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Dissertation

Critical Points of Kirchhoff-Routh-Type Functions

Autor	Betreuer	2. Betreuer
T. Fiernkranz	T. Bartsch	M. Ahmedou

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Abstract

For $2 \leq N \in \mathbb{N}$ and $\Gamma_i \in \mathbb{R} \setminus \{0\}$ we proof that functions of the form

$$H_\Gamma(p_1, \ldots, p_N) = \sum_{i \neq j} \Gamma_i \Gamma_j G(p_i, p_j) + \sum_{i=1}^{N} \Gamma_i^2 R(p_i),$$

admit critical points under various circumstances. The p_i will either belong to an open, bounded subset $\Omega \subset \mathbb{R}^d$ with smooth boundary for $d \geq 3$ or to a compact, two dimensional, riemanian manifold (Σ, g). Furthermore, G is a (Dirichlet) Green's function of the negative Laplacian $-\Delta$ associated to Ω or (Σ, g) and R is its Robin's function.
For the case of an open set, we also consider the function ϱ that is the least eigenvalue of the matrix

$$(M(x_1, \ldots, x_N))_{i,j=1}^{N} := \begin{cases} -G(x_i, x_j), & i \neq j \\ R(x_i), & i = j. \end{cases}$$

To achieve the critical points, we also calculate appropriate approximations of the Green's function and Robin's function when close to their singularities.

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Contents

Chapter 1

Introduction

General notation

- (Σ, g) will be a two dimensional compact riemanian manifold. In some cases, Σ will have boundary.
- $\Omega \subset \mathbb{R}^d$ for some $3 \leq d \in \mathbb{N}$.
- $2 \leq N \in \mathbb{N}$ and $\Gamma_i \in \mathbb{R} \setminus \{0\}$ for $1 \leq i \leq N$.
- For a set X, we define

$$\mathcal{F}_N X := \left\{x = (x_1, \dots, x_N) \in X^N : x_i \neq x_j \,\forall i \neq j\right\} \subset X^N,$$

- In a metric space, $U_\varepsilon(x) = \{y \in M : d(x, y) < \varepsilon\}$ is the open ball and $B_\varepsilon(x) = \{y \in M : d(x, y) \leq \varepsilon\}$ is the closed ball.
- Δ_g is the Laplace-Beltrami-Operator of a riemanian manifold, i.e.

$$\Delta_g f = \operatorname{div}(\nabla f),$$

  where $\nabla f(p) \in T_p\Sigma$ is the gradient of f with respect to g. We suppress in the notation that ∇ depends on g. When $\Sigma = \Omega \subset \mathbb{R}^d$ then, Δ_g is the usual Laplacian

$$\Delta f = \sum_{i=1}^{d} \partial_{ii} f.$$

- $\mathcal{C}^k(X) = \{f : X \to \mathbb{R} : f \text{ is } k\text{-times continously differentiable}\}$ for $k \in \mathbb{N} \cup \{\infty\}$.

Functions of Kirchhoff-Routh-type

We are interested in critical points of functions of the form

$$H_{KR} : \mathcal{F}_N X \to \mathbb{R}, \qquad H_{KR}(x) := \sum_{i\neq j} \Gamma_i \Gamma_j G(x_i, x_j) + \sum_{i=1}^{N} \Gamma_i^2 R(x_i),$$

where G is the (Dirichlet) Green's function to the negative Laplace-Beltrami-Operator $-\Delta_g$ and

$$R(x_i) := \lim_{y\to x_i} G(x_i, y) - \Psi\left(d_g(x_i, y)\right).$$

Here, either $X = \Omega \subset \mathbb{R}^d$ for $d \geq 3$ is bounded and has smooth boundary or $X = \Sigma$ and

$$\Psi : (0,\infty) \to \mathbb{R}, \qquad \Psi(r) := \begin{cases} -\frac{1}{2\pi}\ln(r), & d = 2 \\ c_d r^{2-d}, & d \geq 3, \end{cases}$$

with $c_d := \frac{1}{d(d-2)vol_d(B_1(0))}$.
Functions of this form arise in various areas of Mathematics. In his work [21], Kirchhoff connected this function to fluid dynamics by using a point vortex ansatz to the Euler equation for an incompressible and non-viscous fluid. Routh generalized this to open subsets of $\mathbb{R}^2$, whereas it was Lin, who wrote this generalization in a rigorous way [23, 24]. For the planar case, this gave reason to study H_{KR}. While H_{KR} is defined in a finite dimensional space there still exist some difficulties in the study, i.e even when the Γ_i have the same sign, H_{KR} is unbounded from above and below, plus G is generally only given by a partial differential equation. In [6, 22, 7] various critical points of H_{KR} have been obtained. Furthermore in [5], first general results on the dynamics of H_{KR} appear, i.e. the existence of periodic orbits.
Research also considered the case where H_{KR} is not defined in an open subset, but on a surface. In [8], Boatto and Koiler generalize the Theorem of Lin to two dimensional, compact, orientable riemannian manifolds without boundary. In [15], the equations of motion are formulated for surfaces with genus 0. They also study some explicit examples. Moreover, Kimura investigated the vortex motion in surfaces with constant curvature in [20]. In addition, in [1], the authors search for so called vortex crystals, that is a configuration which may move, whereas the form of the configuration does not change. Results on the planar case, but also various surfaces are included. There are also results on critical points of a function similar to H_{KR}, i.e. where all points that relate to a negative Γ_i are kept fixed, see [14]. Finally in the context of vortex motion, there is the book

of Newton [26], which provides a good starting point into the matter of vortex dynamics.
Leaving the vortex motion aside, there are other fields in which critical points of H_{KR} arise. For example, in the papers [19, 13, 7] critical points of H_{KR} are used to construct blow-up solutions to the sinh-equation or the Lane-Emden-Fowler problem with Dirichlet conditions.
Beside being also interested in H_{KR} when $d \geq 3$, we are moreover interested in the function ϱ, which is the least eigenvalue of the matrix

$$(M(x))_{i,j=1}^N = \begin{cases} -G(x_i, x_j), & i \neq j \\ h(x_i, x_i), & i = j, \end{cases}$$

when $G(x_i, x_j) = c_d \left| x_i - x_j \right|^{2-d} - h(x_i, x_j)$. Then ϱ can be written as

$$\begin{aligned} \varrho(x) &= \inf_{|\Gamma|=1} \langle M(x)\Gamma, \Gamma \rangle = \langle M(x)\Gamma(x), \Gamma(x) \rangle \\ &= \sum_{i=1}^N \Gamma_i(x)^2 h(x_i, x_i) - \sum_{i \neq j} \Gamma_i(x)\Gamma_j(x) G(x_i, x_j). \end{aligned}$$

Thus, with $R(x_i) = -h(x_i, x_i)$, we recognize the similarity to H_{KR}. In [4, 3], critical points of ϱ are related to blow-up points of solutions to $-\Delta u = u^p$ for the critical exponent $p = \frac{d+2}{d-2}$ and, therefore, are of interest as well.

Outline of this thesis

In chapter 2, we will discuss everything related to the cases where $\Omega \subset \mathbb{R}^d$ for $d \geq 3$. We will establish an approximation of G near the boundary, i.e.

$$G(x, y) \approx c_d \left(\left| x - y \right|^{2-d} - \left| \bar{x} - y \right|^{2-d} \right),$$

where $\bar{x}$ is the orthogonal reflection of x at the boundary $\partial\Omega$. This will yield critical points of H_{KR}, when $\Gamma_i > 0$, i.e. the following theorem:

Theorem 1.0.1. *For $\Gamma \in \left(\mathbb{R}^+\right)^N$ the Kirchhoff-Routh function H_{KR} has at least $cat_{\Omega^N} \left(\Omega^N, \Delta_N\Omega\right)$ critical points, where $\Delta_N\Omega := \Omega^N \setminus \mathcal{F}_N\Omega$.*
If Ω is not contractible, H_{KR} has at least one critical point.

In chapter 3, we investigate the Green's function on a surface (Σ, g). When Σ is closed, approximations of G are well known, but we could not find anything similar to the approximation in the case when Σ has boundary. This means we restate known approximations of the Green's function on closed manifolds according to [2] (see (C)). Furthermore, we give the following approximation

$$G(p,q) \approx \frac{1}{2\pi} \ln\left(\frac{d_g(\bar{p},q)}{d_g(p,q)}\right),$$

where $\bar{p}$ is the orthogonal reflection of p at $\partial\Sigma$ and d_g is the distance induced by g.
In chapter 4, we look into surfaces for which the corresponding boundary-less surface is not homeomorphic to the sphere or $\mathbb{R}P^2$. In the case without boundary, we establish the following theorem:

Theorem 1.0.2. *If Σ is closed and not homeomorphic to the sphere, $\mathbb{R}P^2$ nor the Klein bottle, and for Γ holds*

$$\sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j \neq 0 \qquad \text{for every } J \subset \{1,\dots,N\} \text{ with } |J| \geq 3, \tag{1.0.1}$$

then H_{KR} has a critical point.

These conditions on the Γ_i seem to be optimal even though we do not have a proof that they are. When boundary is involved, we are able to generalize the theorems of [22, 6] to surfaces. We have to exclude the sphere, $\mathbb{R}P^2$ and the Klein bottle, because they lack topology. To express that we exclude these manifolds we use the expression *closed manifold belonging to* Σ. We will define this properly in chapter 3.2. Briefly it is the surface that arises from glueing a disc onto every boundary component of Σ. We prove the following theorems:

Theorem 1.0.3. *If Σ has boundary and the closed manifold belonging to Σ is neither homeomorphic to the sphere, $\mathbb{R}P^2$ nor the Klein bottle and for Γ holds*

$$\sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j \neq 0 \qquad \text{for every } J \subset \{1,\dots,N\} \text{ with } |J| \geq 3 \tag{1.0.2}$$

as well as

$$\sum_{i\in J} \Gamma_i^2 > \sum_{\substack{i,j\in J\\ i\neq j}} |\Gamma_i\Gamma_j| \qquad \text{for every } J \subset \{1,\dots,N\}, \text{ with } |J| \geq 2, \tag{1.0.3}$$

then H_{KR} has a critical point.

Theorem 1.0.4. *If Σ has boundary and the closed manifold belonging to Σ is neither homeomorphic to the sphere, $\mathbb{R}P^2$ nor the Klein bottle, $N \in \{3,4\}$ and for Γ holds*

$$\begin{array}{ll} \Gamma_i\Gamma_{i+1} < 0 & \forall i = 1,\dots,N-1, \\ \sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j < 0 & \forall J \subset \{1,\dots,N\} : |J| \geq 3, \end{array} \tag{1.0.4}$$

then H_Γ has a critical point.

Because we excluded the sphere in chapter 4, we focus on surfaces that admit some symmetry in chapter 5. This gives a tool to overcome the lack of compactness of $\mathcal{F}_N\Sigma$ as well as the lack of topology of the sphere. We are also able to find some conditions for critical points. In those cases, we have an idea on how the critical points have to lie on the surface. So, let Σ admit an isometry $\tau : \Sigma \to \Sigma$ with

$$\{p \in \Sigma : \tau(p) = p\} \cong \underbrace{S^1 \dot{\cup} \ldots \dot{\cup} S^1}_{l-times},$$

for some $l \in \mathbb{N}$. For example τ could be a reflection along some plane. Then, the following two theorems are proven in chapter 5:

Theorem 1.0.5. *i) Let N be even i.e. $N = 2k$ for $k \in \mathbb{N}$, $\Gamma_{\sigma(i)} = (-1)^i$ for some $\sigma \in Sym(N)$ for all $i = 1, \ldots, N$, then H_{KR} has at least $l \cdot k$ critical points.*

ii) Let $N = 4$, $\Gamma_1, \Gamma_3 > 0 > \Gamma_2, \Gamma_4$ and

$$\begin{aligned}
&\sum_{\substack{J \\ i \neq j}} \Gamma_i \Gamma_j < 0 && \forall\, |J| = 3 \\
&|\Gamma_i| < |\Gamma_1| + |\Gamma_3| && i = 2, 4 \\
&|\Gamma_i| < |\Gamma_2| + |\Gamma_4| && i = 1, 3,
\end{aligned}$$

then H_{KR} has at least $2l$ critical points.

Theorem 1.0.6. *If $\Gamma_1 = \Gamma_3 > 0 > \Gamma_2$ and $\Gamma_1 > -2\Gamma_2$, then H_{KR} has a critical point.*

In the Appendices, we provide already known facts, which are still important, and easy calculations from already known facts, which we use.

Chapter 2

Euclidean space

In this chapter, we will look at our Hamiltonian H_{KR} in the case of an open bounded set $\Omega \subset \mathbb{R}^d$ for $3 \leq d \in \mathbb{N}$. We will start with some properties of the Dirichlet Green's function for that case. Furthermore, in this chapter, for $i \in \mathbb{N}$, the numbers $C_i = C_i(\cdot) > 0$ will be constants depending on $\cdot$.

2.1 Green's function in euclidean space

Let $3 \leq d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be open and bounded with at least $\mathcal{C}^3$-boundary. Furthermore, we define $c_d := \frac{1}{d(d-2)vol_d(B_1(0))}$ and the function

$$\Psi : \mathcal{F}_2\Omega \to \mathbb{R}, \qquad \Psi(x,y) := c_d \left|x-y\right|^{2-d}.$$

We also choose $\varepsilon_0 > 0$ such that

$$U_{\varepsilon_0}(\partial\Omega) = \left\{y \in \mathbb{R}^d : dist(y, \partial\Omega) < \varepsilon_0\right\}$$

is a tubular neighborhood of $\partial\Omega$. Now let $\Omega_0 := U_{\varepsilon_0}(\partial\Omega) \cap \Omega$, then for $x \in \Omega_0$ the orthogonal projection $p(x) = p_x \in \partial\Omega$ onto $\partial\Omega$ is well defined and $\mathcal{C}^2$. Moreover, the maps $x \mapsto d(x) = d_x := dist(x, \partial\Omega)$ the distance to the boundary, $x \mapsto \nu(x) = \nu_x$ the inner normal at p_x and $x \mapsto \bar{x}$ the reflection of x at $\partial\Omega$ are all well defined and $\mathcal{C}^2$. For $x \in \Omega_0$, the following identities hold:

$$d_x = |x - p_x|; \qquad \nabla d(x) = \nu_x = \frac{x - p_x}{|x - p_x|}$$

$$\bar{x} = x - 2d_x\nu_x; \qquad p_x = x - d_x\nu_x.$$

Now a *generalized Green's function*

$$G : \mathcal{F}_2\Omega \to \mathbb{R}, \qquad G(x,y) := \Psi(x,y) - h(x,y)$$

is a function for which the following axioms hold:

(A1) G is symmetric, i.e. $G(x,y) = G(y,x)$ and $h(x,y) = h(y,x)$, and G is non-negative, i.e. $G \geq 0$.

(A2) The function $h : \Omega \times \Omega \to \mathbb{R}$ is $\mathcal{C}^\infty$ and

$$R(x) := h(x,x) \to \infty, \qquad \text{if } d_x \to 0.$$

(A3) For all $\varepsilon > 0$ there exists $C_1 = C_1(\Omega, \varepsilon) > 0$ such that

$$\begin{aligned} &|R(x)| + |\nabla R(x)| \leq C_1 && \text{for all } x \text{ with } d_x \geq \epsilon \\ &|G(x,y)| + |\nabla_x G(x,y)| + |\nabla_y G(x,y)| \leq C_1 && \text{for all } x, y \in \Omega \text{ with } |x-y| \geq \epsilon. \end{aligned}$$

(A4) There exists $C_2 = C_2(\Omega) > 0$ such that the map

$$\psi : \Omega_0 \times \Omega_0 \to \mathbb{R}, \qquad \psi(x,y) := \Psi(\bar{x}, y) - h(x,y)$$

satisfies

$$\begin{aligned} |\psi(x,y)| &\leq C_2 d_x \left|\bar{x} - y\right|^{2-d}, \\ |\nabla_x \psi(x,y)| + |\nabla_y \psi(x,y)| &\leq C_2 \left|\bar{x} - y\right|^{2-d}. \end{aligned}$$

(A5) For all $C > 0$ exists $\varepsilon_C > 0$ such that

$$\frac{d_x}{|x-y|} \leq C, d_x \leq d_y, d_x \leq \varepsilon_C \Rightarrow \langle \partial_1 G(x,y), \nu_x \rangle > 0.$$

This results in our first theorem.

Theorem 2.1.1. *The Dirichlet Green's function of Ω is a generalized Green's function.*

Proof: Let G be the Dirichlet Green's function. Then, for $x \in \Omega$, we set $h(x,\cdot)$ as the solution of

$$\begin{cases} \Delta h(x,\cdot) = 0 & \text{in } \Omega \\ h(x,\cdot) = \Psi(x,\cdot) & \text{on } \partial\Omega. \end{cases}$$

Thus, $G(x,y) = \Psi(x,y) - h(x,y)$. The Axioms (A1)-(A3) are well known facts of G and will not be shown here. More information on these axioms can be found in [18]. We will start with (A4). The proof follows [7], where they handle the case $d = 2$ to reach similar axioms. Before we start proving (A4), we show some general inequalities where $C_i = C_i(\Omega) > 0$.

$$\frac{|x-y|}{|\bar{x}-y|} \leq 3, \qquad \forall x \in \Omega_0, \forall y \in \Omega. \tag{2.1.1}$$

$$|\bar{x}-y| \geq d_x, \qquad \forall x \in \Omega_0, \forall y \in \Omega. \tag{2.1.2}$$

$$|\langle z-w, \nu_z\rangle| \leq C_3 |z-w|^2, \qquad \forall w, z \in \partial\Omega. \tag{2.1.3}$$

$$\left| |\bar{x}-y|^2 - |\bar{y}-x|^2 \right| \leq C_4(d_x+d_y) |p_x - p_y|^2, \qquad \forall x, y \in \Omega_0. \tag{2.1.4}$$

$$|\bar{x}-y|^2 \geq C_5 |p_x - p_y|^2, \qquad \forall x, y \in \Omega_0. \tag{2.1.5}$$

$$C_6 \leq \frac{|\bar{x}-y|}{|x-\bar{y}|} \leq C_7, \qquad \forall x, y \in \Omega_0. \tag{2.1.6}$$

$$\left| |\bar{x}-y|^{-d} - |\bar{y}-x|^{-d} \right| \leq C_8(d_x+d_y) |\bar{x}-y|^{-d}, \qquad \forall x, y \in \Omega_0. \tag{2.1.7}$$

We start with (2.1.1) and (2.1.2): Since $U_{\varepsilon_0}(\partial\Omega)$ is a tubular neighborhood, we have

$$d_x = dist(\bar{x}, \partial\Omega) = \inf_{z\in\Omega} |\bar{x}-z| \leq |\bar{x}-y|.$$

Furthermore, using this, we have

$$|x-y| \leq |x-\bar{x}| + |\bar{x}-y| = 2d_x + |\bar{x}-y| \leq 3|\bar{x}-y|.$$

Next, we show (2.1.3). For this, we will show that for every $p \in \partial\Omega$ exists an open neighborhood U_p of p and $C(p) > 0$ such that

$$|\langle z-w, \nu_z\rangle| \leq C(p) |z-w|^2 \qquad \forall z, w \in U_p. \tag{$*$}$$

Then, the compactness of $\partial\Omega$ yields a finite covering $(U_{p_i})_{i=1}^k$ and a Lebesgue number $\delta > 0$ with

$$w, z \in \partial\Omega, |w-z| < \delta \Rightarrow \exists 1 \leq i \leq k : w, z \in U_{p_i}.$$

Defining $C_3 := \max\left\{(C(p_i))_{i=1}^k, \frac{1}{\delta}\right\}$ we achieve

$$|\langle z-w, \nu_z\rangle| \leq C_3 |z-w|^2$$

if $|z-w| < \delta$. And if $|z-w| \geq \delta$, we have

$$|\langle z-w, \nu_z\rangle| \leq |z-w| = \frac{\delta |z-w|}{\delta} \leq C_3 |z-w|^2.$$

It remains to show $(*)$.
Without loss of generality we have $0 = p \in \partial\Omega$ and there exists an open neighborhood $\tilde{U}$ of 0, $\varepsilon > 0$ and a $\mathcal{C}^3$ function $F : \left\{x \in \mathbb{R}^{d-1} : |x| < 2\varepsilon\right\} \to [0, \infty)$ such that

$$\tilde{U} \cap \Omega = \left\{(x,t) \in \tilde{U} : t > F(x)\right\}.$$

Then, there is $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\nu_{(x,F(x))} = \alpha \begin{pmatrix} -\nabla F(x) \\ 1 \end{pmatrix}$. For $z, w \in \tilde{U} \cap \partial\Omega$, we set $z = (x, F(x))$ and $w = (y, F(y))$ and we define

$$f_1(x,y) := \left\langle (y, F(y)) - (x, F(x)), \nu_{(x,F(x))} \right\rangle,$$

which is $\mathcal{C}^2$ since F is at least $\mathcal{C}^3$. Taylor's theorem now yields

$$f_1(x,y) = \underbrace{f_1(x,x)}_{=0} + \langle \nabla_2 f_1(x,x), y - x\rangle + (D_y)^2 f_1(x, \xi_y)[y - x, y - x].$$

We will show that

$$\langle \nabla_2 f_1(x,x), y - x\rangle = 0.$$

Then, using the compactness of $\overline{B_\varepsilon(0) \times B_\varepsilon(0)}$ we establish

$$|f_1(x,y)| \le C(p)\,|x - y|^2.$$

Especially for $z, w \in B_\varepsilon(0) \cap \partial\Omega \subset \mathbb{R}^d$, we have $x, y \in B_\varepsilon(0) \subset \mathbb{R}^{d-1}$ and it follows

$$\begin{aligned} |\langle z - w, \nu_z\rangle| &= |f_1(x,y)| \\ &\le C(p)\,|x - y|^2 \\ &\le C(p)\left(|x - y|^2 + |F(x) - F(y)|^2\right) = C(p)\,|z - w|^2. \end{aligned}$$

It remains to show that $\langle \nabla_y f_1(x,x), y - x\rangle = 0$. We have

$$f_1(x,y) = \sum_{i=1}^{d-1} (y_i - x_i)\nu(x, F(x))_i + (F(y) - F(x))\nu(x, F(x))_d.$$

Thus, for $1 \le i \le d-1$

$$\partial_{y_i} f_1(x,y) = \nu(x, F(x))_i + \partial_i F(y)\nu(x, F(x))_d.$$

Bringing this together, we calculate

$$\begin{aligned}
\langle \nabla_2 f_1(x,x), y-x\rangle &= \sum_{i=1}^{d-1}(y_i - x_i)\nu(x,F(x))_i + \nu(x,F(x))_d \langle \nabla F(x), y-x\rangle \\
&= \left\langle \nu(x,F(x)), \begin{pmatrix} y-x \\ \langle \nabla F(x), y-x\rangle \end{pmatrix} \right\rangle \\
&= \alpha \left\langle \begin{pmatrix} -\nabla F(x) \\ 1 \end{pmatrix}, \begin{pmatrix} y-x \\ \langle \nabla F(x), y-x\rangle \end{pmatrix} \right\rangle = 0.
\end{aligned}$$

With this, we showed (2.1.3).
We move on to (2.1.4). With $\bar{x} = p_x - d_x\nu_x$ and $y = p_y + d_y\nu_y$, we have

$$|\bar{x}-y|^2 = |p_x - p_y|^2 + |d_x\nu_x + d_y\nu_y|^2 - 2\langle d_x\nu_x + d_y\nu_y, p_x - p_y\rangle .$$

Thus, with (2.1.3), we get

$$\begin{aligned}
\left| |\bar{x}-y|^2 - |x-\bar{y}|^2 \right| &\le 4d_x |\langle \nu_x, p_x - p_y\rangle| + 4d_y |\langle \nu_y, p_y - p_x\rangle| \\
&\le 4C_3(d_x+d_y)\,|p_y - p_x|^2 .
\end{aligned}$$

This proves (2.1.4) with $C_4 := 4C_3$.
Going on, we show (2.1.5). Again using (2.1.3), we see

$$\begin{aligned}
|\bar{x}-y|^2 &= |p_x - p_y|^2 + |d_x\nu_x + d_y\nu_y|^2 - 2\langle d_x\nu_x + d_y\nu_y, p_x - p_y\rangle \\
&\ge |p_x - p_y|^2 - 2C_3(d_x+d_y)\,|p_x-p_y|^2 \ge (1-4C_3\epsilon_0)\,|p_x - p_y|^2 .
\end{aligned}$$

If now $\varepsilon_0 < \frac{1}{4C_3}$ (2.1.5) follows.
Next we prove (2.1.6). Using (2.1.4) and (2.1.5), we establish

$$\left| 1 - \left(\frac{|\bar{y}-x|}{|\bar{x}-y|}\right)^2 \right| \le C_4(d_x+d_y)\frac{|p_x-p_y|^2}{|\bar{x}-y|^2} \le \frac{2C_4}{C_5}\epsilon_0.$$

Thus, (2.1.6) follows.
Finally we show (2.1.7). First note that (2.1.7) is equivalent to

$$\left| 1 - \left(\frac{|\bar{x}-y|}{|x-\bar{y}|}\right)^d \right| = \left| 1 - \left(\frac{|\bar{x}-y|^2}{|x-\bar{y}|^2}\right)^{\frac{d}{2}} \right| \le C_7(d_x+d_y).$$

The identity

$$\begin{aligned}|\bar{x}-y|^2 &= |p_x-p_y|^2+|d_x\nu_x+d_y\nu_y|^2-2\langle d_x\nu_x+d_y\nu_y,p_x-p_y\rangle\\ &= |x-\bar{y}|^2-4\langle d_x\nu_x+d_y\nu_y,p_x-p_y\rangle\end{aligned}$$

yields

$$\frac{|\bar{x}-y|^2}{|x-\bar{y}|^2}=1-\frac{4\langle d_x\nu_x+d_y\nu_y,p_x-p_y\rangle}{|x-\bar{y}|^2}.$$

Using (2.1.3) and (2.1.5), we see

$$\frac{4\langle d_x\nu_x+d_y\nu_y,p_x-p_y\rangle}{|x-\bar{y}|^2}=O(d_x+d_y).$$

With (2.1.2) and (2.1.6), we see that if $|x-\bar{y}|\to 0$, then $d_x,d_y\to 0$. Thus

$$\frac{4\langle d_x\nu_x+d_y\nu_y,p_x-p_y\rangle}{|x-\bar{y}|^2}\to 0 \qquad \text{if } |x-\bar{y}|\to 0.$$

Finally, we have $(1+p)^z=1+O(z)$ for $z\to 0$ if $p\geq 1$, with Taylor's theorem. We conclude

$$\begin{aligned}\left|1-\left(\frac{|\bar{x}-y|^2}{|x-\bar{y}|^2}\right)^{\frac{d}{2}}\right| &= \left|1-\left(1-\frac{4\langle d_x\nu_x+d_y\nu_y,p_x-p_y\rangle}{|x-\bar{y}|^2}\right)^{\frac{d}{2}}\right|\\ &= O\left(\left|\frac{4\langle d_x\nu_x+d_y\nu_y,p_x-p_y\rangle}{|x-\bar{y}|^2}\right|\right)=O(d_x+d_y),\end{aligned}$$

which proves (2.1.7).

Equipped with these tools we can prove (A4). First, we show

$$|\psi(x,y)|\leq C_2 d_x\,|\bar{x}-y|^{2-d}\,. \tag{2.1.8}$$

We show this with the maximum principle. With

$$\Delta_y\psi(x,y)=\Delta_y h(x,y)-\Delta_y\Psi(x,y)=0,$$

we see that $y\mapsto\psi(x,y)$ is harmonic and the maximum principle holds. Thus, for every $x\in\Omega_0$

$$\max_{y\in\Omega_0}|\psi(x,y)|=\max_{y\in\partial\Omega_0}|\psi(x,y)|\,.$$

Clearly (2.1.8) holds in $\partial\Omega_0\setminus\partial\Omega$, so we have to check it for $y\in\partial\Omega$. If $y\in\partial\Omega$, we have $y = \bar{y} = p_y$ and we calculate

$$
\begin{aligned}
|\psi(x,y)| &= |\Psi(\bar{x},y) - h(x,y)| \\
&= c_d \left| |\bar{x}-y|^{2-d} - |x-y|^{2-d} \right| \\
&= c_d \left| |x-\bar{y}|^{2-d} - |\bar{x}-y|^{2-d} \right| \\
&= c_d \, |\bar{x}-y|^{-d} \left| |x-\bar{y}|^2 \left(\frac{|\bar{x}-y|}{|\bar{y}-x|} \right)^d - |\bar{x}-y|^2 \right| .
\end{aligned}
$$

Furthermore, again because of $y \in \partial\Omega$, the equations

$$
\begin{aligned}
|\bar{x}-y|^2 &= |p_x - p_y|^2 - 2d_x \langle p_x - p_y, \nu_x \rangle + d_x^2, \\
|x-\bar{y}|^2 &= |p_x - p_y|^2 + 2d_x \langle p_x - p_y, \nu_x \rangle + d_x^2
\end{aligned}
$$

hold. We conclude

$$
\begin{aligned}
|\psi(x,y)| \leq c_d \, |\bar{x}-y|^{-d} & \left| \left(1 - \left(\frac{|\bar{x}-y|}{|\bar{y}-x|} \right)^d \right) \left(|p_x - p_y|^2 + d_x^2 \right) \right| \\
&+ c_d \, |\bar{x}-y|^{-d} \left| 2d_x \langle p_y - p_x, \nu_x \rangle \left(1 + \left(\frac{|\bar{x}-y|}{|\bar{y}-x|} \right)^d \right) \right| .
\end{aligned}
$$

Using (2.1.3), (2.1.5) and (2.1.6), we estimate the second term

$$
c_d \, |\bar{x}-y|^{-d} \left| 2d_x \langle p_y - p_x, \nu_x \rangle \left(1 + \left(\frac{|\bar{x}-y|}{|\bar{y}-x|} \right)^d \right) \right| \leq C_9 d_x \, |\bar{x}-y|^{2-d} .
$$

For the first term, we see with (2.1.2) and (2.1.5) that

$$
|p_x + p_y|^2 + d_x^2 \leq C_9 \, |\bar{x}-y|^2 .
$$

Combining this with

$$
\left| 1 - \left(\frac{|\bar{x}-y|}{|x-\bar{y}|} \right)^d \right| \leq C_8 d_x,
$$

yields (2.1.8) for $y \in \partial\Omega$ and thus in general. Next, we show

$$
|\nabla_x \psi(x,y)| \leq \frac{C_2}{2} \, |\bar{x}-y|^{2-d} . \tag{2.1.9}
$$

Therefore, we show

$$
|\Delta_x \psi(x,y)| \leq \frac{C_{10}}{d_x} \, |\bar{x}-y|^{2-d} . \tag{2.1.10}
$$

Before we show (2.1.10), we will use it to prove (2.1.9). For this we need the following theorem.

