_t - Z_t^{(2)}\eta_t\sqrt{1-\overline{r}_t^2} - Y_t\mu_t - \sigma_t Z_t^{(1)}.
\end{aligned}
\tag{3.19}
$$

Recall that $\Gamma_s^{-1} = \frac{\gamma_s^{-1}}{\gamma_t^{-1}}$, $s \in [t,T]$, and therefore

$$
d\Gamma_s^{-1} = -\Gamma_s^{-1}\left(\mu_s - \sigma_s^2\right)d[M^{(1)}]_s - \Gamma_s^{-1}\sigma_s dM_s^{(1)}, \quad s\in[t,T].
$$

We compute that

$$\begin{aligned}
d\left(Y_s\Gamma_s^{-1}\right) &= -Y_s\Gamma_s^{-1}\left(\mu_s-\sigma_s^2\right)d[M^{(1)}]_s - Y_s\Gamma_s^{-1}\sigma_s dM_s^{(1)} + \Gamma_s^{-1}a_s d[M^{(1)}]_s \\
&\quad + \Gamma_s^{-1}Z_s^{(1)}dM_s^{(1)} + \Gamma_s^{-1}Z_s^{(2)}dM_s^{(2)} + \Gamma_s^{-1}dM_s^{\perp} - Z_s^{(1)}\Gamma_s^{-1}\sigma_s d[M^{(1)}]_s \\
&= \left(-Y_s\Gamma_s^{-1}\left(\mu_s-\sigma_s^2\right) + \Gamma_s^{-1}a_s - Z_s^{(1)}\Gamma_s^{-1}\sigma_s\right)d[M^{(1)}]_s \\
&\quad + \left(\Gamma_s^{-1}Z_s^{(1)} - Y_s\Gamma_s^{-1}\sigma_s\right)dM_s^{(1)} + \Gamma_s^{-1}Z_s^{(2)}dM_s^{(2)} + \Gamma_s^{-1}dM_s^{\perp}, \quad s\in[t,T].
\end{aligned} \tag{3.20}$$

Moreover, we have that

$$\begin{aligned}
d(\mathcal{E}(-R)_{t,s}-\Gamma_s)^2 &= -2\mathcal{E}(-R)_{t,s}(\mathcal{E}(-R)_{t,s}-\Gamma_s)dR_s - 2(\mathcal{E}(-R)_{t,s}-\Gamma_s)d\Gamma_s \\
&\quad + d\left[\mathcal{E}(-R)_{t,\cdot}\right]_s - 2d\left[\Gamma,\mathcal{E}(-R)_{t,\cdot}\right]_s + d\left[\Gamma\right]_s \\
&= -2\mathcal{E}(-R)_{t,s}(\mathcal{E}(-R)_{t,s}-\Gamma_s)\rho_s d[M^{(1)}]_s \\
&\quad - 2\mathcal{E}(-R)_{t,s}(\mathcal{E}(-R)_{t,s}-\Gamma_s)\eta_s\overline{r}_s dM_s^{(1)} \\
&\quad - 2\mathcal{E}(-R)_{t,s}(\mathcal{E}(-R)_{t,s}-\Gamma_s)\eta_s\sqrt{1-\overline{r}_s^2}dM_s^{(2)} \\
&\quad - 2\Gamma_s\mu_s(\mathcal{E}(-R)_{t,s}-\Gamma_s)d[M^{(1)}]_s - 2\Gamma_s\sigma_s(\mathcal{E}(-R)_{t,s}-\Gamma_s)dM_s^{(1)} \\
&\quad + \mathcal{E}(-R)_{t,s}^2 d[R]_s + 2\mathcal{E}(-R)_{t,s}d[\Gamma,R]_s + \Gamma_s^2\sigma_s^2 d[M^{(1)}]_s \\
&= \Big(-2(\mathcal{E}(-R)_{t,s}-\Gamma_s)(\mathcal{E}(-R)_{t,s}\rho_s+\Gamma_s\mu_s) + \Gamma_s^2\sigma_s^2 + \mathcal{E}(-R)_{t,s}^2\eta_s^2 \\
&\qquad + 2\mathcal{E}(-R)_{t,s}\eta_s\overline{r}_s\sigma_s\Gamma_s\Big)d[M^{(1)}]_s \\
&\quad - 2(\mathcal{E}(-R)_{t,s}-\Gamma_s)\left(\mathcal{E}(-R)_{t,s}\eta_s\overline{r}_s+\Gamma_s\sigma_s\right)dM_s^{(1)} \\
&\quad - 2(\mathcal{E}(-R)_{t,s}-\Gamma_s)\mathcal{E}(-R)_{t,s}\eta_s\sqrt{1-\overline{r}_s^2}dM_s^{(2)}, \quad s\in[t,T].
\end{aligned} \tag{3.21}$$

It follows from (3.20) and (3.21) that it holds for all $h\in(0,T-t)$ that

$$\begin{aligned}
&E_t\left[Y_{t+h}\Gamma_{t+h}^{-1}\left(\mathcal{E}(-R)_{t,t+h}-\Gamma_{t+h}\right)^2\right] \\
&= E_t\Bigg[\int_t^{t+h} Y_s\Gamma_s^{-1}\Big(\Gamma_s^2\sigma_s^2 - 2(\mathcal{E}(-R)_{t,s}-\Gamma_s)(\mathcal{E}(-R)_{t,s}\rho_s+\Gamma_s\mu_s) + \mathcal{E}(-R)_{t,s}^2\eta_s^2 \\
&\qquad\qquad + 2\mathcal{E}(-R)_{t,s}\Gamma_s\sigma_s\eta_s\overline{r}_s\Big) \\
&\qquad - 2(\mathcal{E}(-R)_{t,s}-\Gamma_s)\Gamma_s^{-1}\Big((\mathcal{E}(-R)_{t,s}\eta_s\overline{r}_s+\Gamma_s\sigma_s)(Z_s^{(1)}-Y_s\sigma_s) \\
&\qquad\qquad + \mathcal{E}(-R)_{t,s}Z_s^{(2)}\eta_s\sqrt{1-\overline{r}_s^2}\Big) \\
&\qquad + (\mathcal{E}(-R)_{t,s}-\Gamma_s)^2\Gamma_s^{-1}(-Y_s(\mu_s-\sigma_s^2)+a_s-Z_s^{(1)}\sigma_s)\,d[M^{(1)}]_s\Bigg].
\end{aligned}$$

Therefore, we obtain heuristically that

$$\frac{E_t\left[Y_{t+h}\Gamma_{t+h}^{-1}\left(\mathcal{E}(-R)_{t,t+h}-\Gamma_{t+h}\right)^2\right]}{E_t\left[[M^{(1)}]_{t+h}\right]-[M^{(1)}]_t} \xrightarrow[h\to 0]{} Y_t(\sigma_t^2+\eta_t^2+2\sigma_t\eta_t\overline{r}_t). \tag{3.22}$$

Since

$$\begin{aligned} d\mathcal{E}(-R)^2_{t,s} &= \mathcal{E}(-R)^2_{t,s}(\eta_s^2 - 2\rho_s)d[M^{(1)}]_s - 2\mathcal{E}(-R)^2_{t,s}\eta_s\overline{r}_s dM_s^{(1)} \\ &\quad - 2\mathcal{E}(-R)^2_{t,s}\eta_s\sqrt{1-\overline{r}_s^2}dM_s^{(2)}, \quad s \in [t,T], \end{aligned}$$

we have that

$$\begin{aligned} d(\Gamma_s^{-1}\mathcal{E}(-R)^2_{t,s}) &= \Gamma_s^{-1}\mathcal{E}(-R)^2_{t,s}(\eta_s^2 - 2\rho_s - \mu_s + \sigma_s^2 + 2\sigma_s\eta_s\overline{r}_s)d[M^{(1)}]_s \\ &\quad - \Gamma_s^{-1}\mathcal{E}(-R)^2_{t,s}(2\eta_s\overline{r}_s + \sigma_s)dM_s^{(1)} \\ &\quad - 2\Gamma_s^{-1}\mathcal{E}(-R)^2_{t,s}\eta_s\sqrt{1-\overline{r}_s^2}dM_s^{(2)}, \quad s \in [t,T], \end{aligned}$$

and we further derive heuristically that

$$\begin{aligned} &\frac{E_t\left[\frac{1}{2}\left(1 - \Gamma_{t+h}^{-1}\mathcal{E}(-R)^2_{t,t+h}\right)\right]}{E_t\left[[M^{(1)}]_{t+h}\right] - [M^{(1)}]_t} \\ &= \frac{E_t\left[\int_t^{t+h}\frac{1}{2}\left(\Gamma_s^{-1}\mathcal{E}(-R)^2_{t,s}\left(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s\eta_s\overline{r}_s\right)\right)d[M^{(1)}]_s\right]}{E_t\left[\int_t^{t+h}d[M^{(1)}]_s\right]} \\ &\xrightarrow[h\to 0]{} \frac{1}{2}\left(2\rho_t + \mu_t - \sigma_t^2 - \eta_t^2 - 2\sigma_t\eta_t\overline{r}_t\right). \end{aligned} \tag{3.23}$$

We conclude from (3.15), (3.17), (3.19), (3.22), and (3.23) that

$$a_t = -Y_t\mu_t - \sigma_t Z_t^{(1)} + \frac{\left(-Y_t(\rho_t+\mu_t) - Z_t^{(1)}(\sigma_t + \eta_t\overline{r}_t) - Z_t^{(2)}\eta_t\sqrt{1-\overline{r}_t^2}\right)^2}{Y_t(\sigma_t^2 + \eta_t^2 + 2\sigma_t\eta_t\overline{r}_t) + \frac{1}{2}(2\rho_t + \mu_t - \sigma_t^2 - \eta_t^2 - 2\sigma_t\eta_t\overline{r}_t)}. \tag{3.24}$$

4

A Riccati-type BSDE

In the previous chapter we have motivated the importance of a certain quadratic BSDE for the continuous-time (semimartingale) problem. Now, we study existence and uniqueness for this BSDE (we additionally incorporate a risk coefficient process λ into the driver (3.24)).

To this end, assume the framework of Section 3.1. We introduce the BSDE

$$Y_t = \frac{1}{2} + \int_t^T f(s, Y_s, Z_s) d[M^{(1)}]_s - \int_t^T Z_s^{(1)} dM_s^{(1)} - \int_t^T Z_s^{(2)} dM_s^{(2)} - \left(M_T^{\perp} - M_t^{\perp}\right), \quad t \in [0, T], \tag{4.1}$$

with terminal condition $Y_T = \frac{1}{2}$ and driver $f\colon \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ defined by, in the case when $(\mathbf{C_{>0}})$ is satisfied,

$$\begin{aligned} f(s, y, z) = &-\frac{\left((\rho_s + \mu_s) y + (\sigma_s + \eta_s \overline{r}_s) z^{(1)} + \eta_s \sqrt{1 - \overline{r}_s^2} z^{(2)} + \lambda_s\right)^2}{(\sigma_s^2 + \eta_s^2 + 2\sigma_s \eta_s \overline{r}_s)(y \vee 0) + \kappa_s + \lambda_s} \\ &+ \mu_s y + \sigma_s z^{(1)} + \lambda_s, \end{aligned} \tag{4.2}$$

for $s \in [0, T]$, $y, z^{(1)}, z^{(2)} \in \mathbb{R}$, $z = (z^{(1)}, z^{(2)})^{\top}$. In differential notation, (4.1) reads

$$dY_s = -f(s, Y_s, Z_s) d[M^{(1)}]_s + Z_s^{(1)} dM_s^{(1)} + Z_s^{(2)} dM_s^{(2)} + dM_s^{\perp}, \quad s \in [0, T], \quad Y_T = \frac{1}{2}.$$

We are interested in solutions of BSDE (4.1) in the following sense[1].

Definition 4.0.1. A solution of BSDE (4.1) is a triple $(Y, Z, M^{\perp})$ of processes where

[1] Note that this notion of a solution of BSDE (4.1) summarizes the definition of a solution of the corresponding BSDE in [AKU21a] and the properties required in condition $(\mathbf{C_{BSDE}})$ of [AKU21a], except for the fact that we in this thesis only assume boundedness of Y and not the specific bound $\frac{1}{2}$. Nevertheless, the upper bound $\frac{1}{2}$ comes out as part of our results in this section (or as a consequence of Theorem 5.2.6) and is natural in view of Section 3.3 and Theorem 2.2.1.

- $M^\perp$ is a càdlàg local martingale with $M_0^\perp = 0$, $[M^\perp, M^{(j)}] = 0$ for $j \in \{1,2\}$, and $E[[M^\perp]_T] < \infty$,
- $Z = (Z^{(1)}, Z^{(2)})^\top$, where $Z^{(1)}, Z^{(2)} \in \mathcal{L}_0^2$, i.e., $Z^{(1)}$, $Z^{(2)}$ are progressively measurable processes such that $E[\int_0^T (Z_s^{(j)})^2 d[M^{(1)}]_s] < \infty$ for $j \in \{1,2\}$,
- Y is an adapted, càdlàg, nonnegative, and bounded process,

such that $\int_0^T |f(s, Y_s, Z_s)|\, d[M^{(1)}]_s < \infty$ a.s. and (4.1) is satisfied a.s.

Remark 4.0.2. (i) We assume $(\mathbf{C_{>0}})$ and write $y \vee 0$ (instead of simply y) in (4.2) only to make sure that the denominator in the definition of the driver is strictly positive (at least $\mathcal{D}_{M^{(1)}}$-a.e.). At some places (e.g., in Proposition 4.3.2 and Section 8.2) we also want to consider BSDE (4.1) without assuming $(\mathbf{C_{>0}})$. In this case, we implicitly understand under $\int_0^T |f(s, Y_s, Z_s)|\, d[M^{(1)}]_s < \infty$ a.s. and (4.1) being satisfied a.s. by $(Y, Z, M^\perp)$ that moreover the fraction

$$\frac{\Big((\rho+\mu)Y + (\sigma + \eta\overline{r})Z^{(1)} + \eta\sqrt{1-\overline{r}^2}Z^{(2)} + \lambda\Big)^2}{(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})Y + \kappa + \lambda}$$

in the driver is well-defined $\mathcal{D}_{M^{(1)}}$-a.e.

(ii) Notice that Y from a solution $(Y, Z, M^\perp)$ of BSDE (4.1) is necessarily a special semimartingale (see [JS03, Section I.4c]).

(iii) In a setting where $(M^{(1)}, \dots, M^{(m)})^\top = (W^{(1)}, \dots, W^{(m)})^\top = W$ is an m-dimensional Brownian motion and $\mathcal{F}_s = \mathcal{F}_s^W$ for all $s \in [0,T]$, (4.1) is equivalent to the formulation

$$Y_t = \frac{1}{2} + \int_t^T f(s, Y_s, (Z_s^{(1)}, Z_s^{(2)})^\top) ds - \sum_{j=1}^m \int_t^T Z_s^{(j)} dW_s^{(j)}, \quad t \in [0,T], \tag{4.3}$$

(with f as in (4.2)). The reason is as follows.

If $(Y, (Z^{(1)}, \dots, Z^{(m)})^\top)$ is a solution of (4.3) (in the sense similar to Definition 4.0.1, but with $Z^{(j)} \in \mathcal{L}_0^2$ for all $j \in \{1, \dots, m\}$), then $(Y, (Z^{(1)}, Z^{(2)})^\top, M^\perp)$ with $M_\cdot^\perp = \sum_{j=3}^m \int_0^\cdot Z_s^{(j)} dW_s^{(j)}$ clearly is a solution of BSDE (4.1).

Suppose now that $(Y, (Z^{(1)}, Z^{(2)})^\top, M^\perp)$ is a solution of BSDE (4.1). By the martingale representation theorem, and since $M^\perp$ is an $(\mathcal{F}_s^W)_{s\in[0,T]}$-local martingale with $M_0^\perp = 0$ and $E[[M^\perp]_T] < \infty$, there exist unique $L^{(j)} \in \mathcal{L}_0^2$, $j \in \{1, \dots, m\}$, such that $M_\cdot^\perp = \sum_{j=1}^m \int_0^\cdot L_s^{(j)} dW_s^{(j)}$. For $j \in \{1,2\}$, we have $\int_0^\cdot (L_s^{(j)})^2 ds = \int_0^\cdot L_s^{(j)} d[M^\perp, W^{(j)}]_s = 0$, and thus $L^{(1)} = 0 = L^{(2)}$ $\mathcal{D}_{W^{(1)}}$-a.e. It follows that $M_\cdot^\perp = \sum_{j=3}^m \int_0^\cdot L_s^{(j)} dW_s^{(j)}$, which shows that $(Y, (Z^{(1)}, Z^{(2)}, L^{(3)}, \dots, L^{(m)})^\top)$ is a solution of (4.3).

In particular, if $m = 2$, then BSDE (4.1) reduces to

$$Y_t = \frac{1}{2} + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s^{(1)} dW_s^{(1)} - \int_t^T Z_s^{(2)} dW_s^{(2)}, \quad t \in [0,T].$$

We in this chapter discuss existence and uniqueness for BSDE (4.1) in three subsettings. Typically, we suppose that $(\mathbf{C}_{\geq\varepsilon})$, $(\mathbf{C}_{\mathbf{bdd}})$, and $(\mathbf{C}_{\mathbf{nonneg}})$ (see Section 3.1) are satisfied.

In the first setting (see Section 4.2) we do not impose restrictions on the filtration but assume $\sigma \equiv 0 \equiv \eta$ in order to meet a Lipschitz condition in some place. We further assume $(\mathbf{C}_{[\boldsymbol{M}^{(1)}]})$. For existence and uniqueness in this first setting, we apply results from [PPS18].

Subsequently (in Section 4.3), we consider a setting with general σ and η, where we assume $[M]_T \leq c_1$ for some deterministic $c_1 \in (0,\infty)$ and that $(\mathcal{F}_s)_{s\in[0,T]}$ is a continuous filtration in the sense that any $(\mathcal{F}_s)_{s\in[0,T]}$-martingale is continuous[2]. We derive an existence result using [Mor09]. When $(M^{(1)},\dots,M^{(m)})^\top = (W^{(1)},\dots,W^{(m)})^\top = W$ is an m-dimensional Brownian motion and $(\mathcal{F}_s)_{s\in[0,T]} = (\mathcal{F}_s^W)_{s\in[0,T]}$, we provide an existence and uniqueness result based on an application of [SXY21] and [KT02].

Finally, we study the BSDE in a setting where the continuous local martingales $M^{(1)}$, $M^{(2)}$ are Brownian motions $W^{(1)}$, $W^{(2)}$ and the input processes are independent of the filtration generated by these Brownian motions (see Section 4.4 for the precise assumptions). To show existence in this framework, we employ [KR21].

The approach common to all three subsettings (except for Proposition 4.3.2) is that we first consider a variant of BSDE (4.1) with a truncated driver. For the BSDE with truncated driver and under appropriate additional assumptions, we then verify that the conditions in relevant literature on existence of BSDEs are satisfied. This provides us with existence of a solution to the BSDE with truncated driver. Subsequently, we use comparison arguments to show that such a solution is actually a solution of BSDE (4.1).

This chapter is based on and uses material from the publication [AKU21b] (joint work with Thomas Kruse and Mikhail Urusov). Furthermore, Proposition 4.3.2 is related to the preprint [AKU22a], and Section 4.4 comes from Section 3.1 of the publication [AKU22b] (both joint work with Thomas Kruse and Mikhail Urusov).

4.1 Preparations

Before we consider the aforementioned subsettings, we establish some helpful results.

The following technical lemma is used, e.g., in the proofs of Lemma 4.1.2 and Lemma 5.2.10. It provides conditions which ensure that the conditional expectation of the supremum of a process with a certain exponential structure is a.s. finite.

Lemma 4.1.1. *Suppose that $(\boldsymbol{C}_{[\boldsymbol{M}^{(1)}]})$ is satisfied. Let $\tau^{(j)} = (\tau_s^{(j)})_{s\in[0,T]}$, $j \in \{1,2\}$, and $\nu = (\nu_s)_{s\in[0,T]}$ be progressively measurable processes such that $|\tau^{(j)}| \leq c_j$ $\mathcal{D}_{M^{(1)}}$-a.e., $j \in \{1,2\}$, and $\nu \leq c_3$ $\mathcal{D}_{M^{(1)}}$-a.e. for some constants $c_j \in (0,\infty)$, $j \in \{1,2,3\}$.*

[2] This condition is for example satisfied for a Brownian filtration.

Let $t \in [0,T]$, and define $N = (N_s)_{s\in[t,T]}$ by

$$N_s = \exp\left(\int_t^s \tau_r^{(1)} dM_r^{(1)} + \int_t^s \tau_r^{(2)} dM_r^{(2)} + \int_t^s \nu_r d[M^{(1)}]_r\right), \quad s \in [t,T].$$

It then holds that

$$E_t\left[\sup_{s\in[t,T]} N_s\right] \leq \frac{16}{9}\left(E_t\left[e^{28c_1^2\left([M^{(1)}]_T-[M^{(1)}]_t\right)}\right]\right)^{\frac{1}{8}}\left(E_t\left[e^{28c_2^2\left([M^{(1)}]_T-[M^{(1)}]_t\right)}\right]\right)^{\frac{1}{8}}$$
$$\cdot\left(E_t\left[e^{(2c_3+c_1^2+c_2^2)\left([M^{(1)}]_T-[M^{(1)}]_t\right)}\right]\right)^{\frac{1}{2}} < \infty \text{ a.s.}$$

Proof. We introduce the continuous local martingales $U^{(j)} = (U_s^{(j)})_{s\in[t,T]}$, $j \in \{1,2\}$, defined by

$$U_s^{(j)} = \exp\left(\int_t^s \tau_r^{(j)} dM_r^{(j)} - \frac{1}{2}\int_t^s (\tau_r^{(j)})^2 d[M^{(j)}]_r\right), \quad s \in [t,T],\ j \in \{1,2\}.$$

We then have that

$$N_s = U_s^{(1)} U_s^{(2)} \exp\left(\int_t^s \left(\nu_r + \frac{1}{2}\left((\tau_r^{(1)})^2 + (\tau_r^{(2)})^2\right)\right) d[M^{(1)}]_r\right), \quad s \in [t,T],$$

and thus, by applying the Cauchy-Schwarz inequality twice,

$$E_t\left[\sup_{s\in[t,T]} N_s\right] \leq \left(E_t\left[\sup_{s\in[t,T]} (U_s^{(1)})^4\right]\right)^{\frac{1}{4}}\left(E_t\left[\sup_{s\in[t,T]} (U_s^{(2)})^4\right]\right)^{\frac{1}{4}}$$
$$\cdot\left(E_t\left[\sup_{s\in[t,T]} \exp\left(\int_t^s \left(2\nu_r + (\tau_r^{(1)})^2 + (\tau_r^{(2)})^2\right) d[M^{(1)}]_r\right)\right]\right)^{\frac{1}{2}}. \tag{4.4}$$

Since $2\nu + (\tau^{(1)})^2 + (\tau^{(2)})^2$ is bounded from above by $2c_3 + c_1^2 + c_2^2$, it holds that

$$E_t\left[\sup_{s\in[t,T]} \exp\left(\int_t^s \left(2\nu_r + (\tau_r^{(1)})^2 + (\tau_r^{(2)})^2\right) d[M^{(1)}]_r\right)\right] \leq E_t\left[e^{(2c_3+c_1^2+c_2^2)\left([M^{(1)}]_T-[M^{(1)}]_t\right)}\right]. \tag{4.5}$$

Next, observe that for $j \in \{1,2\}$

$$E\left[\exp\left(\frac{1}{2}\int_0^T (\tau_r^{(j)})^2 d[M^{(j)}]_r\right)\right] < \infty \text{ a.s.}$$

because $(\tau^{(j)})^2$ is bounded, $[M^{(2)}] = [M^{(1)}]$, and $(\mathbf{C}_{[M^{(1)}]})$ is assumed to hold. Therefore, we obtain by Novikov's criterion that $U^{(j)}$, $j \in \{1,2\}$, are true martingales. Thus, it follows from Doob's maximal inequality that

$$\left(E_t\left[\sup_{s\in[t,T]} (U_s^{(j)})^4\right]\right)^{\frac{1}{4}} \leq \frac{4}{3}\left(E_t\left[(U_T^{(j)})^4\right]\right)^{\frac{1}{4}}, \quad j \in \{1,2\}. \tag{4.6}$$

For $j \in \{1,2\}$, we define $\widetilde{U}^{(j)} = (\widetilde{U}^{(j)}_s)_{s\in[t,T]}$ by

$$\widetilde{U}^{(j)}_s = \exp\left(\int_t^s 8\tau^{(j)}_r dM^{(j)}_r - \frac{1}{2}\int_t^s \left(8\tau^{(j)}_r\right)^2 d[M^{(j)}]_r\right), \quad s \in [t,T],$$

and observe that by the Cauchy-Schwarz inequality it holds that

$$\begin{aligned} E_t\left[(U^{(j)}_T)^4\right] &= E_t\left[(\widetilde{U}^{(j)}_T)^{\frac{1}{2}} \exp\left(\int_t^T 14(\tau^{(j)}_r)^2 d[M^{(j)}]_r\right)\right] \\ &\le \left(E_t\left[\widetilde{U}^{(j)}_T\right]\right)^{\frac{1}{2}} \left(E_t\left[\exp\left(\int_t^T 28(\tau^{(j)}_r)^2 d[M^{(j)}]_r\right)\right]\right)^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

For $j \in \{1,2\}$, as a nonnegative local martingale, $\widetilde{U}^{(j)}$ is a supermartingale, hence $E_t[\widetilde{U}^{(j)}_T] \le \widetilde{U}^{(j)}_t = 1$. Recall moreover that $[M^{(2)}] = [M^{(1)}]$, that $(\tau^{(1)})^2$ is bounded by c_1^2, and that $(\tau^{(2)})^2$ is bounded by c_2^2. We thus obtain from (4.7) that, for $j \in \{1,2\}$,

$$E_t\left[(U^{(j)}_T)^4\right] \le \left(E_t\left[e^{28c_j^2\left([M^{(1)}]_T - [M^{(1)}]_t\right)}\right]\right)^{\frac{1}{2}}. \tag{4.8}$$

It finally follows from (4.4), (4.5), (4.6), and (4.8) that

$$\begin{aligned} E_t\left[\sup_{s\in[t,T]} N_s\right] \le \frac{16}{9} &\left(E_t\left[e^{28c_1^2\left([M^{(1)}]_T - [M^{(1)}]_t\right)}\right]\right)^{\frac{1}{8}} \left(E_t\left[e^{28c_2^2\left([M^{(1)}]_T - [M^{(1)}]_t\right)}\right]\right)^{\frac{1}{8}} \\ &\cdot \left(E_t\left[e^{(2c_3+c_1^2+c_2^2)\left([M^{(1)}]_T - [M^{(1)}]_t\right)}\right]\right)^{\frac{1}{2}}. \end{aligned}$$

This is a.s. finite due to $(\mathbf{C}_{[\boldsymbol{M}^{(1)}]})$. □

We next provide a representation for the first component of solutions of some linear BSDEs[3]. Representations of this kind are classical (as is our proof) and play an important role for comparison principles. We use Lemma 4.1.2 in the proofs of Proposition 4.1.3 and Proposition 4.1.4.

Lemma 4.1.2. *Suppose that $(\boldsymbol{C}_{[\boldsymbol{M}^{(1)}]})$ is satisfied. Assume that $g^{(i)} \colon \Omega \times [0,T] \to \mathbb{R}$, $i \in \{0,1,2,3\}$, are progressively measurable, that $g^{(0)}$ is $\mathcal{D}_{M^{(1)}}$-a.e. bounded from above, that $g^{(1)}$ and $g^{(2)}$ are $\mathcal{D}_{M^{(1)}}$-a.e. bounded, and that $\int_0^T |g^{(3)}_s|\, d[M^{(1)}]_s < \infty$ a.s. Define $g \colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$g(s,y,(z^{(1)},z^{(2)})^\top) = g^{(0)}_s y + g^{(1)}_s z^{(1)} + g^{(2)}_s z^{(2)} + g^{(3)}_s, \quad s \in [0,T],\ y, z^{(1)}, z^{(2)} \in \mathbb{R}.$$

Let $A = (A_s)_{s\in[0,T]}$ be an adapted càdlàg process of finite variation. Let $M^\perp$ be a càdlàg local martingale with $M^\perp_0 = 0$, $[M^\perp, M^{(j)}] = 0$ for $j \in \{1,2\}$, and $E[[M^\perp]_T] <$

[3] We also include an additional finite-variation process A in the BSDE in Lemma 4.1.2 because such a situation appears in the proof of Proposition 4.1.4.

∞. Let $Z^{(1)}$, $Z^{(2)} \in \mathcal{L}^2_0$. Let ξ be an $\mathcal{F}_T$-measurable random variable. Suppose that $Y = (Y_s)_{s\in[0,T]}$ is an adapted, càdlàg process with $g^{(i)}Y \in \mathcal{L}^2_0$ for $i \in \{1,2\}$ and $\int_0^T |g^{(0)}_s Y_s|\, d[M^{(1)}]_s < \infty$ a.s. that satisfies a.s.

$$\begin{aligned} dY_s &= -g(s, Y_s, (Z^{(1)}_s, Z^{(2)}_s)^\top) d[M^{(1)}]_s + Z^{(1)}_s dM^{(1)}_s + Z^{(2)}_s dM^{(2)}_s + dM^\perp_s - dA_s, \\ &\qquad s \in [0,T], \\ Y_T &= \xi. \end{aligned}$$

Let $\Gamma = (\Gamma_t)_{t\in[0,T]}$ be defined by

$$\begin{aligned} \Gamma_t = \exp\Bigg(&\int_0^t g^{(0)}_s d[M^{(1)}]_s + \int_0^t g^{(1)}_s dM^{(1)}_s + \int_0^t g^{(2)}_s dM^{(2)}_s \\ &- \frac{1}{2}\int_0^t (g^{(1)}_s)^2 + (g^{(2)}_s)^2\, d[M^{(1)}]_s \Bigg), \quad t \in [0,T]. \end{aligned}$$

It then holds that Y admits the representation

$$Y_t = \Gamma_t^{-1} E_t\left[\Gamma_T Y_T + \int_t^T \Gamma_s g^{(3)}_s d[M^{(1)}]_s + \int_{(t,T]} \Gamma_s dA_s\right], \quad t \in [0,T]. \tag{4.9}$$

Proof. Note that

$$d\Gamma_s = \Gamma_s g^{(0)}_s d[M^{(1)}]_s + \Gamma_s g^{(1)}_s dM^{(1)}_s + \Gamma_s g^{(2)}_s dM^{(2)}_s, \quad s \in [0,T], \quad \Gamma_0 = 1.$$

We have by integration by parts that

$$\begin{aligned} d(\Gamma_s Y_s) = \Gamma_s &\left(g^{(0)}_s Y_s - g(s, Y_s, (Z^{(1)}_s, Z^{(2)}_s)^\top) + g^{(1)}_s Z^{(1)}_s + g^{(2)}_s Z^{(2)}_s\right) d[M^{(1)}]_s \\ &+ \Gamma_s \left(g^{(1)}_s Y_s + Z^{(1)}_s\right) dM^{(1)}_s + \Gamma_s \left(g^{(2)}_s Y_s + Z^{(2)}_s\right) dM^{(2)}_s \\ &+ \Gamma_s dM^\perp_s - \Gamma_s dA_s, \quad s \in [0,T]. \end{aligned}$$

We thus obtain for all $t \in [0,T]$ that

$$\begin{aligned} &\Gamma_t Y_t + \int_t^T \Gamma_s \left(g^{(1)}_s Y_s + Z^{(1)}_s\right) dM^{(1)}_s + \int_t^T \Gamma_s \left(g^{(2)}_s Y_s + Z^{(2)}_s\right) dM^{(2)}_s + \int_{(t,T]} \Gamma_s dM^\perp_s \\ &= \Gamma_T Y_T + \int_t^T \Gamma_s g^{(3)}_s d[M^{(1)}]_s + \int_{(t,T]} \Gamma_s dA_s. \end{aligned}$$

If the local martingales $N^{(j)} = \int_0^\cdot \Gamma_s (g^{(j)}_s Y_s + Z^{(j)}_s) dM^{(j)}_s$, $j \in \{1,2\}$, and $U = \int_{(0,\cdot]} \Gamma_s dM^\perp_s$ are true martingales, then it follows that

$$\Gamma_t Y_t = E_t\left[\Gamma_T Y_T + \int_t^T \Gamma_s g^{(3)}_s d[M^{(1)}]_s + \int_{(t,T]} \Gamma_s dA_s\right], \quad t \in [0,T],$$

which yields (4.9). To show that for each $j \in \{1,2\}$, $N^{(j)}$ is a martingale, observe that it holds by the Cauchy-Schwarz inequality for $j \in \{1,2\}$ that

$$\begin{aligned} E\left[[N^{(j)}]_T^{\frac{1}{2}}\right] &= E\left[\left(\int_0^T \Gamma_s^2 \left(g_s^{(j)} Y_s + Z_s^{(j)}\right)^2 d[M^{(1)}]_s\right)^{\frac{1}{2}}\right] \\ &\le E\left[\left(\sup_{t\in[0,T]} \Gamma_t^2\right)^{\frac{1}{2}} \left(\int_0^T \left(g_s^{(j)} Y_s + Z_s^{(j)}\right)^2 d[M^{(1)}]_s\right)^{\frac{1}{2}}\right] \\ &\le \left(E\left[\sup_{t\in[0,T]} \Gamma_t^2\right]\right)^{\frac{1}{2}} \left(E\left[\int_0^T \left(g_s^{(j)} Y_s + Z_s^{(j)}\right)^2 d[M^{(1)}]_s\right]\right)^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

Recall the assumptions $Z^{(j)}$, $g^{(j)}Y \in \mathcal{L}_0^2$, $j \in \{1,2\}$. Therefore, the second factor in the last line of (4.10) is finite. For the first factor, observe that, since $g^{(1)}$, $g^{(2)}$ are $\mathcal{D}_{M^{(1)}}$-a.e. bounded and $2g^{(0)} - (g^{(1)})^2 - (g^{(2)})^2$ is $\mathcal{D}_{M^{(1)}}$-a.e. bounded from above, Lemma 4.1.1 implies that $E[\sup_{t\in[0,T]} \Gamma_t^2] < \infty$. Finiteness in (4.10) and the Burkholder-Davis-Gundy inequality show for $j \in \{1,2\}$ that $E[\sup_{t\in[0,T]} |N_t^{(j)}|] < \infty$, and thus that $N^{(j)}$, $j \in \{1,2\}$, are martingales. Similarly, we can show, using $E[[M^{\perp}]_T] < \infty$, that U is a martingale as well, which completes the proof. □

In the next Proposition 4.1.3, we by standard techniques derive a comparison result that is used in Section 4.2 and Section 4.4. In this proposition, we are interested in a BSDE of the form

$$dY_s = -g(s, Y_s) d[M^{(1)}]_s + Z_s^{(1)} dM_s^{(1)} + Z_s^{(2)} dM_s^{(2)} + dM_s^{\perp}, \quad s \in [0,T], \quad Y_T = \xi, \tag{4.11}$$

with a progressively measurable driver $g\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ and an $\mathcal{F}_T$-measurable terminal value ξ, and we denote such a BSDE by BSDE(g,ξ). By a solution of BSDE(g,ξ) we in Proposition 4.1.3 understand a triple $(Y, Z, M^{\perp})$ where $M^{\perp}$ is a càdlàg local martingale with $M_0^{\perp} = 0$, $[M^{\perp}, M^{(j)}] = 0$ for $j \in \{1,2\}$, and $E[[M^{\perp}]_T] < \infty$, $Z = (Z^{(1)}, Z^{(2)})^{\top}$ with $Z^{(1)}$, $Z^{(2)} \in \mathcal{L}_0^2$, and Y is an adapted, càdlàg process, such that $\int_0^T |g(s, Y_s)|\, d[M^{(1)}]_s < \infty$ a.s. and (4.11) holds a.s. We do not consider dependence of g on Z in (4.11) since in the proofs of Proposition 4.2.1 and Proposition 4.4.1 the driver does not depend on Z.

Proposition 4.1.3. *Assume* $(\boldsymbol{C_{[M^{(1)}]}})$. *Let* $g, \widetilde{g}\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *be progressively measurable, and let* $\xi, \widetilde{\xi}$ *be* $\mathcal{F}_T$*-measurable. Suppose that* $(Y, Z, M^{\perp})$ *is a solution of BSDE*(g,ξ) *and that* $(\widetilde{Y}, \widetilde{Z}, \widetilde{M}^{\perp})$ *is a solution of BSDE*$(\widetilde{g}, \widetilde{\xi})$. *Assume that* b *defined by*

$$b_s = 1_{\{Y_s \neq \widetilde{Y}_s\}} \frac{g(s, Y_s) - g(s, \widetilde{Y}_s)}{Y_s - \widetilde{Y}_s}, \quad s \in [0,T],$$

is $\mathcal{D}_{M^{(1)}}$-a.e. bounded from above[4] *and that* $\int_0^T |(Y_s - \widetilde{Y}_s)b_s|\, d[M^{(1)}]_s < \infty$ *a.s. Further, introduce the process* $\Gamma = (\Gamma_t)_{t\in[0,T]}$ *given by* $\Gamma_t = \exp(\int_0^t b_s d[M^{(1)}]_s)$, $t \in [0,T]$.

Then, $Y - \widetilde{Y}$ *admits the representation*

$$Y_t - \widetilde{Y}_t = \Gamma_t^{-1} E_t \left[\Gamma_T \left(\xi - \widetilde{\xi}\right) + \int_t^T \Gamma_s \left(g(s,\widetilde{Y}_s) - \widetilde{g}(s,\widetilde{Y}_s) \right) d[M^{(1)}]_s \right], \quad t \in [0,T]. \tag{4.12}$$

In particular:

(i) If $\xi \geq \widetilde{\xi}$ *a.s. and* $g(s,\widetilde{Y}_s) \geq \widetilde{g}(s,\widetilde{Y}_s)$ $\mathcal{D}_{M^{(1)}}$*-a.e., then* $Y_t \geq \widetilde{Y}_t$ *a.s. for all* $t \in [0,T]$.

(ii) If $\xi \leq \widetilde{\xi}$ *a.s. and* $g(s,\widetilde{Y}_s) \leq \widetilde{g}(s,\widetilde{Y}_s)$ $\mathcal{D}_{M^{(1)}}$*-a.e., then* $Y_t \leq \widetilde{Y}_t$ *a.s. for all* $t \in [0,T]$.

Proof. Denote $\delta Y_t = Y_t - \widetilde{Y}_t$, $\delta g_t = g(t,\widetilde{Y}_t) - \widetilde{g}(t,\widetilde{Y}_t)$, $\delta Z_t^{(j)} = Z_t^{(j)} - \widetilde{Z}_t^{(j)}$, $j \in \{1,2\}$, and $\delta M_t^{\perp} = M_t^{\perp} - \widetilde{M}_t^{\perp}$ for all $t \in [0,T]$. It then holds for all $t \in [0,T]$ that

$$\delta Y_t = \delta Y_T + \int_t^T \left(g(s,Y_s) - \widetilde{g}(s,\widetilde{Y}_s) \right) d[M^{(1)}]_s - \int_t^T \delta Z_s^{(1)} dM_s^{(1)} - \int_t^T \delta Z_s^{(2)} dM_s^{(2)} - \left(\delta M_T^{\perp} - \delta M_t^{\perp} \right).$$

Since

$$g(s,Y_s) - \widetilde{g}(s,\widetilde{Y}_s) = g(s,Y_s) - g(s,\widetilde{Y}_s) + g(s,\widetilde{Y}_s) - \widetilde{g}(s,\widetilde{Y}_s) = b_s\, \delta Y_s + \delta g_s, \quad s \in [0,T],$$

it follows that

$$d\,\delta Y_s = -\left(b_s\, \delta Y_s + \delta g_s \right) d[M^{(1)}]_s + \delta Z_s^{(1)} dM_s^{(1)} + \delta Z_s^{(2)} dM_s^{(2)} + d\delta M_s^{\perp}, \quad s \in [0,T].$$

We can now apply Lemma 4.1.2 to obtain representation (4.12) of $\delta Y = Y - \widetilde{Y}$. The claims (i) and (ii) then are straightforward consequences of (4.12). □

Comparison principles often use some kind of assumptions on the dependence of the driver on Z (e.g., Lipschitz-continuity in [KP16a], or condition (H_2) in [Mor09]) which do not fit well with the structure of our driver (4.2) (or truncated in Y variants of this driver) when σ or η are present. This is why, for the proof of Proposition 4.3.1, we now compute the bounds in our specific situation by hand. The upper bound is also used in the proof of Proposition 4.3.2.

Observe that the upper bound holds in a general setting, whereas for the lower bound we assume a continuous filtration. This ensures that Y is continuous, so that when we apply Itô's formula to $h(Y)$ in the proof of the lower bound, we avoid additional jump terms. In Proposition 4.3.1 we have to assume a continuous filtration anyways as this is part of the setting in [Mor09].

[4] A sufficient condition for b to be $\mathcal{D}_{M^{(1)}}$-a.e. bounded from above is existence of some $c \in (0,\infty)$ such that for all $y, y' \in \mathbb{R}$ with $y \neq y'$ it holds $\frac{g(s,y)-g(s,y')}{y-y'} \leq c$ $\mathcal{D}_{M^{(1)}}$-a.e. This is for example satisfied whenever g is Lipschitz continuous in y uniformly in s.

Proposition 4.1.4. *Assume* $(\boldsymbol{C_{bdd}})$ *and* $(\boldsymbol{C_{[M^{(1)}]}})$. *Let* $c \in [1/2, \infty)$ *and*

$$L\colon \mathbb{R} \to [0, c],\ L(y) = (y \vee 0) \wedge c, \quad y \in \mathbb{R}.$$

Let $M^{\perp}$ *be a càdlàg local martingale with* $M^{\perp}_0 = 0$, $[M^{\perp}, M^{(j)}] = 0$ *for* $j \in \{1, 2\}$, *and* $E[[M^{\perp}]_T] < \infty$. *Let* $Z^{(1)}$, $Z^{(2)} \in \mathcal{L}^2_0$, *and denote* $Z = (Z^{(1)}, Z^{(2)})^{\top}$. *Suppose that* Y *is an adapted, càdlàg, bounded process such that*[5]

$$\exists\, \overline{c} \in (0, \infty)\colon (\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})L(Y) + \kappa + \lambda \geq \overline{c} \quad \mathcal{D}_{M^{(1)}}\text{-a.e.} \tag{4.13}$$

and that satisfies a.s.

$$dY_s = -\overline{f}(s, Y_s, Z_s)d[M^{(1)}]_s + Z^{(1)}_s dM^{(1)}_s + Z^{(2)}_s dM^{(2)}_s + dM^{\perp}_s, \quad s \in [0, T], \quad Y_T = \frac{1}{2},$$

where

$$\begin{aligned}\overline{f}(s, Y_s, Z_s) = &-\frac{\Big((\rho_s + \mu_s)L(Y_s) + (\sigma_s + \eta_s\overline{r}_s)Z^{(1)}_s + \eta_s\sqrt{1 - \overline{r}^2_s}Z^{(2)}_s + \lambda_s\Big)^2}{(\sigma^2_s + \eta^2_s + 2\sigma_s\eta_s\overline{r}_s)L(Y_s) + \kappa_s + \lambda_s} + \mu_s L(Y_s)\\ &+ \sigma_s Z^{(1)}_s + \lambda_s.\end{aligned}$$

(i) It then holds that $Y \leq \frac{1}{2}$.

(ii) Assume in addition $(\boldsymbol{C_{\geq\varepsilon}})$, $(\boldsymbol{C_{nonneg}})$, *and that the filtration* $(\mathcal{F}_s)_{s\in[0,T]}$ *is continuous in the sense that any* $(\mathcal{F}_s)_{s\in[0,T]}$*-martingale is continuous. Then,* $Y \geq 0$.

Proof. (i) For the upper bound, let

$$\widehat{Y} = \frac{1}{2} - Y, \quad \widehat{Z}^{(j)} = -Z^{(j)},\ j \in \{1, 2\}, \quad \text{and} \quad \widehat{M}^{\perp} = -M^{\perp}.$$

Then it holds that

$$d\widehat{Y}_s = \overline{f}(s, Y_s, Z_s)d[M^{(1)}]_s + \widehat{Z}^{(1)}_s dM^{(1)}_s + \widehat{Z}^{(2)}_s dM^{(2)}_s + d\widehat{M}^{\perp}_s, \quad s \in [0, T], \quad \widehat{Y}_T = 0. \tag{4.14}$$

We want to express $\widehat{Y}$ using a driver that is linear in $\widehat{Y}$, $\widehat{Z}^{(1)}$, and $\widehat{Z}^{(2)}$ with bounded coefficients and nonnegative offset. Note that

$$\frac{1}{2} - L(Y_s) = \widehat{Y}\frac{L(Y_s) - \frac{1}{2}}{Y_s - \frac{1}{2}}, \quad s \in [0, T],$$

[5] Condition (4.13) is in particular satisfied whenever $(\mathbf{C}_{\geq\boldsymbol{\varepsilon}})$ holds true.

where here and below we use the convention that $0/0 := 0$. Observe that $|L(y) - \frac{1}{2}| \leq |y - \frac{1}{2}|$ for all $y \in \mathbb{R}$, hence $\frac{L(Y)-\frac{1}{2}}{Y-\frac{1}{2}}$ is bounded. For the driver in (4.14), we have that

$$
\begin{aligned}
-\overline{f}(s, Y_s, Z_s) = & \frac{((\rho_s + \mu_s)L(Y_s) + \lambda_s)^2 + \left((\sigma_s + \eta_s\overline{r}_s)Z_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}Z_s^{(2)}\right)^2}{(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)L(Y_s) + \kappa_s + \lambda_s} \\
& - \frac{2\left((\rho_s + \mu_s)L(Y_s) + \lambda_s\right)\left((\sigma_s + \eta_s\overline{r}_s)\widehat{Z}_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}\widehat{Z}_s^{(2)}\right)}{(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)L(Y_s) + \kappa_s + \lambda_s} \\
& - \frac{1}{2}\mu_s + \widehat{Y}_s\mu_s\frac{L(Y_s) - \frac{1}{2}}{Y_s - \frac{1}{2}} + \sigma_s\widehat{Z}_s^{(1)} - \lambda_s, \quad s \in [0, T].
\end{aligned}
\tag{4.15}
$$

Further, we compute for all $s \in [0, T]$ that

$$
\begin{aligned}
& ((\rho_s + \mu_s)L(Y_s) + \lambda_s)^2 - \left(\frac{1}{2}\mu_s + \lambda_s\right)\left((\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)L(Y_s) + \kappa_s + \lambda_s\right) \\
&= \rho_s^2L(Y_s)^2 + \mu_s(2\rho_s + \mu_s)L(Y_s)^2 + 2\lambda_s(\rho_s + \mu_s)L(Y_s) + \lambda_s^2 \\
&\quad - \left(L(Y_s) - \frac{1}{2}\right)\left(\frac{1}{2}\mu_s + \lambda_s\right)(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s) \\
&\quad - \left(\frac{1}{2}\mu_s + \lambda_s\right)\left(\frac{1}{2}(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s) + \kappa_s + \lambda_s\right) \\
&= \rho_s^2L(Y_s)^2 - \left(\frac{1}{4} - L(Y_s)^2\right)\mu_s(2\rho_s + \mu_s) + \frac{1}{4}\mu_s(2\rho_s + \mu_s) + \lambda_s(\rho_s + \mu_s) \\
&\quad + \left(\frac{1}{2} - L(Y_s)\right)\left(\left(\frac{1}{2}\mu_s + \lambda_s\right)(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s) - 2\lambda_s(\rho_s + \mu_s)\right) + \lambda_s^2 \\
&\quad - \left(\frac{1}{2}\mu_s + \lambda_s\right)\left(\frac{1}{2}(2\rho_s + \mu_s) + \lambda_s\right) \\
&= \rho_s^2L(Y_s)^2 - \left(\frac{1}{4} - L(Y_s)^2\right)\mu_s(2\rho_s + \mu_s) \\
&\quad + \left(\frac{1}{2} - L(Y_s)\right)\left(\left(\frac{1}{2}\mu_s + \lambda_s\right)(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s) - 2\lambda_s(\rho_s + \mu_s)\right) \\
&= \left(\frac{1}{2} - L(Y_s)\right)\left(\left(\frac{1}{2}\mu_s + \lambda_s\right)(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s) - 2\lambda_s(\rho_s + \mu_s)\right) + \rho_s^2L(Y_s)^2 \\
&\quad - \left(\frac{1}{2} - L(Y_s)\right)\left(L(Y_s) + \frac{1}{2}\right)\mu_s(2\rho_s + \mu_s).
\end{aligned}
\tag{4.16}
$$

Define $g^{(i)}\colon \Omega\times[0,T]\to\mathbb{R}$, $i\in\{0,1,2,3\}$, by

$$\begin{aligned}
g_s^{(0)} &= \frac{L(Y_s)-\frac12}{Y_s-\frac12}\Bigg(\mu_s+\frac{\left(\frac12\mu_s+\lambda_s\right)(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)-2\lambda_s(\rho_s+\mu_s)}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s}\\
&\qquad\qquad -\frac{\left(L(Y_s)+\frac12\right)\mu_s(2\rho_s+\mu_s)}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s}\Bigg),\\
g_s^{(1)} &= \sigma_s-\frac{2(\sigma_s+\eta_s\overline{r}_s)\left((\rho_s+\mu_s)L(Y_s)+\lambda_s\right)}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s},\\
g_s^{(2)} &= \frac{-2\eta_s\sqrt{1-\overline{r}_s^2}\left((\rho_s+\mu_s)L(Y_s)+\lambda_s\right)}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s},\\
g_s^{(3)} &= \frac{\left((\sigma_s+\eta_s\overline{r}_s)Z_s^{(1)}+\eta_s\sqrt{1-\overline{r}_s^2}Z_s^{(2)}+\lambda_s\right)^2+\rho_s^2L(Y_s)^2}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s},\quad s\in[0,T].
\end{aligned}$$

It then follows from (4.14), (4.15), and (4.16) that

$$d\widehat{Y}_s=-\widehat{f}(s,\widehat{Y}_s,\widehat{Z}_s)d[M^{(1)}]_s+\widehat{Z}_s^{(1)}dM_s^{(1)}+\widehat{Z}_s^{(2)}dM_s^{(2)}+d\widehat{M}_s^{\perp},\quad s\in[0,T],\quad \widehat{Y}_T=0,$$

where $\widehat{f}\colon\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ is defined by

$$\widehat{f}(s,y,z)=g_s^{(0)}y+g_s^{(1)}z^{(1)}+g_s^{(2)}z^{(2)}+g_s^{(3)},\quad s\in[0,T],\ y,z^{(1)},z^{(2)}\in\mathbb{R}.$$

Observe that, due to (4.13), $(\mathbf{C_{bdd}})$, boundedness of λ, $[-1,1]$-valued $\overline{r}$, and definition of L, the coefficients $g^{(0)}$, $g^{(1)}$, and $g^{(2)}$ are $\mathcal{D}_{M^{(1)}}$-a.e. bounded. Moreover, $\widehat{Y}$ is bounded, and we have $\int_0^T|g_s^{(3)}|\,d[M^{(1)}]_s<\infty$ a.s. Therefore, Lemma 4.1.2 applies. Since $g^{(3)}\geq 0$ $\mathcal{D}_{M^{(1)}}$-a.e. and $\widehat{Y}_T=0$, we deduce from representation (4.9) of $\widehat{Y}$ that $\widehat{Y}\geq 0$, i.e., $Y\leq\frac12$.

(ii) Now, we show that, under the additional assumptions, Y is nonnegative. To this end we first choose $\delta\in(0,\infty)$ such that

$$\frac{\delta}{2}\geq\frac{2(\sigma^2+\eta^2+2\sigma\eta\overline{r})}{\kappa+\lambda}\quad \mathcal{D}_{M^{(1)}}\text{-a.e.},$$

which is possible due to $(\mathbf{C_{\geq\varepsilon}})$ and $(\mathbf{C_{bdd}})$. Let

$$h\colon\mathbb{R}\to\mathbb{R},\ h(y)=1-e^{-\delta y},\quad y\in\mathbb{R},$$

and let $\widetilde{Y}=(\widetilde{Y}_s)_{s\in[0,T]}$ be the process

$$\widetilde{Y}_s=h(Y_s),\quad s\in[0,T].$$

Observe that h is, in particular, twice continuously differentiable and that $M^\perp$, $[M^\perp]$, and Y are continuous. Then it holds by Itô's formula for all $s \in [0,T]$ that

$$\begin{aligned} d\widetilde{Y}_s = dh(Y_s) &= h'(Y_s)dY_s + \frac{1}{2}h''(Y_s)d[Y]_s \\ &= -\left(\overline{f}(s,Y_s,Z_s)h'(Y_s) - \frac{(Z_s^{(1)})^2 + (Z_s^{(2)})^2}{2}h''(Y_s)\right)d[M^{(1)}]_s \\ &\quad + Z_s^{(1)}h'(Y_s)dM_s^{(1)} + Z_s^{(2)}h'(Y_s)dM_s^{(2)} + h'(Y_s)dM_s^\perp + \frac{1}{2}h''(Y_s)d[M^\perp]_s. \end{aligned} \tag{4.17}$$

Let $\widetilde{Z}^{(j)} = (\widetilde{Z}_s^{(j)})_{s\in[0,T]}$, $j \in \{1,2\}$, $\widetilde{M}^\perp = (\widetilde{M}_s^\perp)_{s\in[0,T]}$, and $A = (A_s)_{s\in[0,T]}$ be the processes

$$\begin{aligned} &\widetilde{Z}_s^{(j)} = h'(Y_s)Z_s^{(j)},\ j \in \{1,2\}, \\ &\widetilde{M}_s^\perp = \int_0^s h'(Y_r)dM_r^\perp, \quad \text{and } A_s = -\frac{1}{2}\int_0^s h''(Y_r)d[M^\perp]_r, \quad s \in [0,T]. \end{aligned}$$

Observe that it holds for all $y \in \mathbb{R}$ that

$$\begin{aligned} h'(y) &= \delta e^{-\delta y} = \delta(1-h(y)), \\ h''(y) &= -\delta^2 e^{-\delta y} = -\delta h'(y). \end{aligned}$$

In particular, the process A is nondecreasing due to $h'' \le 0$. We compute that

$$\begin{aligned} &\overline{f}(s,Y_s,Z_s)h'(Y_s) - \frac{(Z_s^{(1)})^2 + (Z_s^{(2)})^2}{2}h''(Y_s) \\ &= -h'(Y_s)\frac{((\rho_s+\mu_s)L(Y_s)+\lambda_s)^2}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s} \\ &\quad - h'(Y_s)\frac{2\left((\rho_s+\mu_s)L(Y_s)+\lambda_s\right)\left((\sigma_s+\eta_s\overline{r}_s)Z_s^{(1)}+\eta_s\sqrt{1-\overline{r}_s^2}Z_s^{(2)}\right)}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s} \\ &\quad - h'(Y_s)\frac{\left((\sigma_s+\eta_s\overline{r}_s)Z_s^{(1)}+\eta_s\sqrt{1-\overline{r}_s^2}Z_s^{(2)}\right)^2}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s} \\ &\quad + \delta(1-\widetilde{Y}_s)\mu_s L(Y_s) + \sigma_s\widetilde{Z}_s^{(1)} + h'(Y_s)\lambda_s + \frac{\delta}{2}h'(Y_s)\left((Z_s^{(1)})^2+(Z_s^{(2)})^2\right) \\ &= \widetilde{g}_s^{(0)}\widetilde{Y}_s + \widetilde{g}_s^{(1)}\widetilde{Z}_s^{(1)} + \widetilde{g}_s^{(2)}\widetilde{Z}_s^{(2)} + \widetilde{g}_s^{(3)}, \quad s \in [0,T], \end{aligned}$$

where we defined $\widetilde{g}^{(i)}\colon \Omega \times [0,T] \to \mathbb{R}$, $i \in \{0,1,2,3\}$, by

$$\begin{aligned}
\widetilde{g}^{(0)}_s &= \frac{L(Y_s)\delta(1-\widetilde{Y}_s)}{\widetilde{Y}_s}\left(\mu_s - \frac{(\rho_s+\mu_s)^2 L(Y_s) + 2(\rho_s+\mu_s)\lambda_s}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s}\right),\\
\widetilde{g}^{(1)}_s &= \sigma_s - \frac{2(\sigma_s+\eta_s\overline{r}_s)\left((\rho_s+\mu_s)L(Y_s)+\lambda_s\right)}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s},\\
\widetilde{g}^{(2)}_s &= -\frac{2\eta_s\sqrt{1-\overline{r}_s^2}\left((\rho_s+\mu_s)L(Y_s)+\lambda_s\right)}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s},\\
\widetilde{g}^{(3)}_s &= \frac{\delta}{2}h'(Y_s)\left((Z^{(1)}_s)^2+(Z^{(2)}_s)^2\right) - h'(Y_s)\frac{\left((\sigma_s+\eta_s\overline{r}_s)Z^{(1)}_s+\eta_s\sqrt{1-\overline{r}_s^2}Z^{(2)}_s\right)^2}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s}\\
&\quad + h'(Y_s)\lambda_s\left(1-\frac{\lambda_s}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(Y_s)+\kappa_s+\lambda_s}\right), \quad s\in[0,T].
\end{aligned}$$

This, (4.17), and the definitions of A, $\widetilde{Z}^{(j)}$, $j \in \{1,2\}$, and $\widetilde{M}^{\perp}$ imply that

$$d\widetilde{Y}_s = -\widetilde{f}(s,\widetilde{Y}_s,\widetilde{Z}_s)d[M^{(1)}]_s + \widetilde{Z}^{(1)}_s dM^{(1)}_s + \widetilde{Z}^{(2)}_s dM^{(2)}_s + d\widetilde{M}^{\perp}_s - dA_s, \quad s\in[0,T],$$

where $\widetilde{f}\colon \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ is defined by

$$\widetilde{f}(s,y,z) = \widetilde{g}^{(0)}_s y + \widetilde{g}^{(1)}_s z^{(1)} + \widetilde{g}^{(2)}_s z^{(2)} + \widetilde{g}^{(3)}_s, \quad s\in[0,T],\ y, z^{(1)}, z^{(2)}\in\mathbb{R}.$$

Note that the process

$$\frac{L(Y_s)\delta(1-\widetilde{Y}_s)}{\widetilde{Y}_s} = \frac{\delta L(Y_s)e^{-\delta Y_s}}{1-e^{-\delta Y_s}}, \quad s\in[0,T],$$

is bounded. This is clear in the case $Y_s \le 0$ because of $L(Y_s)=0$ (recall also the current convention $0/0=0$). For $0<Y_s\le\frac{1}{2}$, it follows from $L(Y_s)=Y_s$ and $0<1-e^{\delta Y_s}\le\delta Y_s$. We then use boundedness of $\frac{L(Y)\delta(1-\widetilde{Y})}{\widetilde{Y}}$ together with $(\mathbf{C}_{\geq\varepsilon})$, $(\mathbf{C}_{\mathbf{bdd}})$, boundedness of λ, $[-1,1]$-valued $\overline{r}$, and $0\le L\le c$ to see that $\widetilde{g}^{(0)}$, $\widetilde{g}^{(1)}$, and $\widetilde{g}^{(2)}$ are $\mathcal{D}_{M^{(1)}}$-a.e. bounded. Furthermore, $(\mathbf{C}_{\geq\varepsilon})$, nonnegativity of L and h', and Jensen's inequality imply that $\mathcal{D}_{M^{(1)}}$-a.e.

$$\begin{aligned}
\widetilde{g}^{(3)} &\ge h'(Y)(Z^{(1)})^2\left(\frac{\delta}{2}-\frac{2(\sigma+\eta\overline{r})^2}{\kappa+\lambda}\right) + h'(Y)(Z^{(2)})^2\left(\frac{\delta}{2}-\frac{2\eta^2(1-\overline{r}^2)}{\kappa+\lambda}\right)\\
&\quad + h'(Y)\lambda\left(1-\frac{\lambda}{\kappa+\lambda}\right).
\end{aligned}$$

Observe moreover that $(\sigma+\eta\overline{r})^2+\eta^2(1-\overline{r}^2)=\sigma^2+\eta^2+2\sigma\eta\overline{r}$. Using nonnegativity of λ, κ, and h' together with our choice of δ, we thus obtain that $\widetilde{g}^{(3)}\ge 0$ $\mathcal{D}_{M^{(1)}}$-a.e. Besides,

$\int_0^T |\widetilde{g}_s^{(3)}|\, d[M^{(1)}]_s < \infty$ a.s. due to $Z^{(1)}, Z^{(2)} \in \mathcal{L}_0^2$, boundedness of λ and Y, $(\mathbf{C_{bdd}})$, and (4.13). We further remark that by boundedness of Y, we obtain for $j \in \{1,2\}$ from $Z^{(j)} \in \mathcal{L}_0^2$ that $\widetilde{Z}^{(j)} \in \mathcal{L}_0^2$, and from $E[[M^\perp]_T] < \infty$ that

$$E\left[[\widetilde{M}^\perp]_T\right] = E\left[\int_0^T (h'(Y_r))^2 d[M^\perp]_r\right] < \infty.$$

It moreover holds for all $t \in [0,T]$, $j \in \{1,2\}$, that

$$[\widetilde{M}^\perp, M^{(j)}]_t = \int_0^t h'(Y_r) d[M^\perp, M^{(j)}]_r = 0.$$

Since $\widetilde{g}^{(3)}$ is nonnegative $\mathcal{D}_{M^{(1)}}$-a.e., A is nondecreasing, and $\widetilde{Y}$ has nonnegative terminal value $1 - e^{-\frac{\delta}{2}}$, the representation (4.9) of $\widetilde{Y}$ in Lemma 4.1.2 shows that $\widetilde{Y} \geq 0$, and hence $Y \geq 0$. □

When we have a solution $(Y, Z, M^\perp)$ of BSDE (4.1), we can say about Y that it does not jump at terminal time, see the next lemma. This is exploited later in the proof of Theorem 5.2.1 and in the proof of Proposition 6.1.6.

