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Dissertation

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# An equivariant degree and periodic solutions of the $N$-vortex problem

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Gießen, February 28, 2018

# Abstract

We examine the $N$-vortex problem on general domains $\Omega \subset \mathbb{R}^2$ concerning the existence of nonstationary collision-free periodic solutions. The problem in question is a first order Hamiltonian system of the form

$$\Gamma_k \dot{z}_k = J\nabla_{z_k} H(z_1, \dots, z_N), \quad k = 1, \dots, N,$$

where $\Gamma_k \in \mathbb{R} \setminus \{0\}$ is the strength of the $k$th vortex at position $z_k(t) \in \Omega$, $J \in \mathbb{R}^{2\times 2}$ is the standard symplectic matrix and

$$H(z_1, \dots, z_N) = -\frac{1}{2\pi} \sum_{\substack{k,j=1 \\ k\neq j}}^{N} \Gamma_j \Gamma_k \log |z_k - z_j| - \sum_{k,j=1}^{N} \Gamma_j \Gamma_k g(z_k, z_j)$$

with some regular and symmetric, but not explicitely known function $g : \Omega \times \Omega \to \mathbb{R}$.

We present two types of periodic solutions that can be found in general domains. The first one is based on the idea to superpose a stationary solution of a system of less than $N$ vortices and several clusters of vortices that are close to rigidly rotating configurations of the whole-plane system. The second type consists of choreographic solutions following approximately a boundary component of the domain. The proofs in both cases rely on a suitable rescaling of the problem, investigation of the limiting system and implicit-function-like methods for a local continuation of existing solutions. Moreover, the modification of a $S^1$-equivariant degree theory allows us to prove that the continuation occurs globally.

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# Notation

## Miscellaneous

Let $U, V$ be subsets of a Banachspace $X$. Then $\overline{U} = \operatorname{clos}(U)$, $U^\circ = \operatorname{int}(U)$, $\partial U$ denote the closure, the interior and the boundary of $U$ and $\operatorname{dist}(U, V) = \inf \{ \|u - v\|_X : u \in U, v \in V \}$. An inner product on $X$ is usually denoted by $\langle \cdot, \cdot \rangle_X$. In the context of fluid dynamics we also use the notation $x \cdot y = x_1 y_1 + x_2 y_2$ for the euclidian scalar product $\langle x, y \rangle_{\mathbb{R}^2}$. The transpose of a matrix $A \in \mathbb{R}^{m \times n}$ is written $A^T$. Commonly used matrices are the $n \times n$ identity matrix $\mathrm{id}_{\mathbb{R}^n}$, as well as

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the rotation by $-\frac{\pi}{2}$ on $\mathbb{R}^2$ and the $2N \times 2N$ block matrices

$$J_N = \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}, \qquad M_\Gamma = \begin{pmatrix} \Gamma_1 \, \mathrm{id}_{\mathbb{R}^2} & & \\ & \ddots & \\ & & \Gamma_N \, \mathrm{id}_{\mathbb{R}^2} \end{pmatrix},$$

where $\Gamma_1, \ldots, \Gamma_N \in \mathbb{R} \setminus \{0\}$ denote the vorticities of the point vortices.

## Derivatives

Partial derivatives are written like $\partial_t \omega(x,t)$, $\partial_1 \omega(x,t) = \partial_{x_1} \omega(x,t)$ or $\partial_r F(r,u)$, $D_u F(r,u)$. As usual $\nabla \Phi$ denotes the gradient of a real valued function $\Phi$ defined on a Hilbertspace $X$ and $\nabla^2 \Phi$ the Hessian matrix or more general the linear and continuous map $X \to X$, such that $\left\langle \nabla^2 \Phi(x) u, v \right\rangle_X = D^2 \Phi(x)[u, v]$. Higher order derivatives are for example written like $D_u^k F(r,u)$, $\nabla^k F_r(u)$ or $F_r^{(k)}(u)$, and for derivatives with respect to time we use $\dot{u}(t)$ or $\frac{d}{dt} u(t)$.

Concerning classical differential operators we have divergence $\operatorname{div} v = \nabla \cdot v = \partial_1 v_1 + \partial_2 v_2$, Laplace $\Delta u = \operatorname{div}(\nabla u)$ and rotation $\operatorname{curl} v = \partial_1 v_2 - \partial_2 v_1$, if $v$ is a two-dimensional vector field.

For a smooth function $G : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto G(x, y)$ we write $\nabla_1 G$, $\nabla_1^2 G$ for the gradient and Hessian with respect to $x$ and

$$\nabla_2 \nabla_1 G = \begin{pmatrix} \partial_{y_1} \partial_{x_1} G & \partial_{y_2} \partial_{x_1} G \\ \partial_{y_1} \partial_{x_2} G & \partial_{y_2} \partial_{x_2} G \end{pmatrix}.$$

In a similar way one has to understand $\nabla_2 G$, $\nabla_2^2 G$ and $\nabla_1 \nabla_2 G$. Note that if $G$ is symmetric, i.e. $G(x, y) = G(y, x)$, then

$$\nabla_2 \nabla_1 G(x, y) = \left( \nabla_2 \nabla_1 G(y, x) \right)^T. \tag{0.1}$$

In the context of fluid dynamics we use $\nabla^\perp \Phi = J \nabla \Phi$ and $\nabla^\perp G(x, y) = J \nabla_1 G(x, y)$.

## Spaces

| | |
|---|---|
| $L^2_T$ | square integrable functions $u : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^{2N}$, $\langle u, v\rangle_{L^2_T} = \int_0^T \langle u(t), v(t)\rangle_{\mathbb{R}^{2N}}\ dt$ |
| $H^s_T$ | Sobolev space $H^s(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^{2N})$ of $T$-periodic functions, for $s = 1$: $\langle u, v\rangle_{H^1_T} = \langle u, v\rangle_{L^2_T} + \langle \dot{u}, \dot{v}\rangle_{L^2_T}$ |
| $C^0_T; C^k_T; C^\infty_T$ | continuous; $k$-times continuously differentiable; smooth $T$-periodic functions $u : \mathbb{R} \to \mathbb{R}^{2N}$, $\|u\|_{C^k_T} = \sup\left\{ \left\|u^{(j)}(t)\right\| : 0 \le j \le k,\ t \in \mathbb{R} \right\}$ |
| $L^2, H^s, C^k$ | $= L^2_{2\pi}$, $= H^s_{2\pi}$, $= C^k_{2\pi}$ |
| $\mathcal{L}(X, Y)$ | linear and continuous operators between two Banach spaces $X, Y$, $\|L\|_{\mathcal{L}(X,Y)} = \sup\{ \|Lx\|_Y : \|x\|_X \le 1 \}$ |
| $\mathcal{L}(X)$ | $= \mathcal{L}(X, X)$ |
| $C^k(U, Y)$ | $k$-times continuously differentiable functions from $U \subset X$ into $Y$ |
| $C^{k,\alpha}(U, Y)$ | $C^k(U, Y)$ with $k$th derivative being $\alpha$-Hölder continuous |

## Topological degrees

| | |
|---|---|
| deg | Brouwer degree or Leray-Schauder degree |
| $S^1$-deg, $\mathrm{d}_k$ | $S^1$-equivariant degree by Dylawerski, Gęba, Jodel, Marzantowicz, see 2.3.1 |
| $S^1$-$\mathrm{deg}^\perp$, $\mathrm{d}^\perp_k$ | degree for $S^1$-orthogonal maps by Rybicki, see 2.3.2 |
| $S^1$-$\mathrm{deg}^\nabla$, $\mathrm{d}^\nabla_k$ | modification of $S^1$-$\mathrm{deg}^\perp$ to $S^1$-equivariant gradients, see 2.1.2 |

# Chapter 1

# Introduction

## 1.1 The $N$-vortex problem

In order to describe an incompressible fluid contained in a domain $\Omega$ one uses partial differential equations like the Navier-Stokes equation or in the nonviscous case the Euler equation

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla p, \\ \operatorname{div} v = 0 \end{cases} \tag{1.1}$$

as most sophisticated models. Here $v(x,t)$ denotes the velocity of the fluid and $p(x,t)$ the pressure at the point $x \in \Omega$ and at time $t \in \mathbb{R}$. Since in general the Navier-Stokes and Euler equation are quite complicated to deal with – we just like to mention [33] – simplified models based on these equations are used in the hope of gaining information about the original equations and/or to describe phenomena in applications in a precise enough manner. One of these simplified models is the so called $N$-vortex problem.

Here as a first simplification one considers a two-dimensional fluid, i.e. $\Omega \subset \mathbb{R}^2$ and $v : \Omega \times \mathbb{R} \to \mathbb{R}^2$. The restriction of the fluid to two dimensions is a reasonable approximation when one of the dimensions is comparably small in relation to the other two, or more generally when the three-dimensional flow is confined to two-dimensional layers due to stratification or rotation. Assuming the two-dimensional fluid to be nonviscous and contained in a smooth domain with an impenetrable boundary, it is described by the Euler equations (1.1) and an additional boundary condition that requires the fluid to be tangential to $\partial\Omega$. A solution $v$ of this boundary value problem can be found by solving the $2D$-Euler equations in vorticity-stream formulation, i.e. finding scalar functions $\omega(x,t)$, $\Psi(x,t)$ satisfying

$$\begin{cases} \partial_t \omega + \nabla^\perp \Psi \cdot \nabla \omega = 0, & \text{in } \Omega \\ -\Delta \Psi = \omega, & \text{in } \Omega \\ \Psi = 0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

and setting $v = \nabla^\perp \Psi$, see Section C.1. One next assumes that the whole velocity field $v(\cdot,t)$ at any time $t$ is solely determined by the position of finitely many vortices. Speaking in terms of vorticity this means that $\omega(\cdot,t)$ is highly concentrated in finitely many points $z_1(t), \dots, z_N(t) \in \Omega$. Let $\Gamma_j \in \mathbb{R} \setminus \{0\}$ be the amount of vorticity located around $z_j(t)$. The corresponding formal ansatz $\omega(x,t) = \sum_j \Gamma_j \delta(x - z_j(t))$ for (1.2) leads to a system of ordinary differential equations describing the motion of the vortex positions in time. This so called $N$-vortex system is of the form

$$\Gamma_j \dot{z}_j(t) = J \nabla_{z_j} H_\Omega(z_1(t), \dots, z_N(t)), \quad j = 1, \dots, N, \tag{1.3}$$

where $J : \mathbb{R}^2 \to \mathbb{R}^2$ denotes rotation around the origin by $-\frac{\pi}{2}$, and $H_\Omega$ defined on an open subset of $\Omega^N$ is a real valued function determined by the stated singular ansatz for the vorticity profile. The derivation of the $N$-vortex problem has its origin in the 19th century and is, depending on the considered case, due to Kirchhoff [47], Routh [68] and Lin [53, 54], but can also be found in more modern books, e.g. [34, 56, 57, 64]. An historic overview of the derivation of the point vortex system together with advanced models can be found in [55]. In appendix C we present the localization theorem of Marchioro and Pulvirenti [59] as a rigorous justification for the point vortex system.

The simplification of the dynamics of a fluid to the motion of finitely many point vortices has a wide range of applications. In Geophysics system (1.3) serves as a simple model for the interaction of ocean eddies with coastlines, see [21, 24, 73], or for an explanation of vortex configurations in the eye of hurricane Isabel (2003), see [48] and chapter 3 of [9]. Point vortex models, typically with a high number of vortices, are also used in numerical simulations of liquids and gases of various kinds, for example to simulate the locomotion of a fish or insect [30] or to create computer animations in video games [41].

Furthermore, the $N$-vortex problem does not only occur as a singular limit of the Euler equations, but also as a limit of other partial differential equations from mathematical physics. If for example the initial data $u_0^\varepsilon = u^\varepsilon(\cdot,0) : \Omega \longrightarrow \mathbb{C}$ of the Gross-Pitaevskii (also called Ginzburg-Landau-Schrödinger) equation

$$\begin{cases} i\partial_t u^\varepsilon - \Delta u^\varepsilon = \varepsilon^{-2}\left(1-|u^\varepsilon|^2\right)u^\varepsilon, & \text{in } \Omega \\ u^\varepsilon(\cdot,t) = f, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

$f : \partial\Omega \longrightarrow \mathbb{C}$ some prescribed function and $t \in \mathbb{R}$, has independent of $\varepsilon > 0$ only isolated zeroes, say $a_1,\ldots,a_N \in \Omega$, with local Brouwer indices having modulus 1, then as $\varepsilon \longrightarrow 0$ the corresponding solution $u^\varepsilon(\cdot,t)$ has zeroes $z_1^\varepsilon(t),\ldots,z_N^\varepsilon(t)$ following the solution of (1.3) with initial data $z_j^\varepsilon(0) = a_j$, $j = 1,\ldots,N$. So the zeroes of the solution of (1.4) behave in the limit like point vortices. The corresponding vorticities $\Gamma_j$ are here given by the local Brouwer degrees $\deg\left(u_0^\varepsilon, B_\rho(a_j)\right) \in \{\pm 1\}$, $\rho > 0$ sufficiently small. Details concerning this motivation for point vortex dynamics can be found in [22, 45] and the references therein. Another equation giving rise to point vortex like dynamics is the Landau-Lifshitz-Gilbert equation, for which we just like to refer to [51].

## 1.2 Central question and related results

The $N$-vortex problem (1.3) is a first order Hamiltonian system with Hamilton function

$$H_\Omega(z_1,\ldots,z_N) = -\frac{1}{2\pi}\sum_{\substack{k,j=1\\k\neq j}}^{N}\Gamma_j\Gamma_k \log\left|z_j - z_k\right| - \sum_{k,j=1}^{N}\Gamma_j\Gamma_k\, g_\Omega(z_j,z_k).$$

Since the first term, which models direct vortex-vortex interactions, becomes singular when $z_j = z_k$, $j \neq k$, $H_\Omega$ is only defined on

$$\mathcal{F}_N(\Omega) = \left\{ z = (z_1,\ldots,z_N) \in \Omega^N : z_j \neq z_k,\ j \neq k \right\}.$$

Besides these logarithmic singularities, a second main difficulty of the Hamiltonian lies in the fact that the function $g_\Omega : \Omega \times \Omega \longrightarrow \mathbb{R}$ contained in the seond term and modelling vortex-boundary interactions is, except for a few special cases, not explicitly known. More

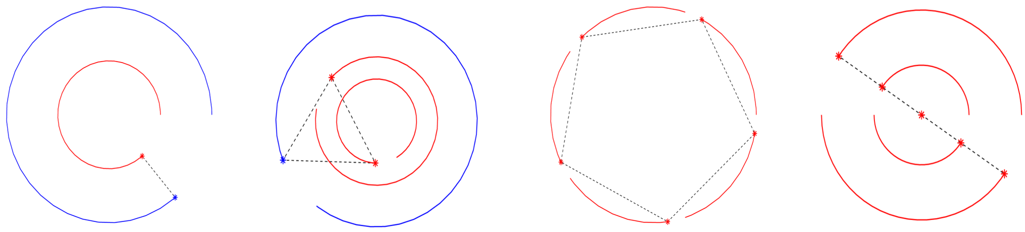

FIGURE 1.1: Relative equilibria solutions on $\Omega = \mathbb{R}^2$: Vortices with $\Gamma_j < 0$ are blue, vortices with $\Gamma_j > 0$ are red, all configurations rotate in counterclockwise direction.

precisely $g_\Omega(\cdot, y)$ satisfies for every $y \in \Omega$ the boundary value problem

$$\begin{cases} \Delta_x g_\Omega(x,y) = 0, & x \in \Omega \\ g_\Omega(x,y) = -\frac{1}{2\pi} \log|x-y|, & x \in \partial\Omega, \end{cases}$$

such that

$$G_\Omega(x,y) = -\frac{1}{2\pi} \log|x-y| - g_\Omega(x,y)$$

is the Green's function of the Dirichlet-Laplace operator. Contrary to the singular part of $G_\Omega$ the evaluation of $g_\Omega$ at the same point is allowed. The function defined by this evaluation $h_\Omega : \Omega \to \mathbb{R}$,

$$h_\Omega(x) = g_\Omega(x,x)$$

is called Robin function and determines the motion of a single vortex inside the domain $\Omega$. Due to $h_\Omega(x) \to \infty$ as $x \to \partial\Omega$ singularities of $H_\Omega$ not only occur when different vortices collide, but also when vortices approach the boundary of $\Omega$. A summary concerning further properties of the Green's function and the Robin function can be found in Appendix B. In particular Section B.4 treats so called hydrodynamic Green's functions, which appear naturally in point vortex dynamics as a generalization of the Dirichlet Green's function.

In this thesis we will investigate the $N$-vortex problem in general domains with respect to a classic aspect of Hamiltonian dynamics: Periodic solutions. We will mainly focus on the question of existence of periodics, but also elaborate a little on their structure as a set of solutions. The discussion of natural related questions like stability is currently work in progress and not included in the thesis. The periodic solutions investigated here are all nontrivial, i.e. have a positive minimal period, nonetheless an overview about stationary solutions can be found in section 3.1.1.

There is a vast amount of literature concerning periodic solutions of the $N$-vortex problem, dating back to the 19th century when Thomson investigated in [75] regular polygon configurations of identical vortices. Nevertheless, almost all of those publications treat special cases, in which the Green's function and hence the Hamiltonian $H_\Omega$ is explicitly known. This is for example the case for $\Omega$ being the whole plane $\mathbb{R}^2$, the upper half-plane $\mathbb{R}^2_+$ or the unit disc $B_1(0)$. Most of these solutions form relative equilibrium solutions, also called vortex crystals, which are vortex configurations that rigidly rotate around a central point. Examples on the whole plane include vortex pairs, equilateral triangles, Thomson's regular $N$-Gons and straight line configurations, see figure 1.1 for now and more detailed section 3.1.2. Even more complicated nested configurations can be found on $\mathbb{R}^2$ or $B_1(0)$ due to the fact that the Green's function is explicit and invariant with respect to rotations. We refer to [4, 5, 64] for the illustrated examples and a general overview.

Contrary to these cases both advantages are lost when one is interested in the dynamics in an arbitrary domain $\Omega \subset \mathbb{R}^2$. Moreover, in general the Hamiltonian $H_\Omega$ is unbounded from both sides, not integrable and has non compact, not metrically complete energy surfaces which makes the search for periodic solutions difficult. An exception is given by the two following cases, in which the definite asymptotic behaviour $H_\Omega(z) \to -\infty$ as $z \to \partial\mathcal{F}_N(\Omega)$ implies the compactness of energy levels:

- $\Omega$ bounded, $N = 1$, $\Gamma_1 \neq 0$: Almost all solutions of the 1-vortex system are periodic, see section 15.5 in [34],
- $\Omega$ bounded, $N = 2$, $\Gamma_1\Gamma_2 < 0$: Almost all non empty level sets $H_\Omega^{-1}(c)$ contain a periodic solution, [74].

The first result going beyond these two cases is due to Bartsch and Dai [11] and treats the case of arbitrarily many identical vortices located near a topological stable critical point of the Robin function $h_\Omega$. A critical point of $h_\Omega$, which is nothing but an equilibrium of the 1-vortex system, can be found in every bounded domain. Moreover, at least after an arbitrarily small deformation of the domain it is nondegenerate and hence topological stable, see [15]. The theorem in a formulation not including all details then reads:

**Theorem 1.1** (Bartsch, Dai [11]). *Let $\Gamma_1 = \ldots = \Gamma_N \neq 0$, $a \in \Omega$ be a topological stable critical point of $h_\Omega$. Then there exists a family $\left(z^{(r)}\right)$, $0 < r < r_0$ of periodic solutions of* (1.3) *with the following properties: For any $r \in (0, r_0)$ all $N$ vortices $z_1^{(r)}(t), \ldots, z_N^{(r)}(t)$ follow the same curve, i.e. the solutions form a choreography. In the limit $r \to 0$ the minimal period $T_r$ tends to $0$ and all vortices converge towards the critical point $a$, whereas the geometrical shape of the configuration approaches a scaled version of Thomson's $N$-Gon configuration.*

This result can be interpreted as the superposition of a stationary solution of the 1-vortex system on $\Omega$ and the $N$-Gon solution on $\mathbb{R}^2$. A natural question arises if similar solutions can be found when the stationary solution $a$ is replaced by a nontrivial periodic trajectory $a(t)$ of the 1-vortex system. For two vortices Bartsch and Sacchet could prove the following result, which we will here again formulate only in a rough way:

**Theorem 1.2** (Bartsch, Sacchet [18]). *Let $N = 2$, $\Gamma_1, \Gamma_2 \neq 0$ with $\Gamma_1 + \Gamma_2 = 0$ and $a(t)$ be a $T$-periodic solution of the 1-vortex system on $\Omega$. Under a geometric condition on $a(t)$ and nearby periodic trajectories of the 1-vortex system, the 2-vortex system on $\Omega$ has infinitely many $T$-periodic solutions, in which the two vortices rotate around their center of vorticity while the center itself approximately follows the trajectory $a(t)$.*

Contrary to the methods used in [11] and the methods that will be used in this thesis, Theorem 1.2 requires a somewhat different approach due to the time-dependent trajectory $a(t)$. In fact it relies on a generalized Poincaré-Birkhoff theorem [36].

## 1.3 Outline of the thesis

In chapter 3 we continue the search for periodic solutions via a superposition of two kinds of solutions. In particular, we generalize Theorem 1.1 in three aspects. First of all, we show that the Thomson $N$-Gons are not the only relative equilibrium solutions of the whole plane system that give rise to a family of periodic solutions $\left(z^{(r)}\right)_{r\in(0,r_0)}$ near a critical point of the Robin function $h_\Omega$. This way also configurations with different vorticities are shown to induce periodic solutions of (1.3). Next using an appropriate equivariant degree theory, which is based on the degree by Rybicki [69] and which represents in chapter 2 besides the periodic solutions the second main part of this thesis, we show that these families of solutions are not only local families but also part of a global connected set of periodic solutions. Moreover,

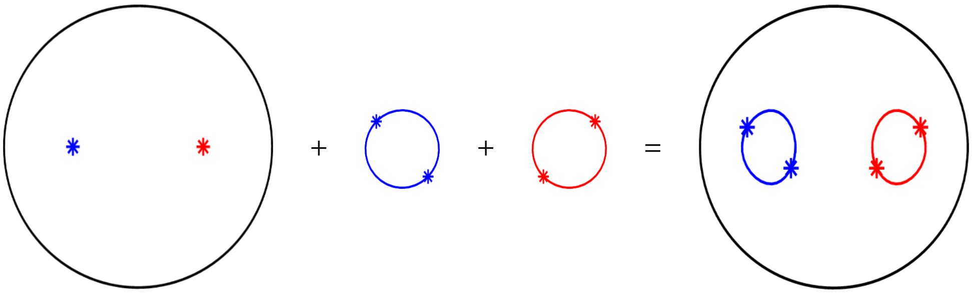

FIGURE 1.2: The 2-vortex problem in the unit disc admits a stationary solution with $\Gamma_1 = -\Gamma_2$, cf. Example 3.2. Combining this solution with rigidly rotating vortex pairs we obtain a periodic solution of the 4-vortex system in the disc, where each pair of vortices moves along a deformed circle. The shown trajectory is the actual numerically computed trajectory of the 4-vortex problem in the unit disc.

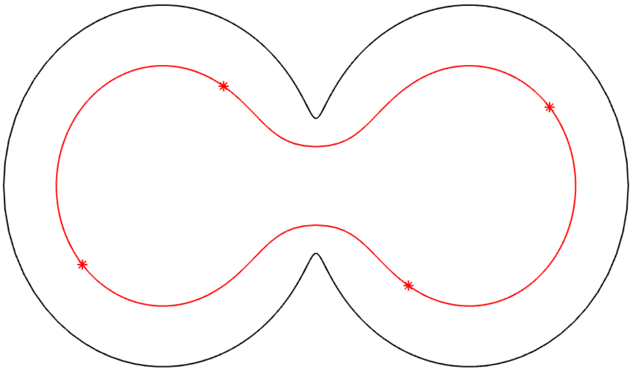

FIGURE 1.3: Four identical vortices in a Neumann oval domain following the red curve in a counterclockwise orientation. The numerical computation is based on code by Tom Ashbee [6].

by replacing the stationary solution $a \in \Omega$ of the 1-vortex system by a stationary solution of a system of $m \in \mathbb{N}$ vortices and placing a rigidly rotating configuration of the whole plane system near each of the $m$ point vortices, we obtain periodic solutions consisting of $m$ clusters. Figure 1.2 shows this idea in the easiest case of $m = 2$ clusters, each consisting of two identical vortices and $\Omega$ being the unit disc. This way we obtain periodic solutions in the unit disc, in which the vortices are not rotating rigidly around the center of the disc. The precise statement of the described results is given in Theorems 3.1, 3.8 and 3.9.

The existence of the presented solutions so far is based on the fact that vortex-vortex interactions dominate the dynamics when vortices come close together. Contrary to that we will exploit vortex-boundary interactions in Chapter 4 in order to obtain periodic solutions for an arbitrary number of identical vortices in a simply connected domain. Here the vortices move separated by time shifts along the same curve close to the domain boundary, i.e. the solutions are also choreographies. As an illustration, the trajectory of 4 identical vortices close to the boundary of a Neumann oval domain is presented in figure 1.3. The rigorous result can be found in Theorem 4.3.

Both types of results are obtained by continuation of existing periodic solutions. Let

$$M_\Gamma = \begin{pmatrix} \Gamma_1 \operatorname{id}_{\mathbb{R}^2} & & \\ & \ddots & \\ & & \Gamma_N \operatorname{id}_{\mathbb{R}^2} \end{pmatrix} \in \mathbb{R}^{2N\times 2N}, \quad J_N = \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix} \in \mathbb{R}^{2N\times 2N},$$

such that the $N$-vortex system (1.3) can be written in the more compact way

$$M_\Gamma \dot{z} = J_N \nabla H_\Omega(z), \tag{1.5}$$

where $z = (z_1(t), \dots, z_N(t)) \in \mathcal{F}_N(\Omega)$. Depending on the situation a suitable rescaling shows that (1.5) is equivalent to

$$M_\Gamma \dot{u} = J_N \nabla H_r(u) \tag{1.6}$$

with a parameter dependent family $(H_r)_{r\in(0,r_0)}$ of Hamiltonians. In particular a $2\pi$-periodic solution of (1.6) corresponds to a periodic solution of (1.5) with period $T = T(r)$ depending on the introduced parameter $r$. So the here presented existence results rely on the investigation of the limiting problem $r \longrightarrow 0$, for which $2\pi$-periodic solutions are known or shown to exist, as well as appropriate continuation methods. More precisely we will find for $r > 0$ critical points of the associated action functional

$$\Phi_r(u) = \frac{1}{2}\int_0^{2\pi} \langle M_\Gamma \dot{u}, J_N u\rangle_{\mathbb{R}^{2N}}\ dt - \int_0^{2\pi} H_r(u)\, dt$$

emanating from a critical manifold of $\Phi_0$ by applying implicit-function-like theorems, e.g. Theorem 2.7 and/or degree arguments, see Theorem 2.3 and Corollary 2.9, to the gradient $\nabla\Phi_r$. Although we do not use variational methods based for example on Morse-theory or a Linking structure, which seem quite hard to apply due to the indefinite behaviour of $H_\Omega(z)$ when $z \longrightarrow \partial\mathcal{F}_N(\Omega)$, the proofs rely on the existence of a variational structure associated to (1.6). The functional $\Phi_r$ will be defined on an open subset of the Sobolev space $H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^{2N})$ consisting of $2\pi$-periodic functions having a square-integrable weak derivative. Since the natural domain of the quadratic form

$$u \mapsto \frac{1}{2}\int_0^{2\pi} \langle M_\Gamma \dot{u}, J_N u\rangle_{\mathbb{R}^{2N}}\ dt$$

is the space of $2\pi$-periodic $H^{1/2}$ functions, the choice of $H^1$ as underlying function space will cause some technical difficulties, in particular when it comes to the task of finding an applicable degree theory for the gradient $\nabla\Phi_r$, cf. the discussion in Section 2.3. But on the other hand the Hamiltonian $H_r$ is only defined for functions $u(t)$ satisfying among other conditions $u_j(t) \neq u_k(t)$ for all $j \neq k$. These conditions do not define an open subset of $H^{1/2}$ due to the fact that $H^{1/2}$ does not embed into the space of continuous, $2\pi$-periodic functions. So it is unclear how to work with $\Phi_r$ on $H^{1/2}$. Contrary to that, the space $H^1$ embeds continuously into the space of continuous functions. More details on the variational structure of Hamiltonian systems and the needed Sobolev spaces are given in section 2.2 and appendix A.

## 1.4 Further comments

This is a declaration about which parts of the thesis have been published with whom before and which parts are new:

- Chapter 2: The construction of the $S^1$-equivariant degree for the action functional of Hamiltonian systems and the proof of the abstract global continuation theorem (Thm. 2.3) is done as in the joint paper with T. Bartsch, [13]. For the construction a little more details have been added. The formula for the computation of the degree for a nondegenerate solution (Thm. 2.24) is new. In [13] we only needed and used that the degree is nonzero, provided the solution is contained in one of the finite-dimensional subspaces $X_n$. As a consequence now Corollary 2.9 is available for a general nondegenerate periodic solution of a Hamiltonian system like the ones obtained in Theorem 4.3.

- Chapter 3: This chapter has so far not been published. Up to minor changes it can be found in the preprint [38].

- Chapter 4: The local part of the result itself (Thm. 4.3 except (4)) is joint work with T. Bartsch and Q. Dai, [12]. The here presented proof however, differs from the original one in the sense that a symplectic transformation onto the unit disc is used. Due to this transformation the result here is formulated for a $C^4$ boundary component contrary to a $C^3$ component in the original formulation, cf. Remark 4.4 d). The global aspect, i.e. property (4) of Thm. 4.3, is new and possible due to Theorem 2.24.

- Chapter 5: This chapter contains a discussion of the obtained results with the aid of known solutions in the unit disc, as well as some open questions. In [12] the regular $N$-Gon in the unit disc has already served as an illustration of the choreographic solutions near the boundary.

- Appendix A: This is just a collection of known facts about Hamiltonian systems and the associated action functional.

- Appendix B: Corollary B.2, Lemma B.3 and Lemma B.4 concerning the boundary behaviour of Green's and Robin function have been taken from [12]. For the remaining facts references are given.

- Appendix C: References are given as well.