Theorem 2.1.2. *If $u \in \mathcal{C}^2(\Omega)$ and $\Delta u = f$ in the open set $\Omega' \subset \Omega \subset \mathbb{R}^d$, there exists $C_{11} = C_{11}(d)$ such that*

$$\sup_{\Omega'} d_x \left|\nabla u(x)\right| \leq C_{11} \left(\sup_{\Omega'} \left|u(x)\right| + \sup_{\Omega'} d_x^2 f(x) \right).$$

Proof: This is Theorem 3.9 in [18].

□

Combining this with (2.1.8) and (2.1.10), we let $y \in \Omega_0$ and compute

$$\begin{aligned} \sup_{x \in \Omega'} d_x \left|\nabla_x \psi(x,y)\right| &\leq C_{11} \left(\sup_{x \in \Omega'} \left|\psi(x,y)\right| + \sup_{x \in \Omega'} d_x^2 \left|\Delta_x \psi(x,y)\right| \right) \\ &\leq C_{11} \left(\sup_{x \in \Omega'} C_2 d_x \left|\bar{x} - y\right|^{2-d} + \sup_{x \in \Omega'} C_{10} d_x \left|\bar{x} - y\right|^{2-d} \right) \\ &\leq C_{12} \sup_{x \in \Omega'} d_x \left|\bar{x} - y\right|^{2-d} \end{aligned}$$

for any open set $\Omega' \subset \Omega_0$. Thus we conclude (2.1.9). It only remains to show (2.1.10).

Because $x \mapsto h(x,y)$ is harmonic, we define

$$f_y : \Omega_0 \to \mathbb{R}, \qquad f_y(x) := \left|\bar{x} - y\right|^{2-d}$$

and prove

$$\left|\Delta f_y(x)\right| \leq \frac{C_{13}}{d_x} \left|\bar{x} - y\right|^{2-d}.$$

For this we calculate Δf_y. If $1 \leq i \leq d$, we have

$$\begin{aligned} \partial_i f(x) = \partial_i \left(\left|\bar{x} - y\right|^{2-d} \right) &= \partial_i \left(\sum_{j=1}^d \left(\bar{x}_j - y_j\right)^2 \right)^{\frac{2-d}{2}} \\ &= \frac{2-d}{2} \left(\sum_{j=1}^d \left(\bar{x}_j - y_j\right)^2 \right)^{-\frac{d}{2}} \cdot 2 \cdot \sum_{j=1}^d \left(\bar{x}_j - y_j\right) \partial_i \bar{x}_j \\ &= (2-d) \left|\bar{x} - y\right|^{-d} \sum_{j=1}^d \left(\bar{x}_j - y_j\right) \partial_i \left(\bar{x}_j\right). \end{aligned}$$

We further see

$$\partial_i\partial_i f(x) = d(d-2)\,|\bar{x}-y|^{-d-2}\left(\sum_{j=1}^{d}(\bar{x}_j-y_j)\partial_i\left(\bar{x}_j\right)\right)^2 \\ + (2-d)\,|\bar{x}-y|^{-d}\sum_{j=1}^{d}\left(\partial_i\left(\bar{x}_j\right)\right)^2 + \left(\bar{x}_j-y_j\right)\partial_{ii}\left(\bar{x}_j\right).$$

Thus,

$$\Delta f(x) = d(d-2)\,|\bar{x}-y|^{-d-2}\sum_{i=1}^{d}\left(\sum_{j=1}^{d}(\bar{x}_j-y_j)\partial_i\left(\bar{x}_j\right)\right)^2 \\ + (2-d)\,|\bar{x}-y|^{-d}\sum_{i=1}^{d}\sum_{j=1}^{d}\Big(\left(\partial_i\left(\bar{x}_j\right)\right)^2 + \left(\bar{x}_j-y_j\right)\partial_{ii}\left(\bar{x}_j\right)\Big).$$

Now let $\nu_i := (\nu_x)_i$. With the identity $\bar{x} = x - 2d_x\nu_x$ and $\nabla d_x = \nu_x$, we infer the following two equations

$$\partial_i\bar{x}_j = \delta_{i,j} - 2\partial_i(d_x)\nu_j - 2d_x\partial_i\nu_j = \delta_{i,j} - 2\nu_i\nu_j - 2d_x\partial_i\nu_j \tag{2.1.11}$$

and

$$\partial_{ii}\bar{x}_j = 2\partial_i\nu_i\nu_j - 4\nu_i\partial_i\nu_j - 2d_x\partial_{ii}\nu_j. \tag{2.1.12}$$

Furthermore, we also will use

$$-\,d\,|\bar{x}-y|^{-2}\sum_{i=1}^{d}\left(\sum_{j=1}^{d}(\bar{x}_j-y_j)(\delta_{ij}-2\nu_i\nu_j)\right)^2 + \sum_{i,j=1}^{d}(\delta_{ij}-2\nu_i\nu_j)^2 = 0. \tag{2.1.13}$$

We show (2.1.13) before we return to the calculations of Δf_y. First, we see

$$\begin{aligned}
\sum_{i,j=1}^{d}(\delta_{ij}-2\nu_i\nu_j)^2 &= \sum_{i=1}^{d}(1-2\nu_i^2)^2+\sum_{i\neq j}4\nu_i^2\nu_j^2 \\
&= \sum_{i=1}^{d}\left((\underbrace{1-2\nu_i^2}_{=\nu_1^2+\cdots-\nu_i^2+\cdots+\nu_d^2})^2+\sum_{\substack{j=1\\ j\neq i}}^{d}4\nu_i^2\nu_j^2. \right) \\
&= \sum_{i=1}^{d}\left(\sum_{l=1}^{d}\nu_l^4+\sum_{\substack{l_1\neq l_2\\ l_1\neq i\neq l_2}}\nu_{l_1}^2\nu_{l_2}^2-2\sum_{i\neq l=1}^{d}\nu_i^2\nu_l^2+4\sum_{i\neq l=1}^{d}\nu_i^2\nu_l^2\right) \\
&= \sum_{i=1}^{d}\left(\sum_{j=1}^{d}\nu_j^2\right)^2=\sum_{i=1}^{d}1=d.
\end{aligned}$$

Thus, we only need to show

$$\sum_{i=1}^{d}\left(\sum_{j=1}^{d}(\bar{x}_j-y_j)(\delta_{ij}-2\nu_i\nu_j)\right)^2=|\bar{x}-y|^2.$$

We see

$$\begin{aligned}
\left(\sum_{j=1}^{d}(\bar{x}_j-y_j)(\delta_{ij}-2\nu_i\nu_j)\right)^2 &= \sum_{j=1}^{d}(\bar{x}_j-y_j)^2(\delta_{ij}-2\nu_i\nu_j)^2 \\
&\quad+\sum_{\substack{j,l=1\\ j\neq l}}^{d}(\bar{x}_j-y_j)(\bar{x}_l-y_l)(\delta_{ij}-2\nu_i\nu_j)(\delta_{il}-2\nu_i\nu_l).
\end{aligned}$$

So, like before, we conclude

$$\begin{aligned}
\sum_{i,j=1}^{d}(\bar{x}_j-y_j)^2(\delta_{ij}-2\nu_i\nu_j)^2 &= \sum_{i=1}^{d}(\bar{x}_i-y_i)^2\left((1-2\nu_i^2)^2+\sum_{\substack{j=1\\ j\neq i}}^{d}4\nu_i^2\nu_j^2\right) \\
&= \sum_{i=1}^{d}(\bar{x}_i-y_i)^2=|\bar{x}-y|^2.
\end{aligned}$$

For $l \neq j$, we continue with

$$\sum_{i,j=1}^{d} (\bar{x}_j - y_j)^2 (\delta_{ij} - 2\nu_i\nu_j)^2 = \sum_{i=1}^{d} (\bar{x}_i - y_i)^2 \left((1 - 2\nu_i^2)^2 + \sum_{\substack{j=1 \\ j \neq i}}^{d} 4\nu_i^2 \nu_j^2 \right)$$
$$= \sum_{i=1}^{d} (\bar{x}_i - y_i)^2 = |\bar{x} - y|^2 .$$

Thus, we finally established

$$\sum_{i=1}^{d} \left(\sum_{j=1}^{d} (\bar{x}_j - y_j)(\delta_{ij} - 2\nu_i\nu_j) \right)^2 = |\bar{x} - y|^2 ,$$

what concludes (2.1.13). Using (2.1.11) and (2.1.13), we calculate

$$\Delta f(x) = d(d-2) |\bar{x} - y|^{-2-d} \sum_{i=1}^{d} 2 \left(\sum_{j=1}^{d} (\bar{x}_j - y_j)(\delta_{ij} - 2\nu_i\nu_j) \right) \left(\sum_{j=1}^{d} (\bar{x}_j - y_j)(-2d_x \partial_i \nu_j) \right)$$
$$+ d(d-2) |\bar{x} - y|^{-2-d} \sum_{i=1}^{d} \left(\sum_{j=1}^{d} (\bar{x}_j - y_j)(-2d_x \partial_i \nu_j) \right)^2$$
$$- (d-2) |\bar{x} - y|^{-d} \sum_{i=1}^{d} \sum_{j=1}^{d} \left(2(\delta_{ij} - 2\nu_i\nu_j)(-2d_x \partial_i \nu_j) + 4d_x^2 (\partial_i \nu_j)^2 + (\bar{x}_j - y_j) \partial_{ii} \bar{x}_j \right) .$$

Having in mind that $\nu, \nabla\nu, D^2\nu$ are bounded, and using (2.1.12) and (2.1.2), we prove

$$|\Delta f(x)| \leq \frac{C_{13}}{d_x} |\bar{x} - y|^{2-d} ,$$

and thus showed (2.1.10). To conclude (A4), it remains to show

$$|\nabla_y \psi(x, y)| \leq \frac{C_2}{2} |\bar{x} - y|^{2-d} . \tag{2.1.14}$$

Because of

$$\psi(x, y) = -h(x, y) + \Psi(\bar{x}, y) + \Psi(x, \bar{y}) - \Psi(x, \bar{y})$$
$$= \psi(y, x) - \Psi(x, \bar{y}) + \Psi(\bar{x}, y),$$

we need to show

$$|\nabla_x (\Gamma(x, \bar{y}) - \Gamma(\bar{x}, y))| \leq C_{13} |\bar{x} - y|^{2-d} .$$

With (2.1.11), we calculate

$$
\begin{aligned}
&\partial_{x_i}\left(|\bar{x}-y|^{2-d}-|x-\bar{y}|^{2-d}\right)\\
=&(2-d)\,|\bar{x}-y|^{-d}\sum_{j=1}^{d}(\bar{x}_j-y_j)(\delta_{i,j}-2(\nu_x)_i(\nu_x)_j-2d_x\partial_i(\nu_x)_j)\\
&-(2-d)(x_i-\bar{y}_i)\,|x-\bar{y}|^{-d}\\
=&-(2-d)(x_i-\bar{y}_i)\,|x-\bar{y}|^{-d}\\
&+(2-d)\,|\bar{x}-y|^{-d}\left(\bar{x}_i-y_i-2(\nu_x)_i\left\langle\bar{x}-y,\nu_x\right\rangle-2d_x\left\langle\bar{x}-y,\partial_i\nu_x\right\rangle\right)\\
=&(2-d)(x_i-\bar{y}_i)\left(|\bar{x}-y|^{-d}-|x-\bar{y}|^{-d}\right)\\
&-2(2-d)(\nu_x)_i\,|\bar{x}-y|^{-d}\left\langle p_x-p_y,\nu_x\right\rangle\\
&-2(2-d)d_x\,|\bar{x}-y|^{-d}\left\langle\bar{x}-y,\partial_i\nu_x\right\rangle\\
&+(2-d)\,|\bar{x}-y|^{-d}\left(2(\nu_x)_i\left\langle d_x\nu_x+d_y\nu_y,\nu_x\right\rangle-2d_x(\nu_x)_i-2d_y(\nu_y)_i\right).
\end{aligned}
$$

With this, we conclude (2.1.14) by using (2.1.2), (2.1.3), (2.1.4), (2.1.5), (2.1.6), (2.1.7) as well as $|\nu_x-\nu_y|=O(|x-y|)$ and

$$
1-\langle\nu_x,\nu_y\rangle=O(|x-y|^2). \tag{2.1.15}
$$

This follows from the following calculation:

$$
\begin{aligned}
2\left(1-\langle\nu_x,\nu_y\rangle\right)&=\langle\nu_x-\nu_y,\nu_x\rangle+\langle\nu_y-\nu_x,\nu_y\rangle\\
&=\sum_{i=1}^{d}\left((\nu_x)_i-(\nu_y)_i\right)(\nu_x)_i+\left((\nu_y)_i-(\nu_x)_i\right)(\nu_y)_i\\
&=\sum_{i=1}^{d}(\nu_x)_i^2-2(\nu_x)_i(\nu_y)_i+(\nu_y)_i^2\\
&=\sum_{i=1}^{d}\left((\nu_x)_i-(\nu_y)_i\right)^2\\
&=|\nu_x-\nu_y|^2.
\end{aligned}
$$

Thus, we proved (A4). To finish our proof, we need to show (A5). This is done in Appendix A

□

Before we finish this section, we give the following lemma, which contains useful expressions concerning the singularities when Green's functions are involved.

Lemma 2.1.3. *For a function G satisfying (A1)-(A4) the following holds:*

(i) $R(x) = 2^{2-d} c_d d_x^{2-d} + O(d_x^{3-d})$, *if* $x \in \Omega_0$.

(ii) $\nabla R(x) = 2^{2-d}(2-d) c_d d_x^{1-d} \nu_x + O(d_x^{2-d})$, *if* $d_x \longrightarrow 0$.

(iii) $G(x,y) = \Psi(x,y) - \Psi(\bar{x},y) + O(d_x \left|\bar{x}-y\right|^{2-d})$, *if* $x \in \Omega_0$.

(iv) $\partial_1 G(x,y) = (d-2)c_d \left(\frac{y-x}{|x-y|^d} + \frac{x-\bar{y}}{|x-\bar{y}|^d} \right) + O(|x-\bar{y}|^{2-d})$, *if* $y \in \Omega_0$.

(v) $|\nabla_x h(x,y)| = O(|\bar{x}-y|^{1-d})$, *if* $x \in \Omega_0$.

(vi) $\langle \partial_1 G(x,y), \nu_x \rangle + \langle \partial_1 G(y,x), \nu_y \rangle = (d-2)c_d(d_x+d_y) \left(|\bar{x}-y|^{-d} + |x-\bar{y}|^{-d} \right) + O(|x-y|^{2-d})$, *if* $x, y \in \Omega_0$.

(vii) $|\bar{x}-y|^2 = |x-y|^2 + 4d_x d_y + o(|x-y|^2)$, *if* $x, y \longrightarrow x^* \in \partial\Omega$.

Proof: We go through every point in the following list.

(i) and *(ii)*: With (A4) and $\bar{x} = x - 2d_x \nu_x$, we have

$$R(x) = \Psi(\bar{x}, x) - \psi(x,x) = 2^{2-d} c_d d_x^{2-d} - \psi(x,x).$$

With (A4), we directly see (i) and (ii), because of $|\bar{x}-x| = 2d_x$ and $\nabla \left(d_x^{2-d}\right) = (2-d) d_x^{1-d} \nu_x$.
(iii) and *(iv)*: We have

$$G(x,y) = \Psi(x,y) - h(x,y) = \Psi(x,y) - \Psi(\bar{x},y) + \psi(x,y).$$

Thus, we directly see (iii) and (iv).
(v): We directly use (A4).
(vi): The proof of (vi) is a more involved. First, we see with (iv) that

$$\begin{aligned}\frac{1}{c_d(d-2)} \left(\langle \partial_1 G(x,y), \nu_x \rangle + \langle \partial_1 G(y,x), \nu_y \rangle \right) &= \left(\frac{\langle y-x, \nu_x \rangle}{|x-y|^d} + \frac{\langle x-\bar{y}, \nu_x \rangle}{|\bar{y}-x|^d} \right) \\ &+ \left(\frac{\langle x-y, \nu_y \rangle}{|x-y|^d} + \frac{\langle y-\bar{x}, \nu_y \rangle}{|y-\bar{x}|^d} \right) + O(|\bar{x}-y|^{2-d}).\end{aligned}$$

Moreover, we see

$$|x-y|^{-d} \left(\langle y-x, \nu_x \rangle + \langle x-y, \nu_y \rangle \right) = |x-y|^{-d} \langle x-y, \nu_y - \nu_x \rangle = O(|x-y|^{2-d}).$$

Therefore, we need to calculate

$$\frac{\langle x-\bar{y}, \nu_x \rangle}{|\bar{y}-x|^d} + \frac{\langle y-\bar{x}, \nu_y \rangle}{|y-\bar{x}|^d}.$$

Consequently, the following two identities hold

$$\left\langle |x-\bar{y}|^{-d}(x-\bar{y}),\nu_x\right\rangle = \left\langle |x-\bar{y}|^{-d}(x-p_x),\nu_x\right\rangle + \left\langle |x-\bar{y}|^{-d}(p_x-\bar{y}),\nu_x\right\rangle,$$
$$\left\langle |y-\bar{x}|^{-d}(y-\bar{x}),\nu_y\right\rangle = \left\langle |y-\bar{x}|^{-d}(y-p_y),\nu_y\right\rangle + \left\langle |y-\bar{x}|^{-d}(p_y-\bar{x}),\nu_y\right\rangle.$$

Using $p_y-\bar{y}=y-p_y$, we conclude

$$\begin{aligned}
|x-\bar{y}|^{-d}\langle p_x-\bar{y},\nu_x\rangle &= |x-\bar{y}|^{-d}\left(\langle p_x-\bar{y},\nu_y\rangle + \langle p_x-\bar{y},\nu_x-\nu_y\rangle\right)\\
&= |x-\bar{y}|^{-d}\left(\langle p_y-\bar{y},\nu_y\rangle + \underbrace{\langle p_x-p_y,\nu_y\rangle}_{=O(|x-y|^2)} + \langle p_x-\bar{y},\nu_x-\nu_y\rangle\right)\\
&= |x-\bar{y}|^{-d}\left(\langle y-p_y,\nu_y\rangle + \langle p_x-\bar{y},\nu_x-\nu_y\rangle\right) + O\left(|x-y|^{2-d}\right).
\end{aligned}$$

In the same way, we derive

$$|y-\bar{x}|^{-d}\langle p_y-\bar{x},\nu_y\rangle = |y-\bar{x}|^{-d}\left(\langle x-p_x,\nu_x\rangle + \langle p_y-\bar{x},\nu_x-\nu_y\rangle\right) + O(|x-y|^{2-d}).$$

Furthermore, using the identity $d_x=\langle x-p_x,\nu_x\rangle$, we have

$$\begin{aligned}
&(d_x+d_y)\left(|\bar{x}-y|^{-d}+|\bar{y}-x|^{-d}\right)\\
=&\left\langle |\bar{x}-y|^{-d}(x-p_x),\nu_x\right\rangle + \left\langle |\bar{y}-x|^{-d}(x-p_x),\nu_x\right\rangle\\
&+\left\langle |\bar{x}-y|^{-d}(y-p_y),\nu_y\right\rangle + \left\langle |\bar{y}-x|^{-d}(y-p_y),\nu_y\right\rangle.
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
&\langle \partial_1 G(x,y),\nu_x\rangle + \langle \partial_1 G(y,x),\nu_y\rangle\\
=&(d-2)c_d\left(\frac{\langle x-\bar{y},\nu_x\rangle}{|x-\bar{y}|^d} + \frac{\langle x-\bar{x},\nu_y\rangle}{|y-\bar{x}|^d}\right) + O\left(|x-y|^{2-d}\right)\\
=&(d-2)c_d\left((d_x+d_y)\left(|\bar{x}-y|^{-d}+|\bar{y}-x|^{-d}\right)\right)\\
&+(d-2)c_d\left(\frac{\langle p_x-\bar{y},\nu_x-\nu_y\rangle}{|x-\bar{y}|^d} + \frac{\langle p_y-\bar{x},\nu_y-\nu_x\rangle}{|\bar{x}-y|^d}\right)\\
&+O\left(|x-y|^{2-d}\right).
\end{aligned}$$

With (2.1.2) and (2.1.6), we calculate

$$\begin{aligned}
|\langle p_x-\bar{y},\nu_y-\nu_x\rangle| &\le |p_x-\bar{y}|\,|\nu_y-\nu_x| \le (|x-\bar{y}|+d_x)\,|\nu_y-\nu_x|\\
&\le C_{15}\,|x-\bar{y}|\,|\nu_y-\nu_x|.
\end{aligned}$$

This, then yields

$$|x-\bar{y}|^{-d}\langle p_x-\bar{y},\nu_x-\nu_y\rangle = O(|x-y|^{2-d}).$$

It remains to show

$$|\bar{x}-y|^{-d}\langle p_y-\bar{x},\nu_y-\nu_x\rangle = O(|x-y|^{2-d}),$$

which can be done in an analogous way. Whit this, we proved (vi).
(vii): With

$$\begin{aligned}
|\bar{x}-y|^2 &= \langle \bar{x}-y,\bar{x}-y\rangle = \langle p_x-d_x\nu_x-p_y-d_y\nu_y, p_x-d_x\nu_x-p_y-d_y\nu_y\rangle\\
&= |p_x-p_y|^2 - 2\langle p_x-p_y, d_x\nu_x+d_y\nu_y\rangle + \langle d_x\nu_x+d_y\nu_y, d_x\nu_x+d_y\nu_y\rangle\\
&= |p_x-p_y|^2 + d_x^2+d_y^2+2d_yd_x\langle \nu_x,\nu_y\rangle - \underbrace{2\langle p_x-p_y, d_x\nu_x+d_y\nu_y\rangle}_{=o\left(|x-y|^2\right)}
\end{aligned}$$

and

$$\begin{aligned}
|x-y|^2 &= \langle x-y,x-y\rangle = \langle p_x+d_x\nu_x-p_y-d_y\nu_y, p_x+d_x\nu_x-p_y-d_y\nu_y\rangle\\
&= |p_x-p_y|^2 + 2\langle p_x-p_y, d_x\nu_x-d_y\nu_y\rangle + \langle d_x\nu_x-d_y\nu_y, d_x\nu_x-d_y\nu_y\rangle\\
&= |p_x-p_y|^2 + d_x^2+d_y^2-2d_xd_y\langle \nu_x,\nu_y\rangle + \underbrace{2\langle p_x-p_y, d_x\nu_x-d_y\nu_y\rangle}_{=o((|x-y|^2)},
\end{aligned}$$

we see

$$|\bar{x}-y|^2-|x-y|^2 = 4d_xd_y\langle\nu_x,\nu_y\rangle + o\left(|x-y|^2\right).$$

We finish with showing

$$4d_xd_y\left(\langle\nu_x,\nu_y\rangle-1\right) = o(\left(|x-y|^2\right),$$

if $x,y\to x^*\in\partial\Omega$. This follows from

$$\langle\nu_x,\nu_y\rangle-1 = O\left(|x-y|^2\right),$$

because $d_x,d_y\to 0$. The last equation holds as a result of (2.1.15)

□

Remark. According to [6], this lemma holds even in the case of $d=2$.

2.2 Critical points in euclidean space

Let G be a generalized Green's function. Let $M : \mathcal{F}_N\Omega \to \mathbb{R}^{N\times N}$ be defined by

$$(M(x))_{i,j} := \begin{cases} -G(x_i, x_j), & \text{if } i \neq j \\ R(x_i), & \text{if } i = j. \end{cases}$$

Furthermore, we let

$$\Gamma : \mathcal{F}_N\Omega \to \mathbb{R}^N$$

be a $\mathcal{C}^1$-function. We then define

$$H_\Gamma : \mathcal{F}_N\Omega \to \mathbb{R}, \qquad H_\Gamma(x) := \langle M(x)\Gamma(x), \Gamma(x)\rangle = \sum_{i=1}^{N} \Gamma_i^2(x_i)R(x_i) - \sum_{i\neq j} \Gamma_i(x)\Gamma_j(x)G(x_i, x_j)$$

and are interested in critical points of H_Γ.

Theorem 2.2.1. *If Γ is bounded, $\inf \Gamma_i > 0$ for all i and*

$$0 = \langle M(x)\Gamma(x), D\Gamma(x)[v]\rangle \qquad \text{for all } x \in \mathcal{F}_N\Omega,\ v \in \mathbb{R}^{dN},$$

then H_Γ has at least $cat_{\Omega^N}\left(\Omega^N, \Delta_N\Omega\right)$ critical points, where $\Delta_N\Omega := \Omega^N \backslash \mathcal{F}_N\Omega$.

Proof: Before we start with the proof, we remind the definition of the Lusternik-Schnirlemann-category (in the following LS-category) and some convenient properties.
For a topological space X and subsets $B \subset A \subset X$ the LS-category $cat_X(A, B)$ is the infimum of all $n \in \mathbb{N}_0$ such that there exist open subsets $U_0, \ldots, U_n \subset X$ with the following properties.

(LS1) $A \subset \bigcup_{i=0}^{n} U_i$, $B \subset U_0$.

(LS2) $U_1, \ldots, U_n$ are contractible in X.

(LS3) There exists $h : U_0 \times [0, 1] \to X$ continuous with $h(x, 0) = x$, $h(x, 1) \in B$ and $h(b, t) \in B$ for all $x \in U_0$, $b \in B$ and $t \in [0, 1]$.

If X is an ANR, i.e. there exists an imbedding $i : X \to Z$ into a metric space Z such that $i(X)$ is closed and there exists a neighborhood $U \subset Z$ such that $i(X)$ is a retract of U, the following holds

i) If $B \subset A \subset A' \subset X$, then

$$cat_X(A, B) \leq cat_X(A', B).$$

ii) If $B \subset A \subset X$, $C \subset X$, then

$$cat_X(A \cup C, B) \leq cat_X(A, B) + cat_X(C).$$

iii) If $B \subset A \subset X$, $C \subset X$ and there exists $h : A \times [0,1] \to X$ with $h(x,0) = x$, $h(x,1) \in C$ and $h(b,t) \in B$ for all $x \in A$, $b \in B$ and $t \in [0,1]$, then

$$cat_X(A, B) \leq cat_X(C, h_1(B)).$$

iv) If $B \subset A \subset X' \subset X$, then

$$cat_X(A, B) \leq cat_{X'}(A, B).$$

v) If $B \subset A \subset X$, then there exist neigborhoods $U \subset X$ of A and $V \subset X$ of B such that

$$cat_X(A, B) = cat_X(U, B) = cat_X(A, V \cap A) = cat_X(U, V).$$

These properties follow by using the definition of the LS-category. We omit the proofs.