Lemma 4.1.5. *Let $(Y, Z, M^\perp)$ be a solution of BSDE (4.1). Then $Y_{T-} = \frac{1}{2}$ a.s., i.e., Y does not jump at terminal time.*

Proof. We have, with f defined in (4.2), that

$$Y_t = \frac{1}{2} + E_t\left[\int_t^T f(s, Y_s, Z_s) d[M^{(1)}]_s\right] = \frac{1}{2} + E_t[A_T] - A_t, \quad t \in [0,T], \tag{4.18}$$

where $A_t = \int_0^t f(s, Y_s, Z_s) d[M^{(1)}]_s$, $t \in [0,T]$. As $A = (A_t)_{t\in[0,T]}$ is a continuous process, it holds that $\lim_{t\uparrow T} A_t = A_T$, hence A_T is $\mathcal{F}_{T-}$-measurable. Therefore,

$$\lim_{t\uparrow T} E_t[A_T] = E\left[A_T | \mathcal{F}_{T-}\right] = A_T \text{ a.s.}$$

The result now follows from (4.18). □

To close this section, we show that uniqueness of a solution in the first component already implies uniqueness of the solution triple. This is referenced in the proof of Corollary 5.2.8. We remark that Lemma 4.1.6 in fact does not only hold for BSDE (4.1), but also for a BSDE of the same structure with possibly different driver and terminal value.

Lemma 4.1.6. *Assume $(\boldsymbol{C_{>0}})$. Suppose that $(Y, Z, M^\perp)$ and $(\hat{Y}, \hat{Z}, \hat{M}^\perp)$ are solutions of BSDE (4.1) such that Y and $\hat{Y}$ are indistinguishable. It then holds that $Z^{(j)} = \hat{Z}^{(j)}$ $\mathcal{D}_{M^{(1)}}$-a.e. for $j \in \{1,2\}$, and that $M^\perp$ and $\hat{M}^\perp$ are indistinguishable.*

Proof. Compare the following canonical decompositions (see [JS03, Section I.4c]) of the special semimartingale $Y = \hat{Y}$:

$$\begin{aligned} Y_t &= Y_0 - \int_0^t f(s, Y_s, Z_s) d[M^{(1)}]_s + \int_0^t Z_s^{(1)} dM_s^{(1)} + \int_0^t Z_s^{(2)} dM_s^{(2)} + M_t^{\perp} \\ &= Y_0 - \int_0^t f(s, Y_s, \hat{Z}_s) d[M^{(1)}]_s + \int_0^t \hat{Z}_s^{(1)} dM_s^{(1)} + \int_0^t \hat{Z}_s^{(2)} dM_s^{(2)} + \hat{M}_t^{\perp}, \quad t \in [0, T]. \end{aligned}$$

For the local martingale parts we have that

$$\int_0^{\cdot} Z_s^{(1)} dM_s^{(1)} + \int_0^{\cdot} Z_s^{(2)} dM_s^{(2)} + M_{\cdot}^{\perp} = \int_0^{\cdot} \hat{Z}_s^{(1)} dM_s^{(1)} + \int_0^{\cdot} \hat{Z}_s^{(2)} dM_s^{(2)} + \hat{M}_{\cdot}^{\perp}. \quad (4.19)$$

This implies that

$$\begin{aligned} [M^{\perp} - \hat{M}^{\perp}]_t &= \left[M^{\perp} - \hat{M}^{\perp}, \int_0^{\cdot} (\hat{Z}_s^{(1)} - Z_s^{(1)}) dM_s^{(1)} + \int_0^{\cdot} (\hat{Z}_s^{(2)} - Z_s^{(2)}) dM_s^{(2)} \right]_t \\ &= \int_0^t (\hat{Z}_s^{(1)} - Z_s^{(1)})\, d[M^{\perp} - \hat{M}^{\perp}, M^{(1)}]_s \\ &\quad + \int_0^t (\hat{Z}_s^{(2)} - Z_s^{(2)})\, d[M^{\perp} - \hat{M}^{\perp}, M^{(2)}]_s \\ &= 0, \quad t \in [0, T]. \end{aligned}$$

Thus, $M^{\perp} - \hat{M}^{\perp}$ is a local martingale starting in 0 with $[M^{\perp} - \hat{M}^{\perp}] = 0$. It follows from the Burkholder-Davis-Gundy inequality that $M^{\perp}$ and $\hat{M}^{\perp}$ are indistinguishable. Then, (4.19) implies further that

$$\int_0^{\cdot} (\hat{Z}_s^{(1)} - Z_s^{(1)}) dM_s^{(1)} + \int_0^{\cdot} (\hat{Z}_s^{(2)} - Z_s^{(2)}) dM_s^{(2)} = 0.$$

Using $[M^{(1)}] = [M^{(2)}]$, we obtain that

$$\begin{aligned} 0 &= \left[\int_0^{\cdot} (\hat{Z}_s^{(1)} - Z_s^{(1)}) dM_s^{(1)} + \int_0^{\cdot} (\hat{Z}_s^{(2)} - Z_s^{(2)}) dM_s^{(2)} \right] \\ &= \int_0^{\cdot} (\hat{Z}_s^{(1)} - Z_s^{(1)})^2 d[M^{(1)}]_s + \int_0^{\cdot} (\hat{Z}_s^{(2)} - Z_s^{(2)})^2 d[M^{(2)}]_s \\ &= \int_0^{\cdot} (\hat{Z}_s^{(1)} - Z_s^{(1)})^2 + (\hat{Z}_s^{(2)} - Z_s^{(2)})^2\, d[M^{(1)}]_s. \end{aligned}$$

It follows that $Z^{(1)} = \hat{Z}^{(1)}$ $\mathcal{D}_{M^{(1)}}$-a.e. and that $Z^{(2)} = \hat{Z}^{(2)}$ $\mathcal{D}_{M^{(1)}}$-a.e. □

4.2 General filtration and $\sigma \equiv 0 \equiv \eta$

Proposition 4.2.1. *Assume that $\sigma \equiv 0 \equiv \eta$. Let $(\boldsymbol{C_{\geq\varepsilon}})$, $(\boldsymbol{C_{bdd}})$, $(\boldsymbol{C_{nonneg}})$, and $(\boldsymbol{C_{[M^{(1)}]}})$ hold true. Then, there exists a unique solution $(Y, Z, M^{\perp})$ of BSDE (4.1). Furthermore, it holds that $Y \leq \frac{1}{2}$.*

Proof. We define the truncation function $L\colon \mathbb{R} \to [0, 1/2]$ by $L(y) = (y \vee 0) \wedge \frac{1}{2}$, $y \in \mathbb{R}$, and consider BSDE (4.1) with the truncated driver $\overline{f}\colon \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$,

$$\overline{f}(s, y) = -\frac{((\rho_s + \mu_s)L(y) + \lambda_s)^2}{\kappa_s + \lambda_s} + \mu_s L(y) + \lambda_s, \quad s \in [0, T],\ y \in \mathbb{R},$$

instead of f defined in (4.2); i.e., with the notation of Proposition 4.1.3, we consider BSDE$(\overline{f}, \frac{1}{2})$. Our aim is to first obtain a unique solution $(Y, Z, M^{\perp})$ (in the sense of Proposition 4.1.3) of BSDE$(\overline{f}, \frac{1}{2})$ via [PPS18, Theorem 3.5] and then show that Y is $[0, 1/2]$-valued.

To this end, we first check that conditions (F1)–(F5) in [PPS18, Section 3.1] are satisfied in our situation. (F1) follows from the Burkholder-Davis-Gundy inequality and $(\mathbf{C_{[M^{(1)}]}})$. Due to $(\mathbf{C_{\geq\varepsilon}})$, $(\mathbf{C_{bdd}})$, boundedness of λ, and the definition of L, it holds for all $y, y' \in \mathbb{R}$ that

$$\begin{aligned}
&|\overline{f}(s, y) - \overline{f}(s, y')| \\
&= \left| \frac{(\rho_s + \mu_s)^2 (L(y')^2 - L(y)^2) + 2(\rho_s + \mu_s)(L(y') - L(y))\lambda_s}{\kappa_s + \lambda_s} + \mu_s (L(y) - L(y')) \right| \\
&= \left| \frac{(\rho_s + \mu_s)^2 (L(y') + L(y)) + 2(\rho_s + \mu_s)\lambda_s}{\kappa_s + \lambda_s} + \mu_s \right| \cdot |L(y') - L(y)| \\
&\leq \left(\frac{2(c_\rho^2 + c_\mu^2) + 2(c_\rho + c_\mu)c_\lambda}{\varepsilon} + c_\mu \right) |y - y'| \quad \mathcal{D}_{M^{(1)}}\text{-a.e.,}
\end{aligned} \tag{4.20}$$

where c_λ denotes the $\mathcal{D}_{M^{(1)}}$-a.e. bound for λ. Therefore, assumption (F3) in [PPS18] is satisfied. Since $M^{(1)}$ is continuous, (F4) holds for all $\Phi > 0$. From $(\mathbf{C_{[M^{(1)}]}})$, and since our terminal value of the BSDE is deterministic, we obtain (F2) for all $\hat{\beta} > 0$. Observe that, due to $(\mathbf{C_{\geq\varepsilon}})$, $(\mathbf{C_{bdd}})$, boundedness of λ, and the definition of L, it holds that there exists a constant $\widetilde{c} \in (0, \infty)$ such that

$$\sup_{y \in \mathbb{R}} |\overline{f}(s, y)| \leq \frac{c_\rho^2 + c_\mu^2 + 2c_\lambda^2}{\varepsilon} + \frac{1}{2}c_\mu + c_\lambda \leq \widetilde{c} \quad \mathcal{D}_{M^{(1)}}\text{-a.e..} \tag{4.21}$$

By the Cauchy-Schwarz inequality, this implies for all $\hat{\beta} > 0$ that

$$\begin{aligned}
E\left[\int_0^T e^{\hat{\beta}[M^{(1)}]_s} \overline{f}(s, 0)^2 d[M^{(1)}]_s\right] &\leq \widetilde{c}^2 E\left[e^{\hat{\beta}[M^{(1)}]_T} [M^{(1)}]_T\right] \\
&\leq \widetilde{c}^2 \left(E\left[e^{2\hat{\beta}[M^{(1)}]_T}\right]\right)^{\frac{1}{2}} \left(E\left[e^{2[M^{(1)}]_T}\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

(F5) now follows from $(\mathbf{C}_{[M^{(1)}]})$.

Thus, by [PPS18, Theorem 3.5] (see also Corollary 3.6 therein) there exists a unique solution $(Y, Z, M^\perp)$ of BSDE$(\overline{f}, \frac{1}{2})$. In particular, the norm in [PPS18, Theorem 3.5] being finite implies that $E[[M^\perp]_T] < \infty$ and $E[\int_0^T (Z_s^{(j)})^2 d[M^{(1)}]_s] < \infty$, $j \in \{1, 2\}$.

In order to show that Y is $[0, 1/2]$-valued, we apply the comparison result Proposition 4.1.3 (recall also (4.20) and (4.21)).

Observe that $(\widetilde{Y}, \widetilde{Z}, \widetilde{M^\perp}) = (\frac{1}{2}, 0, 0)$ is a solution of BSDE$(\widetilde{f}, \frac{1}{2})$ for $\widetilde{f} \equiv 0$, which clearly satisfies $E[[\widetilde{M}^\perp]_T] < \infty$ and $E[\int_0^T (\widetilde{Z}_s^{(j)})^2 d[M^{(1)}]_s] < \infty$, $j \in \{1, 2\}$. Moreover, it holds by $(\mathbf{C}_{\geq\varepsilon})$ that

$$\overline{f}(s, \widetilde{Y}_s) = \overline{f}(s, \frac{1}{2}) = \frac{-\rho_s^2}{4(\kappa_s + \lambda_s)} \leq 0 = \widetilde{f}(s, \widetilde{Y}_s) \quad \mathcal{D}_{M^{(1)}}\text{-a.e.}$$

and $Y_T = \frac{1}{2} = \widetilde{Y}_T$. Therefore, case (ii) of Proposition 4.1.3 yields that $Y \leq \widetilde{Y} = \frac{1}{2}$.

For the other bound, note that $(\hat{Y}, \hat{Z}, \hat{M}^\perp) = (0, 0, 0)$ is a solution of BSDE$(\hat{f}, 0)$ for $\hat{f} \equiv 0$ with $E[[\hat{M}^\perp]_T] < \infty$ and $E[\int_0^T (\hat{Z}_s^{(j)})^2 d[M^{(1)}]_s] < \infty$, $j \in \{1, 2\}$. Further, we have by nonnegativity of λ and κ that

$$\overline{f}(s, \hat{Y}_s) = \overline{f}(s, 0) = \frac{\lambda_s \kappa_s}{\kappa_s + \lambda_s} \geq 0 = \hat{f}(s, \hat{Y}_s) \quad \mathcal{D}_{M^{(1)}}\text{-a.e.}$$

Since moreover $Y_T = \frac{1}{2} \geq 0 = \hat{Y}_T$, it follows from case (i) of Proposition 4.1.3 that $Y \geq \hat{Y} = 0$.

We have thus shown that $(Y, Z, M^\perp)$ is a solution of (4.1). Finally, to see uniqueness, suppose that there is another solution $(Y', Z', (M^\perp)')$ of (4.1). Then, by Proposition 4.1.4, we have that $Y' \leq \frac{1}{2}$. It follows that $(Y', Z', (M^\perp)')$ is also a solution of BSDE$(\overline{f}, \frac{1}{2})$, which (by uniqueness of the solution in [PPS18, Theorem 3.5]) implies that $(Y', Z', (M^\perp)') = (Y, Z, M^\perp)$. □

We further mention that in the setting of Proposition 4.2.1, for any solution $(Y,Z,M^\perp)$ of BSDE (4.1), the appurtenant process $\widetilde{\vartheta}$ defined in (5.22) is bounded, and we could thus obtain uniqueness also via Corollary 5.2.8 of the main theorem on the solution of the semimartingale control problem.

4.3 Continuous filtration and general σ and η

Proposition 4.3.1. *Assume that the filtration $(\mathcal{F}_s)_{s\in[0,T]}$ is continuous in the sense that any $(\mathcal{F}_s)_{s\in[0,T]}$-martingale is continuous. Let $[M^{(1)}]_T \leq c_1$ a.s. for some deterministic $c_1 \in (0, \infty)$. Suppose $(\boldsymbol{C_{\geq\varepsilon}})$, $(\boldsymbol{C_{bdd}})$, and $(\boldsymbol{C_{nonneg}})$. Then, there exists a solution of BSDE (4.1). Furthermore, any solution $(Y, Z, M^\perp)$ of BSDE (4.1) satisfies $Y \leq \frac{1}{2}$.*

Proof. We first consider BSDE (4.1) with its driver replaced by the truncated driver $\overline{f}\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$,

$$\overline{f}(s,y,z) = -\frac{\left((\rho_s+\mu_s)L(y) + (\sigma_s+\eta_s\overline{r}_s)z^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}z^{(2)} + \lambda_s\right)^2}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)L(y)+\kappa_s+\lambda_s} + \mu_s L(y)$$
$$+ \sigma_s z^{(1)} + \lambda_s, \quad s \in [0,T],\, y, z^{(1)}, z^{(2)} \in \mathbb{R},\, z = (z^{(1)}, z^{(2)})^\top,$$

where $L\colon \mathbb{R} \to [0,1/2]$, $L(y) = (y \vee 0) \wedge \frac{1}{2}$, $y \in \mathbb{R}$. Note that $\overline{f}$ is continuous in (y,z).

For this BSDE, we now show that condition (H_1') in [Mor09] is satisfied. We denote the $\mathcal{D}_{M^{(1)}}$-a.e. bound for λ by c_λ. Observe that $(\mathbf{C}_{\geq\varepsilon})$, $L \geq 0$, and $\sigma^2+\eta^2+2\sigma\eta\overline{r} \geq 0$ imply that $|(\sigma^2+\eta^2+2\sigma\eta\overline{r})L(y)+\kappa+\lambda| \geq \varepsilon$ $\mathcal{D}_{M^{(1)}}$-a.e. for all $y \in \mathbb{R}$. We further use $(\mathbf{C_{bdd}})$, $-1 \leq \overline{r} \leq 1$, and $0 \leq L \leq \frac{1}{2}$ to obtain that there exist deterministic constants $c_2, c_3 \in (0,\infty)$ such that for all $y, z^{(1)}, z^{(2)} \in \mathbb{R}$ it holds that

$$\begin{aligned}|\overline{f}(s,y,z)| &\leq \frac{2(\rho_s+\mu_s)^2L(y)^2 + 2\left((\sigma_s+\eta_s\overline{r}_s)z^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}z^{(2)} + \lambda_s\right)^2}{\varepsilon} \\ &\quad + \frac{1}{2}c_\mu + c_\sigma|z^{(1)}| + c_\lambda \\ &\leq \frac{c_\rho^2 + c_\mu^2 + 16\left((c_\sigma^2+c_\eta^2)(z^{(1)})^2 + c_\eta^2(z^{(2)})^2\right) + 4c_\lambda^2}{\varepsilon} + \frac{1}{2}c_\mu + c_\lambda \\ &\quad + c_\sigma\left(1+(z^{(1)})^2\right) \\ &\leq c_2 + \frac{c_3}{2}\left((z^{(1)})^2 + (z^{(2)})^2\right) \quad \mathcal{D}_{M^{(1)}}\text{-a.e.}\end{aligned}$$

Furthermore, it holds that $\int_0^T c_2 d[M^{(1)}]_s \leq c_1c_2$. Hence, assumption (H_1') in [Mor09] is satisfied.

Step 3 and 4 in the proof of [Mor09, Theorem 2.5] show that there exists a solution $(Y,Z,M^\perp)$ (in the sense of Definition 4.0.1, but without the nonnegativity condition on Y) of BSDE (4.1) with driver $\overline{f}$.

We conclude from Proposition 4.1.4 that Y is $[0,1/2]$-valued and that $(Y,Z,M^\perp)$ is also a solution of BSDE (4.1) with the original driver f (as defined in (4.2)).

Moreover, since any solution $(\hat{Y},\hat{Z},\hat{M}^\perp)$ of BSDE (4.1), by Definition 4.0.1, is bounded, we in the current setting can apply Proposition 4.1.4(i) to obtain that $\hat{Y} \leq \frac{1}{2}$. □

In the setting of Proposition 4.3.1 we only provide an existence result and do not claim uniqueness. This is due to the fact that [Mor09, Theorem 2.6] (uniqueness) requires stronger assumptions than [Mor09, Theorem 2.5] (existence). More precisely, the issue is the monotonicity assumption in y uniformly in z on the driver in condition (H_2) in [Mor09]. However, we can obtain uniqueness (and also existence) via [SXY21] or [KT02] in a Brownian setting, as we state next. Proposition 4.3.2 is in particular relevant in Section 8.2 when we solve our continuous-time trade execution problem for progressively measurable strategies.

Proposition 4.3.2. *Let $M^{(j)} = W^{(j)}$, $j \in \{1, \dots, m\}$, for an m-dimensional Brownian motion $W = (W^{(1)}, \dots, W^{(m)})^\top$, and assume that the filtration $(\mathcal{F}_s)_{s\in[0,T]}$ for the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\in[0,T]}, P)$ is the augmented natural filtration of W. Suppose $(\boldsymbol{C_{bdd}})$. Furthermore, assume that at least one of the following conditions holds:*

(a) There exists $\delta \in (0,\infty)$ such that, for all $u \in \mathcal{L}_0^2$ and the associated process H^u defined in (8.1) *with $H_0^u = 0$, the uniform convexity assumption*

$$E\left[\frac{1}{2}(H_T^u)^2 + \int_0^T (\kappa_s + \lambda_s)\, u_s^2 + \lambda_s (H_s^u)^2 - 2\lambda_s H_s^u u_s \, ds\right] \geq \delta E\left[\int_0^T u_s^2 ds\right] \tag{4.22}$$

is satisfied, or

(b) $(\boldsymbol{C_{nonneg}})$ and $(\boldsymbol{C_{\geq\varepsilon}})$, or

(c) $(\boldsymbol{C_{nonneg}})$ and $(\boldsymbol{C_s})$.

Then, there exists a unique solution $(Y, Z, M^\perp)$ of BSDE (4.1)*. Furthermore, there exists $\overline{c} \in (0,\infty)$ such that $(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})Y + \kappa + \lambda \geq \overline{c}\ \mathcal{D}_{W^{(1)}}$-a.e., and it holds that $Y \leq \frac{1}{2}$.*

Proof. **1.** Assume first that (a) holds true, i.e., the uniform convexity assumption (4.22) is satisfied. Observe that BSDE (4.1), in the form of (4.3), corresponds to [SXY21, SRE (92)] for the underlying standard LQ stochastic control problem with state process (8.1) and cost functional (8.2) if we set $\hat{\xi} = 0$ and $\zeta \equiv 0$ in these definitions. Since $\rho, \mu, \sigma, \eta, \overline{r}, \lambda$ are assumed to be bounded and progressively measurable, (A1)' and (A2) of [SXY21] are satisfied. Moreover, condition (4.22) is just the uniform convexity condition in [SXY21] in our situation (see, e.g., their assumption in Theorem 9.1). Furthermore, the filtration by assumption in the current proposition is generated by the Brownian motion $(W^{(1)}, \dots, W^{(m)})^\top$. Therefore, we can indeed apply the results of [SXY21] in our setting. By [SXY21, Theorem 9.1] (see also [SXY21, Theorem 6.3]), there exist unique processes Y, $Z^{(j)}$, $j \in \{1, \dots, m\}$, such that Y is an adapted, continuous, nonnegative[6], bounded process, $Z^{(j)} \in \mathcal{L}_0^2$ for all $j \in \{1, \dots, m\}$, and (4.3) is satisfied P-a.s. Moreover, there exists $\overline{c} \in (0,\infty)$ such that $(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})Y + \kappa + \lambda \geq \overline{c}\ \mathcal{D}_{W^{(1)}}$-a.e. It follows from Remark 4.0.2 that $(Y, (Z^{(1)}, Z^{(2)})^\top, M^\perp)$ with $M_\cdot^\perp = \sum_{j=3}^m \int_0^\cdot Z_s^{(j)} dW_s^{(j)}$ is the unique solution of BSDE (4.1). Due to $(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})Y + \kappa + \lambda \geq \overline{c}\ \mathcal{D}_{W^{(1)}}$-a.e. and boundedness of Y, we can use Proposition 4.1.4(i) to get $Y \leq \frac{1}{2}$.

[6] Although [SXY21] does not seem to state it explicitly, the first solution component of the BSDE in [SXY21] is always nonnegative. This comes from the uniform convexity assumption on the cost functional together with the equivalence [SXY21, Theorem 4.2] of their problems (SLQ) and $(\widehat{SLQ})$ and the representation of the value function in terms of the first solution component of the BSDE in [SXY21, Corollary 5.7].

2. Assume now that $(\mathbf{C_{nonneg}})$ is satisfied. In view of the standard LQ stochastic control problem with state process (8.4) and cost functional (8.5), we can show by some computations that [KT02, BSRDE (9)] for appropriately defined coefficients (see Table 8.1) is the same as BSDE (4.1) in the form of (4.3). These coefficients, due to $(\mathbf{C_{bdd}})$ and $(\mathbf{C_{nonneg}})$, are bounded, and we have that $\frac{1}{2} \geq 0$, $\lambda + \kappa \geq 0$, and $\frac{\lambda\kappa}{\lambda+\kappa} \geq 0$ $\mathcal{D}_{W^{(1)}}$-a.e. (see also Remark 8.2.1, and for $\frac{\lambda}{\lambda+\kappa}$, note the convention of Section 8.1.2). Thus, they satisfy the conditions in [KT02]. Moreover, our filtration is generated by the Brownian motion $(W^{(1)}, \dots, W^{(m)})^\top$, as demanded in [KT02].

If $(\mathbf{C_{\geq\varepsilon}})$ holds in addition to $(\mathbf{C_{nonneg}})$ (i.e., if (b) is satisfied), then we are in the "regular case" and can apply [KT02, Theorem 2.1]. It follows that there exist unique processes Y, $Z^{(j)}$, $j \in \{1, \dots, m\}$, such that Y is an adapted, continuous, nonnegative, bounded process, $Z^{(j)} \in \mathcal{L}_0^2$ for all $j \in \{1, \dots, m\}$, and (4.3) is satisfied P-a.s. Observe that $(\mathbf{C_{\geq\varepsilon}})$ and nonnegativity of Y imply that $(\sigma^2+\eta^2+2\sigma\eta\overline{r})Y+\kappa+\lambda \geq \varepsilon\ \mathcal{D}_{W^{(1)}}$-a.e. From Remark 4.0.2, we have that $(Y, (Z^{(1)}, Z^{(2)})^\top, M^\perp)$ with $M_\cdot^\perp = \sum_{j=3}^m \int_0^\cdot Z_s^{(j)} dW_s^{(j)}$ is the unique solution of BSDE (4.1), and Proposition 4.1.4(i) provides the specific bound $Y \leq \frac{1}{2}$.

Let now $(\mathbf{C_s})$ be satisfied in addition to $(\mathbf{C_{nonneg}})$ (i.e., (c) holds). This corresponds to the "singular case" that is treated in [KT02, Theorem 2.2]. This theorem implies that there exist unique processes Y, $Z^{(j)}$, $j \in \{1, \dots, m\}$, such that Y is an adapted, continuous, nonnegative, bounded process, $Z^{(j)} \in \mathcal{L}_0^2$ for all $j \in \{1, \dots, m\}$, and (4.3) is satisfied P-a.s. Moreover, Y is uniformly positive (in the sense of [KT02, Lemma 4.4]). The fact that Y is uniformly positive together with $(\mathbf{C_s})$ and $(\mathbf{C_{nonneg}})$ yields that there exists $\overline{c} \in (0, \infty)$ such that $(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})Y + \kappa + \lambda \geq \overline{c}\ \mathcal{D}_{W^{(1)}}$-a.e. Again, Remark 4.0.2 and Proposition 4.1.4(i) show that $(Y, (Z^{(1)}, Z^{(2)})^\top, M^\perp)$ with $M_\cdot^\perp = \sum_{j=3}^m \int_0^\cdot Z_s^{(j)} dW_s^{(j)}$ is the unique solution of BSDE (4.1) and that $Y \leq \frac{1}{2}$. □

Note that to prove the second and third case in Proposition 4.3.2, we could also have used [SXY21], where in both cases [SXY21, Proposition 7.1] shows that the uniform convexity condition on the cost functional (8.5) (with $\hat{\xi} = 0$ and $\zeta \equiv 0$) is met so that [SXY21, Theorem 9.1] applies.

4.4 Brownian motion with independent input processes

We now consider yet another subsetting where we can guarantee existence of a solution of BSDE (4.1), and which will (for $\eta \equiv 0$ and $\lambda \equiv 0$) be the setting of Chapter 6.

We assume that $M^{(1)} = W^{(1)}$ and $M^{(2)} = W^{(2)}$ are independent Brownian motions on $(\Omega, \mathcal{F}_T, (\mathcal{F}_s)_{s\in[0,T]}, P)$. Let $(\mathcal{F}_s^W)_{s\in[0,T]}$ be the filtration generated by the two-dimensional Brownian motion $W = (W^{(1)}, W^{(2)})^\top$. We suppose that $(\mathcal{F}_s)_{s\in[0,T]}$ has the structure

$$\mathcal{F}_s = \bigcap_{\varepsilon>0}(\mathcal{F}^W_{s+\varepsilon} \vee \mathcal{F}^\perp_{s+\varepsilon}), \quad s \in [0,T), \quad \mathcal{F}_T = \mathcal{F}^W_T \vee \mathcal{F}^\perp_T,$$

where $(\mathcal{F}^\perp_s)_{s\in[0,T]}$ is a right-continuous complete filtration such that $\mathcal{F}^W_T$ and $\mathcal{F}^\perp_T$ are independent. Furthermore, we assume that $\rho, \mu, \sigma, \eta, \overline{r}$, and λ are $(\mathcal{F}^\perp_s)_{s\in[0,T]}$-progressively measurable.

To obtain a solution of BSDE (4.1), we first consider, under $(\mathbf{C_{>0}})$ and with f defined as in (4.2), the BSDE

$$dY_s = -f(s, Y_s, 0)ds + dM^\perp_s, \quad s \in [0,T], \quad Y_T = \frac{1}{2}, \tag{4.23}$$

on the filtered probability space $(\Omega, \mathcal{F}^\perp_T, (\mathcal{F}^\perp_s)_{s\in[0,T]}, P|_{\mathcal{F}^\perp_T})$. Note that $P|_{\mathcal{F}^\perp_T}$ denotes the probability measure P restricted to the sigma algebra $\mathcal{F}^\perp_T$, and that the expressions "P-a.s." and "$P|_{\mathcal{F}^\perp_T}$-a.s." have the same meaning.

We establish the following result on BSDE (4.23) and BSDE (4.1) in the setting of the current section.

Proposition 4.4.1. *Let $(\boldsymbol{C_{\geq\varepsilon}})$, $(\boldsymbol{C_{bdd}})$, $(\boldsymbol{C_{nonneg}})$, and the assumptions of this section be in force.*

(i) There exists a unique pair $(Y, M^\perp)$ such that Y is a càdlàg, $(\mathcal{F}^\perp_s)_{s\in[0,T]}$-adapted, nonnegative, bounded process, $M^\perp$ is a càdlàg $(\mathcal{F}^\perp_s)_{s\in[0,T]}$-martingale with $M^\perp_0 = 0$ and $E[[M^\perp]_T] < \infty$, $\int_0^T |f(s, Y_s, 0)|\,ds < \infty$ a.s., and BSDE (4.23) is satisfied a.s.

(ii) $(Y, 0, M^\perp)$ with $(Y, M^\perp)$ from (i) is a solution of BSDE (4.1) with $Y \leq \frac{1}{2}$.

Proof. Let $L\colon \mathbb{R} \to [0, 1/2]$ be the truncation function defined by $L(y) = (y \vee 0) \wedge \frac{1}{2}$, $y \in \mathbb{R}$. Let $\overline{f}\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\overline{f}(s, y) = -\frac{((\rho_s + \mu_s)L(y) + \lambda_s)^2}{(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)L(y) + \kappa_s + \lambda_s} + \mu_s L(y) + \lambda_s, \quad s \in [0,T],\, y \in \mathbb{R}.$$

We begin by studying, on the filtered probability space $(\Omega, \mathcal{F}^\perp_T, (\mathcal{F}^\perp_s)_{s\in[0,T]}, P|_{\mathcal{F}^\perp_T})$, BSDE (4.23) with its driver replaced by $\overline{f}$. In the calculations below we assume without loss of generality that ρ, μ, σ, η, $\overline{r}$, and λ satisfy $(\mathbf{C_{\geq\varepsilon}})$, $(\mathbf{C_{bdd}})$, $|\lambda| \leq c_\lambda$ (for a constant $c_\lambda \in (0,\infty)$), and $(\mathbf{C_{nonneg}})$ not only $\mathcal{D}_{W^{(1)}}$-a.e., but for all $(\omega, s) \in \Omega \times [0,T]$, as we can otherwise replace them in $\overline{f}$ with $(\mathcal{F}^\perp_s)_{s\in[0,T]}$-progressively measurable processes $\overline{\rho}$, $\overline{\mu}$, $\overline{\sigma}$, $\overline{\eta}$, $\overline{\overline{r}}$, and $\overline{\lambda}$ that satisfy $(\mathbf{C_{\geq\varepsilon}})$, $(\mathbf{C_{bdd}})$, $|\overline{\lambda}| \leq c_\lambda$, and $(\mathbf{C_{nonneg}})$ for all $(\omega, s) \in \Omega \times [0,T]$ and such that $\overline{\rho} = \rho$ $\mathcal{D}_{W^{(1)}}$-a.e., $\overline{\mu} = \mu$ $\mathcal{D}_{W^{(1)}}$-a.e., $\overline{\sigma} = \sigma$ $\mathcal{D}_{W^{(1)}}$-a.e., $\overline{\eta} = \eta$ $\mathcal{D}_{W^{(1)}}$-a.e., $\overline{\overline{r}} = \overline{r}$ $\mathcal{D}_{W^{(1)}}$-a.e., and $\overline{\lambda} = \lambda$ $\mathcal{D}_{W^{(1)}}$-a.e. To apply [KR21, Proposition 5.1], we justify that conditions (H1)–(H5) in [KR21, Section 2] are satisfied.

Observe that there exists a constant $\widetilde{c} \in (0,\infty)$ such that

$$\begin{aligned}\sup_{y\in\mathbb{R}}|\overline{f}(s,y)| &\leq \frac{2\left((\rho_s+\mu_s)^2\frac{1}{4}+\lambda_s^2\right)}{\varepsilon}+\frac{c_\mu}{2}+c_\lambda \\ &\leq \frac{c_\rho^2+c_\mu^2+2c_\lambda^2}{\varepsilon}+\frac{c_\mu}{2}+c_\lambda \\ &\leq \widetilde{c}, \quad s\in[0,T].\end{aligned} \tag{4.24}$$

In particular, we have for all $s\in[0,T]$ that $|\overline{f}(s,0)|\leq\widetilde{c}$. Since moreover $Y_T=\frac{1}{2}$, this shows that (H1) holds true. Furthermore, $\sup_{y\in\mathbb{R}}|\overline{f}(s,y)-\overline{f}(s,0)|\leq 2\widetilde{c}$ for all $s\in[0,T]$ implies that, in particular, (H5) holds. (H3) is trivially satisfied. Observe that for all $s\in[0,T]$ the function $\mathbb{R}\ni y\mapsto\overline{f}(s,y)$ is continuous, i.e., (H4) is satisfied. Moreover, it holds for all $s\in[0,T]$ that this function is constant on $(-\infty,0]$ and constant on $[1/2,\infty)$. Furthermore, for all $s\in[0,T]$, $(0,1/2)\ni y\mapsto\overline{f}(s,y)$ is twice differentiable with, for $y\in(0,1/2)$,

$$\begin{aligned}\partial_y\overline{f}(s,y) &= \frac{((\rho_s+\mu_s)y+\lambda_s)^2(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)}{a(y)^2}-\frac{2\left((\rho_s+\mu_s)y+\lambda_s\right)(\rho_s+\mu_s)}{a(y)}+\mu_s, \\ \partial_{yy}^2\overline{f}(s,y) &= -2\left(\frac{\rho_s+\mu_s}{a(y)^{\frac{1}{2}}}-\frac{((\rho_s+\mu_s)y+\lambda_s)\left(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s\right)}{a(y)^{\frac{3}{2}}}\right)^2,\end{aligned}$$

where we abbreviated $a(y)=(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)y+\kappa_s+\lambda_s$. Since $\partial_{yy}^2\overline{f}(s,y)\leq 0$ for all $y\in(0,1/2)$, $s\in[0,T]$, we have for all $s\in[0,T]$ that $(0,1/2)\ni y\mapsto\overline{f}(s,y)$ is concave. We therefore obtain for all $s\in[0,T]$ and $y,y'\in\mathbb{R}$ with $y'\neq y$ that

$$\frac{\overline{f}(s,y)-\overline{f}(s,y')}{y-y'}\leq\max\left\{\left|\partial_y^+\overline{f}(s,0)\right|,\left|\partial_y^-\overline{f}(s,1/2)\right|\right\}.$$

Due to $(\mathbf{C}_{\geq\varepsilon})$, $(\mathbf{C}_{\mathbf{bdd}})$, and boundedness of λ, there exists a constant $\overline{c}\in(0,\infty)$ such that it holds for all $s\in[0,T]$ and $y,y'\in\mathbb{R}$ with $y'\neq y$ that

$$\frac{\overline{f}(s,y)-\overline{f}(s,y')}{y-y'}\leq\overline{c}. \tag{4.25}$$

It follows for all $s\in[0,T]$ and $y,y'\in\mathbb{R}$ that $\left(\overline{f}(s,y)-\overline{f}(s,y')\right)(y-y')\leq\overline{c}(y-y')^2$; hence, also (H2) is satisfied. We can thus apply [KR21, Proposition 5.1], which yields that there exists a unique pair $(Y,M^\perp)$ such that BSDE (4.23) with its driver replaced by $\overline{f}$ is satisfied a.s., Y is a càdlàg $(\mathcal{F}_s^\perp)_{s\in[0,T]}$-adapted process with $E[\sup_{s\in[0,T]}Y_s^2]<\infty$, and $M^\perp$ is a càdlàg $(\mathcal{F}_s^\perp)_{s\in[0,T]}$-martingale with $M_0^\perp=0$ and $E[[M^\perp]_T]<\infty$.

We next show that $(Y,0,M^\perp)$ is a solution of BSDE$(\overline{f},1/2)$ (on the filtered probability space $(\Omega,\mathcal{F}_T,(\mathcal{F}_s)_{s\in[0,T]},P)$), where the notation and the notion of a solution are as in Proposition 4.1.3. Since $\mathcal{F}_T^W$ and $\mathcal{F}_T^\perp$ are independent and $\mathcal{F}_s = \bigcap_{\varepsilon>0}(\mathcal{F}_{s+\varepsilon}^W \vee \mathcal{F}_{s+\varepsilon}^\perp)$ for all $s \in [0,T)$, we have that $M^\perp$ is not only an $(\mathcal{F}_s^\perp)_{s\in[0,T]}$-martingale, but also an $(\mathcal{F}_s)_{s\in[0,T]}$-martingale. Furthermore, we can show for $j \in \{1,2\}$ that $M^\perp W^{(j)}$ is an $(\mathcal{F}_s)_{s\in[0,T]}$-martingale. It follows that $\langle M^\perp, W^{(j)}\rangle = 0$, $j \in \{1,2\}$. For $j \in \{1,2\}$, we have from continuity of $W^{(j)}$ that $[M^\perp, W^{(j)}]$ is continuous, and hence $[M^\perp, W^{(j)}] = \langle M^\perp, W^{(j)}\rangle = 0$. Therefore, $(Y, 0, M^\perp)$ is a solution of BSDE$(\overline{f}, 1/2)$.

It remains to justify that Y is $[0,1/2]$-valued. Due to (4.24) and (4.25), we can apply Proposition 4.1.3. For the lower bound, note that $(0,0,0)$ is a solution of BSDE$(0,0)$, and that $\overline{f}(s,0) = \frac{\lambda_s\kappa_s}{\kappa_s+\lambda_s} \geq 0$, $s \in [0,T]$, due to $(\mathbf{C_{nonneg}})$. We thus obtain from Proposition 4.1.3(i) that $Y \geq 0$. For the upper bound, we consider BSDE$(0,1/2)$ with solution $(1/2,0,0)$, and the fact that

$$\overline{f}(s,\frac{1}{2}) = -\frac{\rho_s^2}{4\left(\kappa_s + \lambda_s + \frac{1}{2}(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)\right)} \leq 0, \quad s \in [0,T].$$

The upper bound $Y \leq \frac{1}{2}$ then follows from Proposition 4.1.3(ii). This completes the proof of (i) and (ii). $\square$

5

Càdlàg semimartingale strategies for optimal trade execution in a continuous-time model

In this chapter we examine optimal trade execution in continuous time using semimartingale strategies. We restrict our problem to the case of terminal positions 0. That means, we consider an agent who holds an initial position $x \in \mathbb{R}$ in the asset, and at terminal time T needs to possess exactly $|x|$ shares less (if $x \geq 0$) or more (if $x < 0$) than at the beginning. Starting at time $t \in [0, T]$, the agent has the time interval $[t, T]$ at disposal for trading. The agent may penalize large (in square) positions over the course of the trading period, which is new compared to [AKU21a] (as is the possible diffusion part in the resilience).

The underlying market conditions are described by the price impact process γ and the resilience process R of Section 3.1. We still need to specify how trading according to a semimartingale strategy affects the price, and what costs this incurs. As in Chapter 2, we suppose that the actual prize is the sum of an unaffected price, which we assume to be a martingale, and a price deviation, and we only focus on the price deviation (see also Remark 5.1.1). The definitions (5.1) and (5.2) that we give in Section 5.1.1 for the deviation and for the cost functional are motivated by Section 3.2. We further discuss these definitions in Section 5.1.2, where we compare with relevant literature and show by counterexamples that the conventional definitions (5.7) and (5.8) can lead to arbitrarily large negative costs when optimizing over our class $\mathcal{A}_t^{\mathrm{sem}}(x,d)$ of càdlàg semimartingales. In Remark 5.1.3, we explain that (in the risk-neutral case) our definitions of the deviation and of the expected costs associated to a strategy X coincide with (5.7) and (5.8) whenever X has finite variation. Furthermore, observe that we integrate with respect to the strategy X in both formulations. This is still possible in the present setting due to our choice of the set of admissible strategies: strategies can have infinite variation, but are still càdlàg semimartingales.

We solve the semimartingale stochastic control problem of Section 5.1.1 in Section 5.2. Our approach is based on a solution of the BSDE that we analyzed in Chapter 4. We first rewrite the cost functional with the help of a solution of BSDE (4.1) (see

Section 5.2.1, in particular Theorem 5.2.1), before we come to a representation for the value function, a characterization of the existence of an optimal strategy, and, in this case, an explicit formula for the optimal strategy in the main theorem (Theorem 5.2.6) in Section 5.2.2.

We find several interesting effects, which we discuss in examples in Section 5.3 and Section 5.4[1]. These include optimal strategies that indeed have infinite variation as well as optimal strategies that do not only have jumps at the beginning and at the end of the trading period, but also in between.

Throughout the chapter, we assume the set-up of Section 3.1.

This chapter is based on and uses material from the publication [AKU21a] (joint work with Thomas Kruse and Mikhail Urusov). The examples in Section 5.3 and Section 5.4 also incorporate parts of Section 4 of the preprint [AKU22a] (joint work with Thomas Kruse and Mikhail Urusov).

5.1 The semimartingale stochastic control problem

We formulate our continuous-time stochastic control problem for semimartingale strategies (within the set-up of Section 3.1) in Section 5.1.1 and discuss the definitions of the deviation and of the cost functional in Section 5.1.2.

5.1.1 Problem formulation

Given an initial time $t \in [0,T]$ and $d \in \mathbb{R}$, we associate to a càdlàg semimartingale $X = (X_s)_{s\in[t-,T]}$ a càdlàg semimartingale $D^X = (D^X_s)_{s\in[t-,T]}$ defined by

$$dD^X_s = -D^X_s dR_s + \gamma_s dX_s + d[\gamma, X]_s, \quad s \in [t,T], \quad D^X_{t-} = d. \tag{5.1}$$

By, e.g., [Pro05, Theorem V.7], there indeed exists a unique solution of (5.1), and by, e.g., [Pro05, Theorem V.52], it admits the representation

$$\begin{aligned} D^X_s = \bigg(& d + \int_{[t,s]} e^{R_r - R_t + \frac{1}{2}([R]_r - [R]_t)} \gamma_r dX_r + \int_{[t,s]} e^{R_r - R_t + \frac{1}{2}([R]_r - [R]_t)} \gamma_r d[\gamma, X]_r \\ & + \int_{[t,s]} e^{R_r - R_t + \frac{1}{2}([R]_r - [R]_t)} \gamma_r d[R, X]_r \bigg) e^{-(R_s - R_t) - \frac{1}{2}([R]_s - [R]_t)}, \quad s \in [t,T], \\ D^X_{t-} = d. & \end{aligned}$$

If we have a sequence of càdlàg semimartingales $X^n = (X^n_s)_{s\in[t-,T]}$, $n \in \mathbb{N}$, we usually write D^n instead of D^{X^n} for $n \in \mathbb{N}$.

For $t \in [0,T]$, $d \in \mathbb{R}$, and a càdlàg semimartingale $(X_s)_{s\in[t-,T]}$ with associated $(D^X_s)_{s\in[t-,T]}$ defined by (5.1), we formulate the conditions

[1] We also provide further examples in Section 6.2 and Section 6.3, where we focus on the effect of a negative resilience coefficient.

(A1) $$E_t\left[\sup_{s\in[t,T]}\left(\gamma_s^2\left(X_s-\gamma_s^{-1}D_s^X\right)^4\right)\right]<\infty \text{ a.s.},$$

(A2) $$E_t\left[\left(\int_t^T\gamma_s^2\left(X_s-\gamma_s^{-1}D_s^X\right)^4\sigma_s^2d[M^{(1)}]_s\right)^{\frac{1}{2}}\right]<\infty \text{ a.s.},$$

(A3) $$E_t\left[\left(\int_t^T\left(\gamma_s^{-\frac{1}{2}}D_s^X\right)^4\sigma_s^2d[M^{(1)}]_s\right)^{\frac{1}{2}}\right]<\infty \text{ a.s.},$$

(A4) $$E_t\left[\left(\int_t^T\gamma_s^2\left(X_s-\gamma_s^{-1}D_s^X\right)^4\eta_s^2d[M^{(1)}]_s\right)^{\frac{1}{2}}\right]<\infty \text{ a.s.},$$

(A5) $$E_t\left[\left(\int_t^T\left(\gamma_s^{-\frac{1}{2}}D_s^X\right)^4\eta_s^2d[M^{(1)}]_s\right)^{\frac{1}{2}}\right]<\infty \text{ a.s.}$$

Note that if $E_t[\int_t^T\sigma_s^2d[M^{(1)}]_s]<\infty$ a.s., then, by the Cauchy-Schwarz inequality, **(A2)** follows from **(A1)**. Similarly, if $E_t[\int_t^T\eta_s^2d[M^{(1)}]_s]<\infty$ a.s., then **(A4)** follows from **(A1)**.

For $x,d\in\mathbb{R}$ and $t\in[0,T]$, let $\mathcal{A}_t^{\text{sem}}(x,d)$ be the set of all càdlàg semimartingales $X=(X_s)_{s\in[t-,T]}$ with $X_{t-}=x$, $X_T=0$, and satisfying conditions **(A1)**–**(A5)**. Any element $X\in\mathcal{A}_t^{\text{sem}}(x,d)$ is called a *semimartingale execution strategy*, and x is the initial position. The process D^X defined via (5.1) is called its associated *deviation process*, and d is the initial deviation.

We consider the cost functional J^{sem} defined by

$$\begin{aligned}J_t^{\text{sem}}(x,d,X)&=E_t\left[\int_{[t,T]}D_{s-}^XdX_s+\int_{[t,T]}\frac{\gamma_s}{2}d[X]_s-\int_t^TD_s^Xd[X,R]_s\right]\\&\quad+E_t\left[\int_t^T\gamma_s\lambda_sX_s^2d[M^{(1)}]_s\right]\end{aligned}\tag{5.2}$$

for $t\in[0,T]$, $x,d\in\mathbb{R}$, $X\in\mathcal{A}_t^{\text{sem}}(x,d)$ and associated D^X. Conditions under which the cost functional is well-defined for all $x,d\in\mathbb{R}$, $t\in[0,T]$, and $X\in\mathcal{A}_t^{\text{sem}}(x,d)$ are provided in Theorem 5.2.1. The last summand in the cost functional (5.2) represents a risk term. The choice $\lambda\equiv0$ corresponds to a risk-neutral investor who only experiences the expected (at time t) execution costs (over the trading period $[t,T]$) given by the first line in (5.2). The value function of our control problem is given by

$$V_t^{\text{sem}}(x,d)=\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{\text{sem}}(x,d)}J_t^{\text{sem}}(x,d,X),\quad x,d\in\mathbb{R},t\in[0,T].\tag{5.3}$$

Remark 5.1.1. In the problem setting introduced above we focused on the price deviation only. However, the considerations above also allow to explicitly include an unaffected price into the picture, provided that the unaffected price is a (local) martingale.

To this end, assume that the unaffected price is modeled by a càdlàg local martingale $S^0 = (S^0_r)_{r\in[0-,T]}$. Fix an initial time $t \in [0,T]$, initial position $x \in \mathbb{R}$, and initial deviation $d \in \mathbb{R}$. Consider a càdlàg semimartingale $X = (X_r)_{r\in[t-,T]}$ satisfying $X_{t-} = x$, $X_T = 0$, and **(A1)**–**(A5)**, i.e., $X \in \mathcal{A}_t^{\text{sem}}(x,d)$. When we take the unaffected price S^0 into account, the execution costs (including the risk term) generated by X over $[t,T]$ are given by the formula

$$\begin{aligned}&\int_{[t,T]} S^0_{r-}\,dX_r + \int_{[t,T]} d[S^0,X]_r + \int_{[t,T]} D^X_{r-}\,dX_r + \int_{[t,T]} \frac{\gamma_r}{2}\,d[X]_r \\ &- \int_t^T D^X_r\,d[X,R]_r + \int_t^T \gamma_r\lambda_r X_r^2\,d[M^{(1)}]_r.\end{aligned} \tag{5.4}$$

The first and the second cost terms in (5.4) are due to the unaffected price process S^0. It was first observed in [LS13] via a limiting argument from discrete time that, in continuous time and for semimartingale strategies, the expression for the cost terms due to the unaffected price is

$$\int_{[t,T]} S^0_{r-}\,dX_r + \int_{[t,T]} d[S^0,X]_r \tag{5.5}$$

(see [LS13, Lemma 2.5]).[2] Using integration by parts for the semimartingales X and S^0 together with $X_{t-} = x$ and $X_T = 0$ we obtain that

$$\begin{aligned}\int_{[t,T]} S^0_{r-}\,dX_r + \int_{[t,T]} d[S^0,X]_r &= X_T S^0_T - X_{t-}S^0_{t-} - \int_{[t,T]} X_{r-}\,dS^0_r \\ &= -X_{t-}S^0_{t-} - \int_{(t,T]} X_{r-}\,dS^0_r - X_{t-}\Delta S^0_t \\ &= -xS^0_t - \int_{(t,T]} X_{r-}\,dS^0_r.\end{aligned}$$

It follows from the Burkholder-Davis-Gundy inequality that $E_t[\int_{(t,T]} X_{r-}\,dS^0_r] = 0$ whenever the condition

(A6) $$E_t\left[\left(\int_{(t,T]} X^2_{r-}\,d[S^0]_r\right)^{\frac{1}{2}}\right] < \infty \text{ a.s.}$$

[2] We notice that in the literature preceding [LS13] the execution strategies X were always assumed to be of finite variation (often just monotone), while the part of execution costs coming from the unaffected price was given by the expression $\int_{[t,T]} S^0_r\,dX_r$. This is consistent with (5.5), as $\int_{[t,T]} S^0_{r-}\,dX_r + \int_{[t,T]} d[S^0,X]_r = \int_{[t,T]} S^0_r\,dX_r$ whenever X is of finite variation (see [JS03, Proposition I.4.49a]).

is satisfied. Therefore, under **(A6)**, the expected (at time t) costs (over $[t,T]$) due to the unaffected price S^0 are equal to $-xS_t^0$ and thus do not depend on the execution strategy. Hence, the minimization of the expected (at time t) total costs in (5.4) reduces to the minimization of $J_t^{\mathrm{sem}}(x,d,X)$.

We summarize the discussion as follows. In our work, we minimize $J_t^{\mathrm{sem}}(x,d,X)$ over $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$, i.e., in particular **(A1)**–**(A5)** hold true for X. Given a local martingale unaffected price S^0, a pertinent optimization problem is to minimize $J_t^{\mathrm{sem}}(x,d,X)$ over strategies $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$ satisfying **(A1)**–**(A6)**. Given an optimal strategy $X^* \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$ one thus needs to examine additionally whether X^* satisfies **(A6)**, which in general is not automatically true. However, if S^0 is a square-integrable martingale, then, under the assumptions of Theorem 5.2.6, the optimal strategy $X^* \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$ provided in (5.36) satisfies **(A6)**. Namely, under the assumptions of Theorem 5.2.6, for X^* of (5.36), it holds that $E_t[\sup_{r\in[t,T]}(X_{r-}^*)^2] < \infty$ a.s. As S^0 is a square-integrable martingale, we have $E[[S^0]_T] < \infty$, hence $E_t[[S^0]_T - [S^0]_t] < \infty$ a.s. Condition **(A6)** for X^* of (5.36) now follows from the Cauchy-Schwarz inequality.

Remark 5.1.2. Furthermore, in the problem setting introduced above, we can incorporate a constant permanent price impact coefficient in addition to the transient price impact coefficient γ (compare also with Remark 2.1.5). To this end, let $\hat{c} \in (0,\infty)$ be the permanent price impact coefficient, and note that the order book depth now is described by $q_s = (\gamma_s + \hat{c})^{-1}$, $s \in [0,T]$. Fix an initial time $t \in [0,T]$ and $x, d \in \mathbb{R}$. We add to the deviation process (5.1) of a strategy the additional, permanent shift that is incurred by the permanent price impact when trading according to this strategy. That is, for $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$, we consider the deviation process $\hat{D}^X = (\hat{D}_s^X)_{s\in[t-,T]}$ defined by

$$\hat{D}_s^X = D_s^X + \int_{[t,s]} \hat{c}\, dX_r = D_s^X + (X_s - x)\hat{c}, \quad s \in [t-,T].$$

The (risk-neutral, but we could also include a risk term) costs from trading under transient and permanent price impact are given by, for $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$,

$$\int_{[t,T]} \hat{D}_{s-}^X dX_s + \int_{[t,T]} \frac{1}{2q_s} d[X]_s - \int_t^T D_s^X d[X,R]_s. \tag{5.6}$$

We use the original deviation process D^X in $\int_t^T D_s^X d[X,R]_s$ since this component of the costs is tied to the resilience. For further motivation concerning the definition of the deviation process and the costs with permanent price impact included, combine Remark 2.1.5 and Section 3.2. The costs (5.6) for any $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$ decompose into the sum of the costs in the purely transient case,

$$\int_{[t,T]} D_{s-}^X dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s - \int_t^T D_s^X d[X,R]_s,$$

and the costs due to permanent price impact,

$$\hat{c}\int_{[t,T]}(X_{s-}-x)dX_s+\hat{c}\int_{[t,T]}\frac{1}{2}d[X]_s.$$

Integration by parts implies for all $X\in\mathcal{A}_t^{\mathrm{sem}}(x,d)$ that

$$\begin{aligned}\int_{[t,T]}(X_{s-}-x)dX_s+\int_{[t,T]}\frac{1}{2}d[X]_s&=\int_{[t,T]}X_{s-}dX_s+x^2+\frac{1}{2}\int_{[t,T]}d[X]_s\\&=\frac{1}{2}(X_T^2-X_{t-}^2)+x^2=\frac{x^2}{2}.\end{aligned}$$

Therefore, the added costs due to permanent price impact are the same for all strategies. This means that, effectively, the minimization of the expected (at time t) total costs in the model with transient and constant permanent price impact reduces to the minimization of the expected (at time t) total costs in the model with only transient price impact that we treat in this chapter.

5.1.2 On the deviation process and the cost functional

In this subsection, we consider a risk-neutral investor, i.e., $\lambda\equiv 0$. The conventional definitions of the deviation dynamics and the cost functional for finite-variation strategies X, given $t\in[0,T]$ and $x,d\in\mathbb{R}$, would read

$$d\widetilde{D}_s^X=-\widetilde{D}_s^X dR_s+\gamma_s dX_s,\quad s\in[t,T],\quad \widetilde{D}_{t-}^X=d,\tag{5.7}$$

and

$$\widetilde{J}_t(x,d,X)=E_t\left[\int_{[t,T]}\left(\widetilde{D}_{s-}^X+\frac{\gamma_s}{2}\Delta X_s\right)dX_s\right],\tag{5.8}$$

respectively (see, e.g., [FSU19, equations (2) and (5)]). Tildes in (5.7) and (5.8) are to distinguish these from our setting. Note that for finite-variation strategies X, definitions (5.1) and (5.7) coincide, and the same applies to (5.2) and (5.8). This is the content of the next remark.

Remark 5.1.3. Let $t\in[0,T]$, $x,d\in\mathbb{R}$, and suppose that $X\in\mathcal{A}_t^{\mathrm{sem}}(x,d)$ has finite variation.

Recall that, for two càdlàg semimartingales $K=(K_s)_{s\in[t-,T]}$ and $L=(L_s)_{s\in[t-,T]}$, it holds for all $s\in[t,T]$ that $[K,L]_s=\langle K^c,L^c\rangle_s+\sum_{r\in[t,s]}\Delta K_r\Delta L_r$ (see [JS03, Theorem I.4.52]), where K^c and L^c denote the continuous martingale parts of K and L (see [JS03, Proposition I.4.27]).

In particular, for our strategy $X\in\mathcal{A}_t^{\mathrm{sem}}(x,d)$ of finite variation it holds that $X^c\equiv 0$, and thus $[X]_s=\sum_{r\in[t,s]}(\Delta X_r)^2$, $s\in[t,T]$, and $d[X]_s=\Delta X_s dX_s$, $s\in[t,T]$ (see also [JS03, Proposition I.4.49]). Furthermore, as γ and R are continuous, we have for all $s\in[t,T]$ that $[\gamma,X]_s=\langle\gamma,X^c\rangle_s=0$ and $[R,X]_s=\langle R,X^c\rangle_s=0$.

Therefore, if in our setting an execution strategy X is monotone or, more generally, of finite variation, then (5.1) reduces to (5.7), while (5.2) reduces to (5.8).

However, it is in general not possible to use definitions (5.7) and (5.8) also in our set-up. We show in Example 5.1.6 and in Example 5.1.4 that a change for the deviation dynamics and costs in comparison with the usual set-up for finite-variation strategies indeed can be necessary when optimizing over our set of admissible strategies.

Specifically, using cost functional (5.8) for strategies X of infinite variation can lead to arbitrarily large negative costs even with constant deterministic price impact γ (in which case (5.1) and (5.7) are the same) and with resilience $dR_s = \rho ds$, $s \in [0,T]$, where ρ is a deterministic constant, see the next Example 5.1.4. With the right cost functional (5.2) we recover a well-posed problem, see Section 5.4.2.

Example 5.1.4. Let $m = 2$ and assume that $(M^{(1)}, M^{(2)})^\top = (W^{(1)}, W^{(2)})^\top = W$ is a two-dimensional Brownian motion and $\mathcal{F}_s = \mathcal{F}_s^W$ for all $s \in [0,T]$. Consider the situation where the price impact $\gamma > 0$ and the resilience coefficient $\rho > 0$ are positive deterministic constants (that is, $\mu \equiv \sigma \equiv 0$ in terms of our model parameters) and $\eta \equiv 0$. Since $\eta \equiv 0$, we do not need to specify $\overline{r}$. Recall furthermore that $\lambda \equiv 0$ in the current subsection. Let $t = 0$ and fix the initial position $x = 0$ and the initial deviation $d = 0$.

As γ is constant, for all $X \in \mathcal{A}_0^{\mathrm{sem}}(0,0)$ the associated deviation process D^X satisfies

$$dD_s^X = -D_s^X dR_s + \gamma_s dX_s + d[\gamma, X]_s = -D_s^X dR_s + \gamma dX_s, \quad s \in [0,T].$$

In particular, in this setting the dynamics of D^X is of type (5.7).