# Chapter 2

# An $S^1$-equivariant degree for Hamiltonian systems

This chapter provides an $S^1$-equivariant degree theory for the $H^1$-gradient of the action functional of a first order Hamiltonian system. The first section 2.1 introduces the notation and summarizes the properties of the degree in an abstract setting of potential operators on a Hilbert space, while section 2.2 with Corollary 2.9 as the central statement addresses the application of it to Hamiltonian systems. After that we discuss in section 2.3 other existing equivariant degree theories. In particular we introduce the degrees by Dylawerski, Gęba, Jodel, Marzantowicz (=DGJM) [29] and by Rybicki [69], on which our modification, carried out in section 2.4, is based. Finally section 2.5 is devoted to the proof of the global continuation theorem and section 2.6 to the calculation of the degree in a nondegenerate case.

## 2.1 Summary of results

### 2.1.1 Notation

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\rho : S^1 \longrightarrow \mathcal{L}(X)$ be an orthogonal $S^1$-action, i.e. $\rho$ is a continuous homomorphism between $S^1$ and the group of bounded linear operators $\mathcal{L}(X)$, such that every $\rho(e^{i\theta})$ preserves the inner product $\langle\cdot,\cdot\rangle$. Instead of $\rho(e^{i\theta})u$, $e^{i\theta} \in S^1$, $u \in X$ we just write $\theta * u$ and $\theta \in S^1$ using the identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For the orthogonality of the action this means $\langle\theta * u_1, \theta * u_2\rangle = \langle u_1, u_2\rangle$ for every $u_1, u_2 \in X$, $\theta \in S^1$.

A $S^1$-invariant subset of $X$ is a set, that whenever it contains a point $u$, it also contains the whole orbit $S^1 * u$. A map $f : X_1 \longrightarrow X_2$ between two $S^1$-representations is called $S^1$-equivariant provided $f(\theta * u) = \theta * f(u)$ for all $\theta \in S^1$, $u \in X_1$. Furthermore, $f$ is called invariant, if $f$ is constant along each orbit.

An example of an orthogonal $S^1$-action or in other words a $S^1$-representation is given by $X = \mathbb{R}^2$, $\rho^m : S^1 \longrightarrow SO(2)$, $m \in \mathbb{N}_0$,

$$\rho^m(\theta) = \begin{pmatrix} \cos(m\theta) & -\sin(m\theta) \\ \sin(m\theta) & \cos(m\theta) \end{pmatrix}.$$

We denote by $R[k, m]$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ the direct sum of $k$ copies of the representation $(\mathbb{R}^2, \rho^m)$.

Two representations $X_1$, $X_2$ are said to be equivalent, if there exists an equivariant isomorphism $T : X_1 \longrightarrow X_2$. In this situation we write $X_1 \cong X_2$. For finite-dimensional representations the following classification theorem is available:

**Theorem 2.1** (see [1]). *If $V$ is a finite-dimensional representation of $S^1$, then*

$$V \cong \bigoplus_{i=1}^{r} R[k_i, m_i]$$

*with unique numbers $r \in \mathbb{N}$, $k_i \in \mathbb{N}$, $m_i \in \mathbb{N}_0$, $i = 1, \ldots, r$ satisfying $m_1 < m_2 < \ldots < m_r$.*

Given a subset $U \subset X$ and a closed subgroup $K \leq S^1$, i.e. $K \in \left\{ S^1, \mathbb{Z}_1, \mathbb{Z}_2, \ldots \right\}$ with $\mathbb{Z}_k$ being the group of the $k$th roots of unity, the set of fixed points under $K$ is denoted by

$$U^K = \{ u \in U : \theta * u = u \text{ for all } \theta \in K \}.$$

For the isotropy group of $u \in X$ we write

$$I_u = \left\{ \theta \in S^1 : \theta * u = u \right\}.$$

Whenever the following limit exists the tangent vector to the orbit $S^1 * u$ at $u \in X$ is defined by

$$E(u) = \lim_{\theta \to 0} \frac{1}{\theta}(\theta * u - u) = \frac{d}{d\theta}_{|\theta=0}(\theta * u).$$

If $X$ is infinite-dimensional, then $E(u)$ might not exist for all $u \in X$, cf. $X$ being the Sobolev space $H^1$ in Section 2.2.1, but in the finite-dimensional case the classification Theorem 2.1 guarantees that $E(u)$ is well-defined for all $u \in X$. Moreover, the definition shows that in that case the vector field $E : X \to X$ is linear and $S^1$-equivariant, i.e. $E(\theta * u) = \theta * E(u)$ for all $\theta \in S^1$ and $u \in X$.

The degree theories we are looking at have values in $\bigoplus_{i=0}^{\infty} \mathbb{Z}$. For $\alpha, \beta \in \bigoplus_{i=0}^{\infty} \mathbb{Z}$ we define a multiplication by

$$\begin{aligned} \alpha \star \beta &= (\alpha_0\beta_0, \alpha_0\beta_1 + \beta_0\alpha_1, \alpha_0\beta_2 + \beta_0\alpha_2, \alpha_0\beta_3 + \beta_0\alpha_3, \ldots) \\ &= \alpha_0 \cdot \beta + \beta_0 \cdot \alpha - (\alpha_0\beta_0, 0, 0, \ldots). \end{aligned}$$

### 2.1.2 Degree setting

Now let $X$ be infinite-dimensional, but admitting a Hilbert space decomposition

$$X = \operatorname{clos}\left( \bigoplus_{k \in \mathbb{N}_0} E_k \right), \qquad E_j \perp E_k \text{ for } j \neq k.$$

consisting of finite-dimensional subspaces. For $n \in \mathbb{N}_0$ we set $X_n := \bigoplus_{k=0}^{n} E_k$ and write $P_n : X \to X_n$ for the orthogonal projection, so that $P_n u \to u$ as $n \to \infty$ for every $u \in X$. We consider $S^1$-equivariant maps

$$L - \Psi : \Lambda \to X$$

defined on an open and $S^1$-invariant subset $\Lambda \subset X$, such that the decomposition of $X$ and the maps $L$ and $\Psi$ satisfy

(A1) $E_k$ is a finite-dimensional, $S^1$-invariant linear subspace of $X$, and the isotropy group of $u \in E_k \setminus \{0\}$ is $\mathbb{Z}_k$ for $k \in \mathbb{N}$.

(A2) $L \in \mathcal{L}(X)$ is a bounded, self-adjoint, equivariant operator with $\operatorname{Kern}(L) = E_0$ and $L(E_k) = E_k$ for $k \neq 0$.

(A3) The map $L + P_0$ defines an isomorphism $X \to Y$ onto a Banach space $Y$ that embeds continuously into $X$.

(A4) $\Psi : \Lambda \to X$ is the gradient of an $S^1$-invariant $C^1$-function $\mathfrak{K} : \Lambda \to \mathbb{R}$.

(A5) The image of $\Psi$ is contained in $Y$ and for any bounded, invariant set $\mathcal{O}$ with $\overline{\mathcal{O}} \subset \Lambda$ the restriction $\Psi : \overline{\mathcal{O}} \to Y$ is a compact map.

Note that in (A1) there is no restriction on the isotropy groups of the elements in $E_0$, but $X^{S^1} \subset E_0$. The condition says that $E_k$ is the isotypical component of $E_0^\perp$ corresponding to the representation $(\mathbb{R}^2, \rho^k)$.

Given $X$ together with a decomposition into subspaces $E_k$ satisfying condition (A1), we write $f \in C^0_\nabla(\mathcal{O})$, if $f = L - \Psi : \Lambda \to X$ satisfies (A2)–(A5), $\mathcal{O}$ is an open, bounded, invariant set with closure contained in $\Lambda$ and $f(\partial\mathcal{O}) \subset X \setminus \{0\}$.

**Theorem 2.2.** *For $f \in C^0_\nabla(\mathcal{O})$ there exists a degree*

$$S^1\text{-deg}^\nabla(f, \mathcal{O}) = \left(\mathrm{d}_k^\nabla(f, \mathcal{O})\right)_{k\in\mathbb{N}_0} \in \bigoplus_{k=0}^{\infty} \mathbb{Z}$$

*with the following properties:*

(D1) *(Existence) If $\mathrm{d}_k^\nabla(f, \mathcal{O}) \neq 0$ for some $k \in \mathbb{N}_0$, then there exists $u \in \mathcal{O}^K$ with $f(u) = 0$ where $K = S^1$ if $k = 0$, resp. $K = \mathbb{Z}_k$ if $k \geq 1$.*

(D2) *(Excision and additivity) If $f^{-1}(0) \cap \mathcal{O} \subset \mathcal{O}_1 \cup \mathcal{O}_2$ for two disjoint open $S^1$-invariant subsets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}$ then*

$$S^1\text{-deg}^\nabla(f, \mathcal{O}) = S^1\text{-deg}^\nabla(f, \mathcal{O}_1) + S^1\text{-deg}^\nabla(f, \mathcal{O}_2).$$

(D3) *(Homotopy) Let $\mathcal{U} \subset [0,1] \times X$ be open and bounded, and let $h : (\overline{\mathcal{U}}, \partial\mathcal{U}) \to (X, X \setminus \{0\})$ be continuous. If $h_t = h(t, \cdot) : \mathcal{U}_t = \{\, u \in X : (t,u) \in \mathcal{U} \,\} \to X$, $t \in [0,1]$ lies in $C^0_\nabla(\mathcal{U}_t)$ for each $t \in [0,1]$, then $S^1\text{-deg}^\nabla(h_t, \mathcal{U}_t)$ is independent of $t \in [0,1]$.*

(D4) *(Multiplicativity) If $f_i : (\overline{\mathcal{O}_i}, \partial\mathcal{O}_i) \to (X_i, X_i \setminus \{0\})$, $i = 1, 2$, are in $C^0_\nabla(\mathcal{O}_i)$, then so is $f_1 \times f_2 \in C^0_\nabla(\mathcal{O}_1 \times \mathcal{O}_2)$ and*

$$S^1\text{-deg}^\nabla(f_1 \times f_2, \mathcal{O}_1 \times \mathcal{O}_2) = S^1\text{-deg}^\nabla(f_1, \mathcal{O}_1) \star S^1\text{-deg}^\nabla(f_2, \mathcal{O}_2).$$

The proof will be delayed until section 2.4.

### 2.1.3 Global continuation

Having a degree theory with the typical properties at hand one can prove the existence of connected sets of solutions for parameter dependent equations, see for example the classic continuation theorems by Leray and Schauder [52] or Rabinowitz [66]. We will formulate now a version that will suit the continuation of periodic solutions in our application. Consider a family of equations of the form

$$Lu - \Psi(r, u) = 0, \quad (r, u) \in \mathcal{D} \subset \mathbb{R}^+ \times X. \tag{2.1}$$

Here $S^1$ acts trivially on $\mathbb{R}$ and $X$, $L$ satisfy (A1)–(A3). Concerning the nonlinear map $\Psi$ we replace (A4),(A5) by corresponding parameter dependent assumptions:

(A6) $\Psi : \mathcal{D} \to X$ is defined on an open and invariant subset $\mathcal{D} \subset \mathbb{R}^+ \times X$, it is continuous, equivariant, and $\Psi(r, \cdot)$ is the gradient of $\mathfrak{K}(r, \cdot)$, where $\mathfrak{K} : \mathcal{D} \to \mathbb{R}$ is $S^1$-invariant, continuous and differentiable with respect to the $u$ component.

(A7) The image of $\Psi$ is contained in $Y$. If $B \subset \mathbb{R} \times X$ is bounded, closed, and $B \subset \mathcal{D}$, then the set $\Psi(B)$ is relatively compact in $Y$.

The set of solutions of (2.1) will be denoted by $\mathcal{S} = \{(r,u) \in \mathcal{D} : Lu - \Psi(r,u) = 0\}$. Observe that if $B \subset \mathbb{R} \times X$ is $S^1$-invariant, closed, bounded and satisfies $B \subset \mathcal{D}$ then $\mathcal{S} \cap B$ is compact. This follows easily from (A7). For $M \subset \mathbb{R}^+ \times X$ and $r \in \mathbb{R}^+$ we use the notation $M_r = \{u \in X : (r,u) \in M\}$.

**Theorem 2.3.** *Suppose (A1)-(A3), (A6), (A7) hold, and suppose there exist $r_0 > 0$ and a relatively open, $S^1$-invariant subset $\mathcal{U} \subset (0, r_0] \times X$ such that:*

*(i) For every $r \in (0, r_0]$: $\mathcal{U}_r \neq \emptyset$, bounded, $\overline{\mathcal{U}}_r \subset \mathcal{D}_r$.*

*(ii) $\mathcal{S} \cap \partial\mathcal{U} = \emptyset$ where $\partial\mathcal{U}$ is the relative boundary of $\mathcal{U}$ in $(0, r_0] \times X$.*

*If $S^1\text{-}\deg^\nabla(L - \Psi(r_0, \cdot), \mathcal{U}_{r_0}) \neq 0$ then there exists a connected component $C \subset \mathcal{S}$ with the following properties:*

*a) $(C \cap \mathcal{U})_r \neq \emptyset$ for every $r \in (0, r_0]$,*

*b) $C \setminus \mathcal{U}$ is not contained in a compact subset of $\mathcal{D}$.*

Of course, if needed, we can here replace the interval $(0, r_0]$ by an interval $(r_1, r_0]$ with $0 < r_1 < r_0$.

### 2.1.4 Nontrivial degree for nondegenerate solutions

Theorem 2.3 clearly relies on a nontrivial degree $S^1\text{-}\deg^\nabla(L - \Psi(r_0, \cdot), \mathcal{U}_{r_0})$ for some parameter value $r_0$. As for the classical Brouwer degree we will obtain in 2.6 a formula for the degree in the nondegenerate case. Admittedly this formula will be not very handy for the actual computation, but it allows us to conclude $S^1\text{-}\deg^\nabla(L - \Psi(r_0, \cdot), \mathcal{U}_{r_0}) \neq 0$.

So consider again a map $L - \Psi : \Lambda \to X$ satisfying (A1)–(A5) and additionally the following two assumptions:

(A8) $\Psi : \Lambda \to Y$ is $C^1$.

(A9) For any $u \in Y$ the tangent vector $E(u) \in X$ is defined and $E : Y \to X$ is a bounded linear operator.

Assumption (A8) implies $\mathfrak{K}$ is $C^2$ with $\langle D\Psi(u)v, w\rangle = D^2\mathfrak{K}(u)[v,w] = \langle v, D\Psi(u)w\rangle$. The notion of a nondegenerate solution $u_0 \in \Lambda$ of $L - \Psi = 0$ has to be adapted to the equivariant setting, since every element of the orbit $S^1 * u_0$ is a solution as well.

**Definition 2.4.** *A solution $u_0 \in \Lambda$ of $Lu - \Psi(u) = 0$ is called nondegenerate provided $u_0 \in Y$ and $\mathrm{Kern}(L - D\Psi(u_0)) = \mathbb{R}E(u_0)$, i.e. the kernel of the derivative at $u_0$ is as small as possible.*

**Theorem 2.5.** *Let $u_0 \in \Lambda$ be a nondegenerate solution of $L - \Psi = 0$ with $I_{u_0} = \mathbb{Z}_k$. Then there exists an invariant neighborhood $O$ of $S^1 * u_0$ such that $(L - \Psi)^{-1}(0) \cap \overline{O} = S^1 * u_0$ and*

$$\mathrm{d}_k^\nabla(L - \Psi, O) \in \{\pm 1\}.$$

*Similar if $I_{u_0} = S^1$, then $\mathrm{d}_0^\nabla(L - \Psi, O) \in \{\pm 1\}$.*

The detailed formula can be found in Theorem 2.24.

## 2.2 Application to Hamiltonian systems

Here we will check that first order Hamiltonian systems give rise to the setting presented in section 2.1 via the associated action functional. In order to not get lost in all the details we have moved parts of the elaboration that are considered to be well known into appendix A.

### 2.2.1 The basic degree setting

Let $\mathbb{R}^{2N}$ be equipped with a symplectic form $\omega : \mathbb{R}^{2N} \times \mathbb{R}^{2N} \to \mathbb{R}$, $U \subset \mathbb{R}^{2N}$ open and $H : U \to \mathbb{R}$ be a $C^2$ Hamilton function. Imagine that we are interested in a $2\pi$-periodic solution of the Hamiltonian system

$$\dot{z} = X_H(z). \tag{2.2}$$

Here $X_H$ is the associated Hamiltonian vector field, i.e. $\omega(X_H(z), \cdot) = DH(z)$. By Lemma A.1 there exists a skew-symmetric, regular matrix $A$ that allows us to rewrite (2.2) in the equivalent way:

$$A\dot{z} = \nabla H(z).$$

For example we have for the $N$-vortex system (1.5) $H = H_\Omega$, $U = \mathcal{F}_N(\Omega)$, $A = -J_N M_\Gamma$ and $\omega(v, w) = \langle v, M_\Gamma J_N w\rangle_{\mathbb{R}^{2N}}$.

Next we turn to the functional setting. A square-integrable function $u : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^{2N}$ can be written (with respect to $L^2$-norm) in terms of its Fourier series

$$u(t) = \sum_{k\in\mathbb{Z}} e^{-J_N kt}\alpha_k, \quad \alpha_k \in \mathbb{R}^{2N}.$$

We abbreviate $B_k(t) = e^{-J_N kt} \in \mathbb{R}^{2N\times 2N}$ and define for $s \in [0, \infty)$ the Sobolev spaces

$$H^s = \left\{ \sum_{k\in\mathbb{Z}} B_k\alpha_k \in L^2 : \sum_{k\in\mathbb{Z}} |k|^{2s} |\alpha_k|^2 < \infty \right\}.$$

In particular we need $X = H^1$, which is equipped with the usual scalar product

$$\langle u, v\rangle_X = \int_0^{2\pi} \langle u, v\rangle_{\mathbb{R}^{2N}} + \langle \dot{u}, \dot{v}\rangle_{\mathbb{R}^{2N}}\, dt = 2\pi \sum_{k\in\mathbb{Z}} (1 + k^2)\langle \alpha_k, \beta_k\rangle_{\mathbb{R}^{2N}}$$

for $u = \sum B_k\alpha_k$, $v = \sum B_k\beta_k$. The group $S^1$ acts on $X$ via time shifts, i.e.

$$\theta * u = u(\cdot + \theta) = \sum_{k\in\mathbb{Z}} B_k B_k(\theta)\alpha_k, \quad \theta \in S^1,\ u = \sum_{k\in\mathbb{Z}} B_k\alpha_k.$$

For $k \in \mathbb{N}_0$ let

$$E_k = \left\{ B_k\alpha_k + B_{-k}\alpha_{-k} : \alpha_k, \alpha_{-k} \in \mathbb{R}^{2N} \right\}.$$

Then $E_j \perp E_k$ for $j \neq k$ and $X = \operatorname{clos}\left(\bigoplus_{k\in\mathbb{N}_0} E_k\right)$. Moreover, each $E_k$ is a finite-dimensional, $S^1$-invariant subspace with elements having isotropy group $\mathbb{Z}_k$ for $k \geq 1$ as desired by (A1). Furthermore, we need

$$X_n = \bigoplus_{k=0}^{n} E_k = \left\{ \sum_{|k|\leq n} B_k\alpha_k : \alpha_k \in \mathbb{R}^{2N} \right\}$$

and the orthogonal projections $P_n : X \to X_n$.

The linear map $L : X \longrightarrow X$ is defined by $Lu = (\mathrm{id} - \Delta)^{-1} A\dot{u}$, where $(\mathrm{id} - \Delta) : H^{s+2} \longrightarrow H^s$ is the isomorphism

$$u = \sum_{k \in \mathbb{Z}} B_k \alpha_k \mapsto u - \ddot{u} = \sum_{k \in \mathbb{Z}} (1 + k^2) B_k \alpha_k,$$

such that for $u \in H^1 = X$, $v \in H^0 = L^2$ the relation

$$\left\langle u, (\mathrm{id} - \Delta)^{-1} v \right\rangle_X = \int_0^{2\pi} \langle u, v \rangle_{\mathbb{R}^{2N}} \; dt = \langle u, v \rangle_{L^2}$$

holds true. Then clearly $L$ is a bounded, $S^1$-equivariant operator with $\mathrm{Kern}(L) = E_0$ and $L(E_k) = E_k$ for $k \neq 0$. Moreover, the skew-symmetry of $A$ implies

$$\langle Lu, v \rangle_X = \langle A\dot{u}, v \rangle_{L^2} = \langle u, A\dot{v} \rangle_{L^2} = \langle u, Lv \rangle_X \, ,$$

so $L$ is self-adjoint and thus assumption (A2) holds.

For (A3) observe that $L + P_0$ is an isomorphism between $X$ and the Banach space $Y = H^2$, which is equipped with the usual $H^2$-norm, and clearly $H^2 \hookrightarrow H^1$ in a continuous way.

So we can turn to the nonlinear part. Since $H^1$ contrary to $H^{1/2}$ embeds into the space of continuous $2\pi$-periodic functions, the set $\Lambda = \{ u \in X : u(t) \in U \text{ for all } t \in \mathbb{R} \}$ defines an open subset of $X$. Let $\mathfrak{K} : \Lambda \longrightarrow \mathbb{R}$,

$$\mathfrak{K}(u) = \int_0^{2\pi} H(u(t)) \, dt.$$

Then $H \in C^2(U, \mathbb{R})$ implies that $\mathfrak{K}$ is of class $C^2$ as well. Additionally both $\Lambda$ and $\mathfrak{K}$ are invariant with respect to the $S^1$-action on $X$. So the gradient $\Psi : \Lambda \longrightarrow X$,

$$\Psi(u) = \nabla \mathfrak{K}(u) = (\mathrm{id} - \Delta)^{-1} \nabla H(u)$$

is $S^1$-equivariant and satisfies (A4).

In order to see that (A5) is valid observe that $\Psi$ splits

$$X \supset \Lambda \xrightarrow{\nabla H} X = H^1 \xrightarrow{(\mathrm{id} - \Delta)^{-1}} H^3 \hookrightarrow H^2 = Y.$$

where $\nabla H : \Lambda \longrightarrow X$ maps bounded subsets with closure contained in $\Lambda$ into bounded subsets, since $H$ is $C^2$. Therefore the compactness of the embedding $H^3 \hookrightarrow H^2$ shows that (A5) holds. We'd like to recall that – if needed – more details and references are given in appendix A.

So far we can conclude that the equivariant degree of section 2.1 can be applied to the $H^1$-gradient of the action functional $\Phi : \Lambda \longrightarrow \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int_0^{2\pi} \langle A\dot{u}, u \rangle_{\mathbb{R}^{2N}} \; dt - \int_0^{2\pi} H(u) \, dt = \frac{1}{2} \langle Lu, u \rangle_X - \mathfrak{K}(u).$$

### 2.2.2 Nondegenerate solutions and nontrivial degree

Critical points of $\Phi$, i.e. solutions of $\nabla\Phi = L - \Psi = 0$, are $2\pi$-periodic solutions of (2.2) in the classical sense. Hence $Lu_0 - \Psi(u_0) = 0$ automatically implies $u_0 \in Y$. In order to be able to obtain by Theorem 2.5 a nontrivial degree for a solution $u_0$ we need to convince ourselves that (A8) and (A9) are satisfied. Indeed $\nabla H \in C^1(\Lambda, L^2)$ clearly gives $\Psi \in C^1(\Lambda, Y)$ as required in (A8) and the tangent vector field $E : Y \longrightarrow X$ is just given by $u \mapsto \dot{u}$. Thus (A9) holds.

Let now $u_0 \in Y$ be a solution of $L - \Psi = 0$. By Definition 2.4 in the abstract setting, $u_0$ is called nondegenerate, if $Lv - D\Psi(u_0)v = 0$ implies $v \in \mathbb{R}E(u_0)$. Translated to the

Hamiltonian setting this means that $\dot{u}_0$ is up to scalar multiples the only $2\pi$-periodic solution of the linearization of (2.2) along $u_0(t)$, i.e. of

$$A\dot{v} = \nabla^2 H(u_0(t))v. \tag{2.3}$$

So for Hamiltonian systems we do not have to verify the nondegenerateness condition of Theorem 2.5 by investigating $\mathrm{Kern}(L - D\Psi(u_0))$ in the Hilbert space setting, rather we can use equation (2.3) and especially spectral properties of the associated monodromy operator. The monodromy operator is the matrix $M(2\pi)$, where $M : \mathbb{R} \to \mathbb{R}^{2N\times 2N}$ solves

$$\begin{cases} A\dot{M} = \nabla^2 H(u_0(t))M \\ M(0) = \mathrm{id}_{\mathbb{R}^{2N}} . \end{cases}$$

Its eigenvalues are called Floquet multipliers of the solution $u_0$. Of course 1 is always a multiplier, but note also that, since we are dealing with Hamiltonian systems, the algebraic multiplicity of 1 is at least 2, see for example [61].

By Theorem 2.5 and our discussion we can summarize:

**Proposition 2.6.** *Whenever $u_0$ is a $2\pi$-periodic solution of (2.2), such that the Floquet multiplier 1 has geometric multiplicity one, then the associated local degree $S^1\text{-}\mathrm{deg}^\nabla(L - \Psi, B_\varepsilon(u_0))$, $\varepsilon > 0$ sufficiently small is nontrivial.*

The considerations above of course remain valid, if the period $2\pi$ is replaced by $T > 0$ and spaces and maps are adapted to this period.

### 2.2.3 Continuation of periodic solutions

Consider now on $(\mathbb{R}^{2N}, \omega)$ a family of Hamiltonian systems

$$\dot{z} = X_{H_r}(z), \tag{2.4}$$

where $H : D \to \mathbb{R}$, $(r, z) \mapsto H_r(z)$ is defined on an open subset $D$ of $\mathbb{R}^+ \times \mathbb{R}^{2N}$, twice differentiable with respect to $z$ and $H$ itself is continuous as well as the derivatives $D_z H$, $D_z^2 H$ are. We will obtain such families by a suitable rescaling of the $N$-vortex Hamiltonian $H_\Omega$. Suppose we know that (2.4) has a $2\pi$-periodic solution $u_{r^*}$ for some parameter value $r^*$. It is then natural to ask if there are periodic solutions for other parameter values $r \neq r^*$ emanating from $u_{r^*}$?

This question can be answered be means of the global continuation theorem 2.3. Let $X$, $Y$ and $L$ be defined as before and set

$$\mathcal{D} = \left\{ (r, u) \in \mathbb{R}^+ \times X : (r, u(t)) \in D \text{ for all } t \in \mathbb{R} \right\}.$$

Then one can see as in the discussion before that $\Psi : \mathcal{D} \to X$ defined as the gradient $\nabla_u \mathfrak{K}$ of $\mathfrak{K} : \mathcal{D} \to \mathbb{R}$,

$$\mathfrak{K}(r, u) = \int_0^{2\pi} H_r(u)\, dt$$

satisfies (A6) and (A7). So the equations $L - \Psi(r, \cdot) = 0$ are accessible for Theorem 2.3.

Before we investigate the global aspect we first prove a local continuation Theorem.

**Theorem 2.7.** *Let $H$, $D$ and $\mathcal{D}$ be as just described and suppose that $u_{r^*}$ is a $2\pi$-periodic solution of (2.4) with $r = r^*$, such that the Floquet multiplier 1 has geometric multpilicity one. Then there exists a continuous map $I \ni r \mapsto u^{(r)} \in X$ with $I \subset \mathbb{R}^+$ being an interval around $r^*$, $u^{(r^*)} = u_{r^*}$, $\langle u^{(r)}, \dot{u}_{r^*} \rangle_X = 0$, $\dot{u}^{(r)} = X_{H_r}\left(u^{(r)}\right)$ and the Floquet multiplier 1 of $u^{(r)}$ has geometric* *multiplicity one. Moreover, if $(r, z) \mapsto DH_r(z)$ is of class $C^1$, then the maps $I \ni r \mapsto u^{(r)} \in X$ and $I \times \mathbb{R} \ni (r, t) \mapsto u^{(r)}(t) \in \mathbb{R}^{2N}$ are $C^1$ as well. Also the second order derivatives $\partial_r \partial_t u^{(r)}(t)$, $\partial_t \partial_r u^{(r)}(t)$ exist, are equal and continuous.*

*Proof.* Clearly $u$ is a $2\pi$-periodic solution of (2.4), if and only if $(r, u)$ is a zero of the map $L - \Psi : \mathcal{D} \to X$, i.e. iff $Lu - \Psi(r, u) = (\mathrm{id} - \Delta)^{-1} (A\dot{u} - \nabla H_r(u)) = 0$. Recall also that $u_{r^*}$ is contained in $Y$ as a solution of (2.4).

By our assumption we now that $\mathrm{Kern}\,(L - D_u\Psi(r^*, u_{r^*})) = \mathbb{R}\dot{u}_{r^*}$. Hence if we consider $f : \{ (r, u) \in \mathcal{D} : \langle u, \dot{u}_{r^*} \rangle_X = 0 \} \to \{ u \in Y : \langle u, \dot{u}_{r^*} \rangle_X = 0 \}$,

$$f(r, u) = Lu - \Psi(r, u) - \frac{\langle Lu - \Psi(r, u), \dot{u}_{r^*} \rangle_X}{\|\dot{u}_{r^*}\|^2_{L^2}} (\mathrm{id} - \Delta)^{-1} \dot{u}_{r^*},$$

where the domain is equipped with $\|\cdot\|_X$ and the range with $\|\cdot\|_Y$, then $u_{r^*}$ is indeed orthogonal to $\dot{u}_{r^*}$, $f(r^*, u_{r^*}) = 0$ and $\mathrm{Kern}\, D_u f(r^*, u_{r^*}) = \{0\}$. The latter uses that $L - D_u \Psi(r^*, u_{r^*})$ is as a second derivative selfadjoint and therefore $D_u f(r^*, u_{r^*})[v] = Lv - D_u\Psi(r^*, u_{r^*})[v]$. But $D_u f(r^*, u_{r^*})$ is also onto, since by (A5) $L - D_u\Psi(r^*, u_{r^*}) : X \to Y$ is an index 0 Fredholm operator. Thus the derivative $D_u f(r^*, u_{r^*})$ is an isomorphism.

The implicit function theorem implies the existence of a continuous local family $\left(u^{(r)}\right)_{r \in I}$ contained in the $X$-orthogonal complement of $\dot{u}_{r^*}$ satisfying $u^{(r^*)} = u_{r^*}$ and the equation $Lu^{(r)} = \Psi\left(r, u^{(r)}\right) + \lambda_r (\mathrm{id} - \Delta)^{-1} \dot{u}_{r^*} \in H^3$ for some $\lambda_r \in \mathbb{R}$. Hence $u^{(r)} \in Y$.

By shrinking $I$ if necessary we can assume that $\left\langle \dot{u}_{r^*}, \dot{u}^{(r)} \right\rangle_{L^2} \neq 0$ for $r \in I$. The invariance of the action functional under time translations then implies

$$0 = \frac{\left\langle Lu^{(r)} - \Psi\left(r, u^{(r)}\right), \dot{u}^{(r)} \right\rangle_X}{\left\langle \dot{u}_{r^*}, \dot{u}^{(r)} \right\rangle_{L^2}} = \lambda_r$$

and therefore $Lu^{(r)} - \Psi\left(r, u^{(r)}\right) = 0$ as desired.