The proof of Theorem 2.2.1 follows from standard methods of critical point theory. Essentially if $c_1 < c_2$ are regular values of H_Γ and H_Γ has no critical points in $H_\Gamma^{-1}(c_1, c_2)$, the negative gradient flow of H_Γ can be used to contract

$$H_\Gamma^{c_2} := \{x \in \mathcal{F}_N\Omega : H_\Gamma(x) \leq c_2\}$$

down to $H_\Gamma^{c_1}$. Furthermore, there exists a neighborhood of $\Delta_N\Omega$ that itself can be deformed into $\Delta_N\Omega$. Even though this is one way we could proof this, we will give a more abstract proof. To simplify our notation, we let $x = (x_1, \ldots, x_N) \in \Omega^N$ and $x_i \in \Omega$. Before we can use the standard method which gives us at least $cat_\Omega(H_\Gamma^{c_2}, H_\Gamma^{c_1})$ many critical points, we need to set up the involved sets properly such the negative gradient flow will do as we please. Explicitly, we need to establish that we only have to handle situations where $x \in \mathcal{F}_N\Omega$ stays away from the singularities of H_Γ, when we deform it with the negative gradient flow, i.e. the flow needs to stay away from $\partial\mathcal{F}_N\Omega$. We will start with this now. For $\varepsilon > 0$, we define

$$\Omega^\varepsilon := \{x \in \Omega : d_{x_i} := dist(x_i, \partial\Omega) \geq \varepsilon\}.$$

Again, let $\varepsilon_0 > 0$ such that $U_{\varepsilon_0}(\partial\Omega)$ is a tubular neighborhood of $\partial\Omega$. Then, like in section 1.1, the maps $p_{x_i} =: p_i$, $\nu_{x_i} =: \nu_i$ and $d_{x_i} =: d_i$ are well defined and $\mathcal{C}^2$. We know that $x_i = p_i + d_i\nu_i$ and let

$$\alpha : [0,\varepsilon] \times [0,1] \to \mathbb{R}, \quad \alpha(s,t) := (1-t)s + t\varepsilon.$$

We define

$$D_\varepsilon : \overline{\Omega} \times [0,1] \to \overline{\Omega}, \qquad D_\varepsilon(x_i,t) := \begin{cases} x_i, & x_i \in \Omega^\varepsilon, \\ p_i + \alpha(d_i,t)\nu_i, & x_i \in \overline{\Omega} \setminus \Omega^\varepsilon. \end{cases}$$

Then, D_ε deforms $\overline{\Omega}$ into Ω^ε and we have

$$|D_\varepsilon(x_i,t) - D_\varepsilon(x_j,t)| \le L_\varepsilon\,|x_i - x_j| \tag{2.2.1}$$

for some $L_\varepsilon > 0$. Thus,

$$D : \Omega^N \times [0,1] \to \Omega^N, \qquad H(x,t) := (D_\varepsilon(x_1,t), \dots, D_\varepsilon(x_N,t))$$

is a deformation from Ω^N into $(\Omega_\varepsilon)^N$. Furthermore, if $x \in \Delta_N\Omega$, there exist $i \neq j$ with $x_i = x_j$ and thus (2.2.1) infers $D(x,t) \in \Delta_N\Omega$ for all $t \in [0,1]$. Property iii) and iv) of the LS-category yield

$$cat_{\Omega^N}(\Omega^N, \Delta_N\Omega) \overset{iii)}{\le} cat_{\Omega^N}\left((\Omega_\varepsilon)^N, \Delta_N\Omega^\varepsilon\right) \overset{iv)}{\le} cat_{(\Omega_\varepsilon)^N}\left((\Omega_\varepsilon)^N, \Delta_N\Omega^\varepsilon\right),$$

for all $\varepsilon < \varepsilon_0$. With this estimate, we can guarantee that we do not move close to $(\partial\Omega)^N$. Moreover, we will show later that $(\Omega^\varepsilon)^N$ is positive invariant with respect to the negative gradient flow.
With the next part, we handle the rest of $\partial\mathcal{F}_N\Omega$, i.e. $\Delta_N\Omega$. Because we only need to rely on the negative gradient flow until we reach a certain sublevel set, this boundary will not be a problem. We set this up in the following: There exists $\mu_0 > 0$ such that

$$\Delta_N^\mu\Omega := \left\{x \in \Omega^N : \exists i \neq j \text{ with } |x_i - x_j| < \mu\right\}$$

can be deformed into $\Delta_N\Omega$ for all $\mu < \mu_0$. For this see [12]. For $c \in \mathbb{R}$ we let

$$H_\Gamma^c := \{x \in \mathcal{F}_N\Omega : H_\Gamma(x) \le c\}$$

be the sublevel set of H_Γ. Now assume that H_Γ has only finitely many critical points. Then there exists a regular value $a(\mu) \in \mathbb{R}$ of H_Γ such that

$$x \in H_\Gamma^{a(\mu)} \Rightarrow x \in \Delta_N^\mu\Omega.$$

We prove this in the following. Assume the contrary, then there exists $x^n \in \Omega^N \setminus \Delta_N^\mu \Omega$ such that

$$\begin{aligned} -n \geq H_\Gamma(x^n) &= \sum_{i=1}^N \underbrace{\Gamma_i(x^n) R(x_i^n)}_{\geq C} - \underbrace{\sum_{i \neq j} \Gamma_i(x^n) \Gamma_j(x^n) G(x_i^n, x_j^n)}_{=O(1)} \\ &\geq \tilde{C} \end{aligned}$$

for some $C, \tilde{C} \in \mathbb{R}$, which is a contradiction. Furthermore, without loss of generality $a(\mu)$ may be assumed as a regular value, because there are only finitely many critical points. Thus, we have

$$cat_{\Omega_\varepsilon^N}\left(\Omega_\varepsilon^N, \Delta_N \Omega^\varepsilon\right) = cat_{\Omega_\varepsilon^N}\left(\Omega_\varepsilon^N, H_\Gamma^{a(\mu)}\right), \qquad \forall \varepsilon < \varepsilon_0, \mu < \mu_0,$$

because we just saw that $H_\Gamma^{a(\mu)}$ can be deformed into $\Delta_N \Omega^\varepsilon$. We fix $\mu > 0$ and with it $a := a(\mu)$. We will apply (A5) now. With this, we can guarantee that $(\Omega^\varepsilon)^N$ is positive invariant with respect to the negative gradient flow ϕ of $\dot{x} = -\nabla H_\Gamma(x)$ as long as $\phi(x,t) \notin H_\Gamma^{a(\mu)}$. We apply (A5) next. Therefore, we need to show that there exists a $C > 0$ such that

$$H_\Gamma(x) > a \Rightarrow \frac{d_{i_0}}{|x_j - x_{i_0}|} \leq C, \quad \forall j \neq i_0;\ d_{i_0} = \min_{i=1,\dots,N} d_i.$$

Assume the opposite: This means we assume that there exists $x^n \in \mathcal{F}_N \Omega$, $H_\Gamma(x^n) > a$ and $k \neq i_0$ such that $\frac{d_{i_0}^n}{|x_k - x_{i_0}|} \geq n$. Then,

$$\begin{aligned} a < H_\Gamma(x^n) &= \sum_{i=1}^N \Gamma_i^2(x^n) R(x_i^n) - \sum_{i \neq j} \Gamma_i(x^n)\Gamma_j(x^n) G(x_i^n, x_j^n) \\ &= \left(d_{i_0}^n\right)^{2-d} \left(\underbrace{\sum_{i=1}^N \Gamma_i^2(x^n) \left(d_{i_0}^n\right)^{d-2} R(x_i^n)}_{=O(1)} - \underbrace{\sum_{i \neq j} \underbrace{\Gamma_i(x^n)\Gamma_j(x^n)}_{\geq \alpha > 0} \left(d_{i_0}^n\right)^{d-2} G(x_i^n, x_j^n)}_{\to \infty} \right) \to -\infty, \end{aligned}$$

which is a contradiction. We used (i) and (iii) of Lemma 2.1.3 here. Now, with (A5), we choose $\varepsilon < \varepsilon_0$ such that

$$d_{i_0} = \min_{i=1,\dots,N} d_i = \varepsilon \Rightarrow \partial_{\nu_{i_0}} G(x_{i_0}, x_j) > 0, \quad \forall j \neq i_0, H_\Gamma(x) > a. \tag{2.2.2}$$

Let ϕ be the flow of

$$\dot{x} = -\nabla H_\Gamma(x).$$

If $x \in \Omega_\varepsilon^N$, $a < H_\Gamma(\phi(x,t))$ for all $0 \le t \le T$ then $\phi(x,t) \in \Omega_\varepsilon^N$. We prove this statement now. Let $x \in \Omega_\varepsilon^N$, $a < H_\Gamma(x)$ and $d_{i_0} = \varepsilon$, then we also have $d_{i_0} = \min_{i=1,\dots N} d_i$. Thus, (2.2.2) and (ii) of Lemma 2.1.3 yield

$$-\left\langle (\nabla H_\Gamma(x))_{i_0}, \nu_{i_0} \right\rangle = \sum_{i_0 \neq k=1}^{N} \Gamma_{i_0}(x)\Gamma_k(x)\partial_{\nu_{i_0}} G(x_{i_0}, x_k) - \Gamma_{i_0}^2(x)\partial_{\nu_{i_0}} R(x_{i_0}) > 0. \tag{2.2.3}$$

Before we continue and use this fact, we explain the first equality which uses the assumption

$$0 = \langle M(x)\Gamma(x), D\Gamma(x)[v] \rangle .$$

Having in mind that $M(x) = M(x)^T$, we derive that

$$\begin{aligned} DH_\Gamma(x)[v] &= \langle DM(x)[v]\Gamma(x), \Gamma(x) \rangle + 2 \langle M(x)\Gamma(x), D\Gamma(x)[v] \rangle \\ &= \langle DM(x)[v]\Gamma(x), \Gamma(x) \rangle . \end{aligned}$$

Now $v = (\delta_{k,i_0}\nu_k)_{k=1}^N$ leads to the equation in (2.2.3). Using (2.2.3), we see for small $t > 0$ that

$$\begin{aligned} dist\,(\phi(x,t)_{i_0}, \partial\Omega) - \varepsilon &= d_{\phi(x,s)_{i_0}}\big|_{s=0}^{t} \\ &= \int_0^t \frac{d}{ds} d_{\phi(x,s)_{i_0}}\, ds \\ &= \int_0^t \underbrace{\left\langle -\left(\nabla H_\Gamma(\phi(x,s))\right)_{i_0}, \nu_{\phi(x,s)_{i_0}} \right\rangle}_{>0} ds > 0. \end{aligned}$$

Thus, we proved the claim, that if $x \in \mathcal{F}_N\Omega^\varepsilon$, then $\phi(x,t)$ does not leave $(\Omega^\varepsilon)^N$ as long as $a < H_\Gamma(\phi(x,t))$.
We now are able to finish our proof. For $c \in \mathbb{R}$, we define the critical sets

$$K_c := \{x \in \mathcal{F}_N\Omega : H_\Gamma(x) = c \text{ and } \nabla H_\Gamma(x) = 0\} \cap (\Omega^\varepsilon)^N .$$

For $j = 1, \dots, cat_{(\Omega^\varepsilon)^N}\left((\Omega^\varepsilon)^N, H_\Gamma^a \right)$ let

$$c_j := \inf \left\{ c \ge a : cat_{(\Omega^\varepsilon)^N}\,(H_\Gamma^c, H_\Gamma^a) \ge j \right\} \in (a, \infty).$$

Note here that $c_j < \infty$, because $H_\Gamma|_{\mathcal{F}_N\Omega^\varepsilon}$ is bounded from above. We further define

$$K_j := K_{c_j}.$$

We will show that there exists a $\varepsilon_j > 0$ such that

$$cat_{(\Omega^\varepsilon)^N}\left(H_\Gamma^{c_j+\varepsilon_j}, H_\Gamma^a\right) \leq cat_{(\Omega^\varepsilon)^N}\left(H_\Gamma^{c_j-\varepsilon_j}, H_\Gamma^a\right) + cat_{(\Omega^\varepsilon)^N}\left(K_j\right). \tag{2.2.4}$$

With v), we have an open set $U_j \subset \mathcal{F}_N\Omega^\varepsilon$ such that

$$cat_{(\Omega^\varepsilon)^N}\left(K_j\right) = cat_{(\Omega^\varepsilon)^N}\left(U_j\right).$$

For this U_j, there exists $\varepsilon_j > 0$ such that

$$H_\Gamma(x) \leq c_j + \varepsilon_j,\ x \notin U_j \Rightarrow \exists t \geq 0 : H_\Gamma(\phi(x,t)) \leq c_j - \varepsilon_j. \tag{2.2.5}$$

We prove this. Assume the opposite, then there exist $x^n \notin U_j$ such that

$$H_\Gamma(x^n) \leq c_j + \frac{1}{n} \text{ and } H_\Gamma(\phi(x^n,t)) > c_j - \frac{1}{n} \text{ for all } t \geq 0 \text{ where } \phi(x,t) \text{ exists.}$$

First, we see that $\phi(x^n,t)$ has to be defined for all $t \geq 0$. This is due to the fact that $\phi(x^n,t)$ does not leave $(\Omega^\varepsilon)^N$ and because of $H_\Gamma(\phi(x,t)) > c_j - \frac{1}{n}$ it has to stay away from $\Delta_N\Omega^\varepsilon$. This means $\phi(x^n,t)$ belongs to a compact set of $\mathcal{F}_N\Omega$ and thus has to be defined globally. Further, following the same argument, we see that there exists a compact set $K \subset \mathcal{F}_N\Omega$ such that $x^n \in K$ for all n. Thus there exists $\tilde{x} \in K$ and a subsequence (again declared with n) with $x_n \to \tilde{x}$. For $\tilde{x}$, we have

$$H_\Gamma(\tilde{x}) = c_j \text{ and } \nabla H_\Gamma(\tilde{x}) = 0.$$

In other words $\tilde{x} \in K_j$, but this contradicts that $x^n \notin U_j$ and U_j being a neighborhood of K_j.
With (2.2.5), iii) and iv), we see

$$\begin{aligned} cat_{(\Omega^\varepsilon)^N}\left(H_\Gamma^{c_j+\varepsilon_j}, H_\Gamma^a\right) &\leq cat_{(\Omega^\varepsilon)^N}\left(H_\Gamma^{c_j+\varepsilon_j} \setminus U_j, H_\Gamma^a\right) + cat_{(\Omega^\varepsilon)^N}\left(U_j\right) \\ &\leq cat_{(\Omega^\varepsilon)^N}\left(H_\Gamma^{c_j-\varepsilon_j}, H_\Gamma^a\right) + cat_{(\Omega^\varepsilon)^N}\left(K_j\right). \end{aligned}$$

Finally, we let $c_{j-1} < c_j = c_{j+1} = \cdots = d_{j+p}$ for some $p \geq 0$ and conclude with (2.2.4) that

$$cat_{(\Omega^\varepsilon)^N}\left(K_j\right) \geq \underbrace{cat_{(\Omega^\varepsilon)^N}\left(H_\Gamma^{c_j+\varepsilon_j}, H_\Gamma^a\right)}_{\geq p+j} - \underbrace{cat_{(\Omega^\varepsilon)^N}\left(H_\Gamma^{c_j-\varepsilon_j}, H_\Gamma^a\right)}_{\leq j-1} \geq p+1.$$

So $cat_{(\Omega^\varepsilon)^N}\left(K_j\right) \geq 1$ which means $K_j \neq \emptyset$. If further there exist $c_j = c_{j+1}$, we have $cat_{(\Omega^\varepsilon)^N}\left(K_j\right) \geq 2$ what implies $|K_j| = \infty$.

□

Corollary 2.2.2. *For $\Gamma \in (\mathbb{R}^+)^N$ the Kirchhoff-Routh function H_Γ has at least $cat_{\Omega^N}\left(\Omega^N, \Delta_N\Omega\right)$ critical points.*
If Ω is not contractible, H_Γ has at least one critical point.

Proof: We will apply Theorem 1.4. The last statement is due to [11], i.e. if Ω is not contractible we have

$$cat_{\Omega^N}\left(\Omega^N, \Delta_N\Omega\right) \geq 1.$$

Further, the application to H_Γ is immediate, because $\Gamma(x) \equiv \Gamma$.

□

Remark. Originally we also aimed for critical points of ϱ, the least eigenvalue of $M(x)$. But Theorem 1.4 is a bit to weak to also hold for ϱ, i.e. the assumption $\inf \Gamma_i >$ is too strong. Nonetheless, we show in the following, that the Theorem holds if we could weaken this assumption to $\Gamma_i > 0$ for all i.
We have the following identity

$$\varrho(x) = \inf_{\Gamma \in S^{N-1}} \langle M(x)\Gamma, \Gamma \rangle .$$

In Appendix A of [4], it is proven that $\varrho(x)$ is simple and achieved at an eigenvector with only positive components. We rewrite the proof here.
Let $\Gamma \in S^{N-1}$ be an eigenvector of $\varrho(x)$ such that we have

$$M(x)v = \varrho(x)v.$$

Then $\overline{\Gamma} := (|\Gamma_1|, \ldots, |\Gamma_N|)$ also is an eigenvector of $\varrho(x)$. For this, we calculate

$$\begin{aligned}\varrho(x) \leq \left\langle M(x)\bar{\Gamma}, \bar{\Gamma}\right\rangle &\leq \sum_{i=1}^{N} \Gamma_i^2 R(x_i) - \sum_{i \neq j} |\Gamma_i \Gamma_j| \underbrace{G(x_i, x_j)}_{\geq 0} \\ &\leq \sum_{i=1}^{N} \Gamma_i^2 R(x_i) - \sum_{i \neq j} \Gamma_i \Gamma_j G(x_i, x_j) = \langle M(x)\Gamma, \Gamma \rangle = \varrho(x).\end{aligned}$$

Therefore, we may assume without loss of generality that $\Gamma_i \geq 0$.
We moreover assume there exists $\Gamma_k = 0$. For $r > 0$ and $(e_k)_i = \delta_{i,k}$, we

calculate

$$\begin{aligned}\frac{\langle M(x)\left(\Gamma+re_k\right),\Gamma+re_k\rangle}{\left|\Gamma+re_k\right|^2} &= \frac{\langle M(x)\Gamma,\Gamma\rangle+2r\left\langle M(x)e_k,\Gamma\right\rangle+r^2\left\langle M(x)e_k,e_k\right\rangle}{\left|\Gamma+re_k\right|^2}\\ &= \frac{\langle M(x)\Gamma,\Gamma\rangle}{\left|\Gamma+re_k\right|^2}-2r\frac{\sum_{k\neq i=1}^N \Gamma_i G(x_k,x_i)}{\left|\Gamma+re_k\right|^2}+O(r^2)\\ &\leq \varrho(x)-2r\frac{\sum_{k\neq i=1}^N \Gamma_i G(x_k,x_i)}{\left|\Gamma+re_k\right|^2}+O(r^2)<\varrho(x)\end{aligned}$$

for r small enough. This is a contradiction to

$$\varrho(x)=\inf_{\Gamma\in S^{N-1}}\langle M(x)\Gamma,\Gamma\rangle .$$

This proves that $\varrho(x)$ has an eigenvector Γ where all components are positiv. Because $M(x)=M(x)^T$, there exists an orthonormal basis $\{\Gamma,v^1,\ldots,v^{N-1}\}$. Thus, we have

$$0=\left\langle \Gamma,v^i\right\rangle .$$

Because all components of Γ are positive, there must exist positive and negative components in every vector v^i. With the same calculations as before, we see

$$\varrho(x)\leq\left\langle M(x)\bar{v}^i,\bar{v}^i\right\rangle<\left\langle M(x)v^i,v^i\right\rangle=EV(v^i).$$

This results in the proof of $\varrho(x)$ being simple. Thus we have an unique map

$$\Gamma:\mathcal{F}_N\Omega\to S^{N-1}\cap\left\{v\in\mathbb{R}^N : v_i>0\forall i=1,\ldots,N\right\},x\mapsto\Gamma(x)$$

where $\Gamma(x)$ is the unique eigenvector of $\varrho(x)$ in $S^{N-1}\cap\left\{v\in\mathbb{R}^N : v_i>0\forall i=1,\ldots,N\right\}$. The maps ϱ and Γ both are $\mathcal{C}^1$. To see this, we define

$$f:\mathcal{F}_N\Omega\times\mathbb{R}\times S^{N-1}\cap\left\{v\in\mathbb{R}^N : v_i>0\forall i=1,\ldots,N\right\}\to\mathbb{R}^N,\quad f(x,s,v):=M(x)v-sv.$$

We want to apply the Implicit Function Theorem. It is clear that $f(x,\varrho(x),\Gamma(x))=0$. Thus, we calculate

$$\frac{\partial}{\partial s}f(x,s,v)[r]=-rv \text{ and } \frac{\partial}{\partial v}f(x,s,v)[w]=M(x)w-sw.$$

Furthermore, we have

$$T_{\varrho(x),\Gamma(x)}\left(\mathbb{R}\times S^{N-1}\right)=T_{\varrho(x)}\mathbb{R}\oplus T_{\Gamma(x)}S^{N-1}=\mathbb{R}\oplus\Gamma(x)^{\perp}.$$

We see

$$(-\Gamma(x)r,M(x)v-\varrho(x)v)=0\Leftrightarrow r=0 \text{ and } v=\lambda\Gamma(x)$$

for some $\lambda \in \mathbb{R}$. This means $(r, v) = 0$, therefore $\frac{\partial}{\partial(s,v)} f(x, \varrho(x), \Gamma(x))$ is injective. Thus, we conclude that ϱ and Γ are both $\mathcal{C}^1$. Finally,

$$1 \equiv \langle \Gamma(x), \Gamma(x) \rangle$$

implies

$$0 = \frac{\partial}{\partial x} \langle \Gamma(x), \Gamma(x) \rangle [v] = 2 \langle \Gamma(x), D\Gamma(x)[v] \rangle \qquad \text{for all } x \in \mathcal{F}_N \Omega, v \in \mathbb{R}^{dN}.$$

Thus, we proved the weakened assumptions of Theorem 2.2.1, when we set $\inf \Gamma_i > 0$ to $\Gamma_i(x) > 0$.

Chapter 3

The Green's Function on Surfaces

In this section, we consider (Σ, g) to be a compact two dimensional Riemannian manifold. Before we look for critical points of Kirchhoff-Routh-functions again, we will establish some approximations of the (Dirichlet) Green's function belonging to the negative Laplace-Beltrami operator $-\Delta_g$ with Dirichlet boundary conditions if $\partial\Sigma \neq \emptyset$.

3.1 The Green's function on surfaces without boundary

We start with the case that Σ is closed, so that $\partial\Sigma = \emptyset$. In Appendix C, we see the existence of a Green's function and if $\partial\Sigma = \emptyset$, we have

$$G(p,q) = -\frac{1}{2\pi} \ln\left(d_g(p,q)\right) + h_\Sigma(p,q),$$

where d_g is the metric induced by g and h_Σ is in $\mathcal{C}^\infty\left(\Sigma^2\right)$. We combine this with the fact that every surface is locally conformally flat, see [9], and want to construct a chart $\varphi : U \to V$ such that for $x = \varphi(p)$, $y = \varphi(q)$ we have

$$G(p,q) = -\frac{1}{2\pi} \ln|x-y| + h_\varphi(x,y),$$

where again h_φ is a $\mathcal{C}^\infty$-function. Therefore, we need to see how G changes when we change the metric by a conformal factor.

Lemma 3.1.1. *Let $\tilde{g} := e^{2u} g$ be a metric conformal to g and $\tilde{G}$ a Green's function associated with the negative Laplace-Beltrami-Operator $-\Delta_{\tilde{g}}$. Then,* *there exists a function* $W \in \mathcal{C}^\infty\left(\Sigma^2\right)$ *such that*

$$\tilde{G} = G + W.$$

Proof: In Einstein convention, we have

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_i \left(g^{i,j} \sqrt{|g|} \partial_j f \right),$$

in a local chart, where $g(x) = (g_{i,j}(x))_{i,j=1}^d$, $g^{-1}(x) = \left(g^{i,j}(x)\right)_{i,j=1}^d$ and $|g| = det(g) > 0$. We calculate

$$\begin{aligned}
\Delta_{\tilde{g}} f &= \frac{1}{\sqrt{|\tilde{g}|}} \partial_i \left(\tilde{g}^{i,j} \sqrt{|\tilde{g}|} \partial_j f \right) \\
&= e^{-un} \frac{1}{\sqrt{|g|}} \partial_i \left(e^{u(d-2)} \sqrt{|g|} \partial_j f \right) \\
&= e^{-2u} \Delta_g f + \frac{\partial_i e^{u(d-2)}}{e^{ud}} g^{i,j} \partial_j f \\
&= e^{-2u} \Delta_g f + (d-2) e^{-2u} g^{i,j} \partial_i u \partial_j f.
\end{aligned}$$

Now, with $d = 2$, we see

$$\Delta_{\tilde{g}} f = e^{-2u} \Delta_g f. \tag{3.1.1}$$

Furthermore, in a chart $\varphi = (x_1, x_2) : U \to \mathbb{R}^2$, we have

$$(dV_g)_x = \sqrt{|g(x)|} dx_1 \wedge dx_2.$$

Thus,

$$\int_\Sigma f dV_{\tilde{g}} = \int_\Sigma f e^{2u} dV_g. \tag{3.1.2}$$

A Green's function of the Laplace-Beltrami Operator is defined with the condition

$$(\Delta_g)_q G(p,q) = \delta(p,q) - \frac{1}{vol_g(\Sigma)} \qquad \forall p \in \Sigma$$

in a distributional sense, where $\delta(p,q)$ is the Dirac measure at p. Note that G is defined up to a constant, when we require $G(p,q) = G(q,p)$. With (3.1.1) and (3.1.2), we see that for every f and $A \subset \Sigma$ we have

$$\int_A (\Delta_g)_q G(p,q) f(q)\, dV_g(q) = \int_A (\Delta_{\tilde{g}})_q G(p,q) f(q)\, dV_{\tilde{g}}(q).$$

Thus, again with (3.1.2), we have

$$(\Delta_g)_Q G(p,q) - (\Delta_{\tilde{g}})_q G(p,q) = -\frac{1}{vol_g(\Sigma)} + \frac{e^{2u}}{vol_{\tilde{g}}(\Sigma)},$$

because of

$$\int_A \frac{f}{vol_{\tilde{g}}} dV_{\tilde{g}} = \int_A f \cdot \frac{e^{2u}}{vol_{\tilde{g}}} dV_g.$$

We define $W_1 \in \mathcal{C}^\infty(\Sigma)$ as a function which satisfies

$$\Delta_g W_1 = -\frac{1}{vol_g(\Sigma)} + \frac{e^{2u}}{vol_{\tilde{g}}(\Sigma)}.$$

Note that, because of

$$\int_\Sigma -\frac{1}{vol_g(\Sigma)} + \frac{e^{2u}}{vol_{\tilde{g}}(\Sigma)} dV_g = 0,$$

this has a solution. Therefore the function $(p,q) \mapsto G(p,q) - W_1(q) - W_1(p)$ is a Green's function associated with the negative Laplace-Beltrami Operator $-\Delta_{\tilde{g}}$ and we conclude the proof of this Lemma. □

This yields the following.

Proposition 3.1.2. *For every $p_0 \in \Sigma$, there exists a chart $\varphi : U \to V$ around p_0 with $\varphi(p_0) = 0$, a $\mathcal{C}^\infty$-function $h_\varphi : U \times U \to \mathbb{R}$ such that*

$$G(p,q) = -\frac{1}{2\pi} \ln |\varphi(p) - \varphi(q)| + h_\varphi(\varphi(p), \varphi(q)) \qquad \forall p, q \in U.$$

Proof: According to [9], there exists a conformally flat chart around p_0. We elaborate this briefly. With [9], we achieve a chart $\tilde{\varphi} : U \to V$ and $\lambda(x,y) : U \to (0, \infty)$ such that

$$g_{\tilde{\varphi}}(x,y) = \lambda(x,y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \lambda(x,y) I_2.$$

Without loss of generality, we can assume that λ is bounded away from 0, otherwise we use an open set $\tilde{U}$ with $\overline{\tilde{U}} \subset U$. Then, the metric $\tilde{g} := \frac{1}{\lambda} g$, which is defined on U, can be expanded to a to g conformal metric on the whole surface Σ. In the chart $\varphi := \tilde{\varphi} - \tilde{\varphi}(p_0)$, we then have

$$\tilde{g}_\varphi(x,y) = \frac{1}{\lambda(x,y)} g_{\tilde{\varphi}}(x,y) = I_2.$$

Thus, $\varphi(p_0) = 0$ and

$$d_{\tilde{g}}(p,q) = |\varphi(p) - \varphi(q)| \qquad \forall p, q \in U.$$

For $p, q \in U$, we conclude the following:

$$\begin{aligned} G(p,q) &= \tilde{G}(p,q) + W(p,q) \\ &= -\frac{1}{2\pi} \ln\left(d_{\tilde{g}}(p,q)\right) + h_{\tilde{\Sigma}}(p,q) + W(p,q) \\ &= -\frac{1}{2\pi} \ln\left(|\varphi(p) - \varphi(q)|\right) + h_{\varphi}(\varphi(p), \varphi(q)), \end{aligned}$$

where $h_{\varphi}(x,y) := h_{\tilde{\Sigma}}(\varphi^{-1}(x), \varphi^{-1}(y)) + W(\varphi^{-1}(x), \varphi^{-1}(y))$.

□

Remark. In the following, we need the behavior of Δ_g in a conformally flat metric. In a flat chart, we see that

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_i \left(g^{i,j} \sqrt{|g|} \partial_j f \right) = \sum_{i=1}^{d} \partial_i \partial_i f = \Delta f,$$

where Δ is the usual Laplacian on $\mathbb{R}^d$. Thus, if $\tilde{g} = e^{2u} g$ and $\tilde{g}$ is flat in the chart $\varphi : U \to V$, with (3.1.1), we derive the equation

$$\Delta_g f(p) = e^{2u} \Delta_{\tilde{g}} f(p) = e^{2\left(u \circ \varphi^{-1}\right)(\varphi(p))} \Delta \left(f \circ \varphi^{-1} \right) (\varphi(p)) = e^{2u_{\varphi}(x)} \Delta f_{\varphi}(x)$$

in that chart. In particular we will use that if f is a harmonic function in (Σ, g), i.e. $\Delta_g f = 0$, then $f_{\varphi} := f \circ \varphi^{-1}$ will be a harmonic function in V and vice versa.

3.2 The Green's function on surfaces with boundary

In this subsection, we achieve a good approximation of G when $\partial\Sigma \neq \emptyset$. We want to make use of an approximation of G when Ω is an open and bounded set in $\mathbb{R}^2$. More specifically, in [7, 6] a generalized Green's function is defined as a function

$$G : \mathcal{F}_2\Omega \to \mathbb{R}, \qquad G(x,y) = -\frac{1}{2\pi} \ln|x-y| + h(x,y),$$

which satisfies the following conditions:

(A1) $G \geq 0$ and symmetric.

(A2) h is $\mathcal{C}^\infty$, bounded from above, and $R(x) := h(x,x) \to -\infty$ if $d_x = dist(x, \partial\Omega) \to 0$.

(A3) For every $\varepsilon > 0$, there exists $C_1 = C_1(\Omega, \varepsilon) > 0$ such that

$$\begin{aligned} &|R(x)| + |\nabla R(x)| \le C_1 && \text{for every } x \in \Omega \text{ with } d_x \ge \varepsilon \\ &|G(x,y)| + |\nabla_x G(x,y)| + |\nabla_y G(x,y)| \le C_1 && \text{for every } x, y \in \Omega \text{ with } |x-y| \ge \varepsilon. \end{aligned}$$

(A4) There exists $C_2 = C_2(\Omega) > 0$ such that $\psi(x,y) := -\frac{1}{2\pi} \ln |\bar{x} - y| + h(x,y)$ satisfies

$$|\psi(x,y)| + |\nabla_x \psi(x,y)| + |\nabla_y \psi(x,y)| \le C_2 \qquad \text{for every } x, y \in \Omega_0.$$

Here $\Omega_0 \subset \Omega$ is a tubular neighborhood of $\partial\Omega$ and $\bar{x}$ is the reflection of x at the boundary.

In [7], it is proven that the Dirichlet Green's function satisfies these axioms. Our aim is to generalize these axioms to surfaces. As in chapter 1, the Axioms (A1)-(A3) are well known. Thus, only a generalization of (A4) is missing. However, before we explicitly derive a generalization of (A4), we discuss the basic principals of our generalization approach. The following statement provides a simplified summary of our generalization's aim: "around every point of Σ, there exists a chart φ such that G_φ satisfies (A1)-(A4)". With this, we will be able to generalize theorems that where proven in open sets of $\mathbb{R}^2$. However, it does not provide a good approximation, when considering the manifold itself. Next, we outline how we generalize (A4) such that we have a good notion of G and can make use of the proof in [7].
First, the function $\tilde{G}(x,y) := -\frac{1}{2\pi} \ln |x-y|$ is important. But on a surface, $|\cdot|$ has to be adapted. The fact we will use is that $\tilde{G}$ is the Green's function of the negative Laplacian in $\mathbb{R}^2$. Put differently $\tilde{G}$ is the Green's function of the ambient space of Ω. In addition, the map $h = h_{\tilde{G}}$ is defined by a partial differential equation. We could say that with $h_{\tilde{G}}$ the map

$$G_\Omega = \tilde{G} + h_{\tilde{G}}$$

is a projection of $\tilde{G}$ onto Ω. In the same way, the manifold Σ has an ambient manifold $\tilde{\Sigma}$, which is compact and closed. Following this, $\tilde{\Sigma}$ has a Green's function $\tilde{G}$ and, again with a partial differential equation, we will have that

$$G_\Sigma := \tilde{G} + h_{\tilde{G}}$$

is the Green's function of Σ. This leads to $\psi(p,q) := \tilde{G}(\bar{p},q) + h_{\tilde{G}}(p,q)$ and with Proposition 3.1.2 will yield our generalization of G.
By the classification theorem of compact manifolds with boundary, see [25, Thm. 10.1], there exists a compact and closed 2-dimensional manifold $\tilde{\Sigma}$, $k \in \mathbb{N}$, $\tilde{\Sigma} \supset D_i \cong U_1(0) \subset \mathbb{R}^2$ for $i = 1, \dots, k$ with $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$ such that

$$\Sigma = \tilde{\Sigma} \setminus \bigcup_{i=1}^{k} D_i.$$

Furthermore, we extend $\tilde{g}$ to a smooth riemanian metric on $\tilde{\Sigma}$ such that $\tilde{g}|_{\Sigma} = g$. We call such a surface $\tilde{\Sigma}$ the *closed surface belonging to* Σ.
Let $\tilde{\Sigma}_0 \subset \tilde{\Sigma}$ be a tubular neighborhood of $\partial\Sigma$ in $\tilde{\Sigma}$ as well as $\Sigma_0 := \tilde{\Sigma}_0 \cap (\Sigma \setminus \partial\Sigma)$. Then, like in chapter one, $P(p) := P_p$ is the orthogonal projection of p onto $\partial\Sigma$, $\nu(p) =: \nu_p \in T_{P_p}\Sigma$ the interior normal at P_p and $d_p = dist(p, \partial\Sigma)$ are well defined on Σ_0 and $\mathcal{C}^\infty$ when Σ is a $\mathcal{C}^\infty$ manifold. Now, let $\tilde{G}$ be a Green's function of the negative Laplace-Beltrami-Operator $-\Delta_{\tilde{g}}$ on $\tilde{\Sigma}$. For $p \in int(\Sigma)$, we define $h(q,p) = h_p(q)$ to be the solution of the boundary-value problem

$$\begin{cases} \Delta_g h_p = 0 & \text{in } int(\Sigma) \\ h_p(q) = -\tilde{G}(p,q) & \text{on } \partial\Sigma. \end{cases}$$

Lemma 3.2.1. *The map*

$$G : \mathcal{F}_2\Sigma \to \mathbb{R}, \qquad G(p,q) := \tilde{G}(p,q) + h(p,q)$$

is the Dirichlet Green's function of the negative Laplace-Beltrami-Operator $-\Delta_g$ *on* (Σ, g).