For $n \in \mathbb{N}$ consider the càdlàg semimartingale $X^n = (X_s^n)_{s\in[0-,T]}$ defined by

$$X_{0-}^n = X_0^n = 0, \quad X_s^n = nW_s^{(1)} \text{ for } s \in [0,T), \quad X_T^n = 0,$$

i.e., X^n follows a scaled Brownian motion on $[0,T)$ and has a block trade at time T. For each $n \in \mathbb{N}$, let $D^n = (D_s^n)_{s\in[0-,T]}$ be given by

$$dD_s^n = -\rho D_s^n ds + \gamma n dW_s^{(1)} \text{ for } s \in [0,T), \quad D_{0-}^n = 0, \quad D_T^n = D_{T-}^n - \gamma X_{T-}^n.$$

Note that for each $n \in \mathbb{N}$, D^n is an Ornstein-Uhlenbeck process. One can therefore show that **(A1)** is satisfied, and due to $\sigma \equiv 0$ and $\eta \equiv 0$, **(A2)**–**(A5)** are satisfied as well, thus $X^n \in \mathcal{A}_0^{\mathrm{sem}}(0,0)$ for all $n \in \mathbb{N}$. Observe that it holds for all $n \in \mathbb{N}$ that $\int_0^\cdot D_s^n dW_s^{(1)}$ is a martingale, that X^n is continuous on $(0,T)$, that $\Delta X_T^n = -X_{T-}^n$, and that $\Delta X_0^n = 0$. Therefore, we obtain that

$$\begin{aligned}
\widetilde{J}_0(0,0,X^n) &= E\left[\int_{[0,T)} D_{s-}^n dX_s^n + D_{T-}^n \Delta X_T^n + \frac{\gamma}{2}(\Delta X_T^n)^2\right] \\
&= E\left[n\int_0^T D_s^n dW_s^{(1)} - D_{T-}^n X_{T-}^n + \frac{\gamma}{2} n^2 (W_T^{(1)})^2\right] \\
&= -E\left[D_{T-}^n X_{T-}^n\right] + \frac{\gamma}{2} n^2 T, \quad n \in \mathbb{N}.
\end{aligned}$$

We have for all $n \in \mathbb{N}$ that, by integration by parts,

$$d(D^n X^n)_s = n D^n_s dW^{(1)}_s - \rho X^n_s D^n_s ds + \gamma n^2 W^{(1)}_s dW^{(1)}_s + \gamma n^2 ds, \quad s \in [0,T),$$

and hence

$$E\left[X^n_s D^n_s\right] = -\rho \int_0^s E\left[X^n_r D^n_r\right] dr + \gamma n^2 s, \quad s \in [0,T).$$

It follows for all $n \in \mathbb{N}$ that

$$E\left[X^n_s D^n_s\right] = \frac{\gamma n^2}{\rho}\left(1 - e^{-\rho s}\right), \quad s \in [0,T),$$

and further that

$$E\left[X^n_{T-} D^n_{T-}\right] = \frac{\gamma n^2}{\rho}\left(1 - e^{-\rho T}\right), \quad n \in \mathbb{N}.$$

This implies that

$$\widetilde{J}_0(0,0,X^n) = -\frac{\gamma n^2}{\rho}\left(1 - e^{-\rho T}\right) + \frac{\gamma}{2} n^2 T = \frac{\gamma n^2}{\rho}\left(e^{-\rho T} - 1 + \frac{\rho T}{2}\right), \quad n \in \mathbb{N}.$$

Now we see that, if $\rho > 0$ is chosen in the way that $e^{-\rho T} - 1 + \frac{\rho T}{2} < 0$ (it is enough to take $\rho \in (0, 1/T)$), then

$$\widetilde{J}_0(0,0,X^n) \to -\infty \quad \text{as } n \to \infty.$$

Thus, the cost functional $\widetilde{J}$ leads to an ill-posed optimization problem.

Note that in the setting of the previous example, the cost functional

$$J_t(x,d,X) = E_t\left[\int_{[t,T]} D^X_{s-} dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s\right], \quad x,d \in \mathbb{R}, t \in [0,T], X \in \mathcal{A}^{\text{sem}}_t(x,d), \tag{5.9}$$

from [AKU21a, equation (3)] and the cost functional J^{sem} from (5.2) coincide since R in this setting has finite variation (also the deviation dynamics from [AKU21a, equation (2)] and (5.1) are the same since $\eta \equiv 0$ in the previous example). We now illustrate in Example 5.1.5 that (5.9) in general requires an additional modification when we allow for a diffusion term in the resilience R. With the right cost functional (5.2) we recover a well-posed problem, see Example 5.3.1.

Example 5.1.5. Let $m = 2$ and suppose that $(M^{(1)}, M^{(2)})^\top = (W^{(1)}, W^{(2)})^\top = W$ is a two-dimensional Brownian motion, $\mathcal{F}_s = \mathcal{F}^W_s$ for all $s \in [0,T]$, and $\overline{r} \equiv 1$. Note that $M^R = W^{(1)}$. Assume that $\gamma > 0$ (i.e., $\mu \equiv \sigma \equiv 0$), $\rho > 0$, and $\eta > 0$ are positive deterministic constants such that $2\rho - \eta^2 > 0$ (i.e., condition $(\mathbf{C_{>0}})$ holds). Recall that $\lambda \equiv 0$. We take $t = 0$ and fix some initial position $x \in \mathbb{R} \setminus \{0\}$ and the initial deviation $d = 0$.

For $n \in \mathbb{N}$ we define $X^n = (X^n_s)_{s\in[0-,T]}$ by

$$X^n_{0-} = X^n_0 = x, \quad dX^n_s = -nX^n_s dW^{(1)}_s \text{ for } s \in [0,T), \quad X^n_T = 0,$$

i.e., X^n follows a geometric Brownian motion on $[0,T)$ and has a block trade at time T. For each $n \in \mathbb{N}$, let $D^n = (D^n_s)_{s\in[0-,T]}$ be given by

$$\begin{aligned} &dD^n_s = -\rho D^n_s ds - \eta_s D^n_s dW^{(1)}_s - \gamma n X^n_s dW^{(1)}_s \text{ for } s \in [0,T), \\ &D^n_{0-} = 0, \quad D^n_T = D^n_{T-} - \gamma X^n_{T-}. \end{aligned}$$

We then have for $s \in [0,T)$ and $n \in \mathbb{N}$ that

$$\begin{aligned} X^n_s &= xe^{\left(-nW^{(1)}_s - \frac{1}{2}n^2 s\right)}, \\ D^n_s &= -n\gamma \int_0^s e^{-\left(\rho+\frac{1}{2}\eta^2\right)(s-r)-\eta\left(W^{(1)}_s - W^{(1)}_r\right)} X^n_r \left(\eta dr + dW^{(1)}_r\right). \end{aligned}$$

Since $\sigma \equiv 0$, we only need to verify **(A1)**, **(A4)**, and **(A5)** for X^n to be in $\mathcal{A}^{\text{sem}}_0(x,0)$ for all $n \in \mathbb{N}$. Clearly,

$$E\left[\sup_{s\in[0,T]} |X^n_s|^p\right] < \infty \tag{5.10}$$

for all $n \in \mathbb{N}$ and all $p \in [1,\infty)$. Furthermore, using (5.10), the Burkholder-Davis-Gundy inequality, and the Cauchy-Schwarz inequality, one can show that it holds $E[\sup_{s\in[0,T)} |D^n_s|^p] < \infty$ for all $n \in \mathbb{N}$ and all $p \in [1,\infty)$. It then moreover holds for all $n \in \mathbb{N}$ that $E[|D^n_T|^p] < \infty$ due to $D^n_T = D^n_{T-} - \gamma X^n_{T-}$. Hence,

$$E\left[\sup_{s\in[0,T]} |D^n_s|^p\right] < \infty \tag{5.11}$$

for all $n \in \mathbb{N}$ and all $p \in [1,\infty)$. It follows that **(A1)** is satisfied for all $n \in \mathbb{N}$. Together with $E[\int_0^T \eta^2 ds] = \eta^2 T < \infty$, this further implies that **(A4)** holds true for all $n \in \mathbb{N}$. Finally, we obtain **(A5)** for all $n \in \mathbb{N}$ from (5.11) and the fact that γ, η are deterministic constants. To summarize, we have that $X^n \in \mathcal{A}^{\text{sem}}_0(x,0)$ for all $n \in \mathbb{N}$.

For the cost functional (5.9) it holds in the current setting for all $n \in \mathbb{N}$ that

$$\begin{aligned} J_0(x,0,X^n) &= E\left[\int_{[0,T)} D^n_{s-} dX^n_s + D^n_{T-}\Delta X^n_T + \frac{\gamma}{2}\int_{[0,T)} d[X^n]_s + \frac{\gamma}{2}(\Delta X^n_T)^2\right] \\ &= E\left[-n\int_0^T D^n_s X^n_s dW^{(1)}_s - D^n_{T-}X^n_{T-} + \frac{\gamma}{2}n^2\int_0^T (X^n_s)^2 ds + \frac{\gamma}{2}(X^n_{T-})^2\right]. \end{aligned}$$

From the Burkholder-Davis-Gundy inequality, the Cauchy-Schwarz inequality, (5.10), and (5.11), we have for all $n \in \mathbb{N}$ that $\int_0^\cdot D^n_s X^n_s dW^{(1)}_s$ is a martingale. Moreover, it holds for all $n \in \mathbb{N}$, $s \in [0,T)$, that

$$E\left[(X^n_s)^2\right] = x^2 e^{n^2 s}. \tag{5.12}$$

We use these facts and Fubini to obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned} J_0(x,0,X^n) &= -E\left[D^n_{T-}X^n_{T-}\right] + \frac{\gamma}{2}n^2x^2\int_0^T e^{n^2s}ds + \frac{\gamma}{2}x^2e^{n^2T} \\ &= \frac{\gamma}{2}x^2\left(2e^{n^2T}-1\right) - E\left[D^n_{T-}X^n_{T-}\right]. \end{aligned} \tag{5.13}$$

It holds by integration by parts that, for all $n \in \mathbb{N}$, $s \in [0,T)$,

$$\begin{aligned} d\left(D^n_sX^n_s\right) = &-\rho D^n_sX^n_sds - \eta D^n_sX^n_sdW^{(1)}_s - \gamma n(X^n_s)^2dW^{(1)}_s - nD^n_sX^n_sdW^{(1)}_s \\ &+ n\eta D^n_sX^n_sds + n^2\gamma(X^n_s)^2ds. \end{aligned}$$

Due to the stochastic integrals on the right-hand side being martingales and (5.12), it follows for all $n \in \mathbb{N}$, $s \in [0,T)$, that

$$E\left[D^n_sX^n_s\right] = -(\rho-n\eta)\int_0^s E\left[D^n_rX^n_r\right]dr + \gamma x^2\left(e^{n^2s}-1\right). \tag{5.14}$$

Note that there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}\cap[n_0,\infty)$ it holds that $\rho - n\eta < -1$ and $\rho - n\eta + n^2 > 0$. The integral equation (5.14) yields for all $n \in \mathbb{N}\cap[n_0,\infty)$, $s \in [0,T)$, that

$$E\left[D^n_sX^n_s\right] = e^{-(\rho-n\eta)s}\frac{\gamma x^2n^2}{\rho-n\eta+n^2}\left(e^{(\rho-n\eta+n^2)s}-1\right),$$

from which we further obtain that (recall also (5.10) and (5.11)), for all $n \in \mathbb{N}\cap[n_0,\infty)$,

$$E\left[D^n_{T-}X^n_{T-}\right] = e^{-(\rho-n\eta)T}\frac{\gamma x^2n^2}{\rho-n\eta+n^2}\left(e^{(\rho-n\eta+n^2)T}-1\right).$$

We insert this into (5.13) to get, for all $n \in \mathbb{N}\cap[n_0,\infty)$,

$$\begin{aligned} J_0(x,0,X^n) &= \gamma x^2\left(e^{n^2T}\left(1-\frac{n^2}{\rho-n\eta+n^2}\right)-\frac{1}{2}+\frac{n^2}{\rho-n\eta+n^2}e^{-(\rho-n\eta)T}\right) \\ &\le \gamma x^2\left(e^{n^2T}\frac{-1}{\rho-n\eta+n^2}+\frac{n^2}{\rho-n\eta+n^2}e^{-(\rho-n\eta)T}\right) \\ &= \gamma x^2e^{-(\rho-n\eta)T}\left(\frac{n^2}{\rho-n\eta+n^2}-\frac{1}{\rho-n\eta+n^2}e^{(\rho-n\eta+n^2)T}\right). \end{aligned}$$

Observe that $\frac{n^2}{\rho-n\eta+n^2} \to 1$, $\frac{1}{\rho-n\eta+n^2}e^{(\rho-n\eta+n^2)T} \to \infty$, and $e^{-(\rho-n\eta)T} \to \infty$ as $n \to \infty$. We therefore conclude that

$$J_0(x,0,X^n) \to -\infty \quad \text{as } n \to \infty.$$

In the next Example 5.1.6, we show that even with the right cost functional (5.2) (which in the setting of Example 5.1.6 coincides with (5.9)), the dynamics (5.7) for the deviation process can lead to arbitrarily large negative costs. With the right dynamics (5.1) we recover a well-posed problem, see Example 5.3.1.

Example 5.1.6. Let $m = 2$ and suppose that $(M^{(1)}, M^{(2)})^\top = (W^{(1)}, W^{(2)})^\top = W$ is a two-dimensional Brownian motion and $\mathcal{F}_s = \mathcal{F}_s^W$ for all $s \in [0, T]$. We assume that $\mu \equiv 0$, $\eta \equiv 0$, $\lambda \equiv 0$, and that $\sigma > 0$ and $\rho > 0$ are positive deterministic constants such that $2\rho - \sigma^2 > 0$ (i.e., condition $(\mathbf{C}_{>0})$ holds). As in Example 5.1.4, we do not need to specify the correlation process $\overline{r}$.

Observe that, in our current setting, the price impact process γ is a geometric Brownian motion $\gamma_s = \gamma_0 \exp(\sigma W_s^{(1)} - \frac{\sigma^2}{2} s)$, $s \in [0, T]$.

We consider the starting time $t = 0$ and fix some initial position $x \in \mathbb{R} \setminus \{0\}$ and the initial deviation $d = 0$.

For $n \in \mathbb{N}$ let $(X_s^n)_{s \in [0-, T]}$ be defined by (this is as in Example 5.1.5)

$$X_{0-}^n = X_0^n = x, \quad dX_s^n = -n X_s^n dW_s^{(1)} \text{ for } s \in [0, T), \quad X_T^n = 0.$$

For each $n \in \mathbb{N}$, we assume that $D^n = (D_s^n)_{s \in [0-, T]}$ is given by (5.7), which here reads

$$\begin{aligned} dD_s^n &= -\rho D_s^n ds - n \gamma_s X_s^n dW_s^{(1)}, \quad s \in [0, T), \\ D_{0-}^n &= 0, \quad D_T^n = D_{T-}^n - \gamma_T X_{T-}^n. \end{aligned}$$

In particular, $D_s^n = -\int_0^s n e^{-\rho(s-r)} \gamma_r X_r^n dW_r^{(1)}$ for $s \in [0, T)$ and $n \in \mathbb{N}$.

We first verify that $X^n \in \mathcal{A}_0^{\text{sem}}(x, 0)$ for all $n \in \mathbb{N}$. Notice that in the current setting we have for all $p \in [1, \infty)$ and $n \in \mathbb{N}$ that

$$E\left[\sup_{s \in [0,T]} \gamma_s^p\right] < \infty, \quad E\left[\sup_{s \in [0,T]} \gamma_s^{-p}\right] < \infty, \text{ and } E\left[\sup_{s \in [0,T]} |X_s^n|^p\right] < \infty. \tag{5.15}$$

This, the Burkholder-Davis-Gundy inequality, and the Hölder inequality imply that it holds for all $p \in [2, \infty)$ and $n \in \mathbb{N}$ that there exists $c \in [1, \infty)$ such that

$$\begin{aligned} E\left[\sup_{s \in [0,T)} |D_s^n|^p\right] &\leq c E\left[\left(\int_0^T n^2 e^{-2\rho(T-r)} \gamma_r^2 \left(X_r^n\right)^2 dr\right)^{\frac{p}{2}}\right] \\ &\leq c n^p T^{\frac{p}{2}} E\left[\sup_{r \in [0,T]} \gamma_r^p |X_r^n|^p\right] < \infty. \end{aligned}$$

Furthermore, as $D_T^n = D_{T-}^n - \gamma_T X_{T-}^n$, we also get $E[|D_T^n|^p] < \infty$ for all $n \in \mathbb{N}$ and $p \in [2, \infty)$. We thus obtain for all $p \in [1, \infty)$ and $n \in \mathbb{N}$ that

$$E\left[\sup_{s \in [0,T]} |D_s^n|^p\right] < \infty. \tag{5.16}$$

It now follows from the Hölder inequality, the Minkowski inequality, (5.15), and (5.16) that **(A1)** is satisfied. Since σ^2 is a deterministic constant, **(A2)** then also holds true. Furthermore, the Hölder inequality, (5.15), and (5.16) prove that **(A3)** is satisfied. Moreover, **(A4)** and **(A5)** are trivially satisfied due to $\eta \equiv 0$. Hence, it holds $X^n \in \mathcal{A}_0^{\text{sem}}(x, 0)$ for all $n \in \mathbb{N}$.

We next consider the cost functional J^{sem} defined by (5.2). Note that, since $\eta \equiv 0$ (and $\lambda \equiv 0$), this is the same as (5.9). We obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned} J_0^{\text{sem}}(x, 0, X^n) &= J_0(x, 0, X^n) \\ &= E\left[\int_{[0,T)} D^n_{s-} dX^n_s + D^n_{T-}\Delta X^n_T + \int_{[0,T)} \frac{\gamma_s}{2} n^2 (X^n_s)^2 ds + \frac{\gamma_T}{2}\left(\Delta X^n_T\right)^2\right] \\ &= -nE\left[\int_0^T D^n_s X^n_s dW^{(1)}_s\right] - E\left[D^n_{T-} X^n_{T-}\right] + \frac{n^2}{2}\int_0^T E\left[\gamma_s (X^n_s)^2\right] ds \\ &\quad + \frac{1}{2} E\left[\gamma_{T-}(X^n_{T-})^2\right]. \end{aligned} \tag{5.17}$$

By the Burkholder-Davis-Gundy inequality, the Hölder inequality, (5.15), and (5.16), the stochastic integral $\int_0^\cdot D^n_s X^n_s dW^{(1)}_s$ is a martingale for all $n \in \mathbb{N}$, hence its expectation vanishes. Moreover, it holds for all $n \in \mathbb{N}$ that

$$\gamma_s (X^n_s)^2 = \gamma_0 x^2 e^{\left((\sigma - 2n)W^{(1)}_s - \left(\frac{\sigma^2}{2} + n^2\right)s\right)}, \quad s \in [0, T),$$

and thus

$$E\left[\gamma_s (X^n_s)^2\right] = \gamma_0 x^2 e^{(n^2 - 2\sigma n)s}, \quad s \in [0, T). \tag{5.18}$$

Besides this, we have for all $n \in \mathbb{N}$ and $s \in [0, T)$ that, by integration by parts,

$$d\left(D^n_s X^n_s\right) = -\rho D^n_s X^n_s ds - n\gamma_s (X^n_s)^2 dW^{(1)}_s - nD^n_s X^n_s dW^{(1)}_s + n^2 \gamma_s (X^n_s)^2 ds. \tag{5.19}$$

Again by the Burkholder-Davis-Gundy inequality, the Hölder inequality, and (5.15), one can show that $\int_0^\cdot \gamma_s (X^n_s)^2 dW^{(1)}_s$ is a martingale for all $n \in \mathbb{N}$. Therefore, it follows from (5.18) and (5.19) for all $n \in \mathbb{N} \cap (2\sigma, \infty)$ and $s \in [0, T)$ that

$$\begin{aligned} E\left[D^n_s X^n_s\right] &= -\rho \int_0^s E\left[D^n_r X^n_r\right] dr + n^2 \int_0^s E\left[\gamma_r (X^n_r)^2\right] dr \\ &= -\rho \int_0^s E\left[D^n_r X^n_r\right] dr + \frac{n^2 \gamma_0 x^2}{n^2 - 2\sigma n}\left(e^{(n^2 - 2\sigma n)s} - 1\right). \end{aligned}$$

We thus obtain for all $n \in \mathbb{N} \cap (2\sigma, \infty)$ and $s \in [0, T)$ that

$$E\left[D^n_s X^n_s\right] = e^{-\rho s} \frac{n^2 \gamma_0 x^2}{\rho + n^2 - 2\sigma n}\left(e^{(\rho + n^2 - 2\sigma n)s} - 1\right). \tag{5.20}$$

(5.15), (5.16), (5.18), and (5.20) imply that the cost functional (5.17) for all $n \in \mathbb{N} \cap (2\sigma, \infty)$ becomes

$$\begin{aligned}
&J_0^{\mathrm{sem}}(x, 0, X^n) \\
&= -e^{-\rho T} \frac{n^2 \gamma_0 x^2}{\rho + n^2 - 2\sigma n} \left(e^{(\rho+n^2-2\sigma n)T} - 1 \right) + \frac{n^2}{2} \int_0^T \gamma_0 x^2 e^{(n^2-2\sigma n)s} ds + \frac{\gamma_0 x^2}{2} e^{(n^2-2\sigma n)T} \\
&= \frac{\gamma_0 x^2}{2} \left(\frac{2n^2}{\rho + n^2 - 2\sigma n} \left(e^{-\rho T} - e^{(n^2-2\sigma n)T} \right) + \frac{n^2}{n^2 - 2\sigma n} \left(e^{(n^2-2\sigma n)T} - 1 \right) + e^{(n^2-2\sigma n)T} \right) \\
&= \frac{\gamma_0 x^2}{2} \left(I_1(n) - I_2(n) \right),
\end{aligned}$$

where $I_1(n) = e^{(n^2-2\sigma n)T}(\frac{n^2}{n^2-2\sigma n} - \frac{2n^2}{\rho+n^2-2\sigma n} + 1)$ and $I_2(n) = \frac{n^2}{n^2-2\sigma n} - \frac{2n^2 e^{-\rho T}}{\rho+n^2-2\sigma n}$. Observe that

$$\begin{aligned}
\frac{n^2}{n^2 - 2\sigma n} - \frac{2n^2}{\rho + n^2 - 2\sigma n} + 1 &= \frac{1}{1 - \frac{2\sigma}{n}} - \frac{2}{\frac{\rho}{n^2} + 1 - \frac{2\sigma}{n}} + 1 \\
&= -\frac{2}{n} \frac{\sigma - \frac{\rho}{n} + \frac{\rho\sigma}{n^2} - \frac{2\sigma^2}{n}}{\left(1 - \frac{2\sigma}{n}\right)\left(\frac{\rho}{n^2} + 1 - \frac{2\sigma}{n}\right)}, \quad n \in \mathbb{N} \cap (2\sigma, \infty),
\end{aligned} \tag{5.21}$$

i.e., this term behaves as $-\frac{2\sigma}{n}$ for large $n \in \mathbb{N}$ (in particular, this term is strictly negative provided n is sufficiently large). It follows that $I_1(n) \to -\infty$ as $n \to \infty$, whereas $I_2(n) \to 1 - 2e^{-\rho T}$ as $n \to \infty$, hence

$$J_0^{\mathrm{sem}}(x, 0, X^n) \to -\infty \quad \text{as } n \to \infty.$$

Thus, dynamics (5.7) lead to an ill-posed optimization problem.

Finally, to complement the above discussion on the necessity of some adjustments in our setting, we discuss related literature. Note that the cost functional (5.9) and the counterexample Example 5.1.4, as well as the counterexample Example 5.1.6 for the deviation dynamics, are taken from [AKU21a].

A modification of the cost functional similar to (5.8)→(5.9) already appeared in a closely related setting in [LS13]. Lorenz and Schied in [LS13] consider an optimal trade execution problem in an Obizhaeva-Wang model with drift in the unaffected price. When deriving costs for a semimartingale strategy from the costs of a discrete-time strategy, they obtain $\frac{1}{2}[X]_T$ as one term in the costs, which for more general price impact processes γ corresponds to our term $\int_{[t,T]} \frac{\gamma_s}{2} d[X]_s$. In a less related setting in [GP16], Gârleanu and Pedersen use a term in their cost functional containing the quadratic variation $[X]$ and justify it via limiting arguments from discrete time. Moreover, Horst and Kivman in [HK21] prove that the limiting strategy in the case of vanishing instantaneous price impact in their model can be viewed as the optimal strategy in a problem of optimal execution with semimartingale strategies, where the cost functional is in the spirit of (5.9).

The adjustment (5.9)→(5.2) in the present thesis is, aside from the motivation from discrete-time, inspired by [AKU22a], but not explicitly stated there in the form of (5.2). It can neither arise in the setting of [LS13], where the resilience R is given in terms of a constant resilience coefficient ρ, nor in the setting of [AKU21a] or [HK21], where the resilience R is given in terms of a randomly evolving resilience coefficient ρ, since in all these cases R is a continuous process of finite variation leading to $[X, R] \equiv 0$.

The additional term in the dynamics of the deviation process in (5.1) compared to (5.7) is to the best of our knowledge a new aspect in [AKU21a]. It does not emerge in the aforementioned papers because they consider constant γ, in which case $[\gamma, X] \equiv 0$. In order to see the need for the adjustment (5.7)→(5.1), it is necessary to consider the price impact itself (i.e., the process γ) to be of infinite variation (or discontinuous). We also mention that, although the price impact in [FSU19] can have infinite variation, the additional term in the deviation containing $[\gamma, X]$ does not show up there as only strategies X of finite variation are allowed.

5.2 Optimal strategies and minimal costs

In this section, we state and prove the main results and some of their consequences. The main results include an alternative representation of the cost functional, a representation of the value function (in terms of a solution to BSDE (4.1)), a characterization of existence of an optimal strategy, and an explicit expression for the optimal strategy (when it exists). In Section 5.2.1, we obtain the alternative representation of the cost functional. We also use this result to present first examples of optimal strategies. We then provide general results on optimal strategies and minimal costs in Section 5.2.2.

To state and prove these results, we introduce an auxiliary process based on a solution (in the sense of Definition 4.0.1) of BSDE (4.1). Note that existence and uniqueness for BSDE (4.1) are discussed in Chapter 4, and complemented by Corollary 5.2.8. If $(\mathbf{C_{>0}})$ holds (see Section 3.1), and $(Y, Z, M^\perp)$ is a solution of BSDE (4.1), we define the progressively measurable process $\widetilde{\vartheta} = (\widetilde{\vartheta}_s)_{s\in[0,T]}$ pertaining to (Y, Z) by

$$\widetilde{\vartheta}_s = \frac{(\rho_s + \mu_s)Y_s + (\sigma_s + \eta_s \overline{r}_s)Z_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}Z_s^{(2)} + \lambda_s}{(\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)Y_s + \kappa_s + \lambda_s}, \quad s \in [0, T]. \tag{5.22}$$

5.2.1 Representation of the cost functional based on the BSDE

We now introduce the first solution component Y of BSDE (4.1) (and the auxiliary process $\widetilde{\vartheta}$ from (5.22)) into the cost functional (5.2). This is done (see the proof of Theorem 5.2.1) by splitting the integrals over $[t, T]$ in the cost functional (5.2) up into integrals over $[t,T)$ and the contribution on $\{T\}$, and then exploiting the terminal condition $Y_T = \frac{1}{2}$, which also holds immediately prior to T (see Lemma 4.1.5). Along the way, we obtain that the cost functional (5.2), under **($\mathbf{C_{>0}}$)** and existence of a solution to BSDE (4.1), is well-defined. It turns out that it can be represented as the sum of a term that involves Y and does not depend on the strategy, and a conditional expectation of an integral with respect to $d[M^{(1)}]$ that has an integrand which is nonnegative $\mathcal{D}_{M^{(1)}}$-a.e. This provides us with a lower bound for the value function. The precise results are stated in the next Theorem 5.2.1. This theorem constitutes an important step towards the solution of the control problem in Theorem 5.2.6.

Theorem 5.2.1. *Let ($C_{>0}$) be satisfied. Assume that there exists a solution $(Y, Z, M^\perp)$ of BSDE (4.1), and let $\widetilde{\vartheta}$ pertaining to (Y, Z) be defined by (5.22). For all $x, d \in \mathbb{R}$, $t \in [0, T]$, and $X \in \mathcal{A}_t^{sem}(x, d)$ it then holds that the cost functional (5.2) is well-defined and admits the a.s. representation*

$$J_t^{sem}(x,d,X) = \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t} + E_t\left[\int_t^T \frac{1}{\gamma_s}\left(\widetilde{\vartheta}_s(\gamma_s X_s - D_s^X) + D_s^X\right)^2 \left((\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)Y_s + \kappa_s + \lambda_s\right) d[M^{(1)}]_s\right]. \tag{5.23}$$

In particular, for all $x, d \in \mathbb{R}$ and $t \in [0, T]$ it holds that

$$V_t^{sem}(x,d) \geq \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t} \quad a.s. \tag{5.24}$$

As a consequence of Theorem 5.2.1, we obtain that it is optimal to close the position immediately whenever the initial position $x \in \mathbb{R}$ and the initial deviation $d \in \mathbb{R}$ are related via $x = \frac{d}{\gamma_t}$, or when the resilience vanishes (i.e., $\rho \equiv 0$ and $\eta \equiv 0$). We study these situations in Lemma 5.2.2 and Proposition 5.2.3, respectively.

Lemma 5.2.2. *Let ($C_{>0}$) be satisfied and assume that there exists a solution of BSDE (4.1). Suppose that $t \in [0, T]$ and $x, d \in \mathbb{R}$ with $x = \frac{d}{\gamma_t}$. It then holds that $V_t^{sem}(x, d) = -\frac{d^2}{2\gamma_t}$, and that the strategy $X^* = (X_s^*)_{s\in[t-,T]}$ defined by $X_{t-}^* = x$, $X_s^* = 0$, $s \in [t, T]$, which closes the position immediately, is optimal in $\mathcal{A}_t^{sem}(x, d)$. Moreover, this optimal strategy is unique up to $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-null sets.*

We now treat the case of vanishing resilience, which means that the impact of trading on the price is permanent. Note that we do not need to assume existence of a solution to BSDE (4.1) as we derive an explicit solution of this BSDE in the proof.

Proposition 5.2.3. *Assume ($C_{>0}$), and that $\rho \equiv 0$ and $\eta \equiv 0$. Furthermore, let $E[\sup_{s\in[0,T]} \gamma_s^{-2}] + E[\int_0^T \sigma_s^2 d[M^{(1)}]_s] < \infty$. Then, for all $t \in [0, T]$ and $x, d \in \mathbb{R}$,* *the value function is given by* $V_t^{sem}(x,d) = -x\left(d - \frac{\gamma_t}{2}x\right)$*, and the strategy* $X^* = (X_s^*)_{s\in[t-,T]}$ *defined by* $X_{t-}^* = x$*,* $X_s^* = 0$*,* $s \in [t,T]$*, which closes the position immediately, is optimal in* $\mathcal{A}_t^{sem}(x,d)$*. Moreover, this optimal strategy is unique up to* $\mathcal{D}_{M^{(1)}}|_{[t,T]}$*-null sets.*

Proofs

In this part, we prove Theorem 5.2.1, Lemma 5.2.2, and Proposition 5.2.3.

We first introduce the following lemma that we employ in the proofs of Theorem 5.2.1, Lemma 5.2.5, Theorem 5.2.6, and Lemma 5.2.10. It provides helpful representations for the dynamics of the process $A = X - \gamma^{-1}D^X$ where X is a càdlàg semimartingale with associated D^X.

Lemma 5.2.4. *Let* $d \in \mathbb{R}$ *and* $t \in [0,T]$*. Suppose that* $X = (X_s)_{s\in[t-,T]}$ *is a càdlàg semimartingale, and let* $D^X = (D_s^X)_{s\in[t-,T]}$ *be given by* (5.1)*. Define* $A = (A_s)_{s\in[t,T]}$ *by* $A_s = X_s - \gamma_s^{-1}D_s^X$*,* $s \in [t,T]$*.*

(i) It holds for all $s \in [t,T]$ *that* $d[\gamma^{-1}, D^X]_s = -D_s^X d[\gamma^{-1}, R]_s - \gamma_s^{-1} d[\gamma, X]_s$*.*

(ii) It holds that A *is continuous.*

(iii) It holds for all $s \in [t,T]$ *that*

$$\begin{aligned} dA_s &= -D_s^X d\gamma_s^{-1} + \gamma_s^{-1} D_s^X dR_s + D_s^X d[\gamma^{-1}, R]_s \\ &= (A_s - X_s)\left(\gamma_s d\gamma_s^{-1} - dR_s - \gamma_s d[\gamma^{-1}, R]_s\right) \\ &= (A_s - X_s)\Bigg(-(\mu_s + \rho_s - \sigma_s^2 - \sigma_s\eta_s\overline{r}_s) d[M^{(1)}]_s - (\sigma_s + \eta_s\overline{r}_s) dM_s^{(1)} \\ &\qquad - \eta_s\sqrt{1-\overline{r}_s^2}\, dM_s^{(2)}\Bigg), \quad s \in [t,T]. \end{aligned}$$

Proof. (i) It follows from (5.1) that

$$d[\gamma^{-1}, D^X]_s = -D_s^X d[\gamma^{-1}, R]_s + \gamma_s d[\gamma^{-1}, X]_s, \quad s \in [t,T].$$

Furthermore, we have by (3.3) and (3.2) that

$$\gamma_s d[\gamma_s^{-1}, X]_s = -\sigma_s d[M^{(1)}, X]_s = -\gamma_s^{-1} d[\gamma, X]_s, \quad s \in [t,T].$$

Together, this shows the claim in (i).

(ii) Since $\Delta D_s^X = \gamma_s \Delta X_s$, $s \in [t,T]$, it holds that $\Delta A_s = \Delta X_s - \gamma_s^{-1}\Delta D_s^X = 0$, $s \in [t,T]$.

(iii) Using (5.1) and (i), we obtain by integration by parts for all $s \in [t,T]$ that

$$\begin{aligned} dA_s &= dX_s - \gamma_s^{-1} dD_s^X - D_s^X d\gamma_s^{-1} - d[\gamma^{-1}, D^X]_s \\ &= dX_s + \gamma_s^{-1} D_s^X dR_s - dX_s - \gamma_s^{-1} d[\gamma, X]_s - D_s^X d\gamma_s^{-1} + D_s^X d[\gamma^{-1}, R]_s + \gamma_s^{-1} d[\gamma, X]_s \\ &= \gamma_s^{-1} D_s^X dR_s - D_s^X d\gamma_s^{-1} + D_s^X d[\gamma^{-1}, R]_s. \end{aligned}$$

The second equality in the claim then follows from the fact that $-D^X_s = (A_s - X_s)\gamma_s$, $s \in [t,T]$, by definition of A. For the third equality, observe that (3.3) and (3.1) imply for all $s \in [t,T]$ that

$$-\gamma_s d[\gamma^{-1}, R]_s = \sigma_s d[M^{(1)}, R]_s = \sigma_s \eta_s d[M^{(1)}, M^R]_s = \sigma_s \eta_s \overline{r}_s d[M^{(1)}]_s$$

and that

$$\begin{aligned}\gamma_s d\gamma_s^{-1} - dR_s &= -(\mu_s - \sigma_s^2) d[M^{(1)}]_s - \sigma_s dM^{(1)}_s - \rho_s d[M^{(1)}]_s - \eta_s \overline{r}_s dM^{(1)}_s \\ &\quad - \eta_s \sqrt{1 - \overline{r}_s^2} dM^{(2)}_s.\end{aligned}$$

□

We are now prepared to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. We fix $x, d \in \mathbb{R}$, $t \in [0,T]$, and $X \in \mathcal{A}^{\text{sem}}_t(x,d)$ throughout the proof.

Observe that

$$\int_{[t,T]} D^X_{s-} dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s = \int_{[t,T)} D^X_{s-} dX_s + \int_{[t,T)} \frac{\gamma_s}{2} d[X]_s - D^X_{T-} X_{T-} + \frac{\gamma_T}{2} X^2_{T-}. \tag{5.25}$$

Since $Y_{T-} = \frac{1}{2}$ by Lemma 4.1.5, it holds that

$$\begin{aligned}-D^X_{T-} X_{T-} + \frac{\gamma_T}{2} X^2_{T-} &= \frac{\gamma_T}{2} \left(X_{T-} - \gamma_T^{-1} D^X_{T-} \right)^2 - \frac{\gamma_T^{-1} (D^X_{T-})^2}{2} \\ &= \gamma_T Y_{T-} \left(X_{T-} - \gamma_T^{-1} D^X_{T-} \right)^2 - \frac{\gamma_T^{-1} (D^X_{T-})^2}{2}.\end{aligned} \tag{5.26}$$

We first consider the term $\gamma_T Y_{T-} (X_{T-} - \gamma_T^{-1} D^X_{T-})^2$. We have by integration by parts, (4.1), and (3.2) for all $s \in [0,T]$ that

$$\begin{aligned}d(\gamma_s Y_s) &= -\gamma_s f(s, Y_s, Z_s) d[M^{(1)}]_s + \gamma_s Z^{(1)}_s dM^{(1)}_s + \gamma_s Z^{(2)}_s dM^{(2)}_s + \gamma_s dM^{\perp}_s \\ &\quad + \gamma_s \mu_s Y_s d[M^{(1)}]_s + \gamma_s \sigma_s Y_s dM^{(1)}_s + \gamma_s \sigma_s Z^{(1)}_s d[M^{(1)}]_s \\ &= \gamma_s \left(\widetilde{\vartheta} \left((\rho_s + \mu_s) Y_s + (\sigma_s + \eta_s \overline{r}_s) Z^{(1)}_s + \eta_s \sqrt{1 - \overline{r}_s^2} Z^{(2)}_s + \lambda_s \right) - \lambda_s \right) d[M^{(1)}]_s \\ &\quad + \gamma_s \left(Z^{(1)}_s + \sigma_s Y_s \right) dM^{(1)}_s + \gamma_s Z^{(2)}_s dM^{(2)}_s + \gamma_s dM^{\perp}_s.\end{aligned} \tag{5.27}$$

Denote $A_s = X_s - \gamma_s^{-1} D^X_s$, $s \in [t,T]$. Part (iii) of Lemma 5.2.4 shows for all $s \in [t,T]$ that

$$\begin{aligned}dA_s &= \gamma_s^{-1} D^X_s (\rho_s + \mu_s - \sigma_s^2 - \sigma_s \eta_s \overline{r}_s) d[M^{(1)}]_s + \gamma_s^{-1} D^X_s (\sigma_s + \eta_s \overline{r}_s) dM^{(1)}_s \\ &\quad + \gamma_s^{-1} D^X_s \eta_s \sqrt{1 - \overline{r}_s^2} dM^{(2)}_s.\end{aligned}$$

It follows for all $s \in [t,T]$ that (recall that $[M^{(1)}] = [M^{(2)}]$)

$$d[A]_s = \gamma_s^{-2}(D_s^X)^2(\sigma_s^2 + 2\sigma_s\eta_s\overline{r}_s + \eta_s^2)d[M^{(1)}]_s$$

and

$$\begin{aligned} dA_s^2 &= 2A_s dA_s + d[A]_s \\ &= \gamma_s^{-1}D_s^X\Big(2A_s(\rho_s+\mu_s-\sigma_s^2-\sigma_s\eta_s\overline{r}_s) + \gamma_s^{-1}D_s^X(\sigma_s^2 + 2\sigma_s\eta_s\overline{r}_s + \eta_s^2)\Big)d[M^{(1)}]_s \\ &\quad + 2\gamma_s^{-1}D_s^X A_s(\sigma_s + \eta_s\overline{r}_s)dM_s^{(1)} + 2\gamma_s^{-1}D_s^X A_s\eta_s\sqrt{1-\overline{r}_s^2}dM_s^{(2)}. \end{aligned} \tag{5.28}$$

We can combine (5.27) and (5.28) to obtain by integration by parts that

$$\begin{aligned} \gamma_T Y_{T-}\left(X_{T-} - \gamma_T^{-1}D_{T-}^X\right)^2 &= \gamma_t Y_t\left(x - \gamma_t^{-1}d\right)^2 + \int_{(t,T)} \gamma_s Y_s dA_s^2 + \int_{(t,T)} A_s^2 d(\gamma_s Y_s) \\ &\quad + \int_t^T d[\gamma Y, A^2]_s \\ &= \gamma_t Y_t\left(x - \gamma_t^{-1}d\right)^2 + \int_t^T L_s\, d[M^{(1)}]_s \\ &\quad + \int_t^T 2Y_s D_s^X A_s(\sigma_s + \eta_s\overline{r}_s) + A_s^2\gamma_s(Z_s^{(1)} + \sigma_s Y_s)\, dM_s^{(1)} \\ &\quad + \int_t^T 2Y_s D_s^X A_s\eta_s\sqrt{1-\overline{r}_s^2} + A_s^2\gamma_s Z_s^{(2)}\, dM_s^{(2)} \\ &\quad + \int_{(t,T)} \gamma_s A_s^2\, dM_s^{\perp}, \end{aligned} \tag{5.29}$$

where, for $s \in [t,T]$,

$$\begin{aligned} L_s &= Y_s D_s^X\left(2A_s(\rho_s + \mu_s - \sigma_s^2 - \sigma_s\eta_s\overline{r}_s) + \gamma_s^{-1}D_s^X(\sigma_s^2 + 2\sigma_s\eta_s\overline{r}_s + \eta_s^2)\right) \\ &\quad + A_s^2\gamma_s\left(\widetilde{\vartheta}\left((\rho_s + \mu_s)Y_s + (\sigma_s + \eta_s\overline{r}_s)Z_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}Z_s^{(2)} + \lambda_s\right) - \lambda_s\right) \\ &\quad + 2A_s D_s^X\left((Z_s^{(1)} + \sigma_s Y_s)(\sigma_s + \eta_s\overline{r}_s) + Z_s^{(2)}\eta_s\sqrt{1-\overline{r}_s^2}\right). \end{aligned}$$

Next, we consider the term $\gamma_T^{-1}(D_{T-}^X)^2$. Note that

$$d(D_s^X)^2 = 2D_{s-}^X dD_s^X + d[D^X]_s, \quad s \in [t,T],$$

and therefore by part (i) of Lemma 5.2.4,

$$d[\gamma^{-1}, (D^X)^2]_s = -2(D_s^X)^2 d[\gamma^{-1}, R]_s - 2D_s^X\gamma_s^{-1}d[\gamma, X]_s, \quad s \in [t,T].$$

Moreover, it holds that

$$d[D^X]_s = (D_s^X)^2 d[R]_s - 2D_s^X \gamma_s d[R,X]_s + \gamma_s^2 d[X]_s, \quad s \in [t,T].$$

We use integration by parts and the previous three equations to show for all $s \in [t,T]$ that

$$\begin{aligned}
\gamma_T^{-1}(D_{T-}^X)^2 &= \gamma_t^{-1} d^2 + \int_{[t,T)} \gamma_s^{-1} d(D_s^X)^2 + \int_{[t,T)} (D_s^X)^2 d\gamma_s^{-1} + \int_{[t,T)} d[\gamma^{-1}, (D^X)^2]_s \\
&= \gamma_t^{-1} d^2 + 2\int_{[t,T)} \gamma_s^{-1} D_{s-}^X dD_s^X + \int_{[t,T)} \gamma_s^{-1} d[D^X]_s + \int_{[t,T)} (D_s^X)^2 d\gamma_s^{-1} \\
&\quad - 2\int_{[t,T)} (D_s^X)^2 d[\gamma^{-1}, R]_s - 2\int_{[t,T)} D_s^X \gamma_s^{-1} d[\gamma, X]_s \\
&= \gamma_t^{-1} d^2 - 2\int_{[t,T)} \gamma_s^{-1} (D_s^X)^2 dR_s + 2\int_{[t,T)} D_{s-}^X dX_s + 2\int_{[t,T)} \gamma_s^{-1} D_s^X d[\gamma, X]_s \\
&\quad + \int_{[t,T)} \gamma_s^{-1} (D_s^X)^2 d[R]_s - 2\int_{[t,T)} D_s^X d[R,X]_s + \int_{[t,T)} \gamma_s d[X]_s \\
&\quad + \int_{[t,T)} (D_s^X)^2 d\gamma_s^{-1} - 2\int_{[t,T)} (D_s^X)^2 d[\gamma^{-1}, R]_s - 2\int_{[t,T)} D_s^X \gamma_s^{-1} d[\gamma, X]_s \\
&= \gamma_t^{-1} d^2 - 2\int_t^T \gamma_s^{-1} (D_s^X)^2 dR_s + 2\int_{[t,T)} D_{s-}^X dX_s + \int_t^T \gamma_s^{-1} (D_s^X)^2 d[R]_s \\
&\quad -2\int_t^T D_s^X d[R,X]_s + \int_{[t,T)} \gamma_s d[X]_s + \int_t^T (D_s^X)^2 d\gamma_s^{-1} - 2\int_t^T (D_s^X)^2 d[\gamma^{-1}, R]_s.
\end{aligned}$$

By (3.1) and (3.3) this becomes

$$\begin{aligned}
\gamma_T^{-1}(D_{T-}^X)^2 &= \gamma_t^{-1} d^2 - \int_t^T \gamma_s^{-1} (D_s^X)^2 (2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s \eta_s \overline{r}_s) d[M^{(1)}]_s \\
&\quad - \int_t^T \gamma_s^{-1} (D_s^X)^2 (2\eta_s \overline{r}_s + \sigma_s) dM_s^{(1)} - 2\int_t^T \gamma_s^{-1} (D_s^X)^2 \eta_s \sqrt{1 - \overline{r}_s^2} dM_s^{(2)} \\
&\quad + 2\int_{[t,T)} D_{s-}^X dX_s - 2\int_t^T D_s^X d[R,X]_s + \int_{[t,T)} \gamma_s d[X]_s.
\end{aligned} \tag{5.30}$$

It now follows from (5.25), (5.26), (5.29), and (5.30) that

$$\begin{aligned}
&\int_{[t,T]} D^X_{s-} dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s - \int_t^T D^X_s d[X,R]_s + \int_t^T \gamma_s \lambda_s X_s^2 d[M^{(1)}]_s \\
&= \int_t^T L_s + \gamma_s\lambda_s X_s^2 + \frac{1}{2}\gamma_s^{-1}(D^X_s)^2(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s\eta_s\overline{r}_s) d[M^{(1)}]_s \\
&\quad + \int_t^T 2Y_s D^X_s A_s(\sigma_s + \eta_s\overline{r}_s) + A_s^2\gamma_s(Z^{(1)}_s + \sigma_s Y_s) + \frac{1}{2}\gamma_s^{-1}(D^X_s)^2(2\eta_s\overline{r}_s + \sigma_s)\, dM^{(1)}_s \\
&\quad + \int_t^T 2Y_s D^X_s A_s\eta_s\sqrt{1-\overline{r}_s^2} + A_s^2\gamma_s Z^{(2)}_s + \gamma_s^{-1}(D^X_s)^2\eta_s\sqrt{1-\overline{r}_s^2}\, dM^{(2)}_s \\
&\quad + \int_{(t,T)} \gamma_s A_s^2\, dM^\perp_s + \gamma_t Y_t\left(x - \gamma_t^{-1}d\right)^2 - \frac{1}{2}\gamma_t^{-1}d^2.
\end{aligned} \tag{5.31}$$

For the integrand in the first term, we observe that by definition of L, κ, and $\widetilde{\vartheta}$, it holds that

$$\begin{aligned}
&L + \gamma\lambda X^2 + \frac{1}{2}\gamma^{-1}(D^X)^2(2\rho + \mu - \sigma^2 - \eta^2 - 2\sigma\eta\overline{r}) \\
&= 2AD^X Y(\mu+\rho) + \gamma^{-1}(D^X)^2 Y(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r}) + \gamma\lambda X^2 + \gamma^{-1}(D^X)^2\kappa \\
&\quad + A^2\gamma\left(\widetilde{\vartheta}^2\left((\sigma^2+\eta^2+2\sigma\eta\overline{r})Y + \kappa + \lambda\right) - \lambda\right) \\
&\quad + 2AD^X\left(Z^{(1)}(\sigma + \eta\overline{r}) + Z^{(2)}\eta\sqrt{1-\overline{r}^2}\right) \\
&= \gamma^{-1}(D^X)^2\left((\sigma^2+\eta^2+2\sigma\eta\overline{r})Y + \kappa + \lambda\right) + \lambda\left(\gamma X^2 - \gamma A^2 - 2AD^X - \gamma^{-1}(D^X)^2\right) \\
&\quad + A^2\gamma\widetilde{\vartheta}^2\left((\sigma^2+\eta^2+2\sigma\eta\overline{r})Y + \kappa + \lambda\right) + 2AD^X\widetilde{\vartheta}\left((\sigma^2+\eta^2+2\sigma\eta\overline{r})Y + \kappa + \lambda\right) \\
&= \left((\sigma^2+\eta^2+2\sigma\eta\overline{r})Y + \kappa + \lambda\right)\left(\gamma^{-1}(D^X)^2 + 2AD^X\widetilde{\vartheta} + A^2\gamma\widetilde{\vartheta}^2\right) \\
&\quad + \lambda\left(\gamma X^2 - \gamma A^2 - 2AD^X - \gamma^{-1}(D^X)^2\right).
\end{aligned}$$

Since

$$\gamma X^2 - \gamma A^2 - 2AD^X - \gamma^{-1}(D^X)^2 = 0$$

and

$$\gamma^{-1}(D^X)^2 + 2AD^X\widetilde{\vartheta} + A^2\gamma\widetilde{\vartheta}^2 = \gamma^{-1}\left(\widetilde{\vartheta}(\gamma X - D^X) + D^X\right)^2,$$

it follows that

$$\begin{aligned}
&\int_t^T L_s + \gamma_s\lambda_s X_s^2 + \frac{1}{2}\gamma_s^{-1}(D^X_s)^2(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s\eta_s\overline{r}_s) d[M^{(1)}]_s \\
&= \int_t^T \gamma_s^{-1}(\widetilde{\vartheta}_s(\gamma_s X_s - D^X_s) + D^X_s)^2\left((\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)Y_s + \kappa_s + \lambda_s\right) d[M^{(1)}]_s.
\end{aligned}$$

To prove (5.23), it therefore remains to show that the conditional expectation of the stochastic integrals with respect to $dM^{(1)}$, $dM^{(2)}$, and $dM^{\perp}$ in (5.31) vanishes.

Consider first the stochastic integral $\int_t^T \gamma_s A_s^2 Z_s^{(1)} dM_s^{(1)}$. By the Burkholder-Davis-Gundy inequality and the Cauchy-Schwarz inequality, it holds that for some constant $c \in (0,\infty)$,

$$\begin{aligned}
E_t\left[\sup_{r\in[t,T]}\left|\int_t^r \gamma_s A_s^2 Z_s^{(1)} dM_s^{(1)}\right|\right] &\le cE_t\left[\left(\int_t^T \gamma_s^2 A_s^4 (Z_s^{(1)})^2 d[M^{(1)}]_s\right)^{\frac{1}{2}}\right] \\
&\le cE_t\left[\sup_{s\in[t,T]}\left(\gamma_s A_s^2\right)\cdot\left(\int_t^T (Z_s^{(1)})^2 d[M^{(1)}]_s\right)^{\frac{1}{2}}\right] \\
&\le c\left(E_t\left[\sup_{s\in[t,T]}\left(\gamma_s^2 A_s^4\right)\right]\right)^{\frac{1}{2}}\left(E_t\left[\int_t^T (Z_s^{(1)})^2 d[M^{(1)}]_s\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

This is finite due to $E_t[\int_t^T (Z_s^{(1)})^2 d[M^{(1)}]_s] < \infty$ and **(A1)**. Therefore, $\int_t^\cdot \gamma_s A_s^2 Z_s^{(1)} dM_s^{(1)}$ is a true martingale, and hence

$$E_t\left[\int_t^T \gamma_s A_s^2 Z_s^{(1)} dM_s^{(1)}\right] = 0.$$

Using the same reasoning (with $E_t[\int_t^T (Z_s^{(2)})^2 d[M^{(2)}]_s] < \infty$), we obtain that

$$E_t\left[\int_t^T \gamma_s A_s^2 Z_s^{(2)} dM_s^{(2)}\right] = 0$$

as well. Similarly, $E[[M^{\perp}]_T] < \infty$ and **(A1)** imply that

$$E_t\left[\int_{(t,T)} \gamma_s A_s^2 dM_s^{\perp}\right] = 0.$$

Furthermore, **(A2)** and boundedness of Y yield that $E_t[(\int_t^T \gamma_s^2 A_s^4 \sigma_s^2 Y_s^2 d[M^{(1)}]_s)^{\frac{1}{2}}] < \infty$, and hence

$$E_t\left[\int_t^T \gamma_s A_s^2 \sigma_s Y_s dM_s^{(1)}\right] = 0.$$

To show that

$$E_t\left[\int_t^T \sigma_s D_s^X Y_s A_s dM_s^{(1)}\right] = 0,$$

observe that it holds by Young's inequality that

$$(D_s^X)^2 A_s^2 = (D_s^X)^2 (X_s - \gamma_s^{-1} D_s^X)^2 \le \frac{1}{2}((D_s^X)^4 \gamma_s^{-2} + \gamma_s^2 (X_s - \gamma_s^{-1} D_s^X)^4), \quad s \in [t,T].$$

This together with boundedness of Y (denote the bound by the constant $c_Y \in (0,\infty)$) yields that

$$E_t\left[\left(\int_t^T \sigma_s^2 (D_s^X)^2 Y_s^2 A_s^2 d[M^{(1)}]_s\right)^{\frac{1}{2}}\right] \leq \frac{c_Y}{\sqrt{2}} E_t\left[\left(\int_t^T \sigma_s^2 (D_s^X)^4 \gamma_s^{-2} d[M^{(1)}]_s\right)^{\frac{1}{2}}\right] + \frac{c_Y}{\sqrt{2}} E_t\left[\left(\int_t^T \sigma_s^2 \gamma_s^2 A_s^4 d[M^{(1)}]_s\right)^{\frac{1}{2}}\right],$$

which is finite by **(A3)** and **(A2)**. Similarly, we can argue that

$$E_t\left[\int_t^T \eta_s \overline{r}_s D_s^X Y_s A_s dM_s^{(1)} + \int_t^T \eta_s \sqrt{1-\overline{r}_s^2} D_s^X Y_s A_s dM_s^{(2)}\right] = 0$$

due to $[M^{(1)}] = [M^{(2)}]$, **(A5)**, and **(A4)**.

Moreover, it follows from **(A3)** that

$$E_t\left[\int_t^T \gamma_s^{-1} (D_s^X)^2 \sigma_s dM_s^{(1)}\right] = 0,$$

and from **(A5)** and $[M^{(1)}] = [M^{(2)}]$ that

$$E_t\left[\int_t^T \gamma_s^{-1} (D_s^X)^2 \eta_s \overline{r}_s dM_s^{(1)} + \int_t^T \gamma_s^{-1} (D_s^X)^2 \eta_s \sqrt{1-\overline{r}_s^2} dM_s^{(2)}\right] = 0.$$

We have thus shown (5.23).

Observe that $\frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$ does not depend on the strategy, and that

$$\frac{1}{\gamma_s}\left(\widetilde{\vartheta}_s(\gamma_s X_s - D_s^X) + D_s^X\right)^2 \left((\sigma_s^2 + \eta_s^2 + 2\sigma_s \eta_s \overline{r}_s) Y_s + \kappa_s + \lambda_s\right) \geq 0 \quad \mathcal{D}_{M^{(1)}}\text{-a.e.}$$

due to $Y \geq 0$ and **($\mathbf{C_{>0}}$)**. This explains the inequality (5.24). □

The following result states that, under certain conditions, an optimal strategy is unique. Although the conditions and the proof involve a (possibly nonunique) solution of BSDE (4.1), the uniqueness of optimal strategies holds in a general sense. The lemma is used to prove that the optimal strategies obtained in Lemma 5.2.2 and Proposition 5.2.3 are unique. It is also relevant for the proof of Theorem 5.2.6.

Lemma 5.2.5. *Let* **($C_{>0}$)** *be satisfied. Let $x, d \in \mathbb{R}$, $t \in [0,T]$, and suppose that there exist optimal strategies $X^*, X \in \mathcal{A}_t^{sem}(x,d)$. Assume that there exists a solution $(Y, Z, M^\perp)$ of BSDE (4.1) such that $V_t^{sem}(x,d) = \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$. Then, $X^* = X$ up to $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-null sets.*

Proof. Let $\widetilde{\vartheta}$ (pertaining to (Y,Z)) be defined by (5.22). Combine the assumption $V_t^{\text{sem}}(x,d) = \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$ with Theorem 5.2.1 to obtain that, a.s.,

$$E_t\left[\int_t^T \frac{1}{\gamma_s}\left(\widetilde{\vartheta}_s(\gamma_s X_s - D_s^X) + D_s^X\right)^2 \left((\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)Y_s + \kappa_s + \lambda_s\right) d[M^{(1)}]_s\right] = 0.$$

By taking expectations, it follows that

$$E\left[\int_t^T \frac{1}{\gamma_s}\left(\widetilde{\vartheta}_s(\gamma_s X_s - D_s^X) + D_s^X\right)^2 \left((\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)Y_s + \kappa_s + \lambda_s\right) d[M^{(1)}]_s\right] = 0.$$

Since $(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})Y + \kappa + \lambda > 0$ $\mathcal{D}_{M^{(1)}}$-a.e., this implies that

$$\widetilde{\vartheta}(\gamma X - D^X) + D^X = 0 \quad \mathcal{D}_{M^{(1)}}|_{[t,T]}\text{-a.e.} \tag{5.32}$$

This further yields for the process $A = (A_s)_{s\in[t,T]}$ defined by $A_s = X_s - \gamma_s^{-1}D_s^X$, $s \in [t,T]$, that

$$A - X = -\gamma^{-1}D^X = \widetilde{\vartheta}\gamma^{-1}(\gamma X - D^X) = \widetilde{\vartheta}A \quad \mathcal{D}_{M^{(1)}}|_{[t,T]}\text{-a.e.}$$

By Lemma 5.2.4 and $[M^{(1)}] = [M^{(2)}]$, we thus have that

$$dA_s = \widetilde{\vartheta}_s A_s\left(\gamma_s d\gamma_s^{-1} - dR_s - \gamma_s d[\gamma^{-1}, R]_s\right), \quad s \in [t,T].$$

For X^*, D^{X^*}, and $A^* = X^* - \gamma^{-1}D^{X^*}$ we analogously obtain

$$\widetilde{\vartheta}(\gamma X^* - D^{X^*}) + D^{X^*} = 0 \quad \mathcal{D}_{M^{(1)}}|_{[t,T]}\text{-a.e.} \tag{5.33}$$

and

$$dA_s^* = \widetilde{\vartheta}_s A_s^*\left(\gamma_s d\gamma_s^{-1} - dR_s - \gamma_s d[\gamma^{-1}, R]_s\right), \quad s \in [t,T].$$

Hence, A and A^* satisfy the same dynamics and have the same starting point $A_t = x - \gamma_t^{-1}d = A_t^*$. It follows that A and A^* are indistinguishable. Together with (5.32) and (5.33) this yields that $D^X = -\widetilde{\vartheta}\gamma A = -\widetilde{\vartheta}\gamma A^* = D^{X^*}$ $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-a.e. Finally, it follows from the definition of A and A^* that $X = X^*$ $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-a.e. □

We next give the proof of Lemma 5.2.2, which provides the optimal strategy and the optimal costs in the case $x = \frac{d}{\gamma_t}$.

Proof of Lemma 5.2.2. Suppose that $x = \frac{d}{\gamma_t}$. Let $X^* = (X_s^*)_{s\in[t-,T]}$ be defined by $X_{t-}^* = x$, $X_s^* = 0$, $s \in [t,T]$. Then, X^* is a càdlàg semimartingale with $X_{t-}^* = x$ and $X_T^* = 0$. The associated process $D^{X^*} = (D_s^{X^*})_{s\in[t-,T]}$ of (5.1) satisfies

$$D_t^{X^*} = d + \Delta D_t^{X^*} = d + \gamma_t \Delta X_t^* = d - \gamma_t x = 0,$$

and hence $D_s^{X^*} = 0$ for all $s \in [t, T]$ (cf. (5.1) and the definition of X^*). It follows that $X_s^* - \gamma_s^{-1} D_s^{X^*} = 0$, $s \in [t, T]$. From this, $D_s^{X^*} = 0$ for all $s \in [t, T]$, and the fact that $M^{(1)}$ is continuous, we obtain that the conditions **(A1)**–**(A5)** are satisfied, i.e., $X^* \in \mathcal{A}_t^{\text{sem}}(x, d)$. Since $D_s^{X^*} = 0$ and $\gamma_s X_s^* - D_s^{X^*} = 0$ for all $s \in [t, T]$, Theorem 5.2.1 yields that X^* is optimal and that, using any solution $(Y, Z, M^\perp)$ of BSDE (4.1), it holds that

$$V_t^{\text{sem}}(x, d) = \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} = -\frac{d^2}{2\gamma_t}.$$

Uniqueness of X^* up to $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-null sets follows from Lemma 5.2.5. □

The next proof is the one of Proposition 5.2.3 on the case of vanishing resilience.