By continuity we have $\mathrm{Kern}\, D_u f\left(r, u^{(r)}\right) = \{0\}$ for $r$ close to $r^*$. So if we suppose that $w$ is linear independent to $\dot{u}^{(r)}$ and satisfies $Lw - D_u\Psi\left(r, u^{(r)}\right) w = 0$, then a suitable linear combination $w^\perp = \alpha w + \beta \dot{u}^{(r)}$ is orthogonal to $\dot{u}_{r^*}$ and satisfies $D_u f\left(r, u^{(r)}\right) w^\perp = 0$. It follows $w^\perp = 0$, which contradicts the linear independence of $w$ and $\dot{u}^{(r)}$. Therefore $\mathrm{Kern}\left(L - D_u\Psi\left(r, u^{(r)}\right)\right) = \mathbb{R}\dot{u}^{(r)}$, which is equivalent to saying that 1 is a geometrically simple Floquet multiplier of $u^{(r)}$.

The regularity of the map $I \times \mathbb{R} \ni (r, t) \mapsto u^{(r)}(t) \in \mathbb{R}^{2N}$ follows from the regularity of $I \ni r \mapsto u^{(r)} \in X$, which implies the same regularity for $I \ni r \mapsto u^{(r)}(0) \in \mathbb{R}^{2N}$, and the smooth parameter dependence of the flow associated to (2.4), see Thm. 9.2 of [3]. □

In general the question of continuation of a periodic solution arises also for a fixed, parameter independent Hamiltonian system as in (2.2) and of course also for arbitrary ODEs. In the Hamiltonian setting we can apply Theorem 2.7 to give an answer to this question. Indeed if $H \in C^2(U, \mathbb{R})$ is a fixed Hamiltonian, we introduce the period as a parameter via $\hat{H} : \mathbb{R}^+ \times U \to \mathbb{R}$, $\hat{H}(r, z) = rH(z)$. Clearly $\hat{H}$ is $C^2$ and if $u$ is a $2\pi$-periodic solution of the system $\dot{u} = X_{\hat{H}_r}(u)$, then $u(\cdot/r)$ is a $2\pi r$-periodic solution of the original equation (2.2). This relation also holds in the other direction. So Theorem 2.7 applies if $z^*(t) = u_0(t/r_0)$ is a $2\pi r_0$-periodic solution of (2.2) with geometrically simple Floquet multiplier 1.

Of course local continuation results for parameter dependent and independent systems can be found in the literature. With the flow $\varphi(t, z)$ of the fixed system $\dot{z} = X_H(z)$, such a Theorem for example reads:

**Theorem 2.8** (Prop. 9.1.1 of [61] or Thm. 2.4 of [63])**.** *Let $u_0(t) = \varphi(t, z_0)$ be a $T_0$-periodic solution of (2.2), such that the Floquet multiplier 1 has algebraic multiplicity 2. Then there* *exist $C^1$ maps $T : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, $z : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{2N}$ with $T(0) = T_0$ and $z(0) = z_0$, such that $u_\varepsilon(t) = \varphi(t, z(\varepsilon))$ is a $T(\varepsilon)$-periodic solution of (2.2) contained in the same energy surface as $u_0$.*

Recall that $\varphi$ is $C^1$, since $H$ is $C^2$. The proof relies on the application of the implicit function theorem to a symplectic Poincaré section associated to $u_0$, see section 2.1 of [63] for a sophisticated discussion of these type of continuation theorems. The advantage of the access via a Poincaré section is that it not only applies to Hamiltonian systems, but also to ODEs, and that if first integrals like the Hamiltonian exist, the continuation happens to be in the same level sets of the integrals.

On the other hand in the more abstract point of view we have additionally the degree theory with Theorem 2.3 at hand allowing us to obtain even global continua of periodic solutions. Recall that $\Lambda = \{\, u \in X : u(t) \in U \text{ for all } t \in \mathbb{R} \,\}$.

**Corollary 2.9.** *Let $0 \le r_1 < r_2$ and $(r_1, r_2) \ni r \mapsto u^{(r)} \in \Lambda$ be $C^1$ and such that every $u^{(r)}(\cdot/r)$ is a $2\pi r$-periodic solution of (2.2) with Floquet multiplier 1 having geometric multiplicity 1. Define $C_l = \left\{ \left(r, \theta * u^{(r)}\right) : r \in (r_1, r_2),\ \theta \in S^1 \right\}$. Then there exists an equivariant, connected set $C \subset \mathbb{R}^+ \times \Lambda$ with*

*(i)* $C_l \subset C$,

*(ii)* $(r, u) \in C \Rightarrow u(\cdot/r)$ *is a $2\pi r$-periodic solution of* (2.2),

*(iii)* $C_g = (C \setminus C_l) \cap ((r_1, \infty) \times X)$ *satisfies at least one of the following properties:*

- *a)* $C_g$ *is unbounded,*
- *b)* $\operatorname{dist}(C_g, \partial\Lambda) = 0$,
- *c)* $\inf \left\{ r : (r, u) \in C_g \right\} = r_1$.

*Proof.* Let $f : \mathbb{R}^+ \times \Lambda \to X$, $f(r, u) = Lu - r\Psi(u)$ with $L$, $\Psi$ as in section 2.2.1 and $\mathcal{S} = f^{-1}(0)$. We know that $C_l \subset \mathcal{S}$. Let $C \subset \mathcal{S}$ denote the connected component of $C_l$. Then (i) and (ii) are trivially satisfied. Define $C_g = (C \setminus C_l) \cap ((r_1, \infty) \times X)$ and assume that the three options a), b) and c) of (iii) are wrong. Thus we can choose $r_1 < r_0 < \inf \left\{ r : (r, u) \in C_g \right\}$. The implicit function theorem, cf. proof of Theorem 2.7, provides $\varepsilon > 0$ and $\delta > 0$ such that the closure of $\mathcal{U} = (r_0 - \delta, r_0] \times \left(S^1 * B_\varepsilon(u^{(r_0)})\right)$ is contained in $(r_1, \infty) \times \Lambda$ and

$$f^{-1}(0) \cap \overline{\mathcal{U}} = \left\{ \theta * u^{(r)} : r \in [r_0 - \delta, r_0],\ \theta \in S^1 \right\}.$$

By Proposition 2.6 we also have $S^1\text{-}\deg^\nabla(L - r_0\Psi, \mathcal{U}_{r_0}) \neq 0$. Thus $C_g$ is not contained in a compact subset of $\mathcal{D} = (r_0 - \delta, \infty) \times \Lambda$ by Theorem 2.3. But since (A5) is satisfied by $\Psi$ this contradicts our assumption that a), b) and c) are wrong. □

Typically this Corollary will be applied to a local family of periodic solutions having arbitrarily small periods, i.e. $r_1 = 0$.

**Remark 2.10.** *In the situation of Corollary 2.9 let $\mathcal{E}_g = \left\{ u(\cdot/r) : (r, u) \in C_g \right\}$ denote the set of the actual solutions of (2.2). The options a) – c) say that at least one of the following properties is true:*

*a) The periods of the solutions are unbounded,* $\sup \left\{ \text{period of } z : z \in \mathcal{E}_g \right\} = \infty$, *for example the solutions might merge into a heteroclinic orbit or a stationary solution, which is nothing but a periodic solution having any period. Another option here is that the solutions are unbounded in space, i.e.* $\sup \left\{ |z(t)| : z \in \mathcal{E}_g,\ t \in \mathbb{R} \right\} = \infty$, *or in terms of their velocity,* $\sup \{\, |\dot z(t)| : z \in \mathcal{E},\ t \in \mathbb{R} \,\} = \infty$.

*b) The solutions approach the boundary $\partial U$ of the domain of the Hamiltonian $H$, that is* $\inf\left\{ \operatorname{dist}(z(\mathbb{R}), \partial U) : z \in \mathcal{E}_g \right\} = 0$.

*c) For every $r \in (r_1, r_2)$ there exists a $2\pi r$-periodic solution $z \in \mathcal{E}_g$ of (2.2). Together with the solutions induced by the local graph $\mathcal{E}_l = \left\{ u(\cdot/r) : (r,u) \in C_g \right\}$ we then have at least two distinct periodic orbits for every period in $(2\pi r_1, 2\pi r_2)$.*

An illustration of a part of such a global set of solutions can be seen in Figures 5.2, 5.3.

**Definition 2.11.** *We say that a family of periodic solutions $\left(z^{(s)}\right)_{s\in(s_1,s_2)}$ of (2.2) having periods $(T(s))_{s\in(s_1,s_2)}$ gives rise to a global continuum of periodic solutions, if there exists a connected set $C \subset \mathbb{R}^+ \times \Lambda$ as in Corollary 2.9 containing the set*

$$C_l = \left\{ \left( \frac{T(s)}{2\pi}, \theta * z^{(s)} \left( \frac{T(s)}{2\pi} \cdot \right) \right) : s \in (s_1, s_2),\ \theta \in S^1 \right\} \subset \mathbb{R}^+ \times \Lambda.$$

Note that by the combination of Theorem 2.7 and Corollary 2.9 already a single periodic solution with geometrically simple Floquet multiplier 1 gives rise to a global continuum.

### 2.2.4 Choreographic setting

Here we will discuss the application of our degree to periodic solutions with a special symmetry – so called choreographic solutions. Of course this requires a certain symmetry of the Hamiltonian $H : D \subset \mathbb{R}^{2N} \longrightarrow \mathbb{R}$ and the symplectic form. Let $N = dl$ with $d, l \geq 1$ and consider $\mathbb{R}^{2N}$ as the product $\mathbb{R}^{2N} = \left(\mathbb{R}^{2d}\right)^l$. For the $N$-vortex case we have $d = 1$, $l = N$. As another example $d = 3$, $l \in \mathbb{N}$ can be used for the classical $l$-body problem of celestial mechanics.

The permutation group $\Sigma_l$ of $l$ symbols acts orthogonally on $(\mathbb{R}^{2d})^l$ via

$$\sigma * z = \left(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(l)}\right), \quad z = (z_1, \ldots, z_l) \in (\mathbb{R}^{2d})^l,\ \sigma \in \Sigma_l.$$

We assume that $H$ and the skew-symmetric matrix $A$ associated to $\omega$ are equivariant with respect to a certain permutation $\sigma_0 \in \Sigma_l$, i.e. we assume $\sigma_0 * D \subset D$, $H(\sigma_0 * z) = H(z)$ and $A(\sigma_0 * z) = \sigma_0 * (Az)$ for any $z \in D$.

**Definition 2.12.** *A $T$-periodic solution $z(t) \in D$ of $A\dot{z} = \nabla H(z)$ is called $\sigma_0$-choreographic or just choreographic, if there exists $\theta_0 \in \mathbb{R}$ such that $\sigma_0 * z(t + \theta_0) = z(t)$ for every $t \in \mathbb{R}$. Moreover, $z(t)$ is called $\sigma_0$-nondegenerate provided $\mathbb{R}\dot{z}$ are the only $T$-periodic, $\sigma_0$-choreographic solutions of the linearization $A\dot{v} = \nabla^2 H(z)v$.*

If $\operatorname{ord}(\sigma_0)$ denotes the order of the permutation of $\sigma_0$, then $\operatorname{ord}(\sigma_0)\theta_0$ is necessarily a multiple of the minimal period of $z$. Note also that the notion of $\sigma_0$-nondegenerateness of a solution will be adapted, if additional symmetries are present, see Definition 3.6 and Example 3.7.

Let us assume that we are again interested in $2\pi$-periodic solutions. By our symmetry assumption the action functional $\Phi : \Lambda \longrightarrow \mathbb{R}$,

$$\Phi(u) = \frac{1}{2} \int_0^{2\pi} \langle A\dot{u}, u \rangle_{\mathbb{R}^{2N}}\ dt - \int_0^{2\pi} H(u)\, dt$$

is invariant under the action of $\sigma_0$ induced on $X = H^1$, i.e. $(\sigma_0 * u)(t) := \sigma_0 * (u(t))$, $\sigma_0 * \Lambda \subset \Lambda$ and $\Phi(\sigma_0 * u) = \Phi(u)$ for any $u \in \Lambda$. Combining this with the invariance with respect to time translations, we get $\Phi((\sigma_0 * u)(\cdot + \theta_0)) = \Phi(u)$ for any $u \in \Lambda$. Differentiation therefore implies the equivariance

$$(\sigma_0 * \nabla\Phi(u))(\cdot + \theta_0) = \nabla\Phi((\sigma_0 * u)(\cdot + \theta_0)).$$

Hence, if we define

$$X_{\mathrm{chor}} = \left\{ u \in H^1 : (\sigma_0 * u)(\cdot + \theta_0) = u \right\}, \quad \Lambda_{\mathrm{chor}} = \Lambda \cap X_{\mathrm{chor}},$$

then $\nabla\Phi(\Lambda_{\mathrm{chor}}) \subset X_{\mathrm{chor}}$. This shows that every critical point of the restriction $\Phi_{|\Lambda_{\mathrm{chor}}}$ is also a critical point of $\Phi$ and therefore a $2\pi$-periodic solution of $A\dot{z} = \nabla H(z)$ which in addition is $\sigma_0$-choreographic.

The requirements of the abstract degree setting are induced in a canonical way: The space $X_{\mathrm{chor}}$ is a complete subspace of $X$, decomposition into finite-dimensional subspaces is given by $E_k^{\mathrm{chor}} = E_k \cap X_{\mathrm{chor}}$ and instead of $Y = H^2$ we of course have to take the space $Y_{\mathrm{chor}} := \left\{ u \in H^2 : (\sigma_0 * u)(\cdot + \theta_0) = u \right\}$. By the equivariance of $A$, $L_{\mathrm{chor}} := L_{|X_{\mathrm{chor}}}$ maps $X_{\mathrm{chor}}$ into $Y_{\mathrm{chor}}$. So the gradient $\nabla\Phi^{\mathrm{chor}}$ of the restriction $\Phi^{\mathrm{chor}} = \Phi_{|\Lambda_{\mathrm{chor}}}$ satisfies conditions (A1)–(A5) and (A8),(A9).

In analogy to Proposition 2.6 we have

**Proposition 2.13.** *Whenever $u_0$ is a $2\pi$-periodic, $\sigma_0$-nondegenerate solution of $A\dot{z} = \nabla H(z)$, then the associated local degree $S^1\text{-}\deg^\nabla(\nabla\Phi^{\mathrm{chor}}, B_\varepsilon(u_0))$, $\varepsilon > 0$ sufficiently small is nontrivial.*

Moreover, conditions (A6) and (A7) for the global continuation remain true if we consider a continuous family $H : D \subset \mathbb{R}^+ \times \mathbb{R}^{2N} \to \mathbb{R}$ of Hamiltonians with the same regularity assumptions as in 2.2.3 and additionally $H_r(\sigma_0 * z) = H_r(z)$ for any $(r, z) \in D$.

**Remark 2.14.** *The local and global continuation theorems 2.7, 2.9 and Definition 2.11 have to be adapted to the choreographic context in the following way:*

- *A $2\pi$-periodic, $\sigma_0$-nondegenerate solution of $A\dot{z} = \nabla H_{r^*}(z)$ gives rise to a continuous local family $I \ni r \mapsto u^{(r)} \in X_{\mathrm{chor}}$ consisting of $\sigma_0$-nondegenerate solutions of $A\dot{z} = \nabla H_r(z)$ and satisfying the remaining properties of Theorem 2.7.*

*Now we consider a fixed Hamiltonian system $A\dot{z} = \nabla H(z)$:*

- *Let $(r_1, r_2) \ni r \mapsto u^{(r)} \in \Lambda_{\mathrm{chor}}$ be $C^1$ and such that every $u^{(r)}(\cdot/r)$ is a $2\pi r$-periodic, $\sigma_0$-nondegenerate solution. Then there exists an equivariant, connected set $C \subset \mathbb{R}^+ \times \Lambda_{\mathrm{chor}}$ of solutions satisfying the properties of Corollary 2.9.*

- *A family $\left(z^{(r)}\right)_{r \in (r_1, r_2)}$ of choreographic solutions is said to give rise to a global continuum of choreographic solutions, if there exists a connected set $C \subset \mathbb{R}^+ \times \Lambda_{\mathrm{chor}}$ satisfying the properties of Corollary 2.9.*

Summarizing we can say that a choreographic solution, which is nondegenerate in the choreographic sense, gives rise to a local graph and a global continuum of periodic solutions sharing the same choreographic pattern.

## 2.3 Equivariant degree theories and their application to ODEs

There exists a vast amount of equivariant degree theories that have been used to study differential equations. In this section we briefly introduce some of them. In particular we provide in 2.3.1, 2.3.2 the degree theories needed for our modification. For a better overview we refer to the books [7, 44].

First of all we would like to demonstrate that the usual Leray-Schauder degree is of limited help in the equivariant setting. Consider $L - \Psi : \Lambda \to X$ as in Section 2.1.4, let $u_0 \in \Lambda$ be a nondegenerate solution of $L - \Psi = 0$ with $E(u_0) \neq 0$ and suppose that $U \subset \Lambda$ is a bounded, open and $S^1$-invariant neighborhood of $u_0$, such that $\partial U$ contains no solution of $L - \Psi = 0$ and such that the fixed point set $U^{S^1}$ is empty. By assumption (A5) the map

$\mathrm{id}_X - K := (L + P_0)^{-1} \circ (L - \Psi) : \Lambda \to X$ is a compact perturbation of identity and thus accessible for the Leray-Schauder degree. But since $\mathrm{id}_X - K$ is equivariant, Thm. 2 of [14] implies

$$\deg(\mathrm{id}_X - K, U) = \deg(\mathrm{id}_{X^{S^1}} - K^{S^1}, U^{S^1}) = \deg(\mathrm{id}_{X^{S^1}} - K^{S^1}, \emptyset) = 0,$$

where $K^{S^1} = K_{|X^{S^1}} : \Lambda^{S^1} \to X^{S^1}$. Thus the Leray-Schauder degree only detects solutions with isotropy group $S^1$, i.e. solutions that correspond in the Hamiltonian setting to stationary solutions.

So one has to use a degree theory that takes the $S^1$-symmetry into account. Doing this in the case of Hamiltonian systems one possibility is the application of such an equivariant degree after a finite-dimensional reduction – the so called Amann-Zehnder saddle point reduction. This has been done for example by Dancer [27] with the degree theory developed in the same paper or by Rybicki [71] with the degree presented below. But this reduction relies on the boundedness of the second derivative of the Hamiltonian and thus in general can not be done globally. An example with a global finite-dimensional reduction can be found in [37]. In this paper García-Azpeitia and Ize actually study the $N$-vortex problem on the whole plane in a rotating coordinate frame. In particular they prove the existence of global continua of periodic solutions bifurcating from the $N+1$-Gon configuration.

In order to avoid finite-dimensional reductions Rybicki develops in [70] a degree for $S^1$-equivariant strongly indefinite functionals. This degree applies to the gradient of a $S^1$-invariant functional

$$\nabla F : \mathcal{H} \to \mathcal{H}, \quad \nabla F(u) = Au + K(u),$$

where $\mathcal{H}$ is a Hilbert space equipped with a suitable approximation scheme, $A$ a self-adjoint equivariant Fredholm operator and $K$ a compact map. The situation above arises when looking at the action functional of a Hamiltonian system defined on the space $H^{\frac{1}{2}}$, i.e. the gradient of $F : H^{\frac{1}{2}} \to \mathbb{R}$,

$$F(u) = \frac{1}{2}\int_0^{2\pi} \langle \dot{u}, J_N u\rangle_{\mathbb{R}^{2N}}\, dt - \int_0^{2\pi} H(u)\, dt$$

with a suitable Hamiltonian $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$. But as already mentioned before in the case of the $N$-vortex Hamiltonian $H_\Omega$, which is defined on the open subset $\mathcal{F}_N(\Omega)$ it is not clear how to work on $H^{\frac{1}{2}}$, since $H^{\frac{1}{2}}$ does not embed into the space of continuous functions. While working in the space $H^1$, where this problem does not occur, we have seen that the gradient of the action functional $\Phi$ has the form $\nabla\Phi = L - \Psi$. But the linear map $L$ is no longer Fredholm, because $L$ maps $H^1$ into $H^2$. Hence we can not use the degree from [70].

Neither we can use the infinite-dimensional version of the $S^1$-orthogonal degree, cf. Section 2.3.2, which requires $\nabla\Phi$ to be a compact perturbation of identity. On the other hand if we instead pass to $(L + P_0)^{-1} \circ (L - \Psi) = \mathrm{id} - K$, we loose the orthogonality condition stated in (2.5) below.

Also the degree theory introduced by Dylawerski et al., cf. Section 2.3.1, has been applied to periodic solutions of ordinary differential equations, but the notion of an "elementary periodic point" in Thm. 7.3 of [29] is never satisfied in the case of Hamiltonian systems.

The modification of the degree for $S^1$-orthogonal maps (Theorem 2.2) allows us to handle the action functional of a first order Hamiltonian system on the space $H^1$.

### 2.3.1 The degree of Dylawerski et al.

Let $V = (\mathbb{R}^n, \rho)$ be a finite dimensional, orthogonal representation of $S^1$, i.e. $\rho : S^1 \to SO(n)$ is a continuous homomorphism. We usually do not distinguish between $V$ and $\mathbb{R}^n$ and write as before $\theta * v$ instead of $\rho(e^{i\theta})v$.

Add to $V$ a trivial representation of $S^1$, i.e. we consider $(\mathbb{R}^n \oplus \mathbb{R}, \rho \oplus \mathrm{id}_{\mathbb{R}})$, and denote for any closed subgroup $K \leq S^1$ and any subset $U \subset V \oplus \mathbb{R}$ the corresponding fixedpoint set by $U^K = \{ x \in U : K * x = x \}$. Note again that $K \leq S^1$ closed implies $K = S^1$ or $K \cong \mathbb{Z}_k$.

**Theorem 2.15** (Thm. 1.2 of [29]). *Let $\Omega$ run through the family of all open, bounded, invariant subsets of $V \oplus \mathbb{R}$, $f : \overline{\Omega} \to V$ through $S^1$-equivariant, continuous maps with $f(\partial\Omega) \subset V \setminus \{0\}$. Then there exists a function*

$$S^1\text{-deg}(f, \Omega) = \big(\mathrm{d}_0(f, \Omega), (\mathrm{d}_k(f, \Omega))_{k=1}^{\infty}\big) \in \mathbb{Z}_2 \times \mathbb{Z}^{\mathbb{N}},$$

*called the $S^1$-degree, satisfying the following conditions:*

*(a) If $\mathrm{d}_k(f, \Omega) \neq 0$ then $f^{-1}(0) \cap \Omega^K \neq \emptyset$ with $K = S^1$ if $k = 0$, $K = \mathbb{Z}_k$ if $k > 0$.*

*(b) If $\Omega_1, \Omega_2 \subset \Omega$ are open, invariant with $\Omega_1 \cap \Omega_2 = \emptyset$ and $f^{-1}(0) \subset \Omega_1 \cup \Omega_2$ then $S^1\text{-deg}(f, \Omega) = S^1\text{-deg}(f, \Omega_1) + S^1\text{-deg}(f, \Omega_2)$.*

*(c) If $h : ([0,1] \times \overline{\Omega}, [0,1] \times \partial\Omega) \to (V, V \setminus \{0\})$ is a $S^1$-equivariant homotopy then $S^1\text{-deg}(h_0, \Omega) = S^1\text{-deg}(h_1, \Omega)$.*

*(d) Suppose $W$ is another representation of $S^1$ and let $U$ be an open, bounded, invariant subset of $W$ such that $0 \in U$. Define $F : \overline{U} \times \overline{\Omega} \to W \oplus V$ by $F(w, x) = (w, f(x))$. Then $S^1\text{-deg}(F, U \times \Omega) = S^1\text{-deg}(f, \Omega)$.*

The next theorem tells us how to calculate the $S^1$-degree in a special case. Let $\Omega \subset V \oplus \mathbb{R}$ be as in Theorem 2.15. Suppose $f : (\overline{\Omega}, \partial\Omega) \to (V, V \setminus \{0\})$ is $S^1$-equivariant and continuously differentiable, such that 0 is a regular value with $f^{-1}(0) = S^1 * x_0$. Suppose further that the isotropy group $K$ of $x_0$ is finite, i.e. $K = \mathbb{Z}_k$ for a $k \in \mathbb{N}$. As in the infinite-dimensional case let $E(x_0) = \frac{d}{d\theta}_{|\theta=0}(\theta * x_0)$, which is a tangent vector to the submanifold $S^1 * x_0$ at $x_0$.

The derivative $A = Df(x_0) : V \oplus \mathbb{R} \to V$ is $K$-equivariant and splits into

$$A = A^K + A^{\perp} : (V^K \oplus \mathbb{R}) \oplus (V^K)^{\perp} \to V^K \oplus (V^K)^{\perp}.$$

We choose an arbitrary linear functional $a : V^K \oplus \mathbb{R} \to \mathbb{R}$ satisfying $a(E(x_0)) = 1$ and define $\mathcal{A} : V^K \oplus \mathbb{R} \to V^K \oplus \mathbb{R}$, $x \mapsto (A^K(x), a(x))$. The map $\mathcal{A}$ is an isomorphism, since $E(x_0) \in \mathrm{Kern}\, A^K$ and 0 is a regular value of $f$.

**Theorem 2.16** (Thm. 4.1 of [29]). *(i) If $\det A^{\perp} > 0$ then*

$$\mathrm{d}_j(f, \Omega) = \begin{cases} \operatorname{sign} \det \mathcal{A} & \text{if } j = k \\ 0 & \text{else.} \end{cases}$$

*(ii) If $\det A^{\perp} < 0$ then $k$ is even and*

$$\mathrm{d}_j(f, \Omega) = \begin{cases} \operatorname{sgn} \det \mathcal{A} & \text{if } j = k \\ -\operatorname{sgn} \det \mathcal{A} & \text{if } j = k/2 \\ 0 & \text{else.} \end{cases}$$

### 2.3.2 The degree of Rybicki

Let $V$ still be a finite dimensional orthogonal $S^1$-representation. Rybicki uses in [69] the degree of DGJM to construct a degree theory for $S^1$-orthogonal maps. These are continuous equivariant maps $f : V \to V$ satisfying

$$\langle f(v), E(v) \rangle_V = 0 \tag{2.5}$$

for every $v \in V$, where $E(v) = \frac{d}{d\theta}_{|\theta=0}(\theta * v)$ as before. A subclass of $S^1$-orthogonal maps form gradients of $S^1$-invariant functionals on $V$. For the definition of Rybicki's degree assume that $f : (\overline{\Omega}, \partial\Omega) \to (V, V \setminus \{0\})$ is $S^1$-orthogonal and that there exists $\nu > 0$ such that $v = v_0 + v_\perp \in \left(V^{S^1} \oplus (V^{S^1})^\perp\right) \cap \overline{\Omega}$, $|v_\perp| \leq \nu$ implies $f(v) - v \in V^{S^1}$. A map satisfying this condition is called $S^1$-normal. Now let $U_\nu := \{ v = v_0 + v_\perp \in \Omega : |v_\perp| > \nu \} \times (-1, 1)$, $\hat{f} : U_\nu \to V$, $\hat{f}(v, \lambda) = f(v) + \lambda E(v)$ and define

$$S^1\text{-deg}^\perp(f, \Omega) = (\mathrm{d}_k^\perp(f, \Omega))_{k=0}^\infty \in \mathbb{Z}^{\mathbb{N}_0}$$

via

$$\mathrm{d}_k^\perp(f, \Omega) = \begin{cases} \deg\left(f_{|\Omega^{S^1}}, \Omega^{S^1}\right) & \text{if } k = 0, \\ \mathrm{d}_k(\hat{f}, U_\nu) & \text{else.} \end{cases}$$

Here deg denotes the classical Brouwer degree. Note that $\hat{f}$ is indeed admissible for $U_\nu$ due to the $S^1$-normality of $f$. For a general $S^1$-orthogonal map $g$, Rybicki shows the existence of an admissible homotopy connecting $g$ with a $S^1$-normal map $f$.

The definition $S^1\text{-deg}^\perp(g, \Omega) := S^1\text{-deg}^\perp(f, \Omega)$ turns out to be indeed well-defined and the degree defined in that way has the usual properties:

**Theorem 2.17** (Thm. 3.9 of [69]). *Let $\Omega \subset V$ be an open, bounded and $S^1$-invariant subset and $f : (\overline{\Omega}, \partial\Omega) \to (V, V \setminus \{0\})$ be $S^1$-orthogonal. The degree $S^1\text{-deg}^\perp$ has the following properties:*

*a) if $\mathrm{d}_k^\perp(f, \Omega) \neq 0$ for some $k \in \mathbb{N}_0$, then there exists $v \in \Omega^K$ with $f(v) = 0$ where $K = S^1$ if $k = 0$, resp. $K = \mathbb{Z}_k$ if $k \geq 1$.*

*b) if $\Omega_0 \subset \Omega$ is an open, $S^1$-invariant subset such that $f^{-1}(0) \cap \Omega \subset \Omega_0$, then*

$$S^1\text{-deg}^\perp(f, \Omega) = S^1\text{-deg}^\perp(f, \Omega_0)$$

*c) if $\Omega_1$ and $\Omega_2$ are open $S^1$-invariant subsets of $\Omega$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $f^{-1}(0) \cap \Omega$ is contained in the union $\Omega_1 \cup \Omega_2$, then*

$$S^1\text{-deg}^\perp(f, \Omega) = S^1\text{-deg}^\perp(f, \Omega_1) + S^1\text{-deg}^\perp(f, \Omega_2).$$

*d) if $h : (\overline{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V \setminus \{0\})$ is a $S^1$-orthogonal homotopy, then*

$$S^1\text{-deg}^\perp(h(\cdot, 0), \Omega) = S^1\text{-deg}^\perp(h(\cdot, 1), \Omega)$$

*e) let $W$ be another representation of the group $S^1$ and let $U \subset W$ be an open, bounded and $S^1$-invariant subset such that $0 \in U$. Define a map $F : \overline{U \times \Omega} \to W \oplus V$ by the formula $F(w, v) = (w, f(v))$. Then*

$$S^1\text{-deg}^\perp(F, U \times \Omega) = S^1\text{-deg}^\perp(f, \Omega).$$

Moreover, restricted to the class of gradients he could in Thm. 2.11 of [72] also prove the following multiplication rule.