Remark. Before we prove Lemma 3.2.1, note that

$$R(p) = \lim_{q \to p} \left(G(p,q) + \frac{1}{2\pi} \ln\left(d_g(p,q)\right) \right) = \tilde{R}(p) + h(p,p) = h(p,p) + O(1)$$

holds, where the $O(1)$ is in a $\mathcal{C}^\infty$ sense.

Proof: The Dirichlet Green's function is the unique function which satisfies

$$-\left(\Delta_g\right)_q G(q,p) = \delta_q(p) \qquad \text{in } \Sigma$$

in a distributional sense, and is 0 on the boundary: $G(p,q) = 0$ for $q \in \partial\Sigma$. Hence, for every $\mathcal{C}^2$-function f, there must hold

$$-f(p) = \int_\Sigma G(p,q) \cdot \Delta_g f(q)\, dV_g(q) + \int_{\partial\Sigma} \partial_{N(q)} \left(G(p,q)\right) f(q) ds_g(q)$$

where V_g is the volume element associated with g, the volume element s_g of $\partial\Sigma$ is induced by V_g and $N(q)$ is the exterior unit normal vector at $q \in \partial\Sigma$. This expression is justified by the Green's formulas. Let $p \in \Sigma$, then we have $(\Delta_g)_q G(p,q) = 0$ for every $p \neq q \in \Sigma$. Moreover, if $q \in \partial\Sigma$, we have $G(p,q) = 0$. Thus, for $\varepsilon > 0$, we see with Green's formulas

$$\begin{aligned}
\int_\Sigma G(p,q)\Delta_g f(q) dV(q) &\overset{\varepsilon\to 0}{\leftarrow} \int_{\Sigma\setminus B_\varepsilon(p)} G(p,q)\Delta_g f(q) dV(q) \\
&\overset{GF}{=} -\int_{\partial\Sigma} \partial_N G(p,q) f(q) ds(q) \\
&+ \underbrace{\int_{\partial B_\varepsilon(p)} G(p,q)\partial_N f(q) ds(q)}_{=:I_1\to 0} - \underbrace{\int_{\partial B_\varepsilon(p)} \partial_N G(p,q) f(q) ds(q)}_{=:I_2\to f(p)}.
\end{aligned}$$

Therefore, our claim can be proven by demonstrating the convergence of I_1 and I_2. Because $\varepsilon \to 0$, there exists a local conformal flat chart $\varphi_p : U_p \to V_p$ such that $\partial B_\varepsilon(p) \subset U$. Now, we set $x = \varphi(p) = 0$ and $y = \varphi(q)$. We then have $\varphi(\partial B_\varepsilon(p)) = \partial B_\varepsilon(0)$ and

$$G(p,q) = \tilde{G}(p,q) + h(p,q) = -\frac{1}{2\pi} \ln|y| + h_\varphi(0,y) + h(p,q).$$

Because both maps h_φ and h are $\mathcal{C}^1$, the integrals over them will vanish and there only remain the ln parts. Thus,

$$\lim_{\varepsilon\to 0} I_1 = \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(0)} -\frac{1}{2\pi} \ln(\varepsilon)\partial_N f(y)\, ds(y) = \lim_{\varepsilon\to 0} -\frac{1}{2\pi}\ln(\varepsilon) \underbrace{\int_{\partial B_\varepsilon(0)} \partial_\nu f(y)\, ds(y)}_{\in[-C\varepsilon, C\varepsilon]} = 0,$$

for some $C > 0$. Further, we calculate $\partial_N \ln|y|$ where $N(y) = -\frac{y}{\varepsilon}$ is the exterior normal vector at $y \in \partial B_\varepsilon(0)$. Note that exterior is meant in the sense of exterior to $\Sigma \setminus B_\varepsilon(p)$. We have

$$\partial_N \ln|y| = \left\langle \frac{1}{|y|} \cdot \frac{y}{|y|}, -\frac{y}{\varepsilon} \right\rangle = -\frac{1}{\varepsilon}.$$

Thus, we conclude

$$\lim_{\varepsilon\to 0} I_2 = \lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(0)} -\frac{1}{2\pi} f(y)\partial_N \ln|y|\ ds(y) = \lim_{\varepsilon\to 0} \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon(0)} f(y)\, ds(y) = f(0).$$

$\square$

We now define

$$\psi : \Sigma_0 \times \Sigma_0 \to \mathbb{R}, \qquad \psi(p,q) := \tilde{G}(\bar{p},q) + h(p,q).$$

Proposition 3.2.2. *There exists $C_2 = C_2(\Sigma) > 0$ such that*

$$|\psi| + |\nabla_p \psi| + |\nabla_q \psi| \leq C_2.$$

Furthermore, for every $p_0 \in \Sigma_0 \cup \partial\Sigma$, there exists a chart $\varphi_{p_0} : U_{p_0} \to V_{p_0}$ such that $G_{\varphi_{p_0}} := G \circ \varphi_{p_0}^{-1}$ is a generalized Green's function in the sense of [6], with the small adjustment that only $\partial V_{p_0} \cap \varphi_{p_0}(U_{p_0} \cap \partial\Sigma)$ is the part of ∂V_{p_0} where we reflect at.

Proof: Beside some problems because of the localization, we follow the proof of [7]. For many calculations involving $-\frac{1}{2\pi}\ln|x-y|$, we will refer to [7]. We again use Theorem 2.1.2. However, this theorem only holds in open sets of $\mathbb{R}^d$. Thus, we need to apply it in a chart. Moreover, because $\Sigma_0 \cup \partial\Sigma$ is compact, this local application will also transfer, to the whole surface $\Sigma_0 \cup \partial\Sigma$.
First, we start with

$$|\psi(p,q)| \leq C_3 d_p \qquad \forall p,q \in \Sigma_0. \tag{3.2.1}$$

Let $p \in \Sigma_0$. Then, the map $q \mapsto \psi(p,q)$ is harmonic and, thus the maximum principle yields

$$\max_{q\in\Sigma_0} |\psi(p,q)| = \max_{q\in\partial\Sigma\cup(\partial\Sigma_0\setminus\partial\Sigma)} |\psi(p,q)|\,.$$

For $q \in \partial\Sigma_0 \setminus \partial\Sigma$, the claim follows, because if $P_p \in \partial\Sigma$ is the projection of p onto $\partial\Sigma$, we have $\overline{P}_p = P_p$ and estimate

$$|\psi(p,q)| = |\psi(p,q) - \psi(P_p,q)| \leq C_q d(p,P_p) = C_q d_p.$$

The compactness of $\partial\Sigma_0 \setminus \partial\Sigma$ yields (3.2.1) on that part of the boundary. It remains the other part of the boundary

$$\max_{q\in\partial\Sigma} |\psi(p,q)| = \max_{q\in\partial\Sigma} \left|\tilde{G}(\bar{p},q) - \tilde{G}(p,q)\right|.$$

When p,q belong to the same neighborhood of a conformal flat chart φ, we see in this flat chart

$$\psi_\varphi(x,y) = \underbrace{\frac{1}{2\pi}\ln\left(\frac{|x-y|}{|\bar{x}-y|}\right)}_{=O(d_x)} + \underbrace{h_\varphi(\bar{x},y) - h_\varphi(x,y)}_{=O(d_x)} = O\left(d_x\right) = O\left(d_p\right).$$

For the part with h_φ, we see this because h_φ is $\mathcal{C}^1$ and, thus,

$$\left|\tilde{h}(\bar{x}, y) - \tilde{h}(x, y)\right| \leq C_\varphi \left|\bar{x} - x\right| = 2C_\varphi d_x.$$

The ln part follows from Taylor's Theorem and (2.1.7) with $d = 2$. This finishes (3.2.1).

Our next step is to prove

$$|\nabla_p \psi| \leq C_4. \tag{3.2.2}$$

For this, we will prove that for every $q \in \Sigma_0$ and every $p_0 \in \partial\Sigma$, there exists a neighborhood U_{p_0} such that $\nabla_p \psi|_{U_{p_0}}(\cdot, q)$ is bounded. The claim then follows because $\partial\Sigma$ is compact and $\nabla_p \psi$ is bounded in $(\Sigma_0 \setminus U) \times \overline{\Sigma}_0$ when U is a neighborhood of $\partial\Sigma$. This allows us to do the calculations in a chart.

So let $p_0 \in \partial\Sigma$ and let $\varphi_{p_0} : \tilde{U}_{p_0} \to \tilde{V}_{p_0}$ be a conformally flat chart around p_0 in $\tilde{\Sigma}$. We apply Theorem 2.1.2 to $\psi_{\varphi_{p_0}}$. To avoid problems with the distance, let $p_0 \in U_{p_0} \subset \tilde{U}_{p_0}$ be open such that

$$dist\left(p, \partial\tilde{U}_{p_0}\right) = dist(p, \partial\Sigma) = d_p \qquad \forall p \in U_{p_0}.$$

Now, if $q \notin \tilde{U}_{p_0}$, we see that $\nabla_p \psi|_{U_{p_0}}(\cdot, q)$ is bounded, because q and p are bounded away from each other. So let $q \in \tilde{U}_{p_0}$. Next, we investigate the map

$$f_q : \tilde{V}_{p_0} \to \mathbb{R}, \qquad x \mapsto \psi\left(\varphi_{p_0}^{-1}(x), q\right).$$

We show that $\nabla f_q|_{V_{p_0}}$ is bounded where $V_{p_0} := \varphi_{p_0}\left(U_{p_0}\right)$. This proves (3.2.2). With Theorem 2.1.2 and (3.2.1) the inequality

$$\Delta f_q(x) \leq \frac{C_5}{d_x}$$

remains to prove. We let $y := \varphi(q)$ and conclude with Proposition 3.1.2 that

$$f_q(x) = -\frac{1}{2\pi}\ln|\bar{x} - y| + h_{\varphi_{p_0}}(x, y) + h\left(\varphi_{p_0}^{-1}(x), \varphi_{p_0}^{-1}(y)\right).$$

With

$$\begin{aligned}
0 = (\Delta_g)_p\, G(p, q) &= e^{2u_{\varphi_{p_0}}} \Delta_x G_{\varphi_{p_0}}(x, y) \\
&= e^{2u_{\varphi_{p_0}}} \Delta_x \left(-\frac{1}{2\pi}\ln|x - y| + h_{\varphi_{p_0}}(x, y) + h\left(\varphi_{p_0}^{-1}(x), \varphi_{p_0}^{-1}(y)\right)\right) \\
&= e^{2u_{\varphi_{p_0}}} \Delta_x \left(h_{\varphi_{p_0}}(x, y) + h\left(\varphi_{p_0}^{-1}(x), \varphi_{p_0}^{-1}(y)\right)\right),
\end{aligned}$$

we see

$$\Delta f_q(x) = -\frac{1}{2\pi}\Delta_x \left(\ln |\bar{x} - y|\right).$$

Thus, with [7] the claim $\Delta f_q \leq \frac{C_5}{d_x}$ follows. This means, that $\nabla\psi|_{U_{p_0}}(\cdot, q)$ is bounded for every $q \in \Sigma_0$ and we obtain (3.2.2).
The final part we show is

$$|\nabla_q \psi| \leq C_6.$$

For every $p, q \in \Sigma_0$, we have

$$\begin{aligned}\psi(p,q) &= \tilde{G}(\bar{p},q) + h(p,q) = \tilde{G}(\bar{q},p) + h(q,p) + \left(\tilde{G}(q,\bar{p}) - \tilde{G}(p,\bar{q})\right)\\ &= \psi(q,p) + \left(\tilde{G}(q,\bar{p}) - \tilde{G}(p,\bar{q})\right).\end{aligned}$$

Thus, the claim follows, when

$$\nabla_q \left(\left(\tilde{G}(q,\bar{p}) - \tilde{G}(p,\bar{q})\right)\right) = O(1).$$

To derive this, we utilize a conformally flat chart φ again and see

$$\left(\tilde{G}(q,\bar{p}) - \tilde{G}(p,\bar{q})\right) = \frac{1}{2\pi}\ln\left(\frac{|x-\bar{y}|}{|y-\bar{x}|}\right) - h_\varphi(x,\bar{y}) + h_\varphi(\bar{x},y).$$

Thus, the claim follows, as h_φ is $\mathcal{C}^1$ on $\tilde{\Sigma} \times \tilde{\Sigma}$ and again with [7].

□

Remark. One may think that because in $d = 2$ the approximation of [7] can be generalized to surfaces, that the approximation we did in chapter one may also translate to manifolds. However, the proof from chapter 1 for higher dimensions can not be translated with the methods we used. The most important reasons for the untranslatability of the approximation are the following:

- Not every manifold with dimension $d \geq 3$ is locally conformally flat.
- The Laplace-Beltrami-Operator is not a conformal operator, because when $\tilde{g} = e^{2u}g$, we have

$$\Delta_{\tilde{g}} f = e^{-2u}\Delta_g f + (d-2)e^{-2u}g^{i,j}\partial_i u \partial_j f.$$

- A Green's function of the Laplace-Beltrami-Operator of a closed manifold Σ has an approximation of the form

$$G(p,q) = c_d\,|x-y|^{2-d} + h(x,y),$$

  but h also has some singularities at $x = y$. See more in Appendix C.

Even if we restrict to higher dimensional manifolds that are locally conformal flat, the other points lead to problems. Especially when using Theorem 2.1.2, we have problems calculating $\Delta_x h(x,y)$, as we do not know if h is harmonic with respect to the usual Laplacian in $\mathbb{R}^d$. Moreover, it is not bounded by $\frac{1}{d_x}$, since it has a singularity at $x=y$.

Chapter 4

Critical points on all surfaces excluding the sphere

We remind that we consider the function

$$H_\Gamma : \mathcal{F}_N\Sigma \to \mathbb{R}, \qquad H_\Gamma(p) = \sum_{i\neq j} \Gamma_i\Gamma_j G(p_i, p_j) + \sum_{i=1}^{N} \Gamma_i^N R(p_i) + \Psi(p),$$

where $N \in \mathbb{N}$, $\Gamma = (\Gamma_1, \ldots, \Gamma_N) \in (\mathbb{R} \setminus \{0\})^N$, (Σ, g) is a 2-dimensional, compact, Riemannian manifold, G is a Green's function of the associated Laplace-Beltrami-Operator, R is its Robin's function

$$R(p_i) := \lim_{q\to p_i} \left(G(p_i, q) + \frac{1}{2\pi} \ln\left(d_g(p_i, q)\right) \right)$$

and $\Psi : \Sigma^N \to \mathbb{R}$ shall be $\mathcal{C}^\infty$. The *closed manifold belonging to* Σ is defined in 3.2. We will prove the following three theorems in this chapter:

Theorem 4.0.1. *If Σ is closed and not homeomorphic to the sphere, $\mathbb{R}P^2$ nor the Klein bottle, and for Γ holds*

$$\sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j \neq 0 \qquad \textit{for every } J \subset \{1, \ldots, N\} \textit{ with } |J| \geq 3, \tag{4.0.1}$$

then H_Γ has a critical point.

Theorem 4.0.2. *If Σ has boundary and the closed manifold belonging to Σ is neither homeomorphic to the sphere, $\mathbb{R}P^2$ nor the Klein bottle and for Γ holds*

(4.0.1), as well as

$$\sum_{i\in J}\Gamma_i^2 > \sum_{\substack{i,j\in J\\ i\neq j}} |\Gamma_i\Gamma_j| \qquad \text{for every } J\subset\{1,\dots,N\},\ \text{with } |J|\geq 2, \tag{4.0.2}$$

then H_Γ has a critical point.

Theorem 4.0.3. *If Σ has boundary and the closed manifold belonging to Σ is neither homeomorphic to the sphere, $\mathbb{R}P^2$ nor the Klein bottle, $N\in\{3,4\}$, and for Γ holds*

$$\begin{array}{cc} \Gamma_i\Gamma_{i+1} < 0 & \text{for all } i = 1,\dots,N-1, \\ \sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j < 0 & \text{for all } J\subset\{1,\dots,N\}: |J|\geq 3, \end{array} \tag{4.0.3}$$

then H_Γ has a critical point.

We will use two methods to achieve these theorems. For theorem 4.0.3, we will generalize a method used in [6]. The theorem itself is a generalization of the main theorem in that paper to surfaces. We can weaken the assumptions, because the linking does not need further assumptions on Γ. The method is changing the negative gradient flow in such a way that it will stay away from the boundary of $\mathcal{F}_N\Sigma$.
The proofs of the Theorems 4.0.1 and 4.0.2 will use calculations and the method from [22]. Here, we use a more traditional treatment. Under the assumptions of the theorems, it will be shown that H_Γ satisfies the Palais-Smale-Condition and thus, when combining this with a linking, will achieve critical points. Theorem 4.0.2 is a generalization of a theorem in [22].
We will start with showing the existence of the linking. We will begin with the linking, because this is the main reason we have to exclude the homeomorphism class of the sphere, $\mathbb{R}P^2$ and the Klein bottle, since the linking does not hold in the sphere, $\mathbb{R}P^2$ or the Klein bottle. In chapter five, we will handle the sphere and, in Appendix B, we see that Theorem 4.0.1 is false if $\Sigma = \left(S^2, g_{st}\right)$.

4.1 The linking

With *linking* we mean the existence of $\mathcal{L}\subset\mathcal{F}_N\Sigma$ and a (sequentially) compact topological space S, and a map $\gamma_0 : S\to\mathcal{F}_N\Sigma$ such that

$$-\infty < \inf_{p\in\mathcal{L}} H_\Gamma(p) \leq \sup_{p\in\mathcal{L}} H_\Gamma(p) < \infty \tag{Bound}$$

$$\gamma \simeq \gamma_0 \Rightarrow \gamma(S)\cap\mathcal{L}\neq\emptyset, \tag{Link}$$

where $\gamma \simeq \gamma_0$ means that γ is homotopic to γ_0.
Our linking approach is inspired by the linking in [14]. The rest of this section will be dedicated to the proof of the following theorem, which yields a linking in every situation needed in this thesis.

Theorem 4.1.1. *If Σ is a compact two-dimensional Riemannian manifold where the associated closed manifold is not homeomorphic to the sphere or $\mathbb{R}P^2$, then there exists $\mathcal{L} \subset \mathcal{F}_N\Sigma$, a sequentially compact topological space S, and $\gamma_0 : S \to \Sigma$, such that (Bound) and (Link) are satisfied.*

We start on a more abstract level. After we prove what we need in the abstract sense, we will show the existence of all maps needed in the concrete situation of any surface we consider.
For $i = 1, \dots, N$, let $\gamma^i : S^1 \to \Sigma$ be simple closed curves with

$$\gamma^i\left(S^1\right) \cap \gamma^j\left(S^1\right) = \emptyset \qquad \forall i \neq j.$$

Furthermore, let $P_i : \Sigma \to \gamma^i\left(S^1\right)$ be a retraction and $\xi_i \in \gamma^i\left(S^1\right)$, such that $P_i^{-1}(\xi_i) \subset int(\Sigma)$ is compact and

$$P_i^{-1}(\xi_i) \cap P_j^{-1}(\xi_j) = \emptyset.$$

We define $\xi := (\xi_1, \dots, \xi_N)$,

$$\gamma_0 : \left(S^1\right)^N \to \mathcal{F}_N\Sigma, \qquad \gamma_0(t_1, \dots, t_N) := \left(\gamma^1(t_1), \dots, \gamma^N(t_N)\right)$$
$$P : \mathcal{F}_N\Sigma \to \gamma_0\left(S^1\right)^N, \qquad P(p) := (P_1(p_1), \dots, P_N(p_N))$$

and

$$\mathcal{L} := P^{-1}(\xi) = \prod_{i=1}^{N} P_i^{-1}\left(\xi_i\right).$$

Because $P_i^{-1}(\xi_i) \subset int(\Sigma)$ are compact and they do not intersect, we see that $\mathcal{L} \subset \mathcal{F}_N\Sigma$ is compact. Thus, (Bound) is satisfied. In the next Lemma, we show that also (Link) is satisfied.

Lemma 4.1.2. *Let $\gamma = h(1, \cdot) \simeq h(0, \cdot) = \gamma_0$, then*

$$\gamma\left(\left(S^1\right)^N\right) \cap \mathcal{L} \neq \emptyset.$$

Proof: We will show that for every $\eta \in \gamma_0\left(\left(S^1\right)^N\right)$, there exists $t^\eta \in \left(S^1\right)^N$ such that $P\left(\gamma\left(t^\eta\right)\right) = \eta$. Thus, we also have $P\left(\gamma(t^\xi)\right) = \xi$, which means $\gamma(t^\xi) \in \mathcal{L} \cap \gamma\left(\left(S^1\right)^N\right)$. We use the degree of maps $f : M \to \tilde{M}$ between compact orientable N-dimensional manifolds. Note here, that

$$\deg(f) \neq 0 \Rightarrow f \text{ is onto,}$$

and we want to prove that $P \circ \gamma : \left(S^1\right)^N \to \gamma_0\left(\left(S^1\right)^N\right)$ is onto. Because $\gamma^i : S^1 \to \Sigma$ are simple closed curves, the map

$$\gamma_0 : \left(S^1\right)^N \to \gamma_0\left(\left(S^1\right)^N\right)$$

is a homeomorphism and, thus, $\deg(\gamma_0) \in \{\pm 1\}$ depends on the choosen orientations, on $\left(S^1\right)^N$ and $\gamma_0\left(\left(S^1\right)^N\right)$. Therefore, we get

$$\deg(P \circ \gamma) = \deg(P \circ h(1, \cdot)) = \deg\left(P \circ \gamma_0\right) = \deg\left(\gamma_0\right) \neq 0.$$

We conclude that $P \circ \gamma : \left(S^1\right)^N \to \gamma_0\left(\left(S^1\right)^N\right)$ is onto and, thus, prove our claim. □

Remark. In [14], it is allowed that $\gamma^i\left(S^1\right) \cap \gamma^j\left(S^1\right) \neq \emptyset$. This is possible, because the behavior of the function is known when the points come close together. Thus, an analytical degree is used to prove a similar linking scheme. Because the curves are allowed to intersect, there is only one simple closed curve $\gamma^1 : S^1 \to \Sigma$ to be defined. Then, the application of this abstract scheme in an explicit manifold is easier to achieve.

Now, we are going to define the simple closed curves and the retractions in an explicit closed surface Σ. First, we have to use the classification theorem of closed surfaces. We use it in such a form, that every closed surface is homeomorphic to a sphere, the projective plain, a torus T, the Klein bottle, or to the connected sum of a surface Σ' and a torus. The classification theorem is proven with modern methods of mathematics in [17, Thm. 6.3]. Thus, if Σ is not homeomorphic to S^2, $\mathbb{R}P^2$ or the Klein bottle, it is homeomorphic to the torus T, or to $T\#\Sigma'$, where $\#$ is the connected sum and Σ' is a closed surface. So, up to homeomorphism, there are three cases for which we need to define a retraction and the closed curves. The case $\Sigma = T\#\Sigma'$ is the most difficult one. In this case, the image of the curves will be inside the torus. The retraction will work the same way as the retraction of the torus, beside that Σ' needs to be handled and will be mapped to only one point.
We start with the case $\Sigma \cong T$. We represent the torus with the square $[0,1]^2$, where the parallel edges are identified, i.e. $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$. Considering that $S^1 \cong [0,1]/_{1\sim 0}$, we will define $\gamma_i : [0,1] \to [0,1]^2$. Figure 4.1 shows a sketch of the curves and retractions.

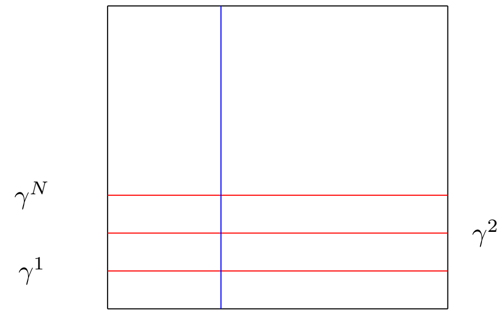

Figure 4.1: Torus

The red lines are the γ_i. The retraction P_i map every point on the blue line to the intersection with the red line, which belongs to the image of γ_i. We write this rigorously in the following. Let $N \in \mathbb{N}$ and set $t_i := \frac{1}{i+1}$ for $i \in \{1, \ldots, N\}$. Moreover, define

$$\gamma^i : [0,1] \to [0,1]^2, \qquad \gamma^i(s) := (s, t_i) .$$

We see that $\gamma^i([0,1]) \cap \gamma^j([0,1]) = \emptyset$ for $i \neq j$. Furthermore, we define

$$P_i : [0,1]^2 \to \gamma^i([0,1]), \qquad P_i(s,t) := (s, t_i) .$$

Then, P_i is a retraction onto $\gamma^i([0,1])$. For $s_0 \in [0,1]$, we have

$$P_i^{-1}(s_0, t_i) = \{s_0\} \times [0,1],$$

which is compact. Thus, if, $s_i \neq s_j \in [0,1]$ for $i \neq j$, we clearly have

$$P_i^{-1}(s_i, t_i) \cap P_j^{-1}(s_j, t_j) = \emptyset.$$

We choose $s_1, \ldots, s_N \in [0,1]$ pairwise distinct and define $\xi_i := (s_i, t_i)$. Then, we pass over to the quotient spaces and have defined everything we aspired.
Let $\Sigma \cong T \# \Sigma'$. We will define one simple closed curve $\gamma : [0,1] \to T \# \Sigma'$ with the retraction $P_\gamma : \Sigma \to \gamma([0,1])$ and then describe what has to be done to get all the curves γ^i. As we explained, we want that $\gamma([0,1]) \subset T$ and that $P_\gamma(\Sigma') \equiv v$. In the process of the connected sum, we cut out a circle in every surface and glue the arising boundaries together. Thus, if we want $P_\gamma : \Sigma \to \gamma([0,1])$ with $P_\gamma(\Sigma') \equiv v$ in a continuous way, we have to construct $P_\gamma : T \setminus B \to \gamma([0,1])$ with $P_\gamma(\partial(T \setminus B)) \equiv v$, where B is the ball we cut out. Furthermore, outside of a neighbourhood of the ball we cut out, we want that P_γ and γ behave as

they do in the case of the torus. In the following three images, we visualize our definitions. We, again, use the square $[0,1]^2$ as a representation of the torus. The cut out ball will also be a square, as this results in easier definitions. We will start with Figure 4.2.

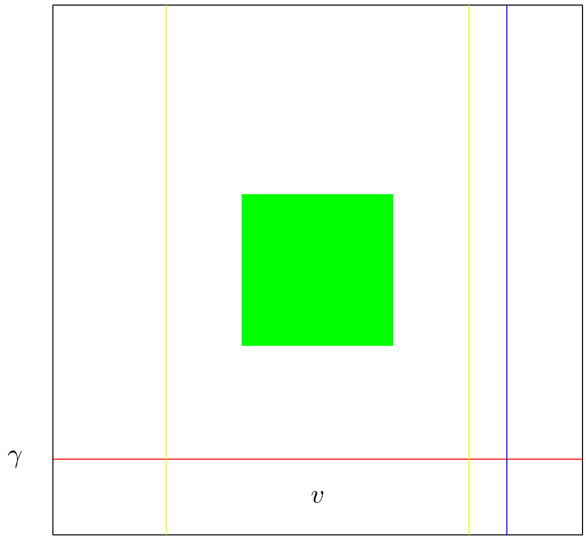

Figure 4.2: Vertical lines not near the square

The red line is our γ. The green square is the cut out square. The point v is located in the exact middle of the horizontal line γ. The yellow lines indicate the part where we want to do the same as in the case of the Torus. Thus, for all blue lines in the area, bounded by the yellow lines, where the square is not located, the retraction P_γ maps every point on a vertical line to the intersection with the red line.
Next, we handle the vertical lines in the area between the yellow lines with the green square inside. This will be explained in the following two pictures.

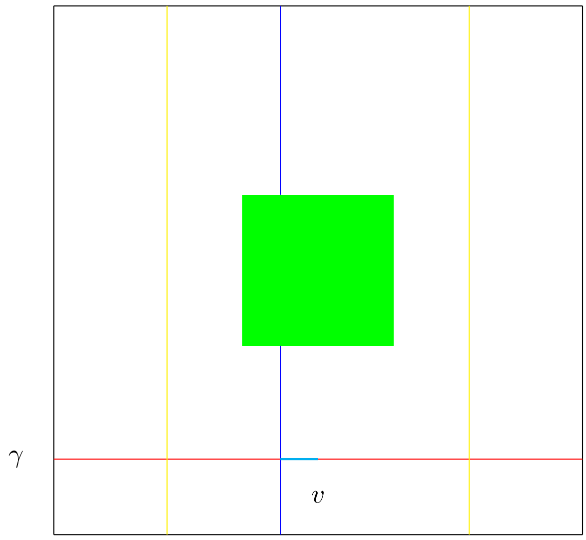

Figure 4.3: Vertical lines intersecting the square

In Figure 4.3, we see the case, where a vertical line intersects with the green square. The green square and the red line split the dark blue line into two lines. P_γ will map these two parts onto the light blue line, which is a path on $\gamma([0,1])$ from v to the intersection point with the dark blue line. This will be done in such a way that the points of the green square are mapped to v and the intersection of the red and dark blue line will be mapped to itself. The remaining vertical lines are the ones between the green square and the yellow lines.

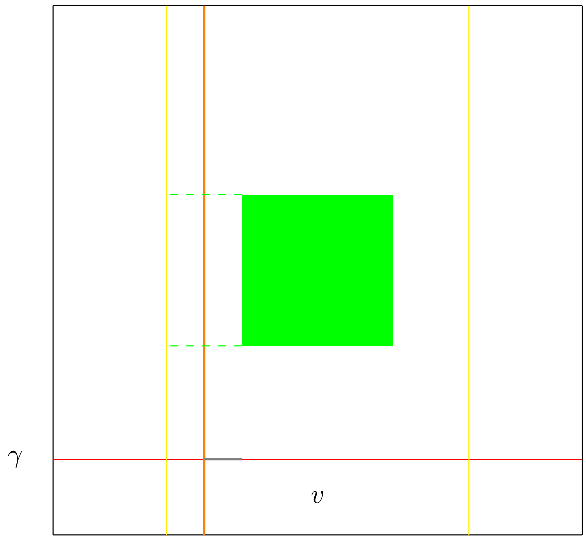

Figure 4.4: Vertical lines near the square not intersecting it

In Figure 4.4, every point of an orange line between the dashed green lines will be mapped to the right endpoint of the gray line. The rest of the orange line will be parametrized onto the gray line, like in the previous picture. Thus, the intersection point of the orange line and the red line will stay fixed and the points exactly at the dashed green line will be mapped to the right endpoint of the gray line. Furthermore, the length of the gray line will depend on how far away the orange line is from the yellow line and the green square. If the orange line intersects the green square, the most right point of the gray line will be v. If the orange line is one of the yellow lines, the length of the gray line will be zero, thus, everything will be mapped to the intersection of the orange line and the yellow line.