Proof of Proposition 5.2.3. In the case $\rho \equiv 0 \equiv \eta$, the driver (4.2) of BSDE (4.1) for $(Y, Z) \equiv (\frac{1}{2}, 0)$ equals

$$f\Big(s, \frac{1}{2}, 0\Big) = -\frac{\left(\frac{1}{2}\mu_s + \lambda_s\right)^2}{\frac{1}{2}\sigma_s^2 + \frac{1}{2}(\mu_s - \sigma_s^2) + \lambda_s} + \frac{1}{2}\mu_s + \lambda_s = 0, \quad s \in [0, T].$$

Hence, $(Y, Z, M^\perp) \equiv (\frac{1}{2}, 0, 0)$ is a solution of BSDE (4.1). For $(Y, Z, M^\perp) \equiv (\frac{1}{2}, 0, 0)$, we further obtain in (5.22) that

$$\widetilde{\vartheta}_s = \frac{\frac{1}{2}\mu_s + \lambda_s}{\frac{1}{2}\sigma_s^2 + \frac{1}{2}(\mu_s - \sigma_s^2) + \lambda_s} = 1, \quad s \in [0, T].$$

Now, fix $x, d \in \mathbb{R}$ and $t \in [0, T]$. By Theorem 5.2.1 it holds for all $X \in \mathcal{A}_t^{\text{sem}}(x, d)$ that

$$J_t^{\text{sem}}(x, d, X) = \frac{1}{2\gamma_t}(d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} + E_t\left[\int_t^T \gamma_s X_s^2 \Big(\frac{1}{2}\mu_s + \lambda_s\Big) d[M^{(1)}]_s\right]. \tag{5.34}$$

Let $X^* = (X_s^*)_{s \in [t-, T]}$ be defined by $X_{t-}^* = x$, $X_s^* = 0$, $s \in [t, T]$. We show that $X^* \in \mathcal{A}_t^{\text{sem}}(x, d)$. First, X^* is a càdlàg semimartingale with $X_{t-}^* = x$ and $X_T^* = 0$. The associated process $D^{X^*} = (D_s^{X^*})_{s \in [t-, T]}$ defined by (5.1) satisfies

$$D_t^{X^*} = d + \Delta D_t^{X^*} = d + \gamma_t \Delta X_t^* = d - \gamma_t x.$$

Since $R_s = 0$ for all $s \in [0, T]$ and $X_s^* = 0$ for all $s \in [t, T]$, it follows that $D_s^{X^*} = d - \gamma_t x$ for all $s \in [t, T]$. From $E[\sup_{s \in [t,T]} \gamma_s^{-2}] < \infty$ we thus obtain **(A1)**. The assumptions $E[\sup_{s \in [t,T]} \gamma_s^{-2}] < \infty$ and $E[\int_0^T \sigma_s^2 d[M^{(1)}]_s] < \infty$, by the Cauchy-Schwarz inequality, imply **(A2)** as well as **(A3)**. Further, **(A4)** and **(A5)** are trivially satisfied because of $\eta \equiv 0$. In summary, it holds that $X^* \in \mathcal{A}_t^{\text{sem}}(x, d)$. Notice that $\frac{1}{2}\mu + \lambda > \frac{1}{2}\sigma^2 \geq 0$ $\mathcal{D}_{M^{(1)}}$-a.e. due to **($\mathbf{C_{>0}}$)**. The optimality of closing the position immediately and the formula for the value function now follow from (5.34). Uniqueness of X^* up to $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-null sets follows from Lemma 5.2.5. □

5.2.2 Main theorem

We now present and prove the main theorem of this chapter. The theorem provides a representation of the value function in terms of a solution to BSDE (4.1), a characterization based on $\widetilde{\vartheta}$ (defined in (5.22)) for existence of an optimal strategy, and, in case of existence, a closed-form representation for the optimal strategy and the associated deviation.

Theorem 5.2.6. *Let* $(\boldsymbol{C_{>0}})$, $(\boldsymbol{C_{bdd}})$, *and* $(\boldsymbol{C_{[M^{(1)}]}})$ *hold true. Assume that there exists a solution* $(Y, Z, M^{\perp})$ *of BSDE* (4.1) *such that* $\widetilde{\vartheta}$, *associated to* (Y, Z) *by* (5.22), *is* $\mathcal{D}_{M^{(1)}}$*-a.e. bounded.*

(i) For all $x, d \in \mathbb{R}$ *and* $t \in [0, T]$ *it holds that*

$$V_t^{sem}(x, d) = \frac{Y_t}{\gamma_t}(d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} \quad a.s.$$

(ii) Let $x, d \in \mathbb{R}$ *and assume that* $x \neq \frac{d}{\gamma_0}$. *Then there exists an optimal strategy* $X^* = (X_s^*)_{s\in[0-,T]} \in \mathcal{A}_0^{sem}(x, d)$ *if and only if there exists a càdlàg semimartingale* $\vartheta = (\vartheta_s)_{s\in[0,T]}$ *such that* $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{M^{(1)}}$*-a.e.*

In this case, the optimal strategy is unique up to $\mathcal{D}_{M^{(1)}}$*-null sets.*

(iii) Suppose that there exists a càdlàg semimartingale $\vartheta = (\vartheta_s)_{s\in[0,T]}$ *such that* $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{M^{(1)}}$*-a.e. Define*

$$\begin{aligned} Q_s = &- \int_0^s \vartheta_r(\sigma_r + \eta_r\overline{r}_r)dM_r^{(1)} - \int_0^s \vartheta_r\eta_r\sqrt{1 - \overline{r}_r^2}dM_r^{(2)} \\ &- \int_0^s \vartheta_r(\mu_r + \rho_r - \sigma_r^2 - \sigma_r\eta_r\overline{r}_r)d[M^{(1)}]_r, \quad s \in [0, T]. \end{aligned} \tag{5.35}$$

Let $x, d \in \mathbb{R}$ *and* $t \in [0, T]$. *Then the optimal strategy* $(X_s^*)_{s\in[t-,T]} \in \mathcal{A}_t^{sem}(x, d)$ *and the associated deviation process* $(D_s^{X^*})_{s\in[t-,T]}$ *(both unique up to* $\mathcal{D}_{M^{(1)}}|_{[t,T]}$*-null sets) are given by the formulas* $X_{t-}^* = x$, $D_{t-}^{X^*} = d$,

$$X_s^* = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q)_{t,s}\,(1 - \vartheta_s), \quad s \in [t, T), \tag{5.36}$$

$$D_s^{X^*} = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q)_{t,s}\,(-\gamma_s\vartheta_s), \quad s \in [t, T), \tag{5.37}$$

and $X_T^* = 0$, $D_T^{X^*} = (x - \frac{d}{\gamma_t})\mathcal{E}(Q)_{t,T}\,(-\gamma_T)$.

Note that by formula (5.36) for the optimal strategy, infinite variation of the optimal strategy can be attributed to the factor $\mathcal{E}(Q)$ (as in Example 5.3.1) or to ϑ (as in Example 5.3.3 or Example 5.3.4), whereas a jump of the optimal strategy inside the trading interval has to correspond to a jump of ϑ (see, e.g., Section 5.4.3).

We observe that the optimal strategy and the optimal deviation process are dynamically consistent.

Corollary 5.2.7. *Under the assumptions of Theorem 5.2.6 consider the case that there exists a càdlàg semimartingale* $\vartheta = (\vartheta_s)_{s\in[0,T]}$ *such that it holds* $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{M^{(1)}}$*-a.e. Define the process* Q *as in* (5.35). *Let* $x, d \in \mathbb{R}$ *and* $t \in [0,T]$. *Then, for the optimal strategy and deviation process given in* (5.36)–(5.37) *and for any* $r \in (t,T)$, *we have that*

$$X^*_s = \left(X^*_{r-} - \frac{D^{X^*}_{r-}}{\gamma_r}\right)\mathcal{E}(Q)_{r,s}\,(1-\vartheta_s), \quad s \in [r,T),$$
$$D^*_s = \left(X^*_{r-} - \frac{D^{X^*}_{r-}}{\gamma_r}\right)\mathcal{E}(Q)_{r,s}\,(-\gamma_s\vartheta_s), \quad s \in [r,T),$$

and $X^*_T = 0$, $D^{X^*}_T = (X^*_{r-} - \frac{D^{X^*}_{r-}}{\gamma_r})\mathcal{E}(Q)_{r,T}\,(-\gamma_T)$.

Under the assumptions of Theorem 5.2.6, it holds that $Y \le \frac{1}{2}$. To explain this, let the assumptions of Theorem 5.2.6 be in force, and let $t \in [0,T]$, $x = 1$, $d = 0$. Then, the process $X = (X_s)_{s\in[t-,T]}$ defined by $X_{t-} = x$, $X_s = 0$, $s \in [t,T]$, is an admissible strategy with associated costs $J^{\mathrm{sem}}_t(1,0,X) = \frac{\gamma_t}{2}$ (cf. (5.2)). The minimal costs for selling $x = 1$ unit given an initial deviation $d = 0$ by Theorem 5.2.6 amount to $V^{\mathrm{sem}}_t(1,0) = \gamma_t Y_t$. Therefore,

$$Y_t = \frac{V^{\mathrm{sem}}_t(1,0)}{\gamma_t} \le \frac{J^{\mathrm{sem}}_t(1,0,X)}{\gamma_t} = \frac{1}{2}.$$

Moreover, we obtain that $2Y_t = V^{\mathrm{sem}}_t(1,0)/J^{\mathrm{sem}}_t(1,0,X)$, and thus the random variable $2Y_t\colon \Omega \to [0,1]$ describes to which percentage the costs of selling one unit immediately at time t can be reduced by executing the position optimally. Hence, under the assumptions of Theorem 5.2.6 (and for $\lambda \equiv 0$), we again (compare with Section 2.4) have the economic interpretation of Y as a savings factor.

The relation $V^{\mathrm{sem}}_t(1,0) = \gamma_t Y_t$, $t \in [0,T]$, from Theorem 5.2.6 can further be used to establish the following uniqueness result.

Corollary 5.2.8. *Assume* $(\boldsymbol{C_{>0}})$, $(\boldsymbol{C_{bdd}})$, *and* $(\boldsymbol{C_{[M^{(1)}]}})$. *Let* $(Y, Z, M^\perp)$, $(\hat{Y}, \hat{Z}, \hat{M}^\perp)$ *be solutions of BSDE* (4.1) *such that the corresponding processes* $\widetilde{\vartheta} = (\widetilde{\vartheta}_s)_{s\in[0,T]}$ *and* $\hat{\vartheta} = (\hat{\vartheta}_s)_{s\in[0,T]}$ *defined by* (5.22) *are* $\mathcal{D}_{M^{(1)}}$*-a.e. bounded. Then,* Y *and* $\hat{Y}$ *are indistinguishable,* $Z^{(j)} = \hat{Z}^{(j)}$ $\mathcal{D}_{M^{(1)}}$*-a.e. for* $j \in \{1,2\}$, *and* $M^\perp$ *and* $\hat{M}^\perp$ *are indistinguishable.*

In particular, if, in the setting of Theorem 5.2.6, the process in (5.22) is $\mathcal{D}_{M^{(1)}}$-a.e. bounded for all solutions of BSDE (4.1), then such a solution of the BSDE is unique.

We finally remark that it is possible to replace the boundedness assumptions in Theorem 5.2.6 by appropriate integrability assumptions. For a more detailed comment on this aspect, we refer to [AKU21a, Remark 3.5(b)].

Proofs

This part contains the proofs of Theorem 5.2.6, Corollary 5.2.7, and Corollary 5.2.8.

As a preparation, we establish helpful results in Lemma 5.2.9 and Lemma 5.2.10. We also use Lemma 4.1.1 and Lemma 4.1.6 from Chapter 4.

We first state and prove an approximation result, based on [KS91, Section 3.2, Lemma 2.7], for any progressively measurable, $\mathcal{D}_{M^{(1)}}$-a.e. bounded process ϑ. This is the content of Lemma 5.2.9 and enables us to exploit Lemma 5.2.10 for the proof of the representation of the value function in Theorem 5.2.6.

Lemma 5.2.9. *Assume that $E[[M^{(1)}]_T] < \infty$, and suppose that $\vartheta = (\vartheta_s)_{s\in[0,T]}$ is a progressively measurable process that is bounded $\mathcal{D}_{M^{(1)}}$-a.e.*

Then there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of càdlàg semimartingales $\vartheta^n = (\vartheta^n_s)_{s\in[0,T]}$ that are $\mathcal{D}_{M^{(1)}}$-a.e. bounded uniformly in n and such that for all $p \in [1,\infty)$ it holds that $E[\int_0^T |\vartheta_s - \vartheta^n_s|^p d[M^{(1)}]_s] \to 0$ as $n \to \infty$.

Proof. It follows from Lemma 2.7 in Section 3.2 of [KS91] that there exists a sequence $(\widehat{\vartheta}^n)_{n\in\mathbb{N}}$ of (càglàd) simple (see [KS91, Def. 2.3]) processes $\widehat{\vartheta}^n = (\widehat{\vartheta}^n_s)_{s\in[0,T]}$ such that

$$E\left[\int_0^T |\vartheta_s - \widehat{\vartheta}^n_s|^2 d[M^{(1)}]_s\right] \to 0 \quad \text{as} \quad n \to \infty.$$

Define

$$\mathring{\vartheta}^n_s(\omega) = \lim_{r\downarrow s} \widehat{\vartheta}^n_r(\omega), \quad s \in [0,T),\, \omega \in \Omega,\, n \in \mathbb{N},$$

and $\mathring{\vartheta}^n_T = 0$, $n \in \mathbb{N}$. Then, $\mathring{\vartheta}^n$ is càdlàg for all $n \in \mathbb{N}$. Let $b \in (0,\infty)$ be such that $|\vartheta| \le b$ $\mathcal{D}_{M^{(1)}}$-a.e., and define, for each $n \in \mathbb{N}$, ϑ^n by

$$\vartheta^n_s(\omega) = \left(\mathring{\vartheta}^n_s(\omega) \wedge b\right) \vee (-b), \quad s \in [0,T],\, \omega \in \Omega.$$

It then holds that $|\vartheta^n_s(\omega)| \le b$ for all $s \in [0,T], \omega \in \Omega, n \in \mathbb{N}$. It follows that $(\vartheta^n)_{n\in\mathbb{N}}$ is a sequence of càdlàg semimartingales that are $\mathcal{D}_{M^{(1)}}$-a.e. bounded uniformly in n. Furthermore, since it holds for all $n \in \mathbb{N}$ that $|\vartheta - \vartheta^n| \le |\vartheta - \mathring{\vartheta}^n|$ and that $\mathring{\vartheta}^n = \widehat{\vartheta}^n$ $\mathcal{D}_{M^{(1)}}$-a.e., we have that

$$\begin{aligned}
E\left[\int_0^T |\vartheta_s - \vartheta^n_s|^2 d[M^{(1)}]_s\right] &\le E\left[\int_0^T |\vartheta_s - \mathring{\vartheta}^n_s|^2 d[M^{(1)}]_s\right] \\
&= E\left[\int_0^T |\vartheta_s - \widehat{\vartheta}^n_s|^2 d[M^{(1)}]_s\right] \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

For $p \in [1,2)$, the convergence

$$E\left[\int_0^T |\vartheta_s - \vartheta^n_s|^p d[M^{(1)}]_s\right] \to 0 \quad \text{as} \quad n \to \infty$$

follows from Jensen's inequality, and for $p \in (2,\infty)$, the convergence holds due to $|\vartheta - \vartheta^n| \le 2b$ $\mathcal{D}_{M^{(1)}}$-a.e. □

In the next lemma we show how to construct from a $\mathcal{D}_{M^{(1)}}$-a.e. bounded sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of càdlàg semimartingales (e.g., coming from Lemma 5.2.9) a sequence of admissible semimartingale strategies $(X^n)_{n\in\mathbb{N}}$ with the additional properties (5.40) and (5.43). We use this result in the proof of Theorem 5.2.6. Note that (5.38) with (5.39) has the structure of the optimal strategy in Theorem 5.2.6.

Lemma 5.2.10. *Suppose that* $(\boldsymbol{C_{[M^{(1)}]}})$ *and* $(\boldsymbol{C_{bdd}})$ *are satisfied. Let* $(\vartheta^n)_{n\in\mathbb{N}}$ *be a sequence of càdlàg semimartingales* $\vartheta^n = (\vartheta^n_s)_{s\in[0,T]}$ *that are* $\mathcal{D}_{M^{(1)}}$*-a.e. bounded uniformly in* n*. Let* $t \in [0,T]$ *and* $x, d \in \mathbb{R}$*. Define for each* $n \in \mathbb{N}$ *the process* $X^n = (X^n_s)_{s\in[t-,T]}$ *by* $X^n_{t-} = x$*,*

$$X^n_s = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,s}\,(1-\vartheta^n_s), \quad s \in [t,T), \tag{5.38}$$

and $X^n_T = 0$*, where*

$$\begin{aligned} Q^n_s = &-\int_0^s \vartheta^n_r(\sigma_r + \eta_r\overline{r}_r)dM^{(1)}_r - \int_0^s \vartheta^n_r\eta_r\sqrt{1-\overline{r}^2_r}dM^{(2)}_r \\ &-\int_0^s \vartheta^n_r(\mu_r + \rho_r - \sigma^2_r - \sigma_r\eta_r\overline{r}_r)d[M^{(1)}]_r, \quad s\in[0,T]. \end{aligned} \tag{5.39}$$

Then, the following properties hold.

(i) $X^n \in \mathcal{A}^{sem}_t(x,d)$ *for all* $n \in \mathbb{N}$*.*

(ii) For all $n \in \mathbb{N}$ *the associated deviation process* D^n *a.s. has the representations*

$$D^n_s = -\vartheta^n_s(\gamma_s X^n_s - D^n_s), \quad s\in[t,T), \tag{5.40}$$

and

$$D^n_s = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,s}\,(-\gamma_s\vartheta^n_s), \quad s \in [t,T), \tag{5.41}$$

and, for the terminal value D^n_T*, we have that*

$$D^n_T = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,T}\,(-\gamma_T). \tag{5.42}$$

(iii) It holds that

$$\sup_{n\in\mathbb{N}} E_t\left[\sup_{s\in[t,T]}\left(\gamma^4_s(X^n_s - \gamma^{-1}_s D^n_s)^8\right)\right] < \infty \ \ a.s. \tag{5.43}$$

Proof. Let $b \in (0,\infty)$ such that for all $n \in \mathbb{N}$ it holds that $|\vartheta^n| \le b$ $\mathcal{D}_{M^{(1)}}$-a.e. Now, fix $n \in \mathbb{N}$.

Since ϑ^n is a càdlàg semimartingale, it holds that X^n defined by (5.38) is also a càdlàg semimartingale. Note that moreover $X^n_{t-} = x$ and $X^n_T = 0$. We first show that D^n defined by (5.1) satisfies (5.40), (5.41), and (5.42). Subsequently, we establish (5.43). Finally, we argue that $X^n \in \mathcal{A}_t^{\text{sem}}(x,d)$.

Let $\widehat{A}^n = (\widehat{A}^n_s)_{s\in[t,T]}$ be the process defined by

$$\widehat{A}^n_s = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,s}, \quad s \in [t,T].$$

Observe that for all $s \in [t,T)$ it holds that $X^n_s = \widehat{A}^n_s(1-\vartheta^n_s)$. This and (5.39) imply for all $s \in [t,T]$ that

$$\begin{aligned} d\widehat{A}^n_s &= \widehat{A}^n_s dQ^n_s \\ &= \vartheta^n_s \widehat{A}^n_s \left(-(\sigma_s + \eta_s\overline{r}_s)dM^{(1)}_s - \eta_s\sqrt{1-\overline{r}_s^2}dM^{(2)}_s - (\mu_s + \rho_s - \sigma_s^2 - \sigma_s\eta_s\overline{r}_s)d[M^{(1)}]_s\right) \\ &= \left(\widehat{A}^n_s - X^n_s\right)\left(-(\sigma_s + \eta_s\overline{r}_s)dM^{(1)}_s - \eta_s\sqrt{1-\overline{r}_s^2}dM^{(2)}_s - (\mu_s + \rho_s - \sigma_s^2 - \sigma_s\eta_s\overline{r}_s)d[M^{(1)}]_s\right). \end{aligned} \tag{5.44}$$

Let $A^n = (A^n_s)_{s\in[t,T]}$ be the process defined by $A^n_s = X^n_s - \gamma_s^{-1}D^n_s$, $s \in [t,T]$. Then it holds by Lemma 5.2.4 and (5.44) that $\widehat{A}^n$ and A^n satisfy the same dynamics. Furthermore, they start in the same point $\widehat{A}^n_t = x - \frac{d}{\gamma_t} = A^n_t$ at time t. Consequently, they are indistinguishable, i.e., almost surely, for all $s \in [t,T]$, it holds that $A^n_s = \widehat{A}^n_s$. This implies that

$$D^n_s = \gamma_s(X^n_s - A^n_s) = \gamma_s(X^n_s - \widehat{A}^n_s) = -\vartheta^n_s\gamma_s\widehat{A}^n_s, \quad s \in [t,T), \tag{5.45}$$

and, proceeding further,

$$D^n_s = -\vartheta^n_s\gamma_s A^n_s = -\vartheta^n_s(\gamma_s X^n_s - D^n_s), \quad s \in [t,T).$$

We thus establish (5.40), while (5.41) follows from (5.45). For the terminal value D^n_T, we have that

$$\begin{aligned} D^n_T &= D^n_{T-} + \gamma_T\Delta X^n_T = D^n_{T-} - \gamma_T X^n_{T-} \\ &= \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,T}\left(-\gamma_T\vartheta^n_{T-} - \gamma_T(1-\vartheta^n_{T-})\right) = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,T}\,(-\gamma_T). \end{aligned}$$

We next show (5.43). It follows from $A^n_s = \widehat{A}^n_s$, $s \in [t,T]$, that

$$\gamma_s^4(X^n_s - \gamma_s^{-1}D^n_s)^8 = \gamma_s^4(x - \gamma_t^{-1}d)^8\left(\mathcal{E}(Q^n)_{t,s}\right)^8, \quad s \in [t,T].$$

Further, it holds for all $s \in [t,T]$ that

$$\begin{aligned}
&\gamma_s^4 \left(\mathcal{E}(Q^n)_{t,s}\right)^8 \\
&= \gamma_t^4 \exp\left(4\int_t^s \mu_r - \frac{1}{2}\sigma_r^2 d[M^{(1)}]_r + 4\int_t^s \sigma_r dM_r^{(1)}\right) \\
&\quad \cdot \exp\Bigg(- 8\int_t^s \vartheta_r^n(\mu_r + \rho_r - \sigma_r^2 - \sigma_r\eta_r\overline{r}_r) + \frac{1}{2}(\vartheta_r^n)^2(\sigma_r^2 + 2\sigma_r\eta_r\overline{r}_r + \eta_r^2) d[M^{(1)}]_r \\
&\qquad\quad - 8\int_t^s \vartheta_r^n(\sigma_r + \eta_r\overline{r}_r) dM_r^{(1)} - 8\int_t^s \vartheta_r^n\eta_r\sqrt{1-\overline{r}_r^2} dM_r^{(2)}\Bigg) \\
&= \gamma_t^4 \exp\left(\int_t^s \nu_r^n d[M^{(1)}]_r + \int_t^s \tau_r^{(1),n} dM_r^{(1)} + \int_t^s \tau_r^{(2),n} dM_r^{(2)}\right),
\end{aligned}$$

where, for all $r \in [t,T]$,

$$\begin{aligned}
\nu_r^n &= 4\mu_r - 2\sigma_r^2 - 8\vartheta_r^n(\mu_r + \rho_r - \sigma_r^2 - \sigma_r\eta_r\overline{r}_r) - 4(\vartheta_r^n)^2(\sigma_r^2 + 2\sigma_r\eta_r\overline{r}_r + \eta_r^2), \\
\tau_r^{(1),n} &= 4\sigma_r - 8\vartheta_r^n(\sigma_r + \eta_r\overline{r}_r), \\
\tau_r^{(2),n} &= -8\vartheta_r^n\eta_r\sqrt{1-\overline{r}_r^2}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&E_t\left[\sup_{s\in[t,T]} \left(\gamma_s^4 (X_s^n - \gamma_s^{-1}D_s^n)^8\right)\right] \\
&= \gamma_t^4 (x - \gamma_t^{-1}d)^8 E_t\left[\sup_{s\in[t,T]} \exp\left(\int_t^s \nu_r^n d[M^{(1)}]_r + \int_t^s \tau_r^{(1),n} dM_r^{(1)} + \int_t^s \tau_r^{(2),n} dM_r^{(2)}\right)\right].
\end{aligned}$$

Since $(\mathbf{C}_{[M^{(1)}]})$ holds and we have (with $c_\mu, c_\rho, c_\sigma, c_\eta$ from $(\mathbf{C_{bdd}})$)

$$\begin{aligned}
|\nu^n| &\le 4c_\mu + 2c_\sigma^2 + 8b\left(c_\mu + c_\rho + c_\sigma^2 + c_\sigma c_\eta\right) + 4b^2\left(c_\sigma^2 + 2c_\sigma c_\eta + c_\eta^2\right), \\
|\tau^{(1),n}| &\le 4c_\sigma + 8b(c_\sigma + c_\eta), \\
|\tau^{(2),n}| &\le 8bc_\eta,
\end{aligned}$$

we obtain (5.43) from Lemma 4.1.1. Observe furthermore that by Jensen's inequality it follows that **(A1)** holds true. $(\mathbf{C}_{[M^{(1)}]})$ and boundedness of σ (respectively, η) then yield **(A2)** (respectively, **(A4)**). Due to (5.40), we have that

$$(\gamma_s^{-\frac{1}{2}}(D_s^n))^4 = (\vartheta_s^n)^4\gamma_s^2(X_s^n - \gamma_s^{-1}D_s^n)^4, \quad s \in [t,T).$$

Since ϑ^n is $\mathcal{D}_{M^{(1)}}$-a.e. bounded, the fact that **(A2)** (respectively, **(A4)**) is satisfied hence already implies that **(A3)** (respectively, **(A5)**) holds true as well. We conclude that $X^n \in \mathcal{A}_t^{\mathrm{sem}}(x,d)$. □

We next prove Theorem 5.2.6. To establish the representation for the value function, we first use Lemma 5.2.9 to obtain an approximating sequence for $\widetilde{\vartheta}$. Subsequently, we employ Lemma 5.2.10 to get an associated sequence of admissible strategies that satisfies helpful properties. We then consider, for these strategies, the representation of the cost functional in Theorem 5.2.1 and show that it tends to $\frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$, which yields an upper bound for the value function. By Theorem 5.2.1, this is also a lower bound.

In order to prove the characterization of existence of an optimal strategy, we first show the direction that existence of an optimal strategy implies existence of a càdlàg semimartingale $\vartheta = (\vartheta_s)_{s\in[0,T]}$ such that $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{M^{(1)}}$-a.e. Main ingredients are the representation of the cost functional (Theorem 5.2.1) and the representation of the value function. We then jointly establish the converse implication and the formula for optimal strategies. Along the way, we also obtain the formula for the associated deviation process. Uniqueness is an immediate consequence of Lemma 5.2.5.

Proof of Theorem 5.2.6. We follow the structure outlined above.

Representation for the value function. Let $t \in [0,T]$ and $x, d \in \mathbb{R}$. Since $\widetilde{\vartheta}$ is $\mathcal{D}_{M^{(1)}}$-a.e. bounded and we assume $(\mathbf{C}_{[M^{(1)}]})$, it follows from Lemma 5.2.9 that there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of càdlàg semimartingales $\vartheta^n = (\vartheta^n_s)_{s\in[0,T]}$ that are $\mathcal{D}_{M^{(1)}}$-a.e. bounded uniformly in n and such that for all $p \in [1,\infty)$ it holds that

$$E_t\left[\int_t^T |\widetilde{\vartheta}_s - \vartheta^n_s|^p d[M^{(1)}]_s\right] \to 0 \quad \text{in } L^1(\Omega,\mathcal{F},P) \text{ as } n\to\infty. \tag{5.46}$$

In particular, by passing to a suitable subsequence, we can obtain almost sure convergence in (5.46). We further obtain from Lemma 5.2.10 that for each $n\in\mathbb{N}$ there exists $X^n \in \mathcal{A}^{\text{sem}}_t(x,d)$ such that $D^n_s = -\vartheta^n_s(\gamma_s X^n_s - D^n_s)$, $s\in[t,T)$, and that

$$\sup_{n\in\mathbb{N}} E_t\left[\sup_{s\in[t,T]} \left(\gamma_s^4 (X^n_s - \gamma_s^{-1} D^n_s)^8\right)\right] < \infty \text{ a.s.} \tag{5.47}$$

It then holds for all $n\in\mathbb{N}$ that

$$\widetilde{\vartheta}_s(\gamma_s X^n_s - D^n_s) + D^n_s = (\widetilde{\vartheta}_s - \vartheta^n_s)(\gamma_s X^n_s - D^n_s), \quad s\in[t,T).$$

Together with Theorem 5.2.1 and $X^n \in \mathcal{A}^{\text{sem}}_t(x,d)$ this implies for all $n\in\mathbb{N}$ that, a.s.,

$$\begin{aligned} V^{\text{sem}}_t(x,d) &\le J^{\text{sem}}_t(x,d,X^n) \\ &= \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t} + E_t\Bigg[\int_t^T \frac{1}{\gamma_s}(\widetilde{\vartheta}_s - \vartheta^n_s)^2(\gamma_s X^n_s - D^n_s)^2 \\ &\qquad \cdot \left((\sigma_s^2 + \eta_s^2 + 2\sigma_s\eta_s\overline{r}_s)Y_s + \kappa_s + \lambda_s\right) d[M^{(1)}]_s\Bigg]. \end{aligned} \tag{5.48}$$

By the Cauchy-Schwarz inequality we have for all $n \in \mathbb{N}$ that

$$\begin{aligned} &E_t\left[\int_t^T \frac{1}{\gamma_s}(\widetilde{\vartheta}_s - \vartheta_s^n)^2(\gamma_s X_s^n - D_s^n)^2 d[M^{(1)}]_s\right] \\ &= E_t\left[\int_t^T \gamma_s(\widetilde{\vartheta}_s - \vartheta_s^n)^2(X_s^n - \gamma_s^{-1}D_s^n)^2 d[M^{(1)}]_s\right] \\ &\leq \left(E_t\left[\int_t^T \gamma_s^2(X_s^n - \gamma_s^{-1}D_s^n)^4 d[M^{(1)}]_s\right]\right)^{\frac{1}{2}}\left(E_t\left[\int_t^T (\widetilde{\vartheta}_s - \vartheta_s^n)^4 d[M^{(1)}]_s\right]\right)^{\frac{1}{2}} \end{aligned} \tag{5.49}$$

and that

$$\begin{aligned} &E_t\left[\int_t^T \gamma_s^2(X_s^n - \gamma_s^{-1}D_s^n)^4 d[M^{(1)}]_s\right] \\ &\leq E_t\left[\sup_{s\in[t,T]}\left(\gamma_s^2(X_s^n - \gamma_s^{-1}D_s^n)^4\right)\left([M^{(1)}]_T - [M^{(1)}]_t\right)\right] \\ &\leq \left(E_t\left[\sup_{s\in[t,T]}\left(\gamma_s^4(X_s^n - \gamma_s^{-1}D_s^n)^8\right)\right]\right)^{\frac{1}{2}}\left(E_t\left[\left([M^{(1)}]_T - [M^{(1)}]_t\right)^2\right]\right)^{\frac{1}{2}}. \end{aligned} \tag{5.50}$$

Since $\rho, \mu, \sigma, \eta, \lambda, \overline{r}$, and Y are bounded, it follows from $(\mathbf{C}_{[M^{(1)}]})$, (5.47), (5.50), (5.46), and (5.49) that, along a suitable subsequence, the right-hand side of (5.48) tends to $\frac{Y_t}{\gamma_t}(d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t}$ a.s., as $n \to \infty$. We obtain the inequality

$$V_t^{\text{sem}}(x,d) \leq \frac{Y_t}{\gamma_t}(d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} \quad \text{a.s.}$$

The reverse inequality is provided in Theorem 5.2.1.

Existence of an optimal strategy implies existence of ϑ. Let $x \neq \frac{d}{\gamma_0}$. Assume that there exists an optimal strategy $X^* = (X_s^*)_{s\in[0-,T]} \in \mathcal{A}_0^{\text{sem}}(x,d)$. It then follows from $V_0^{\text{sem}}(x,d) = \frac{Y_0}{\gamma_0}(d - \gamma_0 x)^2 - \frac{d^2}{2\gamma_0}$ and Theorem 5.2.1, using also $(\mathbf{C_{>0}})$, that

$$\widetilde{\vartheta}\left(\gamma X^* - D^{X^*}\right) + D^{X^*} = 0 \quad \mathcal{D}_{M^{(1)}}\text{-a.e.} \tag{5.51}$$

Let $A^* = (A_s^*)_{s\in[0,T]}$ be defined by $A_s^* = X_s^* - \gamma_s^{-1}D_s^{X^*}$, $s \in [0,T]$. We have by (5.51) that $A^* - X^* = \widetilde{\vartheta}A^*$ $\mathcal{D}_{M^{(1)}}$-a.e. This, Lemma 5.2.4, and $[M^{(1)}] = [M^{(2)}]$ then yield that

$$dA_s^* = \widetilde{\vartheta}_s A_s^*\left(\gamma_s d\gamma_s^{-1} - dR_s - \gamma_s d[\gamma^{-1}, R]_s\right), \quad s \in [0,T].$$

Define $\widetilde{Q} = (\widetilde{Q}_s)_{s\in[0,T]}$ by

$$\widetilde{Q}_s = \int_0^s \widetilde{\vartheta}_r \gamma_r d\gamma_r^{-1} - \int_0^s \widetilde{\vartheta}_r dR_r - \int_0^s \widetilde{\vartheta}_r \gamma_r d[\gamma^{-1}, R]_r, \quad s \in [0,T].$$

Since $\widetilde{Q}$ is a continuous semimartingale, its stochastic exponential $\mathcal{E}(\widetilde{Q})$ is strictly positive. From

$$A_s^* = \left(x - \frac{d}{\gamma_0}\right) \mathcal{E}(\widetilde{Q})_s, \quad s \in [0,T],$$

and the assumption $x \neq \frac{d}{\gamma_0}$ we thus conclude that A^* is nonvanishing. Consequently,

$$\vartheta = -\frac{D^{X^*}}{\gamma A^*}$$

defines a càdlàg semimartingale. By (5.51) and definition of A^* we have that $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{M^{(1)}}$-a.e.

Existence of ϑ implies that the formulas in part (iii) define a unique optimal strategy. Suppose that there exists a càdlàg semimartingale $\vartheta = (\vartheta_s)_{s\in[0,T]}$ such that $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{M^{(1)}}$-a.e., and let $t \in [0,T]$, $x, d \in \mathbb{R}$. It then follows from Lemma 5.2.10 that (5.36) defines a strategy $X^* \in \mathcal{A}_t^{\text{sem}}(x,d)$ such that D^{X^*} has representation (5.37) and, moreover, $D^{X^*} = -\vartheta(\gamma X^* - D^{X^*}) = -\widetilde{\vartheta}(\gamma X^* - D^{X^*})$ $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-a.e. Then Theorem 5.2.1 implies that $J_t^{\text{sem}}(x,d,X^*) = \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$, and since $V_t^{\text{sem}}(x,d) = \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$, the strategy X^* is optimal. The uniqueness up to $\mathcal{D}_{M^{(1)}}|_{[t,T]}$-null sets follows from $V_t^{\text{sem}}(x,d) = \frac{Y_t}{\gamma_t}(d-\gamma_t x)^2 - \frac{d^2}{2\gamma_t}$ and Lemma 5.2.5. □

We next show consistency of the optimal strategy and its deviation.

Proof of Corollary 5.2.7. Notice that the process $X_s^* - \gamma_s^{-1} D_s^{X^*}$, $s \in [t,T]$, is continuous (see also Lemma 5.2.4). Together with (5.36) and (5.37) this yields that, for any $r \in (t,T)$, we have that

$$X_{r-}^* - \gamma_r^{-1} D_{r-}^{X^*} = X_r^* - \gamma_r^{-1} D_r^{X^*} = \left(x - \frac{d}{\gamma_t}\right) \mathcal{E}(Q)_{t,r}. \tag{5.52}$$

Moreover, it holds for all $r \in (t,T)$ and $s \in [r,T]$ that $\mathcal{E}(Q)_{t,r}\,\mathcal{E}(Q)_{r,s} = \mathcal{E}(Q)_{t,s}$. We therefore obtain for all $r \in (t,T)$ and $s \in [r,T]$ that

$$\left(X_{r-}^* - \gamma_r^{-1} D_{r-}^{X^*}\right) \mathcal{E}(Q)_{r,s} = \left(x - \frac{d}{\gamma_t}\right) \mathcal{E}(Q)_{t,s}.$$

The statements of the corollary now follow from the definitions of X^* and D^{X^*} (see part (iii) of Theorem 5.2.6). □

In the final proof of this section, we use Theorem 5.2.6 to obtain the uniqueness result Corollary 5.2.8 for BSDE (4.1).

Proof of Corollary 5.2.8. The assumptions allow us to apply part (i) of Theorem 5.2.6 to both solutions of BSDE (4.1). This yields that

$$\gamma_t Y_t = V_t^{\mathrm{sem}}(1,0) = \gamma_t \hat{Y}_t, \quad t \in [0,T].$$

Since γ is a strictly positive process, this implies that Y and $\hat{Y}$ are indistinguishable. The claim now follows from Lemma 4.1.6. □

5.3 Optimal strategies of infinite variation

In the optimization problem of Section 5.1.1, indeed, strategies of infinite variation can come out. We illustrate this by examples.

Example 5.3.1. Let $m = 2$ and assume that $(M^{(1)}, M^{(2)})^\top = (W^{(1)}, W^{(2)})^\top = W$ is a two-dimensional Brownian motion and $\mathcal{F}_s = \mathcal{F}_s^W$ for all $s \in [0,T]$. Let $\lambda \equiv 0$ and $\mu \equiv 0$. Suppose that $\overline{r} \in [-1,1]$ and $\eta, \rho, \sigma \in \mathbb{R}$ are deterministic constants such that $\kappa = \frac{1}{2}(2\rho - \sigma^2 - \eta^2 - 2\sigma\eta\overline{r}) > 0$ and $\sigma^2 + \eta^2 + 2\sigma\eta\overline{r} > 0$. In particular, we thus need $\rho > 0$. Moreover, notice that σ and η in the current setting can not both be zero[3], but if $\overline{r} \neq -1$ (respectively, $\overline{r} \neq 1$), then σ and η (respectively, $-\eta$) are allowed to take the same nonzero value. Let $t = 0$ and $x, d \in \mathbb{R}$ with $x \neq \frac{d}{\gamma_0}$.

We verify that Theorem 5.2.6 applies and present explicit formulas for the optimal strategy X^* in $\mathcal{A}_0^{\mathrm{sem}}(x,d)$ and the associated deviation process D^{X^*}. Observe that, in the current setting, $(\mathbf{C}_{\geq\varepsilon})$, $(\mathbf{C_{bdd}})$, and $(\mathbf{C}_{[M^{(1)}]})$ are satisfied. BSDE (4.1) takes the form (cf. Remark 4.0.2)

$$\begin{aligned} dY_s &= \left(\frac{\left(\rho Y_s + (\sigma + \eta\overline{r})Z_s^{(1)} + \eta\sqrt{1-\overline{r}^2}Z_s^{(2)}\right)^2}{(\sigma^2+\eta^2+2\sigma\eta\overline{r})(Y_s \vee 0) + \kappa} - \sigma Z_s^{(1)} \right) ds \\ &\quad + Z_s^{(1)} dW_s^{(1)} + Z_s^{(2)} dW_s^{(2)}, \quad s \in [0,T], \\ Y_T &= \frac{1}{2}, \end{aligned} \tag{5.53}$$

and by Proposition 4.3.2 has a unique solution $(Y, Z, 0)$. By solving the ODE corresponding to (5.53) (i.e., (5.53) with $Z \equiv 0$), we obtain that $Z \equiv 0$ and

$$Y_s = \frac{\kappa}{\sigma^2+\eta^2+2\sigma\eta\overline{r}} \mathcal{W}\left(\frac{\kappa}{\sigma^2+\eta^2+2\sigma\eta\overline{r}} \exp\left(c_T - \frac{\rho^2 s}{\sigma^2+\eta^2+2\sigma\eta\overline{r}} \right) \right)^{-1}, \quad s \in [0,T], \tag{5.54}$$

where $\mathcal{W}$ denotes the Lambert W function and

$$c_T = \ln(2) + \frac{2\kappa + \rho^2 T}{\sigma^2+\eta^2+2\sigma\eta\overline{r}}.$$

[3] The case $\eta = 0 = \sigma$ corresponds to the setting for the classical Obizhaeva-Wang model and is covered in Section 5.4.2.

We further have that

$$\widetilde{\vartheta}_s = \frac{\rho Y_s}{(\sigma^2+\eta^2+2\sigma\eta\overline{r})Y_s+\kappa}, \quad s\in[0,T].$$

We can show that Y and $\widetilde{\vartheta}$ both are continuous, deterministic, increasing, $(0,1/2]$-valued functions of finite variation. In particular, we have that $\widetilde{\vartheta}$ is bounded and a càdlàg semimartingale. Hence, Theorem 5.2.6 applies, and the optimal strategy $X^*=(X^*_s)_{s\in[0-,T]}\in\mathcal{A}_0^{\mathrm{sem}}(x,d)$ and its associated deviation process $D^{X^*}=(D^{X^*}_s)_{s\in[0-,T]}$ are given by the formulas $X^*_{0-}=x$, $D^{X^*}_{0-}=d$,

$$\begin{aligned} X^*_s &= \left(x-\frac{d}{\gamma_0}\right)\mathcal{E}(Q)_s\,(1-\widetilde{\vartheta}_s), \quad s\in[0,T),\\ D^{X^*}_s &= \left(x-\frac{d}{\gamma_0}\right)\mathcal{E}(Q)_s\,(-\gamma_s\widetilde{\vartheta}_s), \quad s\in[0,T),\end{aligned}$$

and $X^*_T=0$, $D^{X^*}_T=(x-\frac{d}{\gamma_0})\mathcal{E}(Q)_T\,(-\gamma_T)$, where

$$\begin{aligned}\mathcal{E}(Q)_s &= \exp\left(-(\rho-\sigma^2-\sigma\eta\overline{r})\int_0^s\widetilde{\vartheta}_r dr-\frac{\sigma^2+\eta^2+2\sigma\eta\overline{r}}{2}\int_0^s\widetilde{\vartheta}_r^2 dr\right)\\ &\quad\cdot\exp\left(-(\sigma+\eta\overline{r})\int_0^s\widetilde{\vartheta}_r dW_r^{(1)}-\eta\sqrt{1-\overline{r}^2}\int_0^s\widetilde{\vartheta} dW_r^{(2)}\right), \quad s\in[0,T].\end{aligned}$$

With the help of these representations, we discuss some properties of the optimal strategy. As is typical for optimal strategies in Obizhaeva-Wang type models, X^* has jumps at initial time 0 and terminal time T and is continuous on $(0,T)$. Since $1-\widetilde{\vartheta}$ is positive, X^* has the same sign as $x-\frac{d}{\gamma_0}$ on $(0,T]$. In contrast to the basic Obizhaeva-Wang model (see the case $\sigma\equiv 0\equiv\eta$ in Section 5.4.2), the associated deviation process D^{X^*} is no longer constant on $(0,T)$. Further, as $1-\widetilde{\vartheta}$ is nonvanishing and has finite variation on $[0,T]$, while $\mathcal{E}(Q)$, almost surely, has infinite variation on all subintervals of $[0,T]$, we get that X^*, almost surely, has infinite variation on all subintervals of $[0,T]$. In particular, X^* is in no way monotone on any subinterval of $[0,T]$. The optimal strategy and its associated deviation for a particular choice of the parameters are visualized in Figure 5.1.

We moreover remark that in the current example, all input processes and Y and $\widetilde{\vartheta}$ are deterministic, whereas the optimal strategy and its associated deviation (as well as γ and/or R) are truly stochastic due to nonzero σ and/or nonzero η.

Finally, we point out that the subsetting where $\eta=0$ and $\sigma>0$ (respectively, where $\sigma=0$, $\overline{r}=1$, and $\eta>0$) corresponds to the setting in Example 5.1.6 (respectively, Example 5.1.5), and that now, with the right dynamics for the deviation (respectively, the right cost functional), we were able to solve the optimization problem.

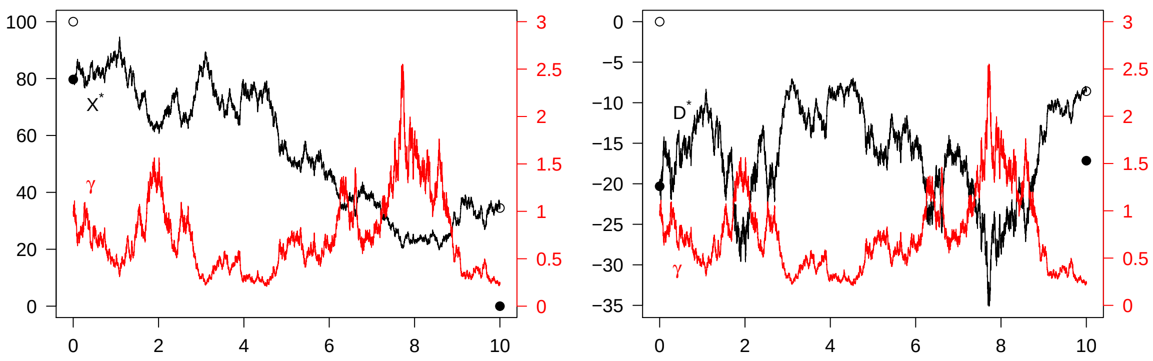


Figure 5.1: Left: A simulation of the optimal strategy X^* (black) and the price impact γ (red) in the setting of Example 5.3.1 for $T = 10$, $x = 100$, $d = 0$, $\gamma_0 = 1$, $\rho = 0.5$, $\eta = 0$, and $\sigma = 0.8$. Note the difference in scales. Right: The associated deviation process $D^* = D^{X^*}$ (black) and the price impact γ (red) for the same situation.

Observe that the price impact process γ in the situation of Example 5.3.1 is given by $\gamma_s = \gamma_0 \exp(\sigma W_s^{(1)} - \frac{\sigma^2}{2} s)$, $s \in [0, T]$, and hence for $\sigma \neq 0$ has infinite variation. Thus, the observation in Example 5.3.1 is in accordance with one of our motivations to include strategies of infinite variation: oscillations of the price impact are reflected in a similarly rough behavior of the optimal strategy (see also Figure 5.1).

It is not surprising that not only the diffusion term in the price impact γ, but also the diffusion term in the resilience R can lead to infinite variation of the optimal strategy.

However, we find that we even do not need infinite variation in the price impact γ nor in the resilience R to obtain strategies of infinite variation that are optimal. E.g., in the next Example 5.3.3, we can choose a smooth price impact process γ, while at the same time ρ is constant and $\eta \equiv 0$, and nevertheless it is optimal to trade with infinite variation.

Before we turn to Example 5.3.3, we first prepare the setting upon which Example 5.3.3, Section 5.4.1, and Section 5.4.3 are based.

Remark 5.3.2. Consider the following set-up. Let $m = 2$, assume that $(M^{(1)}, M^{(2)})^\top = (W^{(1)}, W^{(2)})^\top = W$ is a two-dimensional Brownian motion, and that $(\mathcal{F}_s)_{s\in[0,T]} = (\mathcal{F}_s^W)_{s\in[0,T]}$. Let $t = 0$, $x, d \in \mathbb{R}$ with $x \neq \frac{d}{\gamma_0}$ (for the case $x = \frac{d}{\gamma_0}$, see Lemma 5.2.2). Suppose that $\lambda \equiv 0$. The resilience is taken to be exponential (i.e., $\eta \equiv 0$) with deterministic constant resilience coefficient $\rho \in \mathbb{R} \setminus \{0\}$ (for the case $\rho = 0$, see Proposition 5.2.3). We consider the price impact γ from (3.2) with $\sigma \equiv 0$, i.e., $\gamma_s = \gamma_0 \exp(\int_0^s \mu_r dr)$, $s \in [0, T]$. In particular, γ is continuous and of finite variation. We assume that there exist deterministic constants $\varepsilon, \overline{\mu} \in (0, \infty)$ such that

$$\rho + \frac{\mu}{2} \geq \varepsilon \quad \mathcal{D}_{W^{(1)}}\text{-a.e.} \quad \text{and} \quad \mu \leq \overline{\mu} \quad \mathcal{D}_{W^{(1)}}\text{-a.e.} \tag{5.55}$$

Note that this implies boundedness of μ, and we conclude that $(\mathbf{C}_{\geq\varepsilon})$ and $(\mathbf{C}_{\mathbf{bdd}})$ hold. Our current set-up is a special case of the settings considered in Section 4.2 and Section 4.3. Therefore, it follows from Proposition 4.2.1 (alternatively, from Proposition 4.3.2) that there exists a unique solution $(Y, Z, M^{\perp})$ of BSDE (4.1). We notice that $M^{\perp} \equiv 0$ in our current set-up (cf. Remark 4.0.2(ii)). For the process $\widetilde{\vartheta}$ defined in (5.22) we obtain that

$$\widetilde{\vartheta}_s = \frac{\rho + \mu_s}{2\rho + \mu_s} 2Y_s = \left(1 - \frac{\rho}{2\rho + \mu_s}\right) 2Y_s, \quad s \in [0,T]. \tag{5.56}$$

Notice that, by (5.55), $\widetilde{\vartheta}$ is bounded. That is, in our current set-up, including (5.55), the assumptions of Theorem 5.2.6 are satisfied. Depending on the choice of μ, we have to distinguish between the following two situations.

Situation 1: There exists a càdlàg semimartingale $\vartheta = (\vartheta_s)_{s\in[0,T]}$ such that

$$\widetilde{\vartheta} = \vartheta \quad \mathcal{D}_{W^{(1)}}\text{-a.e.} \tag{5.57}$$

Situation 2: There is no càdlàg semimartingale ϑ such that (5.57) is satisfied.

As we know from Theorem 5.2.6, in Situation 1 there exists a unique (up to $\mathcal{D}_{W^{(1)}}$-null sets) optimal strategy $X^* = (X^*_s)_{s\in[0-,T]} \in \mathcal{A}_0^{\text{sem}}(x,d)$, and it is given by the formulas $X^*_{0-} = x$, $X^*_T = 0$, and

$$X^*_s = \left(x - \frac{d}{\gamma_0}\right) \exp\left(-\int_0^s \vartheta_r(\mu_r + \rho)\, dr\right)(1 - \vartheta_s), \quad s \in [0,T), \tag{5.58}$$

whereas in Situation 2 there does not exist an optimal strategy.

Example 5.3.3. Consider the setting of Remark 5.3.2. To obtain an optimal strategy of infinite variation in Situation 1 of Remark 5.3.2, note that by (5.58), we should construct ϑ of infinite variation. To this end, let μ be a continuous process of finite variation satisfying (5.55) such that

$$\text{a.s. the function } s \mapsto \rho + \mu_s \text{ is nonvanishing on } [0,T]. \tag{5.59}$$

Observe that for a fixed $\omega \in \Omega$, the unique solution to the Bernoulli ODE

$$d\overline{Y}_s(\omega) = \left(\frac{2\,(\rho + \mu_s(\omega))^2\, \overline{Y}_s(\omega)^2}{2\rho + \mu_s(\omega)} - \mu_s(\omega)\overline{Y}_s(\omega)\right) ds, \quad s \in [0,T], \quad \overline{Y}_T(\omega) = \frac{1}{2},$$

is given by the formula

$$\overline{Y}_s(\omega) = e^{\int_s^T \mu_r(\omega)dr} \left(\int_s^T \frac{2\,(\rho + \mu_r(\omega))^2}{2\rho + \mu_r(\omega)} e^{\int_r^T \mu_u(\omega)du} dr + 2\right)^{-1}, \quad s \in [0,T]. \tag{5.60}$$

It follows that it is possible to choose μ such that $\overline{Y}$ is not adapted. Choosing μ in such a way we conclude that the solution $(Y, Z, M^{\perp} \equiv 0)$ of BSDE (4.1) satisfies

$$\mathcal{D}_{W^{(1)}}(Z \neq 0) > 0.$$

This yields that, with positive probability, Y has infinite variation on $[0, T]$. Define

$$\varphi_s = \frac{2(\rho + \mu_s)}{2\rho + \mu_s}, \quad s \in [0, T],$$

which is a nonvanishing (recall (5.59)) continuous process of finite variation. Hence, $\widetilde{\vartheta} = \varphi Y$ (cf. (5.56)) is a continuous semimartingale that, with positive probability, has infinite variation on $[0, T]$. Thus, we are in Situation 1 of Remark 5.3.2 with $\vartheta \equiv \widetilde{\vartheta}$, and the optimal strategy X^*, which is given by (5.58), has, with positive probability, infinite variation on $[0, T]$.

Loosely speaking, infinite variation of the optimal strategy in Example 5.3.3 is due to the incoming information that is reflected in the process Y of the BSDE. We achieved this via our choice of μ as a certain stochastic process (of finite variation). Another possibility, where we can argue similar to Example 5.3.3, is to choose the resilience coefficient ρ (while $\eta \equiv 0$) as an appropriate stochastic process (of finite variation and strictly positive). This is the content of Example 5.3.4. Note that in the setting of Example 5.3.4 the price impact process γ is just a constant. Therefore, Example 5.3.4 is closer to the work [HK21], where, in a related model with a constant deterministic transient price impact coefficient and a time-varying, strictly positive stochastic resilience coefficient, optimal strategies of infinite variation emerge when an instantaneous price impact factor tends to 0.

Example 5.3.4. Let $m = 2$, assume that $(M^{(1)}, M^{(2)})^{\top} = (W^{(1)}, W^{(2)})^{\top} = W$ is a two-dimensional Brownian motion, and suppose that $\mathcal{F}_s = \mathcal{F}_s^W$ for all $s \in [0, T]$. Let $t = 0$, $x, d \in \mathbb{R}$ with $x \neq \frac{d}{\gamma_0}$. Moreover, set $\lambda \equiv 0$, $\eta \equiv 0$, $\sigma \equiv 0$, and $\mu \equiv 0$. The resilience coefficient ρ is assumed to be a continuous process of finite variation such that there exists $\varepsilon, c_\rho \in (0, \infty)$ with $\varepsilon \leq \rho \leq c_\rho$ $\mathcal{D}_{W^{(1)}}$-a.e. Then, $(\mathbf{C}_{\geq \varepsilon})$ and $(\mathbf{C}_{\mathbf{bdd}})$ are satisfied. Again, we are in the settings of Section 4.2 and Section 4.3. Thus, as in Remark 5.3.2, there exists a unique solution $(Y, Z, M^{\perp})$ of BSDE (4.1), and $M^{\perp} \equiv 0$. We further obtain that $\widetilde{\vartheta} \equiv Y$. Clearly, $\widetilde{\vartheta}$ is a bounded, continuous semimartingale. By Theorem 5.2.6, there exists a unique (up to $\mathcal{D}_{W^{(1)}}$-null sets) optimal strategy $X^* = (X_s^*)_{s \in [0-, T]} \in \mathcal{A}_0^{\text{sem}}(x, d)$. The optimal strategy is given by the formulas $X_{0-}^* = x$, $X_T^* = 0$, and $X_s^* = (x - \frac{d}{\gamma_0}) \exp(-\int_0^s Y_r \rho_r \, dr)(1 - Y_s)$, $s \in [0, T)$. For fixed $\omega \in \Omega$, consider the ODE

$$d\overline{Y}_s(\omega) = \rho_s(\omega) \overline{Y}_s(\omega)^2 ds, \quad s \in [0, T], \quad \overline{Y}_T(\omega) = \frac{1}{2},$$

which has the unique solution

$$\overline{Y}_s(\omega) = \left(\int_s^T \rho_r(\omega) dr + 2 \right)^{-1}, \quad s \in [0,T].$$

This shows that we can choose ρ such that $\overline{Y}$ is not adapted. Hence, for such a process ρ, we have that $\mathcal{D}_{W^{(1)}}(Z \neq 0) > 0$. This leads to Y, with positive probability, having infinite variation on $[0,T]$. The same then holds true for X^*.

5.4 Further examples

We here present three more examples.

In Section 5.4.1 we examine a situation where the conditions of Theorem 5.2.6 are satisfied (in particular, the value function is finite), but where a minimizer of J^{sem} within the set $\mathcal{A}_t^{\text{sem}}(x,d)$ of semimartingale strategies does not exist. Together with Section 5.3, this indicates that it is worthwhile to include infinite-variation strategies into the optimization problem as done in this chapter, but that the class of semimartingale strategies considered is not suitable to always find an optimal strategy. We also refer to the discussion in Chapter 9 and to [AKU22a, Section 4.2].

In Section 5.4.2 we observe that infinite variation of the price impact γ and infinite variation of the resilience R may cancel out such that the optimal strategy has finite variation. A particular subsetting (where $\eta \equiv 0 \equiv \sigma$) of the setting in Section 5.4.2 corresponds to the setting for the classical Obizhaeva-Wang model and, moreover, to the setting in Example 5.1.4. As a by-product, Section 5.4.2 shows that we recover the optimal strategy of [OW13, Proposition 3] (although we consider the different, in some sense more general, optimal control problem of Section 5.1.1).

In Section 5.4.3 we illustrate that optimal strategies may also have block trades, i.e., jumps, inside the time interval available for trading. Note that this effect can also be observed in examples in the next Chapter 6 (there, the jumps are due to jumps of ρ, whereas here, the jumps are produced by jumps of μ).

5.4.1 An example where the semimartingale problem does not admit a minimizer

Consider the setting of Remark 5.3.2 and choose any deterministic càdlàg (hence, in particular bounded) function μ such that there exists $\delta \in (0,T)$ with μ having infinite variation on $[0, T-\delta]$. For instance, we could take μ to be the Weierstrass function, or the function $s \mapsto (s \sin \frac{1}{s}) 1_{(0,T]}(s)$, $s \in [0,T]$. We also take $\rho \in \mathbb{R} \setminus \{0\}$ such that (5.55) is satisfied.

Notice that, in this deterministic framework, the process Y is a deterministic continuous function of finite variation explicitly given by (5.60). In particular, Y is non-vanishing.

We now prove that we are in Situation 2 of Remark 5.3.2. To this end, assume by contradiction that there exists a càdlàg semimartingale $\vartheta = (\vartheta_s)_{s\in[0,T]}$ such that $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{W^{(1)}}$-a.e. (ϑ can be stochastic). Then it follows from (5.56) and the fact that Y is nonvanishing that

$$\frac{\rho}{2\rho+\mu} = 1 - \frac{\vartheta}{2Y} \quad \mathcal{D}_{W^{(1)}}\text{-a.e.} \tag{5.61}$$

Set $S = 1 - \frac{\vartheta}{2Y}$ and notice that it is a càdlàg semimartingale. As both sides in (5.61) are càdlàg, they are even indistinguishable on $[0,T)$, i.e., almost surely, it holds that

$$\frac{\rho}{2\rho+\mu_r} = S_r, \quad r \in [0,T). \tag{5.62}$$

Hence, $S \neq 0$ and $S_- \neq 0$ on $[0,T)$, which implies that $\frac{1}{S}$ is also a semimartingale on $[0,T)$. Now (5.62) yields that, almost surely,

$$\mu_r = \frac{\rho}{S_r} - 2\rho, \quad r \in [0,T).$$

Thus, μ is itself a semimartingale on $[0,T)$. As μ is deterministic, this means that μ has finite variation on each compact subinterval of $[0,T)$, in particular, on $[0,T-\delta]$. The obtained contradiction proves that we are in Situation 2.

This example thus shows that an optimal strategy can fail to exist even when the value function is finite.

5.4.2 Cancellation of infinite variation

Let $m=2$ and assume that $(M^{(1)}, M^{(2)})^\top = (W^{(1)}, W^{(2)})^\top = W$ is a two-dimensional Brownian motion and $\mathcal{F}_s = \mathcal{F}_s^W$ for all $s \in [0,T]$. Fix $t=0$ and $x, d \in \mathbb{R}$ with $x \neq \frac{d}{\gamma_0}$. Let $\lambda \equiv 0$ and $\mu \equiv 0$. Suppose that $\overline{r} = -1$ and $\rho > 0$ are deterministic constants, and that η and σ are progressively measurable, $\mathcal{D}_{W^{(1)}}$-a.e. bounded processes such that $\eta = \sigma$ $\mathcal{D}_{W^{(1)}}$-a.e. It then holds $\mathcal{D}_{W^{(1)}}$-a.e. that $\sigma^2 + \eta^2 + 2\sigma\eta\overline{r} = 0$ and $\kappa = \rho > 0$.