**Theorem 2.18** (Thm. 2.11 of [72]). *Let $\Omega_i \subset V_i$, $i = 1, 2$ be open, invariant and bounded subsets of $S^1$-representations $V_i$. If $f_i : (\overline{\Omega}_i, \partial\Omega_i) \to (V_i, V_i \setminus \{0\})$ are gradients of $S^1$-invariant functionals, then*

$$S^1\text{-deg}^\perp(f_1 \times f_2, \Omega_1 \times \Omega_2) = S^1\text{-deg}^\perp(f_1, \Omega_1) \star S^1\text{-deg}^\perp(f_2, \Omega_2).$$

A formula for the computation of Rybicki's degree can also be found in [69]. We will here state it only in the case of a linear isomorphism, which will be needed for our modification. Suppose that the finite-dimensional representation $V$ is given in terms of the classification Theorem 2.1, i.e. $V = \bigoplus_{i=1}^{r} V_{m_i}$ with $V_{m_i} \cong R[k_i, m_i]$, and consider an equivariant, symmetric isomorphism $T : V \longrightarrow V$. By Schur's Lemma, see 3.22 in [1], each restriction $T_{m_i} = T_{|V_{m_i}}$ is an isomorphism $V_{m_i} \longrightarrow V_{m_i}$. Let $\mu_{m_i}$ denote the Morse index of $T_{m_i}$. Due to the equivariance of $T$, the indices $\mu_{m_i}$ are even for $m_i \neq 0$. Indeed if $v$ is an eigenvector, so is $E(v)$. It follows that $\operatorname{sign}\det T = 1$, if $m_i \neq 0$ for all $i = 1, \ldots, r$, and otherwise $\operatorname{sign}\det T = \operatorname{sign}\det T_0$. For equivariant isomorphisms we let

$$S^1\text{-}\deg^{\perp}(T, V) := S^1\text{-}\deg^{\perp}(T, B_1(0)).$$

**Proposition 2.19** (Cor. 4.3 of [69]). *The orthogonal degree $S^1$-$\deg^{\perp}(T, V)$ of an equivariant, symmetric isomorphism $T : V \longrightarrow V$ is given by*

$$\mathrm{d}_j^{\perp}(T, V) = \begin{cases} \operatorname{sign}\det T & \text{if } j = 0, \\ \frac{1}{2}\mu_{m_i} \operatorname{sign}\det T & \text{if } j = m_i, \\ 0 & \text{else.} \end{cases}$$

## 2.4 Construction of the degree

Here we will extend the finite-dimensional degree of Rybicki to our infinite-dimensional setting. We consider the $S^1$-equivariant map $L - \Psi : \Lambda \longrightarrow X$ defined on an open, invariant subset $\Lambda$ of an infinite-dimensional, orthogonal $S^1$-representation $X$ as in section 2.1.2. Let $\mathcal{O} \subset X$ be open, invariant, bounded with $\overline{\mathcal{O}} \subset \Lambda$ and such that $Lu - \Psi(u) = 0$ has no solution on $\partial\mathcal{O}$. By the assumptions (A1)–(A5) the set of solutions $\mathcal{S} = (L - \Psi)^{-1}(0) \cap \mathcal{O}$ is compact.

**Lemma 2.20.** *There exists $n_0 \in \mathbb{N}$ and an invariant neighborhood $B_\varepsilon(\mathcal{S}) \subset \mathcal{O}$ such that for every $m, n \geq n_0$ and every $t \in [0, 1]$ the following holds true*

*a) $u \in \overline{B_\varepsilon(\mathcal{S})}$ implies $(P_m + t(P_n - P_m)) \in \mathcal{O}$,*

*b) $Lu - (P_m + t(P_n - P_m))\, \Psi\left(((P_m + t(P_n - P_m))\, u\right) = 0$ has no solution on $\partial B_\varepsilon(\mathcal{S})$,*

*c) $Lu - P_n\Psi(u) = 0$ has no solution in $\overline{\mathcal{O}} \setminus B_\varepsilon(\mathcal{S})$.*

*Proof.* We prove the properties step-by-step. If the first one is wrong, we can find sequences $m_k, n_k \geq k$, $t_k \in [0, 1]$ and $u_k \in \overline{B_{\frac{1}{k}}(\mathcal{S})}$ satisfying $p_k := \left(P_{m_k} + t_k(P_{n_k} - P_{m_k})\right) u_k \notin \mathcal{O}$. It follows the existence of a solution $w_k \in \mathcal{S}$ satisfying $\|u_k - w_k\|_X \leq \frac{1}{k}$. By the compactness of $\mathcal{S}$ we can assume $w_k \longrightarrow w$, as well as $t_k \longrightarrow t$, for some $w \in \mathcal{S}$, $t \in [0, 1]$. It follows $u_k \longrightarrow w$ and then $X \setminus \mathcal{O} \ni p_k \longrightarrow w \in \mathcal{S}$, which contradicts $\operatorname{dist}(\partial\mathcal{O}, \mathcal{S}) > 0$.

Now we fix $B_\varepsilon(\mathcal{S})$ and an index $n_1 \in \mathbb{N}$, such that a) is true for any $m, n \geq n_1$ and $t \in [0, 1]$. As a consequence $\Psi\left((P_m + t(P_n - P_m))\, u\right)$ is well-defined for such $m, n, t$ and $u \in \overline{B_\varepsilon(\mathcal{S})}$. Next we assume b) to be wrong and find sequences $m_k, n_k \geq k \geq n_1$, $t_k \in [0, 1]$, $u_k \in \partial B_\varepsilon(\mathcal{S})$ such that

$$Lu_k - Q_k(u_k) := Lu_k - \left(P_{m_k} + t_k(P_{n_k} - P_{m_k})\right) \Psi\left(\left(P_{m_k} + t_k(P_{n_k} - P_{m_k})\right) u_k\right) = 0$$

for all $k \geq n_1$. By (A3) this equation can be rewritten as

$$u_k - (L + P_0)^{-1}\left[P_0 u_k + Q_k(u_k)\right] = 0.$$

Since $X_0 = E_0$ is finite-dimensional and since $(L+P_0)^{-1}$ commutes with the projections $P_{m_k}$, $P_{n_k}$, we may by (A5) assume that $t_k \to t \in [0,1]$, $P_0 u_k \to v$ and $(L+P_0)^{-1}[Q_k(u_k)] \to w$. Thus $u_k \in \partial B_\varepsilon(\mathcal{S})$ converges to $u := v + w \in \partial B_\varepsilon(\mathcal{S})$ with $Lu - \Psi(u) = 0$, a contradiction.

Property c) follows with a similar, easier indirect argument, because $L - \Psi = 0$ has no solution in the closed set $\overline{\mathcal{O}} \setminus B_\varepsilon(\mathcal{S})$. □

We abbreviate the Brouwer degree of $L - \Psi$ restricted to the fixed point set $\mathcal{O}^{S^1}$ by

$$\mathrm{d}_0 = \deg(L - \Psi, \mathcal{O}^{S^1}) = \mathrm{d}_0^\perp(L - \Psi, \mathcal{O}).$$

Note that $P_n\Psi : \Lambda \cap X_n \to X_n$ is the gradient of $\mathfrak{R}_{|\Lambda\cap X_n}$ and hence a $S^1$-orthogonal map.

**Lemma 2.21.** *Let $\mathcal{O}$ and $n_0$ be as in Lemma 2.20. Then the difference*

$$S^1\text{-}\deg^\perp(L - P_n\Psi, \mathcal{O} \cap X_n) - \mathrm{d}_0 \cdot S^1\text{-}\deg^\perp(L + P_0, X_n)$$

*is independent of $n \geq n_0$.*

*Proof.* By Lemma 2.20 b) and c) we can make an excision, followed by the use of the homotopy invariance and another excision to get

$$\begin{aligned} S^1\text{-}\deg^\perp(L - P_n\Psi, \mathcal{O} \cap X_n) &= S^1\text{-}\deg^\perp(L - P_n\Psi, B_\varepsilon(\mathcal{S}) \cap X_n) \\ &= S^1\text{-}\deg^\perp(L - P_{n_0}\Psi(P_{n_0}\cdot), B_\varepsilon(\mathcal{S}) \cap X_n). \end{aligned}$$

Note here that the homotopy $h_t : \overline{B_\varepsilon(\mathcal{S})} \cap X_n \to X_n$, $h_t = L - \nabla\big(\mathfrak{R}_{|\Lambda\cap X_n} \circ (P_n + t(P_{n_0} - P_n))\big)$ is indeed well-defined by 2.20 a), $S^1$-orthogonal and admissible by 2.20 b).

Next observe that $\mathcal{S}_0 = (L - P_{n_0} \circ \Psi \circ P_{n_0})^{-1}(0) \cap B_\varepsilon(\mathcal{S})$ is contained in $X_{n_0}$ and compact by 2.20 b). We therefore split $X_n$ into $X_n = X_{n_0} \oplus (X_n \cap X_{n_0}^\perp)$ and find open and invariant neighborhoods $U_1 \subset X_n$ of $\mathcal{S}_0$ and $U_2 \subset X_n \cap X_{n_0}^\perp$ of 0, such that the product $U_1 \times U_2$ is contained in $B_\varepsilon(\mathcal{S}) \cap X_n$. Several excisions and the multiplication formula 2.18 show

$$\begin{aligned} S^1\text{-}\deg^\perp(L - P_{n_0}\Psi(P_{n_0}\cdot), B_\varepsilon(\mathcal{S}) \cap X_n) &= S^1\text{-}\deg^\perp(L - P_{n_0}\Psi(P_{n_0}\cdot), U_1 \times U_2) \\ &= S^1\text{-}\deg^\perp(L - P_{n_0}\Psi, \mathcal{O} \cap X_{n_0}) \star S^1\text{-}\deg^\perp(L, X_n \cap X_{n_0}^\perp). \end{aligned}$$

Now $\mathrm{d}_0^\perp(L, X_n \cap X_{n_0}^\perp) = 1$ by 2.19 and $\mathrm{d}_0^\perp(L - P_{n_0}\Psi, \mathcal{O} \cap X_{n_0}) = \mathrm{d}_0$ yield

$$\begin{aligned} &S^1\text{-}\deg^\perp(L - P_n\Psi, \mathcal{O} \cap X_n) + (\mathrm{d}_0, 0, 0, \ldots) = \\ &\qquad S^1\text{-}\deg^\perp(L - P_{n_0}\Psi, \mathcal{O} \cap X_{n_0}) + \mathrm{d}_0 \cdot S^1\text{-}\deg^\perp(L, X_n \cap X_{n_0}^\perp). \end{aligned}$$

Combining this equation with

$$\begin{aligned} S^1\text{-}\deg^\perp(L + P_0, X_n) &= S^1\text{-}\deg^\perp(L + P_0, X_{n_0}) \star S^1\text{-}\deg^\perp(L, X_n \cap X_{n_0}^\perp) \\ &= S^1\text{-}\deg^\perp(L + P_0, X_{n_0}) + S^1\text{-}\deg^\perp(L, X_n \cap X_{n_0}^\perp) - (1, 0, 0, \ldots) \end{aligned}$$

finally shows that

$$S^1\text{-}\deg^\perp(L - P_n\Psi, \mathcal{O} \cap X_n) - \mathrm{d}_0 \cdot S^1\text{-}\deg^\perp(L + P_0, X_n)$$

does not depend on $n \geq n_0$. □

Having Lemma 2.21 at hand we are ready to define the degree for $S^1$-equivariant gradient maps.

**Definition 2.22.** *For $L-\Psi:\Lambda\to X$ satisfying (A1)–(A5) and bounded, invariant, open subsets $\mathcal{O}\subset X$ with $\overline{\mathcal{O}}\subset\Lambda$ and $(L-\Psi)^{-1}(0)\cap\partial\mathcal{O}=\emptyset$ we define*

$$S^1\text{-}\deg^\nabla(L-\Psi,\mathcal{O})=\left(\mathrm{d}_k^\nabla(L-\Psi,\mathcal{O})\right)_{k\in\mathbb{N}_0}\in\bigoplus_{k=0}^{\infty}\mathbb{Z},$$

*where* $\mathrm{d}_0^\nabla(L-\Psi,\mathcal{O})=\deg(L-\Psi,\mathcal{O}^{S^1})$ *and for* $k\neq 0$*:*

$$\mathrm{d}_k^\nabla(L-\Psi,\mathcal{O})=\lim_{n\to\infty}\left(\mathrm{d}_k^\perp(L-P_n\Psi,\mathcal{O}\cap X_n)-\deg(L-\Psi,\mathcal{O}^{S^1})\cdot\mathrm{d}_k^\perp(L+P_0,X_n)\right).$$

It remains to prove that the degree has the stated properties (D1)–(D4).

*Proof of Thm. 2.2.* (D1) (Existence): Let $\mathrm{d}_k^\nabla(L-\Psi,\mathcal{O})\neq 0$. If $k=0$, we find a solution $u\in\mathcal{O}^{S^1}$ of $L-\Psi=0$ by the corresponding property of the Brouwer degree. If $k\neq 0$, by our definition we can find $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ there holds $\mathrm{d}_k^\perp(L-P_n\Psi,\mathcal{O}\cap X_n)\neq 0$ or $\deg(L-\Psi,\mathcal{O}^{S^1})\neq 0$. In the latter case we are done, since $\mathcal{O}^{S^1}\subset\mathcal{O}^{\mathbb{Z}_k}$. Otherwise we find for any $n\geq n_0$ an element $u_n\in\mathcal{O}^{\mathbb{Z}_k}\cap X_n$ solving $L-P_n\Psi=0$. By our compactness condition (A5) we can conclude that along a subsequence $u_n\to u^*\in\mathcal{O}^{\mathbb{Z}_k}$ with $Lu^*-\Psi(u^*)=0$.

(D2) (Excision and additivity): Replacing $\partial\mathcal{O}$ by $\overline{\mathcal{O}}\setminus(\mathcal{O}_1\cup\mathcal{O}_2)$ in the proof of Lemma 2.20 and using $t=1$ shows that $L-P_n\Psi=0$ does not have a solution in $\overline{\mathcal{O}}\setminus(\mathcal{O}_1\cup\mathcal{O}_2)$ for all $n$ large enough. Therefore (D2) follows from the corresponding properties of the Brouwer degree and Theorem 2.17 b),c).

(D4) (Multiplicativity): This is a straightforward calculation based on the multiplication properties of the degree theories of Brouwer and Rybicki 2.18.

(D3) (Homotopy): First of all we can extend the homotopy invariance stated in Theorem 2.17 d), i.e. we consider a continuous family of $S^1$-gradient maps $L_t-\Psi_t:\Lambda\to X$, $t\in[0,1]$ and a fixed open, invariant and bounded set $\mathcal{O}\subset X$ with $\overline{\mathcal{O}}\subset\Lambda$ and $(L_t-\Psi_t)(\partial\mathcal{O})\subset X\setminus\{0\}$ for all $t\in[0,1]$. As in Lemma 2.20 one can see that there exists $n_0\in\mathbb{N}$ such that the finite-dimensional equation $L_t-P_n\Psi_t=0$ has no solution on $\partial\mathcal{O}\cap X_n$ for all $n\geq n_0$, $t\in[0,1]$. Thus by Theorem 2.17 d), the homotopy invariance of the Brouwer degree and since every $L_t+P_0:X_n\to X_n$ is an isomorphism we obtain that $S^1\text{-}\deg^\nabla(L_t-\Psi_t,\mathcal{O})$ is independent of $t\in[0,1]$.

It remains to prove the generalized homotopy invariance. So let $L_t-\Psi_t:\Lambda\to X$ be as before and consider $\mathcal{U}\subset[0,1]\times X$ (rel.) open, bounded, such that every section $\mathcal{U}_t=\{u\in X:(t,u)\in\mathcal{U}\}$ is invariant and $\overline{\mathcal{U}}_t\subset\Lambda$, $(L_t-\Psi_t)^{-1}(0)\cap\partial\mathcal{U}_t=\emptyset$. We extend $L_t-\Psi_t$ by $L_0-\Psi_0$ for $t<0$ and by $L_1-\Psi_1$ for $t>1$. Furthermore, let $t_0\in[0,1]$, $\tilde{X}=\mathbb{R}\oplus X$ with trivial $S^1$-action on $\mathbb{R}$, $\tilde{\Lambda}=\mathbb{R}\times\Lambda\subset\tilde{X}$ and define $\tilde{L}\in\mathcal{L}(\tilde{X})$, $\tilde{\Psi}:\tilde{\Lambda}\to\tilde{X}$ by

$$\tilde{L}(t,u)=(0,L_tu),\qquad\tilde{\Psi}(t,u)=(t_0-t,\Psi_t(u)).$$

With canonical modifications like $\tilde{E}_0=\mathbb{R}\oplus E_0$ it is easy to see that $\tilde{X},\tilde{L}$ and $\tilde{\Psi}$ satisfy (A1)–(A5). Next we extend $\mathcal{U}$ to an open subset of $\tilde{X}$ via

$$\tilde{\mathcal{U}}=\mathcal{U}\cup(-1,0]\times\mathcal{U}_0\cup[1,2)\times\mathcal{U}_1.$$

Then $\tilde{\mathcal{U}}$ is also invariant, bounded and its closure is contained in $\tilde{\Lambda}$. Clearly the homotopy $\tilde{h}:[0,1]\times\tilde{\Lambda}\to\tilde{X}$,

$$\tilde{h}_\lambda(t,u)=(t-t_0,\lambda(L_{t_0}-\Psi_{t_0})(u)+(1-\lambda)(L_t-\Psi_t)(u))$$

is admissible with respect to $\tilde{\mathcal{U}}$. So we can use the homotopy invariance just shown together with a suitable excision around $\{t_0\}\times((L_{t_0}-\Psi_{t_0})^{-1}(0)\cap\mathcal{U}_{t_0})$ and the mutliplicativity (D4) to conclude

$$\begin{aligned}S^1\text{-deg}^\nabla(\tilde{L}-\tilde{\Psi},\tilde{\mathcal{U}}) &= S^1\text{-deg}^\nabla(\tilde{h}_0,\mathcal{U}) = S^1\text{-deg}^\nabla(\tilde{h}_1,\mathcal{U})\\ &= S^1\text{-deg}^\nabla(\cdot - t_0,(t_0-\varepsilon,t_0+\varepsilon)) \star S^1\text{-deg}^\nabla(L_{t_0}-\Psi_{t_0},\mathcal{U}_{t_0})\\ &= S^1\text{-deg}^\nabla(L_{t_0}-\Psi_{t_0},\mathcal{U}_{t_0}).\end{aligned}$$

Now if we do the same construction with another parameter value $t_1 \in [0,1]$, say we define $\tilde{\Phi}(t,u) = (t_1 - t, \Psi_t(u))$, then we can clearly connect $\tilde{L}-\tilde{\Psi}$ and $\tilde{L}-\tilde{\Phi}$ with an $\tilde{\mathcal{U}}$-admissible homotopy. Therefore

$$S^1\text{-deg}^\nabla(L_{t_0}-\Psi_{t_0},\mathcal{U}_{t_0}) = S^1\text{-deg}^\nabla(\tilde{L}-\tilde{\Psi},\tilde{\mathcal{U}}) = S^1\text{-deg}^\nabla(\tilde{L}-\tilde{\Phi},\tilde{\mathcal{U}}) = S^1\text{-deg}^\nabla(L_{t_1}-\Psi_{t_1},\mathcal{U}_{t_1}).$$

□

## 2.5 The global continuation theorem

The proof of Theorem 2.3 uses a refinement of Whyburn's lemma. Recall that a topological space $S$ is normal provided every two disjoint closed subsets of $S$ have disjoint open neighborhoods. Two subsets $A, B \subset S$ are separated in $S$, if there exist $U, V \subset S$ disjoint, open, nonempty satisfying $S = U \cup V$ and $A \subset U$, $B \subset V$.

**Proposition 2.23** (Prop. 5 of [2]). *Let $S$ be a compact, normal topological space. If $A \subset S$ and $B \subset S$ are closed and not separated, then there exists a connected set $C \subset S \setminus (A \cup B)$ such that $\overline{C} \cap A \neq \emptyset$, $\overline{C} \cap B \neq \emptyset$.*

We consider now the family of equations

$$Lu - \Psi(r,u) = 0, \quad (r,u) \in \mathcal{D} \subset \mathbb{R}^+ \times X,$$

where $X, L$ satisfy (A1)–(A3) and $\Psi$ satisfies (A6),(A7). Recall that we defined $\mathcal{S}$ as the set of solutions

$$\mathcal{S} = \{\, (r,u) \in \mathcal{D} : Lu - \Psi(r,u) = 0 \,\}$$

and that we write $M_r = \{\, u \in X : (r,u) \in M \,\}$ for $M \subset \mathbb{R}^+ \times X$, $r \in \mathbb{R}^+$.

*Proof of Theorem 2.3.* We first add two points at infinity to the set $\mathcal{D} \setminus \partial\mathcal{U}$:

$$\mathcal{D}^* = (\mathcal{D} \setminus \partial\mathcal{U}) \cup \{\, \infty_1, \infty_2 \,\}\,.$$

In order to define the topology of $\mathcal{D}^*$ we set for $0 < \varepsilon < 1$:

$$\mathcal{D}(\varepsilon) = \left\{\, (r,u) \in \mathcal{D} : r \in [\varepsilon, \varepsilon^{-1}],\ \operatorname{dist}(u, \partial\mathcal{D}_r) \geq \varepsilon,\ \|u\|_X \leq \varepsilon^{-1} \,\right\}.$$

A neighborhood basis of $\infty_1$ is given by the family $(\{\infty_1\} \cup \mathcal{U}) \setminus \mathcal{D}(1/n)$, $n \in \mathbb{N}$, and a neighborhood basis of $\infty_2$ is given by $(\{\infty_2\} \cup (\mathcal{D} \setminus \overline{\mathcal{U}})) \setminus \mathcal{D}(1/n)$, $n \in \mathbb{N}$. Then $\mathcal{D}^*$ is a normal topological space and $\mathcal{S}^* := \mathcal{S} \cup \{\infty_1, \infty_2\}$ is a compact subspace of $\mathcal{D}^*$. We need to prove that there exists a connected set $C \subset \mathcal{S}$ such that $\infty_1, \infty_2 \in \overline{C} \subset \mathcal{D}^*$. According to Proposition 2.23 it is sufficient to show that $\infty_1$ and $\infty_2$ are not separated in $\mathcal{S}^*$. Arguing by contradiction suppose that there exist two open subsets $V_1, V_2 \subset \mathcal{D}^*$ such that $V_1 \cap V_2 = \emptyset$, $\infty_1 \in V_1$, $\infty_2 \in V_2$, and $\mathcal{S}^* \subset V_1 \cup V_2$. Then

$$V_1 \subset \{\infty_1\} \cup \mathcal{U} \cup \mathcal{D}(\varepsilon)^\circ \quad \text{and} \quad V_2 \subset \{\infty_2\} \cup \mathcal{D} \setminus \overline{\mathcal{U} \setminus \mathcal{D}(\varepsilon)}$$

for some $0 < \varepsilon < \min\{1, r_0\}$. Since $\mathcal{S}$ and $\mathcal{U}$ are $S^1$-invariant, we can without restriction also assume that $V_j \setminus \{\infty_j\} \subset \mathcal{D}$, $j = 1, 2$ are invariant. By (D2),(D3) it follows that

$$\begin{aligned} &S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_1 \cap \mathcal{U})_{r_0}) + S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_1 \setminus \overline{\mathcal{U}})_{r_0}) \\ &\quad = S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_1)_{r_0}) = S^1\text{-deg}(L - \Psi(1/\varepsilon, \cdot), (V_1)_{1/\varepsilon}) = 0 \end{aligned}$$

and

$$S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_1 \setminus \overline{\mathcal{U}})_{r_0}) = S^1\text{-deg}(L - \Psi(\varepsilon, \cdot), (V_1 \setminus \overline{\mathcal{U}})_{\varepsilon}) = 0,$$

hence

$$S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_1 \cap \mathcal{U})_{r_0}) = 0.$$

Moreover, we have

$$S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_2 \cap \mathcal{U})_{r_0}) = S^1\text{-deg}(L - \Psi(\varepsilon, \cdot), (V_2 \cap \mathcal{U})_{\varepsilon}) = 0.$$

This leads to the contradiction

$$\begin{aligned} 0 &\neq S^1\text{-deg}(L - \Psi(r_0, \cdot), \mathcal{U}_{r_0}) \\ &= S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_1 \cap \mathcal{U})_{r_0}) + S^1\text{-deg}(L - \Psi(r_0, \cdot), (V_2 \cap \mathcal{U})_{r_0}) = 0. \end{aligned}$$

□

## 2.6 Calculation of the degree

The degree of a nondegenerate solution will be expressed in terms of signs of compact linear perturbations of identity. Let $\tilde{X}$ be an arbitrary Banachspace and $Q : \tilde{X} \to \tilde{X}$ compact linear with spectrum $\sigma(Q)$. For $\lambda \in \sigma(Q) \setminus \{0\}$ denote by $G_\lambda = \bigcup_k \operatorname{Kern}(Q - \lambda \operatorname{id})^k$ the generalized eigenspace. The sign of $\operatorname{id} - Q$ is defined by

$$\operatorname{sign}(\operatorname{id} - Q) = (-1)^{m_-}, \quad \text{with} \quad m_- = \dim \bigoplus_{\lambda \in \sigma(K) \cap (1,\infty)} G_\lambda.$$

This is of course possible due to the spectral theorem of Riesz-Schauder.

Let now $L - \Psi : \Lambda \to X$ satisfy (A1)–(A5) and assume that (A8),(A9) hold true as well. In this situation we use the (nonlinear) compact map $K : \Lambda \to X$,

$$K(u) = P_0 u + (L + P_0)^{-1} \Psi(u),$$

such that $(L + P_0)^{-1} \circ (L - \Psi) = \operatorname{id} - K$.

We consider a nondegenerate solution $u_0 \in \Lambda$ of $L - \Psi = 0$ having finite isotropy group $I_{u_0} = \mathbb{Z}_k$. The derivative $\operatorname{id} - DK(u_0)$ is $\mathbb{Z}_k$-equivariant and maps $X^{\mathbb{Z}_k}$ into itself. The same is true for $(X^{\mathbb{Z}_k})^\perp$, since $\operatorname{id} - DK(u_0)$ is up to the isomorphism $(L + P_0)^{-1}$ the second derivative of a functional.

Recall that by (A9) and the definition of a nondegenerate solution in 2.4 $u_0 \in Y$ and $E(u_0) \in X$ – in fact $E(u_0) \in X^{\mathbb{Z}_k}$ – and $\operatorname{Kern}(\operatorname{id} - DK(u_0)) = \mathbb{R}E(u_0)$.

In order to formulate our theorem we define the linear map $Q \in \mathcal{L}(X)$,

$$Qu = \frac{\langle u, E(u_0) \rangle_X}{\|E(u_0)\|_X \, \|(L + P_0)E(u_0)\|_X} (L + P_0) E(u_0)$$

and the following two signs:

$$s_\perp := \operatorname{sign}\left(\operatorname{id} - DK(u_0) : (X^{\mathbb{Z}_k})^\perp \to (X^{\mathbb{Z}_k})^\perp\right),$$
$$s_0 := \operatorname{sign}\left(\operatorname{id} - DK(u_0) + Q : X^{\mathbb{Z}_k} \to X^{\mathbb{Z}_k}\right).$$

**Theorem 2.24.** *Let $u_0 \in \Lambda$ be a nondegenerate solution of $Lu - \Psi(u) = 0$ with isotropy group $I_{u_0} = \mathbb{Z}_k$. There exists an invariant neighborhood $O$ of $S^1 * u_0$ such that $(L-\Psi)^{-1}(0) \cap \overline{O} = S^1 * u_0$ and*

*(i) if $s_\perp = 1$, then*

$$\mathrm{d}_j^\nabla(L - \Psi, O) = \begin{cases} -s_0 & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases}$$

*(ii) if $s_\perp = -1$, then $k$ is even and*

$$\mathrm{d}_j^\nabla(L - \Psi, O) = \begin{cases} -s_0 & \text{if } j = k \\ s_0 & \text{if } j = k/2 \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 2.25.** *Theorem 2.5 directly follows from Theorem 2.24 in the case $I_{u_0} = \mathbb{Z}_k$ and by the analogous property of the Brouwer degree if $I_{u_0} = S^1$.*

**Lemma 2.26.** *In the situation of Theorem 2.24 there exists an invariant neighborhood $O$ of $S^1 * u_0$ with $\overline{O} \cap X^{S^1} = \emptyset$, as well as an index $n_0 \in \mathbb{N}$ and a sequence $(u_n)_{n \geq n_0} \subset O$ such that for all $n \geq n_0$:*

$$(L - \Psi)^{-1}(0) \cap \overline{O} = S^1 * u_0, \quad (L - P_n\Psi)^{-1}(0) \cap \overline{O} = S^1 * u_n, \quad I_{u_n} = \mathbb{Z}_k,$$
$$\operatorname{Kern}(L - P_n D\Psi(u_n)) = \mathbb{R}E(u_n), \quad u_n \to u_0, \quad \frac{E(u_n)}{\|E(u_n)\|_X} \to \frac{E(u_0)}{\|E(u_0)\|_X}.$$

*Proof.* Let $N_{u_0} = \{ u \in X : \langle u, E(u_0) \rangle = 0 \}$, $N_{u_0}^\varepsilon = N_{u_0} \cap B_\varepsilon(u_0)$ and observe that $E(u_0) \neq 0$, $u_0 \in N_{u_0}$ and $B_\varepsilon(S^1 * u_0) = S^1 * N_{u_0}^\varepsilon$ for any $\varepsilon > 0$. The orthogonal projection onto $N_{u_0}$ is denoted by $P_{N_{u_0}} : X \to N_{u_0}$. We abbreviate $E_\star := (L + P_0)E(u_0)$ and denote by $N_{u_0}^\star$ its orthogonal complement and by $P_{N_{u_0}^\star}$ the corresponding projection.

*Claim* 2.26.1. There exists an invariant, bounded neighborhood $O_1$ of $S^1 * u_0$ with $\overline{O}_1 \subset \Lambda$, $\overline{O}_1 \cap X^{S^1} = \emptyset$ and $(L - \Psi)^{-1}(0) \cap \overline{O}_1 = S^1 * u_0$.