Now, we will define this rigorously. We set

$$B := \left(\frac{3}{8}, \frac{5}{8}\right) \times \left(\frac{3}{8}, \frac{5}{8}\right) \subset [0,1]^2.$$

This is the square we cut out. We define

$$\gamma : [0,1] \to [0,1]^2 \setminus B, \qquad t \mapsto (t,0).$$

Next, we define

$$\alpha : [0,1]\times[0,1] \to \mathbb{R}, \qquad \alpha(t,\lambda) = \alpha_t(\lambda) := \begin{cases} t, & t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right] \\ t\lambda + (1-\lambda)\left(2t - \frac{1}{4}\right), & t \in \left[\frac{1}{4}, \frac{3}{8}\right] \\ t\lambda + (1-\lambda)\left(2t - \frac{3}{4}\right), & t \in \left[\frac{5}{8}, \frac{3}{4}\right] \\ \frac{1}{2} + \left(t - \frac{1}{2}\right)\lambda, & t \in \left[\frac{3}{8}, \frac{5}{8}\right]. \end{cases}$$

With the following calculations, we see that α is continuous:

$$\begin{aligned} \frac{1}{4}\lambda + (1-\lambda)\left(2 \cdot \frac{1}{4} - \frac{1}{4}\right) &= \frac{1}{4}, \\ \frac{3}{8}\lambda + (1-\lambda)\left(2 \cdot \frac{3}{8} - \frac{1}{4}\right) &= \frac{1}{2} + \left(\frac{3}{8} - \frac{1}{2}\right)\lambda, \\ \frac{5}{8}\lambda + (1-\lambda)\left(2 \cdot \frac{5}{8} - \frac{3}{4}\right) &= \frac{1}{2} + \left(\frac{5}{8} - \frac{1}{2}\right)\lambda, \\ \frac{3}{4}\lambda + (1-\lambda)\left(2 \cdot \frac{3}{4} - \frac{3}{4}\right) &= \frac{3}{4}. \end{aligned}$$

The map α handles the scaling of the horizontal lines. The yellow lines are the sets $\{(s,t) : t \in [0,1]\}$ with $s = \frac{1}{4}$ or $s = \frac{3}{4}$. The point v is given by $v = \left(\frac{1}{2}, 0\right)$. We still need to parametrize the vertical lines, therefore, we use the following two maps

$$\begin{aligned} \beta_1 &: [0,1] \to \left[\frac{5}{8}, 1\right], & \beta_1(\lambda) &:= \frac{5}{8} + \frac{3}{8}\lambda, \\ \beta_2 &: [0,1] \to \left[0, \frac{3}{8}\right], & \beta_2(\lambda) &:= \frac{3}{8} - \frac{3}{8}\lambda. \end{aligned}$$

Note that β_1 and β_2 are homeomorphisms. We define

$$P_\gamma : T \to T, \quad P_\gamma(t,s) := \begin{cases} \left(\alpha_t\left(\beta_1^{-1}(s)\right), 0\right), & \text{if } s \in \left[\frac{5}{8}, 1\right] \\ \left(\alpha_t\left(\beta_2^{-1}(s)\right), 0\right), & \text{if } s \in \left[0, \frac{3}{8}\right] \\ \left(\alpha_t(0), 0\right), & \text{if } s \in \left[\frac{3}{8}, \frac{5}{8}\right]. \end{cases}$$

We have $P_\gamma\left([0,1]^2\right) \subset \gamma\left([0,1]\right)$. Furthermore, if $(t,s) \in \overline{B}$, we have

$$P_\gamma(t,s) = (\alpha_t(0), 0) = \left(\frac{1}{2} + \left(t - \frac{1}{2}\right) \cdot 0, 0\right) = \left(\frac{1}{2}, 0\right) = v.$$

Thus, we only have to show that P_γ induces a continuous map from $T \to T$ and that $P_\gamma(t,0) = (t,0)$. First, let $s \in [0,1]$, then $(0,s) = (1,s) \in T$. We calculate

$$P_\gamma(0,s) = (\alpha_0(\lambda), 0) = (0,0) \sim (1,0) = (\alpha_1(\lambda), 0) = P_\gamma(1,s).$$

This shows $P_\gamma(0,s) = P_\gamma(1,s) \in T$. Furthermore, with $\beta_1^{-1}(1) = 1$ and $\beta_2^{-1}(0) = 1$, we see that

$$P_\gamma(t,0) = \left(\alpha_t\left(\beta_2^{-1}(0)\right),0\right) = (\alpha_t(1),0) = \left(\alpha_t\left(\beta_1^{-1}(1)\right),0\right) = P_\gamma(t,1).$$

Therefore, we have, $P_\gamma(t,0) = P_\gamma(t,1) \in T$, which implies that $P_\gamma : T \to T$ is well defined. Because of

$$P_\gamma(t,0) = (\alpha_t(1),0) = (t,0),$$

we also have that $\gamma([0,1])$ is kept fixed. It only remains to show that P_γ is continuous. Thus, we show that for $s = \frac{3}{8}$ and $s = \frac{5}{8}$ the possible definitions of P_γ coincide. So, let $s = \frac{3}{8}$, then we have $\beta_2^{-1}\left(\frac{3}{8}\right) = 0$ and, therefore,

$$P_\gamma\left(t,\frac{3}{8}\right) = \left(\alpha_t\left(\beta_2^{-1}\left(\frac{3}{8}\right)\right),0\right) = (\alpha_t(0),0).$$

Furthermore, we have $\beta_1^{-1}\left(\frac{5}{8}\right) = 0$ and, as a result, the definitions of P_γ also coincide for $s = \frac{5}{8}$. By setting

$$P_\gamma : T\#\Sigma' \to \gamma([0,1]), \qquad \begin{cases} P_\gamma(p), & p \in T \\ v, & p \notin T, \end{cases}$$

we have our retraction.
Now, we define γ^i and P_i for $i \in \{1,\ldots,N\}$. We define

$$\gamma^i : [0,1] \to [0,1]^2 \setminus B, \qquad t \mapsto \left(t, \frac{1}{3+i}\right),$$

and $v_i = \left(\frac{1}{2}, \frac{1}{3+i}\right)$. For the retractions P_i, we again use the same horizontal scaling α. Because we changed the height, we have to change the vertical scalings. So, one defines appropriate scalings β_1^i and β_2^i and we can define

$$P_i : [0,1]^2 \to [0,1]^2, \quad P_i(t,s) := \begin{cases} \left(\alpha_t\left(\left(\beta_1^i\right)^{-1}(s)\right), \frac{1}{3+i}\right), & s \in \beta_1^i([0,1]) \\ \left(\alpha_t\left(\left(\beta_2^i\right)^{-1}(s)\right), \frac{1}{3+i}\right), & s \in \beta_2^i([0,1]) \\ \left(\alpha_t(0), \frac{1}{3+i}\right), & s \in \left[\frac{3}{8}, \frac{5}{8}\right]. \end{cases}$$

Then, we can choose $\xi_i := \left(\frac{1}{4+i}, \frac{1}{3+i}\right)$ with which we have

$$P_i^{-1}(\xi_i) = [0,1] \times \left\{\frac{1}{3+i}\right\}.$$

Passing to the quotient spaces, we end up with $\gamma^i : S^1 \to \Sigma$ a retraction P_i onto γ_i with
$$P_i^{-1}(\xi_i) \cap P_j^{-1}(\xi_j) = \emptyset$$
and $\gamma^i\left(S^1\right) \cap \gamma^j\left(S^1\right) = \emptyset$, which are all the properties we required. Therefore, we established the aspired linking, for all manifolds without boundary.

When Σ is a manifold with boundary, then, according to [25], there exists $\tilde{\Sigma}$, $k \in \mathbb{N}$ and $\tilde{\Sigma} \supset D_i \cong U_1(0)$ such that $\overline{D}_i \cap \overline{D}_j = \emptyset$ with
$$\Sigma = \tilde{\Sigma} \setminus \bigcup_{i=1}^{k} D_i.$$

When $\tilde{\Sigma}$ has genus greater or equal than 2, we assume $\Sigma = T\#\Sigma'$ i.e. all boundary components lie in Σ' and we do not have to change anything for our retractions. Thus, the only remaining case is that $\tilde{\Sigma}$ is homeomorphic to the torus T. We use the retraction already defined on the torus. As a reminder on the torus, we defined $t_i = \frac{1}{i+1}$,
$$\gamma_T^i : [0,1] \to [0,1]^2, \qquad \gamma_T^i(s) := (s, t_i),$$
$$P_i^T : [0,1]^2 \to [0,1]^2, \qquad P_i^T(s,t) := (s, t_i).$$
The curves on Σ are just
$$\gamma^i : [0,1] \to \Sigma, \quad \gamma^i(t) := (s, t_i)$$
and the retractions are
$$P_i : [0,1]^2 \to [0,1]^2, \quad P_i(s,t) := (s, t_i).$$
We have to be careful with the choice of ξ_i, because it has to satisfy $\left(P_i^T\right)^{-1}(\xi_i) \subset int\,(\Sigma)$. We will do this , again with drawing a picture and choosing some explicit Σ.

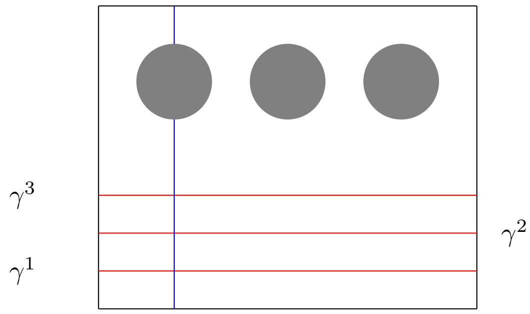

Figure 4.5: Torus with holes

So let $k \in \mathbb{N}$ and for $i = 1, \ldots, k$ let

$$\varepsilon_k := \frac{1}{2^{k+3}} < \frac{1}{2k+4} < \frac{1}{4},\ x_k^i := \left(\frac{i}{k+2}, \frac{3}{4}\right).$$

Then, we define

$$D_k^i := U_{\varepsilon_k}(x_k^i) = \left\{y \in [0,1]^2 : \left|y - x_k^i\right| < \varepsilon_k\right\}.$$

Note that because of $t_i \in \left[0, \frac{1}{2}\right]$, we have $\gamma^i(S^1) \subset int\,(\Sigma)$. Furthermore, we see that $\overline{D}_k^i \cap \overline{D}_k^j = \emptyset$ if $i \neq j$. Furthermore, we see that $\overline{D}_k^i \subset (0,1) \times \left(\frac{1}{2}, 1\right)$. So, nothing strange happens, when we use the quotient map from $[0,1]^2 \to T$. This means, $\Sigma_k := T \setminus \bigcup_{i=1}^k D_k^i$ is one representation of the diffeomorphism class where k discs are cut out. The last step is to define ξ. Thus, we let $s_i \in \left(0, \frac{1}{k+2} - \varepsilon_k\right)$ with $s_i \neq s_j$ for every $i \neq j$ and again define $\xi_i := (s_i, t_i)$. With this, we finished all cases and, thus, established every linking.

4.2 The methods

In this section, we present the methods we will use. Beside the linking, which we just established, we also will need some sort of compactness. For the Theorems 4.0.1 and 4.0.2, the compactness will be the Palais-Smale-condition. For Theorem 4.0.3, we generalize the method of [6] where the Palais-Smale-condition is replaced by another form of compactness.

4.2.1 Using the Palais-Smale-condition

Lemma 4.2.1. *Let*

$$\phi : \bigcup_{q \in \mathcal{F}_N} \left(t^-(q), t^+(q)\right) \times \{q\} \to \mathcal{F}_N \Sigma$$

be the gradient flow of H_Γ. Furthermore, let $\mathcal{L} \subset \mathcal{F}_N\Sigma$ and $\gamma_0 : S \to \mathcal{F}_N\Sigma$ satisfy (Link) and (Bound). Then, for every $\gamma \simeq \gamma_0$, there exists $p \in \gamma(S)$ such that

$$\lim_{t\to T^+(p)} H_\Gamma(\phi(t,p)) < \infty.$$

Proof: This is standard and we refer to [22, Lem 3.5]. Because the proof is short, we will write it down, but will not elaborate most details. Assume the claim is wrong. Then, there exists $\gamma \simeq \gamma_0$ such that for every $p \in \gamma(S)$ we can define $T(p) \in [0, t^+(p))$ by

$$T(p) := \inf\{s \geq 0 : H_\Gamma\left(\phi(s,p)\right) = \underbrace{\sup_{q\in\mathcal{L}} H_\Gamma(q) + 1}_{=:\sigma}\}.$$

Note that $\sigma < \infty$, because of (Bound). Now, for every $p \in \gamma(S)$ the map $t \mapsto H_\Gamma(\phi(t,p))$ is strictly increasing and thus the map

$$\gamma(S) \ni p \mapsto T(p) \in \mathbb{R}$$

is continuous. Then,

$$D : [0,1] \times S \to \mathcal{F}_N\Sigma, \qquad D(t,s) := \phi\left(tT(\gamma(s)), \gamma(s)\right)$$

defines a homotopy and, therefore, $\gamma_1 := D_1 = D(1,\cdot) \simeq D(0,\cdot) = \gamma \simeq \gamma_0$. With (Link), there exists $s \in S$ such that $D_1(s) \in \mathcal{L}$, which means

$$\sigma = \sup_{q\in\mathcal{L}} H_\Gamma(q) \geq H_\Gamma\left(D_1(s)\right) = H_\Gamma\left(\phi(T(\gamma(s)), \gamma(s))\right) = \sigma + 1.$$

$\square$

By Lemma 4.2.1 and the linking, we achieved the existence of a flow line along which H_Γ is bounded. This will yield a critical point, when H_Γ satisfies the Palais-Smale-condition. Note that we also look into the behavior of H_Γ near $\partial\mathcal{F}_N\Sigma$ when we prove the Palais-Smale-condition later.

4.2.2 The other method

We will write this method in an abstract setting. So let M be a Riemannian manifold, let $H, \Phi : M \to \mathbb{R}$ be $\mathcal{C}^\infty$ maps, $\mathcal{L} \subset M$ and $\gamma_0 : S \to M$ satisfy (Bound) with respect to H and (Link).

Theorem 4.2.2. *Let $a < \inf_{\mathcal{L}} H \leq \sup_{\mathcal{L}} H < b$ and assume there exists a regular value $M_0 > 0$ of Φ such that*

$$\Phi(p) \geq M_0, a \leq H(p) \leq b, \nabla H(p) = \lambda\nabla\Phi(p) \Rightarrow \lambda < 0$$

and such that the set

$$\emptyset \neq D := \{p \in M : \Phi(p) \leq M_0\} \subset int(M)$$

is compact, as well as $M_0 > \sup_S \Phi \circ \gamma_0$.
Then, H *has a critical point* $\bar{p}$ *with* $H(\bar{p}) \in [a, b]$.

For the proof of this theorem, we define a vector field with certain properties. Therefore, we define the set

$$D_a^b := \{p \in D : \, a \leq H(p) \leq b\} = \{p \in M : \, \Phi(p) \leq M_0, \, a \leq H(p) \leq b\} \, .$$

Lemma 4.2.3. *Let* $M_0 > 0$ *be a regular value of* Φ *such that* $D = \{p \in M : \Phi(p) \leq M_0\} \subset int(M)$ *is compact and for* $a < b \in \mathbb{R}$ *holds*

$$\Phi(p) \geq M_0, a \leq H(p) \leq b, \nabla H(p) = \lambda \nabla \Phi(p) \Rightarrow \lambda < 0.$$

If H *has no critical points in* $\emptyset \neq D_a^b$*, then there exists a locally Lipschitz continuous vector field* $V : M \to TM$ *(in a sense that it is local Lipschitz continuous in every chart) with the following properties:*

$$\langle \nabla \Phi(p), V(p) \rangle \leq 0 \qquad \forall \, p \in D_a^b \cap \partial D \tag{4.2.1}$$
$$\langle \nabla H(p), V(p) \rangle > 0 \qquad \forall \, p \in D_a^b \tag{4.2.2}$$
$$\langle \nabla H(p), V(p) \rangle \geq 0 \qquad \forall \, p \in M. \tag{4.2.3}$$

Furthermore, we have $V \equiv 0$ *outside of a compact neighbourhood of* D_a^b.

Proof: We first define the vector field on the set $D_a^b \cap \partial D$ and will extend this vector field afterwards. So, we define $V_0 : D_a^b \cap \partial D \to TM$ with

$$V_0(p) := \begin{cases} \nabla H(p) - \frac{\langle \nabla H(p), \nabla \Phi(p) \rangle}{|\nabla \Phi(p)|^2} \nabla \Phi(p), & \text{if } \langle \nabla H(p), \nabla \Phi(p) \rangle \geq 0 \\ \nabla H(p), & \text{if } \langle \nabla H(p), \nabla \Phi(p) \rangle < 0. \end{cases}$$

First, note that V_0 is Lipschitz continuous in any chart because any function involved is $\mathcal{C}^1$. Also, (4.2.1) and (4.2.2) hold for V_0 and every $p \in D_a^b \cap \partial D$. We will show this in the following, beginning with (4.2.1). If $\langle \nabla H(p), \nabla \Phi(p) \rangle < 0$, then this is obvious. So, we let $\langle \nabla H(p), \nabla \Phi(p) \rangle \geq 0$. Then, we calculate

$$\begin{aligned}\langle \nabla \Phi(p), V_0(p) \rangle &= \left\langle \nabla \Phi(p), \nabla H(p) - \frac{\langle \nabla H(p), \nabla \Phi(p) \rangle}{|\nabla \Phi(p)|^2} \nabla \Phi(p) \right\rangle \\ &= \langle \nabla \Phi(p), \nabla H(p) \rangle - \langle \nabla H(p), \nabla \Phi(p) \rangle \frac{\langle \nabla \Phi(p), \nabla \Phi(p) \rangle}{|\nabla \Phi(p)|^2} \\ &= 0.\end{aligned}$$

Thus, we have

$$\langle \nabla \Phi(p), V_0(p) \rangle = \begin{cases} 0, & \text{if } \langle \nabla H(p), \nabla \Phi(p) \rangle \geq 0 \\ \langle \nabla H(p), \nabla \Phi(p) \rangle, & \text{if } \langle \nabla H(p), \nabla \Phi(p) \rangle < 0 \end{cases}$$

and can conclude (4.2.1).
To prove (4.2.2), we have to prove that $V_0(p) \neq 0$ for all $p \in D_a^b \cap \partial D$. Assume the opposite. Because H has no critical points in D_a^b, we conclude that $V_0(p) = 0$ implies $\langle \nabla H(p), \nabla \Phi(p) \rangle \geq 0$. But then, we have

$$0 = V_0(p) = \nabla H(p) - \frac{\langle \nabla H(p), \nabla \Phi(p) \rangle}{|\nabla \Phi(p)|^2} \nabla \Phi(p).$$

With our assumptions, we conclude $\langle \nabla H(p), \nabla \Phi(p) \rangle < 0$ and, thus, a contradiction. As a closed subset of a compact set, $D_a^b \cap \partial D$ is compact and, therefore, we have some $m_1 > 0$ such that

$$|V_0(p)| \geq m_1 > 0 \qquad \forall p \in D_a^b \cap \partial D$$

as well as

$$|\nabla H(p)| \geq m_1 > 0 \qquad \forall p \in D_a^b \cap \partial D.$$

If $\langle \nabla H(p), \nabla \Phi(p) \rangle < 0$, we have

$$\langle \nabla H(p), V_0(p) \rangle = |\nabla H(p)|^2 \geq m_1^2 > 0.$$

If $\langle \nabla H(p), \nabla \Phi(p) \rangle \geq 0$, we calculate

$$\begin{aligned}\langle \nabla H(p), V_0(p) \rangle &= \langle \nabla H(p), V_0(p) \rangle + 0 \\ &= \langle \nabla H(p), V_0(p) \rangle - \left\langle \frac{\langle \nabla H(p), \nabla \Phi(p) \rangle}{|\nabla \Phi(p)|^2} \nabla \Phi(p), V_0(p) \right\rangle \\ &= \langle V_0(p), V_0(p) \rangle = |V_0(p)|^2 \geq m_1^2 > 0.\end{aligned}$$

Thus, we conclude (4.2.2).

The next step is to extend V_0 to V_1, which will be defined on D_a^b. Because M_0 is a regular value of Φ, the set

$$\partial D = \Phi^{-1}(M_0)$$

is a compact submanifold of M. Thus, we have $\delta_0 > 0$ such that every maximal geodesic γ of M with $\gamma(0) = p \in \partial D$ will at least have length $2\delta_0$. For $p \in M$, we define

$$\gamma_p : (-\delta_0, \delta_0) \to M$$

as the geodesic, with initial value condition

$$\begin{cases} \gamma' = \nu_p := -\frac{\nabla\Phi(p)}{|\nabla\Phi(p)|} \\ \gamma(0) = p. \end{cases}$$

Note here that ν_p is the inner normal vector of $p \in \partial D$. Furthermore, the map

$$\chi^{-1} : \partial D \times (-\delta_0, \delta_0) \to M, \qquad (p, t) \mapsto \gamma_p(t)$$

is $\mathcal{C}^\infty$, because ∂D is compact. We define

$$\mathcal{O} := \chi^{-1}\left(\partial D \times (-\delta_0, \delta_0)\right).$$

Without loss of generality, we assume that $\mathcal{O}$ is a tubular neighbourhood of ∂D (if not we choose a smaller $\delta_0 > 0$). Then, $\chi^{-1} : \partial D \times (-\delta_0, \delta_0) \to \mathcal{O}$ is a $\mathcal{C}^\infty$-diffeomorphism and we define $\chi := (\chi_1, \chi_2)$ as its inverse. We have for every $p \in \mathcal{O}$ that

$$p = \gamma_{\chi_1(p)}(\chi_2(p)) \text{ and } d(p, \chi_1(p)) = \chi_2(p).$$

The second equation holds, because γ_p is parametrized by arc length, as $|\nu_p| = 1$. We now define $X_p : (-\delta_0, \delta_0) \to TM$ with $X_p(t) \in T_{\gamma_p(t)}M$, as the parallel transport of $V_0(p)$ along γ_p. So, X_p is uniquely determined by the initial value problem

$$\begin{cases} \nabla_{\gamma_p'(t)} u(t) = 0 \\ u(0) = V_0(p). \end{cases}$$

Before we extend V_0, we look at the map

$$\mathcal{O} \ni p \mapsto X_{\chi_1(p)}(\chi_2(p)) \in T_pM$$

in any local chart. In a local chart, the initial value problem for X_p translates to an initial value problem of the form

$$\begin{cases} u'(t) = A_x(t)u(t) \\ u(0) = g(x), \end{cases}$$

where g is a Lipschitz continuous function, $A_x(t)$ is a matrix, and $(x,t) \mapsto A_x(t)$ is $\mathcal{C}^\infty$. If now $U_x(t)$ is a fundamental system to $u'(t) = A_x(t)u(t)$, the map $(x,t) \mapsto U_x(t)$ is also $\mathcal{C}^\infty$. Then, we have

$$X_x(t) = U_x(t)\left(U_x(0)^{-1}g(x)\right)$$

and conclude that $(x,t) \mapsto X_x(t)$ is Lipschitz continuous. Thus,

$$\mathcal{O} \ni p \mapsto X_{\chi_1(p)}(\chi_2(p)) \in T_pM$$

is Lipschitz continuous in a local chart. For $0 < \delta < \delta_0$, we define $V_\delta : D_a^b \to TM$ in the following way

$$V_\delta(p) := \begin{cases} \frac{\delta - \chi_2(p)}{\delta} X_{\chi_1(p)}(\chi_2(p)) + \frac{\chi_2(p)}{\delta}\nabla H(p), & p \in D_a^b \cap \mathcal{O},\ \chi_2(p) \le \delta \\ \nabla H(p), & p \in D_a^b \setminus \mathcal{O} \text{ or } p \in D_a^b \cap \mathcal{O},\ \chi_2(p) > \delta. \end{cases}$$

First, note that this is well defined, because $X_{\chi_1(p)}(\chi_2(p)) \in T_pM$. If $p \in D_a^b \cap \mathcal{O}$, we have $\chi_2(p) \ge 0$. Furthermore, if $p \in \partial D$, we have $\chi(p) = (p, 0)$ and, thus,

$$V_\delta(p) = X_p(0) = V_0(p).$$

We immediately conclude that (4.2.1) holds for V_δ. Furthermore, the map

$$\mathcal{O} \ni p \mapsto \left\langle \nabla H(p), X_{\chi_1(p)}(\chi_2(p))\right\rangle$$

is continuous and for $p \in \partial D$ we have

$$\left\langle \nabla H(p), X_{\chi_1(p)}(\chi_2(p))\right\rangle = \langle \nabla H(p), V_0(p)\rangle \ge m_1^2 > 0.$$

By continuity and $d(p, \chi_1(p)) = \chi_2(p)$, we choose a fixed $\delta_1 > 0$ such that

$$\left\langle \nabla H(p), X_{\chi_1(p)}(\chi_2(p))\right\rangle > 0 \qquad \forall p \in D_a^b \cap \mathcal{O},\ \chi_2(p) \le \delta_1.$$

We define $V_1 := V_{\delta_1}$. Then, also (4.2.2) holds for V_1, because either

$$\langle \nabla H(p), V_1(p)\rangle = \langle \nabla H(p), \nabla H(p)\rangle > 0$$

or

$$\begin{aligned}\langle \nabla H(p), V_1(p)\rangle &= \left\langle \nabla H(p), \frac{\delta - \chi_2(p)}{\delta} X_{\chi_1(p)}\chi_2(p) + \frac{\chi_2(p)}{\delta}\nabla H(p)\right\rangle \\ &= \frac{\delta_1 - \chi_2(p)}{\delta_1}\left\langle \nabla H(p), X_{\chi_1(p)}(\chi_2(p))\right\rangle + \frac{\chi_2(p)}{\delta_1}\left|\nabla H(p)\right|^2 > 0.\end{aligned}$$

So, (4.2.1) and (4.2.2) hold for V_1. Finally, we define $V : M \to TM$ with

$$V(p) := \begin{cases} V_1(p), & p \in D_a^b \\ \frac{\delta_1 + \chi_2(p)}{\delta_1} X_{\chi_1(p)}(\chi_2(p)), & p \in \mathcal{O},\ 0 > \chi_2(p) \geq -\delta_1 \\ 0, & p \in M \setminus \mathcal{O} \text{ or } p \in \mathcal{O},\ \chi_2(p) < -\delta_1. \end{cases}$$

The last remaining property we have to check is that V is locally Lipschitz continuous. However, this follows, because in a chart V is built from Lipschitz continuous functions, which coincide on the set where we change the definition.

□

We now prove Theorem 4.2.2 . Assume H has no critical point in D_a^b. Let $V : M \to TM$ be the vector field constructed in Lemma 4.2.3 . Because V is locally Lipschitz and vanishes outside of a compact set, there exists a global flow $\phi : M \times \mathbb{R} \to M$ associated with the vector field V. Because of (4.2.1), the flow satisfies

$$p \in D_a^b,\ a \leq H(\phi(p,t)) \leq b\,\forall t \in [0,T] \Rightarrow \phi(p,t) \in D_a^b \quad \forall t \in [0,T].$$

Furthermore, since (Link) holds, for every $n \in \mathbb{N}$, we have $\xi_n \in S$ such that $\phi(\gamma_0(\xi_n), n) \in \mathcal{L}$. Because of (Bound), we have $a < H(\phi(\gamma_0(\xi_n), n)) < b$. Because S is sequentially compact, there exists $\xi \in S$ such that $\xi_n \to \xi$ along a subsequence. As a consequence, we have $\gamma_0(\xi) \in D_a^b$ and $a \leq H(\phi(\gamma_0(\xi)), t) \leq b$ for all $t \geq 0$. This is the contradiction we want to derive: Because of (4.2.2), we have $\langle \nabla H(p), V(p)\rangle \geq m_1 > 0$ for every $p \in D_a^b$. Because of $\phi(\gamma_0(\xi), t) \in D_a^b$, we conclude

$$\begin{aligned} O(1) &= H(\phi(\gamma_0(\xi), t)) - H(\phi(\gamma_0(\xi), 0)) \\ &= \int_0^t \langle \nabla H(\phi(\gamma_0(\xi), s)), \phi'(\gamma_0(\xi), s)\rangle \, ds \\ &= \int_0^t \langle \nabla H(\phi(\gamma_0(\xi), s)), V(\phi(\gamma_0(\xi), s))\rangle \, ds \\ &\geq \int_0^t m_1 \, ds = t \cdot m_1 \overset{t\to\infty}{\to} \infty. \end{aligned}$$

□

4.3 Achieving the compactness

In this section, we will show that our compactness conditions hold and, thus, we will prove the Theorems 4.0.1, 4.0.2 and 4.0.3. We will split this into three parts, whereas each part will handle one theorem.

4.3.1 Theorem 4.0.1

Lemma 4.3.1. *Let Σ be closed and (4.0.1) hold. There exists $\mu > 0$ such that $|\nabla H_\Gamma(p)| > 1$ for every*

$$p \in \Delta_N^\mu \Sigma \cap \mathcal{F}_N \Sigma = \{p \in \mathcal{F}_N \Sigma | \exists i \neq j : d_g(p_i, p_j) \leq \mu\}.$$

In particular, H_Γ satisfies the Palais-Smale-condition.

Before we start the proof, note that this lemma is similar to [22, Prop. 4.1]. We will use the calculations from this paper to prove our lemma. We formulate everything similarly to it to ease the comparison.