Note that $(\mathbf{C}_{\geq\varepsilon})$ and $(\mathbf{C}_{\mathbf{bdd}})$ are satisfied. By Proposition 4.3.2, BSDE (4.1), which here becomes (cf. Remark 4.0.2)

$$dY_s = (\rho Y_s^2 - \sigma_s Z_s^{(1)})ds + Z_s^{(1)} dW_s^{(1)} + Z_s^{(2)} dW_s^{(2)}, \quad s \in [0,T], \quad Y_T = \frac{1}{2},$$

has a unique solution $(Y, Z, 0)$. We find that $Z \equiv 0$ and

$$Y_s = \frac{1}{2 + (T-s)\rho}, \quad s \in [0,T].$$

It then holds that $\widetilde{\vartheta} \equiv Y$. Observe that $\widetilde{\vartheta}$ is a continuous, deterministic, increasing, $(0, 1/2]$-valued function of finite variation. From Theorem 5.2.6 we obtain the existence of a unique optimal strategy $X^* \in \mathcal{A}_0^{\text{sem}}(x,d)$, and that the optimal strategy $X^* = (X^*_s)_{s\in[0-,T]}$ is given by the formulas

$$\begin{aligned} X^*_{0-} &= x, \quad X^*_T = 0, \\ X^*_s &= \left(x - \frac{d}{\gamma_0}\right) \exp\left(-\int_0^s \frac{\rho}{2+(T-r)\rho}\,dr\right) \frac{1+(T-s)\rho}{2+(T-s)\rho} \\ &= \left(x - \frac{d}{\gamma_0}\right) \frac{1+(T-s)\rho}{2+T\rho}, \quad s \in [0,T). \end{aligned} \tag{5.63}$$

Moreover, for the associated deviation process $D^{X^*} = (D_s^{X^*})_{s\in[0-,T]}$ it holds that

$$\begin{aligned} D_{0-}^{X^*} &= d, \quad D_T^{X^*} = -\gamma_0 \left(x - \frac{d}{\gamma_0}\right) \frac{2}{2+T\rho} \exp\left(\int_0^T \eta_r dW_r^{(1)} - \frac{1}{2}\int_0^T \eta_r^2 dr\right), \\ D_s^{X^*} &= -\gamma_0 \left(x - \frac{d}{\gamma_0}\right) \frac{1}{2+T\rho} \exp\left(\int_0^s \eta_r dW_r^{(1)} - \frac{1}{2}\int_0^s \eta_r^2 dr\right), \quad s \in [0,T). \end{aligned} \tag{5.64}$$

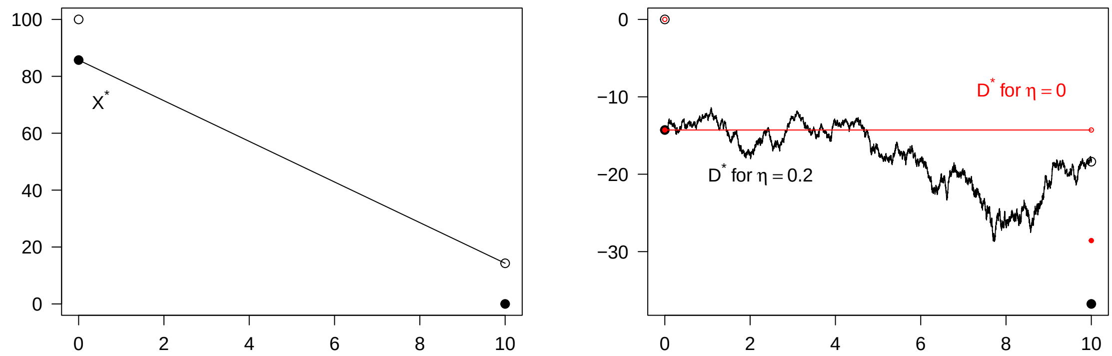


Figure 5.2: Left: The optimal strategy X^* in the setting of Section 5.4.2 for $T = 10$, $x = 100$, $d = 0$, $\gamma_0 = 1$, and $\rho = 0.5$. Right: The associated deviation process $D^* = D^{X^*}$ (red) in the Obizhaeva-Wang case $\sigma = \eta = 0$ and a path of the associated deviation process $D^* = D^{X^*}$ (black) in the case $\sigma = \eta = 0.2$.

We first discuss the case $\sigma = \eta = 0$. In the context of optimal trade execution in a limit order book model, this setting ($\gamma = \gamma_0 > 0$ a deterministic constant and $dR_s = \rho ds$, $s \in [0,T]$, for a deterministic constant $\rho > 0$) is considered in the pioneering work [OW13], and the optimal strategy X^* of (5.63) (for $d = 0$) appears in [OW13, Proposition 3], where the cost functional $\widetilde{J}$ of (5.8) is minimized over a set of strategies which, in particular, have finite variation. We stress again that we obtain optimality of (5.63) in this setting as a result of a different optimization problem (minimization of the cost functional J^{sem} of (5.2) over semimartingale strategies). Notice that the optimal strategy X^* of (5.63) is deterministic, has jumps at times 0 and T (i.e., block trades at the beginning and at the end) and is continuous on $(0, T)$. It is worth noting that the associated deviation process D^{X^*} is constant on $(0, T)$ (but, clearly, has jumps at times 0 and T). In the case $d = 0$ the strategy X^* is monotone. In general, the strategy is monotone only on $(0, T]$. Global monotonicity can fail because of the block trade in the beginning (the size of the block trade depends not only on x but also on d).

Suppose now that $\sigma = \eta$ is nonvanishing. We point out that, with general stochastic $\sigma = \eta$ and negative correlation $\overline{r} = -1$, we still have the same optimal strategy as in the Obizhaeva-Wang case. In particular, the optimal strategy is deterministic and of finite variation, although now the price impact γ and the resilience R are both stochastic and of infinite variation. In some sense, the infinite variation in the price impact process γ is "canceled" by the infinite variation in the resilience process R. While the optimal strategies in the Obizhaeva-Wang case and for general stochastic $\sigma = \eta$ with correlation $\overline{r} = -1$ coincide, this is not true for the associated deviation processes. In contrast to the constant deviation in the Obizhaeva-Wang case, here D^{X^*} has infinite variation (cf. (5.64); see also Figure 5.2).

5.4.3 Intermediate jump

Consider the setting of Remark 5.3.2. In order to construct an optimal strategy with jumps inside $(0, T)$ in Situation 1 of Remark 5.3.2, it is enough to take

a càdlàg semimartingale μ satisfying (5.55) that exhibits jumps in $(0, T)$,

i.e., with positive probability, $\{s \in (0, T) : \Delta\mu_s \neq 0\} \neq \emptyset$, and such that

$$\text{the corresponding process } Y \text{ is nonvanishing.} \tag{5.65}$$

Indeed, in this case, $\widetilde{\vartheta}$ is a càdlàg semimartingale, so we are in Situation 1 of Remark 5.3.2 with $\vartheta \equiv \widetilde{\vartheta}$. Moreover, as Y is continuous and nonvanishing, we readily see from (5.56) that

$$\Delta\mu_s \neq 0 \iff \Delta\widetilde{\vartheta}_s \neq 0,$$

hence the optimal strategy X^*, which is given by (5.58), contains block trades inside $(0, T)$.

We consider a particular example.

Example 5.4.1. To show a specific example of this kind, we take, for some $t_0 \in (0, T)$, a deterministic μ given by the formula $\mu_s = 1_{[t_0,T]}(s)$, $s \in [0, T]$. Observe that (5.55) then is satisfied whenever $\rho > 0$, so we choose some $\rho > 0$. BSDE (4.1) here takes the form (cf. Remark 4.0.2)

$$\begin{aligned} dY_s &= \rho Y_s^2 ds + Z_s^{(1)} dW_s^{(1)} + Z_s^{(2)} dW_s^{(2)}, \quad s \in [0, t_0], \\ dY_s &= \left(\frac{2(\rho+1)^2 Y_s^2}{2\rho+1} - Y_s \right) ds + Z_s^{(1)} dW_s^{(1)} + Z_s^{(2)} dW_s^{(2)}, \quad s \in [t_0, T], \\ Y_T &= \frac{1}{2}, \end{aligned}$$

and its unique solution is given by $(Y, Z \equiv 0, M^\perp \equiv 0)$, where

$$Y_s = \begin{cases} \frac{1}{Y_{t_0}^{-1} + (t_0 - s)\rho}, & s \in [0, t_0), \\ (2\rho+1) \left(2(\rho+1)^2 - 2\rho^2 e^{s-T} \right)^{-1}, & s \in [t_0, T]. \end{cases} \tag{5.66}$$

Notice that Y is deterministic, continuous, strictly increasing, and $(0, 1/2]$-valued. In particular, (5.65) is satisfied, and what is stated after (5.65) applies. Observe that, in this specific example,

$$\vartheta_s = \begin{cases} Y_s, & s \in [0, t_0), \\ Y_s \left(1 + \frac{1}{2\rho+1} \right), & s \in [t_0, T], \end{cases} \tag{5.67}$$

which is a deterministic, strictly increasing, $(0,1)$-valued, càdlàg function with the only jump at time t_0:

$$\Delta \vartheta_{t_0} = \frac{Y_{t_0}}{2\rho+1} > 0.$$

From (5.66) and (5.67) we can compute that

$$\exp\left(- \int_0^s \vartheta_r (\mu_r + \rho) \, dr \right) = \begin{cases} Y_0 Y_s^{-1}, & s \in [0, t_0), \\ e^{t_0 - s} Y_0 Y_s^{-1}, & s \in [t_0, T], \end{cases} \tag{5.68}$$

which, together with (5.66) and (5.67), provides the optimal strategy in closed form (see (5.58)). However, the qualitative structure of the optimal strategy X^*, in fact, follows from (5.58) even without calculating (5.68):

First, X^* is deterministic, and, due to ϑ being strictly increasing and $(0,1)$-valued, X^* is monotone on $(0,T]$. Moreover, the facts that $\vartheta < 1$, $\Delta \vartheta_{t_0} > 0$, and $x \neq \frac{d}{\gamma_0}$ together with (5.58) imply that the optimal strategy necessarily has block trades at the end and at time t_0. Their signs are opposite to the sign of $x - \frac{d}{\gamma_0}$.

Whether or not X^* has a block trade at the beginning depends on the value of the initial deviation d. Namely, X^* has a block trade at the beginning if and only if $x \neq (x - \frac{d}{\gamma_0})(1 - \vartheta_0)$, i.e., if and only if $d \neq -\frac{\vartheta_0}{1-\vartheta_0} \gamma_0 x$.

Likewise, we claim the monotonicity of X^* only on $(0,T]$ because whether or not X^* is monotone on $[0,T]$ also depends on d. More precisely, X^* is monotone on $[0,T]$

if and only if either $x \geq 0$, $d \geq -\frac{\vartheta_0}{1-\vartheta_0}\gamma_0 x$ holds or $x \leq 0$, $d \leq -\frac{\vartheta_0}{1-\vartheta_0}\gamma_0 x$ holds. In particular, if $d = 0$, then X^* is monotone on $[0, T]$.

Between the block trades, the associated deviation process D^{X^*} is constant: It follows from (5.37), (5.67), and (5.68) that

$$D_s^{X^*} = \begin{cases} (d - \gamma_0 x)Y_0, & s \in [0, t_0), \\ (d - \gamma_0 x)Y_0 \left(1 + \frac{1}{2\rho+1}\right), & s \in [t_0, T). \end{cases}$$

Figure 5.3 is an illustration for specific parameter values.

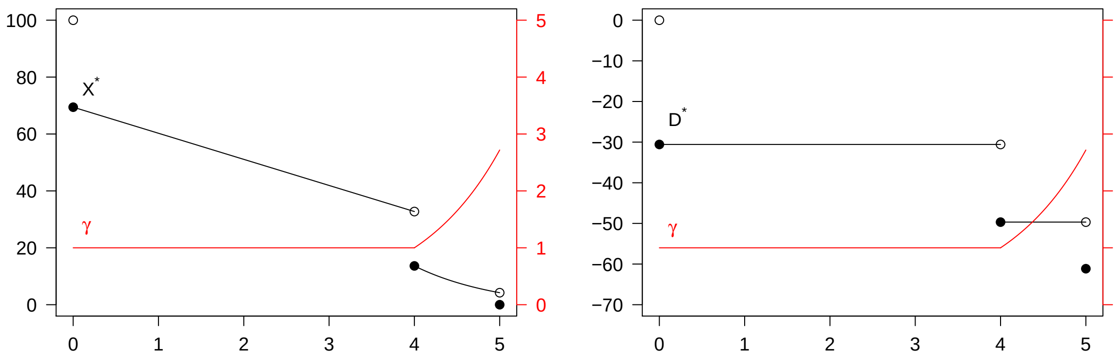


Figure 5.3: Left: The optimal strategy X^* (black) and the price impact γ (red) in the setting of Example 5.4.1 with $\mu_s = 1_{[t_0,T]}(s)$, $s \in [0, T]$, and for $T = 5$, $x = 100$, $d = 0$, $\gamma_0 = 1$, $\rho = 0.3$, and $t_0 = 4$. Note the difference in scales. Right: The associated deviation process $D^* = D^{X^*}$ (black) and the price impact γ (red) for the same situation.

Observe that the reaction of the optimal strategy to changes in the price impact is rather sensitive: here only μ jumps at time t_0 (not the price impact γ itself), but this already causes a jump in X^* at time t_0. Finally, it is worth noting that a model with deterministic time-varying price impact and resilience coefficient was considered in [FSU14, Section 8], but examples of such type are not possible in their framework because the smoothness assumption in [FSU14, Assumption 8.1] excludes the possibility of block trades inside $(0, T)$ (cf. [FSU14, Theorem 8.4]).

6 Negative resilience coefficient

Trade execution models of Obizhaeva-Wang type and related works incorporate some kind of resilience effect. This is often[1] done by having a term $-\rho_s D_s ds$ in the dynamics of the deviation D, i.e, by an exponential resilience $e^{-\int_t^s \rho_r dr}$ described by a resilience coefficient ρ; see, e.g, [OW13, AFS08, AS10, AA14, BF14, FSU14, FSU19], but also articles in the line of [GH17]. This resilience coefficient is typically assumed to be positive. The explanation is that the impact of a trade should decay over time. But a negative resilience coefficient also has a natural interpretation, as it models self-exciting behavior of the price impact, where trading activities of the large investor stimulate other market participants to trade in the same direction. As in [CMK16] and in [FHX22b], we motivate self-exciting price impact by the following reasons. Imagine, for instance, a large trader performing extensive selling. Firstly, a continued selling pressure makes it more and more difficult to find counterparties. Secondly, such an extensive selling by the large trader may trigger stop-loss strategies by other market participants, where they start selling in anticipation of further decrease in the price. Thirdly, extensive selling may also attract predatory traders that employ front-running strategies. In each case, we obtain an increased price impact for subsequent trades.

We point out that there recently appeared several articles on trade execution that, in different ways (often involving Hawkes processes), model self-excitement of the impact of trading on the price (see, e.g., [AB16], [FHX22b], [CJR18], [CMK16]). We propose a negative resilience coefficient as an alternative, simple way of modeling this effect. A “more endogenous” approach is presented by Fu, Horst, and Xia in [FHX22b], who consider liquidation games between several large traders (and the corresponding mean-field limit as well as the single-player subcase) with a self-exciting order flow. There, the large traders’ trading activity triggers child orders, and the strategies come out as Nash equilibria in the game. Despite the differences in the set-up, it is interesting to

[1] There are also works that employ a, typically nonincreasing, decay kernel to model resilience (cf. [GS13, Section 22.4.1]; see also [Gat10], [ASS12], and [GSS12]). The case of constant price impact and resilience coefficient as in [OW13] can be represented by such a decay kernel. In contrast, when the price impact or resilience coefficient are time-varying, this is not covered by the notion of decay kernel in the above-mentioned literature, as also remarked in [FSU14, Remark 8.9].

We further mention that also works such as [BBF18a] include resilience.

observe the following qualitative similarity in the strategies that may result from our approach and from the one in [FHX22b]. In this chapter we, in particular, discuss that, in our framework, it is never optimal to overshoot the execution target whenever the resilience coefficient is positive, but it can be optimal to overshoot the target if we allow the resilience coefficient to take negative values. In other words, in our framework, the possibility to overshoot the target is a qualitative effect of self-excitation via a negative resilience coefficient. In the same vein, in the single-player benchmark model for [FHX22b] without self-excitation, which goes back to [GH17], it is not optimal to overshoot the execution target (this is observed in [HK21, Theorem 2.2]), whereas the resulting strategies in the model with self-excitation in [FHX22b] do sometimes overshoot the target (cf. Figure 1 or Figure 2 in [FHX22b]).

We assume throughout this chapter the framework of Section 3.1 and consider the semimartingale control problem of Section 5.1.1. Furthermore, as we want to focus on the effect of the resilience coefficient ρ taking negative values, we leave aside the diffusion term in the definition of the resilience process R, i.e., we set $\eta \equiv 0$, and we consider a risk-neutral investor, i.e., we set $\lambda \equiv 0$. We moreover assume the setting of Section 4.4, where the local martingales are Brownian motions and the input processes are adapted to a filtration that is orthogonal to the Brownian filtration. For the whole chapter, we also suppose that $(\mathbf{C}_{\geq\varepsilon})$ and $(\mathbf{C}_{\mathbf{bdd}})$ are satisfied, and we fix the initial time $t = 0$.

In Section 6.1 we define and investigate in this framework what we call “overjumping zero” and “premature closure”. Intuitively, overjumping zero is optimal if, at some time, the optimal strategy jumps from a strictly negative position to a strictly positive position, or vice versa. Premature closure is optimal if there is some time point before the end of the trading period when the optimal position already takes the value 0. We complement the theory of Section 6.1 with some case studies in Section 6.2 and Section 6.3. In the latter we study a situation where it is optimal to close the position prematurely, keep it closed during some time interval, and reenter trading again.

This chapter is based on the publication [AKU22b] (joint work with Thomas Kruse and Mikhail Urusov) and in particular contains material of Sections 1, 3.2, 4, and 5 thereof.

6.1 Overjumping zero and premature closure

Recall that, in the present set-up (see the end of the introduction of this chapter), Proposition 4.4.1 ensures existence of a solution $(Y, 0, M^\perp)$ to BSDE (4.1). Fix such a solution $(Y, 0, M^\perp)$. It then holds for the process defined in (5.22) that

$$\widetilde{\vartheta}_s = \frac{(\rho_s + \mu_s)Y_s}{\sigma_s^2 Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma_s^2)}, \quad s \in [0, T]. \tag{6.1}$$

By $(\mathbf{C}_{\geq\varepsilon})$, $(\mathbf{C_{bdd}})$, and the fact that Y is $[0,1/2]$-valued, we have that $\widetilde{\vartheta}$ is $\mathcal{D}_{W^{(1)}}$-a.e. bounded. In particular, Corollary 5.2.8 implies that $(Y,0,M^{\perp})$ is unique (among the solutions of BSDE (4.1) whose second component is 0). Under the condition that

$$\exists \text{ a càdlàg semimartingale } \vartheta \text{ such that } \widetilde{\vartheta} = \vartheta \ \mathcal{D}_{W^{(1)}}\text{-a.e.} \tag{6.2}$$

we obtain from Theorem 5.2.6 for any initial values $x, d \in \mathbb{R}$ (see also Lemma 5.2.2 for the case $x = \frac{d}{\gamma_0}$) the existence of an optimal strategy, which is unique up to $\mathcal{D}_{W^{(1)}}$-null sets. Notice that, in our present context, this is equivalent to uniqueness up to indistinguishability. Indeed, if X^* and X are optimal strategies, then they are indistinguishable, as X^* and X are càdlàg and $X^* = X \ \mathcal{D}_{W^{(1)}}$-a.e. The optimal strategy and its associated deviation (under the condition (6.2)) are given by the formulas (5.36) and (5.37), where, in the present set-up,

$$Q_s = -\int_0^s \vartheta_r \sigma_r dW_r^{(1)} - \int_0^s \vartheta_r(\mu_r + \rho_r - \sigma_r^2)dr, \quad s \in [0,T].$$

We remark that condition (6.2) is in particular guaranteed if ρ, μ, σ are deterministic and of finite variation, as in the examples in Section 6.2 and Section 6.3 below.

In this section we study qualitative effects of a negative resilience coefficient on the optimal strategy. In particular, we examine effects that we call *overjumping zero* and *premature closure*. Roughly speaking, we are interested in market situations where it is optimal to change a buy program into a sell program (or vice versa), or where it is optimal to close the position strictly before the end of the execution period. More precisely, we intend to identify market conditions under which paths of optimal trade execution strategies with positive probability jump over the target level 0 or already take the value 0 prior to T.

To this end recall that under (6.2), given an initial position $x \in \mathbb{R}$ and an initial deviation $d \in \mathbb{R}$, the optimal strategy X^* satisfies

$$X_{0-}^* = x, \quad X_T^* = 0, \quad \text{and} \quad X_s^* = \left(x - \frac{d}{\gamma_0}\right)(1-\vartheta_s)\mathcal{E}(Q)_s, \quad s \in [0,T).$$

This representation allows to disentangle the contributions to the optimal strategy's sign of the initial conditions x and d on the one side and the input processes (recall that, in this chapter, $\lambda \equiv 0$ and $\eta \equiv 0$) ρ, μ, and σ defining the market dynamics on the other side. Indeed, since the stochastic exponential $\mathcal{E}(Q)$ is positive, the sign of X_s^* for $s \in [0,T)$ is determined by the signs of the two factors $(x - \frac{d}{\gamma_0})$ and $(1-\vartheta_s)$.

The first factor $(x - \frac{d}{\gamma_0})$ is determined by the initial conditions, does not depend on time, and thus can only contribute to a change of sign of X^* at time 0. Note that $(x - \frac{d}{\gamma_0})$ has a different sign than the initial condition $X_{0-}^* = x$ if and only if $\gamma_0|x| < \operatorname{sgn}(x)d$. A nonzero initial deviation $d \neq 0$ can thus have the effect that X^* changes its sign directly at time 0. In practice, one would typically assume that $d = 0$, in which case this factor does not contribute to a change of sign.

In the sequel we focus on the contribution of the second factor $(1 - \vartheta)$ and provide definitions of the effects *overjumping zero* and *premature closure* which are only built upon $(1 - \vartheta)$. This factor and hence also these effects are determined by the input processes ρ, μ, and σ driving the market dynamics and are independent of the initial conditions x and d.

For ease of notation, we extend the domain of ϑ to the point $0-$ by setting $\vartheta_{0-} = 0$. In what follows, we denote by π_Ω the projection operator from $\Omega \times [0,T]$ onto Ω (in particular, for $C \in \mathcal{F}_T \otimes \mathcal{B}([0,T])$, $\pi_\Omega(C) = \{\omega \in \Omega \colon \exists s \in [0,T] \text{ s.t.}(\omega, s) \in C\}$).

Definition 6.1.1. Assume that (6.2) holds true. Define

$$\begin{aligned} A_{oj} &= \{(\omega, s) \in \Omega \times [0,T) \colon (1 - \vartheta_{s-}(\omega))(1 - \vartheta_s(\omega)) < 0\}, \\ A_{pc} &= \{(\omega, s) \in \Omega \times [0,T) \colon (1 - \vartheta_{s-}(\omega))(1 - \vartheta_s(\omega)) = 0\}. \end{aligned}$$

(i) We say that *overjumping zero* is optimal in the limit order book model driven by ρ, μ, and σ if $P(\pi_\Omega(A_{oj})) > 0$.

(ii) We say that *premature closure* is optimal if $P(\pi_\Omega(A_{pc})) > 0$.

In relation with Definition 6.1.1 we need to make the following comments.

Remark 6.1.2. (i) $\pi_\Omega(A_{oj}), \pi_\Omega(A_{pc})$ are elements of $\mathcal{F}_T$ by the measurable projection theorem (e.g., [RY99, Theorem I.4.14]): recall that $\mathcal{F}_T$ is complete and notice that, as ϑ is adapted and càdlàg, A_{oj}, A_{pc} are optional sets, and thus in particular $A_{oj}, A_{pc} \in \mathcal{F}_T \otimes \mathcal{B}([0,T])$.

(ii) The terms *overjumping zero* and *premature closure* are well-defined, as ϑ satisfying (6.2) is unique up to indistinguishability.

It is worth noting that the terms *overjumping zero* and *premature closure* could be equivalently defined with the help of stopping times:

Lemma 6.1.3. *Assume that* (6.2) *holds true. Then, overjumping zero (resp., premature closure) is optimal if and only if there exists a stopping time* $\tau \colon \Omega \to [0,T]$ *such that* $P(\tau < T) > 0$ *and*

$$(1 - \vartheta_{\tau-})(1 - \vartheta_\tau) < 0 \ \ (\textit{resp.}, \ = 0) \ \ P\textit{-a.s. on } \{\tau < T\}.$$

Proof. The claims follow from the optional section theorem (e.g., [RY99, Theorem IV.5.5]), which applies because A_{oj} and A_{pc} are optional sets. We here provide more detail on the proof for overjumping zero (the proof for premature closure is analogous).

Suppose first that overjumping zero is optimal. Then, we can choose a constant $\widetilde{\varepsilon} \in (0, P(\pi_\Omega(A_{oj})))$. By the optional section theorem, there exists a stopping time $\tau_{\widetilde{\varepsilon}} \colon \Omega \to [0,T]$ such that $P(\tau_{\widetilde{\varepsilon}} < T) \geq P(\pi_\Omega(A_{oj})) - \widetilde{\varepsilon} > 0$, and for any $\widetilde{\omega} \in \{\omega \in \Omega \colon \tau_{\widetilde{\varepsilon}}(\omega) < T\}$, we have that $(\widetilde{\omega}, \tau_{\widetilde{\varepsilon}}(\widetilde{\omega})) \in A_{oj}$.

For the other direction, assume that there exists a stopping time $\tau \colon \Omega \to [0,T]$ such that $P(\tau < T) > 0$ and $(1 - \vartheta_{\tau-})(1 - \vartheta_\tau) < 0$ P-a.s. on $\{\tau < T\}$. Then, $P(\pi_\Omega(A_{oj})) > 0$ due to $\pi_\Omega(A_{oj}) \supseteq \{\omega \in \Omega \colon (1 - \vartheta_{\tau-(\omega)}(\omega))(1 - \vartheta_{\tau(\omega)}(\omega)) < 0\} \cap \{\omega \in \Omega \colon \tau(\omega) < T\}$. □

We also remark that a simple attempt to define τ as, say, $T \wedge \inf\{s \in [0,T)\colon (1-\vartheta_{s-})(1-\vartheta_s) < 0\}$ does not always work, as, for ω such that $\tau(\omega) < T$ but the infimum is not attained, the expression $(1-\vartheta_{\tau-(\omega)}(\omega))(1-\vartheta_{\tau(\omega)}(\omega))$ will be zero.

We now turn to the question about new qualitative effects we can get if we allow for negative values of the resilience coefficient. Loosely speaking, with a resilience coefficient that is positive everywhere, we will not be able to observe overjumping zero or premature closure in the optimal strategy. On the contrary, if we allow the resilience coefficient to take negative values, then overjumping zero and premature closure in the optimal strategy become possible. Proposition 6.1.4 and Proposition 6.1.6 contain precise mathematical formulations of these statements. At the end of this section we also provide a more detailed informal discussion.

Proposition 6.1.4. *(i) We have that*

$$\widetilde{\vartheta}_{\cdot} \leq \left(1 - \frac{\rho_{\cdot}}{2\rho_{\cdot} + \mu_{\cdot}}\right) 1_{\{\rho_{\cdot}+\mu_{\cdot}>0\}} \leq 1 \;\; \mathcal{D}_{W^{(1)}}\text{-a.e. on } \{(\omega,s) \in \Omega \times [0,T] : \rho_s(\omega) \geq 0\}.$$

(ii) Assume (6.2) *and that* $\rho \geq 0$ $\mathcal{D}_{W^{(1)}}$*-a.e. Then overjumping zero is not optimal.*

(iii) Assume (6.2) *and that there exists an* $\mathcal{F}_T$*-measurable random variable* δ *such that*

$$\delta > 0 \;\; P\text{-a.s. and } \rho \geq \delta \;\; \mathcal{D}_{W^{(1)}}\text{-a.e.} \tag{6.3}$$

Then neither overjumping zero nor premature closure is optimal.

Proof. (i) Define

$$\begin{aligned} B = \{(\omega,s) \in \Omega \times [0,T] : \;\; & Y_s(\omega) \in [0,1/2], \\ & 2\rho_s(\omega) + \mu_s(\omega) - \sigma_s^2(\omega) > 0, \\ & \rho_s(\omega) \geq 0\} \end{aligned}$$

and observe that $B \in \mathcal{F}_T \otimes \mathcal{B}([0,T])$. It is enough to show the claim for every $(\omega,s) \in B$. To this end, we fix an arbitrary $(\omega,s) \in B$. By (6.1) we have to show that

$$\frac{(\rho_s(\omega)+\mu_s(\omega))Y_s(\omega)}{\sigma_s^2(\omega)Y_s(\omega) + \frac{1}{2}(2\rho_s(\omega)+\mu_s(\omega)-\sigma_s^2(\omega))} \leq \left(\frac{\rho_s(\omega)+\mu_s(\omega)}{2\rho_s(\omega)+\mu_s(\omega)}\right) 1_{\{\rho_s(\omega)+\mu_s(\omega)>0\}}. \tag{6.4}$$

If $\rho_s(\omega)+\mu_s(\omega) \leq 0$, this inequality is evident. Therefore, we assume in the sequel that $\rho_s(\omega)+\mu_s(\omega) > 0$. Note that $Y_s(\omega) \leq \frac{1}{2}$ implies that

$$\begin{aligned} Y_s(\omega) - \frac{1}{2}\frac{2\rho_s(\omega)+\mu_s(\omega)-\sigma_s^2(\omega)}{2\rho_s(\omega)+\mu_s(\omega)} &\leq Y_s(\omega)\left(1 - \frac{2\rho_s(\omega)+\mu_s(\omega)-\sigma_s^2(\omega)}{2\rho_s(\omega)+\mu_s(\omega)}\right) \\ &= \frac{\sigma_s^2(\omega)Y_s(\omega)}{2\rho_s(\omega)+\mu_s(\omega)}. \end{aligned}$$

This shows that

$$(\rho_s(\omega)+\mu_s(\omega))Y_s(\omega) \le \frac{\rho_s(\omega)+\mu_s(\omega)}{2\rho_s(\omega)+\mu_s(\omega)}\left(\sigma_s^2(\omega)Y_s(\omega)+\frac{1}{2}(2\rho_s(\omega)+\mu_s(\omega)-\sigma_s^2(\omega))\right),$$

and hence establishes (6.4).

(ii) We first notice that part (i) and (6.2) ensure that $\vartheta \le 1$ $\mathcal{D}_{W^{(1)}}$-a.e. As ϑ has càdlàg paths, by the standard Fubini argument, we infer that P-a.s. it holds: for all $s \in [0,T]$, we have $\vartheta_s \le 1$. This shows that overjumping zero is not optimal.

(iii) It suffices (cf. part (ii)) to show that premature closure is not optimal. Let $\overline{c} \in (0,\infty)$ such that $|\mu| \le \overline{c}$ $\mathcal{D}_{W^{(1)}}$-a-e. and $|\rho| \le \overline{c}$ $\mathcal{D}_{W^{(1)}}$-a-e. (exists due to $(\mathbf{C_{bdd}})$). Define

$$\begin{aligned} C = \{(\omega,s)\in\Omega\times[0,T]:\quad & Y_s(\omega)\in[0,1/2],\\ & 2\rho_s(\omega)+\mu_s(\omega)-\sigma_s^2(\omega)>0,\\ & \max\{|\rho_s(\omega)|,|\mu_s(\omega)|\}\le\overline{c},\\ & \delta(\omega)>0 \text{ and } \rho_s(\omega)\ge\delta(\omega)\} \end{aligned}$$

and notice that $C \in \mathcal{F}_T \otimes \mathcal{B}([0,T])$. It follows from part (i) that

$$\widetilde{\vartheta}_s(\omega) \le \max\left\{1-\frac{\delta(\omega)}{3\overline{c}},0\right\} < 1 \quad \text{for all } (\omega,s)\in C.$$

As $\mathcal{D}_{W^{(1)}}((\Omega\times[0,T])\setminus C)=0$ and ϑ is càdlàg with $\widetilde{\vartheta}=\vartheta$ $\mathcal{D}_{W^{(1)}}$-a.e., we conclude that P-a.s. it holds that

$$\sup_{s\in[0,T]}\vartheta_s \le \max\left\{1-\frac{\delta}{3\overline{c}},0\right\} < 1$$

(again by the Fubini argument). Hence, premature closure is not optimal. □

In relation with Proposition 6.1.4 we make the following comments.

Remark 6.1.5. (a) To discuss the assumption in part (iii) of Proposition 6.1.4 in more detail, we remark that, if

$$\inf_{s\in[0,T]}\rho_s > 0 \quad P\text{-a.s.}, \tag{6.5}$$

then (6.3) is satisfied. Indeed, in this case we can take $\delta = \inf_{s\in[0,T]}\rho_s$ because, by the measurable projection theorem, for all $z\in\mathbb{R}$ we have that

$$\left\{\omega\in\Omega\colon \inf_{s\in[0,T]}\rho_s(\omega)<z\right\} = \pi_\Omega\left(\{(\omega,s)\in\Omega\times[0,T]\colon \rho_s(\omega)<z\}\right)\in\mathcal{F}_T,$$

i.e., $\delta := \inf_{s\in[0,T]}\rho_s$ is $\mathcal{F}_T$-measurable. More precisely, (6.3) is slightly weaker than (6.5) and can be equivalently expressed as follows: there exists an $\mathcal{F}_T\otimes\mathcal{B}([0,T])$-measurable $\widetilde{\rho}$ such that $\widetilde{\rho}=\rho$ $\mathcal{D}_{W^{(1)}}$-a.e. and $\inf_{s\in[0,T]}\widetilde{\rho}_s>0$ P-a.s.

(b) The observation in part (iii) of Proposition 6.1.4 is in line with [HK21], where in a different but related setting (with a strictly positive stochastically varying resilience coefficient) it is observed that the optimal strategy never changes its sign (see [HK21, Theorem 2.2]), which in our terminology means that neither overjumping zero nor premature closure is optimal.

(c) Comparison of (ii) and (iii) poses the question if premature closure can be optimal with nonnegative resilience. The answer is affirmative: e.g., if $\rho \equiv 0$, then $\vartheta_s = 1$ for all $s \in [0,T]$, and the optimal strategy is to close the position immediately (cf. Proposition 5.2.3). This is, however, a rather degenerate example. A much more interesting one, for which we, however, allow the resilience to be negative, is presented in Section 6.3.

In the sequel, for a set $C \subseteq \Omega \times [0,T]$ and $\omega \in \Omega$, we use the notation

$$C_\omega = \{s \in [0,T] \colon (\omega, s) \in C\}$$

for the section of C. We will permanently use the following well-known statements (see, e.g., [Sal16, Lemma 7.2 and Theorem 7.9]): if $C \in \mathcal{F}_T \otimes \mathcal{B}([0,T])$, then

- for any $\omega \in \Omega$, it holds that $C_\omega \in \mathcal{B}([0,T])$,
- the mapping $\Omega \ni \omega \mapsto \mathrm{Leb}(C_\omega)$ is $\mathcal{F}_T$-measurable.

Note that here and in the sequel we use Leb to denote the Lebesgue measure on $([0,T], \mathcal{B}([0,T]))$.

Proposition 6.1.6. *Assume* (6.2), *and that there exists an* $\mathcal{F}_T$*-measurable random variable* δ *such that*

$$P\left(\left\{\omega \in \Omega \colon \forall n \in \mathbb{N},\ \mathrm{Leb}\left(B_\omega \cap \left[T - \frac{1}{n}, T\right]\right) > 0\right\}\right) > 0, \tag{6.6}$$

where

$$B = \{(\omega, s) \in \Omega \times [0,T] \colon \delta(\omega) > 0 \text{ and } \rho_s(\omega) \le -\delta(\omega)\} \quad (\in \mathcal{F}_T \otimes \mathcal{B}([0,T])). \tag{6.7}$$

Then overjumping zero or premature closure is optimal.

Proof. **1.** In the first step of the proof we establish that $(\mathbf{C}_{\geq\varepsilon})$, $(\mathbf{C}_{\mathbf{bdd}})$, and (6.6) imply that $\mathcal{D}_{W^{(1)}}(C) > 0$, where

$$C = \{(\omega, s) \in \Omega \times [0,T] \colon \widetilde{\vartheta}_s(\omega) > 1\} \quad (\in \mathcal{F}_T \otimes \mathcal{B}([0,T])).$$

To this end, we first recall from Lemma 4.1.5 that $\lim_{s \uparrow T} Y_s = Y_T \ (= \frac{1}{2})$ P-a.s., i.e., for the solution $(Y, 0, M^\perp)$ of BSDE (4.1), the orthogonal to $W^{(1)}$ and $W^{(2)}$ martingale

$M^\perp$ does not jump at terminal time T. Let $\bar{c} \in (0,\infty)$ such that $|\mu| \le \bar{c}\ \mathcal{D}_{W^{(1)}}$-a-e. and $|\rho| \le \bar{c}\ \mathcal{D}_{W^{(1)}}$-a-e. (exists due to $(\mathbf{C_{bdd}})$). We define

$$\begin{aligned} A = \{(\omega,s) \in \Omega \times [0,T]\colon \quad & \lim_{r\uparrow T} Y_r(\omega) = Y_T(\omega) = \frac{1}{2}, \\ & Y_s(\omega) \ge 0, \\ & 2\rho_s(\omega) + \mu_s(\omega) - \sigma_s^2(\omega) > 0, \\ & \max\{|\rho_s(\omega)|, |\mu_s(\omega)|\} \le \bar{c}\} \end{aligned}$$

and note that $A \in \mathcal{F}_T \otimes \mathcal{B}([0,T])$, $\mathcal{D}_{W^{(1)}}((\Omega \times [0,T]) \setminus A) = 0$. Now we set

$$K = B \cap A,$$

where B is from (6.7). From $0 = \mathcal{D}_{W^{(1)}}((\Omega \times [0,T]) \setminus A) = \int_\Omega \int_0^T 1_{(A^c)_\omega}(s) ds P(d\omega)$ we have that $1 = P(\{\omega \in \Omega\colon \int_0^T 1_{(A^c)_\omega}(s)ds = 0\}) = P(\{\omega \in \Omega\colon \operatorname{Leb}((A_\omega)^c) = 0\})$. This together with (6.6) implies that

$$\begin{aligned} 0 &< P\left(\left\{\omega \in \Omega\colon \operatorname{Leb}(A_\omega) = \operatorname{Leb}([0,T]) \text{ and } \forall n \in \mathbb{N}, \operatorname{Leb}\left(B_\omega \cap \left[T - \frac{1}{n}, T\right]\right) > 0\right\}\right) \\ &= P\left(\left\{\omega \in \Omega\colon \forall n \in \mathbb{N}, \operatorname{Leb}\left(A_\omega \cap B_\omega \cap \left[T - \frac{1}{n}, T\right]\right) > 0\right\}\right), \end{aligned}$$

i.e., (6.6) holds with B replaced by K. As $\mathcal{D}_{W^{(1)}}(C) = \int_\Omega \operatorname{Leb}(C_\omega)\, P(d\omega)$, we get $\mathcal{D}_{W^{(1)}}(C) > 0$ once we prove that

$$F := \left\{\omega \in \Omega\colon \forall n \in \mathbb{N},\ \operatorname{Leb}\left(K_\omega \cap \left[T - \frac{1}{n}, T\right]\right) > 0\right\} \subseteq \{\omega \in \Omega\colon \operatorname{Leb}(C_\omega) > 0\}. \tag{6.8}$$

To establish (6.8), we fix an arbitrary $\omega_0 \in F$ and make the following simple observation

$$s \in K_{\omega_0} \iff (\omega_0, s) \in A \text{ and } \rho_s(\omega_0) \le -\delta(\omega_0) < 0.$$

This yields that, for $s \in K_{\omega_0}$,

$$\mu_s(\omega_0) - \sigma_s^2(\omega_0) \le |\mu_s(\omega_0)| \le \bar{c}$$

and further

$$0 < 2\rho_s(\omega_0) + \mu_s(\omega_0) - \sigma_s^2(\omega_0) < \rho_s(\omega_0) + \mu_s(\omega_0) - \sigma_s^2(\omega_0) \le \bar{c} - \delta(\omega_0). \tag{6.9}$$

Now we compute from (6.1) that, for $s \in K_{\omega_0}$, we have the equivalence

$$\widetilde{\vartheta}_s(\omega_0) > 1 \iff 2Y_s(\omega_0) > 1 + \frac{\rho_s(\omega_0)}{\rho_s(\omega_0) + \mu_s(\omega_0) - \sigma_s^2(\omega_0)}. \tag{6.10}$$

Moreover, (6.9) and (6.10) reveal that, for $s \in K_{\omega_0}$,

$$\text{if } 2Y_s(\omega_0) > 1 - \frac{\delta(\omega_0)}{\bar{c} - \delta(\omega_0)}, \text{ then } \widetilde{\vartheta}_s(\omega_0) > 1 \quad (\Longleftrightarrow s \in C_{\omega_0}). \tag{6.11}$$

Recalling that $\omega_0 \in F$, the definition of the event F in (6.8), and that $\lim_{r \uparrow T} Y_r(\omega_0) = \frac{1}{2}$ (as $\omega_0 \in F$ implies that there exists $s \in [0, T]$ with $(\omega_0, s) \in K \subseteq A$), we conclude from (6.11) that there exists $n_0 \in \mathbb{N}$ (which depends on ω_0) such that

$$K_{\omega_0} \cap \left[T - \frac{1}{n_0}, T\right] \subseteq C_{\omega_0};$$

hence $\mathrm{Leb}(C_{\omega_0}) \geq \mathrm{Leb}(K_{\omega_0} \cap [T - 1/n_0, T]) > 0$. We thus proved (6.8) and completed the first step of the proof.

2. The first step together with (6.2) yields that $\mathcal{D}_{W^{(1)}}(\vartheta > 1) > 0$. Define the stopping time $\tau = T \wedge \inf\{s \in [0, T] \colon \vartheta_s > 1\}$ (as usual, $\inf \emptyset := \infty$). We get from $\mathcal{D}_{W^{(1)}}(\vartheta > 1) > 0$ that $P(\tau < T) > 0$ (by the Fubini argument). Since $\vartheta_{0-} = 0$ and ϑ is càdlàg, P-a.s. on $\{\tau < T\}$ it holds that $\vartheta_{\tau-} \leq 1$ and $\vartheta_\tau \geq 1$. By Lemma 6.1.3, this yields the result. □

The meaning of (6.6) is that, with positive probability, the resilience coefficient ρ is assumed to be negative with positive Lebesgue measure in any neighborhood of the terminal time T.

It is instructive to compare Proposition 6.1.6 with part (iii) of Proposition 6.1.4. The assumptions are "almost" complementary: compare (6.3) with (6.6)–(6.7). In both cases, we step a little away from 0 (this is the role of δ in (6.3) and (6.7)) but in a "soft" sense (the bound δ can depend on ω).

In view of these interpretations, we informally summarize part (iii) of Proposition 6.1.4 and Proposition 6.1.6 as follows. Positivity of the resilience coefficient implies that neither overjumping zero nor premature closure is optimal; negativity "close to T" implies optimality of overjumping zero or premature closure. There arises the question of whether negativity "far from T" also implies overjumping zero or premature closure. The answer is negative: see Example 6.2.2 below.

6.2 Piecewise constant resilience coefficient

We here analyze the effects of a negative resilience coefficient and discuss the results of Proposition 6.1.4 and Proposition 6.1.6 in a subsetting with N different regimes of resilience. That is to say that ρ is piecewise constant. Moreover, we assume that ρ is deterministic, $\mu > 0$ is constant deterministic, and $\sigma \equiv 0$. These assumptions lead to deterministic optimal strategies. We summarize the results in the following proposition.

Proposition 6.2.1. *Assume that* $\gamma_0 > 0$ *is deterministic and that*[2] $x - \frac{d}{\gamma_0} > 0$. *Suppose furthermore that* $\sigma \equiv 0$, *that* $\mu > 0$ *is a deterministic constant, and that the resilience coefficient* $\rho\colon [0,T] \to (-\mu/2, \infty)$ *is piecewise constant in the sense that there exist* $N \in \mathbb{N}$, $\rho^{(1)}, \ldots, \rho^{(N)} \in (-\mu/2, \infty)$, *and* $0 = T_0 < T_1 < \ldots < T_N = T$ *such that for all* $s \in [0,T)$ *it holds that*

$$\rho_s = \sum_{i=1}^{N} \rho^{(i)} 1_{[T_{i-1}, T_i)}(s).$$

Then, $(\boldsymbol{C}_{\geq \varepsilon})$ *and* $(\boldsymbol{C}_{\mathbf{bdd}})$ *are satisfied. The unique solution of BSDE (4.1) is given by*

$$\begin{aligned} Y_s &= e^{(T-s)\mu} \left(\sum_{i=n(s)+1}^{N} \frac{(\rho^{(i)} + \mu)^2 e^{T\mu}}{\mu(\rho^{(i)} + \frac{1}{2}\mu)} (e^{-(s \vee T_{i-1})\mu} - e^{-T_i \mu}) + 2 \right)^{-1}, \\ Z_s^{(1)} &= 0, \quad Z_s^{(2)} = 0, \quad M_s^{\perp} = 0, \quad s \in [0,T], \end{aligned} \tag{6.12}$$

where $n(s) = \max\{i \in \{0, \ldots, N\} \colon T_i \leq s\}$. *Moreover, (6.2) is satisfied with*

$$\vartheta_s = \widetilde{\vartheta}_s = \frac{\rho_s + \mu}{\rho_s + \frac{1}{2}\mu} Y_s, \quad s \in [0,T].$$

The optimal strategy X^* *and the associated deviation* D^{X^*} *are deterministic, for every* $i \in \{1, \ldots, N\}$ *they are continuous on* (T_{i-1}, T_i), *and for every* $i \in \{1, \ldots, N-1\}$ *they have a jump at* T_i *if and only if* ρ *has a jump at* T_i. *Furthermore, for every* $i \in \{1, \ldots, N\}$ *the deviation* D^{X^*} *is constant on* (T_{i-1}, T_i) *and takes negative values, and the optimal strategy* X^* *is monotone on* (T_{i-1}, T_i)*: more precisely, if* $\rho^{(i)} > 0$ *(resp.,* $\rho^{(i)} < 0$*; resp.,* $\rho^{(i)} = 0$*), then* X^* *is strictly decreasing (resp., strictly increasing; resp., constant) on* (T_{i-1}, T_i).

Proof. Clearly, $(\mathbf{C}_{\geq \varepsilon})$ and $(\mathbf{C}_{\mathbf{bdd}})$ are satisfied. Next note that Y from (6.12) satisfies for all $r \in [0,T]$ that

$$Y_r = e^{(T-r)\mu} \left(\int_r^T \frac{(\rho_s + \mu)^2}{\rho_s + \frac{1}{2}\mu} e^{(T-s)\mu} ds + 2 \right)^{-1}.$$

From this it follows that Y satisfies the Bernoulli ODE

$$dY_s = \left(\frac{(\rho_s + \mu)^2}{\rho_s + \frac{1}{2}\mu} Y_s^2 - \mu Y_s \right) ds, \quad s \in [0,T], \quad Y_T = \frac{1}{2}.$$

Consequently, $(Y, 0, 0)$ is the, by Proposition 4.2.1 unique, solution of BSDE (4.1). Moreover, $\widetilde{\vartheta}$ defined by (6.1) in the current setting reads $\widetilde{\vartheta}_s = \frac{\rho_s + \mu}{\rho_s + \frac{1}{2}\mu} Y_s$, $s \in [0,T]$, and

[2]This assumption is only for ease of exposition. All statements hold also in the case $x - \frac{d}{\gamma_0} < 0$ with the suitable adjustments.

is càdlàg and of finite variation, and thus we have (6.2) with $\vartheta = \widetilde{\vartheta}$. In particular, ϑ is deterministic, and since $\sigma \equiv 0$ and γ_0, ρ, μ are deterministic, we have that the optimal strategy X^* and its deviation D^{X^*} are deterministic as well.

For every $i \in \{1, \dots, N-1\}$ observe also that ϑ has a jump at T_i if and only if ρ has a jump at T_i. This directly translates into jumps of the optimal strategy X^* and jumps of the associated deviation D^{X^*} via (5.36) and (5.37). To show that the deviation D^{X^*} is constant on each (T_{i-1}, T_i), $i \in \{1, \dots, N\}$, observe that for all $i \in \{1, \dots, N\}$ and $s \in (T_{i-1}, T_i)$ it holds that

$$d\vartheta_s = \frac{\rho^{(i)} + \mu}{\rho^{(i)} + \frac{1}{2}\mu} \left(\frac{(\rho^{(i)} + \mu)^2}{\rho^{(i)} + \frac{1}{2}\mu} Y_s^2 - \mu Y_s \right) ds = \vartheta_s^2(\rho^{(i)} + \mu) ds - \mu \vartheta_s ds, \tag{6.13}$$

and hence

$$\begin{aligned} d\left(\gamma_s \vartheta_s \mathcal{E}(Q)_s\right) &= \vartheta_s d(\gamma_s \mathcal{E}(Q)_s) + \gamma_s \mathcal{E}(Q)_s d\vartheta_s \\ &= \vartheta_s \gamma_s \mathcal{E}(Q)_s \left(\mu - \vartheta_s(\mu + \rho^{(i)})\right) ds + \gamma_s \mathcal{E}(Q)_s \left(\vartheta_s^2(\rho^{(i)} + \mu) - \mu\vartheta_s\right) ds \\ &= 0. \end{aligned}$$

It thus follows from (5.37) that D^{X^*} is constant on (T_{i-1}, T_i) for $i \in \{1, \dots, N\}$. Moreover, since $\rho > -\frac{1}{2}\mu$, $\mu > 0$, and $Y > 0$, it holds that $\vartheta > 0$, and therefore $D^{X^*} < 0$ (recall that we assume $x - \frac{d}{\gamma_0} > 0$).

Next note that we have for all $i \in \{1, \dots, N\}$ and $s \in (T_{i-1}, T_i)$ that, using (6.13),

$$\begin{aligned} d((1-\vartheta_s)\mathcal{E}(Q)_s) &= (1-\vartheta_s) d\mathcal{E}(Q)_s - \mathcal{E}(Q)_s d\vartheta_s \\ &= -\mathcal{E}(Q)_s (1-\vartheta_s)\vartheta_s \left(\mu + \rho^{(i)}\right) ds - \mathcal{E}(Q)_s \left(\vartheta_s^2(\rho^{(i)} + \mu) - \mu\vartheta_s\right) ds \\ &= -\mathcal{E}(Q)_s \vartheta_s \rho^{(i)} ds. \end{aligned}$$

Since $\vartheta > 0$ and $x - \frac{d}{\gamma_0} > 0$, we conclude that if $\rho^{(i)}$ is positive, then X^* in (2.10) is decreasing on (T_{i-1}, T_i), and if $\rho^{(i)}$ is negative, then X^* is increasing on (T_{i-1}, T_i), $i \in \{1, \dots, N\}$. Clearly, if $\rho^{(i)} = 0$, then X^* is constant on (T_{i-1}, T_i), $i \in \{1, \dots, N\}$. □

In Example 6.2.2, Example 6.2.3, and Example 6.2.4 below we consider the setting of Proposition 6.2.1 with $N = 3$ different regimes of resilience. More precisely, we assume in the remainder of this section the setting of Proposition 6.2.1 with $N = 3$, $x = 1$, $d = 0$, $\gamma_0 = 1$, $\mu = 0.5$, and $T_i = i$ for $i \in \{1, 2, 3\}$.

We already know from Proposition 6.1.6 that overjumping zero or premature closure is optimal if we have a negative resilience coefficient in the last regime (i.e., $\rho^{(3)} < 0$). In the three examples below we want to analyze under what conditions these effects occur when the resilience coefficient is positive in the last (and also the first) regime. We choose $\rho^{(1)} = 0.1$ and $\rho^{(3)} = 1$. Proposition 6.1.4 entails that we necessarily need $\rho^{(2)} < 0$ to see these effects. Therefore, we choose a different negative value for $\rho^{(2)}$ in each example.

For these choices of $\rho^{(i)}$, $i \in \{1,2,3\}$, Proposition 6.2.1 shows that it is optimal to first sell during $(0,1)$, change this to a buy program on $(1,2)$ to profit from the negative resilience coefficient during that time interval, and then sell again during $(2,3)$. Moreover, since $\rho^{(1)}$ and $\rho^{(3)}$ are positive, we can already derive (e.g., by Proposition 6.1.4) that $\vartheta < 1$ on $[0,1)$ and on $[2,3)$, and hence that X^* is strictly positive on $[0,1)$ and on $[2,3)$ due to $x - \frac{d}{\gamma_0} = 1$.

Between Example 6.2.2, Example 6.2.3, and Example 6.2.4 we vary the size of $\rho^{(2)} < 0$. This then determines if we get overjumping zero or premature closure for the optimal strategy. Recall that ϑ in all examples has jumps at $s=1$ and $s=2$ and is continuous on $(0,1)$, $(1,2)$, and $(2,3)$, with values strictly smaller than 1 on $[0,1)$ and $[2,3)$. The facts that $\rho^{(1)} = 0.1$, $\mu = 0.5$, and $Y_1 \in (0, 1/2]$ yield that also $\vartheta_{1-} < 1$. We moreover have that $(1-\vartheta_{s-})(1-\vartheta_s) > 0$ for all $s \in [0,1) \cup (2,3)$. This, continuity of ϑ on $(1,2)$, $\vartheta_{1-} < 1$, and $\vartheta_2 < 1$ imply that overjumping zero is optimal if and only if at least one of

$$\vartheta_1 = \frac{\rho^{(2)} + \frac{1}{2}}{\rho^{(2)} + \frac{1}{4}} Y_1 > 1 \tag{6.14}$$

and

$$\vartheta_{2-} = \frac{\rho^{(2)} + \frac{1}{2}}{\rho^{(2)} + \frac{1}{4}} Y_2 > 1 \tag{6.15}$$

is satisfied. Premature closure is optimal if and only if

$$\exists\, s \in [1,2] \text{ such that } (1-\vartheta_{s-})(1-\vartheta_s) = 0. \tag{6.16}$$

The function ϑ and the optimal strategy X^* for each of the examples below are shown in Figure 6.1.

Example 6.2.2. We choose $\rho^{(2)} = -0.05$. The first row in Figure 6.1 shows that ϑ stays strictly smaller than one also on $[1,2)$, and hence the optimal strategy X^* is strictly positive on the time interval $[0,3)$. We conclude that, in general, a period with negative resilience coefficient does not necessarily lead to overjumping zero or premature closure.

Example 6.2.3. We next provide an example where a negative resilience coefficient indeed leads to overjumping zero and premature closure. To this end we choose $\rho^{(2)} = -0.085$ in the above set-up. From the second row of Figure 6.1 we observe that ϑ jumps above 1 at time $s=1$, but then decays continuously below 1 already before its next jump at $s=2$. It therefore holds that (6.14) and (6.16) are satisfied. We thus have overjumping zero as well as premature closure for the optimal strategy. This implies (recall $x - \frac{d}{\gamma_0} = 1$) that the optimal strategy jumps to a negative value at time $s=1$ and crosses 0 within the time interval $(1,2)$ to become positive again. Note that the set of points in time $s \in [0,T)$ for which we have $\vartheta_s > 1$ is strictly included in the set where $\rho_s < 0$ (which is $[1,2)$).

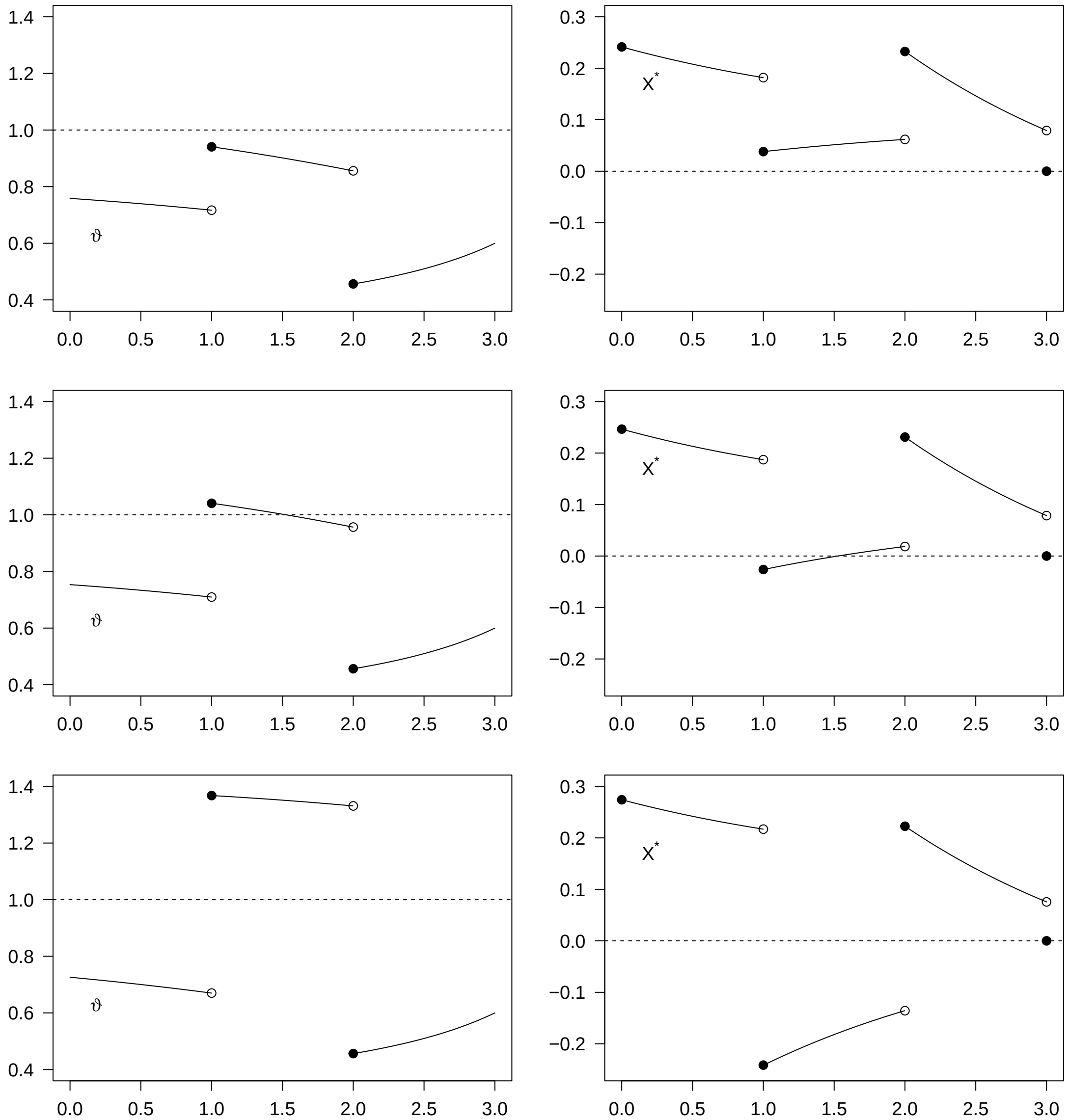


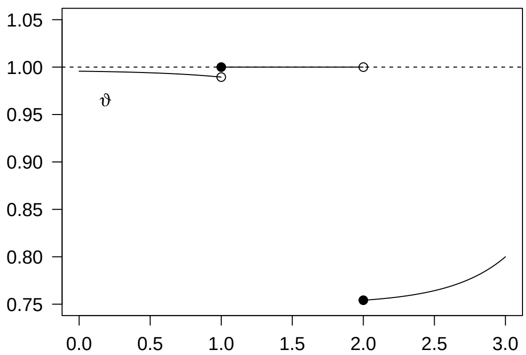
Figure 6.1: Top row: ϑ and X^* in Example 6.2.2 ($\rho^{(2)} = -0.05$). Middle row: ϑ and X^* in Example 6.2.3 ($\rho^{(2)} = -0.085$). Bottom row: ϑ and X^* in Example 6.2.4 ($\rho^{(2)} = -0.15$). The initial positions are not depicted.

Example 6.2.4. We finally provide an example where the set of points in time $s \in [0, T)$ for which we have $\vartheta_s > 1$ is equal to the set where $\rho_s < 0$. This means that the time periods with negative resilience coefficient exactly coincide with the time periods where the optimal strategy is negative. We achieve this for example for $\rho^{(2)} = -0.15$ in the above set-up (see the third row of Figure 6.1). In particular, (6.14) is satisfied, i.e., overjumping zero is optimal. Furthermore, we can compute that (6.15) holds true as well. It follows that condition (6.16) is not met, and therefore, premature closure is not optimal. Note that the optimal strategy changes its sign twice, but does not continuously cross 0.

Remark 6.2.5. We can produce the main effects discussed in Example 6.2.2, Example 6.2.3, and Example 6.2.4 also in the case with nonzero σ, see [AKU22b, Section 5.2] for more detail. Observe that for deterministic constant $\sigma \neq 0$, although BSDE (4.1) still has a deterministic solution and the process $\vartheta = \widetilde{\vartheta}$ still is deterministic, the optimal strategy X^* and its associated deviation D^{X^*} in general become stochastic. Moreover, the properties derived in Proposition 6.2.1 that D^{X^*} is constant between jumps and that X^* is monotone between jumps no longer hold when $\sigma \neq 0$.

6.3 Premature closure over a time interval

In Example 6.2.3 the optimal strategy entails to close the position at a certain point in time and to reopen it immediately. On the other hand, in the case $\rho \equiv 0$, it is optimal to close the position immediately and to not reenter trading (cf. Proposition 5.2.3). In the same way we can show that if, say, $\rho = 0$ on (T_1, T), for some $T_1 \in (0, T)$, then the optimal strategy X^* satisfies $X^* = 0$ on $[T_1, T]$ (and it can involve nontrivial trading on $[0, T_1]$ depending on the behavior of the model parameters on $(0, T_1)$). Keeping the position closed during a time interval and reopening again is more tricky, but also possible, as we show next. For an illustration, we refer to Figure 6.2.



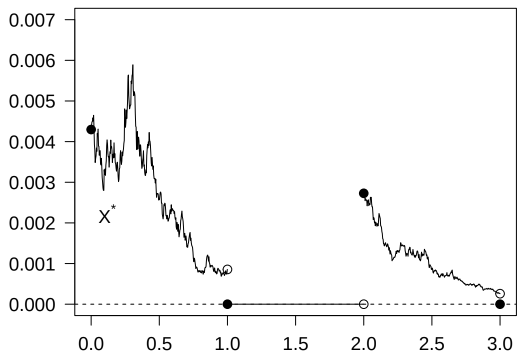


Figure 6.2: ϑ and a path of the optimal strategy X^* in the setting where σ and $\mu = \sigma^2 + 2$ are deterministic constants and ρ is defined as in (6.17). The specific parameter values are $x = 1$, $d = 0$, $\gamma_0 = 1$, $\sigma = 1$, $T = 3$, $T_1 = 1$, $T_2 = 2$, $\rho^{(1)} = 0.01$, $\rho^{(3)} = 1$, and $c = 2.416$. The initial value $X^*_{0-} = x = 1$ is not depicted. Observe that $\vartheta = 1$ and $X^* = 0$ between $s = 1$ and $s = 2$.