*Proof.* Clearly $L - \Psi = 0$ if and only if $\operatorname{id} - K = 0$. The derivative $\operatorname{id} - DK(u_0)$ is a compact perturbation of identity on $X$ and therefore an index 0 Fredholm-operator, i.e.

$$\operatorname{codim} \operatorname{Range}(\operatorname{id} - DK(u_0)) = \dim \operatorname{Kern}(\operatorname{id} - DK(u_0)) = 1.$$

Moreover, we have, since $L - D\Psi(u_0)$ is as the second derivative of a functional self-adjoint, the inclusion $\operatorname{Range}(\operatorname{id} - DK(u_0)) \subset N_{u_0}^\star$. Thus by comparing dimensions equality holds and therefore $\operatorname{id} - DK(u_0) : N_{u_0} \to N_{u_0}^\star$ is an isomorphism. Hence the inverse function theorem implies the existence of $\varepsilon_0 > 0$ such that $P_{N_{u_0}^\star} \circ (\operatorname{id} - K) : \overline{N_{u_0}^{\varepsilon_0}} \to N_{u_0}^\star$ is a diffeomorphism onto its image. We set $O_1 := S^1 * N_{u_0}^{\varepsilon_0}$. Then $u \in \overline{O_1}$, $Lu - \Psi(u) = 0$ implies $\theta * u \in \overline{N_{u_0}^{\varepsilon_0}}$ for some $\theta \in S^1$ and $P_{N_{u_0}^\star}(\theta * u - K(\theta * u)) = 0$. Therefore $\theta * u = u_0$. Due to $I_{u_0} = \mathbb{Z}_k$ we can without restriction also assume that $\overline{O_1} \cap X^{S^1} = \emptyset$. ∎

Next we want to show that there exists an invariant subset $O \subset O_1$ such that for all $n \in \mathbb{N}$ big enough $(L - P_n\Psi)^{-1}(0) \cap O \cap X_n$ contains exactly one nondegenerate orbit. In order to do this we define $g_n : \overline{N_{u_0}^{\varepsilon_0}} \longrightarrow N_{u_0}^{\star}$,

$$g_n(u) = P_{N_{u_0}^{\star}}\left(u - P_nK(u)\right).$$

*Claim* 2.26.2. There exists $\varepsilon_1 \in (0, \varepsilon_0)$ and $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ one can find in $\overline{N_{u_0}^{\varepsilon_1}}$ a unique zero $u_n$ of $g_n$. Moreover, $u_n \longrightarrow u_0$ as $n \longrightarrow \infty$ and the derivative $Dg_n(u_n)$ has trivial kernel.

*Proof.* Clearly $u \in N_{u_0}^{\varepsilon_0}$ is a zero of $g_n$ if and only if $u$ is a fixpoint of $T_n : N_{u_0}^{\varepsilon_0} \longrightarrow N_{u_0}$,

$$T_n(u) = u - (\mathrm{id} - DK(u_0))^{-1}[g_n(u)].$$

Let $c_0 := \left\|(\mathrm{id} - DK(u_0))^{-1}\right\|_{\mathcal{L}(N_{u_0}^{\star}, N_{u_0})}$ and choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$\|DK(u) - DK(u_0)\|_{\mathcal{L}(X)} \leq \frac{1}{4c_0} \quad \text{for all } u \in \overline{N_{u_0}^{\varepsilon_1}}.$$

Then choose $n_0 \in \mathbb{N}$ with

$$\|u_0 - P_nu_0\|_X \leq \frac{\varepsilon_1}{2c_0} \quad \text{and} \quad \|DK(u_0) - P_nDK(u_0)\|_{\mathcal{L}(X)} \leq \frac{1}{4c_0} \quad \text{for all } n \geq n_0.$$

The latter is possible since $DK(u_0)$ is compact. With these choices we have for $u, v \in \overline{N_{u_0}^{\varepsilon_1}}$ and $n \geq n_0$

$$\begin{aligned}
\|T_n(u) - T_n(v)\|_X &= \left\|u - v - (\mathrm{id} - DK(u_0))^{-1} \circ P_{N_{u_0}^{\star}}\left[u - v - P_n(K(u) - K(v))\right]\right\|_X \\
&\leq c_0 \left\|(\mathrm{id} - DK(u_0))[u - v] - P_{N_{u_0}^{\star}}\left[u - v - P_n(K(u) - K(v))\right]\right\|_X \\
&\leq c_0 \left\|P_n(K(u) - K(v)) - DK(u_0)[u - v]\right\|_X \\
&\leq c_0 \left\|K(u) - K(v) - DK(u_0)[u - v]\right\|_X \\
&\qquad\qquad + c_0 \left\|(DK(u_0) - P_nDK(u_0))[u - v]\right\|_X \\
&\leq c_0 \int_0^1 \|DK(u_0) - DK(su + (1 - s)v)\|_{\mathcal{L}(X)}\, ds\ \|u - v\|_X + \frac{1}{4}\|u - v\|_X \\
&\leq \frac{1}{2}\|u - v\|_X,
\end{aligned}$$

as well as

$$\begin{aligned}
\|T_n(u) - u_0\|_X &\leq \|T_n(u_0) - u_0\|_X + \frac{1}{2}\|u - u_0\|_X \\
&\leq c_0 \|u_0 - P_nu_0\|_X + \frac{1}{2}\varepsilon_1 \leq \varepsilon_1.
\end{aligned}$$

So $T_n : \overline{N_{u_0}^{\varepsilon_1}} \longrightarrow \overline{N_{u_0}^{\varepsilon_1}}$, $n \geq n_0$ is a contraction and thus has a unique fixpoint, which we call $u_n$. Next

$$\|u_n - u_0\|_X = \|T_n(u_n) - u_0\|_X \leq c_0 \|u_0 - P_nu_0\|_X + \frac{1}{2}\|u_n - u_0\|_X$$

implies $u_n \longrightarrow u_0$ as $n \longrightarrow \infty$. Furthermore, observe that $v \in \mathrm{Kern}\, Dg_n(u_n)$ gives $DT_n(u_n)v = v$ and therefore $v = 0$, since $T_n$ is a contraction. $\square$

In a next step we need to get rid of the projection $P_{N^\star_{u_0}}$ in $g_n$.

*Claim* 2.26.3. There exists $n_1 \geq n_0$ such that $u_n - P_n K(u_n) = 0$ for every $n \geq n_1$.

*Proof.* Since we already have $g_n(u_n) = 0$, it is enough to find a direction $v_n \in X \setminus N^\star_{u_0}$ with $\langle u_n - P_n K(u_n), v_n\rangle = 0$. Using $g_n(u_n) = 0$ we can write $u_n$ as

$$u_n = P_n K(u_n) + \frac{\langle u_n - P_n K(u_n), E_\star\rangle_X}{\|E_\star\|_X^2} E_\star =: y_n + \lambda_n E_\star,$$

where $y_n \in X_n \subset Y$, $\lambda_n \in \mathbb{R}$ and $E_\star = (L + P_0)E(u_0) \in Y$. So $u_n \in Y$ and $E(u_n) \in X$ exists by (A9). We abbreviate $E_\star(u_n) := (L + P_0)E(u_n)$, use the identity

$$\langle u_n - K(u_n), E_\star(u_n)\rangle_X = \langle Lu_n - \Psi(u_n), E(u_n)\rangle_X = 0$$

and obtain

$$\begin{aligned}
\langle u_n - P_n K(u_n), E_\star(u_n)\rangle_X &= \langle (\mathrm{id} - P_n)K(u_n), E_\star(u_n)\rangle_X \\
&= \langle (\mathrm{id} - P_n)K(u_n), (L + P_0)E(y_n) + \lambda_n (L + P_0)E(E_\star)\rangle_X \\
&= \lambda_n \langle K(u_n), (\mathrm{id} - P_n)(L + P_0)E(E_\star)\rangle_X =: \lambda_n \beta_n.
\end{aligned}$$

Thus with $v_n := \|E_\star\|_X^2\, E_\star(u_n) - \beta_n E_\star$ there holds

$$\langle u_n - P_n K(u_n), v_n\rangle_X = \|E_\star\|_X^2\, \lambda_n\beta_n - \beta_n\lambda_n\, \|E_\star\|_X^2 = 0.$$

By (A3),(A9) $E_\star(u_n) = (L+P_0)E(u_n) = E((L+P_0)u_n) \longrightarrow E_\star$ as $n \longrightarrow \infty$ and $\beta_n \longrightarrow 0$. Therefore we can find $n_1 \geq n_0$, such that $v_n \notin N^\star_{u_0}$ for $n \geq n_1$. □

So far we have found with $O := S^1 * N^{\varepsilon_1}_{u_0}$ an invariant, bounded neighborhood of $S^1 * u_0$ satisfying $(L - P_n\Psi)^{-1}(0) \cap \overline{O} = S^1 * u_n \subset X_n$. By Claim 2.26.2 and the equivariance of $L - P_n\Psi$, $\mathrm{Kern}(L - P_n D\Psi(u_n)) = \mathbb{R}E(u_n)$ holds. We also know $u_n \longrightarrow u_0$.

*Claim* 2.26.4. The isotropy group $I_{u_n}$ of $u_n$ satisfies $I_{u_n} = \mathbb{Z}_k$ for all $n$ big enough.

*Proof.* We first show that $\mathbb{Z}_k = I_{u_0}$ is a subgroup of every $I_{u_n}$. Therefore observe that $\mathbb{Z}_k$ leaves $\overline{N^{\varepsilon_1}_{u_0}}$ invariant. So by the uniqueness of the solution of $L - P_n\Psi = 0$ in $\overline{N^{\varepsilon_1}_{u_0}}$ it follows $\theta * u_n = u_n$ for every $\theta \in \mathbb{Z}_k$. Thus $\mathbb{Z}_k \leq I_{u_n}$.

For the other inclusion recall that $E_j$ denotes the isotypical component of $E_0^\perp$ corresponding to $(\mathbb{R}^2, \rho^j)$. Define $\tilde{E}_j \leq E_0 \oplus E_j$ to be the full isotypical component of $X$ and $P_{\tilde{E}_j}$ to be the orthogonal projection $X \longrightarrow \tilde{E}_j$. Note that $P_{\tilde{E}_j} u_0 \neq 0$ implies $j \in k\mathbb{N}_0$, because $I_{u_0} = \mathbb{Z}_k$. We define $A$ to be the set of all indices $l \in \mathbb{N}$ satisfying $P_{\tilde{E}_{lk}} u_0 \neq 0$. Then $I_{u_0} = \mathbb{Z}_k$ implies

$$\bigcap_{l \in A} \left( \frac{1}{l}\mathbb{Z} \cap [0,1) \right) = \{\, 0 \,\}.$$

We can replace $A$ by a finite subset $\tilde{A} \subset A$ such that the equation above still remains valid. Since $u_n \longrightarrow u_0$, we find an index $n_2 \geq n_1$ with $P_{\tilde{E}_{lk}} u_n \neq 0$ for every $l \in \tilde{A}$, $n \geq n_2$. This yields

$$I_{u_n} \leq \bigcap_{l \in \tilde{A}} \left( \frac{2\pi}{lk}\mathbb{Z} \cap [0, 2\pi) \right) = \left\{ 0, \frac{2\pi}{k}, \ldots, \frac{(k-1)2\pi}{k} \right\} = \mathbb{Z}_k.$$

□

It remains to prove

*Claim* 2.26.5. The normed tangent vectors $e(u_n) := \frac{E(u_n)}{\|E(u_n)\|_X}$ converge in $X$ towards the normed tangent vector $e(u_0) := \frac{E(u_0)}{\|E(u_0)\|_X}$.

*Proof.* This would be clear, if $u_n \to u_0$ in $Y$, but we only know $u_n \to u_0$ in $X$. Since $DK(u_0)$ is compact one has $\|P_nDK(u_n) - DK(u_0)\|_{\mathcal{L}(X)} \to 0$. By

$$e(u_n) = P_nDK(u_n)e(u_n) = DK(u_0)e(u_n) + o(1),$$

the convergence (along a subsequence) of $e(u_n)$ to an element in the kernel of $\mathrm{id} - DK(u_0)$ with length 1 follows. Hence $e(u_n) \to \pm e(u_0)$. The correct sign is obtained by using the fact that $(L + P_0)u_n \to (L + P_0)u_0$ in $Y$ and therefore $(L + P_0)E(u_n) \to (L + P_0)E(u_0)$ in $X$. □

With that we have shown all properties of the Lemma. □

*Proof of Theorem 2.24.* Take everything as in Lemma 2.26. The fact that $\overline{O} \cap X^{S^1} = \emptyset$ implies $\mathrm{d}_0^\nabla(L - \Psi, O) = 0$ and

$$\mathrm{d}_j^\nabla(L - \Psi, O) = \lim_{n\to\infty} \mathrm{d}_j^\perp(L - P_n\Psi, O \cap X_n)$$

for $j \geq 1$. So we need to calculate the degree $S^1\text{-}\deg^\perp(L - P_n\Psi, O \cap X_n)$ for $n$ sufficiently large. Luckily the map $L - P_n\Psi : O \cap X_n \to X_n$ is already $S^1$-normal, since $\overline{O} \cap X^{S^1} = \emptyset$, such that the definition of Rybicki's degree (cf. section 2.3.2) directly transfers us to the degree of DGMJ, i.e.

$$\mathrm{d}_j^\nabla(L - \Psi, O) = \lim_{n\to\infty} \mathrm{d}_j(\hat{f}_n, U_n),$$

where $U_n := (O \cap X_n) \times (-1, 1)$, $\hat{f}_n : U_n \to X_n$, $\hat{f}_n(v, \lambda) = Lv - P_n\Psi(v) + \lambda E(v)$. Due to the orthogonality of $L - P_n\Psi$ and $E$, the orbit $S^1 * (u_n, 0)$ is the only orbit of zeroes of $\hat{f}_n$. Moreover, Lemma 2.26 implies

$$\mathrm{Kern}\, D\hat{f}_n(u_n, 0) = \mathbb{R}(E(u_n), 0),$$

so we can apply Theorem 2.16. In order to use this Theorem we define the needed linear maps $A_n^\perp : (X_n^{\mathbb{Z}_k})^\perp \to (X_n^{\mathbb{Z}_k})^\perp$, $\mathcal{A}_n : X_n^{\mathbb{Z}_k} \oplus \mathbb{R} \to X_n^{\mathbb{Z}_k} \oplus \mathbb{R}$,

$$A_n^\perp v = D\hat{f}_n(u_n, 0)(v, 0) = Lv - P_nD\Psi(u_n)v$$

$$\mathcal{A}_n(v, \mu) = \left(Lv - P_nD\Psi(u_n)v + \mu E(u_n), \frac{\langle v, E(u_n)\rangle_X}{\|E(u_n)\|_X^2}\right).$$

Then according to 2.16 it remains to show $\operatorname{sign}\det A_n^\perp \to s^\perp$ and $\operatorname{sign}\det \mathcal{A}_n \to -s_0$. And indeed due to the equivariance of $L + P_0$ one has

$$\begin{aligned}\operatorname{sign}\det A_n^\perp &= \operatorname{sign}\det(L + P_0) \cdot \operatorname{sign}\det\left(\mathrm{id} - P_nDK(u_n) : (X_n^{\mathbb{Z}_k})^\perp \to (X_n^{\mathbb{Z}_k})^\perp\right) \\ &= 1 \cdot \operatorname{sign}\left(\mathrm{id} - P_nDK(u_n) : (X^{\mathbb{Z}_k})^\perp \to (X^{\mathbb{Z}_k})^\perp\right) \to s^\perp.\end{aligned}$$

For $\mathcal{A}_n$ we need some auxiliary maps. We abbreviate in a similar way as before the tangent vectors $E_n = E(u_n)$, $E_{\star n} = (L + P_0)E(u_n)$ and the orthogonal projection $P_{N^\star_{u_n}} : X_n^{\mathbb{Z}_k} \to E_{\star n}^\perp$. Observe that $v - DK(u_n)v \in E_{\star n}^\perp$ for every $v \in X_n$. Therefore, if we define the maps

$I_n, Q_n : X_n^{\mathbb{Z}_k} \oplus \mathbb{R} \to X_n^{\mathbb{Z}_k} \oplus \mathbb{R}$,

$$I_n(v,\mu) = \left( P_{N^\star_{u_n}} v + \mu \frac{\|E_{\star n}\|_X}{\|E_n\|_X}(L+P_0)^{-1}E_n, \frac{\langle v, E_{\star n}\rangle_X}{\|E_n\|_X \, \|E_{\star n}\|_X} \right),$$
$$Q_n(v,\mu) = \left( v - P_n DK(u_n)v + \left\langle v, \frac{E_n}{\|E_n\|_X} \right\rangle_X \frac{E_{\star n}}{\|E_{\star n}\|_X}, \mu \frac{\|E_n\|_X}{\|E_{\star n}\|_X} \right),$$

there holds

$$\begin{aligned}(L+P_0, \mathrm{id}_{\mathbb{R}}) \circ I_n \circ Q_n(v,\mu) &= \left( (L+P_0)\,(v - P_n DK(u_n)v) + \mu E_n, \frac{\langle v, E_n\rangle_X}{\|E_n\|_X^2} \right) \\ &= \mathcal{A}_n(v,\mu).\end{aligned}$$

Again by the equivariance of $L + P_0$ we have $\operatorname{sign}\det(L+P_0, \mathrm{id}_{\mathbb{R}}) = 1$ and by Lemma 2.26 there holds $\operatorname{sign}\det Q_n \longrightarrow s_0$. So it remains to show that $I_n$ is not orientation preserving. To see this decompose $X_n^{\mathbb{Z}_k}$ into $E_{\star n}^\perp \oplus \mathbb{R}E_{\star n}$ and write according to this decomposition $v = v^\perp + \alpha \frac{E_{\star n}}{\|E_{\star n}\|_X}$, such that in blockmatrix form

$$I_n(v^\perp, \alpha, \mu) = \begin{pmatrix} & & & 0 & * \\ & \mathrm{id}_{E_{\star n}^\perp} & & \vdots & \vdots \\ & & & 0 & * \\ 0 & \cdots & 0 & 0 & \|E_n\|_X \\ 0 & \cdots & 0 & \frac{1}{\|E_n\|_X} & 0 \end{pmatrix} \begin{pmatrix} v^\perp \\ \alpha \\ \mu \end{pmatrix}.$$

Therefore $\det I_n = -1$ and the proof of theorem 2.24 is finished. □

# Chapter 3

# Periodic solutions consisting of clusters

In this chapter we will establish the existence of periodic solutions for the $N$-vortex system consisting of several vortex clusters located at the points of an equilibrium solution of an $m$-vortex system, $m < N$. Each of the clusters is in its shape close to a rigidliy rotating configuration of the whole-plane system.

We would like to mention that the general idea of grouping vortices into different clusters plays a role in establishing the existence of quasi-periodic solutions via KAM theory, see [46, 58].

## 3.1 Statement of results

Let $\Omega \subset \mathbb{R}^2$ be a domain and fix a symmetric $C^2$ function $g : \Omega \times \Omega \to \mathbb{R}$, for example the regular part of the Dirichlet (or more generally a hydrodynamic) Green's function of $\Omega$. We will investigate a point vortex like system similar to (1.3), which is induced by the generalized Green's and Robin functions

$$G(x, y) = -\frac{1}{2\pi} \log |x - y| - g(x, y), \qquad h(x) = g(x, x).$$

At first we consider on the domain $\Omega$ a system of $m \in \mathbb{N}$ vortices with vorticities $\Gamma^1, \ldots, \Gamma^m \in \mathbb{R} \setminus \{0\}$ and Hamiltonian

$$\mathcal{H}(a) = \sum_{\substack{k,k'=1 \\ k \neq k'}}^{m} \Gamma^k \Gamma^{k'} G(a^k, a^{k'}) - \sum_{k=1}^{m} \Gamma^k \Gamma^k h(a^k)$$

defined on $\mathcal{F}_m(\Omega) = \left\{ a = (a^1, \ldots, a^m) \in \Omega^m : a^k \neq a^{k'} \text{ for all } k \neq k' \right\}$. We require that the corresponding $m$-vortex system admits a stationary solution, cf. section 3.1.1. To be more precise we assume

(A1) $\mathcal{H}$ has a nondegenerate critical point $\alpha \in \mathcal{F}_m(\Omega)$.

Next we fix a number $l \in \{1, \ldots, m\}$, which will be the number of vortices that are splitted into configurations consisting of more than a single vortex. Without restriction we take the first $l$ vortices. I.e. for $k = 1, \ldots, l$ choose $N_k \geq 2$ vorticities $\Gamma_1^k, \ldots, \Gamma_{N_k}^k \in \mathbb{R} \setminus \{0\}$, such that

(A2) $\sum_{j=1}^{N_k} \Gamma_j^k = \Gamma^k$.

We then define the Hamiltonian $H^k_{\mathbb{R}^2} : \mathcal{F}_{N_k}(\mathbb{R}^2) \to \mathbb{R}$,

$$H^k_{\mathbb{R}^2}(z) = -\frac{1}{2\pi} \sum_{\substack{j,j'=1 \\ j\neq j'}}^{N_k} \Gamma^k_j \Gamma^k_{j'} \log \left| z_j - z_{j'} \right|$$

inducing the $N_k$-vortex system

$$\Gamma^k_j \dot{z}_j = J \nabla_{z_j} H^k_{\mathbb{R}^2}(z), \quad j = 1, \ldots, N_k \tag{3.1}$$

on $\mathbb{R}^2$.

As mentioned in the introduction a $\tilde{N}$-vortex system on $\mathbb{R}^2$ allows rigidly rotating solutions, also called relative equilibria, of the form $Z(t) = e^{\omega J_{\tilde{N}} t} z$, $\omega \neq 0$, cf. section 3.1.2 for examples. Here $J_{\tilde{N}} = \operatorname{diag}(J, J, \ldots, J) \in \mathbb{R}^{2\tilde{N} \times 2\tilde{N}}$. Due to scaling $Z(t) \to \lambda Z(t/\lambda^2)$, $\lambda > 0$, we can assume $\omega = \pm 1$. The corresponding $2\pi$-periodic relative equilibrium is called nondegenerate, if the linearized equation

$$\Gamma_j \dot{w}_j = J(\nabla^2 H_{\mathbb{R}^2}(Z(t))w)_j, \quad j = 1, \ldots, \tilde{N} \tag{3.2}$$

has only 3 linear independent $2\pi$-periodic solutions. This is the minimal possible number due to the invariance under rotations and translations. Our third requirement is:

(A3) For $k \in \{ 1, \ldots, l \}$ there exists a $2\pi$-periodic nondegenerate relative equilibrium solution $Z^k(t) = e^{\pm J_{N_k} t} z^k$ of (3.1).

Note that condition (A2) can always be achieved by a change of time scale provided one has a relative equilibrium solution of (3.1) with $\sum_j \Gamma^k_j \neq 0$.

The remaining $m - l$ vortices – which may be none – are not splitted into configurations. I.e. for $k = l + 1, \ldots, m$ we let $N_k = 1$, $\Gamma^k_1 = \Gamma^k$, $H^k_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}$, $H^k_{\mathbb{R}^2} \equiv 0$ and $Z^k : \mathbb{R} \to \mathbb{R}^2$, $Z^k(t) \equiv 0$.

The system under investigation is the generalized $N := \sum_{k=1}^m N_k$-vortex system

$$\Gamma^k_j \dot{z}^k_j = J \nabla_{z^k_j} H(z), \quad k = 1, \ldots, m, \; j = 1, \ldots, N_k, \tag{3.3}$$

with Hamiltonian

$$H(z) = \sum_{(k,j)\neq(k',j')} \Gamma^k_j \Gamma^{k'}_{j'} G(z^k_j, z^{k'}_{j'}) - \sum_{(k,j)} \Gamma^k_j \Gamma^k_j h(z^k_j).$$

Here $z = (z^1_1, \ldots, z^1_{N_1}, \ldots, z^m_1, \ldots, z^m_{N_m}) \in \mathcal{F}_N(\Omega)$ and the indices of the sums run through $\{ (k, j) : 1 \leq k \leq m, 1 \leq j \leq N_k \}$. We equivalently write for (3.3)

$$M_\Gamma \dot{z} = J_N \nabla H(z)$$

with $M_\Gamma = \operatorname{diag}\left(\Gamma^1_1, \Gamma^1_1, \ldots, \Gamma^1_{N_1}, \Gamma^1_{N_1}, \ldots, \Gamma^m_1, \Gamma^m_1, \ldots, \Gamma^m_{N_m}, \Gamma^m_{N_m}\right) \in \mathbb{R}^{2N \times 2N}$ and the symplectic matrix $J_N = \operatorname{diag}\left(J, \ldots, J\right) \in \mathbb{R}^{2N \times 2N}$.

As described before we will use the Sobolev spaces $H^1_T = H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^{2N})$, $T > 0$ of continuous $T$-periodic functions with square-integrable derivative, equipped with the scalar product

$$\langle u, v \rangle_{H^1_T} = \int_0^T \langle u, v \rangle_{\mathbb{R}^{2N}} \; dt + \int_0^T \langle \dot{u}, \dot{v} \rangle_{\mathbb{R}^{2N}} \; dt$$

and induced norm $\|\cdot\|_{H^1_T}$. For $Z^1, \ldots, Z^m$ as defined before let

$$\mathcal{M} = \left\{ \left(Z^1(\cdot + \theta_1), \ldots, Z^m(\cdot + \theta_m)\right) : \theta_1, \ldots, \theta_m \in \mathbb{R} \right\} \subset H^1_{2\pi}, \tag{3.4}$$

which is a $l$-dimensional submanifold, since $Z^{l+1} = \ldots = Z^m = 0$. And for an element $a = (a^1, \ldots, a^m) \in \mathbb{R}^{2m}$ we define

$$\hat{a} = (a^1, \ldots, a^1, a^2, \ldots, a^2, \ldots, a^m, \ldots, a^m) \in \mathbb{R}^{2N_1} \times \ldots \times \mathbb{R}^{2N_m} = \mathbb{R}^{2N}.$$

Now we are ready to formulate a first version of our theorem.

**Theorem 3.1.** *Under the assumptions (A1)–(A3) there exists $T_0 > 0$ such that for every $T \in (0, T_0)$ the $N$-vortex type system (3.3) has $l$ distinct $T$-periodic solutions that are in the following sense close to $\alpha$ and $(Z^1, \ldots, Z^m)$: Let $(z_n)_{n \in \mathbb{N}}$ be a sequence consisting of these periodic solutions with periods $T_n \to 0$ as $n \to \infty$, then the $k$th components $[z_n]^k_j$, $j = 1, \ldots, N_k$ converge to $\alpha^k$ as $n \to \infty$, $k = 1, \ldots, m$. Moreover, if we rescale $z_n$, such that*

$$z_n(t) = r_n u_n\left(\frac{t}{r_n^2}\right) + \hat{\alpha}, \quad r_n = \sqrt{\frac{T_n}{2\pi}}, \quad u_n \in H^1_{2\pi},$$

*then* $\operatorname{dist}(u_n, \mathcal{M}) \to 0$ *with respect to* $\|\cdot\|_{H^1_{2\pi}}$ *as* $n \to \infty$.

So roughly speaking we can split vortices of a stationary solution into suitable rigidly rotating configurations and obtain periodic solutions. For fixed $T \in (0, T_0)$ the multiplicity of the $T$-periodic solutions is based on the relative orientation of the $l$ nontrivial configurations to each other.

Note that the conditions (A1), (A3) are only related to each other in the sense that the vorticities need to add up as stated in (A2). Also the specific relative equilibrium solutions can be choosen independently of each other. Under an additional technical assumption, that couples the critical point of $\mathcal{H}$ and the relative equilibria $Z^k$, one could improve the multiplicity from $l$ to $2^{l-1}$ $T$-periodic solutions, cf. Section 3.4.1. This would also lead to global continua of solutions as in Theorem 3.9 below.

Next we will discuss and improve assumptions (A1), (A3) with respect to their applicability to the classical $N$-vortex system (1.3). Whenever we provide a function with an index $\Omega$, like $\mathcal{H}_\Omega$, we refer to the corresponding function induced by the regular part of the Dirichlet Green's function.

### 3.1.1 Critical points of $\mathcal{H}_\Omega$

The search for stationary solutions in general domains itself is not an easy task. Of course there is one trivial case: If $m = 1$ the 1-vortex Hamiltonian $\mathcal{H}_\Omega$ coincides up to a factor with the Robin function $h_\Omega$, which always has a minimum in bounded domains.

Concerning more vortices only in the last years some results on the existence of critical points of the $N$-vortex – in our case $m$-vortex – Hamiltonian for bounded domains could be achieved, examples include:

- $m \in \mathbb{N}$, $\Gamma^1 = \ldots = \Gamma^m \neq 0$ and $\Omega$ not simply connected [28] or dumbell shaped [31],
- $m \in \{2, 3, 4\}$, conditions on $\Gamma^k$, e.g. $m = 2$ and $\Gamma^1\Gamma^2 < 0$, $\Omega$ arbitrary [16],
- $m \in \mathbb{N}$, conditions on $\Gamma^k$ (different from the ones in [16]) for $\Omega$ arbitrary and for $\Omega$ not simply connected [49],
- $m \in \mathbb{N}$, $\Gamma^k = (-1)^{k+1}\Gamma^1$, $\Omega$ symmetric with respect to reflection at a line [17] or the action of a dihedral group [50].

None of the mentioned results addresses the question of nondegeneracy of the critical points, on which our proof relies. Indeed condition (A1) is for these solutions hard to check, since the Hamiltonian $\mathcal{H}_\Omega$ and the critical point $\alpha$ are not explicitely known. However, a recent result of Bartsch, Micheletti and Pistoia [15] shows that $\mathcal{H}_\Omega$ has only nondegenerate critical points for a generic bounded domain $\Omega$.

So if the vorticities $\Gamma^1, \dots, \Gamma^m$ allow the existence of a critical point of $\mathcal{H}_\Omega$, as for example in one of the listed cases, then condition (A1) is satisfied at least after an arbitrarily small deformation of the domain.

In some cases also explicit stationary configurations are known, for example if $\Omega = \mathbb{R}^2$ or $\Omega = B_1(0)$. But these are all degenerate due to the symmetries of the domain, i.e. if $\alpha \in \mathcal{F}_m(B_1(0))$ is a critical point of $\mathcal{H}_{B_1(0)}$, then every $e^{\lambda J_m}\alpha$, $\lambda \in \mathbb{R}$ is a critical point as well. Thus $J_m\alpha \in \operatorname{Kern} \nabla^2 \mathcal{H}_{B_1(0)}(\alpha)$ and condition (A1) is violated. But we will see that degeneracy induced by symmetries can still be handled, i.e. we may replace assumption (A1) by

(A1′) $\mathcal{H}$ has a critical point $\alpha \in \mathcal{F}_m(\Omega)$ and one of the following properties holds:
  (i) $\alpha$ is nondegenerate,
  (ii) $\Omega$ and $g$ are radial $\left(e^{\lambda J}\Omega = \Omega,\ g\left(e^{\lambda J}x, e^{\lambda J}y\right) = g(x,y) \text{ for every } \lambda \in \mathbb{R},\ x, y \in \Omega\right)$ and $\dim \operatorname{Kern} \nabla^2 \mathcal{H}(\alpha) = 1$,
  (iii) $\Omega$ and $g$ are in one direction translational invariant $\left(\text{there exists } v \in \mathbb{R}^2 \setminus \{0\} \text{ with } \lambda v + \Omega = \Omega,\ g(x+\lambda v, y+\lambda v) = g(x,y) \text{ for every } \lambda \in \mathbb{R},\ x, y \in \Omega\right)$ and $\dim \operatorname{Kern} \nabla^2 \mathcal{H}(\alpha) = 1$,
  (iv) $\Omega = \mathbb{R}^2$, $g(x,y) = \tilde{g}(|x-y|)$ and $\dim \operatorname{Kern} \nabla^2 \mathcal{H}(\alpha) = 3$.