Proof: In [22], the author looks at the behavior of H_Γ for some clusters $C \subset \{1, \dots, N\}$ with $|C| \geq 2$, i.e. when $d_g(p_i, p_j) < \mu$ for every $i, j \in C$. Without loss of generality, we assume $\mu > 0$ to be small enough, such that if $d_g(p_i, p_j) < \mu$, there exists a locally conformal flat chart $\varphi_C : U_C \to V_C$ such that $p_i \in U_C$ if $i \in C$. Now, we look at H_Γ in a chart $\varphi : U \to V$ where $\varphi_i = \varphi_C$, if $i \in C$ and φ_j is some arbitrary chart if $j \notin C$. With $x = \varphi(p) = (\varphi_1(p_1), \dots \varphi_N(p_N))$, we then have the following decomposition of H_Γ :

$$\left(H_\Gamma\right)_\varphi(x) = -\frac{1}{2\pi} J_C(x) + K(x),$$

where

$$J_C(x) := \sum_{\substack{i,j \in C \\ i \neq j}} \Gamma_i \Gamma_j \ln |x_i - x_j|$$

and

$$K_C(x) := -\frac{1}{2\pi} \sum_{i \in C \not\ni j} \Gamma_i \Gamma_j G_\varphi\left(x_i, x_j\right) - \frac{1}{2\pi} \sum_{\substack{i \neq j \\ i,j \notin C}} \Gamma_i \Gamma_j G_\varphi(x_i, x_j) + \tilde{\Psi}(x).$$

Note that $\tilde{\Psi}$ is smooth and bounded. Next, we define $|x|_C := |\pi_C x|$, where

$$\pi_C : \left(\mathbb{R}^2\right)^N \to \{x = (x_1, \dots, x_N) : x_i = 0 \text{ for all } i \notin C\}$$

is the orthogonal projection. Now, following the calculations of [22, Lem. 4.2; Lem. 4.3], we derive

$$\begin{aligned}|\nabla H_\Gamma(x)| &\geq |\nabla H_\Gamma(x)|_C \geq \frac{1}{2\pi}|\nabla J_C(x)| - |\nabla K_C(x)|_C \\ &\geq \frac{C_\Gamma}{2\pi}\left(\sum_{i\in C}|x_i - x_C|^2\right)^{-\frac{1}{2}} - \tilde{C} \geq \frac{C_\Gamma}{4\pi}\left(\sum_{i\in C}|x_i - x_C|^2\right)^{-\frac{1}{2}},\end{aligned}$$

where x_C is some cluster point, i.e. there exists $\tilde{\mu} \in (0,\mu)$ such that $|x_i - x_C| < \tilde{\mu}$ for every $i \in C$, $\tilde{C} > 0$ is just a constant and C_Γ is defined in [22] by

$$C_\Gamma := \min_{\substack{\mathcal{P}\,partition\,of\,\{1,\dots,N\}\\ \mathcal{C}(\mathcal{P})\neq\emptyset}} \min_{C\in\mathcal{C}(\mathcal{P})} \left|\sum_{\substack{i,j\in C\\ i\neq j}}\Gamma_i\Gamma_j\right|,$$

where $\mathcal{C}(\mathcal{P}) = \{I \in \mathcal{P} : |I| \geq 2\}$. Because of (4.0.1), we have $C_\Gamma > 0$. Now, we finish the lemma with a contradiction. Assume there exists $0 < \mu_n \to 0$, $p^n \in \Delta_N^{\mu_n}\Sigma \cap \mathcal{F}_N\Sigma$ such that $\nabla H_\Gamma(p^n) \leq 1$. Without loss of generality, we assume $\mu_n \leq 1$, which implies

$$p^n \in \Delta_N^{\mu_n}\Sigma \cap \mathcal{F}_N\Sigma \subset \Delta_N^1.$$

So there exists a $p \in \Delta_N^1$ and a convergent subsequence (again denotet by p^n) with $p^n \to p$. Because $p^n \in \Delta_N^{\mu_n}\Sigma$, there exists $i \neq j$ such that $p_0 := p_i = p_j$. Thus, there exists a cluster

$$C := \{l \in \{1,\dots,N\} : p_l = p_0\}.$$

Now, going into a conformal flat chart φ around that cluster and defining $x_i := \varphi_i(p_i)$, $x_0 := \varphi_C(p_0)$, we then deduce

$$1 \geq \left|\nabla\left(H_\Gamma\right)_\varphi(x^n)\right| \geq \frac{C_\Gamma}{4\pi}\left(\sum_{i\in C}|x_i^n - x_0|^2\right)^{-\frac{1}{2}}.$$

We claim that, $d_g(\varphi_C^{-1}(x_i^n), \varphi_C^{-1}(x_0)) \leq \mu_n$, implies $|x_i^n - x_0| \leq C_1\mu_n := \tilde{\mu}_n$, where $C_1 > 0$ is a constant. In order to see this let $\tilde{g} = e^{2u}g$ be a to g conformal metric and let $\gamma = \gamma_{p,q} : [0,1] \to \Sigma$ be a path with $\gamma(0) = p$ and $\gamma(1) = q$. We remind, that the length of γ is defined by

$$L_{\bar{g}}(\gamma) := \int_0^1 \sqrt{\bar{g}\left(\gamma'(t),\gamma'(t)\right)}\,dt \qquad \bar{g} \in \{g,\tilde{g}\}.$$

Now, with $e^u \leq C_1$, we see

$$L_{\tilde{g}}(\gamma) \leq C_1 L_g(\gamma).$$

This implies

$$d_{\tilde{g}}(p,q) = \inf_{\gamma_{p,q}} L_{\tilde{g}}(\gamma_{p,q}) \leq C_1 \inf_{\gamma_{p,q}} L_g(\gamma_{p,q}) = C_1 d_g(p,q).$$

Now, with $|x_i^n - x_0| = d_{\tilde{g}}(\varphi_C^{-1}(x_i^n), \varphi_C^{-1}(x_0))$, the claim follows. Finally, we observe $\tilde{\mu}_n \to 0$ and, thus,

$$1 \geq \left| \nabla \left(H_\Gamma\right)_\varphi (x^n) \right| \geq \frac{C_\Gamma}{4\pi\sqrt{|C|}\tilde{\mu}_n} \to \infty.$$

The fact that H_Γ satisfies the Palais-Smale-condition follows, because $\Delta_N^\mu \Sigma \cap \mathcal{F}_N\Sigma$ is a neighborhood of $\partial\mathcal{F}_N\Sigma$. Thus, every Palais-Smale sequence stays inside some compact subset of $\mathcal{F}_N\Sigma$ and must have a convergent subsequence.

□

Proof of Theorem 4.0.1: According to the linking and Lemma 4.2.1, there exists a flow line along which H_Γ is bounded, i.e. there exists $p \in \mathcal{F}_N\Sigma$ such that

$$\lim_{t\to T^+(p)} H_\Gamma(\phi(t,p)) \leq C_0 < \infty,$$

where ϕ is the gradient flow of H_Γ and $\phi : (\cdot, p) : (T^-(p), T^+(p)) \to \mathcal{F}_N\Sigma$. First, we show that $T^+(p) = \infty$. This is done like in [22, Lem 4.7]. Because $[t_0, t_1] \ni s \mapsto \phi(s,p)$ is a path from $\phi(t_0,p)$ to $\phi(t_1,p)$, we see that

$$\begin{aligned}
d_g(\phi(t_0,p), \phi(t_1,p)) &\leq \int_{t_0}^{t_1} |\nabla H_\Gamma(\phi(s,p))| \, ds \\
&\leq \sqrt{t_1 - t_0} \sqrt{\int_{t_0}^{t_1} |\nabla H_\Gamma(\phi(s,p))|^2 \, ds} \\
&= \sqrt{t_1 - t_0} \sqrt{H_\Gamma(\phi(t_1,p)) - H_\Gamma(\phi(t_0,p))} \\
&\leq \sqrt{|t_1 - t_0|} \sqrt{C_0 - H_\Gamma \phi((t_0,p))}.
\end{aligned}$$

Now, assuming $T^+(p) < \infty$, we see that there exists some $\bar{p} \in \overline{\mathcal{F}_N\Sigma}$ such that $\phi(s,p) \to \bar{p}$ for $s \to T^+(p)$. If $\bar{p} \in \mathcal{F}_N\Sigma$, we are done, because then $\phi(\cdot,p)$ remains in a compact set of $\mathcal{F}_N\Sigma$. If $\bar{p} \in \partial\mathcal{F}_N\Sigma$, there exists a cluster C such that $\bar{p}_i = \bar{p}_j = \bar{p}_C$ for every $i,j \in C$ and $|C| \geq 2$. We choose a chart φ around $\bar{p}$ such that $\varphi_i = \varphi_C$ is a conformal flat chart around $\bar{p}_C$ for every $i \in C$. In that chart, we follow [22, Lem 4.7] and see

$$(H_\Gamma)_\varphi(\phi(t_1,p)) - (H_\Gamma)_\varphi(\phi(t_0,p)) \to \infty \qquad \text{for } t_1 \to T^+(p).$$

This is a contradiction, hence $T^+(p) = \infty$. Now, $p^n := \phi(n,p)$ is a Palais-Smale-sequence and, therefore has a convergent subsequence. The limit of this subsequence then is a critical point of H_Γ. □

Remark. One may wonder, if Theorem 4.2.2 could be used to achieve Theorem 4.0.1. For this case we change the condition $\sum_{\substack{i,j\in I\\ i\neq j}} \Gamma_i\Gamma_j \neq 0$ to $\sum_{\substack{i,j\in I\\ i\neq j}} \Gamma_i\Gamma_j < 0$. In this case we are able to give a proof. We assume that

$$\Phi(p) \geq M_0, a \leq H(p) \leq b, \nabla H(p) = \lambda\nabla\Phi(p) \Rightarrow \lambda < 0$$

does not hold, where

$$\Phi(p) = \sum_{i\neq j} |\Gamma_i\Gamma_j|\, G(p_i,p_j).$$

This leads to $p_i^n \to p^* \in \Sigma$ for some $i \in I$. Going into a conformal flat chart and defining $z_i^n = x_i^n$ if $i \in I$ and $z_j^n = 0$ if $j \notin I$, leads to

$$\begin{aligned}\langle \nabla H(x^n), z^n\rangle &= -\frac{1}{2\pi}\sum_{i\in I}\sum_{i\neq j\in I}\Gamma_i\Gamma_j\frac{\langle x_i^n - x_j^n, x_i^n\rangle}{\left|x_i^n - x_j^n\right|^2} + o(1)\\ &= -\frac{1}{4\pi}\sum_{\substack{i,j\in I\\ i\neq j}}\Gamma_i\Gamma_j + o(1),\end{aligned}$$

as well as

$$\langle \nabla\Phi(x^n), z^n)\rangle = -\frac{1}{4\pi}\sum_{\substack{i,j\in I\\ i\neq j}} |\Gamma_i\Gamma_j| + o(1).$$

Thus, one can conclude

$$0 \leq \lambda_n = \frac{\langle \nabla H(x^n), z^n\rangle}{\langle \nabla\Phi(x^n), z^n)\rangle} = \frac{\sum_{\substack{i,j\in I\\ i\neq j}}\Gamma_i\Gamma_j}{\sum_{\substack{i,j\in I\\ i\neq j}}|\Gamma_i\Gamma_j|} + o(1) < 0.$$

These calculations also appear in [6]. With Theorem 4.2.2, this establishes a critical point.
To keep our previous assumptions, it is necessary to show that also $\sum_{\substack{i,j\in I\\ i\neq j}}\Gamma_i\Gamma_j > 0$ yields a contradiction. This will be a lot harder, because now one ends up to sort the points by how fast they approach to p^* and further by which Γ_i is positive or negative. When $\Gamma_i > 0$ for $1 \leq i \leq N-1$ and $\Gamma_N < 0$, the calcula-

tions for this are similar to [14], because only one moving vortex with negative sign behaves like a fixed one. But, when there are more Γ_i involved, the proper sorting becomes more complicated.
Because we can rely on the calculations of [22], we prefer to use them instead.

4.3.2 Theorem 4.0.2

Besides our derivation of Theorem 4.0.1, we need to consider clusters around points at the boundary of Σ. Most important for this, will be [22, Lem. 4.5].

Lemma 4.3.2. *Assume Σ has boundary and (4.0.1), (4.0.2) hold. Further, let $\tilde{p} \in \partial\mathcal{F}_N\Sigma$ with $\tilde{p}_i = \tilde{p}_j$ for $i \neq j$ implies $\tilde{p}_i \in \partial\Sigma$. Moreover, let C be a cluster, i.e. $\tilde{p}_i = p_C \in \partial\Sigma$ for all $i \in C$ and $\tilde{p}_k \neq p_C$ for all $k \notin C$. Then, there exists $\delta > 0$ such that*

$$|\nabla_\Gamma H(p)| \geq \frac{\varepsilon_C}{2\pi}\left(\sum_{j\in C} d_{p_j}^2\right)^{-\frac{1}{2}} \qquad \text{for every } p \in U_\delta(\tilde{p}) \cap \mathcal{F}_N\Sigma,$$

where

$$\varepsilon_C := \frac{1}{2}\left(\sum_{i\in C}\Gamma_i^2 - \sum_{\substack{i,j\in C\\ i\neq j}}\Gamma_i\Gamma_j\right) > 0.$$

Proof: For every $i \in C$, we use a conformal flat chart φ_C around p_C, for every $k \notin C$, we use some chart φ_k. This yields a chart φ such that for $i \neq j \in C$, we have

$$G_\varphi(x_i, x_j) = \frac{1}{2\pi}\ln\frac{|\bar{x}_i - x_j|}{|x_i - x_j|} + O(1),$$

where the $O(1)$ is in a $\mathcal{C}^1$-sense. Further, for $i \in C \not\ni k$, we have that $G_\varphi(x_i, x_k) = O(1)$ again in a $\mathcal{C}^1$-sense. The rest is the same as in [22, Lem. 4.4; Lem. 4.5].

□

Proof of Theorem 4.0.2: First, we prove that for every

$$p \in M_\delta := \{q \in \mathcal{F}_N\Sigma : d_g(q_i, q_j) \leq \delta \text{ or } d_{q_i} \leq \delta \text{ for some } 1 \leq i < j \leq N\},$$

we have $|\nabla H_\Gamma(p)| > 1$. Arguing by contradiction suppose there exist $\delta^n \to 0$, $p^n \in M_{\delta_n}$ such that $|\nabla H_\Gamma(p^n)| \leq 1$. Because Σ^N is compact, there exists a convergent subsequence (again denoted with p^n) and $p^0 \in \Sigma^N$ such that $p^n \to p^0$. Because $\delta_n \to 0$, there exists $i \neq j$ such that $p_i^0 = p_j^0$ or $d_{p_i^0} = 0$.

If there exists a cluster C with $int(\Sigma) \ni p_C = p_i^0$ for all $i \in C$, we reach a contradiction like in the previous section. And for a cluster with $\partial\Sigma \ni p_C = p_i$ for $i \in C$, with lemma 4.3.2, we obtain the contradiction

$$1 \geq |\nabla H_\Gamma(p^n)| \geq \frac{\varepsilon_C}{2\pi} \left(\sum_{j \in C} d^2_{p_j^n} \right)^{\frac{1}{2}} \geq \frac{\varepsilon_C}{2\sqrt{|C|}\delta_n} \to \infty.$$

Thus, again, we see that H_Γ satisfies the Palais-Smale-condition. With Lemma 4.2.1, we again find a $p \in \mathcal{F}_N\Sigma$ such that

$$\lim_{t \to T^+(p)} H_\Gamma(\phi(p,t)) \leq C_0 < \infty,$$

where ϕ is the gradient flow of H_Γ and $\phi(\cdot, p) : (T^-(p), T^+(p)) \to \mathcal{F}_N\Sigma$. With [22, Lem. 4.7], and the calculations we saw earlier, we deduce $T^+(p) = \infty$. Then, $p^n := \phi(n,p)$ is Palais-Smale-sequence and the corresponding limit of the convergent subsequence is a critical point of H_Γ.

□

4.3.3 Theorem 4.0.3

Proof of Theorem 4.0.3: Let Σ have boundary and let (4.0.3) hold. We will use Theorem 4.2.2 to prove it. We define

$$\Phi : \mathcal{F}_N\Sigma \to \mathbb{R}, \qquad \Phi(p) := -\sum_{i \neq j} |\Gamma_i \Gamma_j| \, G(p_i, p_j) + \sum_{i=1}^N \Gamma_i^2 h(p_i, p_i).$$

Proposition 4.3.3. *For every $a < b \in \mathbb{R}$ exists $M_0 > 0$ such that*

$$\Phi(p) \leq -M_0, \; a \leq H_\Gamma(p) \leq b, \; \nabla H_\Gamma(p) = \lambda \nabla \Phi(p) \Rightarrow \lambda > 0.$$

Proof: This is [6, Prop. 3.1] generalized to a surface. Thus, we only need to do the localization. Every further necessary calculation can be found in [6]. Therefore, assume the opposite holds. Then, there exists $a < b \in \mathbb{R}$, $p^n \in \mathcal{F}_N\Sigma$, $\lambda_n \leq 0$ such that $\Phi(p^n) \to -\infty$, $a \leq H_\Gamma(p^n) \leq b$ and $\nabla H_\Gamma(p^n) = \lambda_n \nabla \Phi(p^n)$. Let $p^0 \in \Sigma^N$ be the limit of p^n (which exists along a subsequence, because Σ is compact). Because $\Phi(p^n) \to -\infty$ and $H(p^n) = O(1)$, there exists $i \neq j$ such that $p_i^0 = p_j^0$. Now, let φ be a chart around p^0 where $\varphi_i = \varphi_j$ for all $i \neq j$ with $p_i^0 = p_j^0$ and each φ_i being a conformal flat chart. Then, G_φ is a generalized Green's functions in the sense of [6], i.e. every point of [6, Lem 3.2] holds for G_φ. Hence, with the calculations of [6], we reach a contradiction.

□

Now, let $a < b \in \mathbb{R}$ satisfy (Bound). With the Proposition 4.3.3, we see that for $-\Phi$, there exists $M_0 > 0$ such that

$$-\Phi(p) \geq M_0,\, a \leq H_\Gamma(p) \leq b,\, \nabla H_\Gamma(p) = -\lambda \nabla\Phi(p) \Rightarrow \lambda < 0.$$

Note that this will hold true for any $M_1 > M_0$. Thus, according to the Lemma of Sard, we choose M_0 to be a regular value of $-\Phi$ and with $M_0 > \sup_S -\Phi \circ \gamma_0$. The only remaining step to apply Theorem 4.2.2 is to prove that

$$\emptyset \neq D = \{p \in \mathcal{F}_N\Sigma : -\Phi(p) \leq M_0\} \subset int(\mathcal{F}_N\Sigma) = \mathcal{F}_N\left(int(\Sigma)\right)$$

is compact. That $\emptyset \neq D \subset int(\mathcal{F}_N\Sigma)$ is obvious. Furthermore, we see that D is closed and that there exists $\delta_1, \delta_2 > 0$ such that

$$D \subset \left\{p \in \Sigma^N : d_g(p_i, p_j) \geq \delta_1,\, dist(p_i, \partial\Sigma) \geq \delta_2\right\}.$$

Hence, as a closed subset of a compact set, D itself is compact. We briefly show that δ_1, δ_2 exist. Assume the contrary, then there exists $p^n \in D$ such that $d_g(p_i^n, p_j^n) \to 0$ or $d_{p_i^n} \to 0$. However, this means $M_0 \geq -\Phi(p^n) \to \infty$.
With Theorem 4.2.2 H_Γ has a critical point.

Remark. Because so many methods and results, for critical points of H_Γ, of open sets in $\mathbb{R}^2$ translate to surfaces, it is possible that results on dynamics also translate to surfaces. As this is not included in this thesis, it is a potential route to continue research from this work onwards.

Chapter 5

Critical points under symmetries

In this chapter, we assume that the closed d-dimensional Riemanian manifold (Σ, g) is symmetric in the sense that there exists a $\mathcal{C}^\infty$ isometry $\tau : \Sigma \to \Sigma$, i.e.τ is a $\mathcal{C}^\infty$-diffeomorphism that satisfies

$$g_p(X,Y) = g_{\tau(p)}\left(D_p\tau(X), D_p\tau(Y)\right) \qquad \text{for all } p \in \Sigma; X, Y \in T_p\Sigma. \tag{Sym}$$

5.1 The Green's function under symmetries

Theorem 5.1.1. *If $G : \mathcal{F}_2\Sigma \to \mathbb{R}$ is a Green's function of the negative Laplace-Beltrami Operator $-\Delta_g$, then*

$$G_\tau : \mathcal{F}_2\Sigma \to \mathbb{R}, \qquad G_\tau(p_1, p_2) = G(\tau(p_1), \tau(p_2))$$

also is a Green's function to the negative Laplace-Beltrami-Operator, hence $G - G_\tau$ is constant. If there exists $p_1 \neq p_2 \in \Sigma$ such that $\tau(p_i) = p_i$ for $i = 1, 2$ then

$$G = G_\tau.$$

For the proof of this, we need to show that

$$-f(p) = \frac{1}{vol_g\Sigma}\int_\Sigma f\, dV_g + \int_\Sigma G_\tau(p,q)\Delta_g f(q)\, dV_g(q), \qquad \forall f \in \mathcal{C}^2(\Sigma).$$

Lemma 5.1.2. *For every measurable set $A \subset \Sigma$ with $\tau(A) = A$ and all contin-*

uous maps $f : A \to \mathbb{R}$, there holds

$$\int_A f\, dV_g = \int_A f \circ \tau\, dV_g.$$

Proof: Let $\varphi = (x_1, \dots, x_d) : U \to \mathbb{R}^d$ be a chart of Σ. Further, let $g_\varphi(x) = (g_{i,j}(x))_{i,j=1}^d$ be the local representation of g in φ. Also, let $|g_\varphi(x)| := \det(g_\varphi(x)) > 0$. Per definition, we have

$$\int_U f\, dV_g = \int_{\varphi(U)} f \circ \varphi^{-1} \sqrt{|g_\varphi|}\, dx_1 \dots dx_d.$$

Furthermore, the map $\varphi_\tau := \varphi \circ \tau^{-1} : \tau(U) \to \mathbb{R}^d$ is a chart of Σ. Because of (Sym), we have

$$\sqrt{|g_\varphi|} = \sqrt{|g_{\varphi_\tau}|}.$$

We calculate

$$\begin{aligned}
\int_{\tau(U)} f\, dV_g &= \int_{\varphi \circ \tau^{-1}(\tau(U))} f \circ (\varphi \circ \tau^{-1})^{-1} \cdot \sqrt{|g_{\varphi_\tau}|}\, dx_1 \dots dx_n \\
&= \int_{\varphi(U)} (f \circ \tau) \circ \varphi^{-1} \sqrt{|g_\varphi|}\, dx_1 \dots dx_n \\
&= \int_U f \circ \tau\, dV.
\end{aligned}$$

Now, let $(U_i)_{i=1}^\infty$ be a disjoint family of sets $U_i \subset \Sigma$ such that for every U_i there exists a chart $\varphi_i : U_i \to \mathbb{R}^d$ and

$$A = N \cup \bigcup_{i=1}^\infty U_i,$$

where N is a zero set. Then,

$$\begin{aligned}
\int_A f \circ \tau\, dV_g &= \sum_{i=1}^\infty \int_{U_i} f \circ \tau\, dV_g \\
&= \sum_{i=1}^\infty \int_{\tau(U_i)} f\, dV_g = \int_{\tau(A)} f\, dV_g = \int_A f\, dV_g.
\end{aligned}$$

□

Lemma 5.1.3. *Let f be $\mathcal{C}^2$, then*

$$\Delta_g (f \circ \tau)(p) = \Delta_g f(\tau(p)).$$

Proof: In the chart φ, the Laplace-Beltrami Operator is given by

$$\Delta_g f = \frac{1}{\sqrt{|g_\varphi|}}\partial_i \left(g_\varphi^{i,j} \sqrt{|g_\varphi|}\partial_j \left(f \circ \varphi^{-1} \right) \right),$$

where $g_\varphi^{-1}(x) = \left(g_\varphi^{i,j}(x)\right)_{i,j=1}^d$ and we have used the Einstein summation convention. Next, let φ be a chart around $p \in \Sigma$ with $\varphi(p) = 0$. Again, let $\varphi_\tau := \varphi \circ \tau^{-1}$. This is a chart around $\tau(p)$ and, with (Sym), there holds $g_\varphi = g_{\varphi_\tau} =: g$. Hence, we calculate

$$\Delta_g f(\tau(p)) = \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|} g^{i,j} \partial_j (f \circ (\varphi \circ \tau^{-1})^{-1}) \right)(0)$$

as well as

$$\begin{aligned}
\Delta_g (f \circ \tau)(p) &= \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|} g^{i,j} \partial_j (f \circ \tau \circ \varphi^{-1}) \right)(0) \\
&= \frac{1}{\sqrt{|g|}}\partial_i \left(\sqrt{|g|} g^{i,j} \partial_j (f \circ (\varphi \circ \tau^{-1})^{-1}) \right)(0).
\end{aligned}$$

$\square$

We are now ready to prove Theorem 5.1.1. Using the two preceding lemmas, we calculate

$$\begin{aligned}
-(f\circ\tau)(p) = -f(\tau(p)) &= \frac{1}{vol_g \Sigma}\int_\Sigma f + \int_\Sigma G(\tau(p), q)\Delta_g f(q)\, dV_g(q) \\
&= \frac{1}{vol_g \Sigma}\int_\Sigma f \circ \tau + \int_\Sigma G(\tau(p), \tau(q))\Delta_g f(\tau(q))\, dV_g(q) \\
&= \frac{1}{vol_g \Sigma}\int_\Sigma f \circ \tau + \int_\Sigma G_\tau(p, q)\Delta_g (f\circ\tau)(q)\, dV_g(q).
\end{aligned}$$

We see that for G_τ the equation we need to show holds for $f\circ\tau$ if $f : \Sigma \to \mathbb{R}$ is $\mathcal{C}^2$. Since τ is one to one, and τ as well as τ^{-1} are $\mathcal{C}^\infty$, any $\mathcal{C}^2$ map $\tilde{f} : \Sigma \to \mathbb{R}$ can be written in the form $\tilde{f} = f \circ \tau$, where $f := \tilde{f} \circ \tau^{-1}$, then is $\mathcal{C}^2$. Hence, G_τ is a Green's function to the negative Laplace-Beltrami operator.
Since G_τ is a Green's function, we have $G - G_\tau \equiv const$ and with $p_i = \tau(p_i)$ we have

$$G(p_1, p_2) - G_\tau(p_1, p_2) = 0.$$

5.2 The Principle of Symmetric Criticality

In this section, we want to use the Principle of Symmetric Criticality to achieve critical points of H_Γ. The following theorem states this principle.

Theorem 5.2.1. *Let Σ be a closed riemanian manifold and $\mathcal{G}$ be a group of isometries on Σ. Further, let $f : \Sigma \to \mathbb{R}$ be invariant under $\mathcal{G}$ i.e.*

$$f(p) = f(\tau p) \qquad \forall \tau \in \mathcal{G}.$$

When $Fix_{\mathcal{G}}$ denotes fixed points of $\mathcal{G}$, then $Fix_{\mathcal{G}}$ is a totally geodesic submanifold of Σ and, if $p \in Fix_{\mathcal{G}}$ is a critical point of $f|_{Fix_{\mathcal{G}}}$, then p is a critical point of f.

Proof: See [27] □

Now, let $d = 2$ and $\tau : \Sigma \to \Sigma$ be an isometry with $\tau^2 = id$ such that

$$Fix_\tau = \bigcup_{i=1}^{l} D_i$$

for $l \in \mathbb{N}$, and $D_i \cong S^1$ are the connected components of Fix_τ for all $1 \leq i \leq l$. For example, τ can be a reflection along some plane, when Σ is imbedded in $\mathbb{R}^3$. Moreover, let $2 \leq N \in \mathbb{N}$ and

$$H : \mathcal{F}_N\Sigma \to \mathbb{R}, \qquad H(p) := \sum_{i \neq j} \Gamma_i \Gamma_j G(p_i, p_j) + \Psi(p)$$

with $\Psi : \Sigma^N \to \mathbb{R}$ being $\mathcal{C}^\infty$ and with $\Psi \circ \tau = \Psi$. We make no difference between $\tau : \Sigma \to \Sigma$ and

$$\tau : \Sigma^N \to \Sigma^N, \qquad \tau(p) := (\tau(p_1), \ldots, \tau(p_N)).$$

In Theorem 5.1.1, we showed that $G \circ \tau = G$, hence $H \circ \tau = H$. This means we can apply Theorem 5.2.1. In Addition, note that, because of (Sym), we have $d_g = d_g \circ \tau$ for the distance $d_g : \Sigma^2 \to \mathbb{R}$. Furthermore, the map

$$R(p_i) := \lim_{q \to p_i} G(p_i, q) + \frac{1}{2\pi} \ln\left(d_g(p_i, q)\right)$$

satisfies $R \circ \tau = R$. Hence, the map H_Γ is contained in the class of maps that are included here.

Theorem 5.2.2. *i) Let N be even, i.e. $N = 2k$ for $k \in \mathbb{N}$, and $\Gamma_{\sigma(i)} = (-1)^i$ for some $\sigma \in Sym(N)$ for all $i = 1, \ldots, N$. Then, H has at least $l \cdot k$ critical points.*

ii) Let $N = 4$, $\Gamma_1, \Gamma_3 > 0 > \Gamma_2, \Gamma_4$ and

$$\begin{aligned} &\sum_{\substack{J \\ i \neq j}} \Gamma_i \Gamma_j < 0 && \text{for all } J \subset \{1,2,3,4\} \text{ with } |J| = 3 \\ &|\Gamma_i| < |\Gamma_1| + |\Gamma_3| && i = 2,4 \\ &|\Gamma_i| < |\Gamma_2| + |\Gamma_4| && i = 1,3. \end{aligned}$$

Then, H has at least $2l$ critical points.

Proof: Without loss of generality, we assume that $\sigma = id$. Due to Theorem 5.2.1, we only need to find critical points of $H|_{Fix_\tau}$. We will search for sets $\mathcal{L}_i \subset Fix_\tau \subset \mathcal{F}_N\Sigma$ for $1 \leq i \leq l \cdot k$ where in ii) $k = 2$. Thereafter, we will show that $\inf H|_{\mathcal{L}_i} > -\infty$ and that the negative gradient flow of $H|_{Fix_\tau}$ is invariant in $\mathcal{L}_i$ and exists for an infinite time. This, will correspond to a critical point $p \in \mathcal{L}_i$. We will see that the $\mathcal{L}_i$ are distinct and, thus, yield the number of critical points as claimed. Essentially, we will write $p = (p_1, \ldots, p_{2k}) = (q_1, \ldots, q_k)$ where $q_i = (p_{(2i-1)}, (p_{2i}))$. We derive the factor $l \cdot k$ by placing the $q_i \in D_i \times D_i$ and, thus, have kl different possibilities. In ii), its the same, but with $N = 4 = 2 \cdot 2$ we have $2 \cdot l = k \cdot l$.
Let $\alpha \in \{1, \ldots, l\}^k$ be a multiindex. We first define $\mathcal{L}_\alpha \subset Fix_\tau \subset \mathcal{F}_N\Sigma$. A point $p = (q_1, \ldots, q_k) \in \mathcal{F}_N\Sigma$ belongs to $\mathcal{L}_\alpha$, if the following holds:

- $q_i \in D^2_{\alpha_i}$ for every $1 \leq i \leq k$.
- For every $1 \leq j \leq l$, there exists a parametrization $\gamma_j : [0,1] \to D_j$ of D_j such that $\gamma_j(0) = \gamma_j(1) \neq p_i$ for every $i = 1, \ldots, N$, γ_j be one to one as a map $(0,1) \to D_j \setminus \{\gamma_j(0)\}$. Now, let $\{j_1, \ldots, j_{2s}\}$ be the set of indices such that $p_{j_i} \in D_j$ for every $1 \leq i \leq 2s$. Then, there exists $t_{j_i} \in (0,1)$ such that $\gamma_j(t_{j_i}) = p_{j_i}$. Without loss of generality, we assume $t_{j_i} < t_{j_{i+1}}$ for $i = 1, \ldots, 2s-1$. Then, there must hold

  $$\Gamma_{j_i}\Gamma_{j_{i+1}} < 0 \qquad \text{for every } i = 1, \ldots, 2s-1.$$

  Note that because there is an even amount of points on D_j, this also yields $\Gamma_{j_1}\Gamma_{j_{2s}} < 0$.
  This means that we locate an even amount of points on D_j, the connected component of Fix_τ. The points are arranged such that the adjacent points have a different sign in the vorticities.