Let $T_1, T_2 \in (0, T)$ such that $T_1 < T_2$. Suppose that $\sigma^2 > 0$ is a deterministic constant and that $\mu = \sigma^2 + 2$. For deterministic $\rho^{(1)} > -1$, $\rho^{(3)} > 0$, and $c > 0$ let

$$\rho_s = \begin{cases} \rho^{(1)}, & s \in [0, T_1), \\ \left(ce^{2(s-T)} + 1\right)^{-1/2} - 1, & s \in [T_1, T_2), \\ \rho^{(3)}, & s \in [T_2, T]. \end{cases} \tag{6.17}$$

Note that $(\mathbf{C}_{\geq \varepsilon})$ and $(\mathbf{C_{bdd}})$ are satisfied. Let Y be the unique solution[3] of the ODE

$$dY_s = \left(\frac{(\rho_s + \sigma^2 + 2)^2 Y_s^2}{\sigma^2 Y_s + \rho_s + 1} - (\sigma^2 + 2) Y_s \right) ds, \quad s \in [0, T], \quad Y_T = \frac{1}{2}. \tag{6.18}$$

We have (6.2) with

$$\vartheta_s = \widetilde{\vartheta}_s = \frac{(\rho_s + \sigma^2 + 2) Y_s}{\sigma^2 Y_s + \rho_s + 1}, \quad s \in [0, T].$$

This implies that

$$\{s \in [0, T] \colon \vartheta_s = 1\} = \left\{ s \in [0, T] \colon Y_s = \frac{\rho_s + 1}{\rho_s + 2} \right\}.$$

In the sequel we establish that if c is chosen such that $\lim_{s \uparrow T_2} \frac{\rho_s + 1}{\rho_s + 2} = Y_{T_2}$, then $\frac{\rho+1}{\rho+2} = Y$ on (T_1, T_2). To this end, suppose[4] that $\lim_{s \uparrow T_2} \frac{\rho_s+1}{\rho_s+2} = Y_{T_2}$ and define $\widetilde{Y} = \frac{\rho+1}{\rho+2}$ on (T_1, T_2). We show that $\widetilde{Y}$ is a solution of (6.18) on (T_1, T_2). It holds for all $s \in (T_1, T_2)$ that

$$\frac{d\widetilde{Y}_s}{ds} = \frac{1}{(\rho_s + 2)^2} \frac{d\rho_s}{ds} = -ce^{2(s-T)} \frac{(\rho_s + 1)^3}{(\rho_s + 2)^2}.$$

On the other hand, we obtain for all $s \in (T_1, T_2)$ that

$$\begin{aligned} \frac{(\rho_s + \sigma^2 + 2)^2 \widetilde{Y}_s^2}{\sigma^2 \widetilde{Y}_s + \rho_s + 1} - (\sigma^2 + 2) \widetilde{Y}_s &= \left(\frac{(\rho_s + \sigma^2 + 2)^2 (\rho_s + 1)}{\sigma^2 (\rho_s + 1) + (\rho_s + 1)(\rho_s + 2)} - (\sigma^2 + 2) \right) \frac{\rho_s + 1}{\rho_s + 2} \\ &= \rho_s \frac{\rho_s + 1}{\rho_s + 2}. \end{aligned}$$

In order to show that

$$-ce^{2(s-T)} \frac{(\rho_s + 1)^3}{(\rho_s + 2)^2} = \rho_s \frac{\rho_s + 1}{\rho_s + 2}, \quad s \in (T_1, T_2), \tag{6.19}$$

[3] E.g., consider a trivial filtration $(\mathcal{F}_s^{\perp})_{s \in [0,T]}$ in Proposition 4.4.1 to see existence and uniqueness.

[4] Observe that to determine Y_{T_2} it suffices to consider ρ only on $[T_2, T]$. In particular, Y_{T_2} does not depend on the choice of c. Moreover, as $\rho^{(3)} \neq 0$, we have that $Y_{T_2} \in (0, 1/2)$ (via a straightforward comparison argument for (6.18)). Therefore, we can set $c = e^{2(T-T_2)}(1 - 2Y_{T_2}) Y_{T_2}^{-2} > 0$. It follows for this c that $\lim_{s \uparrow T_2} \frac{\rho_s+1}{\rho_s+2} = Y_{T_2}$.

note first that this is equivalent to

$$-ce^{2(s-T)}(\rho_s+1)^2 = \rho_s(\rho_s+2), \quad s \in (T_1,T_2).$$

Denoting $a_s = ce^{2(s-T)}$, $s \in (T_1,T_2)$, and using $\rho_s + 1 = (a_s+1)^{-\frac{1}{2}}$, $s \in (T_1,T_2)$, we can rewrite this as

$$-a_s(a_s+1)^{-1} = \left((a_s+1)^{-\frac{1}{2}} - 1\right)\left((a_s+1)^{-\frac{1}{2}} + 1\right), \quad s \in (T_1,T_2).$$

The right-hand side equals $(a_s+1)^{-1} - 1$, $s \in (T_1,T_2)$. We thus obtain the equivalent equation

$$-a_s = 1 - (a_s+1), \quad s \in (T_1,T_2),$$

which clearly holds true. This proves (6.19). Thus, by uniqueness of the solution of (6.18) and $\lim_{s\uparrow T_2} \frac{\rho_s+1}{\rho_s+2} = Y_{T_2}$, we have that $Y = \frac{\rho+1}{\rho+2}$ on (T_1,T_2).

This implies that $\vartheta = 1$ on (T_1,T_2). It follows that for all $x,d \in \mathbb{R}$, almost all paths of the optimal strategy X^* (cf. (5.36)) equal 0 on $[T_1,T_2)$. Finally, observe that if $x,d \in \mathbb{R}$ with $x \neq \frac{d}{\gamma_0}$, then almost all paths of X^* are nonzero everywhere on $[T_2,T)$ because, on $[T_2,T)$, we have $Y \leq \frac{1}{2} < \frac{\rho^{(3)}+1}{\rho^{(3)}+2}$, as $\rho^{(3)} > 0$, i.e., $Y = \frac{\rho+1}{\rho+2}$ holds nowhere on $[T_2,T)$.

7 Continuous extension from finite-variation to progressively measurable strategies

In the current and the subsequent chapter we study optimal trade execution in continuous time using progressively measurable strategies[1]. As in Chapter 2, we allow for an $\mathcal{F}_T$-measurable terminal position in the definition of the set of admissible strategies and for a risk term with stochastic target process in the cost functional. We use the price impact process γ and the resilience process R of Section 3.1 for independent Brownian motions $M^{(1)} = W^{(1)}$, $M^{(2)} = W^{(2)}$. As in Chapter 2 and Chapter 5, we only include a price deviation into our considerations and do not explicitly deal with an unaffected price.

To set up the stochastic control problem for progressively measurable strategies, we start from the finite-variation stochastic control problem of Section 7.1. Control problems of the kind of Section 7.1 are typical for continuous-time models of Obizhaeva-Wang type. We also refer to the discussion in Section 5.1.2 and to the basic example of an Obizhaeva-Wang type model in Section 1.1. The aim in the present chapter is to establish a continuous extension of the cost functional (7.4) from finite-variation strategies to progressively measurable strategies.

In a first step, we in Section 7.2 provide alternative representations for the cost functional and for the deviation associated to a finite-variation strategy of Section 7.1 that do not contain the strategy in the integrator anymore. This makes it feasible to, more generally, consider progressively measurable strategies in Section 7.3. In Section 7.4 we introduce the scaled hidden deviation process as a tool for the proof of Theorem 7.5.2 and for Section 8.1. Finally, we in Section 7.5 present and prove the main result Theorem 7.5.2 of this chapter on the continuous extension of the cost functional.

Throughout this chapter, we assume the setting of Section 3.1 and let $M^{(j)} = W^{(j)}$, $j \in \{1, \dots, m\}$, be independent Brownian motions.

[1] We refer to Chapter 9 for a discussion of the relation between Chapter 5 (cf. [AKU21a]) and Chapter 7–Chapter 8 (cf. [AKU22a])

This chapter makes extensive use of material from Section 1 and Section 5 of the preprint [AKU22a] (joint work with Thomas Kruse and Mikhail Urusov).

7.1 The finite-variation stochastic control problem

In addition to the setting of Section 3.1 with $M^{(j)} = W^{(j)}$, $j \in \{1, \dots, m\}$, we assume that $\hat{\xi}$ is an $\mathcal{F}_T$-measurable random variable satisfying

$$E[\gamma_T \hat{\xi}^2] < \infty, \tag{7.1}$$

and that $\zeta = (\zeta_s)_{s\in[0,T]}$ is a progressively measurable process satisfying

$$E\left[\int_0^T \gamma_s \zeta_s^2 ds\right] < \infty. \tag{7.2}$$

As in Chapter 2, we interpret $\hat{\xi}$ as the target position to be necessarily reached at terminal time (see also the definition of the set of admissible strategies), and ζ as a target process that describes a target position to be preferably followed over the trading period (see also the definition of the cost functional in (7.4)). Note that ζ can only become relevant if λ is not chosen equivalent to zero.

We next introduce the finite-variation strategies that we consider in the sequel. Given $t \in [0,T]$ and $d \in \mathbb{R}$ we associate to an adapted, càdlàg, finite-variation process $X = (X_s)_{s\in[t-,T]}$ a process $D^X = (D_s^X)_{s\in[t-,T]}$ defined by

$$dD_s^X = -D_s^X dR_s + \gamma_s dX_s, \quad s \in [t,T], \quad D_{t-}^X = d. \tag{7.3}$$

If we have a sequence of adapted, càdlàg, finite-variation processes $X^n = (X_s^n)_{s\in[t-,T]}$, $n \in \mathbb{N}$, we usually write D^n instead of D^{X^n} for $n \in \mathbb{N}$. For $t \in [0,T]$, $x, d \in \mathbb{R}$ we denote by $\mathcal{A}_t^{\mathrm{fv}}(x,d)$ the set of all adapted, càdlàg, finite-variation processes $X = (X_s)_{s\in[t-,T]}$ satisfying

$$X_{t-} = x, \quad X_T = \hat{\xi},$$

and

(B1) $E\left[\int_t^T \gamma_s^{-1} (D_s^X)^2 ds\right] < \infty$,

(B2) $E\left[\left(\int_t^T (D_s^X)^4 \gamma_s^{-2} \eta_s^2 ds\right)^{\frac{1}{2}}\right] < \infty$,

(B3) $E\left[\left(\int_t^T (D_s^X)^4 \gamma_s^{-2} \sigma_s^2 ds\right)^{\frac{1}{2}}\right] < \infty$.

Any element $X \in \mathcal{A}_t^{\mathrm{fv}}(x,d)$ is called a *finite-variation execution strategy*. The process D^X defined via (7.3) is called the associated *deviation process*.

For $t \in [0,T]$, $x,d \in \mathbb{R}$, $X \in \mathcal{A}_t^{\mathrm{fv}}(x,d)$ and associated D^X, the cost functional J^{fv} is given by

$$J_t^{\mathrm{fv}}(x,d,X) = E_t\left[\int_{[t,T]} \left(D_{s-}^X + \frac{\gamma_s}{2}\Delta X_s\right) dX_s\right] + E_t\left[\int_t^T \gamma_s \lambda_s \left(X_s - \zeta_s\right)^2 ds\right] \quad (7.4)$$

(see the proofs of Proposition 7.2.2 and Proposition 7.4.2 for well-definedness under $(\mathbf{C_{bdd}})$). The finite-variation stochastic control problem consists of minimizing the cost functional J^{fv} over $X \in \mathcal{A}_t^{\mathrm{fv}}(x,d)$.

7.2 Alternative representations for the cost functional and the deviation process

In this section we first in Proposition 7.2.1 provide the alternative pathwise representation (7.7) for the integral with respect to the strategy X that appears in the cost functional J^{fv} in (7.4). This subsequently, in Proposition 7.2.2, leads to the alternative representation (7.11) for the cost functional J^{fv}. Furthermore, we in the first proposition also derive the alternative expression (7.8) for the deviation process D^X. Note that neither X nor D^X show up as an integrator in (7.7) and (7.8). This allows to extend the definition of J^{fv} beyond the set of finite-variation execution strategies and even semimartingale execution strategies. Moreover, the presentation and proof of Proposition 7.2.1 are kept general in the sense that they do not hinge on the specific dynamics of the resilience process R or of the price impact process γ (as long as both are continuous semimartingales and γ is strictly positive and $R_0 = 0$).

For $t \in [0,T]$ we introduce an auxiliary process $\nu = (\nu_s)_{s\in[t,T]}$. It is defined to be the solution of

$$d\nu_s = \nu_s d\left(R_s + [R]_s\right), \quad s \in [t,T], \quad \nu_t = 1. \quad (7.5)$$

Observe that the inverse is given by

$$d\nu_s^{-1} = -\nu_s^{-1} dR_s, \quad s \in [t,T], \quad \nu_t^{-1} = 1. \quad (7.6)$$

Proposition 7.2.1. *Let $t \in [0,T]$ and $x,d \in \mathbb{R}$. Suppose that $X = (X_s)_{s\in[t-,T]}$ is an adapted, càdlàg, finite-variation process with $X_{t-} = x$ and with associated process D^X defined by* (7.3)*. It then holds that*

$$\int_{[t,T]} \left(D_{s-}^X + \frac{\gamma_s}{2}\Delta X_s\right) dX_s = \frac{1}{2}\left(\gamma_T^{-1}(D_T^X)^2 - \int_t^T (D_s^X)^2 \nu_s^2 d\left(\nu_s^{-2}\gamma_s^{-1}\right)\right) - \frac{d^2}{2\gamma_t} \quad (7.7)$$

and

$$D_r^X = \gamma_r X_r + \nu_r^{-1}\left(d - \gamma_t x - \int_t^r X_s d(\nu_s \gamma_s)\right), \quad r \in [t,T]. \quad (7.8)$$

Proof. Observe that integration by parts implies for all $s \in [t,T]$ that

$$\begin{aligned} d(\nu_s D_s^X) &= \nu_s dD_s^X + D_s^X d\nu_s + d[\nu, D^X]_s \\ &= -\nu_s D_s^X dR_s + \nu_s \gamma_s dX_s + \nu_s D_s^X dR_s + \nu_s D_s^X d[R]_s + d[\nu, D^X]_s \\ &= \nu_s \gamma_s dX_s + \nu_s D_s^X d[R]_s + d[\nu, D^X]_s. \end{aligned}$$

Since

$$d[\nu, D^X]_s = \nu_s d[R, D^X]_s = -\nu_s D_s^X d[R]_s, \quad s \in [t,T],$$

it follows that the process $\widetilde{D}_s^X = \nu_s D_s^X$, $s \in [t,T]$, $\widetilde{D}_{t-}^X = d$, satisfies

$$d\widetilde{D}_s^X = d(\nu_s D_s^X) = \nu_s \gamma_s dX_s, \quad s \in [t,T]. \tag{7.9}$$

In particular, $\widetilde{D}^X$ is of finite variation. The facts that $\Delta D_s^X = \gamma_s \Delta X_s$, $s \in [t,T]$, and $d\widetilde{D}_s^X = \nu_s \gamma_s dX_s$, $s \in [t,T]$, imply that

$$\begin{aligned} \int_{[t,T]} \left(2D_{s-}^X + \gamma_s \Delta X_s\right) dX_s &= \int_{[t,T]} \left(2D_{s-}^X + \Delta D_s^X\right) dX_s \\ &= \int_{[t,T]} \left(2D_{s-}^X + \Delta D_s^X\right) \gamma_s^{-1} \nu_s^{-1} d\widetilde{D}_s^X \\ &= \int_{[t,T]} \left(2\widetilde{D}_{s-}^X + \Delta \widetilde{D}_s^X\right) \nu_s^{-2} \gamma_s^{-1} d\widetilde{D}_s^X. \end{aligned} \tag{7.10}$$

Denote moreover $\varphi_s = \nu_s^{-2}\gamma_s^{-1}$, $s \in [t,T]$. It then holds for all $s \in [t,T]$ that

$$\Delta \widetilde{D}_s^X \varphi_s d\widetilde{D}_s^X = d[\widetilde{D}^X \varphi, \widetilde{D}^X]_s$$

due to finite variation of $\widetilde{D}^X$. We therefore obtain by integration by parts for all $s \in [t,T]$ that

$$\begin{aligned} \left(2\widetilde{D}_{s-}^X + \Delta \widetilde{D}_s^X\right) \varphi_s d\widetilde{D}_s^X &= 2\widetilde{D}_{s-}^X \varphi_s d\widetilde{D}_s^X + d[\widetilde{D}^X \varphi, \widetilde{D}^X]_s \\ &= d\left((\widetilde{D}_s^X \varphi_s)\widetilde{D}_s^X\right) - \widetilde{D}_{s-}^X d(\widetilde{D}_s^X \varphi_s) + \varphi_s \widetilde{D}_{s-}^X d\widetilde{D}_s^X. \end{aligned}$$

Furthermore, it holds for all $s \in [t,T]$ that

$$d(\widetilde{D}_s^X \varphi_s) = \varphi_s d\widetilde{D}_s^X + \widetilde{D}_s^X d\varphi_s,$$

and thus, for all $s \in [t,T]$,

$$\left(2\widetilde{D}_{s-}^X + \Delta \widetilde{D}_s^X\right) \varphi_s d\widetilde{D}_s^X = d\left((\widetilde{D}_s^X)^2 \varphi_s\right) - (\widetilde{D}_s^X)^2 d\varphi_s.$$

Together with (7.10) this yields that

$$\int_{[t,T]} \left(D^X_{s-} + \frac{\gamma_s}{2}\Delta X_s\right) dX_s = \frac{1}{2}\left((\widetilde{D}^X_T)^2\varphi_T - (\widetilde{D}^X_{t-})^2\varphi_t - \int_t^T (\widetilde{D}^X_s)^2 d\varphi_s\right)$$
$$= \frac{1}{2}\left(\gamma_T^{-1}(D^X_T)^2 - \gamma_t^{-1}d^2 - \int_t^T (D^X_s)^2\nu_s^2 d\left(\nu_s^{-2}\gamma_s^{-1}\right)\right).$$

This proves (7.7).

In order to show (7.8), we obtain from (7.9) and integration by parts that

$$\nu_r D^X_r - d = \nu_r\gamma_r X_r - \gamma_t x - \int_{[t,r]} X_s d(\nu_s\gamma_s) - \int_{[t,r]} d[\nu\gamma, X]_s, \quad r \in [t,T],$$

which implies that

$$D^X_r = \gamma_r X_r + \nu_r^{-1}\left(d - \gamma_t x - \int_t^r X_s d(\nu_s\gamma_s)\right), \quad r \in [t,T].$$

□

As a consequence of Proposition 7.2.1, and relying on **(B1)**–**(B3)**, we can rewrite the cost functional J^{fv} as follows[2]. Recall from (3.6) that $\kappa = \frac{1}{2}(2\rho+\mu-\sigma^2-\eta^2-2\sigma\eta\overline{r})$.

Proposition 7.2.2. *Assume* $(\boldsymbol{C_{bdd}})$*. Let* $t \in [0,T]$ *and* $x, d \in \mathbb{R}$*. Suppose that* $X \in \mathcal{A}^{fv}_t(x,d)$ *with associated deviation process* D^X *defined by* (7.3)*. It then holds that* $J^{fv}_t(x,d,X)$ *in* (7.4) *admits the representation*

$$J^{fv}_t(x,d,X) = E_t\left[\frac{1}{2}\gamma_T^{-1}(D^X_T)^2 + \int_t^T \gamma_s^{-1}(D^X_s)^2\kappa_s\,ds\right] - \frac{d^2}{2\gamma_t}$$
$$+ E_t\left[\int_t^T \gamma_s\lambda_s\left(X_s - \zeta_s\right)^2 ds\right] \;\; a.s. \tag{7.11}$$

Proof. We first consider the integrator $\nu^{-2}\gamma^{-1}$ on the right-hand side of (7.7). It holds by integration by parts and (7.6) that

$$\begin{aligned} d(\nu_s^{-2}\gamma_s^{-1}) &= \nu_s^{-1}d(\gamma_s^{-1}\nu_s^{-1}) + \gamma_s^{-1}\nu_s^{-1}d\nu_s^{-1} + d[\nu^{-1},\gamma^{-1}\nu^{-1}]_s \\ &= 2\nu_s^{-1}\gamma_s^{-1}d\nu_s^{-1} + \nu_s^{-2}d\gamma_s^{-1} + \nu_s^{-1}d[\gamma^{-1},\nu^{-1}]_s + d[\nu^{-1},\gamma^{-1}\nu^{-1}]_s \\ &= -2\nu_s^{-2}\gamma_s^{-1}dR_s + \nu_s^{-2}d\gamma_s^{-1} - \nu_s^{-2}d[\gamma^{-1},R]_s + d[\nu^{-1},\gamma^{-1}\nu^{-1}]_s, \quad s \in [t,T]. \end{aligned}$$

Note that

$$\begin{aligned} d[\nu^{-1},\gamma^{-1}\nu^{-1}]_s &= -\nu_s^{-1}d[R,\gamma^{-1}\nu^{-1}]_s = -\nu_s^{-1}d\left[R, \int_t^\cdot \gamma^{-1}d\nu^{-1} + \int_t^\cdot \nu^{-1}d\gamma^{-1}\right]_s \\ &= -\nu_s^{-1}\gamma_s^{-1}d[R,\nu^{-1}]_s - \nu_s^{-2}d[R,\gamma^{-1}]_s \\ &= \nu_s^{-2}\gamma_s^{-1}d[R]_s - \nu_s^{-2}d[R,\gamma^{-1}]_s, \quad s \in [t,T]. \end{aligned}$$

[2] Compare (7.11) also with representation (E.1) of the cost functional of the zero-spread two-sided order book model in [FSU19].

It hence follows that

$$d(\nu_s^{-2}\gamma_s^{-1}) = -2\nu_s^{-2}\gamma_s^{-1}dR_s + \nu_s^{-2}d\gamma_s^{-1} - 2\nu_s^{-2}d[\gamma^{-1}, R]_s + \nu_s^{-2}\gamma_s^{-1}d[R]_s, \quad s \in [t, T].$$

Plugged into (7.7) (cf. Proposition 7.2.1), we obtain that

$$\begin{aligned}&\int_{[t,T]} \left(D_{s-}^X + \frac{\gamma_s}{2}\Delta X_s\right) dX_s \\ &= \frac{1}{2}\left(\gamma_T^{-1}(D_T^X)^2 - \int_t^T (D_s^X)^2 \left(d\gamma_s^{-1} + \gamma_s^{-1}d[R]_s - 2\gamma_s^{-1}dR_s - 2d[\gamma^{-1}, R]_s\right)\right) - \frac{d^2}{2\gamma_t}.\end{aligned} \tag{7.12}$$

We further have by (3.1) and (3.3) for all $s \in [t, T]$ that

$$\begin{aligned}&d\gamma_s^{-1} + \gamma_s^{-1}d[R]_s - 2\gamma_s^{-1}dR_s - 2d[\gamma^{-1}, R]_s \\ &= -\gamma_s^{-1}(\mu_s - \sigma_s^2)ds - \gamma_s^{-1}\sigma_s dW_s^{(1)} + \gamma_s^{-1}\eta_s^2 ds - 2\gamma_s^{-1}\rho_s ds - 2\gamma_s^{-1}\eta_s dW_s^R \\ &\quad + 2\gamma_s^{-1}\sigma_s\eta_s\overline{r}_s ds \\ &= -\gamma_s^{-1}\left(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s\eta_s\overline{r}_s\right) ds - \gamma_s^{-1}\sigma_s dW_s^{(1)} - 2\gamma_s^{-1}\eta_s dW_s^R.\end{aligned} \tag{7.13}$$

It follows from **(B1)** and **($\mathbf{C_{bdd}}$)** that

$$E\left[\left|\int_t^T (D_s^X)^2\gamma_s^{-1}\left(2\rho_s + \mu_s - \sigma_s^2 - \eta_s^2 - 2\sigma_s\eta_s\overline{r}_s\right) ds\right|\right] < \infty.$$

The Burkholder-Davis-Gundy inequality together with **(B3)** shows that it holds for some constant $c \in (0, \infty)$ that

$$E\left[\sup_{r\in[t,T]}\left|\int_t^r (D_s^X)^2\gamma_s^{-1}\sigma_s dW_s^{(1)}\right|\right] \le cE\left[\left(\int_t^T (D_s^X)^4\gamma_s^{-2}\sigma_s^2 ds\right)^{\frac{1}{2}}\right] < \infty.$$

We therefore have that

$$E_t\left[\int_t^T (D_s^X)^2\gamma_s^{-1}\sigma_s dW_s^{(1)}\right] = 0.$$

Similarly, **(B2)** and $[W^R] = [W^{(1)}]$ imply that

$$E_t\left[\int_t^T 2(D_s^X)^2\gamma_s^{-1}\eta_s dW_s^R\right] = 0.$$

It thus follows from (7.12), (7.13), and Definition (3.6) of κ that

$$E_t\left[\int_{[t,T]} \left(D_{s-}^X + \frac{\gamma_s}{2}\Delta X_s\right) dX_s\right] = E_t\left[\frac{1}{2}\gamma_T^{-1}(D_T^X)^2 + \int_t^T (D_s^X)^2\gamma_s^{-1}\kappa_s\, ds\right] - \frac{d^2}{2\gamma_t}.$$

By Definition (7.4) of J^{fv} this proves (7.11). □

7.3 The extended stochastic control problem

We point out that the right-hand side of (7.11) is also well-defined for progressively measurable processes X satisfying an appropriate integrability condition and with associated deviation D^X defined by (7.8) for which one assumes **(B1)**. This motivates the following extension of the control problem from Section 7.1.

For $t \in [0,T]$, $x, d \in \mathbb{R}$, and a progressively measurable process $X = (X_s)_{s\in[t-,T]}$ such that $\int_t^T X_s^2 ds < \infty$ a.s. and $X_{t-} = x$, we define the process $D^X = (D_s^X)_{s\in[t-,T]}$ by

$$D_s^X = \gamma_s X_s + \nu_s^{-1}\left(d - \gamma_t x - \int_t^s X_r d(\nu_r \gamma_r)\right), \quad s \in [t,T], \quad D_{t-}^X = d \tag{7.14}$$

(recall ν from (7.5)). Notice that the condition $\int_t^T X_s^2 ds < \infty$ a.s. ensures that the stochastic integral in (7.14) is well-defined. Again, for a sequence of such progressively measurable processes X^n, $n \in \mathbb{N}$, we usually write D^n instead of D^{X^n} for $n \in \mathbb{N}$.

Further, for $t \in [0,T]$, $x, d \in \mathbb{R}$, let $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ be the set of (equivalence classes of) progressively measurable processes $X = (X_s)_{s\in[t-,T]}$ with

$$X_{t-} = x \;\text{ and }\; X_T = \hat{\xi}$$

that satisfy $\int_t^T X_s^2 ds < \infty$ a.s. and such that condition **(B1)** holds true for D^X defined by (7.14). To be precise, we stress that the equivalence classes for $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ are understood with respect to the equivalence relation

$$\begin{aligned} X \sim \widetilde{X} \quad \text{means} \quad & X_\cdot = \widetilde{X}_\cdot \;\; \mathcal{D}_{W^{(1)}}\text{-a.e. on } \Omega \times [t,T], \\ & X_{t-} = \widetilde{X}_{t-} \,(= x), \text{ and } X_T = \widetilde{X}_T \,(= \hat{\xi}). \end{aligned} \tag{7.15}$$

Any element $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ is called a *progressively measurable execution strategy*. Again, the process D^X now defined via (7.14) is called the associated *deviation process*.

Given $t \in [0,T]$, $x, d \in \mathbb{R}$, and $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ with associated D^X (see (7.14)), we (under $(\mathbf{C_{bdd}})$) define the cost functional J^{pm} by

$$J_t^{\mathrm{pm}}(x,d,X) = E_t\left[\frac{1}{2}\gamma_T^{-1}(D_T^X)^2 + \int_t^T \gamma_s^{-1}(D_s^X)^2 \kappa_s \, ds + \int_t^T \gamma_s \lambda_s \left(X_s - \zeta_s\right)^2 ds\right] - \frac{d^2}{2\gamma_t}. \tag{7.16}$$

The extended, or progressively measurable, stochastic control problem is to minimize the cost functional J^{pm} over $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$.

Observe that we have the following corollary of Proposition 7.2.1 and Proposition 7.2.2.

Corollary 7.3.1. *Assume $(\boldsymbol{C_{bdd}})$. Let $t \in [0,T]$, $x, d \in \mathbb{R}$, and $X \in \mathcal{A}_t^{fv}(x,d)$ with associated deviation process D^X given by (7.3). It then holds that $X \in \mathcal{A}_t^{pm}(x,d)$, that D^X satisfies (7.14), and that $J_t^{fv}(x,d,X) = J_t^{pm}(x,d,X)$.*

Proof. By (7.8) of Proposition 7.2.1, D^X satisfies (7.14). Since X is càdlàg, we have that $\int_t^T X_s^2 ds < \infty$ a.s. Clearly, $X_{t-} = x$, $X_T = \hat{\xi}$, and **(B1)** are satisfied, and X is progressively measurable; hence, $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$. From Proposition 7.2.2 and Definition (7.16), we immediately see that $J_t^{\mathrm{fv}}(x,d,X) = J_t^{\mathrm{pm}}(x,d,X)$. □

Put differently, the progressively measurable control problem extends the finite-variation one in the sense that $\mathcal{A}_t^{\mathrm{fv}}(x,d) \subseteq \mathcal{A}_t^{\mathrm{pm}}(x,d)$ and, on $\mathcal{A}_t^{\mathrm{fv}}(x,d)$, the cost functionals J^{fv} and J^{pm} coincide. In Section 7.5 we show that J^{pm} can be considered as a continuous extension of J^{fv} to progressively measurable strategies and that $\mathcal{A}_t^{\mathrm{fv}}(x,d)$ is dense in $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ (with respect to an appropriate metric).

7.4 The scaled hidden deviation process

In this section we introduce the scaled hidden deviation process associated to a strategy X. This process, due to Proposition 7.4.2, plays a key role in Section 8.1.1 as the state process in the LQ stochastic control problem that we are going to construct. Furthermore, this process already appears in the proof of Theorem 7.5.2 in Section 7.5 on the continuous extension of the cost functional. The lemmas in the current section are part of the preparation for this proof.

Suppose that the agent follows a finite-variation execution strategy $X \in \mathcal{A}_t^{\mathrm{fv}}(x,d)$ until time $s \in [t,T]$ and then decides to close the position, i.e., to sell $X_{s-} > 0$ units (respectively, to buy $|X_{s-}|$ units in the case $X_{s-} \le 0$). By (7.3), this results in the price deviation $D_s^X = D_{s-}^X - \gamma_s X_{s-}$ immediately after the trade. The value of[3] $D_{s-}^X - \gamma_s X_{s-} = D_s^X - \gamma_s X_s$ hence represents the hypothetical deviation if the agent decides to close the position at time $s \in [t,T]$. In the following, we consider the process $D^X - \gamma X$ for all $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ and scale it by $\gamma^{-\frac{1}{2}}$ to obtain what we call the scaled hidden deviation process.

For $t \in [0,T]$, $x,d \in \mathbb{R}$, and $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ with associated deviation process D^X, we introduce the *scaled hidden deviation process* $\overline{H}^X = (\overline{H}_s^X)_{s\in[t,T]}$ defined by

$$\overline{H}_s^X = \gamma_s^{-\frac{1}{2}}(D_s^X - \gamma_s X_s) = \gamma_s^{-\frac{1}{2}} D_s^X - \gamma_s^{\frac{1}{2}} X_s, \quad s \in [t,T]. \tag{7.17}$$

For a sequence of strategies $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}_t^{\mathrm{pm}}(x,d)$, we also write $\overline{H}^n$ for the process associated to X^n, $n \in \mathbb{N}$. Note that, due to (7.14), it holds that

$$\overline{H}_s^X = \gamma_s^{-\frac{1}{2}} \nu_s^{-1} \left(d - \gamma_t x - \int_t^s X_r d(\nu_r \gamma_r) \right), \quad s \in [t,T]. \tag{7.18}$$

The dynamics of $\overline{H}_s^X$ that we compute in the following lemma are used in the proofs of Proposition 7.4.2 and Lemma 7.4.4.

[3]Note that the process $D^X - \gamma X$ is continuous for all $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$, which can, e.g., be seen from (7.14) and the fact that R (hence also ν) and γ are continuous.

Lemma 7.4.1. *Let $t \in [0,T]$, $x, d \in \mathbb{R}$. Assume that $X = (X_s)_{s\in[t,T]}$ is a progressively measurable process such that $\int_t^T X_s^2 ds < \infty$ a.s. For $\alpha_s = \gamma_s^{-\frac{1}{2}}\nu_s^{-1}$, $s \in [t,T]$, and $\beta_s = d - \gamma_t x - \int_t^s X_r d(\nu_r\gamma_r)$, $s \in [t,T]$, it then holds for all $s \in [t,T]$ that*

$$\begin{aligned} &d(\alpha_s\beta_s) \\ &= -\gamma_s^{\frac{1}{2}} X_s\Bigg(\Big(\mu_s + \rho_s - \frac{1}{2}\sigma_s\eta_s\overline{r}_s - \frac{1}{2}\sigma_s^2\Big) ds + \big(\sigma_s + \eta_s\overline{r}_s\big)dW_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\Bigg) \\ &\quad + \alpha_s\beta_s\Bigg(\Big(- \rho_s - \frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2 + \frac{1}{2}\sigma_s\eta_s\overline{r}_s\Big)ds + \Big(- \eta_s\overline{r}_s - \frac{1}{2}\sigma_s\Big)dW_s^{(1)} \\ &\qquad - \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\Bigg). \end{aligned} \tag{7.19}$$

Proof. Observe that $\alpha = (\alpha_s)_{s\in[t,T]}$ and $\beta = (\beta_s)_{s\in[t,T]}$ are semimartingales. Integration by parts implies that

$$d(\alpha_s\beta_s) = -\alpha_s X_s d(\nu_s\gamma_s) + \beta_s d(\gamma_s^{-\frac{1}{2}}\nu_s^{-1}) - X_s d[\gamma^{-\frac{1}{2}}\nu^{-1}, \nu\gamma]_s, \quad s \in [t,T]. \tag{7.20}$$

It further holds by integration by parts, (7.5), (3.1), and (3.2) for all $s \in [t,T]$ that

$$\begin{aligned} d(\nu_s\gamma_s) &= \nu_s d\gamma_s + \gamma_s\nu_s dR_s + \gamma_s\nu_s d[R]_s + \nu_s d[R,\gamma]_s \\ &= \nu_s\gamma_s\mu_s ds + \nu_s\gamma_s\sigma_s dW_s^{(1)} + \nu_s\gamma_s\rho_s ds + \nu_s\gamma_s\eta_s\overline{r}_s dW_s^{(1)} + \nu_s\gamma_s\eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)} \\ &\quad + \nu_s\gamma_s\eta_s^2 ds + \nu_s\gamma_s\sigma_s\eta_s\overline{r}_s ds \\ &= \nu_s\gamma_s\left(\big(\mu_s + \rho_s + \eta_s^2 + \sigma_s\eta_s\overline{r}_s\big)ds + \big(\sigma_s + \eta_s\overline{r}_s\big)dW_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\right). \end{aligned} \tag{7.21}$$

Also by integration by parts, and using (7.6), (3.1), and (3.5), we obtain that

$$\begin{aligned} d(\gamma_s^{-\frac{1}{2}}\nu_s^{-1}) &= -\gamma_s^{-\frac{1}{2}}\nu_s^{-1}dR_s + \nu_s^{-1}d\gamma_s^{-\frac{1}{2}} - \nu_s^{-1}d[R,\gamma^{-\frac{1}{2}}]_s \\ &= -\gamma_s^{-\frac{1}{2}}\nu_s^{-1}\rho_s ds - \gamma_s^{-\frac{1}{2}}\nu_s^{-1}\eta_s\overline{r}_s dW_s^{(1)} - \gamma_s^{-\frac{1}{2}}\nu_s^{-1}\eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)} \\ &\quad + \gamma_s^{-\frac{1}{2}}\nu_s^{-1}\left(-\frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2\right)ds - \frac{1}{2}\gamma_s^{-\frac{1}{2}}\nu_s^{-1}\sigma_s dW_s^{(1)} + \frac{1}{2}\gamma_s^{-\frac{1}{2}}\nu_s^{-1}\sigma_s\eta_s\overline{r}_s ds \\ &= \alpha_s\Bigg(\left(-\rho_s - \frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2 + \frac{1}{2}\sigma_s\eta_s\overline{r}_s\right)ds + \left(-\eta_s\overline{r}_s - \frac{1}{2}\sigma_s\right)dW_s^{(1)} \\ &\qquad - \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\Bigg), \quad s \in [t,T]. \end{aligned} \tag{7.22}$$

It follows from (7.21) and (7.22) that

$$\begin{aligned} d[\gamma^{-\frac{1}{2}}\nu^{-1}, \nu\gamma]_s &= \alpha_s\nu_s\gamma_s\Big(-\eta_s\overline{r}_s - \frac{1}{2}\sigma_s\Big)\big(\sigma_s + \eta_s\overline{r}_s\big)ds - \alpha_s\nu_s\gamma_s\eta_s^2(1-\overline{r}_s^2)ds \\ &= -\gamma_s^{\frac{1}{2}}\left(\frac{3}{2}\eta_s\sigma_s\overline{r}_s + \frac{1}{2}\sigma_s^2 + \eta_s^2\right)ds, \quad s\in[t,T]. \end{aligned} \tag{7.23}$$

From (7.21) and (7.23) we get for all $s\in[t,T]$ that

$$\begin{aligned} &-\alpha_s X_s d(\nu_s\gamma_s) - X_s d[\gamma^{-\frac{1}{2}}\nu^{-1}, \nu\gamma]_s \\ &= -\gamma_s^{-\frac{1}{2}}X_s\left(\left(\mu_s + \rho_s - \frac{1}{2}\sigma_s\eta_s\overline{r}_s - \frac{1}{2}\sigma_s^2\right)ds + \big(\sigma_s + \eta_s\overline{r}_s\big)dW_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\right). \end{aligned} \tag{7.24}$$

We then plug (7.22) and (7.24) into (7.20) to obtain (7.19). □

We next use Lemma 7.4.1 to show that, under **($\mathbf{C_{bdd}}$)**, the scaled hidden deviation process satisfies a linear SDE and an L^2-bound. Moreover, we derive a representation of J^{pm} in terms of the scaled hidden deviation process.

Proposition 7.4.2. *Assume* **(C_{bdd})**. *Let $t\in[0,T]$, $x,d\in\mathbb{R}$, and $X\in\mathcal{A}_t^{pm}(x,d)$. Then it holds that*

$$\begin{aligned} d\overline{H}_s^X &= \left(\frac{1}{2}\left(\mu_s - \frac{1}{4}\sigma_s^2\right)\overline{H}_s^X - \frac{1}{2}\left(2(\rho_s+\mu_s) - \sigma_s^2 - \sigma_s\eta_s\overline{r}_s\right)\gamma_s^{-\frac{1}{2}}D_s^X\right)ds \\ &\quad + \left(\frac{1}{2}\sigma_s\overline{H}_s^X - (\sigma_s + \eta_s\overline{r}_s)\gamma_s^{-\frac{1}{2}}D_s^X\right)dW_s^{(1)} - \eta_s\sqrt{1-\overline{r}_s^2}\gamma_s^{-\frac{1}{2}}D_s^X dW_s^{(2)},\ s\in[t,T], \\ \overline{H}_t^X &= \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, \end{aligned} \tag{7.25}$$

that $E[\sup_{s\in[t,T]}(\overline{H}_s^X)^2]<\infty$, and that

$$\begin{aligned} J_t^{pm}(x,d,X) &= E_t\left[\frac{1}{2}\big(\overline{H}_T^X + \sqrt{\gamma_T}\hat{\xi}\big)^2 + \int_t^T(\kappa_s+\lambda_s)\gamma_s^{-1}(D_s^X)^2ds\right] - \frac{d^2}{2\gamma_t} \\ &\quad + E_t\left[\int_t^T \lambda_s\left(\overline{H}_s^X + \sqrt{\gamma_s}\zeta_s\right)^2 - 2\lambda_s\left(\overline{H}_s^X + \sqrt{\gamma_s}\zeta_s\right)\gamma_s^{-\frac{1}{2}}D_s^X ds\right]. \end{aligned} \tag{7.26}$$

Proof. We denote $\alpha_s = \gamma_s^{-\frac{1}{2}}\nu_s^{-1}$, $s\in[t,T]$, and $\beta_s = d - \gamma_t x - \int_t^s X_r d(\nu_r\gamma_r)$, $s\in[t,T]$. It then holds that $\overline{H}_s^X = \alpha_s\beta_s$, $s\in[t,T]$. We use Lemma 7.4.1 and substitute $-\gamma^{\frac{1}{2}}X =$

$\overline{H}^X - \gamma^{-\frac{1}{2}} D^X$ in (7.19) to obtain for all $s \in [t, T]$ that

$$
\begin{aligned}
d\overline{H}_s^X &= \big(\overline{H}_s^X - \gamma_s^{-\frac{1}{2}} D_s^X\big)\bigg(\Big(\mu_s + \rho_s - \frac{1}{2}\sigma_s\eta_s\overline{r}_s - \frac{1}{2}\sigma_s^2\Big)\, ds + \big(\sigma_s + \eta_s\overline{r}_s\big) dW_s^{(1)} \\
&\qquad\qquad + \eta_s\sqrt{1-\overline{r}_s^2}\, dW_s^{(2)}\bigg) \\
&\quad + \overline{H}_s^X \bigg(\Big(-\rho_s - \frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2 + \frac{1}{2}\sigma_s\eta_s\overline{r}_s\Big) ds + \Big(-\eta_s\overline{r}_s - \frac{1}{2}\sigma_s\Big) dW_s^{(1)} \\
&\qquad\qquad - \eta_s\sqrt{1-\overline{r}_s^2}\, dW_s^{(2)}\bigg) \\
&= -\gamma_s^{-\frac{1}{2}} D_s^X \bigg(\Big(\mu_s + \rho_s - \frac{1}{2}\sigma_s\eta_s\overline{r}_s - \frac{1}{2}\sigma_s^2\Big)\, ds + \big(\sigma_s + \eta_s\overline{r}_s\big) dW_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}\, dW_s^{(2)}\bigg) \\
&\quad + \overline{H}_s^X \bigg(\Big(\frac{1}{2}\mu_s - \frac{1}{8}\sigma_s^2\Big)\, ds + \frac{1}{2}\sigma_s dW_s^{(1)}\bigg).
\end{aligned}
$$

This proves the dynamics in (7.25).

In particular, $\overline{H}^X$ satisfies an SDE that is linear in $\overline{H}^X$ and $\gamma^{-\frac{1}{2}} D^X$. Furthermore, boundedness of $\rho, \mu, \sigma, \eta, \overline{r}$ implies that the coefficients of the SDE are bounded. Since moreover $E[\int_t^T \big(\gamma_s^{-\frac{1}{2}} D_s^X\big)^2 ds] < \infty$ by **(B1)** and since $\overline{H}_t^X = \gamma_t^{-\frac{1}{2}} d - \gamma_t^{\frac{1}{2}} x$ (cf. (7.17)) is square integrable, we have that $E[\sup_{s\in[t,T]} (\overline{H}_s^X)^2] < \infty$ (see, e.g., [Zha17, Theorem 3.2.2 and Theorem 3.3.1]).

We next prove that the cost functional (7.16) admits the representation (7.26). To this end, note that by (7.17) it holds for all $s \in [t, T]$ that

$$
\begin{aligned}
\gamma_s \left(X_s - \zeta_s\right)^2 &= \Big(\gamma_s^{-\frac{1}{2}} D_s^X - \overline{H}_s^X - \gamma_s^{\frac{1}{2}} \zeta_s\Big)^2 \\
&= \gamma_s^{-1} (D_s^X)^2 - 2\gamma_s^{-\frac{1}{2}} D_s^X \Big(\overline{H}_s^X + \gamma_s^{\frac{1}{2}} \zeta_s\Big) + \Big(\overline{H}_s^X + \gamma_s^{\frac{1}{2}} \zeta_s\Big)^2.
\end{aligned}
$$

Due to Assumption (7.2) on ζ and $E[\sup_{s\in[t,T]} (\overline{H}_s^X)^2] < \infty$, we have that

$$
E_t\left[\int_t^T \Big(\overline{H}_s^X + \gamma_s^{\frac{1}{2}} \zeta_s\Big)^2 ds\right] < \infty.
$$

This, **(B1)**, and the Cauchy–Schwarz inequality imply that also

$$
E_t\left[\int_t^T \left|\gamma_s^{-\frac{1}{2}} D_s^X \Big(\overline{H}_s^X + \gamma_s^{\frac{1}{2}} \zeta_s\Big)\right| ds\right] < \infty.
$$

Since λ is bounded, we conclude that

$$E_t\left[\int_t^T \lambda_s\gamma_s\left(X_s-\zeta_s\right)^2 ds\right] = E_t\left[\int_t^T \lambda_s\gamma_s^{-1}(D_s^X)^2 ds\right] + E_t\left[\int_t^T \lambda_s\left(\overline{H}_s^X+\gamma_s^{\frac{1}{2}}\zeta_s\right)^2 ds\right]$$
$$-2E_t\left[\int_t^T \lambda_s\gamma_s^{-\frac{1}{2}}D_s^X\left(\overline{H}_s^X+\gamma_s^{\frac{1}{2}}\zeta_s\right)ds\right], \tag{7.27}$$

where all conditional expectations are well-defined and finite. Moreover, (7.17) implies that $\gamma_T^{-\frac{1}{2}}D_T^X = \overline{H}_T^X + \gamma_T^{\frac{1}{2}}X_T$, and thus $\gamma_T^{-1}(D_T^X)^2 = (\overline{H}_T^X + \sqrt{\gamma_T}\hat{\xi})^2$. Inserting this and (7.27) into (7.16), we obtain (7.26). □

The next result on the scaled hidden deviation is helpful in the proof of Theorem 7.5.2 in order to show convergence of the cost functional.

Lemma 7.4.3. *Assume (C_{bdd}). Let $t\in[0,T]$, $x,d\in\mathbb{R}$, and $X\in\mathcal{A}_t^{pm}(x,d)$. Suppose that $(X^n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}_t^{pm}(x,d)$ such that*

$$\lim_{n\to\infty} E\left[\int_t^T \left(D_s^n-D_s^X\right)^2\gamma_s^{-1}ds\right]=0.$$

It then holds that

$$\lim_{n\to\infty} E\left[\sup_{s\in[t,T]}\left(\overline{H}_s^n-\overline{H}_s^X\right)^2\right]=0.$$

Proof. Define $\delta\overline{H}^n = \overline{H}^n - \overline{H}^X$, $n\in\mathbb{N}$, and let for $n\in\mathbb{N}$, $s\in[t,T]$, $z\in\mathbb{R}$

$$b_s^n(z) = -\frac{1}{2}\Big(2(\rho_s+\mu_s)-\sigma_s^2-\sigma_s\eta_s\overline{r}_s\Big)\big(\gamma_s^{-\frac{1}{2}}D_s^n-\gamma_s^{-\frac{1}{2}}D_s^X\big)+\frac{1}{2}\left(\mu_s-\frac{1}{4}\sigma_s^2\right)z,$$
$$a_s^n(z) = \left(-(\sigma_s+\eta_s\overline{r}_s)\big(\gamma_s^{-\frac{1}{2}}D_s^n-\gamma_s^{-\frac{1}{2}}D_s^X\big)+\frac{1}{2}\sigma_s z,\ -\eta_s\sqrt{1-\overline{r}_s^2}\big(\gamma_s^{-\frac{1}{2}}D_s^n-\gamma_s^{-\frac{1}{2}}D_s^X\big)\right).$$

In view of (7.25) it then holds for all $n\in\mathbb{N}$ that

$$d(\delta\overline{H}_s^n) = b_s^n(\delta\overline{H}_s^n)ds + a_s^n(\delta\overline{H}_s^n)d\begin{pmatrix}W_s^{(1)}\\W_s^{(2)}\end{pmatrix},\quad s\in[t,T],\quad \delta\overline{H}_t^n=0.$$

The definitions of b^n, a^n, $n\in\mathbb{N}$, and boundedness of μ and σ imply that there exists $c_1\in(0,\infty)$ such that for all $n\in\mathbb{N}$ and all $z_1,z_2\in\mathbb{R}$ it holds $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e. that

$$|b^n(z_1)-b^n(z_2)|+\|a^n(z_1)-a^n(z_2)\|_2 \le \frac{1}{2}\left|\mu-\frac{1}{4}\sigma^2\right||z_1-z_2|+\frac{1}{2}|\sigma||z_1-z_2| \le c_1|z_1-z_2|.$$

By boundedness of $\mu, \rho, \sigma, \eta, \overline{r}$ and Jensen's inequality, we have some $c_2 \in (0,\infty)$ such that for all $n \in \mathbb{N}$, it holds that

$$E\left[\left(\int_t^T |b_s^n(0)|ds\right)^2\right] + E\left[\int_t^T \|a_s^n(0)\|_2^2 ds\right] \leq c_2 E\left[\int_t^T \left(D_s^n - D_s^X\right)^2 \gamma_s^{-1} ds\right].$$

E.g., [Zha17, Theorem 3.2.2] (see also [Zha17, Theorem 3.4.2]) now implies that there exists $c_3 \in (0,\infty)$ such that for all $n \in \mathbb{N}$ we obtain that

$$\begin{aligned} E\left[\sup_{s\in[t,T]} \left|\overline{H}_s^n - \overline{H}_s^X\right|^2\right] &\leq c_3 E\left[\left(\int_t^T |b_s^n(0)|ds\right)^2 + \int_t^T \|a_s^n(0)\|_2^2 ds\right] \\ &\leq c_2 c_3 E\left[\int_t^T \left(D_s^n - D_s^X\right)^2 \gamma_s^{-1} ds\right]. \end{aligned}$$

The claim follows from the assumption that $\lim_{n\to\infty} E[\int_t^T (D_s^n - D_s^X)^2 \gamma_s^{-1} ds] = 0$. □

We next show how to construct an execution strategy $X^0 \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ based on a square-integrable process u^0 and a process H^0 that satisfies SDE (7.25) (with u^0 instead of $\gamma^{-\frac{1}{2}} D^X$). This result is crucial for Lemma 8.1.2. It is also used in the proof of Theorem 7.5.2.

Lemma 7.4.4. *Let $t \in [0,T]$ and $x, d \in \mathbb{R}$. Suppose that $u^0 = (u_s^0)_{s\in[t,T]} \in \mathcal{L}_t^2$, and let $H^0 = (H_s^0)_{s\in[t,T]}$ be given by $H_t^0 = \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t} x$,*

$$\begin{aligned} dH_s^0 = &\left(\frac{1}{2}\left(\mu_s - \frac{1}{4}\sigma_s^2\right) H_s^0 - \frac{1}{2}\left(2(\rho_s + \mu_s) - \sigma_s^2 - \sigma_s \eta_s \overline{r}_s\right) u_s^0\right) ds \\ &+ \left(\frac{1}{2}\sigma_s H_s^0 - (\sigma_s + \eta_s \overline{r}_s) u_s^0\right) dW_s^{(1)} - \eta_s \sqrt{1 - \overline{r}_s^2} u_s^0 dW_s^{(2)}, \quad s \in [t,T]. \end{aligned} \tag{7.28}$$

Define $X^0 = (X_s^0)_{s\in[t-,T]}$ by

$$X_s^0 = \gamma_s^{-\frac{1}{2}}(u_s^0 - H_s^0), \quad s \in [t,T), \quad X_{t-}^0 = x, \quad X_T^0 = \hat{\xi}.$$

Then, $X^0 \in \mathcal{A}_t^{pm}(x,d)$, and for the associated deviation process $D^0 = (D_s^0)_{s\in[t-,T]}$ it holds that $D^0 = \gamma X^0 + \gamma^{\frac{1}{2}} H^0$, and H^0 is the scaled hidden deviation process for X^0.

Proof. First, X^0 is progressively measurable and has initial value $X_{t-}^0 = x$ and terminal value $X_T^0 = \hat{\xi}$. Furthermore, it holds that

$$\int_t^T (X_s^0)^2 ds \leq 2\int_t^T \gamma_s^{-1}(u_s^0)^2 ds + 2\int_t^T \gamma_s^{-1}(H_s^0)^2 ds < \infty \text{ a.s.}$$

since γ^{-1} and H^0 have a.s. continuous paths and $E[\int_t^T (u_s^0)^2 ds] < \infty$. We are therefore able to define D^0 by (7.14). Moreover, denote $\alpha_s = \gamma_s^{-\frac{1}{2}}\nu_s^{-1}$, $s \in [t,T]$, and $\beta_s = d - \gamma_t x - \int_t^s X_r^0 d(\nu_r\gamma_r)$, $s \in [t,T]$. It follows from Lemma 7.4.1 and $-\gamma_s^{\frac{1}{2}} X_s^0 = H_s^0 - u_s^0$, $s \in [t,T)$, that

$$\begin{aligned} d(\alpha_s\beta_s) = (H_s^0 - u_s^0)\bigg(&\Big(\mu_s + \rho_s - \frac{1}{2}\sigma_s\eta_s\overline{r}_s - \frac{1}{2}\sigma_s^2\Big)ds + \big(\sigma_s + \eta_s\overline{r}_s\big)dW_s^{(1)} \\ &+ \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\bigg) \\ + \alpha_s\beta_s\bigg(&\Big(-\rho_s - \frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2 + \frac{1}{2}\sigma_s\eta_s\overline{r}_s\Big)ds + \Big(-\eta_s\overline{r}_s - \frac{1}{2}\sigma_s\Big)dW_s^{(1)} \\ &- \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\bigg), \quad s \in [t,T]. \end{aligned}$$

We combine this with

$$\begin{aligned} dH_s^0 = -u_s^0\bigg(&\Big(\mu_s + \rho_s - \frac{1}{2}\sigma_s\eta_s\overline{r}_s - \frac{1}{2}\sigma_s^2\Big)ds + (\sigma_s + \eta_s\overline{r}_s)dW_s^{(1)} + \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\bigg) \\ + H_s^0\bigg(&\Big(\frac{1}{2}\mu_s - \frac{1}{8}\sigma_s^2\Big)ds + \frac{1}{2}\sigma_s dW_s^{(1)}\bigg), \quad s \in [t,T], \end{aligned}$$

to obtain for all $s \in [t,T]$ that

$$\begin{aligned} d(\alpha_s\beta_s - H_s^0) = (\alpha_s\beta_s - H_s^0)\bigg(&\Big(-\rho_s - \frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2 + \frac{1}{2}\sigma_s\eta_s\overline{r}_s\Big)ds \\ &+ \Big(-\eta_s\overline{r}_s - \frac{1}{2}\sigma_s\Big)dW_s^{(1)} - \eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}\bigg). \end{aligned} \tag{7.29}$$

Note that $\alpha_t\beta_t = \gamma_t^{-\frac{1}{2}}d - \gamma_t^{\frac{1}{2}}x = H_t^0$. We thus conclude that 0 is the unique solution of (7.29), and hence

$$H_s^0 = \gamma_s^{-\frac{1}{2}}\nu_s^{-1}\left(d - \gamma_t x - \int_t^s X_r^0 d(\nu_r\gamma_r)\right), \quad s \in [t,T].$$

This implies that $D^0 = \gamma X^0 + \gamma^{\frac{1}{2}}H^0$, i.e., $D_s^0 = \gamma_s^{\frac{1}{2}}u_s^0$, $s \in [t,T)$, and $D_T^0 = \gamma_T\hat{\xi} + \gamma_T^{\frac{1}{2}}H_T^0$. The fact that $E[\int_t^T (u_s^0)^2 ds] < \infty$ then immediately yields that **(B1)** holds true. This completes the proof. □

7.5 Continuous extension of the cost functional

Corollary 7.3.1 states that for finite-variation execution strategies, the cost functionals J^{fv} and J^{pm} are the same. In this section, we show that J^{pm} can be considered as a continuous extension of J^{fv} to progressively measurable strategies. To this end, we first need to introduce a metric on $\mathcal{A}_t^{\mathrm{pm}}(x,d)$.

For $t \in [0,T]$, $x,d \in \mathbb{R}$, and $X, \widetilde{X} \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ (with associated deviation processes D^X, $D^{\widetilde{X}}$ defined by (7.14)), we define

$$\mathbf{d}(X,\widetilde{X}) = \left(E\left[\int_t^T \left(D_s^X - D_s^{\widetilde{X}}\right)^2 \gamma_s^{-1} ds \right] \right)^{\frac{1}{2}} \quad \left(= \left\| \gamma^{-\frac{1}{2}} \left(D^X - D^{\widetilde{X}}\right) \right\|_{\mathcal{L}_t^2} \right). \tag{7.30}$$

Lemma 7.5.1. *Let $t \in [0,T]$ and $x,d \in \mathbb{R}$. Then,* (7.30) *defines a metric on $\mathcal{A}_t^{pm}(x,d)$ (identifying any processes that are equal $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e.).*

Note that, for fixed $t \in [0,T]$ and $x,d \in \mathbb{R}$, and under $(\mathbf{C_{bdd}})$, we may consider the cost functional (7.16) as a function

$$J_t^{\mathrm{pm}}(x,d,\cdot)\colon (\mathcal{A}_t^{\mathrm{pm}}(x,d), \mathbf{d}) \to (L^1(\Omega,\mathcal{F}_t,P), \|\cdot\|_{L^1}).$$

Indeed, using **(B1)**, Proposition 7.4.2, (7.1), (7.2), and boundedness of the input processes, we see that $J_t^{\mathrm{pm}}(x,d,X) \in L^1(\Omega,\mathcal{F}_t,P)$ for all $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$.

We now come to the main result of this chapter.

Theorem 7.5.2. *Assume $(\boldsymbol{C_{bdd}})$. Let $t \in [0,T]$ and $x,d \in \mathbb{R}$.*

(i) Suppose that $X \in \mathcal{A}_t^{pm}(x,d)$. For every sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}_t^{pm}(x,d)$ with $\lim_{n\to\infty} \mathbf{d}(X^n,X) = 0$ *it holds that* $\lim_{n\to\infty} \| J_t^{pm}(x,d,X^n) - J_t^{pm}(x,d,X) \|_{L^1} = 0$.

(ii) For any $X \in \mathcal{A}_t^{pm}(x,d)$ there exists a sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}_t^{fv}(x,d)$ such that $\lim_{n\to\infty} \mathbf{d}(X^n,X) = 0$.

(iii) For any Cauchy sequence $(X^n)_{n\in\mathbb{N}}$ in $(\mathcal{A}_t^{pm}(x,d), \mathbf{d})$ there exists $X^0 \in \mathcal{A}_t^{pm}(x,d)$ such that $\lim_{n\to\infty} \mathbf{d}(X^n,X^0) = 0$.

This establishes that $J_t^{\mathrm{pm}}(x,d,X)$ is continuous in the strategy $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ (the first part of Theorem 7.5.2), that $\mathcal{A}_t^{\mathrm{fv}}(x,d)$ is dense in $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ (the second part of Theorem 7.5.2) and that the metric space $(\mathcal{A}_t^{\mathrm{pm}}(x,d), \mathbf{d})$ is complete (the third part of Theorem 7.5.2).

The first and the second parts of Theorem 7.5.2 mean that, under the metric $\mathbf{d}$, $J_t^{pm}(x,d,\cdot)$ is a unique continuous extension of $J_t^{fv}(x,d,\cdot)$ from $\mathcal{A}_t^{\mathrm{fv}}(x,d)$ onto $\mathcal{A}_t^{\mathrm{pm}}(x,d)$. The third part of Theorem 7.5.2 means that, under the metric $\mathbf{d}$, $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ is the largest space where such a continuous extension is uniquely determined by $J_t^{fv}(x,d,\cdot)$ on $\mathcal{A}_t^{\mathrm{fv}}(x,d)$. This is because the completeness of $(\mathcal{A}_t^{\mathrm{pm}}(x,d), \mathbf{d})$ is equivalent to the following statement: for any metric space $(\widehat{\mathcal{A}}_t(x,d), \widehat{\mathbf{d}})$ containing $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ and such that $\widehat{\mathbf{d}}|_{\mathcal{A}_t^{\mathrm{pm}}(x,d)\times\mathcal{A}_t^{\mathrm{pm}}(x,d)} = \mathbf{d}$, it holds that the set $\mathcal{A}_t^{\mathrm{pm}}(x,d)$ is closed in $\widehat{\mathcal{A}}_t(x,d)$.

As a corollary of Theorem 7.5.2, we obtain the following equivalence of the finite-variation and the extended stochastic control problem.

Corollary 7.5.3. *Under* (C_{bdd})*, it holds for all* $t \in [0,T]$ *and* $x, d \in \mathbb{R}$ *that*

$$\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{fv}(x,d)} J_t^{fv}(x,d,X) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{pm}(x,d)} J_t^{pm}(x,d,X). \tag{7.31}$$

Proofs

In this part, we prove Lemma 7.5.1, Theorem 7.5.2, and Corollary 7.5.3.