Note that in the classical case $g = g_\Omega$ always inherits the symmetries of the domain.

**Example 3.2.** *Let $\Omega$ be the unit disc $B_1(0)$ and $g = g_{B_1(0)}$ be the regular part of the Dirichlet Green's function of $B_1(0)$, which is given by*

$$g(x,y) = g_{B_1(0)}(x,y) = -\frac{1}{4\pi} \log\left(|x|^2 |y|^2 - 2\langle x, y\rangle_{\mathbb{R}^2} + 1\right).$$

*The 2-vortex Hamiltonian $\mathcal{H}_{B_1(0)}$ with vorticities $\Gamma^1 = 1$, $\Gamma^2 = -1$ satisfies (A1′) with a degenerate critical point $\alpha = \left((\mu,0),(-\mu,0)\right)$, where $\mu = \sqrt{\sqrt{5}-2}$. This will be shown in section 3.6.*

**Remark 3.3.** *If $\Omega = \mathbb{R}^2$, $g = g_{\mathbb{R}^2} \equiv 0$ then critical points of $\mathcal{H}_{\mathbb{R}^2}$ exist depending on the vorticities $\Gamma^1, \dots, \Gamma^m$. In the easiest case $m = 3$ vortices with strengths $\Gamma^1, \Gamma^2, \Gamma^3$ satisfying $\Gamma^1\Gamma^2 + \Gamma^1\Gamma^3 + \Gamma^2\Gamma^3 = 0$ are stationary when placed at certain distances along a fixed line, see Theorem 2.2.1 in [64]. More on stationary configurations can also be found in [5]. However, for every critical point $\alpha$ of $\mathcal{H}_{\mathbb{R}^2}$ the inequality $\dim \operatorname{Kern} \nabla^2 \mathcal{H}_{\mathbb{R}^2}(\alpha) \geq 4$ holds true. Here 3 dimensions of the kernel are induced by translations and rotations of the critical point $\alpha$. A fourth dimension by scaling, since differentiation of $\lambda \mapsto \mathcal{H}_{\mathbb{R}^2}(\lambda\alpha)$ at $\lambda = 1$ shows that $\sum_{k \neq k'} \Gamma^k \Gamma^{k'} = 0$ is a necessary condition for the existence of critical points. Therefore we have $\mathcal{H}_{\mathbb{R}^2}(\lambda\alpha) = \mathcal{H}_{\mathbb{R}^2}(\alpha)$ and*

$$\operatorname{Kern} \nabla^2 \mathcal{H}_{\mathbb{R}^2}(\alpha) \supset \left\{ (a, \dots, a) \in \mathbb{R}^{2m} \right\} \oplus \mathbb{R} J_m \alpha \oplus \mathbb{R}\alpha.$$

*This means that (A1′) never holds for critical points of the classical $m$-vortex Hamiltonian $\mathcal{H}_{\mathbb{R}^2}$, cf. Remark 3.14.*

**Remark 3.4.** *Another idea for the existence of periodic solutions is the Weinstein-Moser Theorem [10, 62, 77] to obtain periodics for the Hamiltonian $\mathcal{H}_\Omega$ itself via bifurcation from the critical point $\alpha$. But here one encounters the difficulties that $\alpha$ and $\mathcal{H}_\Omega$ are not explicitely known as well.*

### 3.1.2 Relative equilibria on $\mathbb{R}^2$

For the $N$-vortex problem on $\Omega = \mathbb{R}^2$ quite a lot of rigidly rotating vortex configurations are known, see [4, 5] for an overview. Checking the nondegeneracy condition of such a configuration is, after writing (3.2) in a rotating coordinate frame, a matter of calculating the spectrum of a $2N \times 2N$ matrix. The spectral properties of this matrix are also of interest in the investigation of the linear stability of the configuration as a periodic solution. So in section 3.5 we will use results of Roberts, [67] to verify the nondegeneracy.

**Example 3.5.** *The following relative equilibrium solutions are nondegenerate after normalization (scaling and translation):*

- $N = 2$, $\Gamma_1 + \Gamma_2 \neq 0$, $Z(0) \in \mathcal{F}_2(\mathbb{R}^2)$ *arbitrary, cf. Example 3.21,*
- $N = 3$, $\Gamma_1 + \Gamma_2 + \Gamma_3 \neq 0$, $0 \neq \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 \neq \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2$, $Z_1(0), Z_2(0), Z_3(0)$ *forming an equilateral triangle, cf. Example 3.22,*
- $N \geq 3$, $\Gamma_1 = \ldots = \Gamma_N$, $Z_j(0) = (x_k, 0)$, $j = 1, \ldots, N$ *with* $x_1, \ldots, x_N$ *being the roots of the* $N$*th Hermitian polynomial, see Example 3.23.*

Observe that the condition for the equilateral triangle configuration excludes the special case $\Gamma_1 = \Gamma_2 = \Gamma_3$. Nonetheless with a second refinement we can also treat this case leading to solutions for (3.3) in which the vortices of a subgroup may form choreographies.

The permutation group $\Sigma_N$ of $N$ symbols acts orthogonally on $\mathbb{R}^{2N}$ via permutation of components, i.e.

$$\sigma * z = \left(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(N)}\right), \quad \sigma \in \Sigma_N,\ z \in \mathbb{R}^{2N}.$$

**Definition 3.6.** *A relative equilibrium solution* $Z(t)$ *of the whole plane system is called* $\sigma$*-nondegenerate, provided* $\sigma * Z(\cdot + 2\pi) = Z$ *and* (3.2) *has only three linear independent solutions satisfying* $\sigma * w(\cdot + 2\pi) = w$.

Note that every nondegenerate relative equilibrium is $\sigma$-nondegenerate with $\sigma = \mathrm{id}_{\Sigma_N}$. As a nontrivial example we have

**Example 3.7.** $N \in \mathbb{N}$ *identical vortices placed at the vertices of a regular* $N$*-Gon form a rigidly rotating configuration, called Thomson's* $N$*-Gon configuration. It is (after scaling) a* $\sigma$*-nondegenerate relative equilibrium solution with* $\sigma = (1\ 2\ \ldots\ N) \in \Sigma_N$*, see Lemma 4.1 in [12].*

Concerning our situation we weaken assumption (A3) to

(A3′) For each $k \in \{1, \ldots, l\}$ there exists $\sigma_k \in \Sigma_{N_k}$ with $\Gamma_j^k = \Gamma_{\sigma_k^{-1}(j)}^k$, $j = 1, \ldots, N_k$, as well as a $\sigma_k$-nondegenerate relative equilibrium $Z^k(t) = \exp\left(\pm J_{N_k} t/\mathrm{ord}(\sigma_k)\right) z^k$ solving (3.1). For consistency in notation let $Z^k \equiv 0$ and $\sigma_k = \mathrm{id}_{\Sigma_1}$ in the case $k \in \{l+1, \ldots, m\}$.

### 3.1.3 Statement of results part 2

For $\left(\sigma_k, Z^k\right)_{k=1}^m$ as in assumption (A3′) let $\tau = 2\pi\,\mathrm{ord}(\sigma)$, where $\mathrm{ord}(\sigma)$ denotes the order of $\sigma = (\sigma_1, \ldots, \sigma_m) \in \prod_k \Sigma_{N_k}$, further on let $\sigma * z = (\sigma_1 * z^1, \ldots, \sigma_m * z^m)$ for a vector $z = (z^1, \ldots, z^m) \in \mathbb{R}^{2N}$. Observe that $\mathcal{M}$ as defined in (3.4) is now contained in $H_\tau^1$. We have the following generalization of Theorem 3.1.

**Theorem 3.8.** *Assume that (A1′), (A2) and (A3′) hold. Then there exists $T_0 > 0$ such that (3.3) has $l$ distinct $T$-periodic orbits for every $T \in (0, T_0)$. Similar to Theorem 3.1 if we rescale a sequence $(z_n)_{n\in\mathbb{N}}$ of these solutions with periods $T_n \to 0$ by*

$$z_n(t) = r_n u_n\left(\frac{t}{r_n^2}\right) + \hat{\alpha}, \quad r_n = \sqrt{\frac{T_n}{\tau}}, \quad u_n \in H^1_\tau,$$

*then* $\mathrm{dist}(u_n, \mathcal{M}) \to 0$ *in $H^1_\tau$. Additionally the $k$th subgroup, $k = 1, \dots, m$ consisting of the $N_k$ vortices $z^k(t) = (z_1^k(t), \dots, z_{N_k}^k(t))$ of one of the $T$-periodic solutions $z(t)$ inherits the symmetry of the relative equilibrium $Z^k(t)$, i.e.*

$$\sigma * z(t + T/\mathrm{ord}(\sigma)) = z(t).$$

In the case that only the first vortex is splitted up into a configuration with at least two vortices, i.e. when $l = 1$, we can slightly improve Theorem 3.8.

**Theorem 3.9.** *Let $l = 1$, $g \in C^k(\Omega \times \Omega, \mathbb{R})$ with $k \geq 2$. If (A1′)-(A3′) hold, then there exists $r_1 > 0$ and a $C^{k-2}$ map $u : [0, r_1) \to H^1_\tau$, $r \mapsto u^{(r)}$ with $u^{(0)} = Z = (Z^1, 0, \dots, 0) \in H^1_\tau$, $\sigma * u^{(r)}(\cdot + 2\pi) = u^{(r)}$ and such that*

$$z^{(r)}(t) = r u^{(r)}\left(\frac{t}{r^2}\right) + \hat{\alpha}$$

*is a $\tau r^2$-periodic solution of (3.3) for every $r \in (0, r_1)$. If $k \geq 3$, then*

$$\partial_r u^{(0)} \in \left\{ \hat{a} : a \in \mathbb{R}^{2m} \right\} \subset H^1_\tau.$$

*Moreover, if in (A1′) (i) or (iii) is true, the family $(z^{(r)})_{r\in(0,r_1)}$ gives rise to a global continuum $\mathcal{C}(\alpha, Z)$ of periodic (choreographic) solutions of (3.3) in the sense of Definition 2.11 (or Remark 2.14).*

**Remark 3.10.** *a) In the cases (A1′) (ii) or (iv) a global continuum of solutions is likely to exist as well, but this seems to require a further development of the degree theory, cf. Remark 3.19.*

*b) If $m = l = 1$ it is possible to obtain a global continuum of periodic solutions also under the weaker assumption that $\alpha \in \Omega$ is only a topological stable critical point of the Robin function $h$, i.e. under the condition* $\deg(\nabla h, B_\varepsilon(\alpha)) \neq 0$ *instead of* $\det \nabla^2 h(\alpha) \neq 0$. *But in that situation the local part is not guaranteed to be a graph, see Thm. 2.1 in [13].*

*c) By our definition the global continuum $\mathcal{C}(\alpha, Z) \subset \mathbb{R}^+ \times H^1$ contains the set*

$$\left\{ \left( \frac{\tau r^2}{2\pi}, r u^{(r)}\left(\frac{\tau}{2\pi}\cdot\right) + \hat{\alpha} \right) : r \in (0, r_1) \right\}.$$

## 3.2 Ansatz and preliminaries

Fix $\alpha, Z^k, \sigma_k, k = 1, \dots, m$ according to (A1′), (A3′) and let $\sigma = (\sigma_1, \dots, \sigma_m)$. We are looking for a solution $z : \mathbb{R} \to \mathcal{F}_N(\Omega)$ where each subgroup of vortices $(z_1^k(t), \dots, z_{N_k}^k(t))$ is located near $\alpha^k$ and forms a configuration close to a scaled version of the relative equilibrium $Z^k(t)$.

In order to reformulate the problem we define

$$F(z) = \sum_{\substack{k,k'=1 \\ k\neq k'}}^{m} \sum_{j=1}^{N_k} \sum_{j'=1}^{N_{k'}} \Gamma_j^k \Gamma_{j'}^{k'} G(z_j^k + \alpha^k, z_{j'}^{k'} + \alpha^{k'}) - \sum_{k=1}^{m} \sum_{j,j'=1}^{N_k} \Gamma_j^k \Gamma_{j'}^k g(z_j^k + \alpha^k, z_{j'}^k + \alpha^k)$$

together with the following Hamiltonians $H_0 : \mathcal{O}_0 := \mathcal{F}_{N_1}(\mathbb{R}^2) \times \ldots \times \mathcal{F}_{N_m}(\mathbb{R}^2) \to \mathbb{R}$,

$$H_0(u) = \sum_{k=1}^{m} H^k_{\mathbb{R}^2}(u^k_1, \ldots, u^k_{N_k})$$

and for $r > 0$, $H_r : \mathcal{O}_r := \left\{ u \in \mathbb{R}^{2N} : ru + \hat{\alpha} \in \mathcal{F}_N(\Omega) \right\} \to \mathbb{R}$,

$$H_r(u) = H_0(u) + F(ru) - \mathcal{H}(\alpha).$$

Observe that $F$ is defined on an open subset of $\mathbb{R}^{2N}$ containing 0.

**Lemma 3.11.** *Let $I$ be an open intervall and $r > 0$. Then $z(t) = ru(t/r^2) + \hat{\alpha}$ solves (3.3) on $I$ if and only if $u$ solves*

$$M_\Gamma \dot{u} = J_N \nabla H_r(u) \tag{3.5}$$

*on $r^2 I$.*

*Proof.* Clearly $z(t)$ as above is a solution of (3.3) if and only if

$$M_\Gamma \dot{u} = r J_N \nabla H(ru + \hat{\alpha})$$

and

$$H(ru + \hat{\alpha}) = H_0(u) + F(ru) - \frac{1}{2\pi} \sum_{k=1}^{m} \sum_{\substack{j,j'=1 \\ j \neq j'}}^{N_k} \Gamma^k_j \Gamma^k_{j'} \log r.$$

□

**Lemma 3.12.** *The set $\mathcal{O} := \bigcup_{r \geq 0} \{r\} \times \mathcal{O}_r$ is open in $[0, \infty) \times \mathbb{R}^{2N}$ and the family of Hamiltonians $H : \mathcal{O} \to \mathbb{R}$, $(r, u) \mapsto H_r(u)$ defines a $C^2$ function, especially $F(0) = \mathcal{H}(\alpha)$. Furthermore,*

$$\begin{aligned} \Gamma^k \nabla_{z^k_j} F(0) &= \Gamma^k_j \nabla_{a^k} \mathcal{H}(\alpha) = 0, \\ \Gamma^k \left( \nabla^2 F(0) \hat{a} \right)^k_j &= \Gamma^k_j \left( \nabla^2 \mathcal{H}(\alpha) a \right)^k \end{aligned} \tag{3.6}$$

*for any $(k, j)$ and $a \in \mathbb{R}^{2m}$.*

*Proof.* Openess and smoothness are easy to check, since by (A2) indeed

$$F(0) = \sum_{\substack{k,k'=1 \\ k \neq k'}}^{m} \Gamma^k \Gamma^{k'} G(\alpha^k, \alpha^{k'}) - \sum_{k=1}^{m} \Gamma^k \Gamma^k g(\alpha^k, \alpha^k) = \mathcal{H}(\alpha).$$

For the derivative of $F$ with respect to $z^k_j$ we have

$$\nabla_{z^k_j} F(z) = 2 \sum_{\substack{k'=1 \\ k' \neq k}}^{m} \sum_{j'=1}^{N_{k'}} \Gamma^k_j \Gamma^{k'}_{j'} \nabla_1 G(z^k_j + \alpha^k, z^{k'}_{j'} + \alpha^{k'}) - 2 \sum_{j'=1}^{N_k} \Gamma^k_j \Gamma^k_{j'} \nabla_1 g(z^k_j + \alpha^k, z^k_{j'} + \alpha^k)$$

and therefore

$$\Gamma^k \nabla_{z^k_j} F(0) = \Gamma^k_j \Gamma^k \left( 2 \sum_{\substack{k'=1 \\ k' \neq k}}^{m} \Gamma^{k'} \nabla_1 G(\alpha^k, \alpha^{k'}) - \Gamma^k \nabla h(\alpha^k) \right) = \Gamma^k_j \nabla_{a^k} \mathcal{H}(\alpha) = 0$$

by (A1′). Now let $a \in \mathbb{R}^{2m}$. The $(k, j)$th component of $\nabla^2 F(0)\hat{a}$ is given by

$$
\begin{aligned}
\left(\nabla^2 F(0)\hat{a}\right)_j^k &= \sum_{k'=1}^{m} \left( \sum_{j'=1}^{N_{k'}} \nabla_{z_{j'}^{k'}} \nabla_{z_j^k} F(0) \right) a^{k'} \\
&= 2\Gamma_j^k \sum_{\substack{k'=1 \\ k'\neq k}}^{m} \Gamma^{k'} \left( \nabla_1^2 G(\alpha^k, \alpha^{k'}) a^k + \nabla_2 \nabla_1 G(\alpha^k, \alpha^{k'}) a^{k'} \right) \\
&\qquad\qquad - \Gamma_j^k \Gamma^k \underbrace{\left(2\nabla_1^2 g(\alpha^k, \alpha^k) + 2\nabla_2 \nabla_1 g(\alpha^k, \alpha^k)\right)}_{=\nabla^2 h(\alpha^k)} a^k \\
&= \frac{\Gamma_j^k}{\Gamma^k} \sum_{k'=1}^{m} \nabla_{a^{k'}} \nabla_{a^k} \mathcal{H}(\alpha) a^{k'} = \frac{\Gamma_j^k}{\Gamma^k} \left(\nabla^2 \mathcal{H}(\alpha) a\right)^k .
\end{aligned}
$$

□

Next we turn to the functional setting. Let $\tau := 2\pi \operatorname{ord}(\sigma)$. In order to find $T$-periodic solutions of (3.3) with $T > 0$ small, we use the variational structure of (3.5) to look for $\tau$-periodic solutions of (3.5) with $r > 0$ small. We work on the Sobolev space $H_\tau^1$ as stated in section 3.1 and will also need the corresponding spaces $L_\tau^2$ and $H_\tau^2$. The action functional associated to (3.5) is given by

$$
\Phi_r(u) = \frac{1}{2} \int_0^\tau \langle M_\Gamma \dot{u}, J_N u \rangle_{\mathbb{R}^{2N}} \, dt - \int_0^\tau H_r(u)\, dt = \Phi_0(u) - \int_0^\tau F(ru)\, dt + \tau \mathcal{H}(\alpha).
$$

Let $\Phi : \Lambda' \to \mathbb{R}$, $(r, u) \mapsto \Phi_r(u)$, where

$$
\Lambda' := \left\{ (r, u) \in [0, \infty) \times H_\tau^1 : (r, u(t)) \in \mathcal{O} \text{ for all } t \in \mathbb{R} \right\}.
$$

Then $\Lambda'$ is open in $[0, \infty) \times H_\tau^1$, since $H_\tau^1$ embeds into $C_\tau^0$, $\Phi \in C^2(\Lambda', \mathbb{R})$ due to Lemma 3.12 and we have to solve $\nabla \Phi_r(u) = 0$ for $(r, u) \in \Lambda'$ with $r > 0$.

The action $\sigma * z = (\sigma_1 * z^1, \ldots, \sigma_m * z^m)$ on $\mathbb{R}^{2N}$, induces an action on $H_\tau^1$. Let

$$
X = \left\{ u \in H_\tau^1 : \sigma * u(\cdot + 2\pi) = u \right\}, \quad \Lambda = \Lambda' \cap (\mathbb{R} \times X), \quad \Lambda_r = \left\{ u : (r, u) \in \Lambda \right\}.
$$

Then $X$ is a complete subspace of $H_\tau^1$ and (A3′) implies $\nabla \Phi_r(u) \in X$ for $(r, u) \in \Lambda$, since indeed $H_r(\sigma * z) = H_r(z)$, $M_\Gamma(\sigma * z) = \sigma * (M_\Gamma z)$ yield $\Phi_r(\sigma * u(\cdot + 2\pi)) = \Phi_r(u)$ for any $(r, u) \in \Lambda'$. So it is enough to find a critical point of the restriction $\Phi_{r|\Lambda_r} : \Lambda_r \to \mathbb{R}$. We denote the restriction $\Phi_{|\Lambda}$ again by $\Phi$. As stated in 2.2.1 one has

$$
\nabla \Phi_r(u) = \nabla \Phi_0(u) - (\mathrm{id} - \Delta)^{-1} r \nabla F(ru) = (\mathrm{id} - \Delta)^{-1} \left( -J_N M_\Gamma \dot{u} - \nabla H_0(u) - r \nabla F(ru) \right),
$$

where $\Delta : H_\tau^2 \to L_\tau^2$, $u \mapsto \ddot{u}$, such that for $v \in H_\tau^1$, $w \in L_\tau^2$ the relation

$$
\left\langle v, (\mathrm{id} - \Delta)^{-1} w \right\rangle_{H_\tau^1} = \int_0^\tau \langle v, w \rangle_{\mathbb{R}^{2N}} \, dt = \langle v, w \rangle_{L_\tau^2}
$$

holds true. Note that actually $\nabla \Phi \in C^1(\Lambda, H_\tau^2 \cap X)$, where $H_\tau^2 \cap X$ is equipped with the norm $\|\cdot\|_{H_\tau^2}$.

## 3.3 Proof of Theorem 3.8

For $r \to 0$ the limiting equation of (3.5) is the decoupled system

$$\Gamma_j^k \dot{u}_j^k = J\nabla_{u_j^k} H_{\mathbb{R}^2}^k(u_1^k, \dots, u_{N_k}^k), \quad j = 1, \dots, N_k, \quad k = 1, \dots, m.$$

So by (A3′), $Z(t) := (Z^1(t), \dots, Z^m(t)) \in X$ is a critical point of $\Phi_0$, which of course is not isolated due to the symmetries of $H_0$. Let $D = \left\{ \hat{a} : a \in \mathbb{R}^{2m} \right\} \subset X$ and for a tuple $\theta = (\theta_1, \dots, \theta_m) \in (\mathbb{R}/\tau\mathbb{Z})^m = \mathbb{T}^m$, $u \in X$ define the shifted version

$$\theta * u \in X \quad \text{by} \quad (\theta * u)_j^k = u_j^k(\cdot + \theta_k).$$

Then $\Phi_0(u + \hat{a}) = \Phi_0(u) = \Phi_0(\theta * u)$ for any $u \in \Lambda_0$, $\hat{a} \in D$, $\theta \in \mathbb{T}^m$ indeed implies that

$$\left\{ \theta * Z + \hat{a} : \theta \in \mathbb{T}^m, a \in \mathbb{R}^{2m} \right\}$$

is a $(l+2m)$-dimensional critical manifold of $\Phi_0$. Since every $Z^k$, $k = 1, \dots, l$ is by assumption (A3′) a $\sigma_k$-nondegenerate solution of (3.1), we have

$$\operatorname{Kern} \nabla^2 \Phi_0(Z) = \operatorname{span}\left\{ \dot{Z}^1, \dots, \dot{Z}^l \right\} \oplus D. \tag{3.7}$$

Here $\dot{Z}^k$ is meant to be the element $(0, \dots, 0, \dot{Z}^k, 0, \dots, 0) \in X$. Whereas this degeneracy is natural for the limiting case $r = 0$, the functionals $\Phi_r$ with $r > 0$ are in general neither invariant with respect to translations by elements of $D$ nor under the action of $\mathbb{T}^m$ - except for synchronous time shifts $\theta = (\theta_1, \dots, \theta_1) \in \mathbb{T}^m$. We would like to mention here that this loss of symmetry for $r > 0$ prevents us from simply using continuation theorems like Thm. 2.4 of [63].

To deal with the degeneracy of the limiting problem we modify our equation $\nabla\Phi_r(u) = 0$. For a subspace $Y \subset X$ we denote by $P_Y : X \to Y$ the orthogonal projection onto $Y$ and by $Y^\perp$ the orthogonal complement of $Y$ in $X$. Let

$$\mathcal{M} = \mathbb{T}^m * Z, \qquad Y = \left\{ \hat{a} : a \in \operatorname{Kern} \nabla^2 \mathcal{H}(\alpha) \right\}^\perp \subset X.$$

Here the space $Y$ is not to be confused with the space $Y = H_\tau^2$ in the degree theoretic setting.

**Lemma 3.13.** *There exist constants $r_0, \rho > 0$ with $[0, r_0) \times B_\rho(\mathcal{M}) \subset \Lambda$, such that the map $\psi : \mathcal{U} := [0, r_0) \times \left(B_\rho(\mathcal{M}) \cap Y\right) \to Y$,*

$$\psi_r(u) = \begin{cases} (\mathrm{id} - P_D)\nabla\Phi_r(u) + \frac{1}{r^2} P_{D\cap Y} \nabla\Phi_r(u), & r > 0, \\ \nabla\Phi_0(u) - P_{D\cap Y}\nabla^2 F(0)u, & r = 0 \end{cases}$$

*is continuous, $C^1$ on $\mathcal{U} \cap ((0, r_0) \times X)$ with $D_u\psi$ continuous up to $r = 0$ and satisfies for $(r, u) \in \mathcal{U}$, $r > 0$:*

$$\nabla\Phi_r(u) = 0 \quad \Leftrightarrow \quad \psi_r(u) = 0.$$

*Moreover, $\mathcal{M}$ is a nondegenerate $l$-dimensional manifold of zeroes of $\psi_0$. I.e. for any $v \in \mathcal{M}$ there holds*

$$\psi_0(v) = 0, \quad \operatorname{Kern} D\psi_0(v) = T_v\mathcal{M} = \operatorname{span}\left\{ \dot{v}^1, \dots, \dot{v}^l \right\}.$$

*Proof.* As a first step observe that for positive $r$, $\bar{\psi}_r : \Lambda_r \to X$,

$$\begin{aligned}\bar{\psi}_r(u) &= (\mathrm{id} - P_D)\nabla\Phi_r(u) + \frac{1}{r^2}P_D\nabla\Phi_r(u) \\ &= \nabla\Phi_0(u) - (\mathrm{id} - P_D)(\mathrm{id} - \Delta)^{-1}r\nabla F(ru) - \frac{1}{r}P_D\nabla F(ru)\end{aligned} \tag{3.8}$$

has the same zeroes as $\nabla\Phi_r$. In the second equation we used that $\nabla\Phi_0$ maps into $D^\perp$, since $\Phi_0$ is invariant with respect to translations. Clearly $\bar{\psi}$ is $C^1$ as long as $r > 0$. Since $F$ is $C^2$ and $\nabla F(0) = 0$, $\bar{\psi}_r$ extends as $r \to 0$ continuously to $\bar{\psi}_0 : \Lambda_0 \to \mathbb{R}$,

$$\bar{\psi}_0(u) = \nabla\Phi_0(u) - P_D\nabla^2 F(0)u.$$

The partial derivative $D_u\bar{\psi} : \Lambda \to \mathcal{L}(X)$ is continuous as well and the regularity of $\bar{\psi}$ will carry over to $\psi$ once we have defined it.

Now let $v \in \mathcal{M}$. Since $Z^k(t) = \exp\big(\pm J_{N_k}t/\mathrm{ord}(\sigma_k)\big)z^k$ or $Z^k(t) \equiv 0$ due to (A3′) , we see that $\nabla^2 F(0)v \in D^\perp \subset Y$. Hence $\bar{\psi}_0(v) = 0$. Next

$$\mathrm{Kern}\, D\bar{\psi}_0(v) \underset{(3.7)}{=} \left(\mathrm{span}\left\{\dot{v}^1, \dots, \dot{v}^l\right\} \oplus D\right) \cap \mathrm{Kern}\, P_D\nabla^2 F(0)$$

and

$$P_D\nabla^2 F(0)\left[\sum_k \lambda_k\dot{v}^k + \hat{a}\right] = P_D\nabla^2 F(0)\hat{a}.$$

By Lemma 3.12, $\nabla^2 F(0)\hat{a} = M_\Gamma\hat{b}$ with $b^k = \frac{1}{\Gamma^k}\big(\nabla^2\mathcal{H}(\alpha)a\big)^k$, which projected onto $D$ gives $P_D M_\Gamma\hat{b} = \hat{c}$ with $c^k = \frac{\Gamma^k}{N_k}b^k$. Hence we see that $\sum_k \lambda_k\dot{v}^k + \hat{a}$ is an element of the kernel of $D\bar{\psi}_0(v)$ if and only if $a \in \mathrm{Kern}\,\nabla^2\mathcal{H}(\alpha)$, which means $\hat{a} \in Y^\perp$. So if we restrict $\bar{\psi}$ to $\psi$ as stated in the Lemma, especially $D\psi_0(v) = P_Y D\bar{\psi}_0(v) : Y \to Y$, we get

$$\mathrm{Kern}\, D\psi_0(v) = \mathrm{span}\left\{\dot{v}^1, \dots, \dot{v}^l\right\} = T_v\mathcal{M}.$$

It remains to prove that $\psi_r(u) = 0$ for $r > 0$ small, $u \in Y$ close to $\mathcal{M}$ implies $\nabla\Phi_r(u) = 0$. Note that $\psi_r(u) = 0$ if and only if $P_Y\nabla\Phi_r(u) = 0$. If $\alpha \in \mathcal{F}_m(\Omega)$ is a nondegenerate critical point of $\mathcal{H}$ as in (A1′)(i), we have $Y = X$ and are done. Otherwise by (A1′), $\Omega$, $g$ and hence also $G$ and $h$ are invariant with respect to translations and/or rotations.

Assume first that (iii) of (A1′) holds, i.e. $\lambda\nu + \Omega = \Omega$, $g(x + \lambda\nu, y + \lambda\nu) = g(x, y)$ for any $x, y \in \Omega$, $\lambda \in \mathbb{R}$ and some $\nu \in \mathbb{R}^2 \setminus \{0\}$. Then $\mathcal{H}(\alpha + \lambda\check{\nu}) = \mathcal{H}(\alpha)$, where $\check{\nu} = (\nu, \dots, \nu) \in \mathbb{R}^{2m}$, and $\Phi_r(u + \lambda\hat{\check{\nu}}) = \Phi_r(u)$ show that $\hat{\check{\nu}} \in Y^\perp$ and $\big\langle\nabla\Phi_r(u), \hat{\check{\nu}}\big\rangle = 0$ for any $u \in \Lambda_r$. So if $\nu$ is the only direction, in which $g$ is invariant, then $X = Y \oplus \mathbb{R}\hat{\check{\nu}}$ by (A1′) and $P_Y\nabla\Phi_r(u) = 0$ automatically gives $\nabla\Phi_r(u) = 0$.