Lemma 5.2.3.

$$(p^n)_n \subset \mathcal{L}_\alpha, p^n \to \partial\mathcal{F}_N Fix_\tau \Rightarrow H(p^n) \to -\infty.$$

Proof: $p^n \to \partial\mathcal{F}_N Fix_\tau$ yields $d_g(p_i^n, p_j^n) \to 0$ for $n \to \infty$. Because $Fix_\tau \subset \Sigma^N$ is compact, there exists a limit $p^0 \in Fix_\tau$ along a subsequence (again denotet with n). This yields a partition $I_0, \dots, I_m$ of $\{1, \dots, N\}$ where $i \in I_0$ iff $p_i^0 \neq p_k^0$ for every $k \neq i$ and $j \in I_j$ iff $p_j^0 = p_k^0 \neq p_s^k$ for every $k \in I_j \not\ni s$. Because Fix_τ is a totally geodesic submanifold, there exists a conformal flat chart around p^0 with $x_i^n = \varphi_i(p_i^n) \in \mathbb{R} \times \{0\}$, such that

$$\begin{aligned} H_\varphi(p^n) &= \sum_{i=1}^{m} -\frac{1}{2\pi} \sum_{\substack{k,j \in I_i \\ k \neq j}} \Gamma_j \Gamma_k \ln\left|x_j^n - x_k^n\right| + O(1) \\ &= -\frac{1}{2\pi} \sum_{i=1}^{m} \ln\left(\prod_{\substack{k,j \in I_i \\ k \neq j}} \left|x_j^n - x_k^n\right|^{\Gamma_j \Gamma_k} \right) + O(1). \end{aligned}$$

The rest follows from $p^n \in \mathcal{L}_\alpha$ and Lemma 5.2.4, which will be provided right after the rest of the proof.

□

So $\max H|_{\mathcal{L}_\alpha}$ exists. That maximum is a critical point of $\max H|_{\mathcal{L}_\alpha}$. Then it also is a critical point of $H|_{Fix_\tau}$ and as a consequence of the principle of symmetric criticality also one of H. As a result, we found $\left|\{1, \dots, l\}^k\right| = k \cdot l$ critical points.

□

Remark. Before we prove the final lemma, we want to remark that the amount of critical points we inferred from this method is not optimal. The reason for this is that when we wrote p as k pairs of points, we could also have written $p = (q_1, \dots, q_k)$ with $q_i = (p_{\sigma(i)}, p_{\sigma(i+1)})$ such that $\Gamma_{\sigma(i)}\Gamma_{\sigma(i+1)} < 0$ for some $\sigma \in Sym(N)$. There are $k!$ ways of doing this. However, when placing $q_i \in D_j^2$, we would have to be more careful. If $q_k \in D_j^2$ we could count a critical point more than once, when using this method. We do not solve this shortcoming in this study. However, it demonstrates potential for future lines of research.

Lemma 5.2.4. *i) Let $k \in \mathbb{N}$, $(t_i^n)_n \subset \mathbb{R}$ and $t_1^n < t_2^n < \dots < t_k^n$ for every $n \in \mathbb{N}$. We define*

$$\alpha_2^n := \frac{1}{|t_1^n - t_2^n|}, \quad \alpha_j^n := \begin{cases} \alpha_{j-1}^n \frac{1}{|t_1^n - t_j^n|} \prod_{i=1}^{\frac{j}{2}-1} \frac{|t_j^n - t_{2i}^n|}{|t_j^n - t_{2i+1}^n|}, & j \text{ even} \\ \alpha_{j-1}^n \prod_{i=1}^{\frac{j-1}{2}} \frac{|t_{2i-1}^n - t_j^n|}{|t_{2i}^n - t_j^n|}, & j \text{ uneven.} \end{cases}$$

Then, $\alpha_k^n \to \infty$, if $|t_1^n - t_k^n| \to 0$. Note that $\alpha_k^n = \prod_{i \neq j} \left|t_i^n - t_j^n\right|^{\Gamma_i \Gamma_j}$, if

$\Gamma_i = (-1)^i$.

ii) Let $0 \le t_1^n < t_2^n < t_3^n < t_4^n$ *and* $\gamma_i > 0$, *then*

$$\frac{|t_3^n - t_1^n|^{\gamma_1\gamma_3} |t_4^n - t_2^n|^{\gamma_2\gamma_4}}{|t_1^n - t_2^n|^{\gamma_1\alpha_2} |t_1^n - t_4^n|^{\gamma_1\gamma_4} |t_3^n - t_2^n|^{\gamma_2\gamma_3} |t_3^n - t_4^n|^{\gamma_3\gamma_4}} \to \infty,$$

if $t_1^n, t_2^n \to 0$ *and if* $\gamma_1\gamma_3 - \gamma_1\gamma_2 - \gamma_3\gamma_2 < 0$, $\gamma_4\gamma_2 - \gamma_1\gamma_4 - \gamma_1\gamma_2 < 0$, $\gamma_i < \gamma_1 + \gamma_3$ *for* $i = 2, 4$ *and* $\gamma_i < \gamma_2 + \gamma_4$ *for* $i = 1, 3$.

Proof: i): We prove this by induction. Its clear that $\alpha_2^n \to \infty$. If k is even, we have

$$\alpha_k^n = \underbrace{\alpha_{k-1}^n \frac{1}{|t_1^n - t_k^n|}}_{\to\infty} \underbrace{\prod_{i=1}^{\frac{k}{2}-1} \frac{|t_{2i} - t_k|}{|t_{2i+1} - t_k|}}_{\ge 1} \to \infty.$$

If k is uneven, we have

$$\alpha_k^n = \underset{\to\infty}{\alpha_{k-1}^n} \cdot \prod_{i=1}^{\frac{k-1}{2}} \underbrace{\frac{|t_{2i-1}^n - t_k^n|}{|t_{2i}^n - t_k^n|}}_{\ge 1} \to \infty.$$

ii): Without loss of generality, we assume $t_1^n \equiv 0$. Then,

$$\begin{aligned} &\frac{|t_3^n - t_1^n|^{\gamma_1\gamma_3} |t_4^n - t_2^n|^{\gamma_2\gamma_4}}{|t_1^n - t_2^n|^{\gamma_1\gamma_2} |t_1^n - t_4^n|^{\gamma_1\gamma_4} |t_3^n - t_2^n|^{\gamma_2\gamma_3} |t_3^n - t_4^n|^{\gamma_3\gamma_4}} \\ =&\frac{|t_3^n|^{\gamma_1\gamma_3} |t_4^n - t_2^n|^{\gamma_2\gamma_4}}{|t_2^n|^{\gamma_1\gamma_2} |t_4^n|^{\gamma_1\gamma_4} |t_3^n - t_2^n|^{\gamma_2\gamma_3} |t_3^n - t_4^n|^{\gamma_3\alpha_4}}. \end{aligned}$$

We have to look into four different cases. If $t_3^n \not\to 0$, we also have that $|t_4^n - t_2^n| \not\to 0$ and the claim follows.
Now, let $\frac{t_3^n}{t_4^n} = o(1)$. Then, we have $\frac{t_2^n}{t_4^n} = o(1)$. In this case, we need $t_2^n < t_3^n$ and $|t_2^n - t_3^n| = t_3^n - t_2^n < t_3^n$ to see that

$$\begin{aligned} &\frac{|t_3^n|^{\gamma_1\alpha_3} |t_4^n - t_2^n|^{\gamma_2\gamma_4}}{|t_2^n|^{\gamma_1\gamma_2} |t_4^n|^{\gamma_1\gamma_4} |t_3^n - t_2^n|^{\gamma_2\gamma_3} |t_3^n - t_4^n|^{\gamma_3\gamma_4}} \\ \ge& (t_3^n)^{\gamma_1\gamma_3 - \gamma_1\gamma_2 - \gamma_2\gamma_3} \cdot \frac{|t_2^n - t_4^n|^{\gamma_2\gamma_4}}{|t_4^n|^{\gamma_1\gamma_4} |t_3^n - t_4^n|^{\gamma_3\gamma_4}}. \end{aligned}$$

Because of $\gamma_1\gamma_3 - \gamma_1\gamma_2 - \gamma_2\gamma_3 < 0$, we have $(t_3^n)^{\gamma_1\gamma_3 - \gamma_1\gamma_2 - \gamma_2\gamma_3} \to \infty$. Thus, we show that

$$\frac{|t_2^n - t_4^n|^{\gamma_2\gamma_4}}{|t_4^n|^{\gamma_1\gamma_4} |t_3^n - t_4^n|^{\gamma_3\gamma_4}} \ge \beta > 0$$

and the claim will follow. For $i = 2, 3$, we have

$$\frac{|t_i^n - t_4^n|}{t_4^n} = 1 - \frac{t_i^n}{t_4^n} = 1 + o(1).$$

Thus, we infer

$$\begin{aligned}&\frac{|t_2^n - t_4^n|^{\gamma_2\gamma_4}}{|t_4^n|^{\gamma_1\gamma_4}\,|t_3^n - t_4^n|^{\gamma_3\gamma_4}}\\ &= (t_4^n)^{\alpha_4(\gamma_2-\gamma_1-\gamma_3)}\underbrace{\left(\frac{|t_2^n - t_4^n|}{t_4^n}\right)^{\gamma_2\gamma_4}\cdot\left(\frac{t_4^n}{|t_3^n - t_4^n|}\right)^{\gamma_3\gamma_4}}_{\to 1}.\end{aligned}$$

With $\gamma_2 - \gamma_1 - \gamma_3 < 0$, we see that $(t_4^n)^{\alpha_4(\gamma_2-\gamma_1-\gamma_3)} \not\to 0$ and our claim follows. We continue with the third case, $\frac{t_4^n}{t_2^n} = O(1)$, note that then also $\frac{t_4^n}{t_3^n} = O(1)$. We see that

$$\frac{|t_i^n - t_j^n|}{|t_k^n - t_l^n|} = O(1) \qquad \forall i \neq j, k \neq l.$$

With this, we can deduce

$$\begin{aligned}&\frac{|t_3^n|^{\gamma_1\gamma_3}\,|t_4^n - t_2^n|^{\gamma_2\gamma_4}}{|t_2^n|^{\gamma_1\gamma_2}\,|t_4^n|^{\gamma_1\gamma_4}\,|t_3^n - t_2^n|^{\gamma_2\gamma_3}\,|t_3^n - t_4^n|^{\gamma_3\gamma_4}}\\ &\geq \beta\,(t_2^n)^{(\gamma_1\gamma_3-\gamma_1\gamma_2-\gamma_2\gamma_3)+\gamma_4(\gamma_2-\gamma_1-\gamma_3)} \to \infty,\end{aligned}$$

for some $\beta > 0$.
The final case is $\frac{t_4^n}{t_3^n} = O(1)$, but $\frac{t_4^n}{t_2^n} = o(1)$. Then, we also have $\frac{t_2^n}{t_3^n} = o(1)$. We see

$$\begin{aligned}&\frac{|t_3^n|^{\gamma_1\gamma_3}\,|t_4^n - t_2^n|^{\gamma_2\gamma_4}}{|t_2^n|^{\gamma_1\gamma_2}\,|t_4^n|^{\gamma_1\gamma_4}\,|t_3^n - t_2^n|^{\gamma_2\gamma_3}\,|t_3^n - t_4^n|^{\gamma_3\gamma_4}}\\ =&\frac{|t_3^n|^{\gamma_1\gamma_3}}{|t_3^n - t_2^n|^{\gamma_3\gamma_2}\,|t_4^n - t_3^n|^{\gamma_4\gamma_3}}\cdot\frac{|t_4^n - t_2^n|^{\gamma_4\gamma_2}}{|t_4^n|^{\gamma_4\gamma_1}\,|t_2^n|^{\gamma_2\gamma_1}}.\end{aligned}$$

We look into both factors separately. Since $\frac{t_3^n - t_2^n}{t_3^n} = 1 + o(1)$ and $\frac{t_4^n - t_3^n}{t_3^n} = O(1)$, we calculate

$$\frac{|t_3^n|^{\gamma_1\gamma_3}}{|t_3^n - t_2^n|^{\gamma_3\gamma_2}\,|t_4^n - t_3^n|^{\gamma_4\alpha_3}} \geq \beta\,(t_3^n)^{\gamma_3(\gamma_1-\gamma_2-\gamma_4)} \to \infty,$$

as $\gamma_1 - \gamma_2 - \gamma_4 < 0$. Finally, with $\frac{t_4^n - t_2^n}{t_4^n} = 1 + o(1)$ and $t_2^n < t_4^n$, we infer

$$\frac{|t_4^n - t_2^n|^{\gamma_4\gamma_2}}{|t_4^n|^{\gamma_4\gamma_1}\,|t_2^n|^{\gamma_2\gamma_1}} \geq \beta\,(t_4^n)^{\gamma_4\gamma_2-\gamma_1\alpha_4-\gamma_1\gamma_2} \to \infty,$$

as $\gamma_4\gamma_2 - \gamma_1\gamma_4 - \gamma_1\gamma_2 < 0$.

□

Before we finish this section, we state how γ_i relate to Γ_i. After going into a chart, there will hold $\gamma_i = \left|\Gamma_{\sigma(i)}\right|$ for a $\sigma \in Sym(4)$. The following chart states all possibilities:

γ_1	γ_2	γ_3	γ_4
Γ_1	Γ_2	Γ_3	Γ_4
Γ_2	Γ_3	Γ_4	Γ_1
Γ_3	Γ_4	Γ_1	Γ_2
Γ_4	Γ_1	Γ_2	Γ_3
Γ_1	Γ_4	Γ_3	Γ_2
Γ_2	Γ_1	Γ_4	Γ_3
Γ_3	Γ_2	Γ_1	Γ_4
Γ_4	Γ_3	Γ_2	Γ_1

Remark. If we want to have better conditions for arbitrary N, we would need to find conditions as in Lemma 5.2.4 ii). The possible arrangements are growing fast and the proof would involve even more cases. Nonetheless, it should be possible to achieve better conditions than $\Gamma_i = (-1)^i$, as we saw better conditions when $N = 4$.

5.3 Another way using symmetry

Again, we assume $\tau : \Sigma \to \Sigma$ to be an isometry with $\tau^2 = id$ and with $Fix_\tau \cong S^1 \dot\cup \dots \dot\cup S^1$. This time, we let $N = 3$. Further, we assume there exists $\sigma \in Sym(3)$ such that $\Gamma_{\sigma(1)} = \Gamma_{\sigma(3)} > 0 > \Gamma_{\sigma(2)}$. Without loss of generality, we assume $\sigma = id$. We then look at

$$H : \mathcal{F}_3\Sigma \to \mathbb{R}, \qquad H(p) = \sum_{i \neq j} \Gamma_i \Gamma_j G(p_i, p_j) + \Psi(p)$$

where $\Psi \circ \tau = \Psi$ and $\Psi(p_1, p_2, p_3) = \Psi(p_3, p_2, p_1)$. Because of $\Gamma_1 = \Gamma_3$, we have $H(p_1, p_2, p_3) = H(p_3, p_2, p_1)$. These assumptions hold for H_Γ, when $\Gamma_1 = \Gamma_3$.

Theorem 5.3.1. *If $\Gamma_1 = \Gamma_3 > 0 > \Gamma_2$ and $\Gamma_1 > -2\Gamma_2$, then H has a critical point.*

Proof: We define the set

$$\mathcal{L} := \{(p_1, p_2, \tau(p_1)) : p_2 \in Fix_\tau \not\ni p_1\} \subset \mathcal{F}_3\Sigma,$$

and use the following lemma.

Lemma 5.3.2.

$$(p^n)_n \subset \mathcal{L}, p^n \to \partial\mathcal{F}_3\Sigma \Rightarrow H(p^n) \to \infty.$$

Proof: If $p_1^n \to Fix_\tau$, but $d_g(p_1^n, p_2^n) \not\to 0$, we have

$$H(p^n) = -\frac{\Gamma_1\Gamma_3}{2\pi} \ln\left(d_g(p_1^n, \tau(p_1^n)\right) + O(1) \to \infty.$$

Thus, let $p_i^n \to p^* \in Fix_\tau$. Therefore, there exists a conformal flat chart φ around $p^* \in \Sigma$. Thus, with $z_i^n := \varphi(p_i^n)$, we have

$$H_\varphi(p) = -\frac{1}{2\pi}\sum_{i\neq j}\Gamma_i\Gamma_j \ln\left|z_i^n - z_j^n\right| + O(1).$$

Because Fix_τ is a totally geodesic submanifold, we further assume, that $z_2^n = (x_2^n, 0) \in \mathbb{R}\times\{0\}$, $z_1^n = (x_1^n, y_1^n)$ and $z_3^n = (x_1^n, -y_1^n)$. Hence, we need to show that

$$\frac{|z_1^n - z_3^n|^{\Gamma_1\Gamma_3}}{|z_1^n - z_2^n|^{-\Gamma_1\Gamma_2}\,|z_2^n - z_3^n|^{-\Gamma_2\Gamma_3}} = \left(\frac{|z_1^n - z_3^n|^{\Gamma_1}}{|z_1^n - z_2^n|^{-\Gamma_2}\,|z_3^n - z_2|^{-\Gamma_2}}\right)^{\Gamma_1} \to 0$$

if $|z_1^n - z_2^n| \to 0$. We see $|z_1^n - z_3^n| = 2\,|x_1^n|$ and $|z_1^n - z_2^n| = |z_3^n - z_2^n|$. Without loss of generality, we further assume $z_2^n \equiv 0$. We calculate

$$\begin{aligned}\frac{|z_1^n - z_3^n|^{\Gamma_1}}{|z_1^n - z_2^n|^{-\Gamma_2}\,|z_3^n - z_2|^{-\Gamma_2}} &= 2^{\Gamma_1}\frac{|x_1^n|^{\Gamma_1}}{|z_1^n|^{-2\Gamma_2}}\\ &\leq 2^{\Gamma_1}\,|z_1^n|^{\Gamma_1+2\Gamma_2} \to 0,\end{aligned}$$

as $\Gamma_1 + 2\Gamma_2 > 0$.

□

Now, let ϕ be the negative gradient flow of H.
We also need the following lemma.

Lemma 5.3.3.
$$p \in \mathcal{L} \Rightarrow \phi(t,p) \in \mathcal{L} \qquad \forall t < T^+(p)$$

where $\phi(\cdot, p) : (T^-(p), T^+(p)) \to \mathcal{F}_3\Sigma$.

First, we note that $H \circ \tau = H$ and, thus, because of the chain rule, we have

$$D_pH = D_{\tau(p)}H \circ D_p\tau.$$

Now, $D_p\tau$ is an isometry on the tangent spaces $T_p\Sigma \to T_{\tau(p)}\Sigma$, i.e. we see

$$D_p\tau\left(\nabla H(p)\right) = \nabla H(\tau(p)),$$

as

$$\begin{aligned}\left\langle D_p\tau\left[\nabla H(p)\right], D_p\tau[X]\right\rangle &= \langle \nabla H(p), X\rangle = D_pH[X] = \left(D_{\tau(p)}H \circ D_p\tau\right)[X] \\ &= \left\langle \nabla H(\tau(p)), D_p\tau[X]\right\rangle.\end{aligned}$$

With this, we deduce $\tau\phi(t,p) = \phi(t,\tau(p))$. We prove this now. We see

$$\tau(\phi(0,p)) = \tau(p) = \phi(0,\tau(p)).$$

Moreover, we calculate

$$\frac{d}{dt}\tau\phi(t,p) = D_{\phi(t,p)}\tau\left[\frac{d}{dt}\phi(t,p)\right] = D_{\phi(t,p)}\tau\left[-\nabla H(\phi(t,p)\right] = -\nabla H(\tau(\phi(t,p)))$$

and

$$\frac{d}{dt}\phi(t,\tau(p)) = -\nabla H(\phi(t,\tau(p))).$$

Thus, $\tau\phi(t,p) = \phi(t,\tau(p))$, because both satisfy the initial value problem

$$\begin{cases}\dot{u} = -\nabla H(u) \\ u(0) = \tau(p).\end{cases}$$

Now, for $p \in \mathcal{L}$, we want

$$\tau\phi_1(t,p) = \phi_3(p,t) \text{ and } \phi_2(t,p) \in Fix_\tau.$$

Because of $H(p_1,p_2,p_3) = H(p_3,p_2,p_1)$, we have $\phi_1(t,(p_1,p_2,p_3)) = \phi_3(t,(p_3,p_2,p_1))$, as they will satisfy the same initial value problem. Thus, if $p = (p_1,p_2,\tau(p_1)) \in \mathcal{L}$, we infer

$$\tau\phi_1(t,p) = \phi_1(t,\tau(p)) = \phi_1(t,(\tau(p_1),p_2,p_1)) = \phi_3(t,(p_1,p_2,\tau(p_1))) = \phi_3(t,p).$$

It remains to show that $\tau\phi_2(p,t) = \phi_2(p,t)$, if $p \in \mathcal{L}$. We see this with the above used facts. Again, let $p = (p_1,p_2,\tau(p_1)) \in \mathcal{L}$:

$$\phi(t,p) = \begin{pmatrix}\phi_3 \\ \phi_2 \\ \phi_1\end{pmatrix}(t,(p_3,p_2,p_1)) = \begin{pmatrix}\phi_3 \\ \phi_2 \\ \phi_1\end{pmatrix}(t,\tau(p)) = \tau\left(\begin{pmatrix}\phi_3(t,p) \\ \phi_2(t,p) \\ \phi_1(t,p)\end{pmatrix}\right) = \begin{pmatrix}\phi_1(t,p) \\ \tau(\phi_2(t,p)) \\ \phi_3(t,p)\end{pmatrix}.$$

This yields $\tau\phi_2(t,p) = \phi_2(t,p)$.

Now, with Lemma 5.3.2, we deduce $T^+(p) = \infty$ for $p \in \mathcal{L}$. Then, the sequence $\phi(n,p)$ converges to a critical point of H.

□

Remark. If we assume $\Gamma_1 = \Gamma_3 > 0 > \Gamma_2$, the assumption $\Gamma_1 > -2\Gamma_2$ is optimal as we see in Appendix B. If $\Gamma_1 \neq \Gamma_3$, the method can not work, because of the loss of $H(p_1,p_2,p_3) = H(p_3,p_2,p_1)$.

Appendix A

The axiom A5

As a remainder, we will show that for all $C > 0$, there exists $\varepsilon_C > 0$ such that

$$\frac{d_x}{|x-y|} \le C, d_x \le d_y, d_x \le \varepsilon_C \Rightarrow \langle \partial_1 G(x,y), \nu_x \rangle > 0.$$

In this Appendix, we continue with $C_i(\cdot)$ being constants depending on $\cdot$.

We prove this by contradiction. Assume there exists $C > 0$, $x^n, y^n \in \mathcal{F}_n\Omega$ such that

$$d_{x^n} := d_x^n \le d_y^n =: d_{y^n},\ d_x^n \to 0 \text{ for } n \to \infty,$$

$$d_x^n \le C\,|x^n - y^n| \text{ and } \langle \partial_1 G(x^n, y^n), \nu_x^n \rangle \le 0,$$

where $\nu_z^n := \nu_{z^n}$ if $z^n \in \{x^n, y^n\}$. Furthermore, let $p_z^n := p_{z^n}$. Along a subsequence we have

$$y_0 := \lim_{n\to\infty} y^n \in \overline{\Omega} \text{ and } x_0 := \lim_{n\to\infty} x^n \in \partial\Omega.$$

According to Hopf's Lemma (see [16] page 330), the following holds

$$\langle \partial_1 G(p,z), \nu_p \rangle > 0 \qquad \forall z \in \Omega, p \in \partial\Omega.$$

To prevent sign problems, note that we use the interior normal and not the exterior normal. This yields

$$\langle \partial_1 G(x_0, y_0), \nu_{x_0} \rangle > 0$$

if $x_0 \ne y_0$, because of continuity and, thus, a contradiction. This yields $x_0 = y_0$.

Now, we have

$$\frac{d_y^n}{|x^n - y^n|} \leq 1 + \frac{d_x^n}{|x^n - y^n|} + \frac{\left|p_x^n - p_y^n\right|}{|x^n - y^n|} = O(1).$$

We investigate two cases, that is $\frac{d_y^n}{|x^n-y^n|} = o(1)$ or not. If this does not hold, we can use (A4) to reach a contradiction, our estimations for ψ where not enough to reach a contradiction with just (A4) in every case. We elaborate which calculations were missing throughout the proof. Nonetheless, we start with $\frac{d_y^n}{|x^n-y^n|} = o(1)$.
In this case, we reach the contradiction similar to [3] in Appendix B. To provide a complete proof, we rewrite it in more detail. We use the transformation

$$T_{S,\lambda,a} : \Omega \to \Omega_{S,\lambda,a}, x \mapsto S(\lambda x - a)$$

where $S \in O(d), \lambda > 0, a \in \mathbb{R}^d$. Note that $T_{S,\lambda,a}$ is one to one, where $T_{S^{-1},\lambda^{-1},b}$ is its inverse, if $b = -\lambda S a$. First, we calculate the change of G. In other words, we will calculate the Dirichlet Green's function $G_{S,\lambda,a}$ of $\Omega_{S,\lambda,a}$ in terms of G. Thus, we calculate

$$\begin{aligned} |T_{S,\lambda,a}x - T_{S,\lambda,a}y|^{2-d} &= |S\left(\lambda x - a - \lambda y + a\right)|^{2-d} \\ &= \lambda^{2-d} |x-y|^{2-d}. \end{aligned}$$

Furthermore, for a function $u : \Omega \to \mathbb{R}$ with $\Delta u = 0$, we have

$$\Delta\left(u\left(T_{S,\lambda,a}^{-1} z\right)\right) = \frac{1}{\lambda^2}\Delta u\left(S^{-1}(\lambda^{-1}z - b)\right) = 0.$$

Thus, we see that

$$G(x,y) = \lambda^{d-2} G_{S,\lambda,a}\left(T_{S,\lambda,a}x, T_{S,\lambda,a}y\right). \tag{A.0.1}$$

We calculate this in detail now. We have

$$\begin{aligned} G_{S,\lambda,a}\left(T_{S,\lambda,a}x, T_{S,\lambda,a}y\right) &= c_d \Psi\left(T_{S,\lambda,a}x, T_{S,\lambda,a}y\right) - h_{S,\lambda,a}\left(T_{S,\lambda,a}x, T_{S,\lambda,a}y\right) \\ &= c_d \lambda^{2-d} \Psi(x,y) - h_{S,\lambda,a}\left(T_{S,\lambda,a}x, T_{S,\lambda,a}y\right). \end{aligned}$$

Further, we just calculated, that for fixed $x \in \Omega_{S,\lambda,a}$ we have $\Delta_y h(T_{S,\lambda,a}^{-1}x, T_{S,\lambda,a}^{-1}y) = 0$. If $y \in \partial\Omega_{S,\lambda,a}$, we see

$$h(T_{S,\lambda,a}^{-1}x, T_{S,\lambda,a}^{-1}y) = \Psi(T_{S,\lambda,a}^{-1}x, T_{S,\lambda,a}^{-1}y) = \lambda^{d-2}\Psi(x,y) = \lambda^{d-2} h_{S,\lambda,a}(x,y).$$

This yields

$$h_{S,\lambda,a}(x,y) = \lambda^{2-d} h(T^{-1}_{S,\lambda,a}x, T^{-1}_{S,\lambda,a}y),$$

what concludes (A.0.1). Now, we see that

$$\partial_1 G(x,y) = \lambda^{d-1}\partial_1 G(T_{S,\lambda,a}x, T_{S,\lambda,a}y).$$

We define $\lambda_n := \frac{1}{|x^n - y^n|}$, $a_n := \lambda_n p^n_y$ and $S_n \in O(d)$ such that

$$T_{S_n,\lambda_n,a_n} =: T_n : \Omega \to \Omega_n := \Omega_{S_n,\lambda_n,a_n}$$

yields $\nu_{T_n p^n_y} = (0,1) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Moreover, we set the following notations:

$$T_n d^n_z := dist(\partial\Omega_n, T_n z^n) \text{ and } T_n \nu^n_z := \nu_{T_n z^n} \text{ for } z^n \in \{x^n, y^n\}.$$

Furthermore, we set $G_n := G_{S_n,\lambda_n,a_n}$ and see

$$G = \frac{G_n}{|x^n - y^n|^{d-2}} \text{ and } 0 \geq \langle \partial_1 G(x^n, y^n), \nu^n_x \rangle = \frac{\langle \partial_1 G_n(T_n x^n, T_n y^n), T_n \nu^n_x \rangle}{|x^n - y^n|^{d-1}}.$$

In addition, note that $T_n p^n_y = 0$ and

$$|T_n x^n - T_n y^n| = \lambda_n |x^n - y^n| = 1.$$

Thus, we see, for $z^n \in \{x^n, y^n\}$, that

$$T_n d^n_z = \frac{d^n_z}{|x^n - y^n|} \leq \frac{d^n_y}{|x^n - y^n|} = o(1).$$

For $t > 0$, let $B_t := B_t(0,t)$ be the closed ball with radius t around $(0,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Next, we choose $t_0 > 0$ such that

$$\sup_{\mathbb{R}^n \ni (z,0): |z| \leq 1} dist\,(z, B_{t_0}) < \frac{1}{16}. \tag{A.0.2}$$

This can be done, because $B_t \to \mathbb{R}^d_+$ for $t \to \infty$. For large enough n, we have

$$B_{t_0} \setminus \{0\} \subset \Omega_n.$$

We prove this last statement. First, note that $B_s \subset B_t$ holds for $t > s$. Also, Ω satisfies an interior ball condition, because $\partial\Omega$ is at least $\mathcal{C}^3$. So, for every $p^n_y \in \partial\Omega$, there exists $w^n \in \Omega$ such that $B_{d^n_w}(w^n) \setminus \{p^n_y\} \subset \Omega$ where w^n can be chosen such that d^n_w is bounded away from 0, because $\partial\Omega$ is compact. Then, we have $T_n w^n = (0, T d_w^n)$ where

$$T_n d_w^n = \frac{d_w^n}{|x^n - y^n|} \to \infty \text{ and } T_n B_{d_w^n}(w^n) = B_{T d_w^n}.$$

Hence, for large enough n, we have $B_{t_0} \subset B_{T d_w^n} \subset \Omega_n \setminus \{0\}$.
We define $B_0 := B_{t_0}$ and G_0 as the Green's function belonging to B_0. Note here that G_0 is explicitly known, because B_0 is a ball (see Appendix D). We further set $R := \frac{1}{8}$ and $z_x^n := T_n p_x^n + R T_n \nu_x^n$. If n is big enough, we again have

$$B_R(z_x^n) \setminus \{T_n p_x^n\} \subset \Omega_n \text{ and } T_n p_x^n \in B_R(z_x^n).$$