Proof of Lemma 7.5.1. Note first that it holds for all $X, \widetilde{X} \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ that $\mathbf{d}(X,\widetilde{X}) \geq 0$, and that $\mathbf{d}(X,\widetilde{X})$ is finite due to **(B1)**. Symmetry of $\mathbf{d}$ is obvious.

To establish the triangle inequality, let $X, \widetilde{X}, \hat{X} \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$. It follows from

$$(D^X - D^{\widetilde{X}})^2 = (D^X - D^{\hat{X}})^2 + 2(D^X - D^{\hat{X}})(D^{\hat{X}} - D^{\widetilde{X}}) + (D^{\hat{X}} - D^{\widetilde{X}})^2$$

and the Cauchy–Schwarz inequality that

$$\begin{aligned}\mathbf{d}(X,\widetilde{X})^2 &= \mathbf{d}(X,\hat{X})^2 + 2E\left[\int_t^T \left(D_s^X - D_s^{\hat{X}}\right)\gamma_s^{-\frac{1}{2}}\left(D_s^{\hat{X}} - D_s^{\widetilde{X}}\right)\gamma_s^{-\frac{1}{2}} ds\right] + \mathbf{d}(\hat{X},\widetilde{X})^2 \\ &\leq \mathbf{d}(X,\hat{X})^2 + 2\mathbf{d}(X,\hat{X})\mathbf{d}(\hat{X},\widetilde{X}) + \mathbf{d}(\hat{X},\widetilde{X})^2.\end{aligned}$$

We thus obtain that $\mathbf{d}(X,\widetilde{X}) \leq \mathbf{d}(X,\hat{X}) + \mathbf{d}(\hat{X},\widetilde{X})$.

We now let $X, \widetilde{X} \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ and show that $X = \widetilde{X}$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e. if and only if $\mathbf{d}(X,\widetilde{X}) = 0$.

If $X = \widetilde{X}$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e., then $\gamma^{-\frac{1}{2}} D^X = \gamma^{-\frac{1}{2}} D^{\widetilde{X}}$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e., and thus

$$\mathbf{d}(X,\widetilde{X}) = \left(E\left[\int_t^T \left(\gamma_s^{-\frac{1}{2}} D_s^X - \gamma_s^{-\frac{1}{2}} D_s^{\widetilde{X}}\right)^2 ds\right]\right)^{\frac{1}{2}} = 0.$$

For the other direction, suppose that $\mathbf{d}(X,\widetilde{X}) = 0$. This implies that $\gamma^{-\frac{1}{2}} D^X - \gamma^{-\frac{1}{2}} D^{\widetilde{X}} = 0$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e. By Definition (7.14) of D^X and $D^{\widetilde{X}}$ it further follows from a multiplication by $\nu\gamma^{\frac{1}{2}}$ that

$$\nu_s\gamma_s(X_s - \widetilde{X}_s) = \int_t^s (X_r - \widetilde{X}_r) d(\nu_r\gamma_r) \quad \mathcal{D}_{W^{(1)}}|_{[t,T]}\text{-a.e.}$$

Observe that $\nu\gamma > 0$ and define $U = (U_s)_{s\in[t,T]}$ by $U_s = \int_0^s (\nu_r\gamma_r)^{-1} d(\nu_r\gamma_r)$, $s \in [t,T]$. Let $K = (K_s)_{s\in[t,T]}$ be defined by $K_s = \int_t^s \nu_r\gamma_r(X_r - \widetilde{X}_r) dU_r$, $s \in [t,T]$. Then, K is a continuous semimartingale with $K = \nu\gamma(X - \widetilde{X})$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e. It follows that

$$\int_t^s K_r dU_r = \int_t^s \nu_r\gamma_r(X_r - \widetilde{X}_r) dU_r = K_s, \quad s \in [t,T].$$

This shows that $K = 0$ (as a stochastic exponential with start in 0). Since $K = \nu\gamma(X - \widetilde{X})$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e. and $\nu\gamma > 0$, we conclude that $X = \widetilde{X}$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e. □

In order to establish existence of an appropriate approximating sequence in Theorem 7.5.2, we rely on the next Lemma 7.5.4, which is based on [KS91, Section 3.2, Lemma 2.7]. For the statement of this lemma and for the proof of the second part of Theorem 7.5.2, we introduce a process $L = (L_s)_{s\in[0,T]}$ defined by

$$L_s = \exp\left(-\int_0^s \left(\frac{1}{2}\sigma_r + \eta_r\overline{r}_r\right) dW_r^{(1)} - \int_0^s \eta_r\sqrt{1-\overline{r}_r^2}dW_r^{(2)}\right), \quad s \in [0,T]. \tag{7.32}$$

Observe that L solves the SDE

$$\begin{aligned} dL_s = {} & L_s\frac{1}{2}\left(\left(\frac{1}{2}\sigma_s + \eta_s\overline{r}_s\right)^2 + \eta_s^2(1-\overline{r}_s^2)\right) ds - L_s\left(\frac{1}{2}\sigma_s + \eta_s\overline{r}_s\right) dW_s^{(1)} \\ & - L_s\eta_s\sqrt{1-\overline{r}_s^2}dW_s^{(2)}, \quad s \in [0,T], \quad L_0 = 1. \end{aligned} \tag{7.33}$$

Lemma 7.5.4. *Assume* $(\boldsymbol{C_{bdd}})$. *Let* $L = (L_s)_{s\in[0,T]}$ *be defined by* (7.32). *Let* $t \in [0,T]$ *and let* $u = (u_s)_{s\in[t,T]} \in \mathcal{L}_t^2$. *Then there exists a sequence of bounded càdlàg finite-variation processes* $(v^n)_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} E\left[\int_t^T \left(\frac{u_s}{L_s} - v_s^n\right)^2 L_s^2 ds\right] = 0.$$

In particular, for the sequence of processes $(u^n)_{n\in\mathbb{N}}$ *defined by* $u^n = v^n L$, $n \in \mathbb{N}$, *it holds for all* $n \in \mathbb{N}$ *that* u^n *is a càdlàg semimartingale with* $E[\sup_{s\in[t,T]}|u_s^n|^p] < \infty$ *for any* $p \geq 2$ *(in particular,* $u^n \in \mathcal{L}_t^2$*), and that* $\lim_{n\to\infty}\|u - u^n\|_{\mathcal{L}_t^2} = 0$.

Proof. Define $A = (A_s)_{s\in[0,T]}$ by $A_s = \int_0^s L_r^2 dr$, $s \in [0,T]$. Moreover, let $v = (v_s)_{s\in[t,T]}$ be defined by $v_s = \frac{u_s}{L_s}$, $s \in [t,T]$. We verify the assumptions of Lemma 2.7 in Section 3.2 of [KS91] on A and v.

The process A is continuous, adapted, and nondecreasing. Since σ, η, and $\overline{r}$ are bounded, Lemma 4.1.1 implies for all $p \geq 2$ that $E[\sup_{s\in[0,T]}|L_s|^p] < \infty$. In particular, it holds that $E[A_T] = E[\int_0^T L_r^2 dr] < \infty$. Moreover, v is progressively measurable and satisfies $E[\int_t^T v_s^2 dA_s] = E[\int_t^T u_s^2 ds] < \infty$ due to $u \in \mathcal{L}_t^2$.

Thus, Lemma 2.7 in Section 3.2 of [KS91] applies and yields that there exists a sequence $(\hat{v}^n)_{n\in\mathbb{N}}$ of (càglàd) simple (see [KS91, Def. 2.3]) processes $\hat{v}^n = (\hat{v}_s^n)_{s\in[t,T]}$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} E\left[\int_t^T (v_s - \hat{v}_s^n)^2 dA_s\right] = 0.$$

Define $v_s^n(\omega) = \lim_{r\downarrow s}\hat{v}_r^n(\omega)$, $s \in [t,T)$, $\omega \in \Omega$, $n \in \mathbb{N}$, and $v_T^n = 0$, $n \in \mathbb{N}$. Then, $(v^n)_{n\in\mathbb{N}}$ is a sequence of bounded càdlàg finite-variation processes such that

$$\lim_{n\to\infty} E\left[\int_t^T \left(\frac{u_s}{L_s} - v_s^n\right)^2 L_s^2 ds\right] = \lim_{n\to\infty} E\left[\int_t^T (v_s - v_s^n)^2 dA_s\right] = 0.$$

Moreover, it holds that for each $n \in \mathbb{N}$, $u^n = (u^n_s)_{s\in[t,T]}$ defined by $u^n_s = v^n_s L_s$, $s \in [t,T]$, is a càdlàg semimartingale. Since v^n is bounded for all $n \in \mathbb{N}$ and $E[\sup_{s\in[0,T]}|L_s|^p]$ is finite for any $p \geq 2$, we have that $E[\sup_{s\in[t,T]}|u^n_s|^p]$ is finite for all $n \in \mathbb{N}$ and any $p \geq 2$. It furthermore holds that

$$\|u - u^n\|^2_{\mathcal{L}^2_t} = E\left[\int_t^T (u_s - u^n_s)^2 ds\right] = E\left[\int_t^T (v_s - v^n_s)^2 dA_s\right] \to 0$$

as $n \to \infty$. □

Lemma 7.5.4 is employed in the proof of Theorem 7.5.2(ii) as follows: given $X \in \mathcal{A}^{\mathrm{pm}}_t(x,d)$, we approximate $u = \gamma^{-\frac{1}{2}} D^X$ by a sequence $u^n = v^n L$, $n \in \mathbb{N}$, from Lemma 7.5.4. Based on this sequence $(u^n)_{n\in\mathbb{N}}$, we define a sequence of progressively measurable strategies $(X^n)_{n\in\mathbb{N}}$ as in Lemma 7.4.4. Subsequently, we argue that finite variation of v^n leads to finite variation of X^n for all $n \in \mathbb{N}$.

Let us now prove Theorem 7.5.2.

Proof of Theorem 7.5.2. (i) Let $(X^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}^{\mathrm{pm}}_t(x,d)$ such that it holds $\lim_{n\to\infty} \mathbf{d}(X^n, X) = 0$. By (7.26) in Proposition 7.4.2 it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned}
&|J^{\mathrm{pm}}_t(x,d,X^n) - J^{\mathrm{pm}}_t(x,d,X)| \\
&= \Bigg| \frac{1}{2} E_t\left[\left(\overline{H}^n_T + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2 - \left(\overline{H}^X_T + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2\right] + E_t\left[\int_t^T (\kappa_s + \lambda_s)\gamma_s^{-1}\left((D^n_s)^2 - (D^X_s)^2\right) ds\right] \\
&\quad - 2E_t\left[\int_t^T \lambda_s \gamma_s^{-\frac{1}{2}} \left(D^n_s\left(\overline{H}^n_s + \gamma_s^{\frac{1}{2}}\zeta_s\right) - D^X_s\left(\overline{H}^X_s + \gamma_s^{\frac{1}{2}}\zeta_s\right)\right) ds\right] \\
&\quad + E_t\left[\int_t^T \lambda_s \left(\left(\overline{H}^n_s + \gamma_s^{\frac{1}{2}}\zeta_s\right)^2 - \left(\overline{H}^X_s + \gamma_s^{\frac{1}{2}}\zeta_s\right)^2\right) ds\right] \Bigg|.
\end{aligned}$$

From $(\mathbf{C_{bdd}})$ and boundedness of λ and $\overline{r}$ we obtain that there exists a constant $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned}
&E\left[|J^{\mathrm{pm}}_t(x,d,X^n) - J^{\mathrm{pm}}_t(x,d,X)|\right] \\
&\leq E\left[\left|\left(\overline{H}^n_T + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2 - \left(\overline{H}^X_T + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2\right|\right] + c\,E\left[\int_t^T \left|\gamma_s^{-1}\left((D^n_s)^2 - (D^X_s)^2\right)\right| ds\right] \\
&\quad + c\,E\left[\int_t^T \left|\gamma_s^{-\frac{1}{2}}\left(D^n_s\left(\overline{H}^n_s + \gamma_s^{\frac{1}{2}}\zeta_s\right) - D^X_s\left(\overline{H}^X_s + \gamma_s^{\frac{1}{2}}\zeta_s\right)\right)\right| ds\right] \\
&\quad + c\,E\left[\int_t^T \left|\left(\overline{H}^n_s + \gamma_s^{\frac{1}{2}}\zeta_s\right)^2 - \left(\overline{H}^X_s + \gamma_s^{\frac{1}{2}}\zeta_s\right)^2\right| ds\right].
\end{aligned} \tag{7.34}$$

We treat the terminal costs first. It holds for all $n \in \mathbb{N}$ that

$$\begin{aligned} & E\left[\left|\left(\overline{H}_T^n + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2 - \left(\overline{H}_T^X + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2\right|\right] \\ &= E\left[\left|(\overline{H}_T^n)^2 + 2\overline{H}_T^n\gamma_T^{\frac{1}{2}}\hat{\xi} - (\overline{H}_T^X)^2 - 2\overline{H}_T^X\gamma_T^{\frac{1}{2}}\hat{\xi}\right|\right] \\ &\leq E\left[\left|(\overline{H}_T^n)^2 - (\overline{H}_T^X)^2\right|\right] + 2E\left[\left|(\overline{H}_T^n - \overline{H}_T^X)\gamma_T^{\frac{1}{2}}\hat{\xi}\right|\right] \\ &\leq E\left[\left|(\overline{H}_T^n)^2 - (\overline{H}_T^X)^2\right|\right] + 2\left(E\left[\left(\overline{H}_T^n - \overline{H}_T^X\right)^2\right]\right)^{\frac{1}{2}}\left(E\left[\gamma_T\hat{\xi}^2\right]\right)^{\frac{1}{2}}. \end{aligned}$$

From

$$\lim_{n\to\infty} E\left[\int_t^T \left(D_s^n - D_s^X\right)^2\gamma_s^{-1}ds\right] = \lim_{n\to\infty} \mathbf{d}(X^n, X)^2 = 0 \tag{7.35}$$

and Lemma 7.4.3 we have that

$$\lim_{n\to\infty} E\left[\sup_{s\in[t,T]}\left(\overline{H}_s^n - \overline{H}_s^X\right)^2\right] = 0. \tag{7.36}$$

Since furthermore $E[\gamma_T\hat{\xi}^2] < \infty$ (cf. (7.1)), we obtain that

$$\lim_{n\to\infty} E\left[\left|\left(\overline{H}_T^n + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2 - \left(\overline{H}_T^X + \gamma_T^{\frac{1}{2}}\hat{\xi}\right)^2\right|\right] = 0.$$

The second term in (7.34) converges to 0 using (7.35). For the third term in (7.34) we have for all $n \in \mathbb{N}$ that

$$\begin{aligned} & E\left[\int_t^T \left|\gamma_s^{-\frac{1}{2}}\left(D_s^n\left(\overline{H}_s^n + \gamma_s^{\frac{1}{2}}\zeta_s\right) - D_s^X\left(\overline{H}_s^X + \gamma_s^{\frac{1}{2}}\zeta_s\right)\right)\right| ds\right] \\ &\leq E\left[\int_t^T |\overline{H}_s^X + \gamma_s^{\frac{1}{2}}\zeta_s|\,|D_s^n - D_s^X|\gamma_s^{-\frac{1}{2}} + \gamma_s^{-\frac{1}{2}}|D_s^n|\,|\overline{H}_s^n - \overline{H}_s^X|ds\right] \\ &\leq \left(E\left[\int_t^T (\overline{H}_s^X + \gamma_s^{\frac{1}{2}}\zeta_s)^2 ds\right]\right)^{\frac{1}{2}}\left(E\left[\int_t^T \left(D_s^n - D_s^X\right)^2\gamma_s^{-1}ds\right]\right)^{\frac{1}{2}} \\ &\quad + \left(E\left[\int_t^T \gamma_s^{-1}(D_s^n)^2 ds\right]\right)^{\frac{1}{2}} T^{\frac{1}{2}}\left(E\left[\sup_{s\in[t,T]}\left(\overline{H}_s^n - \overline{H}_s^X\right)^2\right]\right)^{\frac{1}{2}}. \end{aligned} \tag{7.37}$$

By Proposition 7.4.2 and (7.2) it holds that $E[\int_t^T(\overline{H}_s^X + \gamma_s^{\frac{1}{2}}\zeta_s)^2 ds] < \infty$. Moreover, due to (7.35), we have that $E[\int_t^T \gamma_s^{-1}(D_s^n)^2 ds]$ is uniformly bounded in $n \in \mathbb{N}$. It thus follows from (7.35), (7.36), and (7.37) that the third term in (7.34) converges to 0 as $n \to \infty$. The last term in (7.34) converges to 0 using (7.2) and (7.36). This proves claim (i).

(ii) Suppose that $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$. Let $u = (u_s)_{s\in[t,T]}$ be defined by $u_s = \gamma_s^{-\frac{1}{2}} D_s^X$, $s \in [t,T]$. Then, u is a progressively measurable process, and due to assumption **(B1)** it holds that $E[\int_t^T u_s^2 ds] < \infty$, i.e., $u \in \mathcal{L}_t^2$.

By Lemma 7.5.4 there exists a sequence of bounded càdlàg finite-variation processes $(v^n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} E[\int_t^T (\frac{u_s}{L_s} - v_s^n)^2 L_s^2 ds] = 0$, where L is defined in (7.32). Set $u^n = v^n L$, $n \in \mathbb{N}$. This is a sequence of càdlàg semimartingales in $\mathcal{L}_t^2$ that satisfies $\lim_{n\to\infty} \|u - u^n\|_{\mathcal{L}_t^2} = 0$. Moreover, it holds for all $n \in \mathbb{N}$ and any $p \geq 2$ that $E[\sup_{s\in[t,T]} |u_s^n|^p] < \infty$.

For each u^n, $n \in \mathbb{N}$, let $H^n = (H_s^n)_{s\in[t,T]}$ be the solution of (7.28). We then define $X^n = (X_s^n)_{s\in[t-,T]}$, $n \in \mathbb{N}$, by $X_s^n = \gamma_s^{-\frac{1}{2}}(u_s^n - H_s^n)$, $s \in [t,T)$, $X_{t-}^n = x$, $X_T^n = \hat{\xi}$. Note that this is a sequence of càdlàg semimartingales. Moreover, for all $n \in \mathbb{N}$, Lemma 7.4.4 proves that $X^n \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ and that $D^n = \gamma X^n + \gamma^{\frac{1}{2}} H^n$ for the associated deviation process $D^n = (D_s^n)_{s\in[t-,T]}$.

It follows for all $n \in \mathbb{N}$ that $D_s^n = \gamma_s^{\frac{1}{2}} u_s^n$, $s \in [t,T)$. Therefore, it holds for all $n \in \mathbb{N}$ that

$$\mathbf{d}(X^n, X) = \left(E\left[\int_t^T (D_s^n - D_s^X)^2 \gamma_s^{-1} ds \right] \right)^{\frac{1}{2}} = \left(E\left[\int_t^T (u_s^n - u_s)^2 ds \right] \right)^{\frac{1}{2}}.$$

Due to $\lim_{n\to\infty} \|u - u^n\|_{\mathcal{L}_t^2} = 0$, we thus have that $\lim_{n\to\infty} \mathbf{d}(X^n, X) = 0$.

We next show that for all $n \in \mathbb{N}$, X^n has finite variation. To this end, we observe that for all $n \in \mathbb{N}$ and $s \in [t,T)$ it holds by integration by parts that

$$dX_s^n = \gamma_s^{-\frac{1}{2}} d(u_s^n - H_s^n) + (u_s^n - H_s^n) d\gamma_s^{-\frac{1}{2}} + d[\gamma^{-\frac{1}{2}}, u^n - H^n]_s. \tag{7.38}$$

Again by integration by parts, and using (7.33), we have for all $n \in \mathbb{N}$ and $s \in [t,T]$ that

$$\begin{aligned} du_s^n &= v_s^n dL_s + L_s dv_s^n + d[v^n, L]_s \\ &= u_s^n \frac{1}{2}\left(\left(\frac{1}{2}\sigma_s + \eta_s \overline{r}_s \right)^2 + \eta_s^2 (1 - \overline{r}_s^2) \right) ds - u_s^n \left(\frac{1}{2}\sigma_s + \eta_s \overline{r}_s \right) dW_s^{(1)} \\ &\quad - u_s^n \eta_s \sqrt{1 - \overline{r}_s^2} dW_s^{(2)} + L_s dv_s^n. \end{aligned}$$

This and (7.28) yield for all $n \in \mathbb{N}$ and $s \in [t,T]$ that

$$\begin{aligned} \gamma_s^{-\frac{1}{2}} d(u_s^n - H_s^n) = \gamma_s^{-\frac{1}{2}} \left(\rho_s + \mu_s + \frac{1}{2}\eta_s^2 - \frac{3}{8}\sigma_s^2 \right) u_s^n ds - \gamma_s^{-\frac{1}{2}} \left(\frac{1}{2}\mu_s - \frac{1}{8}\sigma_s^2 \right) H_s^n ds \\ + \gamma_s^{-\frac{1}{2}} \frac{1}{2}\sigma_s (u_s^n - H_s^n) dW_s^{(1)} + \gamma_s^{-\frac{1}{2}} L_s dv_s^n. \end{aligned} \tag{7.39}$$

Moreover, it follows from (3.5) for all $n \in \mathbb{N}$ and $s \in [t,T]$ that

$$(u_s^n - H_s^n) d\gamma_s^{-\frac{1}{2}} = (u_s^n - H_s^n)\gamma_s^{-\frac{1}{2}} \left(-\frac{1}{2}\mu_s + \frac{3}{8}\sigma_s^2\right) ds - (u_s^n - H_s^n)\gamma_s^{-\frac{1}{2}} \frac{1}{2}\sigma_s dW_s^{(1)}. \tag{7.40}$$

We combine (7.38), (7.39), and (7.40) to obtain for all $n \in \mathbb{N}$ and $s \in [t,T)$ that

$$dX_s^n = \gamma_s^{-\frac{1}{2}} u_s^n \left(\rho_s + \frac{1}{2}\mu_s + \frac{1}{2}\eta_s^2\right) ds - \gamma_s^{-\frac{1}{2}} H_s^n \frac{1}{4}\sigma_s^2 ds + \gamma_s^{-\frac{1}{2}} L_s dv_s^n + d[\gamma^{-\frac{1}{2}}, u^n - H^n]_s.$$

Since v^n has finite variation for all $n \in \mathbb{N}$, this representation shows that also X^n has finite variation for all $n \in \mathbb{N}$.

Note that for all $n \in \mathbb{N}$, by Proposition 7.2.1, the process (7.3) associated to the càdlàg finite-variation process X^n is nothing but D^n. Since η is bounded, there exists $c \in (0,\infty)$ such that for all $n \in \mathbb{N}$

$$E\left[\left(\int_t^T (D_s^n)^4 \gamma_s^{-2} \eta_s^2 ds\right)^{\frac{1}{2}}\right] = E\left[\left(\int_t^T (u_s^n)^4 \eta_s^2 ds\right)^{\frac{1}{2}}\right] \leq c\, E\Big[\sup_{s\in[t,T]} (u_s^n)^2\Big] < \infty.$$

This implies **(B2)**. Similarly, by boundedness of σ, we obtain **(B3)**. We thus conclude that $X^n \in \mathcal{A}_t^{\mathrm{fv}}(x,d)$ for all $n \in \mathbb{N}$.

(iii) Let $(X^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{A}_t^{\mathrm{pm}}(x,d), \mathbf{d})$. Then, $(\gamma^{-\frac{1}{2}} D^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{L}_t^2, \|\cdot\|_{\mathcal{L}_t^2})$. Since $(\mathcal{L}_t^2, \|\cdot\|_{\mathcal{L}_t^2})$ is complete (see, e.g., Lemma 2.2 in Section 3.2 of [KS91]), there exists $u^0 \in \mathcal{L}_t^2$ such that $\lim_{n\to\infty} \|\gamma^{-\frac{1}{2}} D^n - u^0\|_{\mathcal{L}_t^2} = 0$. Define $X^0 = (X_s^0)_{s\in[t-,T]}$ by $X_{t-}^0 = x$, $X_T^0 = \hat{\xi}$, $X_s^0 = \gamma_s^{-\frac{1}{2}}(u_s^0 - H_s^0)$, $s \in [t,T)$, where H^0 is given by (7.28). By Lemma 7.4.4 it holds that $X^0 \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$. We furthermore obtain from Lemma 7.4.4 that, for the associated deviation, $D^0 = \gamma X^0 + \gamma^{\frac{1}{2}} H^0$. By definition of X^0, this yields $\gamma_s^{-\frac{1}{2}} D_s^0 = u_s^0$, $s \in [t,T)$. It follows that

$$\mathbf{d}(X^n, X^0) = \left(E\left[\int_t^T \left(\gamma_s^{-\frac{1}{2}} D_s^n - \gamma_s^{-\frac{1}{2}} D_s^0\right)^2 ds\right]\right)^{\frac{1}{2}} = \|\gamma^{-\frac{1}{2}} D^n - u^0\|_{\mathcal{L}_t^2},$$

and hence $\lim_{n\to\infty} \mathbf{d}(X^n, X^0) = 0$. □

We conclude with the proof of the equivalence of the control problems.

Proof of Corollary 7.5.3. For the proof, fix $t \in [0,T]$ and $x, d \in \mathbb{R}$. We know from Corollary 7.3.1 that $\mathcal{A}_t^{\mathrm{fv}}(x,d) \subseteq \mathcal{A}_t^{\mathrm{pm}}(x,d)$ and that $J_t^{\mathrm{fv}}(x,d,X) = J_t^{\mathrm{pm}}(x,d,X)$ for all $X \in \mathcal{A}_t^{\mathrm{fv}}(x,d)$. Hence,

$$\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{\mathrm{fv}}(x,d)} J_t^{\mathrm{fv}}(x,d,X) \geq \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{\mathrm{pm}}(x,d)} J_t^{\mathrm{pm}}(x,d,X).$$

It further follows from Theorem 7.5.2 that, for every $X^0 \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$, there exists a sequence $(X^n)_{n\in\mathbb{N}}$ in $\mathcal{A}_t^{\mathrm{fv}}(x,d)$ such that $J_t^{\mathrm{fv}}(x,d,X^n) \to J_t^{\mathrm{pm}}(x,d,X^0)$ in $L^1(\Omega,\mathcal{F}_t,P)$ as $n \to \infty$ (with a.s. convergence for a subsequence). Therefore, for every $X^0 \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$, it holds that

$$\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{\mathrm{fv}}(x,d)} J_t^{\mathrm{fv}}(x,d,X) \leq J_t^{\mathrm{pm}}(x,d,X^0).$$

This implies that

$$\operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{\mathrm{fv}}(x,d)} J_t^{\mathrm{fv}}(x,d,X) \leq \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{\mathrm{pm}}(x,d)} J_t^{\mathrm{pm}}(x,d,X).$$

□

8 Solution of the extended problem via reduction to a standard LQ stochastic control problem

We here continue our investigations of Chapter 7. The aim of the present chapter is to solve the extended problem of Section 7.3 (and, potentially, the finite-variation problem of Section 7.1).

We do this by reducing the extended problem to a standard LQ stochastic control problem in Section 8.1. Subsequently, in Section 8.2, we apply stochastic control literature (more precisely, Kohlmann and Tang [KT02]) to solve the LQ problem (under additional assumptions). A direct link between the control problems allows us to recover the solution of the extended problem in Corollary 8.2.4. In particular, we find that the unique optimal strategy in general is characterized by the solutions of two BSDEs. The first BSDE is (4.1), while the second one is linear (with in general unbounded coefficients) and enters the solution in case of a nonzero terminal position or when one tries to follow a nonzero target position over the course of the trading period. We provide a formula for the optimal strategy and a representation for the minimal costs. Finally, we in Section 8.3 present the Obizhaeva-Wang model (i.e., γ and ρ are constants, and $\eta \equiv 0$) with random targets $\hat{\xi}$ and ζ.

Throughout this chapter, we assume the setting of Section 3.1 and let $M^{(j)} = W^{(j)}$, $j \in \{1, \dots, m\}$, be independent Brownian motions. $\hat{\xi}$ and ζ are as introduced in Section 7.1. We further suppose that the condition $(\mathbf{C_{bdd}})$ is always in force.

This chapter makes extensive use of material from Sections 2, 3, 4, and 5 of the preprint [AKU22a] (joint work with Thomas Kruse and Mikhail Urusov).

8.1 Reduction to a standard LQ stochastic control problem

We recast the problem of minimizing J^{pm} over $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ as a standard LQ stochastic control problem. That means, we transform the extended problem into a control problem where the state is driven by a controlled SDE and the control acts as one of the arguments in that SDE and as one of the arguments in the integrand of the cost functional. Moreover, the pair of control and state enters linearly into the dynamics of the state and quadratic into the cost functional.

8.1.1 The first reduction

Observe that (7.26) in Proposition 7.4.2 shows that for $t \in [0,T]$, $x,d \in \mathbb{R}$, and $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ the costs $J_t^{\mathrm{pm}}(x,d,X)$ depend in a quadratic way on $(\overline{H}^X, \gamma^{-\frac{1}{2}} D^X)$. Moreover, (7.25) in Proposition 7.4.2 ensures that the dynamics of $\overline{H}^X$ depend linearly on $(\overline{H}^X, \gamma^{-\frac{1}{2}} D^X)$. These two observations suggest to view the minimization problem of J^{pm} over $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ as a standard LQ stochastic control problem with state process $\overline{H}^X$ and control $\gamma^{-\frac{1}{2}} D^X$, and motivate the following definitions.

For every $t \in [0,T]$, $x,d \in \mathbb{R}$, and $u \in \mathcal{L}_t^2$ (i.e., $u=(u_s)_{s\in[t,T]}$ is a progressively measurable process with $E[\int_t^T u_s^2 ds] < \infty$), we consider the state process $H^u = (H_s^u)_{s\in[t,T]}$ defined by

$$
\begin{aligned}
dH_s^u &= \left(\frac{1}{2}\left(\mu_s - \frac{1}{4}\sigma_s^2\right) H_s^u - \frac{1}{2}\left(2(\rho_s+\mu_s) - \sigma_s^2 - \sigma_s\eta_s\overline{r}_s\right) u_s\right) ds \\
&\quad + \left(\frac{1}{2}\sigma_s H_s^u - (\sigma_s + \eta_s\overline{r}_s) u_s\right) dW_s^{(1)} - \eta_s\sqrt{1-\overline{r}_s^2}\, u_s dW_s^{(2)}, \quad s \in [t,T], \qquad (8.1)\\
H_t^u &= \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}\, x,
\end{aligned}
$$

and the cost functional J^{LQ} defined by

$$
\begin{aligned}
J_t^{\mathrm{LQ}}\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}\, x, u\right) &= E_t\Bigg[\frac{1}{2}\left(H_T^u + \sqrt{\gamma_T}\hat{\xi}\right)^2 + \int_t^T (\kappa_s + \lambda_s) u_s^2 ds \\
&\quad + \int_t^T \lambda_s \left(H_s^u + \sqrt{\gamma_s}\zeta_s\right)^2 - 2\lambda_s \left(H_s^u + \sqrt{\gamma_s}\zeta_s\right) u_s ds\Bigg]. \qquad (8.2)
\end{aligned}
$$

Our standard LQ stochastic control problem (with possible cross-terms) is to minimize (8.2) over the set of admissible controls $\mathcal{L}_t^2$.

It holds that for every progressively measurable execution strategy $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ there exists a control $u \in \mathcal{L}_t^2$ such that the cost functional J^{pm} can be rewritten in terms of J^{LQ} (and $-\frac{d^2}{2\gamma_t}$). In fact, this is achieved by taking $u = \gamma^{-\frac{1}{2}} D^X$, as outlined in the motivation above. We state this as Lemma 8.1.1.

Lemma 8.1.1. *Let $t \in [0,T]$ and $x,d \in \mathbb{R}$. Suppose that $X \in \mathcal{A}_t^{pm}(x,d)$ with associated deviation D^X. Define $u=(u_s)_{s\in[t,T]}$ by $u_s = \gamma_s^{-\frac{1}{2}} D_s^X$, $s \in [t,T]$. It then holds that*

$u \in \mathcal{L}_t^2$ *and that*

$$J_t^{pm}(x,d,X) = J_t^{LQ}\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u\right) - \frac{d^2}{2\gamma_t} \quad a.s.$$

Proof. By definition of u we have that u is progressively measurable and, due to assumption **(B1)**, satisfies $E[\int_t^T u_s^2 ds] < \infty$; hence, $u \in \mathcal{L}_t^2$.

Let $\overline{H}_s^X = \gamma_s^{-\frac{1}{2}} D_s^X - \gamma_s^{\frac{1}{2}} X_s$, $s \in [t,T]$, be the scaled hidden deviation (7.17) associated to X. We can substitute $u = \gamma^{-\frac{1}{2}} D^X$ in the cost functional (7.26) and also in the dynamics (7.25) of $\overline{H}^X$. Observe that $\overline{H}^X$ follows the same dynamics as the state process H^u associated to u (see (8.1)), and that $\overline{H}_t^X = \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x = H_t^u$. Therefore, $\overline{H}^X$ and H^u coincide, which completes the proof. □

On the other hand, we may also start with $u \in \mathcal{L}_t^2$ and derive a progressively measurable execution strategy $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ such that the expected costs match.

Lemma 8.1.2. *Let $t \in [0,T]$ and $x, d \in \mathbb{R}$. Suppose that $u = (u_s)_{s\in[t,T]} \in \mathcal{L}_t^2$ and let H^u be the associated solution of* (8.1). *Define $X = (X_s)_{s\in[t-,T]}$ by*

$$X_s = \gamma_s^{-\frac{1}{2}}(u_s - H_s^u), \quad s \in [t,T), \quad X_{t-} = x, \quad X_T = \hat{\xi}.$$

It then holds that $X \in \mathcal{A}_t^{pm}(x,d)$ and that

$$J_t^{pm}(x,d,X) = J_t^{LQ}\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u\right) - \frac{d^2}{2\gamma_t} \quad a.s.$$

Proof. It follows from Lemma 7.4.4 that $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$. Moreover, we have from Lemma 7.4.4 that the associated deviation satisfies $D^X = \gamma X + \gamma^{\frac{1}{2}} H^u$, i.e., $D_s^X = \gamma_s^{\frac{1}{2}} u_s$, $s \in [t,T)$, and H^u is the scaled hidden deviation of X. It thus holds that $J_t^{\mathrm{pm}}(x,d,X)$ is given by (7.26) with $\overline{H}^X = H^u$. In the definition (8.2) of J^{LQ}, we may replace u under the integrals with respect to the Lebesgue measure by $\gamma^{-\frac{1}{2}} D^X$. This shows that $J_t^{\mathrm{pm}}(x,d,X) = J_t^{\mathrm{LQ}}(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u) - \frac{d^2}{2\gamma_t}$. □

Lemma 8.1.1 and Lemma 8.1.2 together with Corollary 7.5.3 establish the following equivalence of the control problems pertaining to J^{fv}, J^{pm}, and J^{LQ}.

Corollary 8.1.3. *For $t \in [0,T]$ and $x, d \in \mathbb{R}$ it holds that*

$$\begin{aligned} \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{fv}(x,d)} J_t^{fv}(x,d,X) &= \operatorname*{ess\,inf}_{X\in\mathcal{A}_t^{pm}(x,d)} J_t^{pm}(x,d,X) \\ &= \operatorname*{ess\,inf}_{u\in\mathcal{L}_t^2} J_t^{LQ}\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u\right) - \frac{d^2}{2\gamma_t} \quad a.s. \end{aligned}$$

Proof. The first equality is just Corollary 7.5.3. The inequality that $J_t^{\mathrm{pm}}(x,d,X) \geq \operatorname{ess\,inf}_{u\in\mathcal{L}_t^2} J_t^{\mathrm{LQ}}(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u) - \frac{d^2}{2\gamma_t}$ for all $X \in \mathcal{A}_t^{\mathrm{pm}}(x,d)$ follows from Lemma 8.1.1, whereas $\operatorname{ess\,inf}_{X\in\mathcal{A}_t^{\mathrm{pm}}(x,d)} J_t^{\mathrm{pm}}(x,d,X) \leq J_t^{\mathrm{LQ}}(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u) - \frac{d^2}{2\gamma_t}$ for all $u \in \mathcal{L}_t^2$ is an immediate consequence of Lemma 8.1.2. □

Furthermore, Lemma 8.1.1, Lemma 8.1.2, and Corollary 8.1.3 provide a method to obtain an optimal progressively measurable execution strategy, and potentially an optimal finite-variation execution strategy, from an optimal control of the standard LQ stochastic control problem and vice versa.

Corollary 8.1.4. *Let $t \in [0,T]$ and $x, d \in \mathbb{R}$.*

(i) Suppose that $X^* = (X_s^*)_{s\in[t-,T]} \in \mathcal{A}_t^{pm}(x,d)$ minimizes J^{pm} over $\mathcal{A}_t^{pm}(x,d)$ and let D^{X^*} be the associated deviation process. Then, $u^* = (u_s^*)_{s\in[t,T]}$ defined by

$$u_s^* = \gamma_s^{-\frac{1}{2}} D_s^{X^*}, \quad s \in [t,T],$$

minimizes J^{LQ} over $\mathcal{L}_t^2$.

(ii) Suppose that $u^* = (u_s^*)_{s\in[t,T]} \in \mathcal{L}_t^2$ minimizes J^{LQ} over $\mathcal{L}_t^2$ and let H^{u^*} be the associated solution of (8.1) *for u^*. Then, $X^* = (X_s^*)_{s\in[t,T]}$ defined by*

$$X_s^* = \gamma_s^{-\frac{1}{2}}(u_s^* - H_s^{u^*}), \quad s \in [t,T), \quad X_{t-}^* = x, \quad X_T^* = \hat{\xi},$$

minimizes J^{pm} over $\mathcal{A}_t^{pm}(x,d)$. Moreover, if $X^* \in \mathcal{A}_t^{fv}(x,d)$ (in the sense that there is an element of $\mathcal{A}_t^{fv}(x,d)$ within the equivalence class of X^*, see (7.15)*), then X^* minimizes J^{fv} over $\mathcal{A}_t^{fv}(x,d)$.*

Proof. Part (i) is an immediate consequence of Corollary 8.1.3 and Lemma 8.1.1. Part (ii) follows directly from Corollary 8.1.3 and Lemma 8.1.2. □

Moreover, we keep uniqueness of a minimizer.

Corollary 8.1.5. *Let $t \in [0,T]$ and $x, d \in \mathbb{R}$. There exists a ($\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e.) unique minimizer of J^{pm} over $\mathcal{A}_t^{pm}(x,d)$ if and only if there exists a ($\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e.) unique minimizer of J^{LQ} over $\mathcal{L}_t^2$.*

Proof. Assume first that X^* uniquely minimizes J^{pm} over $\mathcal{A}_t^{\mathrm{pm}}(x,d)$. Then, by Corollary 8.1.4(i), $u^* = \gamma^{-\frac{1}{2}} D^{X^*}$ defines a minimizer of J^{LQ} over $\mathcal{L}_t^2$. Suppose that $\widetilde{u}^*$ also minimizes J^{LQ} over $\mathcal{L}_t^2$. We have from Corollary 8.1.4(ii) that $\widetilde{X}^*$ defined by $\widetilde{X}_s^* = \gamma_s^{-\frac{1}{2}}(\widetilde{u}_s^* - H_s^{\widetilde{u}^*})$, $s \in [t,T)$, $\widetilde{X}_{t-}^* = x$, $\widetilde{X}_T^* = \hat{\xi}$, minimizes J^{pm} over $\mathcal{A}_t^{\mathrm{pm}}(x,d)$. Lemma 7.4.4 yields that $\widetilde{u}^* = \gamma^{-\frac{1}{2}} D^{\widetilde{X}^*}$. Since $X^* = \widetilde{X}^*$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e., it holds that $D^{X^*} = D^{\widetilde{X}^*}$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e., and we conclude that $u^* = \widetilde{u}^*$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e.

For the other direction, assume that u^* uniquely minimizes J^{LQ} over $\mathcal{L}_t^2$. We obtain from Corollary 8.1.4(ii) that X^* defined by $X_s^* = \gamma_s^{-\frac{1}{2}}(u_s^* - H_s^{u^*})$, $s \in [t,T)$, $X_{t-}^* = x$,

$X_T^* = \hat{\xi}$, minimizes J^{pm} over $\mathcal{A}_t^{\mathrm{pm}}(x,d)$. From Lemma 7.4.4 we have that $u^* = \gamma^{-\frac{1}{2}} D^{X^*}$. Suppose that $\widetilde{X}^*$ minimizes J^{pm} over $\mathcal{A}_t^{\mathrm{pm}}(x,d)$. Then, Corollary 8.1.4(i) implies that $\widetilde{u}^* = \gamma^{-\frac{1}{2}} D^{\widetilde{X}^*}$ defines a minimizer of J^{LQ} over $\mathcal{L}_t^2$. Since $\gamma^{-\frac{1}{2}} D^{X^*} = u^* = \widetilde{u}^* = \gamma^{-\frac{1}{2}} D^{\widetilde{X}^*}$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e., we get that $\mathbf{d}(X^*, \widetilde{X}^*) = 0$, and hence that $X^* = \widetilde{X}^*$ $\mathcal{D}_{W^{(1)}}|_{[t,T]}$-a.e. (recall that $\mathbf{d}$ defined in (7.30) is a metric by Lemma 7.5.1). □

8.1.2 Formulation without cross-terms

The last integral in the definition (8.2) of the cost functional J^{LQ} involves a product between the state process H^u and the control process u. A larger part of the literature on LQ optimal control considers cost functionals that do not contain such cross-terms. In particular, this applies to [KT02], whose results we want to apply in Section 8.2. For this reason we provide in this subsection a reformulation of the control problem (8.1) and (8.2) that does not contain cross-terms. In order to carry out the transformation necessary for this, we need to impose a further condition on our model inputs: we assume in the current subsection that there exists a constant $C \in [0,\infty)$ such that for all $s \in [0,T]$ we have P-a.s. that

$$|\lambda_s| \le C|\lambda_s + \kappa_s|. \tag{8.3}$$

Note that this assumption ensures that the set $\{\lambda_s + \kappa_s = 0\}$ is a subset of $\{\lambda_s = 0\}$ (up to a P-null set). For this reason, in the sequel, we use the following convention:

$$\text{under (8.3) we always understand } \frac{\lambda_s}{\lambda_s+\kappa_s} = 0 \text{ on the set } \{\lambda_s + \kappa_s = 0\}.$$

Now in order to get rid of the cross-term in (8.2) we transform for $t \in [0,T]$ any control process $u \in \mathcal{L}_t^2$ in an affine way to $\hat{u}_s = u_s - \frac{\lambda_s}{\lambda_s+\kappa_s}(H_s^u + \sqrt{\gamma_s}\zeta_s)$, $s \in [t,T]$. This leads to the new controlled state process $\widehat{H}^{\hat{u}} = (\widehat{H}_s^{\hat{u}})_{s\in[t,T]}$ that is defined for every $t \in [0,T]$, $x, d \in \mathbb{R}$, and $\hat{u} \in \mathcal{L}_t^2$ by

$$\begin{aligned}
d\widehat{H}_s^{\hat{u}} &= \left(\frac{\mu_s}{2} - \frac{1}{8}\sigma_s^2 - \frac{\lambda_s}{\lambda_s+\kappa_s}\left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s\eta_s\overline{r}_s}{2} \right) \right) \widehat{H}_s^{\hat{u}} ds \\
&\quad - \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s\eta_s\overline{r}_s}{2} \right) \hat{u}_s ds - \frac{\lambda_s}{\lambda_s+\kappa_s}\left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s\eta_s\overline{r}_s}{2} \right) \sqrt{\gamma_s}\zeta_s ds \\
&\quad + \left(\frac{\sigma_s}{2} - \frac{\lambda_s}{\lambda_s+\kappa_s}(\sigma_s + \eta_s\overline{r}_s) \right) \widehat{H}_s^{\hat{u}} dW_s^{(1)} - (\sigma_s + \eta_s\overline{r}_s)\hat{u}_s dW_s^{(1)} \\
&\quad - \frac{\lambda_s}{\lambda_s+\kappa_s}(\sigma_s + \eta_s\overline{r}_s)\sqrt{\gamma_s}\zeta_s dW_s^{(1)} - \frac{\lambda_s}{\lambda_s+\kappa_s}\eta_s\sqrt{1-\overline{r}_s^2}\,\widehat{H}_s^{\hat{u}} dW_s^{(2)} \\
&\quad - \eta_s\sqrt{1-\overline{r}_s^2}\,\hat{u}_s dW_s^{(2)} - \frac{\lambda_s}{\lambda_s+\kappa_s}\eta_s\sqrt{1-\overline{r}_s^2}\sqrt{\gamma_s}\zeta_s dW_s^{(2)}, \quad s \in [t,T], \\
\widehat{H}_t^{\hat{u}} &= \frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x.
\end{aligned} \tag{8.4}$$

The meaning of (8.4) is that we only reparametrize the control ($u \to \hat{u}$) but not the state variable ($\widehat{H}^{\hat{u}} = H^u$), see Lemma 8.1.6 for the formal statement. For $t \in [0,T]$, $x, d \in \mathbb{R}$, $\hat{u} \in \mathcal{L}_t^2$ and associated $\widehat{H}^{\hat{u}}$ defined by (8.4), we define the cost functional $\hat{J}^{\mathrm{LQ}}$ by

$$\begin{aligned} &\hat{J}_t^{\mathrm{LQ}}\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, \hat{u}\right) \\ &= E_t\left[\frac{1}{2}\left(\widehat{H}_T^{\hat{u}} + \sqrt{\gamma_T}\hat{\xi}\right)^2 + \int_t^T \frac{\lambda_s \kappa_s}{\lambda_s + \kappa_s}\left(\widehat{H}_s^{\hat{u}} + \sqrt{\gamma_s}\zeta_s\right)^2 + (\lambda_s + \kappa_s)\hat{u}_s^2 ds\right]. \end{aligned} \tag{8.5}$$

This cost functional does not exhibit cross-terms and is equivalent to J^{LQ} of (8.2) in the sense of the following lemma.

Lemma 8.1.6. *Assume that* (8.3) *holds true. Let* $t \in [0,T]$ *and* $x, d \in \mathbb{R}$.

(i) Suppose that $u = (u_s)_{s\in[t,T]} \in \mathcal{L}_t^2$ *with associated state process* H^u *defined by* (8.1). *Then,* $\hat{u} = (\hat{u}_s)_{s\in[t,T]}$ *defined by*

$$\hat{u}_s = u_s - \frac{\lambda_s}{\lambda_s + \kappa_s}(H_s^u + \sqrt{\gamma_s}\zeta_s), \quad s \in [t,T],$$

is in $\mathcal{L}_t^2$*, and it holds that* $\widehat{H}^{\hat{u}} = H^u$ *and* $J_t^{LQ}(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u) = \hat{J}_t^{LQ}(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, \hat{u})$.

(ii) Suppose that $\hat{u} = (\hat{u}_s)_{s\in[t,T]} \in \mathcal{L}_t^2$ *with associated state process* $\widehat{H}^{\hat{u}}$ *defined by* (8.4). *Then,* $u = (u_s)_{s\in[t,T]}$ *defined by*

$$u_s = \hat{u}_s + \frac{\lambda_s}{\lambda_s + \kappa_s}(\widehat{H}_s^{\hat{u}} + \sqrt{\gamma_s}\zeta_s), \quad s \in [t,T],$$

is in $\mathcal{L}_t^2$*, and it holds that* $H^u = \widehat{H}^{\hat{u}}$ *and* $J_t^{LQ}(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u) = \hat{J}_t^{LQ}(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, \hat{u})$.

(iii) It holds that

$$\operatorname*{ess\,inf}_{u\in\mathcal{L}_t^2} J_t^{LQ}\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, u\right) = \operatorname*{ess\,inf}_{\hat{u}\in\mathcal{L}_t^2} \hat{J}_t^{LQ}\left(\frac{d}{\sqrt{\gamma_t}} - \sqrt{\gamma_t}x, \hat{u}\right).$$

Proof. Note first that, if $u, \hat{u} \in \mathcal{L}_t^2$ with the relations $\hat{u} = u - \frac{\lambda}{\lambda+\kappa}(H^u + \sqrt{\gamma}\zeta)$ and $\widehat{H}^{\hat{u}} = H^u$, then

$$\begin{aligned} &\lambda_s(H_s^u + \sqrt{\gamma_s}\zeta_s)^2 - 2\lambda_s(H_s^u + \sqrt{\gamma_s}\zeta_s)u_s + (\kappa_s + \lambda_s)u_s^2 \\ &= \lambda_s(H_s^u + \sqrt{\gamma_s}\zeta_s)^2 - (\lambda_s + \kappa_s)\frac{\lambda_s^2}{(\lambda_s + \kappa_s)^2}(H_s^u + \sqrt{\gamma_s}\zeta_s)^2 \\ &\quad + (\lambda_s + \kappa_s)\left(u_s - \frac{\lambda_s}{\lambda_s + \kappa_s}(H_s^u + \sqrt{\gamma_s}\zeta_s)\right)^2 \\ &= \frac{\lambda_s \kappa_s}{\lambda_s + \kappa_s}\left(\widehat{H}_s^{\hat{u}} + \sqrt{\gamma_s}\zeta_s\right)^2 + (\lambda_s + \kappa_s)\hat{u}_s^2, \quad s \in [t,T]. \end{aligned} \tag{8.6}$$

(i) We have that $\hat{u}$ is progressively measurable. Furthermore, the fact that all of $E[\int_t^T u_s^2 ds]$, $E[\sup_{s\in[t,T]}(H_s^u)^2]$, and $E[\int_0^T \gamma_s \zeta_s^2 ds]$ are finite and (8.3) imply that $E[\int_t^T \hat{u}_s^2 ds] < \infty$. Hence, $\hat{u} \in \mathcal{L}_t^2$. Substituting $u_s = \hat{u}_s + \frac{\lambda_s}{\lambda_s+\kappa_s}(H_s^u + \sqrt{\gamma_s}\zeta_s)$, $s \in [t,T]$, in (8.1) leads to (8.4). Equality of the cost functionals follows from (8.6).

(ii) Note that (8.4) is an SDE that is linear in $\widehat{H}^{\hat{u}}$, $\hat{u}$, and $\sqrt{\gamma}\zeta$. Furthermore, boundedness of $\rho, \mu, \sigma, \eta, \overline{r}$ and (8.3) imply that the coefficients of the SDE are bounded. Since moreover $E[\int_t^T (\hat{u}_s)^2 + \gamma_s \zeta_s^2 ds] < \infty$ and $\widehat{H}_t^{\hat{u}}$ is square integrable, we know that $E[\sup_{s\in[t,T]}(\widehat{H}_s^{\hat{u}})^2] < \infty$ (see, e.g., [Zha17, Theorem 3.2.2 and Theorem 3.3.1]). We can thus argue similar to (i) that $u \in \mathcal{L}_t^2$. A substitution of $\hat{u}$ in (8.4) yields (8.1). Equality of the cost functionals again follows from (8.6).

(iii) This is an immediate consequence of (i) and (ii). □

As a corollary, we obtain the following link between an optimal control for $\hat{J}^{\text{LQ}}$ and an optimal control for J^{LQ}.

Corollary 8.1.7. *Assume that* (8.3) *holds true. Let* $t \in [0,T]$ *and* $x, d \in \mathbb{R}$.

(i) Suppose that $u^* = (u_s^*)_{s\in[t,T]} \in \mathcal{L}_t^2$ *is a (unique) optimal control for* J^{LQ}*, and let* H^{u^*} *be the solution of* (8.1) *for* u^**. Then,* $\hat{u}^* = (\hat{u}_s^*)_{s\in[t,T]}$ *defined by*

$$\hat{u}_s^* = u_s^* - \frac{\lambda_s}{\lambda_s + \kappa_s}(H_s^{u^*} + \sqrt{\gamma_s}\zeta_s), \quad s \in [t,T],$$

is a (unique) optimal control in $\mathcal{L}_t^2$ *for* $\hat{J}^{LQ}$*. Moreover,* $\widehat{H}^{\hat{u}^*} = H^{u^*}$.

(ii) Suppose that $\hat{u}^* = (\hat{u}_s^*)_{s\in[t,T]} \in \mathcal{L}_t^2$ *is a (unique) optimal control for* $\hat{J}^{LQ}$*, and let* $\widehat{H}^{\hat{u}^*}$ *be the solution of* (8.4) *for* $\hat{u}^*$*. Then,* $u^* = (u_s^*)_{s\in[t,T]}$ *defined by*

$$u_s^* = \hat{u}_s^* + \frac{\lambda_s}{\lambda_s + \kappa_s}(\widehat{H}_s^{\hat{u}^*} + \sqrt{\gamma_s}\zeta_s), \quad s \in [t,T],$$

is a (unique) optimal control in $\mathcal{L}_t^2$ *for* J^{LQ}*. Moreover,* $H^{u^*} = \widehat{H}^{\hat{u}^*}$.

Proof. This is clear from Lemma 8.1.6. □

8.2 Solving the LQ problem and the extended problem

We now solve the control problems. More precisely, we consider the problem formulation of Section 8.1.2 and obtain, under appropriate assumptions, via [KT02] existence of a unique optimal control in terms of two associated BSDEs and a representation for the minimal costs in Theorem 8.2.3. From this, we derive in Corollary 8.2.4 a unique optimal strategy for the extended problem of Section 7.3 via Corollary 8.1.7, Corollary 8.1.4, and Corollary 8.1.5.

In our general setting (see the end of the introduction of the present chapter) we, for this section, additionally assume that the filtration $(\mathcal{F}_s)_{s\in[0,T]}$ for the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_s)_{s\in[0,T]}, P)$ is the augmented natural filtration of the Brownian motion $(W^{(1)}, \dots, W^{(m)})^\top$. Furthermore, we set the initial time to $t = 0$. We moreover assume that $(\mathbf{C_{nonneg}})$ is in force.

Remark 8.2.1. Note that the assumption $(\mathbf{C_{nonneg}})$ of nonnegativity of λ and κ is necessary to apply the results of [KT02]. Indeed, [KT02] requires that $\lambda+\kappa$ (the coefficient in front of $\hat{u}^2$ in (8.5)) and $\frac{\lambda\kappa}{\lambda+\kappa}$ (the coefficient in front of $(\widehat{H}^{\hat{u}}_s + \sqrt{\gamma_s}\zeta_s)^2$ in (8.5)) are nonnegative and bounded, which implies that λ and κ have to be nonnegative. Moreover, we note that nonnegativity of λ and κ ensures that (8.3) is satisfied, and we observe that the mentioned coefficients $\lambda + \kappa$ and $\frac{\lambda\kappa}{\lambda+\kappa}$ are bounded, as required. Indeed, it clearly holds that $\frac{\lambda\kappa}{\lambda+\kappa} \le \kappa$, and it remains to recall that $\mu, \sigma, \rho, \eta, \overline{r}$, and λ are bounded.

Observe that the standard LQ stochastic control problem without cross-terms of Section 8.1.2, which consists of minimizing $\hat{J}^{\mathrm{LQ}}$ in (8.5) with state dynamics given by (8.4), is of the form considered in [KT02, (79)–(81)] (note also Table 8.1). The solution can be described by the two BSDEs [KT02, (9) and (85)]. In our setting, the BSDE of Riccati-type [KT02, (9)], after some computations, turns out to correspond to BSDE (4.1) (in the form of (4.3)). Recall from Proposition 4.3.2 that, if $(\mathbf{C_{\geq\varepsilon}})$ or $(\mathbf{C_s})$ holds, then we are guaranteed existence of a unique solution $(Y, Z, M^\perp)$ of BSDE (4.1) and it holds that $(\sigma^2 + \eta^2 + 2\sigma\eta\overline{r})Y + \kappa + \lambda \ge \overline{c}$ $\mathcal{D}_{W^{(1)}}$-a.e. for some $\overline{c} \in (0, \infty)$. For such a solution $(Y, Z, M^\perp)$ of BSDE (4.1), we define $\widetilde{\vartheta} = (\widetilde{\vartheta}_s)_{s\in[0,T]}$ as in (5.22), and consider the second BSDE [KT02, (85)], which in our setting reads

$$\begin{aligned} d\psi_s = &-\Bigg[\left(\frac{\mu_s}{2} - \frac{\sigma_s^2}{8} - \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s\eta_s\overline{r}_s}{2}\right)\widetilde{\vartheta}_s\right)\psi_s \\ &\qquad + \left(\frac{\sigma_s}{2} - (\sigma_s + \eta_s\overline{r}_s)\,\widetilde{\vartheta}_s\right)\phi_s^{(1)} - \eta_s\sqrt{1-\overline{r}_s^2}\,\widetilde{\vartheta}_s\,\phi_s^{(2)} + \sqrt{\gamma_s}\zeta_s\lambda_s(\widetilde{\vartheta}_s - 1)\Bigg]ds \\ &+ \sum_{j=1}^m \phi_s^{(j)} dW_s^{(j)}, \quad s \in [0,T], \\ \psi_T = &-\frac{1}{2}\sqrt{\gamma_T}\hat{\xi}. \end{aligned} \tag{8.7}$$

Definition 8.2.2. A pair (ψ, ϕ) with $\phi = (\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(m)})^\top$ is called a solution of BSDE (8.7) if ψ is an adapted continuous process with $E[\sup_{s\in[0,T]} \psi_s^2] < \infty$, ϕ is

[KT02]	Our setting
M	$\frac{1}{2}$
ξ	$-\sqrt{\gamma_T}\hat{\xi}$
Q	$\frac{\lambda\kappa}{\lambda+\kappa}$
q	$-\sqrt{\gamma}\zeta$
N	$\lambda+\kappa$
A	$\frac{\mu}{2}-\frac{\sigma^2}{8}-\frac{\lambda}{\lambda+\kappa}(\rho+\mu-\frac{\sigma^2+\sigma\eta\overline{r}}{2})$
B	$-(\rho+\mu-\frac{\sigma^2+\sigma\eta\overline{r}}{2})$
f	$-\frac{\lambda}{\lambda+\kappa}(\rho+\mu-\frac{\sigma^2+\sigma\eta\overline{r}}{2})\sqrt{\gamma}\zeta$
C_1	$\frac{\sigma}{2}-\frac{\lambda}{\lambda+\kappa}(\sigma+\eta\overline{r})$
C_2	$-\frac{\lambda}{\lambda+\kappa}\eta\sqrt{1-\overline{r}^2}$
D_1	$-(\sigma+\eta\overline{r})$
D_2	$-\eta\sqrt{1-\overline{r}^2}$
g_1	$-\frac{\lambda}{\lambda+\kappa}(\sigma+\eta\overline{r})\sqrt{\gamma}\zeta$
g_2	$-\frac{\lambda}{\lambda+\kappa}\eta\sqrt{1-\overline{r}^2}\sqrt{\gamma}\zeta$

Table 8.1: We make the following identifications of quantities in [KT02] with quantities in our setting.

progressively measurable with $\int_0^T \|\phi_s\|_m^2 ds < \infty$ P-a.s., and BSDE (8.7) is satisfied P-a.s.

Observe that BSDE (8.7) is linear, but existence of a solution is not evident since the coefficients of this BSDE in general are unbounded.

For a solution $(Y, Z, M^\perp)$ of BSDE (4.1) and a corresponding solution (ψ, ϕ) of BSDE (8.7), we define $\vartheta^0 = (\vartheta^0_s)_{s\in[0,T]}$ by

$$\vartheta^0_s = \frac{\left(\rho_s+\mu_s-\frac{\sigma_s^2+\sigma_s\eta_s\overline{r}_s}{2}\right)\psi_s+(\sigma_s+\eta_s\overline{r}_s)\phi_s^{(1)}+\eta_s\sqrt{1-\overline{r}_s^2}\phi_s^{(2)}-\sqrt{\gamma_s}\zeta_s\lambda_s}{(\sigma_s^2+\eta_s^2+2\sigma_s\eta_s\overline{r}_s)Y_s+\kappa_s+\lambda_s}, \; s\in[0,T]. \tag{8.8}$$

We then further introduce for $x, d \in \mathbb{R}$ the SDE

$$d\widehat{H}^*_s = \widehat{H}^*_s\,dK_s + dL_s, \quad s\in[0,T], \quad \widehat{H}^*_0 = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0}x, \tag{8.9}$$

where

$$\begin{aligned} K_r &= \int_0^r \left(\frac{\mu_s}{2} - \frac{\sigma_s^2}{8} - \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \overline{r}_s}{2} \right) \widetilde{\vartheta}_s \right) ds \\ &\quad + \int_0^r \left(\frac{\sigma_s}{2} - (\sigma_s + \eta_s \overline{r}_s) \widetilde{\vartheta}_s \right) dW_s^{(1)} - \int_0^r \eta_s \sqrt{1 - \overline{r}_s^2}\, \widetilde{\vartheta}_s dW_s^{(2)}, \\ L_r &= \int_0^r \left(\rho_s + \mu_s - \frac{\sigma_s^2 + \sigma_s \eta_s \overline{r}_s}{2} \right) \vartheta_s^0 ds + \int_0^r (\sigma_s + \eta_s \overline{r}_s) \vartheta_s^0 dW_s^{(1)} \\ &\quad + \int_0^r \eta_s \sqrt{1 - \overline{r}_s^2}\, \vartheta_s^0 dW_s^{(2)}, \quad r \in [0,T]. \end{aligned} \tag{8.10}$$

We will show that the solution $\widehat{H}^*$ of (8.9) is the optimal state process in the stochastic control problem to minimize $\hat{J}^{\mathrm{LQ}}$ of (8.5). Notice that $\widehat{H}^*$ can be easily expressed via K and L in closed form, e.g.,

$$\widehat{H}_r^* = \mathcal{E}(K)_r \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x + \int_0^r \mathcal{E}(K)_s^{-1} d\left(L_s - [L,K]_s \right) \right), \quad r \in [0,T]. \tag{8.11}$$

In the next theorem, we summarize consequences from [KT02] in our setting to obtain a minimizer of $\hat{J}^{\mathrm{LQ}}$ in (8.5) and a representation of the minimal costs.