If $\Omega$ and $g$ are rotational invariant, i.e. $e^{\lambda J}\Omega = \Omega$, $g(e^{\lambda J}x, e^{\lambda J}y) = g(x, y)$ for any $\lambda \in \mathbb{R}$, $x, y \in \Omega$, we obtain $J_m\alpha \in \mathrm{Kern}\,\nabla^2\mathcal{H}(\alpha)$, since $\mathcal{H}(e^{\lambda J_m}\alpha) = \mathcal{H}(\alpha)$ for any $\lambda \in \mathbb{R}$. For $\Phi_r$ there holds

$$\Phi_r\left(e^{\lambda J_N}\left(u + \frac{1}{r}\hat{\alpha}\right) - \frac{1}{r}\hat{\alpha}\right) = \Phi_r(u)$$

and therefore $\langle\nabla\Phi_r(u), J_N(ru + \hat{\alpha})\rangle = 0$ for any $u \in \Lambda_r$. Assuming that $\Omega$, $g$ have no other symmetry properties leads to the fact that $P_Y\nabla\Phi_r(u) = 0$ implies $\nabla\Phi_r(u) = 0$ as long as $X = Y \oplus \mathbb{R}J_N(ru + \hat{\alpha})$. Due to $J_N\hat{\alpha} \in Y^\perp$ we can find a subset $[0, r_0) \times B_\rho(\mathcal{M}) \subset \Lambda$ on which this condition holds. This settles case (A1′)(ii).

In the remaining case (A1′)(iv), where $\Omega = \mathbb{R}^2$ we have to choose the neighbourhood of $\{0\} \times \mathcal{M}$ such that

$$X = Y \oplus \operatorname{span}\left\{ \hat{\hat{e}}_1, \hat{\hat{e}}_2, J_N(ru + \hat{\alpha}) \right\}.$$

□

**Remark 3.14.** *If $\alpha$ is a critical point of $\mathcal{H}$ not satisfying (A1′), then Lemma 3.13 remains true with the exception that $\psi_r(u) = 0$ only implies $P_Y \nabla\Phi_r(u) = 0$.*

So far we have reduced the degeneracy of the limiting problem by $2m = \dim D$ dimensions. To overcome the remaining degeneracy induced by the $l$ independent time shifts of $Z^1, \dots, Z^l$ we perform a Lyapunov-Schmidt reduction.

For $v \in \mathcal{M}$ denote by $P_v : X \to T_v\mathcal{M} \subset Y$ the orthogonal projection onto $T_v\mathcal{M}$. Moreover, define $\tilde{\psi} : \tilde{\mathcal{U}} := [0, r_0) \times \mathcal{M} \times (B_\rho(0) \cap Y) \to Y$,

$$\tilde{\psi}(r, v, w) = (\operatorname{id} - P_v)\psi_r(v + w) + P_v w.$$

Since $\mathcal{M} \ni v \mapsto P_v \in \mathcal{L}(X)$ is $C^1$, we have $\tilde{\psi} \in C^1$ where $r > 0$, as well as continuity of $\tilde{\psi}$, $D_v\tilde{\psi}$, $D_w\tilde{\psi}$ on all of $\tilde{\mathcal{U}}$. For $(r, v, w) \in \tilde{\mathcal{U}}$ there holds

$$\psi_r(v + w) = 0,\ w \perp T_v\mathcal{M} \quad \Longleftrightarrow \quad \begin{cases} P_v\psi_r(v + w) = 0, \\ \tilde{\psi}(r, v, w) = 0. \end{cases}$$

**Lemma 3.15.** *Shrinking both $r_0 > 0$ and $\rho > 0$ if necessary, we find a continuous map $W : [0, r_0) \times \mathcal{M} \to B_\rho(0) \cap Y$ satisfying $W(r, v) \perp T_v\mathcal{M}$ for any $(r, v) \in [0, r_0) \times \mathcal{M}$ and*

$$\tilde{\psi}(r, v, w) = 0 \quad \Longleftrightarrow \quad w = W(r, v)$$

*on $\tilde{\mathcal{U}}$. Moreover, each $W(r, \cdot) : \mathcal{M} \to B_\rho(0)$ is equivariant with respect to the orthogonal action of $\{\, \theta \in \mathbb{T}^m : \theta_1 = \ldots = \theta_m \,\} \cong S^1$ on $X$. Concerning regularity we have $W \in C^1((0, r_0) \times \mathcal{M})$, and $D_vW$ is as $W$ itself continuous up to $r = 0$.*

*Proof.* Let $v \in \mathcal{M}$. One has $\tilde{\psi}(0, v, 0) = 0$ and

$$T := D_w\tilde{\psi}(0, v, 0) = (\operatorname{id} - P_v)D\psi_0(v) + P_v = D\psi_0(v) + P_v$$

has trivial kernel by Lemma 3.13. But note that $\operatorname{Range}(T) \neq Y$, in fact $T$ is an isomorphism between $Y$ and $H^2_\tau \cap Y$, which, similar to the proof of Theorem 2.7, can be seen in the following way:

Let $P_0 : H^1_\tau \to \mathbb{R}^{2N}$ be the orthogonal projection onto the space of constant functions and $L : H^s_\tau \to H^{s+1}_\tau$, $u \mapsto (\operatorname{id} - \Delta)^{-1}(-J_N M_\Gamma \dot{u}) + P_0 u$. Then $L$ is an isomorphism, also when viewed as a mapping from $Y \to H^2_\tau \cap Y$. Since $v$ is smooth, $L^{-1}\tilde{\psi}(0, v, \cdot) - \operatorname{id} : B_\rho(0) \cap Y \to Y$ is continuously differentiable and maps bounded subsets onto relatively compact subsets. Hence $L^{-1}T : Y \to Y$ is an index 0 Fredholm operator with trivial kernel. Therefore also $T : Y \to LY = H^2_\tau \cap Y$ is an isomorphism.

Note also that $\tilde{\psi}$ viewed as a map into $H^2_\tau \cap Y$ with $\|\cdot\|_{H^2_\tau}$ instead of $Y$ has the same regularity as the original $\tilde{\psi}$. So the implicit function theorem yields local maps $W_v$ solving the stated equation on $[0, r_v) \times U_v \times B_{\rho_v}(0)$, where $U_v \subset \mathcal{M}$ is an open neighbourhood of $v$.

However, the compactness of $\mathcal{M}$ and the uniqueness of the solution allow us to construct a global map $W : [0, r_0) \times \mathcal{M} \to B_\rho(0) \cap Y$ as requested by the Lemma. The equivariance of every $W(r, \cdot)$ with respect to synchronuous time shifts follows from the corresponding equivariance of $\tilde{\psi}$, i.e. $\tilde{\psi}(r, \theta * v, \theta * w) = \theta * \tilde{\psi}(r, v, w)$. □

For $r \in (0, r_0)$, $v \in \mathcal{M}$ it now remains to solve

$$P_v \psi_r(v + W(r, v)) = 0.$$

Therefore let $\varphi : [0, r_0) \times \mathcal{M} \to \mathbb{R}$,

$$\varphi(r, v) = \varphi_r(v) = \Phi_r(v + W(r, v)). \tag{3.9}$$

**Lemma 3.16.** *There exists $r_1 \in (0, r_0)$ such that $r \in (0, r_1)$, $v \in \mathcal{M}$ with $D\varphi_r(v) = 0$ imply $P_v \psi_r(v + W(r, v)) = 0$.*

*Proof.* Differentiation of $P_v W(0, v) = 0$ shows that $P_v D_v W(0, v) = 0$. We therefore have $P_v D_v W(r, v) = o(1)$ uniformly in $v \in \mathcal{M}$ as $r \to 0$ and thus can choose $r_1 \in (0, r_0)$ such that $\|P_v D_v W(r, v)\|_{\mathcal{L}(T_v \mathcal{M})} \leq \frac{1}{2}$ for every $(r, v) \in (0, r_1) \times \mathcal{M}$.

Assume $D\varphi_r(v) = 0$ for some $0 < r < r_1$, $v \in \mathcal{M}$. Using $P_v \circ P_D = 0$ one sees that $\tilde{\psi}(r, v, W(r, v)) = 0$ implies

$$(\mathrm{id} - P_v) P_Y \nabla \Phi_r(v + W(r, v)) = 0. \tag{3.10}$$

Thus we obtain for $v' \in T_v \mathcal{M}$

$$\begin{aligned} 0 = D\varphi_r(v)v' &= \langle \nabla \Phi_r(v + W(r, v)), (\mathrm{id} + D_v W(r, v))v' \rangle \\ &= \langle P_Y \nabla \Phi_r(v + W(r, v)), P_Y(\mathrm{id} + D_v W(r, v))v' \rangle \\ &= \langle P_v P_Y \nabla \Phi_r(v + W(r, v)), (\mathrm{id} + P_v D_v W(r, v))v' \rangle \end{aligned} \tag{3.11}$$

and conclude $P_v \psi_r(v + W(r, v)) = P_v P_Y \nabla \Phi_r(v + W(r, v)) = 0$, since by our choice of $r_1$ the map $\mathrm{id} + P_v D_v W(r, v) : T_v \mathcal{M} \to T_v \mathcal{M}$ is an isomorphism. □

Now it remains to investigate critical points of $\varphi_r$ for $r \in (0, r_1)$.

*Proof of Theorem 3.8.* Let $r \in (0, r_1)$. The reduced functional $\varphi_r$ is invariant with respect to the action of $\{ \theta \in \mathbb{T}^m : \theta_1 = \ldots = \theta_m \}$, which is smooth on $\mathcal{M}$. So every critical point of $\varphi_r$ belongs to a whole orbit of critical points. If $l = 1$, we are done. Otherwise we can find on each of the critical orbits a point of the form $(v^1, \ldots, v^{l-1}, Z^l, 0, \ldots, 0) \in \mathcal{M}$. Therefore the number of critical orbits is given by the number of critical points of $\mathbb{T}^{l-1} \to \mathbb{R}$, $\theta \mapsto \varphi_r\big((\theta_1, \ldots, \theta_{l-1}, 0, \ldots, 0) * Z\big)$, for which the Lusternik-Schnirelmann category of $\mathbb{T}^{l-1}$ provides $l$ as a minimal bound, see for example [23].

This way we have found for every $r \in (0, r_1)$ $l$ critical points of $\Phi_r$ lying on distinct orbits. Let $u = v + W(r, v) \in Y$ be one of them. Then $z(t) = ru(t/r^2) + \hat{\alpha}$ is by construction a $T(r) = \tau r^2 = 2\pi \operatorname{ord}(\sigma) r^2$-periodic solution of (3.3), for which the properties of Theorem 3.8 hold. □

## 3.4 Additional information and the case $l = 1$

For now we just continue our investigation with $l \in \{ 1, \ldots, m \}$ arbitrary.

**Lemma 3.17.** *Let $r \in (0, r_1)$, $\bar{v} \in \mathcal{M}$ be a critical point of $\varphi_r$, which means $\nabla \Phi_r(\bar{u}) = 0$ with $\bar{u} = \bar{v} + W(r, \bar{v})$. Then*

$$\operatorname{Kern} \nabla^2 \Phi_r(\bar{u}) \cap Y = (\mathrm{id} + D_v W(r, \bar{v})) \operatorname{Kern} D^2 \varphi_r(\bar{v}).$$

*Proof.* With the modified function $\psi$ from 3.13 we have $\operatorname{Kern} D\psi_r(\bar{u}) = \operatorname{Kern} \nabla^2 \Phi_r(\bar{u}) \cap Y$. Now $(\mathrm{id} - P_{\bar{v}}) D\psi_r(\bar{u}) + P_{\bar{v}} = D_w \tilde{\psi}(r, \bar{v}, W(r, \bar{v})) : Y \to Y \cap H^2_\tau$ is an isomorphism for every $r > 0$ small enough and $(\mathrm{id} - P_{\bar{v}}) D\psi_r(\bar{u}) : Y \to Y \cap H^2_\tau$ is a Fredholm operator with index 0, cf.

proof of Lemma 3.15. Thus by shrinking $r_1$ if necessary we get $\dim \operatorname{Kern}(\mathrm{id} - P_{\bar{v}}) D\psi_r(\bar{u}) = l$. On the other hand we know from Lemma 3.15, that $(\mathrm{id} - P_v)\psi_r(v + W(r,v)) = 0$ for every $v \in \mathcal{M}$, and hence

$$(\mathrm{id} - P_{\bar{v}}) D\psi_r(\bar{u})(\mathrm{id} + D_v W(r,\bar{v}))v' = 0$$

for $v' \in T_{\bar{v}}\mathcal{M}$. So

$$\operatorname{Kern} \nabla^2 \Phi_r(\bar{u}) \cap Y = \{\, v' + D_v W(r,\bar{v})v' : v' \in T_{\bar{v}}\mathcal{M} \,\} \cap \operatorname{Kern} P_{\bar{v}} D\psi_r(\bar{u}).$$

Next (3.11) shows that $v' \in \operatorname{Kern} D^2\varphi_r(\bar{v})$ iff $P_{\bar{v}} P_Y \nabla^2 \Phi_r(\bar{u})(\mathrm{id} + D_v W(r,\bar{v}))v' = 0$ and we conclude the statement since $P_{\bar{v}} P_Y \nabla^2 \Phi_r(\bar{u})_{|Y} = P_{\bar{v}} D\psi_r(\bar{u})$. □

**Lemma 3.18.** *Let $g \in C^k(\Omega \times \Omega, \mathbb{R})$ with $k \geq 2$. The map $W : [0, r_0) \times \mathcal{M} \to H^1_\tau$ is of class $C^{k-2}$. Furthermore, if $k \geq 3$, $\partial_r W(0,v) \in D$ for any $v \in \mathcal{M}$.*

*Proof.* Since $\mathcal{M} \ni v \mapsto P_v \in \mathcal{L}(X)$ is $C^\infty$ and since $W$ is implicitly defined, the regularity of $W$ is induced by $\psi$. With $g \in C^k$ we also have $F \in C^k$ and hence $\Phi \in C^k$. Then by the definition of $\psi$ in 3.13 one sees that $\psi$ is indeed of class $C^{k-2}$ provided the map $\kappa : \mathcal{U} \to L^2(\mathbb{R}/\tau\mathbb{Z}, \mathbb{R}^{2N})$,

$$\kappa(r,u) = \begin{cases} \frac{1}{r}\nabla F(ru), & r > 0, \\ \nabla^2 F(0)u, & r = 0 \end{cases}$$

is $C^{k-2}$. In order to proove this observe that $\kappa$ is $C^k$ as long as $r > 0$. The continuity up to $r = 0$ follows as in the proof of Lemma 3.13 from the fact that $F$ is $C^2$ and that $\nabla F(0) = 0$. Also the partial dervivatives that include at least one differentiation of $\kappa$ with respect to $u$ are easily seen to extend in a continuous way as $r \to 0$. So we have to look at the partial derivative

$$\partial_r^{k-2}\kappa(r,u) = \sum_{j=0}^{k-2} \frac{(k-2)!}{j!}(-1)^{k-j}\frac{1}{r^{k-1-j}}F^{(j+1)}(ru)[u]^j,$$

where $(r,u) \in \mathcal{U}$ with $r > 0$. Now a (pointwise) expansion of $F^{(j+1)}$ gives

$$F^{(j+1)}(ru)[u]^j = \sum_{l=0}^{k-2-j} \frac{r^l}{l!}F^{(j+1+l)}(0)[u]^{j+l} + \frac{r^{k-1-j}}{(k-1-j)!}F^{(k)}(\xi u)[u]^{k-1}$$

for some $\xi = \xi(j,u,t) \in (0,r)$. But as $r \to 0$ we obtain for the remainder

$$F^{(k)}(\xi u)[u]^{k-1} = F^{(k)}(0)[u]^{k-1} + o(1)$$

with respect to $\|\cdot\|_{L^2_\tau}$ and uniformly in $u \in B_\rho(\mathcal{M})$. Thus

$$\begin{aligned}
\partial_r^{k-2}\kappa(r,u) &= \sum_{j=0}^{k-2}\sum_{l=0}^{k-2-j} \frac{(k-2)!(-1)^{k-j}}{j!l!}\frac{1}{r^{k-1-l-j}}F^{(j+1+l)}(0)[u]^{j+l} \\
&\quad + \sum_{j=0}^{k-2}\frac{(k-2)!(-1)^{k-j}}{j!(k-1-j)!}F^{(k)}(0)[u]^{k-1} + o(1) \\
&= \sum_{n=0}^{k-2}\left(\frac{(k-2)!(-1)^k}{n!r^{k-1-n}}F^{(n+1)}(0)[u]^n \sum_{j=0}^{n}\frac{n!(-1)^j}{j!(n-j)!}\right) \\
&\quad + F^{(k)}(0)[u]^{k-1}\int_0^1 (1-s)^{k-2}\,ds + o(1) \\
&= \frac{1}{k-1}F^{(k)}(0)[u]^{k-1} + o(1).
\end{aligned}$$

So the partial derivatives $\partial_r^j\kappa$, $j = 1, \ldots, k-2$ exist and are continuous on all of $\mathcal{U}$.

For the second part assume that $g \in C^3$. Now $W$ is $C^1$ on all of $[0, r_0) \times \mathcal{M}$ and we know by Lemma 3.15 that

$$(\mathrm{id} - P_v)P_Y\nabla\Phi_r(v + W(r,v)) = 0, \quad P_v W(r,v) = 0$$

for $r > 0$ small, cf. equation (3.10). Differentiation of both equations with respect to $r$ at $r = 0$ and the use of $\partial_r\nabla\Phi_0(v) = 0$ as well as $(\mathrm{id} - P_v)P_Y\nabla^2\Phi_0(v) = \nabla^2\Phi_0(v)$ shows

$$\partial_r W(0,v) \in \operatorname{Kern}\nabla^2\Phi_0(v) \cap (T_v\mathcal{M})^\perp = D.$$

□

*Proof of Theorem 3.9.* Let now $l = 1$. In that case the reduced map $\varphi_r$ is in fact constant. Hence the demanded solutions of $\nabla\Phi_r(u) = 0$ can be parameterized by $u : [0, r_1) \to H^1_\tau$, $r \mapsto u^{(r)} = Z + W(r,Z)$, where $r_1 > 0$ is taken from Lemma 3.16 and $Z = (Z^1, 0 \ldots, 0) \in \mathcal{M}$. By Lemma 3.18 this parametrization is indeed $C^{k-2}$ provided $g \in C^k$, $k \geq 2$ and $\partial_r u^{(0)} \in D$ when $k \geq 3$.

It remains to prove the part concerning the global continuum. Assume that (A1′) (i) holds. Then $X = Y$ and by Lemma 3.17

$$\operatorname{Kern}\nabla^2\Phi_r\left(u^{(r)}\right) = (\mathrm{id} + D_v W(r,Z))\operatorname{Kern} D^2\varphi_r(Z) = (\mathrm{id} + D_v W(r,Z))\mathbb{R}\dot{Z} = \mathbb{R}\dot{u}^{(r)},$$

where the last equality holds due to the equivariance of $W$. By Theorem 2.5 and Theorem 2.3 the local family $\left(u^{(r)}\right)_r$ implies the existence of a global connected set $\tilde{C} \subset \Lambda$, such that $ru(t/r^2) + \hat{\alpha}$ is a $\tau r^2$-periodic solution of (3.3) for every $(r,u) \in \tilde{C}$. Via the rescaling

$$C = \left\{\left(\frac{\tau r^2}{2\pi}, ru\left(\frac{\tau}{2\pi}\cdot\right) + \hat{\alpha}\right) : (r,u) \in \tilde{C}\right\},$$

such that $(s,v) \in C$ implies $v(\cdot/s)$ is a $2\pi s$-periodic solution of 3.3, we translate this continuum into a continuum $C \subset \mathbb{R}^+ \times H^1_{2\pi}$ satisfying the properties of Corollary 2.9 and/or Remark 2.14.

If (A1′) (iii) holds, then the orbits $S^1 * u^{(r)}$ are not isolated critical orbits of $\Phi_r$. But we still can argue in the same way as before with the difference that we now have to work on the space $Y = \left\{u \in X : \left\langle u, \hat{\dot{v}}\right\rangle_X\right\}$ instead of $X$. For this reasoning it is important that $\nabla\Phi$ maps $\Lambda \cap Y$ into $Y$. □

**Remark 3.19.** *In the cases (A1′) (ii) or (iv) the orbits $S^1 * u^{(r)}$ are also not isolated. Additionally, since the symmetry group of rotations is now nonlinear, $\Lambda \cap Y$ is not mapped into $Y$ by $\nabla\Phi$, which prevents us from using the degree. One could try to factor out the rotational symmetry and establish the degree theory in the quotient manifold setting, but this has not been done.*

### 3.4.1 The case $l > 1$

For the case $l > 1$ a corresponding result would be true provided one knows that $\varphi_r$ for every $r > 0$ small is a Morse function except for synchronuous time shifts. This would not only imply that the solution set of $\nabla\Phi_r(u) = 0$ close to $\{0\} \times \mathcal{M}$ is a union of graphs and that these graphs induce global continua of solutions, but also increase for fixed $r > 0$ the number of existing solutions to $2^{l-1}$, which is the bound given by Morse theory.

However, a verification of the Morse property seems quite difficult. Suppose for simplicity that $g$ is $C^\infty$, such that $\varphi : [0, r_1) \times \mathcal{M} \to \mathbb{R}$, $\varphi_r(v) = \Phi_r(v + W(r, v))$ is also of class $C^\infty$. Obviously $\varphi_0(v) = \Phi_0(v)$ is independent of $v \in \mathcal{M}$. So we will expand $\varphi$ with respect to $r$. By (3.10) and since $W(r, v) \in Y \cap (T_v\mathcal{M})^\perp$ we have

$$\left\langle \nabla\Phi_r(v + W(r, v)), \partial_r^k W(r, v) \right\rangle_{H^1_\tau} = \left\langle (\mathrm{id} - P_v) P_Y \nabla\Phi_r(v + W(r, v)), \partial_r^k W(r, v) \right\rangle_{H^1_\tau} = 0. \tag{3.12}$$

Thus $\partial_r\varphi_r(v) = \partial_r\Phi_r(v+W(r, v))$ and especially $\partial_r\varphi_0(v) = 0$. Differentiation of (3.12) shows that the second derivative can be expressed by

$$\partial_r^2\varphi_r(v) = \partial_r^2\Phi_r(v + W(r, v)) - \left\langle \nabla^2\Phi_r(v + W(r, v))\partial_r W(r, v), \partial_r W(r, v) \right\rangle_{H^1_\tau}.$$

In particular, since $\partial_r W(0, v) \in D$,

$$\partial_r^2\varphi_0(v) = \partial_r^2\Phi_0(v) = -\int_0^\tau \left\langle \nabla^2 F(0)v, v \right\rangle_{\mathbb{R}^{2N}} dt.$$

We will show that this function is in fact also independent of $v \in \mathcal{M}$. Recall that

$$\begin{aligned} F(z) &= \sum_{\substack{k,k'=1 \\ k\neq k'}}^{m} \sum_{j=1}^{N_k} \sum_{j'=1}^{N_{k'}} \Gamma_j^k \Gamma_{j'}^{k'} G(z_j^k + \alpha^k, z_{j'}^{k'} + \alpha^{k'}) - \sum_{k=1}^{m} \sum_{j,j'=1}^{N_k} \Gamma_j^k \Gamma_{j'}^k g(z_j^k + \alpha^k, z_{j'}^k + \alpha^k) \\ &=: F_0(z) - \sum_{k=1}^{m} F_k(z). \end{aligned}$$

For $k = 1, \ldots, m$ the map $F_k(z)$ does only depend on the components $z^k$. Thus if we write $v = \theta * Z \in \mathcal{M}$, $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m$ and substitute $\tilde{t} = t + \theta_k$, we see that

$$\int_0^\tau \left\langle \nabla^2 F_k(0)v, v \right\rangle_{\mathbb{R}^{2N}} dt = \partial^2_{\varepsilon|\varepsilon=0} \int_0^\tau F_k(\varepsilon v)\, dt$$

does only depend on $Z^k$ and not on $\theta$ resp. $v$. With the same argument we obtain for some constant $c \in \mathbb{R}$

$$\begin{aligned} \int_0^\tau \left\langle \nabla^2 F_0(0)v, v \right\rangle_{\mathbb{R}^{2N}} dt &= c + 2\sum_{k\neq k'} \sum_j \sum_{j'} \Gamma_j^k \Gamma_{j'}^{k'} \int_0^\tau \left\langle \nabla_2\nabla_1 G(\alpha^k, \alpha^{k'}) v_{j'}^{k'}, v_j^k \right\rangle_{\mathbb{R}^2} dt \\ &= c + 2\sum_{k\neq k'} \int_0^\tau \left\langle \nabla_2\nabla_1 G(\alpha^k, \alpha^{k'}) c_{k'}, c_k \right\rangle_{\mathbb{R}^2} dt, \end{aligned}$$

where $c_k$ denotes the center of vorticity of the whole plane solution $v^k$, i.e.

$$c_k = \sum_{j=1}^{N_k} \Gamma_j^k v_j^k.$$

In general the center of vorticity is preserved along solutions of the whole plane systems, see for example [64]. In our case we have $c_k = 0$, since $v^k(t)$ is a relative equilibrium solution rigidly rotating around the origin. Thus also the second derivative $\partial_r^2 \varphi_0(v)$ is independent of $v$.

One step further one can also show that the third derivative

$$\partial_r^3 \varphi_0(v) = \partial_r^3 \Phi_r(v + W(r, v)) = -\int_0^\tau F^{(3)}(0)[v]^3\, dt$$

does not depend on $v \in \mathcal{M}$. Only the fourth order expansion of $\varphi_r$ at $r = 0$ has a chance to be a Morse function, since it involves among other, less explicit terms containing $\partial_r^2 W(0, v)$ the term $\int_0^\tau F^{(4)}(0)[v]^4\, dt$. At least the latter seems not to be constant, since we can not recover the center of vorticity in "biquadratic" terms like

$$\sum_j \sum_{j'} \Gamma_j \Gamma_{j'} D_1^2 D_2^2 G(\alpha^k, \alpha^{k'})[v_j^k, v_j^k, v_{j'}^{k'}, v_{j'}^{k'}].$$

That the critical points of $\partial_r^4 \varphi_0(v)$ are indeed nondegenerate (up to synchronuous time shifts) has not been investigated further.

## 3.5 Examples of nondegenerate relative equilibria

Let $Z(t) = e^{-\omega J_N t} z$, $z \in \mathbb{R}^{2N}$ fix, be a rigidly rotating solution of the whole plane system

$$M_\Gamma \dot{z} = J_N \nabla H_{\mathbb{R}^2}(z). \tag{3.13}$$

We define the so called stability matrix

$$A = J_N(M_\Gamma^{-1} \nabla^2 H_{\mathbb{R}^2}(z) + \omega \cdot \mathrm{id}) \in \mathbb{R}^{2N \times 2N}.$$

If we rewrite (3.13) in a rotating coordinate frame $w(t) = e^{-\omega J_N t} z(t)$, then $Z$ is a nondegenerate relative equilibrium if and only if

$$\dot{w} = Aw \tag{3.14}$$

has only 3 linear independent $\frac{2\pi}{|\omega|}$-periodic solutions.

In order to check this for concrete examples we shall use results of Roberts [67], who studied the linear stability of relative equilibria and therefore investigated the spectrum of $A$. For the convenience of the reader we recall Lemma 2.4 and some consequences from [67]. For $v \in \mathbb{R}^{2N}$ we use the notation $E_v := \operatorname{span}\{v, J_N v\} \subset \mathbb{R}^{2N}$.

**Lemma 3.20.** *a) Let $\hat{e}_1, \hat{e}_2 \in D$ be the standard basis of $D \subset \mathbb{R}^{2N}$. The spaces $E_z$ and $D$ are invariant subspaces of $A$. The representation of $A$ in the basis $(z, J_N z, \hat{e}_1, J_N \hat{e}_2)$ of the direct sum $E_z \oplus D$ is given by*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{pmatrix}.$$

b) *Suppose $v$ is a real eigenvector of $M_\Gamma^{-1}\nabla^2 H_0(z)$ with eigenvalue $\mu$. Then $E_v$ is an invariant subspace of $A$, on which $A$ is represented by*

$$\begin{pmatrix} 0 & \mu-\omega \\ \mu+\omega & 0 \end{pmatrix}.$$

c) *Suppose $v = v_1 + iv_2$ is a complex eigenvector of $M_\Gamma^{-1}\nabla^2 H_0(z)$ with eigenvalue $\mu = \xi + i\eta$. Then $\operatorname{span}\{ v_1, v_2, J_N v_1, J_N v_2 \} \subset \mathbb{R}^{2N}$ is a real invariant subspace of $A$, on which $A$ is represented by*

$$\begin{pmatrix} 0 & 0 & \xi-\omega & \eta \\ 0 & 0 & -\eta & \xi-\omega \\ \xi+\omega & \eta & 0 & 0 \\ -\eta & \xi+\omega & 0 & 0 \end{pmatrix}.$$

Note that the Hamiltonian in [67] differs by a factor of $\pi^{-1}$ from $H_{\mathbb{R}^2}$ but the corresponding stability matrices coincide, when translating the solution of one system to the other.