Further, we let $r := \frac{1}{32}$ and have $B_r(z_x^n) \subset B_0$ if n is big enough. Note that (A.0.2) is used here. For $w \in B_R(z_x^n)$, we define

$$v_n(w) := e^{-\alpha_n |w - z_x^n|^2} - e^{-\alpha_n R},$$

where $\alpha_n > 0$ is chosen such that

$$0 \le \Delta v(w) = \alpha_n e^{\alpha_n |w - z_x^n|^2} \left(\alpha_n |w - z_x^n|^2 - d \right) \qquad \forall w \in B_R(z_x^n) \setminus B_r(z_x^n).$$

For $w_0, w_1 \in \mathbb{R}^d$, we let

$$[w_0, w_1] := \{ t w_0 + (1-t) w_1 : t \in [0,1] \}.$$

For $w \in B_r(z_x^n)$, we have $w \in B_{\frac{1}{2}}(T_n x^n)$. Thus, we conclude

$$|t - w| > \frac{1}{4} \qquad \forall t \in [0, T_n y^n].$$

This holds, because $[0, T_n y^n] \subset B_{\frac{1}{4}}(T_n y^n)$ and $|T_n x^n - T_n y^n| = 1$. Now, Taylor's theorem implies

$$G_0(w, T_n y^n) = \underbrace{G_0(w,0)}_{=0} + \partial_{T_n \nu_y^n} G_0(w,0) T_n d_y^n + \underbrace{O\left(\sup_{t \in [0, T_n y^n]} \partial^2_{T_n \nu_y^n} G_0(w,t) \, (T_n y^n)^2 \right)}_{=O\left((T_n d_y^n)^2 \right)}.$$

From the exact form of G_0 (see Appendix C), we infer

$$\partial_{T_n \nu_y^n} G_0(w,0) = c_d \frac{t_0 - |w - (0,t_0)|^2}{|w|^d} \ge C(t_0) > 0 \qquad \forall w \in B_r(z_x^n).$$

This, yields

$$G_0(w, T_n y^n) \ge C_{14} T_n d_y^n \qquad \forall w \in B_r(z_x^n).$$

We chose $M_n > 0$ with

$$\max\{\sup v_n, \sup \partial_\nu v_n\} \leq M_n.$$

Our aim is to use the maximum principle on the function

$$B_R(z_x^n) \setminus B_r(z_x^n) \ni w \mapsto G_n(w, T_n y^n) - \frac{C_{14} T_n d_y^n}{M_n} v_n(w).$$

If $w \in \partial B_R(z_x^n)$, we have $v_n(w) = 0$ and, thus,

$$G_n(w, T_n y^n) - \frac{C_{14} T_n d_y^n}{M_n} v_n(w) = G_n(w, T_n y^n) \geq 0.$$

Further, if $w \in \partial B_r(z_x^n)$, we estimate

$$\begin{aligned} G_n(w, T_n y^n) - \frac{C_{14} T_n d_y^n}{M_n} \underbrace{v_n(w)}_{\geq 0} &\geq G_n(w, T_n y^n) - C_{14} T_n d_y^n \\ &\geq G_n(w, T_n y^n) - G_0(w, T_n y^n) \geq 0. \end{aligned}$$

In the last inequality, we use the maximum principle and $B_0 \subset \Omega_n$. Thus, we see

$$G_n(w, T_n y^n) - \frac{C_{14} T_n d_y^n}{M_n} v_n(w) \geq 0 \qquad \text{on } \partial \left(B_R(z_x^n) \setminus B_r(z_x^n)\right).$$

Furthermore, we calculate

$$-\Delta_w \left(G_n(w, T_n y^n) - \frac{C_{14} T_n d_y^n}{M_n} v_n(w) \right) = \frac{C_{14} T_n d_y^n}{M_n} \Delta_w v_n(w) \geq 0,$$

for every $w \in B_R(z_x^n) \setminus B_r(z_x^n)$. Thus, the maximum principles yields

$$G_n(w, T_n y^n) - \frac{C_{14} T_n d_y^n}{M_n} v_n(w) \geq 0 \qquad \text{in } B_R(z_x^n) \setminus B_r(z_x^n).$$

We continue with

$$G_n(T_n p_x^n, T_n y^n) - \frac{C_{14} T_n d_y^n}{M_n} v_n(T_n p_x^n) = 0.$$

Thus,

$$\partial_{T_n \nu_x^n} G_n(T_n p_x^n, T_n y^n) \geq \frac{C_{14} T_n d_y^n}{M_n} \partial_{T_n \nu_x^n} v_n(T_n p_x^n) \geq C_{14} T_n d_y^n,$$

since $T_n \nu_x^n$ is the inner normal of $B_R(z_x^n) \setminus B_r(z_x^n)$ at p_x^n. Taylor's theorem therefore yields

$$\partial_{T_n\nu_x^n} G_n(T_n x^n, T_n y^n) = \partial_{T_n\nu_x^n} G_n(T_n p_x^n, T_n y^n) + O\left(\sup_{t\in[T_n p_x^n, T_n x^n]} \left(\partial_{T_n\nu_x^n}\right)^2 G_n(t, T_n y^n) T_n d_x^n\right).$$

Using Taylor's theorem another time, we reach

$$\left(\partial_{T_n\nu_x^n}\right)^2 G_n(t, T_n y^n) = \underbrace{\left(\partial_{T_n\nu_x^n}\right)^2 G_n(t, 0)}_{=0} + O\left(\sup_{s\in[0, T_n y^n]} \left(\partial_{T_n\nu_x^n}\right)^2 \partial_{T_n\nu_y^n} G_n(t, s) T_n d_y^n\right).$$

Since $t \in [T_n p_x^n, T_n x^n]$ and $s \in [0, T_n y^n]$ are bound away from each other, we finally reach

$$\partial_{T_n\nu_x^n} G(T_n x^n, T_n y^n) = \underbrace{\partial_{T_n\nu_x^n} G_n(T_n p_x^n, T_n y^n)}_{\geq C_{14} T_n d_y^n} + o\left(T_n d_y^n\right).$$

Thus, we conclude

$$0 \geq \langle \partial_1 G(x^n, y^n), \nu_x^n \rangle = \frac{\partial_{T_n\nu_x^n} G(T_n x^n, T_n y^n)}{|x^n - y^n|^{d-1}} > 0.$$

This finishes the case $\frac{d_y^n}{|x^n - y^n|} = o(1)$.

Next, let $\frac{|x^n - y^n|}{d_y^n} = O(1)$. From Lemma 2.1.3, we see that, because of (A4), we have

$$\partial_1 G(x^n, y^n) = (d-2)c_d \left(\frac{x^n - y^n}{|y^n - x^n|^d} + \frac{x^n - \bar{y}^n}{|x^n - \bar{y}^n|^d}\right) + \partial_2 \psi(y, x).$$

We are interested in $\langle \partial_1 G(x^n, y^n), \nu_x^n \rangle$. Thus, we calculate each part of the sum. The identities $x^n = p_x^n + d_x^n \nu_x^n$ and $\bar{x}^n = p_x^n - d_x^n \nu_x^n$ will be used here.

$$\begin{aligned} \frac{\langle y^n - x^n, \nu_x^n \rangle}{|x^n - y^n|^d} &= \frac{\left\langle p_y^n - p_x^n + d_y^n \nu_y^n - d_x^n \nu_x^n, \nu_x^n \right\rangle}{|x^n - y^n|^d} \\ &= \frac{d_y^n - d_x^n}{|x^n - y^n|^d} + \left(1 - \left\langle \nu_x^n, \nu_y^n \right\rangle\right) \frac{d_y^n}{|x^n - y^n|^d} + \frac{\left\langle p_y^n - p_x^n, \nu_x^n \right\rangle}{|x^n - y^n|^d} \end{aligned}$$

and

$$\begin{aligned} \frac{\langle x^n - \bar{y}^n, \nu_x^n \rangle}{|x^n - \bar{y}^n|^d} &= \frac{\left\langle p_x^n - p_y^n + d_y^n \nu_y^n + d_x^n \nu_x^n, \nu_x^n \right\rangle}{|x^n - \bar{y}^n|^d} \\ &= \frac{d_x^n + d_y^n}{|x^n - \bar{y}^n|^d} + \left(1 - \left\langle \nu_x^n, \nu_y^n \right\rangle\right) \frac{d_y^n}{|x^n - \bar{y}^n|^d} + \frac{\left\langle p_x^n - p_y^n, \nu_x^n \right\rangle}{|x^n - \bar{y}^n|^d}. \end{aligned}$$

In the next step, we estimate

$$\frac{\langle p_y^n - p_x^n, \nu_x^n \rangle}{|x^n - y^n|^d} + \frac{\langle p_x^n - p_y^n, \nu_x^n \rangle}{|x^n - \bar{y}^n|^d}.$$

From (2.1.3), we have $\langle p_x^n - p_y^n, \nu_x^n \rangle = O(|x-y|^2)$. Furthermore, again using Lemma 2.1.3, we have

$$\left(\frac{|x^n - \bar{y}^n|}{|x^n - y^n|} \right)^2 = 1 + \frac{4 d_x^n d_y^n}{|x^n - y^n|^2} + o(1).$$

Now, Taylor's theorem applied to $(1+z)^p = 1 + O(z)$ for $p \geq 1$ yields

$$\left(\left(\frac{|x^n - \bar{y}^n|}{|x^n - y^n|} \right)^d - 1 \right) = O\left(\frac{d_x^n d_y^n}{|x^n - y^n|^2} \right).$$

Thus, we reached

$$\frac{\langle p_y^n - p_x^n, \nu_x^n \rangle}{|x^n - y^n|^d} + \frac{\langle p_x^n - p_y^n, \nu_x^n \rangle}{|x^n - \bar{y}^n|^d} = O\left(\frac{d_x^n d_y^n}{|x^n - \bar{y}^n|^d} \right).$$

Combining all of the calculations, also including (2.1.15), we see

$$\langle \partial_1 G(x^n, y^n), \nu_x^n \rangle = \frac{d_y^n - d_x^n}{|x^n - y^n|^d} + \frac{d_x^n + d_y^n}{|x^n - \bar{y}^n|^d} + \langle \partial_2 \psi(y^n, x^n), \nu_x^n \rangle + o\left(\frac{d_y^n}{|x^n - y^n|^d} \right).$$

Because of $\frac{|x^n - y^n|}{d_y^n} = O(1)$, we have

$$\frac{|x^n - \bar{y}^n|^2}{(d_y^n)^2} = \frac{|x^n - y^n|^2}{(d_y^n)^2} + 4\frac{d_x^n}{d_y^n} + o(1) = O(1)$$

and, thus,

$$\langle \partial_2 \psi(y^n, x^n), \nu_x^n \rangle = O\left(|x^n - \bar{y}^n|^{2-d} \right) = o\left(\frac{d_y^n}{|x^n - y^n|^d} \right).$$

We finally see

$$\frac{|x^n - y^n|^d}{d_y^n} \langle \partial_1 G(x^n, y^n), \nu_x^n \rangle = \underbrace{1 - \frac{d_x^n}{d_y^n}}_{\geq 0} + \underbrace{\left(1 + \frac{d_x^n}{d_y^n} \right) \frac{|x^n - y^n|^d}{|x^n - \bar{y}^n|^d}}_{>0} + o(1) > 0.$$

This finishes (A5), because at least one of the two summands will not tend to 0.

Since we were not able to show

$$\langle \partial_2 \psi(y^n, x^n), \nu_x^n \rangle = o\left(\frac{d_y^n}{|x^n - y^n|^d}\right)$$

in the first case, we had to treat it separately.

Appendix B

Some calculations on the round sphere

In this appendix, we explicitly look at the case $(\Sigma, g) = (S^2, g_{st})$. In that case, the Green's function is explicitly known.

B.1 The Green's Function of the round Sphere

According to [10, 20], we have

$$G(p, q) = -\frac{1}{2\pi} \ln \left(\sin \left(\frac{d_g(p, q)}{2} \right) \right).$$

In spherical coordinates, it is possible to show

$$\Delta_p G(p, N) = \frac{1}{4\pi} = -Vol_2(S^2)^{-1},$$

where $N = (0, 0, 1) \in S^2$ is the north pole. Here, it is useful that

$$\Delta_g f(\varphi, \theta) = \frac{1}{\sin(\varphi)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin(\theta)^2} \frac{\partial^2}{\partial^2 \varphi} f$$

and that $d_g(N, p) = \theta$ if $p = \Phi(\varphi, \theta)$ in spherical coordinates. We omit the exact calculations. The rotational invariance of Δ yields $\Delta_q G(p, q) = -\frac{1}{4\pi}$. Furthermore, using $\sin(t) = t + O(t^2)$ for small t, we see for $p, g \in S^2$ with

$d_g(p,q) \to 0$ that

$$\begin{aligned}G(p,q) &= -\frac{1}{2\pi}\ln\left(\sin\left(\frac{d_g(p,q)}{2}\right)\right) = -\frac{1}{2\pi}\ln\left(\frac{d_g(p,q)}{2}\right) + O(1)\\ &= -\frac{1}{2\pi}\ln(d_g(p,q)) + O(1).\end{aligned}$$

This also yields the Dirac property of G.
At this point, our aim is to bring this into an easier form. So, let $p,q \in S^2$. By rotational invariance of (S^2, g), we assume $p = (1,0) \in \mathbb{C}\times\mathbb{R}$ and $q = (e^{i\phi}, 0) \in \mathbb{C}\times\mathbb{R}$, where $\phi \in [0,\pi]$. Then,

$$d_g(p,g) = d_g(e^0, e^{i\phi}) = \phi.$$

Now, if $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$, we see with the Pythagorean Theorem that

$$\begin{aligned}&\left|1 - e^{i\phi}\right|^2 = \sin(\phi)^2 + (1-\cos(\phi))^2 = 2 - 2\cos(\phi)\\ \Leftrightarrow &\phi = \arccos\left(1 - \frac{\left|1-e^{i\phi}\right|^2}{2}\right) \qquad\qquad \text{for all } \phi \in [0,\pi].\end{aligned}$$

Because $|\cdot|$ is rotationally invariant, this leads to

$$d_g(p,q) = \arccos\left(1 - \frac{|p-q|^2}{2}\right).$$

Next, with $d_g(p,q) \in [0,\pi]$ and the identity

$$\sin^2\left(\frac{t}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos(t),$$

we deduce

$$\begin{aligned}G(p,q) &= -\frac{1}{2\pi}\ln\left(\sin\left(\frac{d_g(p,q)}{2}\right)\right)\\ &= -\frac{1}{2\pi}\ln\left(\sin\left(\frac{\arccos\left(1-\frac{|p-q|^2}{2}\right)}{2}\right)\right)\\ &= -\frac{1}{2\pi}\ln\left(\left(\frac{1}{2} - \frac{1}{2}\left(1 - \frac{|p-q|^2}{2}\right)\right)^{\frac{1}{2}}\right)\\ &= -\frac{1}{4\pi}\ln\left(|p-q|^2\right) + \frac{\ln(4)}{4\pi}.\end{aligned}$$

Thus, we see that

$$G(p,q) = -\frac{1}{4\pi}\ln\left(|p-q|^2\right)$$

is a Green's function of the sphere. Moreover, we need the Robin's function:

$$\begin{aligned}
R(p) &= \lim_{q\to p} G(p,q) + \frac{1}{2\pi}\ln(d_g(p,q)) \\
&= \lim_{q\to p} -\frac{1}{2\pi}\left(\ln\left(\underbrace{\sin\left(\frac{d_g(p,q)}{2}\right)}_{=\frac{d_g(p,q)}{2}+O(d_g(p,q)^2)}\right) - \ln(d_q(p,q))\right) \\
&= \lim_{q\to p} -\frac{1}{2\pi}\ln\left(\frac{1}{2}+o(1)\right) = \frac{\ln(2)}{2\pi}.
\end{aligned}$$

Thus, the map

$$H : \mathcal{F}_N S^2 \to \mathbb{R}, \qquad H(p) := \sum_{i\neq j}\Gamma_i\Gamma_j \ln|p_i - p_j|$$

yields the same dynamics as H_Γ.

B.2 Critical points on the Sphere

In [26, Chapter 4], the map H is further investigated. We explicitly calculate critical points of H. When $N = 3$, the critical points of H are completly characterized. We look into the case $\Gamma_1 = \Gamma_3 > 0 > \Gamma_2$. In this case [26, Ch. 4, Thm 4.2.2] implies $(p_1, p_2, p_3) \in \mathcal{F}_3 S^2$ is a critical point of H, iff

$$\Gamma_1(\Gamma_1+\Gamma_2)p_1 + 2\Gamma_1\Gamma_2 p_2 + \Gamma_1(\Gamma_1+\Gamma_2)p_3 = 0.$$

Thus, we see that p_1, p_2, p_3 lie in some plane. With the rotational invariance, we assume $p_i \in S^1 \times \{0\}$. In addition, we further assume $p_2 = (0, 1, 0)$. We let $p_i = (x_i, y_i, 0)$ for $i = 1, 3$ and infer

$$\begin{aligned}
\Gamma_1(\Gamma_1+\Gamma_2)x_1 + \Gamma_1(\Gamma_1+\Gamma_2)x_3 = 0 \\
\Gamma_1(\Gamma_1+\Gamma_2)y_1 + 2\Gamma_1\Gamma_2 + \Gamma_1(\Gamma_1+\Gamma_2)y_3 = 0.
\end{aligned}$$

Thus, we have $x_1 = -x_3$. This yields $|y_1| = |y_3|$. From the second equation, we see $\Gamma_1(\Gamma_1 + \Gamma_2) \neq 0$ and can deduce

$$y_1 + y_3 = -\frac{2\Gamma_1\Gamma_2}{\Gamma_1(\Gamma_1+\Gamma_2)} = -\frac{2\Gamma_2}{\Gamma_1+\Gamma_2} \neq 0.$$

Because $|y_1| = |y_3|$ and $y_1 + y_3 \neq 0$, we see that

$$y_1 = y_3 = -\frac{\Gamma_2}{\Gamma_1+\Gamma_2}.$$

We finally reach

$$x_1 = -x_3 = \pm\sqrt{1 - \left(\frac{\Gamma_2}{\Gamma_1 + \Gamma_2}\right)^2},$$

if $\left|\frac{\Gamma_2}{\Gamma_1+\Gamma_2}\right| < 1$. Now, it is easy to check that this holds iff

$$\Gamma_1 + 2\Gamma_2 > 0.$$

Thus, the only critical point (up to rotation) is given by

$$\bar{p}_1 = \begin{pmatrix} \sqrt{1 - \left(\frac{\Gamma_2}{\Gamma_1+\Gamma_2}\right)^2} \\ -\frac{\Gamma_2}{\Gamma_1+\Gamma_2} \\ 0 \end{pmatrix}, \quad \bar{p}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{p}_3 = \begin{pmatrix} -\sqrt{1 - \left(\frac{\Gamma_2}{\Gamma_1+\Gamma_2}\right)^2} \\ -\frac{\Gamma_2}{\Gamma_1+\Gamma_2} \\ 0 \end{pmatrix}.$$

Remark. This proves that the assumptions in section 5.3 can not be optimized, as in the Theorem 5.3.1 Σ is allowed to be the sphere. Furthermore, we see that theorem 4.0.1 can not include the sphere, because, if $\Gamma_1 = \Gamma_3 > 0 > \Gamma_2$, the condition of this theorem translates to

$$\begin{aligned} &0 \neq \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 = 2\Gamma_1\Gamma_2 + \Gamma_1^2 \\ \Leftrightarrow &0 \neq 2\Gamma_2 + \Gamma_1. \end{aligned}$$

So if $2\Gamma_2 + \Gamma_1 < 0$ then the assumption of theorem 4.0.1 are satisfied but there does not exist any critical point of H_Γ on the sphere.

Appendix C

Existence and approximation of the Green's function

Let (Σ, g) be a compact riemanian manifold. The aim of this Appendix is to outline the proof of the existence of the (Dirichlet) Green's function of the negative Laplace-Beltrami-Operator. Details are provided in [2, p.101-113]. On the pages [2, p. 101-105], Aubin investigates eigenvalues of $-\Delta = -\nabla^v \nabla_v$ and the existence of solutions to $\Delta u = f$.

Theorem C.0.1. *i) If Σ has no boundary there exists a solution $\varphi \in H_1$ to $\Delta u = f$ iff $\int f dV_g = 0$. The solution φ is unique up to a constant and if $f \in \mathcal{C}^{k+\alpha}$ for $k \in \mathbb{N}$ and $\alpha \in (0,1)$ then $\varphi \in \mathcal{C}^{k+2+\alpha}$.*

ii) If Σ has boundary then there exists a unique solution $\varphi \in \mathring{H}_1(\Sigma)$ to $\Delta u = f$. If $f \in \mathcal{C}^\infty(\Sigma)$ then also $\varphi \in \mathcal{C}^\infty(\Sigma)$ and $\varphi|_{\partial\bar{\Sigma}} \equiv 0$.

Definition C.0.2. i) If Σ is closed with volume $vol_g(\Sigma) =: V$ then the Green's function to the negative Laplace-Beltrami operator is a function that satisfies

$$-\Delta_q G(p,q) = \delta_p(q) - V^{-1}$$

in a distributional sense, where δ_p is the Dirac function at p. In this case G is only unique up to a map $p \mapsto w(p)$ or up to a constant if we call for the symmetry $G(p,q) = G(q,p)$.

ii) If $\overline{\Sigma}$ has boundary then the (Dirichlet) Green's function of the negative Laplace-Beltrami-Operator is the unique map G such that

$$\Delta_q G(p,q) = \delta_p(q) \qquad \text{on } \Sigma \times \Sigma$$

and vanishes on the boundary where $p \in \partial\Sigma$ or $q \in \partial\Sigma$.

If we define

$$\Psi = \Psi_d : (0,\infty) \to \mathbb{R}, \qquad \Psi(r) := \begin{cases} c_d r^{2-d}, & \text{if } d \geq 3, \\ -\frac{1}{2\pi}\ln(r), & \text{if } d = 2 \end{cases}$$

for $2 \leq d \in \mathbb{N}$ and $c_d := \frac{1}{d(d-2)vol_d(B_1(0))}$, then the map $(x,y) \mapsto \Psi \circ |x-y|$ is the fundamental solution of the negative Laplacian in $\mathbb{R}^d$. This leads to the idea to define $r(p,q) := d_g(p,q)$ and that $\Psi \circ r$ has to be the leading part of G. The problem here is that r is not $\mathcal{C}^\infty$ on $\mathcal{F}_2\Sigma$ and we thus need to change it a little. We define a positive decreasing cut-off map $\alpha : \mathbb{R} \to \mathbb{R}$ where $\alpha \equiv 1$ in a neighborhood of 0 and $\alpha \equiv 0$ on $[\delta, \infty)$ where $\delta > 0$ is the injectivity radius of Σ. We then define

$$H(p,q) = H_d(p,q) := \Psi_d(d_g(p,q)) \cdot \alpha\left(d_g(p,q)\right).$$

Remark that

$$\Delta f = \frac{1}{\sqrt{\det(g)}}\partial_i\left(g^{i,j}\partial_j f\right)$$

where $g^{-1}(x) = \left(g^{i,j}(x)\right)_{i,j=1}^d$ and we used the Einstein sum convention. With this we can calculate $\Delta_q H(p,q)$ and see that

$$|\Delta_q H(p,q)| \leq C_1 d_g(p,q)^{2-d}. \tag{C.0.1}$$

Before we state the existence theorem we also want to give an important lemma so the construction will work.

Lemma C.0.3. *Let Ω be a bounded open set of $\mathbb{R}^d$ and let $X, Y : \mathcal{F}_2\Omega \to \mathbb{R}$ be maps that satisfy*

$$|X(p,q)| \leq Const \cdot (d(p,q))^{\alpha-d} \ \ and \ \ |Y(p,q)| \leq Const \cdot (d(p,q))^{\beta-d}$$

for some $\alpha, \beta \in (0,d)$. Then the map

$$Z : \mathcal{F}_2\Omega \to \mathbb{R}, \qquad Z(p,q) := \int_\Omega X(p,s)Y(s,q)\,dV(s)$$

is continious and satisfies

$$\begin{aligned} &|Z(p,q)| \leq Const \cdot d(p,q)^{\alpha+\beta-d} && \text{if } \alpha+\beta < n \\ &|Z(p,q)| \leq Const \cdot (1 + |\ln(d(p,q))|) && \text{if } \alpha+\beta = n \\ &|Z(p,q)| \leq Const && \text{if } \alpha+\beta > n. \end{aligned}$$

Theorem C.0.4. *If Σ is a closed manifold then there exists a Green's function* *to the negative Laplace-Beltrami-Operator* $G : \mathcal{F}_2\Sigma \to \mathbb{R}$ *with the following properties.*

a) For all functions $\varphi \in \mathcal{C}^2$ *there holds*

$$\varphi(p) = V^{-1} \int_\Sigma \varphi(q)\, dV(q) - \int_\Sigma G(p,q)\Delta\varphi(q)\, dV(q).$$

b) G *is* $\mathcal{C}^\infty$.

c) There exists a constant $C > 0$ *such that*

$$\begin{aligned} |G(p,q)| &\leq C(1 + |\ln(d_g(p,q))|), \text{ if } d = 2 \\ |G(p,q)| &\leq C d_g(p,q)^{2-d}, \text{ if } d > 2 \\ |\nabla_q G(p,q)| &< C d_g(p,q)^{1-d}, \\ \left|\nabla_q^2 G(p,q)\right| & y C d_g(p,q)^{-d}. \end{aligned}$$

d) There exists a constant A *such that* $G(p,q) \geq A$. *Since* G *is only defined up to a constant we may choose* $A > 0$.

e) $\int_\sigma G(p,q)\, dV(q) \equiv Const.$ *We may choose* G *such that this integral vanishes.*

f) $G(p,q) = G(q,p)$.

Proof: First, we define

$$\Gamma_1(p,q) := \Delta_q H(p,q) \text{ and } \Gamma_{i+1}(p,q) := \int_\Sigma \Gamma_i(p,s)\Gamma_1(s,q)\, dV(s).$$

Then, we choose $\frac{d}{2} < k \in \mathbb{N}$ and define

$$G(p,q) := H(p,q) + \sum_{i=1}^{k} \int_\Sigma \Gamma_i(p,r)H(r,q)\, dV(r) + F(p,q)$$

where F shall satisfy

$$-\Delta_q F(p,q) = \Gamma_{k+1}(p,q) - V^{-1}.$$

With the lemma above and $|\Gamma_1(p,q)| \leq C_1 d_g(p,q)^{2-d}$, we see that Γ_k is bounded and thus, Γ_{k+1} is a $\mathcal{C}^1$ map. This, then leads to the fact that a map F like this exists and is unique up to a constant. The rest of the proof is to verify the points a)-f) which we omit here.

$\square$

Remark. If $d = 2$ we see in this construction that $G(p, q) = H(p, q) + h(p, q)$ for a bounded map h.
In [2] follows a similar result for manifolds with boundary. But because we gave a proof of the existence of G in Lemma 3.2.1 we omit the rest. We may just remark that again $G \geq 0$ and $G(p, q) = G(q, p)$ when $\partial\Sigma \neq \emptyset$.

Appendix D

The Green's function of balls

For $R > 0$, we calculate the Green's function of $U_R(0) \subset \mathbb{R}^d$ for $d \geq 3$. Our calculations are not new and can be found in for instance [18]. We need to find a function $h(x, y)$ that satisfies

$$\begin{cases} \Delta_x h(x,y) = 0 & \text{in } U_R(0) \\ h(x,y) = -c_d \left|x-y\right|^{2-d} & \text{on } \partial U_R(0). \end{cases}$$

We define

$$\tilde{\cdot} : B_R(0) \setminus \{0\} \to \mathbb{R}^2, \qquad \tilde{y} := \frac{R^2}{y^2} y.$$

Then, we let

$$h(x,y) := \begin{cases} c_d \left|\frac{|y|}{R}(x-\tilde{y})\right|^{2-d} & \text{if } y \neq 0 \\ c_d R^{2-d} & \text{if } y = 0. \end{cases}$$

Because of $\Delta \left|x\right|^{2-d} = 0$, we see that $\Delta_x h(x,y) = 0$. If $y = 0$ and $x \in \partial U_R(0)$, we see $h(x,0) = -c_d \left|x-y\right|^{2-d}$. If $y \neq 0$ and $x \in \partial U_R(0)$ we see with $\left|x-y\right|^2 = \left|x\right|^2 + \left|y\right|^2 - 2\langle x, y\rangle$ that

$$\frac{|y|}{R}\left|x-\tilde{y}\right| = \left|x-y\right|.$$

Hence, we have that

$$G(x,y) = c_d \left|x-y\right|^{2-d} + h(x,y) = \begin{cases} c_d \left|x-y\right|^{2-d} - c_d \left(\frac{|y|}{R}\left|x-\tilde{y}\right|\right)^{2-d} & \text{if } y \neq 0 \\ c_d \left|x-y\right|^{2-d} - c_d R^{2-d} & \text{if } y = 0 \end{cases}$$

is the Green's function of $U_R(0)$. Now, let $x \in \partial U_R(0)$. Then, $\nu_x = -\frac{x}{R}$ is the inner normal at x. Next, we calculate

$$\left\langle \nabla_x |x-y|^{2-d}, -\frac{x}{R} \right\rangle = \left\langle (2-d)\frac{x-y}{|x-y|^d}, -\frac{x}{R} \right\rangle = (d-2)\frac{R^2}{R\,|x-y|^d} - (d-2)\frac{\langle x, y\rangle}{R\,|x-y|^d}.$$

For $y = 0$, we see

$$\langle \nabla_x G(x,y), \nu_x \rangle = c_d(d-2)\frac{R^2 - |y|^2}{R\,|x-y|^d}.$$

So, let $y \neq 0$. First, remember that $|x-y| = \frac{|y|}{R}\,|x-\tilde{y}|$, because of $|x| = R$. Thus, we see

$$\begin{aligned}
\left\langle \nabla_x \left(\frac{|y|}{R}\,|x-\tilde{y}|\right)^{2-d}, -\frac{x}{R} \right\rangle &= \left\langle (2-d)\left(\frac{|y|}{R}\right)^{2-d} \frac{x-\tilde{y}}{|x-\tilde{y}|^d}, -\frac{x}{R} \right\rangle \\
&= \left\langle (2-d)\frac{|y|^2}{R^2}\frac{x-\tilde{y}}{|x-y|^d}, -\frac{x}{R} \right\rangle \\
&= (d-2)\frac{|y|^2}{R\,|x-y|^d} - (d-2)\frac{\langle x, y\rangle}{R\,|x-y|^d}.
\end{aligned}$$

We see

$$\langle \nabla_x G(x,y), \nu_x \rangle = c_d(d-2)\frac{R^2 - |y|^2}{R\,|x-y|^d}$$

again.

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