Theorem 8.2.3. *Let the assumptions of this section be in force and assume that* $(\boldsymbol{C_{\geq \varepsilon}})$ *or* $(\boldsymbol{C_s})$ *is satisfied. Let* $(Y, Z, M^{\perp})$ *be the unique solution of BSDE* (4.1) *(cf. Proposition 4.3.2).*

(i) There exists a unique solution (ψ, ϕ) *of BSDE* (8.7)*.*

(ii) Let $x, d \in \mathbb{R}$*, and let* $\widehat{H}^*$ *be the solution of SDE* (8.9)*. Then,* $\hat{u}^* = (\hat{u}_s^*)_{s \in [0,T]}$ *defined by*

$$\hat{u}_s^* = \left(\widetilde{\vartheta}_s - \frac{\lambda_s}{\lambda_s + \kappa_s} \right) \widehat{H}_s^* - \left(\vartheta_s^0 + \sqrt{\gamma_s} \zeta_s \frac{\lambda_s}{\lambda_s + \kappa_s} \right), \quad s \in [0,T], \tag{8.12}$$

is the unique optimal control in $\mathcal{L}_0^2$ *for* $\hat{J}^{LQ}$*, and* $\widehat{H}^*$ *is the corresponding state process (i.e.,* $\widehat{H}^* = \widehat{H}^{\hat{u}^*}$*).*

(iii) Let $x, d \in \mathbb{R}$*. The costs associated to the optimal control* (8.12) *are given by*

$$\begin{aligned} \inf_{\hat{u} \in \mathcal{L}_0^2} \hat{J}_0^{LQ} \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x, \hat{u} \right) &= \hat{J}_0^{LQ} \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x, \hat{u}^* \right) \\ &= Y_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right)^2 - 2\psi_0 \left(\frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0} x \right) + C_0, \end{aligned}$$

where

$$\begin{aligned} C_0 &= \frac{1}{2} E\left[\gamma_T \hat{\xi}^2 \right] + E\left[\int_0^T \gamma_s \lambda_s \zeta_s^2 ds \right] \\ &\quad - E\left[\int_0^T (\vartheta_s^0)^2 \left((\sigma_s^2 + \eta_s^2 + 2\sigma_s \eta_s \overline{r}_s) Y_s + \kappa_s + \lambda_s \right) ds \right]. \end{aligned} \tag{8.13}$$

Proof. Observe that the problem in Section 8.1.2 fits the problem considered in [KT02, Section 5]. In particular, note that the coefficients in SDE (8.4) for $\widehat{H}^{\hat{u}}$ and in the cost functional $\hat{J}^{\mathrm{LQ}}$ (see (8.5)) are bounded, and that the inhomogeneities are in $\mathcal{L}_0^2$. The initial state $\widehat{H}_0^{\hat{u}} = \frac{d}{\sqrt{\gamma_0}} - \sqrt{\gamma_0}x$ is in $\mathbb{R}$. Moreover, we have that $\frac{1}{2}$, $\frac{\lambda\kappa}{\lambda+\kappa}$, and $\lambda + \kappa$ are nonnegative. Furthermore, the filtration by assumption in this section is generated by the Brownian motion $(W^{(1)}, \dots, W^{(m)})^\top$.

(i) This is due to [KT02, Theorem 5.1].

(ii) The first part of [KT02, Theorem 5.2] yields the existence of a unique optimal control $\hat{u}^*$, which is given in feedback form by the formula $\hat{u}^* = \theta \widehat{H}^{\hat{u}^*} + u^0$, where

$$\begin{aligned}\theta &= \frac{\left(\rho + \mu - \frac{\lambda}{\lambda+\kappa}(\sigma^2 + 2\sigma\eta\overline{r} + \eta^2)\right) Y + (\sigma + \eta\overline{r})Z^{(1)} + \eta\sqrt{1-\overline{r}^2}Z^{(2)}}{\lambda + \kappa + (\sigma^2 + 2\sigma\eta\overline{r} + \eta^2)Y} \\ &= \widetilde{\vartheta} - \frac{\lambda}{\lambda+\kappa}\end{aligned}$$

and

$$\begin{aligned}u^0 = &-\Bigg(\left(\rho + \mu - \frac{\sigma^2 + \sigma\eta\overline{r}}{2}\right)\psi + \frac{\lambda}{\lambda+\kappa}\sqrt{\gamma}\zeta(\sigma^2 + 2\sigma\eta\overline{r} + \eta^2)Y \\ &\qquad + (\sigma + \eta\overline{r})\phi^{(1)} + \eta\sqrt{1-\overline{r}^2}\,\phi^{(2)}\Bigg) \cdot \left(\lambda + \kappa + (\sigma^2 + 2\sigma\eta\overline{r} + \eta^2)Y\right)^{-1} \\ = &-\left(\vartheta^0 + \sqrt{\gamma}\zeta\frac{\lambda}{\lambda+\kappa}\right).\end{aligned}$$

We obtain (8.9) by plugging the formula for $\hat{u}^*$ into the dynamics (8.4) for $\widehat{H}^{\hat{u}^*}$.

(iii) The second part of [KT02, Theorem 5.2], after a straightforward computation for C_0, provides us with the optimal costs. □

For our trade execution problem, this implies the following.

Corollary 8.2.4. *Let the assumptions of this section be in force and assume that* $(\boldsymbol{C_{\geq\varepsilon}})$ *or* $(\boldsymbol{C_s})$ *is satisfied. Let* $(Y, Z, M^\perp)$ *be the unique solution of BSDE* (4.1), (ψ, ϕ) *the unique solution of BSDE* (8.7), *and recall definitions* (5.22) *of* $\widetilde{\vartheta}$ *and* (8.8) *of* ϑ^0. *Let* $x, d \in \mathbb{R}$. *Then,* $X^* = (X_s^*)_{s\in[0-,T]}$ *defined by*

$$\begin{aligned}&X_{0-}^* = x, \quad X_T^* = \hat{\xi}, \\ &X_s^* = \gamma_s^{-\frac{1}{2}}\left(\left(\widetilde{\vartheta}_s - 1\right)\widehat{H}_s^* - \vartheta_s^0\right), \quad s \in [0, T),\end{aligned} \tag{8.14}$$

with $\widehat{H}^*$ *from* (8.9)*, is the unique (up to* $\mathcal{D}_{W^{(1)}}$*-null sets) optimal execution strategy in* $\mathcal{A}_0^{pm}(x,d)$ *for* J^{pm}*. The associated costs are given by*

$$\begin{aligned}\inf_{X\in\mathcal{A}_0^{pm}(x,d)} J_0^{pm}(x,d,X) &= J_0^{pm}(x,d,X^*)\\ &= \frac{Y_0}{\gamma_0}(d-\gamma_0 x)^2 - \frac{d^2}{2\gamma_0} - 2\frac{\psi_0}{\sqrt{\gamma_0}}(d-\gamma_0 x) + C_0\end{aligned}$$

with C_0 *from* (8.13)*.*

Proof. By Theorem 8.2.3(ii), $\hat{u}^*$ from (8.12) is the unique optimal control in $\mathcal{L}_0^2$ for $\hat{J}^{\mathrm{LQ}}$, and $\widehat{H}^* = \widehat{H}^{\hat{u}^*}$. Corollary 8.1.7(ii) implies further that $u^* = \hat{u}^* + \frac{\lambda}{\lambda+\kappa}(\widehat{H}^{\hat{u}^*} + \sqrt{\gamma}\zeta) = \widetilde{\vartheta}\widehat{H}^* - \vartheta^0$ is the unique optimal control in $\mathcal{L}_0^2$ for J^{LQ}, and $H^{u^*} = \widehat{H}^{\hat{u}^*} = \widehat{H}^*$. It then follows from Corollary 8.1.4(ii) and Corollary 8.1.5 that X^*_{0-}, $X^*_T = \hat{\xi}$, $X^*_s = \gamma_s^{-\frac{1}{2}}(u^*_s - H^{u^*}_s) = \gamma_s^{-\frac{1}{2}}((\widetilde{\vartheta}_s - 1)\widehat{H}^*_s - \vartheta^0_s)$, $s\in[0,T)$, is the unique optimal strategy in $\mathcal{A}_0^{\mathrm{pm}}(x,d)$ for J^{pm}. The representation for the minimal costs is an immediate consequence of Theorem 8.2.3(iii), Lemma 8.1.6(iii), and Corollary 8.1.3. □

A special case of our setting is when, as in Chapter 5, we require to close the position (i.e., $\hat{\xi} = 0$) and do not try to follow a (nonzero) target process (i.e., ζ or the risk coefficient λ vanishes). We remark that BSDE (4.1) neither contains $\hat{\xi}$ nor ζ. In particular, the solution component Y, the process $\widetilde{\vartheta}$, and the process K from (8.10) do not depend on the choice of $\hat{\xi}$ or ζ (although they depend on the choice of λ). In contrast, BSDE (8.7) involves both $\hat{\xi}$ and ζ. If $\hat{\xi} = 0$ and at least one of λ and ζ is equivalent to 0, we have that (ψ,ϕ) from (8.7), ϑ^0 from (8.8), L from (8.10), and C_0 from (8.13) vanish.

In general, the terminal costs and the running costs in (8.2) (and also (8.5)) contain terms such as $(H^u_T + \sqrt{\gamma_T}\hat{\xi})^2$ and $\lambda_s(H^u_s + \sqrt{\gamma_s}\zeta_s)^2$, which are inhomogeneous. In the case where $\hat{\xi} = 0$ and where at least one of λ and ζ vanishes, the problem becomes homogeneous. In that case, we could also apply results of [SXY21]. For instance, by applying [SXY21] to the problem of Section 8.1.1 (note that [SXY21] allows for cross-terms), it is possible to obtain the results of Corollary 8.2.4 also if we replace the assumptions $(\mathbf{C_{nonneg}})$, $(\mathbf{C_{\geq\varepsilon}})$, and $(\mathbf{C_s})$ by the set of the following assumptions[1]: $\hat{\xi} = 0$, at least one of ζ and λ vanishes, and there exists $\delta\in(0,\infty)$ such that, for all $u\in\mathcal{L}_0^2$ and the associated process H^u defined in (8.1) with $H^u_0 = 0$, the uniform convexity assumption (4.22) is satisfied. The uniform convexity assumption on the cost functional is a weaker requirement than the usually in the LQ literature imposed nonnegativity and positivity assumptions on the coefficients of the cost functional.

If $\hat{\xi} = 0$ and at least one of λ and ζ vanishes, we find that the optimal strategy of Corollary 8.2.4 can be represented by a formula that is very close to the one of Theorem 5.2.6(iii) (see also Chapter 9):

[1] We still require the remaining assumptions, e.g., $(\mathbf{C_{bdd}})$ and that the filtration is generated by the Brownian motion $(W^{(1)},\dots,W^{(m)})^\top$.

Remark 8.2.5. Let the assumptions of Theorem 8.2.3 be in force and define

$$\begin{aligned}\widetilde{Q}_s = & -\int_0^s \widetilde{\vartheta}_r(\sigma_r + \eta_r\overline{r}_r)dW_r^{(1)} - \int_0^s \widetilde{\vartheta}_r\eta_r\sqrt{1-\overline{r}_r^2}dW_r^{(2)} \\ & -\int_0^s \widetilde{\vartheta}_r(\mu_r + \rho_r - \sigma_r^2 - \sigma_r\eta_r\overline{r}_r)dr, \quad s \in [0,T].\end{aligned}$$

We can compute from (3.5), (8.9), and (8.10) that $\gamma^{-\frac{1}{2}}\mathcal{E}(K) = \gamma_0^{-\frac{1}{2}}\mathcal{E}(\widetilde{Q})$. Using (8.11) with (8.14), we then obtain that the optimal strategy in $\mathcal{A}_0^{\text{pm}}(x,d)$ for J^{pm} on $s \in [0,T)$ can be expressed as

$$X_s^* = \left(x - \frac{d}{\gamma_0} - \gamma_0^{-\frac{1}{2}}\int_0^s \mathcal{E}(K)_r^{-1}d(L_r - [L,K]_r)\right)\mathcal{E}(\widetilde{Q})_s(1-\widetilde{\vartheta}_s) - \gamma_s^{-\frac{1}{2}}\vartheta_s^0.$$

In particular, in the subsetting where $\hat{\xi} = 0$ and at least one of ζ and λ vanishes, the optimal strategy (8.14) can be represented as

$$X_{0-}^* = x, \quad X_T^* = 0, \quad X_s^* = \left(x - \frac{d}{\gamma_0}\right)\mathcal{E}(\widetilde{Q})_s\big(1-\widetilde{\vartheta}_s\big), \quad s \in [0,T).$$

8.3 The Obizhaeva-Wang model with random targets

The models developed by Obizhaeva and Wang [OW13] can essentially[2] be considered as special cases of the set-up in Chapter 7. Indeed, we obtain the framework of [OW13, Section 6] by setting $\mu \equiv 0$, $\sigma \equiv 0$, $\eta \equiv 0$, $\overline{r} \equiv 0$, $\lambda \equiv 0$ and choosing $\rho \in (0,\infty)$ and $\hat{\xi} \in \mathbb{R}$ as deterministic constants. Also the extension in [OW13, Section 8.3] including risk aversion can be regarded as a special case of our setting by allowing $\lambda \in (0,\infty)$ to be a positive constant and choosing $\zeta \equiv 0$.

In this section we apply our results (in particular, Corollary 8.2.4) and provide closed-form solutions (see (8.19) below) for optimal progressively measurable execution strategies in versions of these problems which allow for general random terminal targets $\hat{\xi}$ and general running targets ζ.

To this end let $x, d \in \mathbb{R}$ and $t = 0$. Assume that $(\mathcal{F}_s)_{s\in[0,T]}$ is the augmented natural filtration of the Brownian motion $(W^{(1)},\dots,W^{(m)})^\top$. Suppose that $\mu \equiv 0$, $\sigma \equiv 0$, $\eta \equiv 0$, and $\overline{r} \equiv 0$. Furthermore, assume that $\rho \in (0,\infty)$ and $\lambda \in [0,\infty)$ are deterministic constants. We take some $\hat{\xi}$ and ζ as specified in Section 7.1 (in particular, see (7.1) and (7.2)). Note that the conditions of Proposition 4.3.2, Theorem 8.2.3, and Corollary 8.2.4 hold true, and that $\gamma_s = \gamma_0$ for all $s \in [0,T]$. We find the unique

[2]Note that the set of admissible strategies in the continuous-time optimization problem of [OW13] is slightly different even from our finite-variation problem of Section 7.1.

solution (cf. Proposition 4.3.2) of BSDE (4.1) in the current setting by solving the scalar Riccati ODE with constant coefficients

$$dY_s = \left(\frac{\rho^2}{\rho+\lambda}Y_s^2 + \frac{2\lambda\rho}{\rho+\lambda}Y_s - \frac{\lambda\rho}{\rho+\lambda}\right)ds, \quad s \in [0,T], \qquad Y_T = \frac{1}{2}.$$

Such an equation can be solved explicitly, and in our situation we obtain in the case $\lambda > 0$ that

$$Y_s = \frac{1}{2}\frac{\lambda \tanh\left(\frac{\sqrt{\lambda}\rho(T-s)}{\sqrt{\lambda+\rho}}\right) + \sqrt{\lambda(\rho+\lambda)}}{(\frac{\rho}{2}+\lambda)\tanh\left(\frac{\sqrt{\lambda}\rho(T-s)}{\sqrt{\lambda+\rho}}\right) + \sqrt{\lambda(\rho+\lambda)}}, \quad s \in [0,T], \tag{8.15}$$

and in the case $\lambda = 0$ that

$$Y_s = \frac{1}{2+(T-s)\rho}, \quad s \in [0,T]. \tag{8.16}$$

Hence, $(Y, Z, M^\perp)$ with $Z \equiv 0 \equiv M^\perp$ and Y of (8.15) (if $\lambda > 0$), respectively (8.16) (if $\lambda = 0$), is the unique solution of BSDE (4.1) in the present setting. The process $\widetilde{\vartheta}$ from (5.22) here is given by

$$\widetilde{\vartheta}_s = \frac{\rho Y_s + \lambda}{\rho+\lambda}, \quad s \in [0,T].$$

Note that $\widetilde{\vartheta}$ is deterministic and bounded by $|\widetilde{\vartheta}| \leq \frac{\frac{1}{2}\rho+\lambda}{\rho+\lambda}$. BSDE (8.7) becomes

$$\begin{aligned} d\psi_s &= -\left(-\rho\widetilde{\vartheta}_s\psi_s + \sqrt{\gamma_0}\,\zeta_s\lambda(\widetilde{\vartheta}_s - 1)\right)ds + \sum_{j=1}^m \phi_s^{(j)} dW_s^{(j)}, \quad s \in [0,T], \\ \psi_T &= -\frac{1}{2}\sqrt{\gamma_0}\hat{\xi}. \end{aligned} \tag{8.17}$$

By Theorem 8.2.3(i), there exists a unique solution (ψ, ϕ). Let us show that $\phi^{(j)} \in \mathcal{L}_0^2$ for all $j \in \{1, \ldots, m\}$. To this end, consider

$$\sum_{j=1}^m \int_0^r \phi_s^{(j)} dW_s^{(j)} = \psi_r - \psi_0 + \int_0^r \left(-\rho\widetilde{\vartheta}_s\psi_s + \sqrt{\gamma_0}\,\zeta_s\lambda(\widetilde{\vartheta}_s - 1)\right)ds, \quad r \in [0,T],$$

and apply to this continuous local martingale the Burkholder-Davis-Gundy inequality, and subsequently Jensen's inequality, to obtain existence of some $\widetilde{c} \in (0,\infty)$ such that

$$\begin{aligned} E\left[\left[\sum_{j=1}^m \int_0^\cdot \phi_s^{(j)} dW_s^{(j)}\right]_T\right] &\leq \widetilde{c}\,E\left[\sup_{r\in[0,T]}\left(\sum_{j=1}^m \int_0^r \phi_s^{(j)} dW_s^{(j)}\right)^2\right] \\ &\leq 8\widetilde{c}\,E\left[\sup_{r\in[0,T]}\psi_r^2\right] + 4\widetilde{c}T^2\rho^2\left(\frac{\frac{1}{2}\rho+\lambda}{\rho+\lambda}\right)^2 E\left[\sup_{r\in[0,T]}\psi_r^2\right] \\ &\quad + 4\widetilde{c}T\lambda^2\left(\frac{\frac{1}{2}\rho+\lambda}{\rho+\lambda}+1\right)^2 E\left[\int_0^T \gamma_0\zeta_s^2 ds\right]. \end{aligned}$$

Due to Definition 8.2.2 and (7.2), the right-hand side is finite. Since

$$\left[\sum_{j=1}^{m}\int_0^{\cdot}\phi_s^{(j)}dW_s^{(j)}\right]_T=\sum_{j=1}^{m}\int_0^T(\phi_s^{(j)})^2ds,$$

this shows that $\phi^{(j)}\in\mathcal{L}_0^2$ for all $j\in\{1,\dots,m\}$. Observe moreover that the coefficient of ψ in the driver of BSDE (8.17) is bounded. It thus follows, e.g., by Lemma 4.1.2 (with $g^{(0)}=-\rho\widetilde{\vartheta}$, $g^{(1)}=0=g^{(2)}$, $g^{(3)}=\sqrt{\gamma_0}\zeta\lambda(\widetilde{\vartheta}-1)$, $A=0$, $\xi=-\frac{1}{2}\sqrt{\gamma_0}\hat{\xi}$), that the solution component ψ of (8.17) is given by

$$\psi_s=\Gamma_s^{-1}\sqrt{\gamma_0}\left(-\frac{1}{2}\Gamma_T E_s[\hat{\xi}]-\frac{\rho\lambda}{\rho+\lambda}\int_s^T\Gamma_r(1-Y_r)E_s[\zeta_r]dr\right),\quad s\in[0,T],$$

where

$$\Gamma_s=\exp\left(-\rho\int_0^s\widetilde{\vartheta}_r dr\right)=\exp\left(-\frac{\rho}{\rho+\lambda}\left(\lambda s+\rho\int_0^s Y_r dr\right)\right),\quad s\in[0,T]. \tag{8.18}$$

It holds for the process in (8.8) that

$$\vartheta_s^0=\frac{\rho\psi_s-\sqrt{\gamma_0}\,\zeta_s\lambda}{\rho+\lambda},\quad s\in[0,T].$$

Further, the processes K and L for SDE (8.9) in our current setting are given by

$$K_r=-\rho\int_0^r\widetilde{\vartheta}_s ds,\quad L_r=\rho\int_0^r\vartheta_s^0 ds,\quad r\in[0,T].$$

It then follows from Corollary 8.2.4 (see also Remark 8.2.5, and note that $\mathcal{E}(K)=\mathcal{E}(\widetilde{Q})=\Gamma$ in the current setting) that $X^*=(X_s^*)_{s\in[0-,T]}$ defined by $X_{0-}^*=x$, $X_T^*=\hat{\xi}$, and

$$\begin{aligned}X_s^*=&\left(x-\frac{d}{\gamma_0}+\frac{\rho}{\rho+\lambda}\int_0^s\Gamma_r^{-1}\left(\lambda\zeta_r-\frac{\rho}{\sqrt{\gamma_0}}\psi_r\right)dr\right)\Gamma_s\frac{\rho}{\rho+\lambda}(1-Y_s)\\&+\frac{\rho}{\rho+\lambda}\left(\frac{\lambda}{\rho}\zeta_s-\frac{1}{\sqrt{\gamma_0}}\psi_s\right),\quad s\in[0,T),\end{aligned} \tag{8.19}$$

is the (up to $\mathcal{D}_{W^{(1)}}$-null sets unique) execution strategy in $\mathcal{A}_0^{\mathrm{pm}}(x,d)$ that minimizes J^{pm}.

We consider the case $\lambda=0$ as a particular example.

Example 8.3.1. Suppose that $\lambda=0$. If the terminal target $\hat{\xi}\in\mathbb{R}$ is a deterministic constant (and $d=0$), then the optimal strategy[3] from [OW13, Proposition 3] is given by $X_{0-}^*=x$, $X_T^*=\hat{\xi}$, and

$$X_s^*=(x-\hat{\xi})\frac{1+(T-s)\rho}{2+T\rho}+\hat{\xi},\quad s\in[0,T); \tag{8.20}$$

[3] We will see in (8.22) that, for $\hat{\xi}\in\mathbb{R}$ (and $d=0$), this is also the optimal strategy in our extended problem, and moreover in our finite-variation problem (cf. Corollary 8.1.4(ii)).

it consists of potential block trades at times 0 and T and a continuous linear trading program on $[0, T)$. In the following we analyze how this structure changes when we allow for a random terminal target $\hat{\xi}$.

First recall that the solution component Y of BSDE (4.1) is given in this case by (8.16). It follows that Γ from (8.18) simplifies to

$$\Gamma_s = \frac{2 + (T-s)\rho}{2 + T\rho}, \quad s \in [0, T].$$

For the solution component ψ of BSDE (8.17), we thus obtain that

$$\psi_s = -\frac{\sqrt{\gamma_0}}{2 + (T-s)\rho} E_s[\hat{\xi}], \quad s \in [0, T].$$

The optimal strategy from (8.19) for $s \in [0, T)$ becomes

$$\begin{aligned} X_s^* &= \left(x - \frac{d}{\gamma_0} - \frac{\rho}{\sqrt{\gamma_0}} \int_0^s \Gamma_r^{-1} \psi_r dr \right) \Gamma_s (1 - Y_s) - \frac{1}{\sqrt{\gamma_0}} \psi_s \\ &= \left(x - \frac{d}{\gamma_0} + (2 + T\rho)\rho \int_0^s \frac{E_r[\hat{\xi}]}{(2 + (T-r)\rho)^2} dr \right) \frac{1 + (T-s)\rho}{2 + T\rho} + \frac{E_s[\hat{\xi}]}{2 + (T-s)\rho}. \end{aligned} \tag{8.21}$$

Integration by parts implies that (note that $(E_r[\hat{\xi}])_{r \in [0,T]}$ is a continuous martingale)

$$\begin{aligned} &\int_0^s \frac{E_r[\hat{\xi}]}{(2 + (T-r)\rho)^2} dr = \int_0^s E_r[\hat{\xi}] d \frac{1}{(2 + (T-r)\rho)\rho} \\ &= \frac{E_s[\hat{\xi}]}{(2 + (T-s)\rho)\rho} - \frac{E_0[\hat{\xi}]}{(2 + T\rho)\rho} - \int_0^s \frac{1}{(2 + (T-r)\rho)\rho} dE_r[\hat{\xi}], \quad s \in [0, T). \end{aligned}$$

Substituting this into (8.21) yields for $s \in [0, T)$ that

$$\begin{aligned} X_s^* &= \left(x - E_0[\hat{\xi}] - \frac{d}{\gamma_0} \right) \frac{1 + (T-s)\rho}{2 + T\rho} + E_s[\hat{\xi}] - \int_0^s \frac{1 + (T-s)\rho}{2 + (T-r)\rho} dE_r[\hat{\xi}] \\ &= \left(x - E_0[\hat{\xi}] - \frac{d}{\gamma_0} \right) \frac{1 + (T-s)\rho}{2 + T\rho} + E_0[\hat{\xi}] + \int_0^s \left(1 - \frac{1 + (T-s)\rho}{2 + (T-r)\rho} \right) dE_r[\hat{\xi}]. \end{aligned}$$

We finally obtain the alternative representation

$$X_s^* = \left(x - E[\hat{\xi}] - \frac{d}{\gamma_0} \right) \frac{1 + (T-s)\rho}{2 + T\rho} + E[\hat{\xi}] + \int_0^s \frac{1 + (s-r)\rho}{2 + (T-r)\rho} dE_r[\hat{\xi}], \quad s \in [0, T), \tag{8.22}$$

for (8.21). We see that this optimal strategy $X^* \in \mathcal{A}_0^{\mathrm{pm}}(x, d)$ consists of two additive parts: the first part (for $d = 0$) exactly corresponds to the optimal deterministic strategy in (8.20) where the deterministic terminal target is replaced by the expected terminal target $E[\hat{\xi}]$. The second part represents fluctuations around this deterministic strategy which incorporate updates about the random terminal target $\hat{\xi}$. Note that this stochastic integral vanishes in expectation.

9 The semimartingale problem vs. the extended problem

Let us discuss the relation between the semimartingale problem (see Chapter 5) and the extended problem (see Chapter 7–Chapter 8).

Although the semimartingale problem and the extended problem use the same base setting of Section 3.1, the set-ups of both problems exhibit some differences.

For instance, for the semimartingale problem, we work with the independent continuous local martingales $M^{(j)}$, $j \in \{1, \dots, m\}$, of Section 3.1 and in a general filtration, whereas for the extended problem, we assume that $M^{(j)} = W^{(j)}$, $j \in \{1, \dots, m\}$, are independent Brownian motions. In Section 8.2 we additionally require the filtration to be generated by these Brownian motions and we consider initial time $t = 0$.

A difference in the setting where Chapter 7–Chapter 8 is more general than Chapter 5 concerns the possibility to include nonzero, stochastic targets $\hat{\xi}$, ζ, and to consider progressively measurable strategies.

The shared motivation for the definition of the cost functional and of the deviation process in the semimartingale problem and in the extended problem is the usual kind of formulation for finite-variation strategies in Obizhaeva-Wang type models (see Section 1.1, Section 5.1.2, and Section 7.1). The formulation in the semimartingale problem is in addition motivated by a heuristic limit from discrete time (see Section 3.2), while counterexamples show that the conventional formulation here indeed is not sufficient (see Section 5.1.2). In contrast, we give a rigorous justification (Theorem 7.5.2) for the particular formulation that we use in the extended problem.

Having set up the problems, the solution approaches that we take, in both cases, are probabilistic and BSDE (4.1) plays a crucial role. In the semimartingale problem, this BSDE appears already in the alternative representation of the cost functional (see Section 5.2.1), whereas in the extended problem BSDE (4.1) arises as a special case of [KT02, BSRDE (9)] in the context of a standard LQ stochastic control problem.

This is related to the difference that the solution approach in Chapter 5 is more self-contained than the one in Chapter 8, where we eventually rely on literature on LQ optimal control to solve the standard LQ and thus our trade execution problem.

Instead, a large effort in Chapter 7–Chapter 8 goes into showing equivalence of certain control problems.

To show one of these equivalences, namely, that the cost functional J^{pm} can be considered as a continuous extension of J^{fv} from finite-variation strategies to progressively measurable strategies, we in Lemma 7.5.4 exploit [KS91, Section 3.2, Lemma 2.7]. Observe that we rely on [KS91, Section 3.2, Lemma 2.7] also in Lemma 5.2.9 to ultimately prove the main theorem on the semimartingale problem. In the first case, [KS91, Section 3.2, Lemma 2.7] is used to approximate the deviation, whereas in the second case, the same result [KS91, Section 3.2, Lemma 2.7] is used to approximate $\widetilde{\vartheta}$ of (5.22).

Recall that in Chapter 7–Chapter 8, the scaled hidden deviation process $\overline{H}^X = \gamma^{-\frac{1}{2}} D^X - \gamma^{\frac{1}{2}} X$ of Section 7.4 is important for the proof of several results and essentially becomes the state process in the standard LQ stochastic control problem (see Section 8.1). The counterpart in Chapter 5 of $\overline{H}^X$ is the process $A = X - \gamma^{-1} D^X$. In particular, note that we place $\overline{H}^X$ into the cost functional J^{pm} (see Proposition 7.4.2) and that, in fact, we also introduce A into the cost functional J^{sem}; see the proof of Theorem 5.2.1 and observe that $\frac{1}{\gamma}(\widetilde{\vartheta}(\gamma X - D^X) + D^X)^2 = \gamma(\widetilde{\vartheta} A + \gamma^{-1} D^X)^2$ in (5.23). Moreover, A shows up when proving uniqueness of optimal strategies in Lemma 5.2.5, when approximating strategies in Lemma 5.2.10, and in the proof of the main result Theorem 5.2.6.

To be better able to compare the main results Theorem 5.2.6 and Corollary 8.2.4, let us in the sequel consider the following subsetting of Section 3.1: assume that the continuous local martingale $(M^{(1)}, \dots, M^{(m)})^\top = (W^{(1)}, \dots, W^{(m)})^\top = W$ is an m-dimensional Brownian motion, that $\mathcal{F}_s = \mathcal{F}_s^W$ for all $s \in [0,T]$, $\hat{\xi} = 0$, $\zeta \equiv 0$, $t = 0$, and that $(\mathbf{C_{bdd}})$ is satisfied. Note that Theorem 5.2.6 in addition requires $(\mathbf{C_{>0}})$ (and existence of the BSDE (4.1) and boundedness of $\widetilde{\vartheta}$ of (5.22)), whereas in Corollary 8.2.4 we demand the slightly different additional conditions[1] $(\mathbf{C_{nonneg}})$ and at least one of $(\mathbf{C_{\geq\varepsilon}})$, $(\mathbf{C_s})$. Let us now assume that $(\mathbf{C_{>0}})$, $(\mathbf{C_{nonneg}})$, and at least one of $(\mathbf{C_{\geq\varepsilon}})$, $(\mathbf{C_s})$ hold. Then, there exists a unique solution of BSDE (4.1) (cf. Proposition 4.3.2). Moreover, we know that the denominator in definition (5.22) of $\widetilde{\vartheta}$ is strictly positive and bounded away from zero. However, due to $Z^{(1)}$ and $Z^{(2)}$ in (5.22), we can in general not guarantee that $\widetilde{\vartheta}$ is $\mathcal{D}_{W^{(1)}}$-a.e. bounded. Thus, the premises of Theorem 5.2.6 are not yet completely satisfied. We therefore assume now in addition that $\widetilde{\vartheta}$ is $\mathcal{D}_{W^{(1)}}$-a.e. bounded. Then, we can apply both, Theorem 5.2.6 and Corollary 8.2.4.

We find that the optimal costs in the semimartingale problem and in the extended problem (for all $x, d \in \mathbb{R}$) are the same (cf. Theorem 5.2.6(i) and Corollary 8.2.4, since $\psi \equiv 0$, $C_0 = 0$ for $\hat{\xi} = 0 \equiv \zeta$). In the extended problem, there always exists an optimal strategy (cf. Corollary 8.2.4), whereas in the semimartingale problem, for $x \neq \frac{d}{\gamma_0}$, we

[1] In the current case of $\hat{\xi} = 0 \equiv \zeta$, we could obtain the results of Corollary 8.2.4 also under slightly weaker conditions than these when we apply [SXY21] to the standard LQ problem with cross-terms of Section 8.1.1.

have existence of an optimal strategy if and only if we can find a càdlàg semimartingale ϑ such that $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{W^{(1)}}$-a.e.

As an example, we have illustrated in Section 5.4.1 that an optimizer of the semimartingale problem does not exist when we consider the setting of Remark 5.3.2 with μ chosen as a deterministic càdlàg function such that there exists $\delta \in (0,T)$ with μ having infinite variation on $[0, T-\delta]$. Note that the setting in that example is a special case of our current setting. In particular, the conditions of Corollary 8.2.4 are satisfied, and we are able to compute via (8.14) a unique optimal strategy in the extended problem (see also [AKU22a, Section 4.2]).

If existent, the (unique) optimal strategy of the semimartingale problem is given by the formulas in Theorem 5.2.6(iii). The (unique) optimal strategy of the extended problem satisfies the formulas in Remark 8.2.5. The only difference is that in the solution of the extended problem, we keep $\widetilde{\vartheta}$, which we replace in the solution of the semimartingale problem by ϑ. Nevertheless, the solutions coincide, as $\widetilde{\vartheta} = \vartheta$ $\mathcal{D}_{W^{(1)}}$-a.e. and as uniqueness of optimal strategies is up to $\mathcal{D}_{W^{(1)}}$-null sets. In particular[2], for $x, d \in \mathbb{R}$, the optimal semimartingale strategy in $\mathcal{A}_0^{\mathrm{sem}}(x,d)$ from Theorem 5.2.6(iii) is also the optimal progressively measurable strategy in $\mathcal{A}_0^{\mathrm{pm}}(x,d)$. In general, though, we do not have that $X \in \mathcal{A}_0^{\mathrm{sem}}(x,d)$ implies that $X \in \mathcal{A}_0^{\mathrm{pm}}(x,d)$ (nor that $\mathcal{A}_0^{\mathrm{sem}}(x,d)$ is a superset of $\mathcal{A}_0^{\mathrm{fv}}(x,d)$) due to differences in the respective integrability assumptions on admissible strategies.

A natural question that arises is whether for $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d) \cap \mathcal{A}_t^{\mathrm{pm}}(x,d)$ (for $t \in [0,T]$, $x, d \in \mathbb{R}$) the control problems considered in Section 5.1.1 and Section 7.3 coincide. The answer is affirmative. Indeed, for $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d) \cap \mathcal{A}_t^{\mathrm{pm}}(x,d)$, we can show that definitions (5.1) and (7.14) of the associated deviations in the semimartingale, respectively extended, problem coincide and that $J_t^{\mathrm{sem}}(x,d,X) = J_t^{\mathrm{pm}}(x,d,X)$, see Proposition 9.0.1. Moreover, we remark that if $X \in \mathcal{A}_t^{\mathrm{sem}}(x,d) \cap \mathcal{A}_t^{\mathrm{fv}}(x,d)$, it holds that the associated deviations (5.1) and (7.3) in the semimartingale, respectively finite-variation, problem agree and that $J_t^{\mathrm{sem}}(x,d,X) = J_t^{\mathrm{fv}}(x,d,X)$ (see also Remark 5.1.3).

Proposition 9.0.1. *Consider the setting of Section 3.1 and suppose that $M^{(j)} = W^{(j)}$, $j \in \{1, \dots, m\}$, are independent Brownian motions and that $\hat{\xi} = 0$ and $\zeta \equiv 0$. Fix $t \in [0,T]$ and $x, d \in \mathbb{R}$. Let $X \in \mathcal{A}_t^{sem}(x,d) \cap \mathcal{A}_t^{pm}(x,d)$. Let D be defined by* (5.1), *and let $\overline{D}$ be defined by* (7.14). *Assume that $J_t^{sem}(x,d,X)$ of* (5.2) *and $J_t^{pm}(x,d,X)$ of* (7.16) *are well defined.*

(i) It holds that $D = \overline{D}$.

(ii) It holds that $J_t^{sem}(x,d,X) = J_t^{pm}(x,d,X)$.

Proof. (i) Denote $\beta_s = d - \gamma_t x - \int_t^s X_r d(\nu_r \gamma_r)$, $s \in [t,T]$, where we recall that ν is defined in (7.5). Observe that X, β, and $\overline{D} = \gamma X + \nu^{-1}\beta$ are semimartingales. We

[2]Especially, note that the optimal strategies of Example 5.3.1, Example 5.3.3, Example 5.3.4, Section 5.4.2, and Section 5.4.3 are also the optimal strategies in the extended problem (see also [AKU22a, Section 4] for Example 5.3.1 and Section 5.4.2).

compute by integration by parts, and using (7.6), for all $s \in [t,T]$ that

$$\begin{aligned} d\overline{D}_s &= \gamma_s dX_s + X_s d\gamma_s + d[\gamma, X]_s + \nu_s^{-1} d\beta_s + \beta_s d\nu_s^{-1} + d[\nu^{-1}, \beta]_s \\ &= \gamma_s dX_s + X_s d\gamma_s + d[\gamma, X]_s - \nu_s^{-1} X_s d(\nu_s \gamma_s) - \nu_s^{-1} \beta_s dR_s + \nu_s^{-1} X_s d[R, \nu\gamma]_s. \end{aligned} \tag{9.1}$$

Furthermore, it holds by integration by parts for all $s \in [t,T]$ that

$$d(\nu_s \gamma_s) = \nu_s d\gamma_s + \gamma_s \nu_s dR_s + \gamma_s \nu_s d[R]_s + \nu_s d[R, \gamma]_s. \tag{9.2}$$

We obtain from (9.1) and (9.2) for all $s \in [t,T]$ that

$$\begin{aligned} d\overline{D}_s &= \gamma_s dX_s + X_s d\gamma_s + d[\gamma, X]_s - X_s d\gamma_s - X_s \gamma_s dR_s - X_s \gamma_s d[R]_s - X_s d[R, \gamma]_s \\ &\quad - \nu_s^{-1} \beta_s dR_s + X_s d[R, \gamma]_s + X_s \gamma_s d[R]_s \\ &= \gamma_s dX_s + d[\gamma, X]_s - (\gamma_s X_s + \nu_s^{-1} \beta_s) dR_s \\ &= \gamma_s dX_s + d[\gamma, X]_s - \overline{D}_s dR_s. \end{aligned}$$

D also satisfies this SDE with the same initial value $D_{t-} = d = \overline{D}_{t-}$. Since the solution is unique, we have that $\overline{D} = D$.

(ii) From (5.1) we have that

$$dX_s = \gamma_s^{-1} dD_s + \gamma_s^{-1} D_s dR_s - \gamma_s^{-1} d[\gamma, X]_s, \quad s \in [t,T],$$

and further that

$$\begin{aligned} d[X]_s &= \gamma_s^{-2} d[D]_s + \gamma_s^{-2} D_s^2 d[R]_s + 2\gamma_s^{-2} D_s d[D, R]_s \\ &= \gamma_s^{-2} d[D]_s + \gamma_s^{-2} D_s^2 d[R]_s - 2\gamma_s^{-2} D_s^2 d[R]_s + 2\gamma_s^{-1} D_s d[X, R]_s, \quad s \in [t,T]. \end{aligned}$$

Moreover, since

$$d\gamma_s^{-1} = -\gamma_s^{-2} d\gamma_s + \gamma_s^{-3} d[\gamma]_s, \quad s \in [0,T],$$

it holds that

$$-\gamma_s^{-1} d[\gamma, X]_s = \gamma_s d[\gamma^{-1}, X]_s = d[\gamma^{-1}, D]_s + D_s d[\gamma^{-1}, R]_s, \quad s \in [t,T].$$

We then obtain for all $s \in [t,T]$ that

$$\begin{aligned} &D_{s-} dX_s + \frac{\gamma_s}{2} d[X]_s - D_s d[X, R]_s \\ &= D_{s-} \gamma_s^{-1} dD_s + \gamma_s^{-1} D_s^2 dR_s + D_s d[\gamma^{-1}, D]_s + D_s^2 d[\gamma^{-1}, R]_s + \frac{1}{2} \gamma_s^{-1} d[D]_s \\ &\quad + \frac{1}{2} \gamma_s^{-1} D_s^2 d[R]_s - \gamma_s^{-1} D_s^2 d[R]_s + D_s d[X, R]_s - D_s d[X, R]_s \\ &= \gamma_s^{-1} D_{s-} dD_s + \gamma_s^{-1} D_s^2 dR_s + D_s d[\gamma^{-1}, D]_s + D_s^2 d[\gamma^{-1}, R]_s + \frac{1}{2} \gamma_s^{-1} d[D]_s \\ &\quad - \frac{1}{2} \gamma_s^{-1} D_s^2 d[R]_s. \end{aligned} \tag{9.3}$$

Furthermore, it holds by integration by parts that

$$\begin{aligned} d(\gamma_s^{-1} D_s^2) &= \gamma_s^{-1} dD_s^2 + D_s^2 d\gamma_s^{-1} + d[\gamma^{-1}, D^2]_s \\ &= 2\gamma_s^{-1} D_{s-} dD_s + \gamma_s^{-1} d[D]_s + D_s^2 d\gamma_s^{-1} + 2D_s d[\gamma^{-1}, D]_s, \quad s \in [t,T], \end{aligned}$$

and thus

$$\gamma_s^{-1} D_{s-} dD_s = \frac{1}{2} d(\gamma_s^{-1} D_s^2) - \frac{1}{2}\gamma_s^{-1} d[D]_s - \frac{1}{2} D_s^2 d\gamma_s^{-1} - D_s d[\gamma^{-1}, D]_s, \quad s \in [t,T].$$

We insert this into (9.3) and obtain for all $s \in [t,T]$ that

$$\begin{aligned} & D_{s-} dX_s + \frac{\gamma_s}{2} d[X]_s - D_s d[X,R]_s \\ &= \frac{1}{2} d(\gamma_s^{-1} D_s^2) - \frac{1}{2} D_s^2 d\gamma_s^{-1} + \gamma_s^{-1} D_s^2 dR_s + D_s^2 d[\gamma^{-1}, R]_s - \frac{1}{2}\gamma_s^{-1} D_s^2 d[R]_s. \end{aligned}$$

Using the dynamics (3.1) and (3.3) of R and γ^{-1}, it follows for all $s \in [t,T]$ that

$$\begin{aligned} & D_{s-} dX_s + \frac{\gamma_s}{2} d[X]_s - D_s d[X,R]_s \\ &= \frac{1}{2} d(\gamma_s^{-1} D_s^2) + \frac{1}{2}\gamma_s^{-1}(\mu_s - \sigma_s^2) D_s^2 ds + \frac{1}{2}\gamma_s^{-1}\sigma_s D_s^2 dW_s^{(1)} + \gamma_s^{-1} D_s^2 \rho_s ds \\ &\quad + \gamma_s^{-1} D_s^2 \eta_s \overline{r}_s dW_s^{(1)} + \gamma_s^{-1} D_s^2 \eta_s \sqrt{1-\overline{r}_s^2} dW_s^{(2)} - \gamma_s^{-1} D_s^2 \sigma_s \eta_s \overline{r}_s ds - \frac{1}{2}\gamma_s^{-1} D_s^2 \eta_s^2 ds. \end{aligned}$$

This yields that

$$\begin{aligned} & \int_{[t,T]} D_{s-} dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s - \int_t^T D_s d[X,R]_s \\ &= \frac{1}{2}\gamma_T^{-1} D_T^2 - \frac{d^2}{2\gamma_t} + \int_t^T \gamma_s^{-1} D_s^2 \kappa_s ds + \int_t^T \gamma_s^{-1} D_s^2 \left(\frac{1}{2}\sigma_s + \eta_s \overline{r}_s \right) dW_s^{(1)} \\ &\quad + \int_t^T \gamma_s^{-1} D_s^2 \eta_s \sqrt{1-\overline{r}_s^2} dW_s^{(2)}. \end{aligned} \tag{9.4}$$

By the Burkholder-Davis-Gundy inequality, Jensen's inequality, and Minkowski's inequality, there exists $c \in (0,\infty)$ such that

$$\begin{aligned} & E_t \left[\sup_{r \in [t,T]} \left| \int_t^r \gamma_s^{-1} D_s^2 \left(\frac{1}{2}\sigma_s + \eta_s \overline{r}_s \right) dW_s^{(1)} \right| \right] \\ &\leq c\, E_t \left[\left(\int_t^T \gamma_s^{-2} D_s^4 \left(\frac{1}{2}\sigma_s + \eta_s \overline{r}_s \right)^2 ds \right)^{\frac{1}{2}} \right] \\ &\leq c\, E_t \left[\left(\int_t^T \gamma_s^{-2} D_s^4 \frac{1}{2}\sigma_s^2 ds \right)^{\frac{1}{2}} \right] + c\, E_t \left[\left(\int_t^T \gamma_s^{-2} D_s^4\, 2\eta_s^2 ds \right)^{\frac{1}{2}} \right]. \end{aligned}$$

The first and the second conditional expectation on the right-hand side are finite due to **(A3)** and **(A5)** of Section 5.1.1, respectively. It follows that

$$E_t\left[\int_t^T \gamma_s^{-1} D_s^2 \left(\frac{1}{2}\sigma_s + \eta_s \overline{r}_s\right) dW_s^{(1)}\right] = 0. \tag{9.5}$$

Similarly, we can show by the Burkholder-Davis-Gundy inequality and **(A5)** of Section 5.1.1 that

$$E_t\left[\int_t^T \gamma_s^{-1} D_s^2 \eta_s \sqrt{1-\overline{r}_s^2} dW_s^{(2)}\right] = 0. \tag{9.6}$$

(9.4) together with (9.5), (9.6), and $D = \overline{D}$ proves that $J_t^{\mathrm{sem}}(x,d,X) = J_t^{\mathrm{pm}}(x,d,X)$. □

Bibliography

[AA14] Aurélien Alfonsi and José Infante Acevedo. Optimal execution and price manipulations in time-varying limit order books. *Applied Mathematical Finance*, 21(3):201–237, 2014.

[AB16] Aurélien Alfonsi and Pierre Blanc. Dynamic optimal execution in a mixed-market-impact Hawkes price model. *Finance and Stochastics*, 20(1):183–218, 2016.

[ABC01] Hee-Joon Ahn, Kee-Hong Bae, and Kalok Chan. Limit orders, depth, and volatility: Evidence from the stock exchange of Hong Kong. *The Journal of Finance*, 56(2):767–788, 2001.

[AC01] Robert Almgren and Neil Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.

[AFS08] Aurélien Alfonsi, Antje Fruth, and Alexander Schied. Constrained portfolio liquidation in a limit order book model. *Banach Center Publications*, 83:9–25, 2008.

[AFS10] Aurélien Alfonsi, Antje Fruth, and Alexander Schied. Optimal execution strategies in limit order books with general shape functions. *Quantitative Finance*, 10(2):143–157, 2010.

[AJK14] Stefan Ankirchner, Monique Jeanblanc, and Thomas Kruse. BSDEs with singular terminal condition and a control problem with constraints. *SIAM Journal on Control and Optimization*, 52(2):893–913, 2014.

[AK15] Stefan Ankirchner and Thomas Kruse. Optimal position targeting with stochastic linear-quadratic costs. *Banach Center Publications*, 104:9–24, 2015.

[AKU21a] Julia Ackermann, Thomas Kruse, and Mikhail Urusov. Càdlàg semimartingale strategies for optimal trade execution in stochastic order book models. *Finance and Stochastics*, 25(4):757–810, 2021.

[AKU21b] Julia Ackermann, Thomas Kruse, and Mikhail Urusov. Optimal trade execution in an order book model with stochastic liquidity parameters. *SIAM Journal on Financial Mathematics*, 12(2):788–822, 2021.

[AKU22a] Julia Ackermann, Thomas Kruse, and Mikhail Urusov. Reducing Obizhaeva-Wang type trade execution problems to LQ stochastic control problems. *Preprint, arXiv:2206.03772v1*, 2022.

[AKU22b] Julia Ackermann, Thomas Kruse, and Mikhail Urusov. Self-exciting price impact via negative resilience in stochastic order books. *Annals of Operations Research*, DOI:10.1007/s10479-022-04973-00, 2022.

[Alm12] Robert Almgren. Optimal trading with stochastic liquidity and volatility. *SIAM Journal on Financial Mathematics*, 3(1):163–181, 2012.

[AS10] Aurélien Alfonsi and Alexander Schied. Optimal trade execution and absence of price manipulations in limit order book models. *SIAM Journal on Financial Mathematics*, 1(1):490–522, 2010.

[ASS12] Aurélien Alfonsi, Alexander Schied, and Alla Slynko. Order book resilience, price manipulation, and the positive portfolio problem. *SIAM Journal on Financial Mathematics*, 3(1):511–533, 2012.

[BBF18a] Dirk Becherer, Todor Bilarev, and Peter Frentrup. Optimal asset liquidation with multiplicative transient price impact. *Applied Mathematics & Optimization*, 78(3):643–676, 2018.

[BBF18b] Dirk Becherer, Todor Bilarev, and Peter Frentrup. Optimal liquidation under stochastic liquidity. *Finance and Stochastics*, 22(1):39–68, 2018.

[BBF19] Dirk Becherer, Todor Bilarev, and Peter Frentrup. Stability for gains from large investors' strategies in M_1/J_1 topologies. *Bernoulli*, 25(2):1105–1140, 2019.

[BF14] Peter Bank and Antje Fruth. Optimal order scheduling for deterministic liquidity patterns. *SIAM Journal on Financial Mathematics*, 5(1):137–152, 2014.

[BGPW04] Jean-Philippe Bouchaud, Yuval Gefen, Marc Potters, and Matthieu Wyart. Fluctuations and response in financial markets: the subtle nature of 'random' price changes. *Quantitative Finance*, 4(2):176–190, 2004.

[BHS95] Bruno Biais, Pierre Hillion, and Chester Spatt. An empirical analysis of the limit order book and the order flow in the Paris bourse. *The Journal of Finance*, 50(5):1655–1689, 1995.

[Bis76] Jean-Michel Bismut. Linear quadratic optimal stochastic control with random coefficients. *SIAM Journal on Control and Optimization*, 14(3):419–444, 1976.

[Bis78] Jean-Michel Bismut. Contrôle des systemes linéaires quadratiques: applications de l'intégrale stochastique. In *Séminaire de Probabilités XII*, pages 180–264. Springer, 1978.

[BL98] Dimitris Bertsimas and Andrew W. Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1(1):1–50, 1998.

[BL14] Erhan Bayraktar and Michael Ludkovski. Liquidation in limit order books with controlled intensity. *Mathematical Finance*, 24(4):627–650, 2014.

[BSV17] Peter Bank, H. Mete Soner, and Moritz Voß. Hedging with temporary price impact. *Mathematics and Financial Economics*, 11(2):215–239, 2017.

[BV18] Peter Bank and Moritz Voß. Linear quadratic stochastic control problems with stochastic terminal constraint. *SIAM Journal on Control and Optimization*, 56(2):672–699, 2018.

[CJ15] Álvaro Cartea and Sebastian Jaimungal. Optimal execution with limit and market orders. *Quantitative Finance*, 15(8):1279–1291, 2015.

[CJR18] Álvaro Cartea, Sebastian Jaimungal, and Jason Ricci. Algorithmic trading, stochastic control, and mutually exciting processes. *SIAM Review*, 60(3):673–703, 2018. Revised reprint of "Buy low, sell high: a high frequency trading perspective".

[CKW18] Ningyuan Chen, Steven Kou, and Chun Wang. A partitioning algorithm for Markov decision processes with applications to market microstructure. *Management Science*, 64(2):784–803, 2018.

[CL21] René Carmona and Laura Leal. Optimal execution with quadratic variation inventories. *Preprint, arXiv:2104.14615*, 2021.

[CMK16] Thomas Cayé and Johannes Muhle-Karbe. Liquidation with self-exciting price impact. *Mathematics and Financial Economics*, 10(1):15–28, 2016.

[CRS01] Tarun Chordia, Richard Roll, and Avanidhar Subrahmanyam. Market liquidity and trading activity. *The Journal of Finance*, 56(2):501–530, 2001.

[CS14] Patrick Cheridito and Tardu Sepin. Optimal trade execution under stochastic volatility and liquidity. *Applied Mathematical Finance*, 21(4):342–362, 2014.

[CW19] René Carmona and Kevin Webster. The self-financing equation in limit order book markets. *Finance and Stochastics*, 23(3):729–759, 2019.

[FHX22a] Guanxing Fu, Ulrich Horst, and Xiaonyu Xia. A mean-field control problem of optimal portfolio liquidation with semimartingale strategies. *Preprint, arXiv:2207.00446*, 2022.

[FHX22b] Guanxing Fu, Ulrich Horst, and Xiaonyu Xia. Portfolio liquidation games with self-exciting order flow. *Mathematical Finance*, 32(4):1020–1065, 2022.

[Fru11] Antje Fruth. *Optimal order execution with stochastic liquidity*. Doctoral thesis, Technische Universität Berlin, Fakultät II - Mathematik und Naturwissenschaften, 2011.

[FSU14] Antje Fruth, Torsten Schöneborn, and Mikhail Urusov. Optimal trade execution and price manipulation in order books with time-varying liquidity. *Mathematical Finance*, 24(4):651–695, 2014.

[FSU19] Antje Fruth, Torsten Schöneborn, and Mikhail Urusov. Optimal trade execution in order books with stochastic liquidity. *Mathematical Finance*, 29(2):507–541, 2019.

[Gat10] Jim Gatheral. No-dynamic-arbitrage and market impact. *Quantitative Finance*, 10(7):749–759, 2010.

[GH17] Paulwin Graewe and Ulrich Horst. Optimal trade execution with instantaneous price impact and stochastic resilience. *SIAM Journal on Control and Optimization*, 55(6):3707–3725, 2017.

[GHQ15] Paulwin Graewe, Ulrich Horst, and Jinniao Qiu. A non-Markovian liquidation problem and backward SPDEs with singular terminal conditions. *SIAM Journal on Control and Optimization*, 53(2):690–711, 2015.

[GHS18] Paulwin Graewe, Ulrich Horst, and Eric Séré. Smooth solutions to portfolio liquidation problems under price-sensitive market impact. *Stochastic Processes and their Applications*, 128(3):979–1006, 2018.

[GLFT12] Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia. Optimal portfolio liquidation with limit orders. *SIAM Journal on Financial Mathematics*, 3(1):740–764, 2012.

[GP16] Nicolae Gârleanu and Lasse Heje Pedersen. Dynamic portfolio choice with frictions. *Journal of Economic Theory*, 165:487–516, 2016.

[GS13] Jim Gatheral and Alexander Schied. Dynamical models of market impact and algorithms for order execution. In J.-P. Fouque and J. Langsam, editors, *Handbook on Systemic Risk*, chapter 22, pages 579–602. Cambridge University Press, Cambridge, 2013.

[GSS12] Jim Gatheral, Alexander Schied, and Alla Slynko. Transient linear price impact and Fredholm integral equations. *Mathematical Finance*, 22(3):445–474, 2012.

[GZ15] Xin Guo and Mihail Zervos. Optimal execution with multiplicative price impact. *SIAM Journal on Financial Mathematics*, 6(1):281–306, 2015.

[HK21] Ulrich Horst and Evgueni Kivman. Optimal trade execution under small market impact and portfolio liquidation with semimartingale strategies. *Preprint, arXiv:2103.05957*, 2021.

[HQZ16] Ulrich Horst, Jinniao Qiu, and Qi Zhang. A constrained control problem with degenerate coefficients and degenerate backward SPDEs with singular terminal condition. *SIAM Journal on Control and Optimization*, 54(2):946–963, 2016.

[HS04] Gur Huberman and Werner Stanzl. Price manipulation and quasi-arbitrage. *Econometrica*, 72(4):1247–1275, 2004.

[HX19] Ulrich Horst and Xiaonyu Xia. Multi-dimensional optimal trade execution under stochastic resilience. *Finance and Stochastics*, 23(4):889–923, 2019.

[JS03] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes*. A Series of Comprehensive Studies in Mathematics; 288. Springer, Berlin, 2nd edition, 2003.

[KP16a] Thomas Kruse and Alexandre Popier. BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics*, 88(4):491–539, 2016.

[KP16b] Thomas Kruse and Alexandre Popier. Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting. *Stochastic Processes and their Applications*, 126(9):2554–2592, 2016.

[KR21] Tomasz Klimsiak and Maurycy Rzymowski. Nonlinear BSDEs in general filtration with drivers depending on the martingale part of a solution. *Preprint, arXiv:2103.07536*, 2021.

[KS91] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*. Graduate texts in mathematics; 113. Springer, New York, 2nd edition, 1991.

[KT02] Michael Kohlmann and Shanjian Tang. Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean–variance hedging. *Stochastic Processes and their Applications*, 97(2):255–288, 2002.

[Lar07] Jeremy Large. Measuring the resiliency of an electronic limit order book. *Journal of Financial Markets*, 10(1):1–25, 2007.

[LH15] Danny K. Lo and Anthony D. Hall. Resiliency of the limit order book. *Journal of Economic Dynamics and Control*, 61(C):222–244, 2015.

[LO09] Julian Lorenz and Jörg Osterrieder. Simulation of a limit order driven market. *The Journal of Trading*, 4(1):23–30, 2009.

[LS13] Christopher Lorenz and Alexander Schied. Drift dependence of optimal trade execution strategies under transient price impact. *Finance and Stochastics*, 17(4):743–770, 2013.

[May06] Daniel Mayston. *Liquidity risk in limit order book markets*. Doctoral thesis, Universität zu Köln, Wirtschafts- u. Sozialwissenschaftliche Fakultät, 2006.

[Mor09] Marie-Amélie Morlais. Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. *Finance and Stochastics*, 13(1):121–150, 2009.

[OW13] Anna A. Obizhaeva and Jiang Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16:1–32, 2013.

[Pha09] Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*. Stochastic Modelling and Applied Probability; 61. Springer, Berlin Heidelberg, 2009.

[PPS18] Antonis Papapantoleon, Dylan Possamaï, and Alexandros Saplaouras. Existence and uniqueness results for BSDE with jumps: the whole nine yards. *Electronic Journal of Probability*, 23:1–68, 2018.

[Pro05] Philip E. Protter. *Stochastic integration and differential equations*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg, 2005.

[PSS11] Silviu Predoiu, Gennady Shaikhet, and Steven E. Shreve. Optimal execution in a general one-sided limit-order book. *SIAM Journal on Financial Mathematics*, 2(1):183–212, 2011.

[PZ19] Alexandre Popier and Chao Zhou. Second-order BSDE under monotonicity condition and liquidation problem under uncertainty. *The Annals of Applied Probability*, 29(3):1685–1739, 2019.

[RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition, 1999.

[Sal16] Dietmar Salamon. *Measure and integration*. EMS textbooks in mathematics. European Mathematical Society, 2016.

[Sch13] Alexander Schied. A control problem with fuel constraint and Dawson–Watanabe superprocesses. *The Annals of Applied Probability*, 23(6):2472–2499, 2013.

[SS09] Alexander Schied and Torsten Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13(2):181–204, 2009.

[SST10] Alexander Schied, Torsten Schöneborn, and Michael Tehranchi. Optimal basket liquidation for CARA investors is deterministic. *Applied Mathematical Finance*, 17(6):471–489, 2010.

[SXY21] Jingrui Sun, Jie Xiong, and Jiongmin Yong. Indefinite stochastic linear-quadratic optimal control problems with random coefficients: Closed-loop representation of open-loop optimal controls. *The Annals of Applied Probability*, 31(1):460–499, 2021.

[YZ99] Jiongmin Yong and Xun Yu Zhou. *Stochastic controls*. Stochastic Modelling and Applied Probability. Springer New York, 1999.

[Zha17] Jianfeng Zhang. *Backward stochastic differential equations*. Probability Theory and Stochastic Modelling; 86. Springer, New York, 1st edition, 2017.

Declaration of authorship

I declare that I have completed this dissertation single-handedly without the unauthorized help of a second party and only with the assistance acknowledged therein. I have appropriately acknowledged and cited all text passages that are derived verbatim from or are based on the content of published work of others, and all information relating to verbal communications. I consent to the use of an anti-plagiarism software to check my thesis. I have abided by the principles of good scientific conduct laid down in the charter of the Justus Liebig University Giessen "Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" in carrying out the investigations described in the dissertation.