**Example 3.21.** *Let $N = 2$ and $\Gamma_1, \Gamma_2 \neq 0$ with $\Gamma := \Gamma_1 + \Gamma_2 \neq 0$. Any initial position $z_1, z_2$ of the two point vortices gives a relative equilibrium solution of* (3.13) *(see e.g. [64]). Via translation we can assume that they rotate rigidly around the origin with frequency $\omega = \frac{\Gamma}{\pi|z_1-z_2|^2} \neq 0$. Due to Lemma 3.20 the stability matrix $A \in \mathbb{R}^{4\times 4}$ of any such solution is given (in a suitable basis) by*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{pmatrix}.$$

*The linear system* (3.14) *then possesses exactly* 3 *linearly independent $\frac{2\pi}{|\omega|}$-periodic solutions.*

**Example 3.22.** *Now we consider $N = 3$ vortices with vortex strengths, $\Gamma_1, \Gamma_2, \Gamma_3 \neq 0$, and such that $\Gamma := \Gamma_1 + \Gamma_2 + \Gamma_3 \neq 0$. Then every equilateral triangle configuration $z_1, z_2, z_3$ is a relative equilibrium solution of the 3-vortex problem* (3.13) *(see Section 2.2 in [64]). Let $Z(t) = e^{-\omega J_3 t} z$ be an equilateral triangle configuration rotating around the origin. The corresponding stability matrix $A$ is a $6 \times 6$ matrix. In [67] Roberts computed its eigenvalues explicitly in the case when $\omega = \Gamma/3$ – this can always be achieved by a suitable scaling. He showed that in addition to the eigenvalues $0, 0, \pm i\omega$ of the block in 3.20 a) there are two more eigenvalues given by $\pm\sqrt{-L/3}$, where $L = \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3$ is the total vortex angular momentum. Hence the linear system* (3.14) *has more than 3 linearly independent $\frac{2\pi}{|\omega|}$-periodic solutions if $L > 0$ and $\sqrt{L/3} \in \omega\mathbb{Z} = \frac{\Gamma}{3}\mathbb{Z}$, hence if there exists $k \in \mathbb{Z}$ with*

$$3L = k^2\Gamma^2 = k^2\left(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 + 2L\right).$$

*This is only possible if $k^2 = 1$ and $L = \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2$. Therefore the equilateral triangle configuration is nondegenerate provided $\Gamma \neq 0$, $L \neq 0$ and $L \neq \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2$.*

**Example 3.23.** *Let $N \in \mathbb{N}$ be arbitrary and $\Gamma_1 = \ldots = \Gamma_N \neq 0$. If one places all point vortices on a line, say $z_k = (x_k, 0)$ and such that $x_1, \ldots, x_N$ are the roots of the $N$th Hermitian polynomial, then the line of vortices rotates rigidly around the origin, see page 2172 of [4]. From Corollary 3.3 in [67] we can conclude that this configuration is nondegenerate for any $N \geq 3$.*

So far we have shown all the examples stated in 3.5. We would like to mention that [67] provides with rhombus configurations and the trapezoidal configurations two more specific examples consisting of 4 vortices that could be considered for condition (A3).

## 3.6 An explicit stationary solution

With Examples 3.5 and 3.7 we have already seen some relative equilibrium solutions that are $\sigma$-nondegenerate or just nondegenerate and therefore can be choosen in (A3′) for theorem 3.8. Independent of the relative equilibrium solutions we also need for (A1′) a nondegenerate or not too degenerate critical point of the $m$-vortex Hamiltonian $\mathcal{H}$. We will verify this for Example 3.2. I.e. we look at the 2-vortex system in the unit disc $\Omega = B_1(0)$ with vorticities $\Gamma^1 = 1$, $\Gamma^2 = -1$. By combining for example a Thomson $N_1$-Gon configuration with vorticities $\Gamma^1_j = \frac{1}{N_1}$, $j = 1, \ldots, N_1$ and a collinear configuration of $N_2$ vortices of strengths $\Gamma^2_j = -\frac{1}{N_2}$, $j = 1, \ldots, N_2$ or another Thomson configuration we obtain therefore periodic solutions of (3.3) in the unit disc for an arbitrary number of $N = N_1 + N_2 \geq 3$ vortices that are not rigidly rotating around the center of the disc.

The regular part of the Dirichlet Green's function in $B_1(0)$ is given by

$$g(x, y) = g_{B_1(0)}(x, y) = -\frac{1}{4\pi} \log\left(|x|^2 |y|^2 - 2\langle x, y\rangle_{\mathbb{R}^2} + 1\right)$$

and

$$h(x) = h_{B_1(0)}(x) = -\frac{1}{2\pi} \log(1 - |x|^2),$$

such that the Hamiltonian defined on $\mathcal{F}_2(B_1(0))$ is given by

$$\begin{aligned} \mathcal{H}(a^1, a^2) = \frac{1}{\pi}\Big(\log\left|a^1 - a^2\right| - \frac{1}{2}\log\left(\left|a^1\right|^2\left|a^2\right|^2 - 2\left\langle a^1, a^2\right\rangle_{\mathbb{R}^2} + 1\right)\Big) \\ + \frac{1}{2\pi}\left(\log\left(1 - \left|a^1\right|^2\right) + \log\left(1 - \left|a^2\right|^2\right)\right). \end{aligned}$$

Let $R(y) = \frac{y}{|y|^2}$ be the reflection at the unit circle, then

$$\begin{aligned} \pi\nabla_1\mathcal{H}(a^1, a^2) &= \frac{a^1 - a^2}{|a^1 - a^2|^2} - \frac{a^1 - R(a^2)}{|a^1 - R(a^2)|^2} - \frac{a^1}{1 - |a^1|^2}, \\ \pi\nabla_2\mathcal{H}(a^1, a^2) &= \frac{a^2 - a^1}{|a^2 - a^1|^2} - \frac{a^2 - R(a^1)}{|a^2 - R(a^1)|^2} - \frac{a^2}{1 - |a^2|^2}. \end{aligned}$$

The ansatz $\alpha^1 = (\mu, 0)$, $\alpha^2 = (-\mu, 0)$ with $\mu > 0$ shows that $\alpha = (\alpha^1, \alpha^2)$ is a critical point of $\mathcal{H}$ if and only if

$$\mu^4 = 1 - 4\mu^2, \tag{3.15}$$

which means $\mu = \sqrt{\sqrt{5} - 2}$. For the second derivatives at the critical point $\alpha = (\mu, 0, -\mu, 0)$ we get with a repeated use of (3.15)

$$\begin{aligned} \pi\nabla_1^2\mathcal{H}(\alpha) &= \left(\frac{1}{4\mu^2} - \frac{1}{(\mu + \frac{1}{\mu})^2}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{(1 - \mu)^2}\begin{pmatrix} 1 + \mu^2 & 0 \\ 0 & 1 - \mu^2 \end{pmatrix} \\ &= \frac{1}{26\mu^2 - 6}\begin{pmatrix} -6\mu^2 + 1 & 0 \\ 0 & 4\mu^2 - 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \pi\nabla_2\nabla_1\mathcal{H}(\alpha) &= \frac{1}{4\mu^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{(1 + \mu^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &= \frac{1}{20\mu^2 - 4}\begin{pmatrix} \mu^2 + 1 & 0 \\ 0 & 3\mu^2 - 1 \end{pmatrix} \end{aligned}$$

and $\nabla_2^2\mathcal{H}(\alpha) = \nabla_1^2\mathcal{H}(\alpha)$, $\nabla_1\nabla_2\mathcal{H}(\alpha) = \nabla_2\nabla_1\mathcal{H}(\alpha)$. So the Hessian of $\mathcal{H}$ is given by

$$\pi\nabla^2\mathcal{H}(\alpha) = \begin{pmatrix} \frac{-6\mu^2+1}{26\mu^2-6} & 0 & \frac{\mu^2+1}{20\mu^2-4} & 0 \\ 0 & \frac{4\mu^2-1}{26\mu^2-6} & 0 & \frac{3\mu^2-1}{20\mu^2-4} \\ \frac{\mu^2+1}{20\mu^2-4} & 0 & \frac{-6\mu^2+1}{26\mu^2-6} & 0 \\ 0 & \frac{3\mu^2-1}{20\mu^2-4} & 0 & \frac{4\mu^2-1}{26\mu^2-6} \end{pmatrix}.$$

Using (3.15) one can verify that the second and the fourth column are identical. This corresponds to the degeneracy induced by the rotational invariance, which means that the vector $J_2\alpha = (0, -\mu, 0, \mu)$ is contained in Kern $\nabla^2\mathcal{H}(\alpha)$. On the other hand one easily sees that the first three columns are linearly independent. This shows that $\alpha$ is a critical point of the 2 vortex Hamiltonian $\mathcal{H}$ satisfying condition (A2′)(ii) as it has been stated in Example 3.2.

# Chapter 4

# Periodic solutions near $\partial\Omega$

In the previous chapter we have obtained periodic solutions by viewing the influence of the boundary $\partial\Omega$ as a perturbation of several whole-plane systems. Contrary to that we will now exploit vortex-boundary interactions in order to prove the existence of periodic solutions close to $\partial\Omega$. The proof again relies on a suitable singular limit scaling and the continuation of an existing periodic solution of the limiting problem. This time we will also use a symplectic change of coordinates transferring a neighborhood of a boundary component to the unit disc.

## 4.1 Statement of results

Let $\Omega \subset \mathbb{R}^2$ be a domain with nonempty boundary $\partial\Omega$, and let $C \subset \partial\Omega$ be a compact connected component of the boundary of class $C^4$. Clearly $C$ is diffeomorphic to $S^1$. Let $\nu : C \to \mathbb{R}^2$ denote the exterior unit normal and $\kappa : C \to \mathbb{R}$ the curvature of $C$ with respect to $\nu$. Set $d(x) = \operatorname{dist}(x, C)$ and fix $\delta > 0$ sufficiently small such that the orthogonal projection

$$p : \{\, x \in \Omega : d(x) \le \delta \,\} \to C$$

is well defined.

Instead of an actual Green's function we consider as before a general symmetric $C^3$ function $g : \Omega \times \Omega \to \mathbb{R}$, and set

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| - g(x,y) \quad \text{for } x, y \in \Omega,\ x \ne y.$$

We also need the corresponding generalized Robin function $h : \Omega \to \mathbb{R}$ defined as usual by $h(x) = g(x,x)$ and the generalized harmonic radius $\rho : \Omega \to \mathbb{R}$,

$$\rho(x) = \exp(-2\pi h(x)).$$

See appendix B.3 for the definition and some properties of the classical harmonic radius.

Contrary to chapter 3 we consider the case of $N$ identical point vortices, i.e. without limitation $\Gamma_1 = \ldots = \Gamma_N = 1$, such that the Hamiltonian $H : \mathcal{F}_N(\Omega) \to \mathbb{R}$ is given by

$$H(z_1, \ldots, z_N) = \sum_{\substack{j,k=1 \\ j \ne k}}^{N} G(z_j, z_k) - \sum_{k=1}^{N} h(z_k).$$

Therefore the system under investigation simply reads

$$\dot{z} = J_N \nabla H(z). \tag{4.1}$$

Assumption 4.1 contains the required asymptotic behavior of the functions induced by $g$ near $C$. For $y \in \Omega$ with $d(y) \le \delta$ let $Q_y : \mathbb{R}^2 \to \mathbb{R}\nu(p(y))$ denote the orthogonal projection onto the normalspace $N_{p(y)}C$, i.e. $Q_y v = \langle \nu(p(y)), v\rangle_{\mathbb{R}^2}\, \nu(p(y))$.

**Assumption 4.1.** *$\rho$ can be extended to a $C^3$ function on $\Omega \cup C$ by setting $\rho(x) := 0$ for $x \in C$, thus from now on $\rho : \Omega \cup C \to \mathbb{R}$. Moreover, $\nabla\rho(x) = -2\nu(x)$, and $\nabla^2\rho(x) = -2\kappa(x) \cdot \mathrm{id}_{\mathbb{R}^2}$ for every $x \in C$. For every $\varepsilon > 0$ the function $G$ satisfies*

$$|\nabla_1 G(x,y)| + \left|\nabla_1^2 G(x,y)\right| = O(d(y)) \quad \text{and} \quad \nabla_2\nabla_1 G(x,y) = O(1)Q_y + O(d(y))$$

*as $d(y) \to 0$ uniformly on the set $\left\{ (x,y) \in \overline{\Omega} \times \Omega_\delta : |x-y| \ge \varepsilon \right\}$.*

The terms $O(1)$ and $O(d(y)) = d(y) \cdot O(1)$ in the second equation in 4.1 are matrix valued. It follows from Assumption 4.1 that

$$\rho(x) = 2d(x) - \kappa(p(x))d(x)^2 + o(d(x)^2)$$

as $d(x) \to 0$, and $h(x) \to \infty$ as $d(x) \to 0$.

**Proposition 4.2.** *Assumption 4.1 holds for $g = g_\Omega$ being the regular part of the Green function of the Dirichlet Laplace operator in the cases when*

*a) $\Omega$ is a simply connected bounded domain with $C^{3,\alpha}$ boundary.*

*b) $\Omega$ is an annulus $\left\{ x \in \mathbb{R}^2 : a < |x| < b \right\}$, $0 < a < b$.*

*Proof.* See Corollary B.2, Lemma B.3 and Lemma B.4 in Appendix B. □

For more general multiply connected domains it might be possible to prove Assumption 4.1 using analytical formulae for the Green's function based on the Schottky-Klein prime function, see [25].

**Theorem 4.3.** *Let $L$ denote the length of $C$ and suppose that $g : \Omega \times \Omega \to \mathbb{R}$ satisfies Assumption 4.1. Then there exists $\bar{r} > 0$ and a $C^1$ map*

$$(0,\bar{r}) \times \mathbb{R} \ni (r,t) \mapsto z^{(r)}(t) \in \mathcal{F}_N(\Omega),$$

*having also continuous mixed derivatives $\partial_r\partial_t z^{(r)}(t) = \partial_t\partial_r z^{(r)}(t)$ and such that $z^{(r)}$ is a periodic solution of (4.1) with minimal period $Lr$ for each $r \in (0,\bar{r})$. Moreover, these periodic solutions possess the following properties:*

*(1) $z_k^{(r)}(t) = z_1^{(r)}\left(t + \frac{(k-1)Lr}{N}\right)$ for every $k = 1,\dots,N$.*

*(2) The rescaled function $v^{(r)}(t) := z_1^{(r)}(rt)$ converges in the space of $L$-periodic $C^1$ functions towards a parametrization $\gamma$ of $C$ according to arc-length and with $J\dot{\gamma} = \nu \circ \gamma$. More precisely there holds*

$$v^{(r)} = \gamma - \frac{r}{2\pi}\nu \circ \gamma + o(r),$$
$$\dot{v}^{(r)} = \left(1 - \frac{r}{2\pi}\kappa \circ \gamma\right)\dot{\gamma} + o(r)$$

*uniformly in $t$ as $r \to 0$.*

*(3) The distance $d\big(v^{(r)}(t)\big)$ satisfies*

$$d\big(v^{(r)}\big) = \frac{r}{2\pi} + \frac{r^2}{8\pi^2}\kappa \circ \gamma + o(r^2)$$

*as $r \to 0$ uniformly in $t \in \mathbb{R}$.*

(4) *The family $\left(z^{(r)}\right)_r$ induces a global continuum of choreographic solutions in the sense of Definition 2.11 and Remark 2.14.*

**Remark 4.4.** *a) The theorem shows that for $T > 0$ small enough, the system (4.1) has a $T$-periodic solution with all vortices moving on the same trajectory. At first order the trajectory has distance $r/2\pi = T/2\pi L$ from $C$. The second order term in (3) tells us that the trajectory of the vortices comes closer to $C$ in regions where $C$ has negative curvature, and the vortices speed up by (2). On the other hand, near positively curved parts of $C$ the trajectory increases the distance to $C$ and the vortices slow down. In any case the vortices try to use shortcuts near curved parts of the boundary, cf. Figure 1.3.*

*b) The expansions in (2) and (3) are independent of $N$.*

*c) The boundary component $C$ splits $\mathbb{R}^2$ into a bounded component $B_C$ and an unbounded component $U_C$, i.e. $\mathbb{R}^2$ is the disjoint union $B_C \cup C \cup U_C$. Let $\sigma(C) = 1$, if $\Omega \cap B_C \neq \emptyset$ and $\sigma(C) = -1$, if $\Omega \cap U_C \neq \emptyset$. With the use of (2) we obtain the following expansion of the $H^1_L$-norm*

$$\begin{aligned}\left\|v^{(r)}\right\|^2_{H^1_L} &= \|\gamma\|^2_{H^1_L} - \frac{r}{\pi}\left(\int_0^L \langle\gamma, J\dot{\gamma}\rangle_{\mathbb{R}^2}\, dt + \int_0^L \kappa\circ\gamma\, dt\right) + o(r)\\ &= \|\gamma\|^2_{H^1_L} - \frac{2\sigma(C)r}{\pi}\big(vol_2(B_C) + \pi\big) + o(r).\end{aligned}$$

*d) The theorem holds also for a boundary component of class $C^3$ instead of $C^4$, cf. the paper with Q. Dai and T. Bartsch [12]. However, the proof shown here requires the curvature $\kappa$ to be of class $C^2$. One might recover the original theorem by an approximation procedure with $C^4$ boundary curves and a careful control of the maximal parameter value $\bar{r}$, but this has not been examined closer.*

## 4.2 Scaling of the domain

Here we will just prove that it is sufficient to consider the case of a boundary component $C$ with length $2\pi$.

**Lemma 4.5.** *Suppose that Theorem 4.3 holds under the additional condition that $L = 2\pi$. Then it in fact holds for any length $L$ of $C$.*

*Proof.* Let $\Omega \subset \mathbb{R}^2$ be a domain, $C$ a bounded boundary component of class $C^4$ with length $L$ and let $g : \Omega\times\Omega \to \mathbb{R}$ be such that the induced functions $G$ and $\rho$ satisfy the conditions in Assumption 4.1.

Define $\lambda = \frac{2\pi}{L}$, such that the boundary component $\lambda C$ of $\lambda\Omega$ has length $2\pi$. On $\lambda\Omega$ consider $g_\lambda : \lambda\Omega\times\lambda\Omega \to \mathbb{R}$,

$$g_\lambda(x, y) = g(x/\lambda, y/\lambda) - \frac{1}{2\pi}\log\lambda$$

and the induced functions

$$\begin{aligned}\rho_\lambda(z) &= e^{-2\pi g_\lambda(z,z)} = \lambda\rho(z/\lambda),\\ G_\lambda(x, y) &= -\frac{1}{2\pi}\log|x-y| - g_\lambda(x, y) = G(x/\lambda, y/\lambda).\end{aligned}$$

Since the curvatures $\kappa_\lambda$ of $\lambda C$ and $\kappa$ of $C$ are related via $\kappa_\lambda(\lambda p) = \lambda^{-1}\kappa(p)$, $p \in C$, we can verify that $\rho_\lambda, G_\lambda$ satisfy Assumption 4.1.

Thus we obtain a family of $2\pi\tilde{r}$-periodic solutions $\tilde{z}^{(\tilde{r})}$, $\tilde{r} \in (0, \tilde{r}_1)$ of the generalized $N$-vortex system on $\lambda\Omega$ induced by $g_\lambda$ instead of $g$ satisfying the properties of Theorem 4.3. For $r \in (0, \lambda^{-1}\tilde{r}_1)$ we define

$$z^{(r)}(t) = \frac{1}{\lambda}\tilde{z}^{(\lambda r)}(\lambda^2 t).$$

Then each $z^{(r)}$ is indeed a $Lr$-periodic solution of the $N$-vortex system on $\Omega$ induced by $g$ and $(r,t) \mapsto z^{(r)}(t)$ has the regularity required by Theorem 4.3. Concerning the expansions we write $\gamma_\lambda : [0, 2\pi] \longrightarrow \lambda C$ for the parametrization of $\lambda C$ in arclength appearing in Theorem 4.3 and set $\gamma : [0, L] \longrightarrow C$, $\gamma(t) = \lambda^{-1}\gamma_\lambda(\lambda t)$, which is a parametrization of $C$ in arclength. We then have

$$\begin{aligned}
z_1^{(r)}(rt) &= \frac{1}{\lambda}\tilde{z}_1^{(\lambda r)}(\lambda r \cdot \lambda t) = \frac{1}{\lambda}\left(\gamma_\lambda(\lambda t) - \frac{\lambda r}{2\pi}J\dot{\gamma}_\lambda(\lambda t) + o(\lambda r)\right) \\
&= \gamma(t) - \frac{r}{2\pi}J\dot{\gamma}(t) + o(r), \\
\frac{d}{dt}\left(z_1^{(r)}(rt)\right) &= \frac{d}{ds}_{|s=\lambda t}\left(\tilde{z}_1^{(\lambda r)}(\lambda r \cdot s)\right) = \left(1 - \frac{\lambda r}{2\pi}\kappa_\lambda(\gamma_\lambda(\lambda t))\right)\dot{\gamma}_\lambda(\lambda t) + o(\lambda r) \\
&= \left(1 - \frac{r}{2\pi}\kappa(t)\right)\dot{\gamma}(t) + o(r), \\
d\left(z_1^{(r)}(rt)\right) &= \frac{1}{\lambda}\operatorname{dist}\left(\tilde{z}_1^{(\lambda r)}(\lambda r \cdot \lambda t), \lambda C\right) = \frac{1}{\lambda}\left(\frac{\lambda r}{2\pi} + \frac{\lambda^2 r^2}{8\pi^2}\kappa_\lambda(\gamma_\lambda(\lambda t)) + o(\lambda^2 r^2)\right) \\
&= \frac{r}{2\pi} + \frac{r^2}{8\pi^2}\kappa(\gamma(t)) + o(r^2).
\end{aligned}$$

Of course also the choreographic property of the solutions is not lost:

$$z_k^{(r)}(t) = \frac{1}{\lambda}\tilde{z}_1^{(\lambda r)}\left(\lambda^2 t + \frac{(k-1)2\pi\lambda r}{N}\right) = \frac{1}{\lambda}\tilde{z}_1^{(\lambda r)}\left(\lambda^2\left(t + \frac{(k-1)2\pi r}{\lambda N}\right)\right) = z_1^{(r)}\left(t + \frac{(k-1)Lr}{N}\right).$$

Concerning the global continuum we translate the induced set of periodic solutions

$$\tilde{C} \subset \mathbb{R}^+ \times \left\{ u \in H^1_{\text{chor}} : u(t) \in \lambda\mathcal{F}_N(\Omega) \text{ for all } t \in \mathbb{R} \right\}$$

containing the local graph $\left\{ \left(\tilde{r}, \tilde{z}^{(\tilde{r})}(\tilde{r}\cdot)\right) : \tilde{r} \in (0, \tilde{r}_1) \right\}$ via $C := \left\{ (r/\lambda, u/\lambda) : (r,u) \in \tilde{C} \right\}$. Then $C \subset \mathbb{R}^+ \times \left\{ u \in H^1_{\text{chor}} : u(t) \in \Omega,\ t \in \mathbb{R} \right\}$ is a global connected set of periodic solutions as in Corollary 2.9 with $C \supset \left\{ \left(r, z^{(r)}\left(\frac{Lr}{2\pi}\cdot\right)\right) : r \in (0, \lambda^{-1}\tilde{r}_1) \right\}$. □

## 4.3 Symplectic boundary coordinates and scaling

We will introduce a very useful change of coordinates in a neighbourhood of $C$. By the previous section we can assume that the length of the boundary component $C$ is exactly $2\pi$. Let $\gamma : \mathbb{R} \longrightarrow C$ be a $C^4$ $2\pi$-periodic covering of $C$, such that $J\dot{\gamma}(s) = \nu(\gamma(s))$ for every $s \in \mathbb{R}$, in particular $|\dot{\gamma}| \equiv 1$. Then the curvature $\kappa$ depending on the parameter $s$ can be expressed by

$$\kappa(s) := \kappa(\gamma(s)) = \langle J\ddot{\gamma}(s), \dot{\gamma}(s)\rangle_{\mathbb{R}^2}.$$

Furthermore, we define $U_1 = \left\{ (s,q) \in \mathbb{R}^2 : |\kappa(s)q| < \frac{1}{2} \right\}$, $\alpha : U_1 \to \mathbb{R}$, $\varphi : U_1 \to \mathbb{R}^2$,

$$\begin{aligned} \alpha(s,q) &= \begin{cases} \frac{1}{\kappa(s)}\left(1 - \sqrt{1 - 2\kappa(s)q}\right), & \kappa(s) \neq 0, \\ q, & \kappa(s) = 0, \end{cases} \\ \varphi(s,q) &= \gamma(s) - \alpha(s,q)J\dot{\gamma}(s). \end{aligned}$$

**Example 4.6.** *If $\Omega$ is the unit disc $B_1(0)$, then the transformation is given by*

$$\varphi_{B_1(0)}(s,q) = \sqrt{1-2q}\begin{pmatrix}\cos(s)\\ \sin(s)\end{pmatrix}, \quad s \in \mathbb{R},\ -\frac{1}{2} < q < \frac{1}{2}.$$

In general we have $d(\varphi(s,q)) = \alpha(s,q)$ as well as $p(\varphi(s,q)) = \gamma(s)$ for all $q \geq 0$ small enough, since $\partial\Omega$ satisfies the double sided ball condition. The reason for defining $\alpha$ as above is that the harmonic radius composed with $\varphi$ now satisfies uniformly in $s \in \mathbb{R}$ and as $0 < q \to 0$ the expansion

$$\begin{aligned} \rho(\varphi(s,q)) &= 2\alpha(s,q) - \kappa(s)\alpha(s,q)^2 + o(\alpha(s,q)^2) = 2q + o(q^2) \\ &= \rho_{\mathbb{R}^2_+}(s,q) + o(q^2), \end{aligned} \tag{4.2}$$

where $\rho_{\mathbb{R}^2_+}(s,q) = 2q$ is the actual harmonic radius of the Green's function in the upper half-plane $\mathbb{R}^2_+ = \left\{ x \in \mathbb{R}^2 : x_2 > 0 \right\}$, see Section B.2.2. So one could say $\varphi$ flattens the boundary of $\Omega$ up to second order.

The standard symplectic form on $\mathbb{R}^2$ is given by $\omega_{\mathbb{R}^2}(v,w) = \langle v, Jw\rangle_{\mathbb{R}^2} = v_1w_2 - v_2w_1$.

**Lemma 4.7.** *$\varphi$ is of class $C^2$ and symplectic, i.e. $\omega_{\mathbb{R}^2}(D\varphi(u)v, D\varphi(u)w) = \omega_{\mathbb{R}^2}(v,w)$ for all $u = (s,q) \in U_1$, $v,w \in \mathbb{R}^2$.*

*Proof.* The expansion

$$\alpha(s,q) = q + \frac{1}{2}\kappa(s)q^2 + \dots \tag{4.3}$$

shows that $\alpha$ is as smooth as $\kappa$. Therefore $\varphi$ is $C^2$. In dimension 2 a map is symplectic if and only if its derivative has determinant 1 everywhere. One has

$$D\varphi(s,q) = \left(\dot{\gamma}(s) - \partial_s\alpha(s,q)J\dot{\gamma}(s) - \alpha(s,q)J\ddot{\gamma}(s), -\partial_q\alpha(s,q)J\dot{\gamma}(s)\right) \in \mathbb{R}^{2\times 2}.$$

So we calculate

$$\begin{aligned} \det D\varphi(s,q) &= \omega_{\mathbb{R}^2}\left(\dot{\gamma}(s) - \partial_s\alpha(s,q)J\dot{\gamma}(s) - \alpha(s,q)J\ddot{\gamma}(s), -\partial_q\alpha(s,q)J\dot{\gamma}(s)\right) \\ &= \partial_q\alpha(s,q) - \partial_q\alpha(s,q)\alpha(s,q)\left\langle J\ddot{\gamma}(s), \dot{\gamma}(s)\right\rangle_{\mathbb{R}^2} \\ &= \partial_q\alpha(s,q)(1 - \kappa(s)\alpha(s,q)) = 1. \end{aligned}$$

□

By the inverse function theorem and the compactness of $C$ we obtain a small number $r_0 > 0$ such that for any $r \in (0, r_0)$ the map $\varphi$ is locally a diffeomorphism between the stripe $\mathbb{R} \times (0,r) \subset U_1$ and the image $\Omega_{2r} := \varphi(\mathbb{R} \times (0,r)) \subset \Omega$ with the property $\varphi(s_1,q_1) = \varphi(s_2,q_2)$ if and only if $q_1 = q_2$ and $s_1 - s_2 \in 2\pi\mathbb{Z}$.

We prefer to avoid working on the quotient manifold $(\mathbb{R}/2\pi\mathbb{Z}) \times (0, r_0)$ and therefore compose $\varphi$ with a scaled version of the "inverse" of $\varphi_{B_1(0)}$. Let $B_* := B_1(0) \setminus \{0\}$ and for $r \in [0, r_0)$ define

$$\psi_r : B_* \to \Omega_r, \quad \psi_r(x) = \varphi\left(\iota(x), r\frac{1-|x|^2}{2}\right),$$

where $\iota : \mathbb{R}^2 \setminus \{0\} \longrightarrow [0, 2\pi)$ is the argument function.

**Lemma 4.8.** *For $r \in (0, r_0)$ the map $\psi_r$ is a $C^2$ diffeomorphism and*

$$\omega_{\mathbb{R}^2}(D\psi_r(x)v, D\psi_r(x)w) = r\omega_{\mathbb{R}^2}(v, w) \quad \text{for all } x \in B_*, v, w \in \mathbb{R}^2,$$

*i.e. $\psi_r$ is symplectic with multiplier $r$. Moreover, $[0, r_0) \times B_* \ni (r, x) \mapsto \psi_r(x) \in \mathbb{R}^2$ is of class $C^2$.*

*Proof.* Of course $\iota$ is not continuous, but the composition with $\gamma$ defines a diffeomorphism between $S^1$ and $C \subset \partial\Omega$. Observe also that the following diagram commutes

$$\begin{tikzcd} B^* \arrow[r, "\psi_r"] & \Omega_r \\ \mathbb{R} \times \left(0, \frac{1}{2}\right) \arrow[u, "\varphi_{B_1(0)}"] \arrow[r, "A_r"] & \mathbb{R} \times \left(0, \frac{r}{2}\right) \arrow[u, "\varphi"'] \end{tikzcd} ,$$

where $A_r : \mathbb{R}^2 \to \mathbb{R}^2$, $(s, q) \mapsto (s, rq)$. Since $A_r$ is symplectic with multiplier $r$, the statement follows. □

In order to treat $N$-vortices we define $\Psi_r : B_*^N \to \Omega_r^N$,

$$\Psi_r(x_1, \ldots, x_N) = (\psi_r(x_1), \ldots, \psi_r(x_N)) .$$

Recall that the symplectic form associated to the $N$-vortex Hamiltonian system (4.1) is given by $\omega(v, w) = \sum_{j=1}^N \omega_{\mathbb{R}^2}(v_j, w_j)$. Therefore $\Psi : B_*^N \to \Omega_r^N$ is a $C^2$ diffeomorphism and symplectic with multiplier $r$. With a look at Lemma A.2, if needed, we can therefore conclude

**Proposition 4.9.** *Let $r \in (0, r_0)$. A function $u : I \to \mathcal{F}_N(B_*)$ defined on an interval $I \subset \mathbb{R}$ solves*

$$\dot{u} = J_N \nabla(H \circ \Psi_r)(u), \tag{4.4}$$

*if and only if $z : rI \to \mathcal{F}_N(\Omega_r)$, $z(t) = (\Psi_r \circ u)(t/r)$ is a solution of the $N$-vortex system (4.1).*

## 4.4 A single vortex

We start the investigation of (4.4) in the case of having only a single vortex in the domain, i.e. we consider $(H \circ \Psi_r)(u) = -h(\psi_r(u))$, $u \in B_*$. The single vortex case will already cover a major part of the work to do in the multiple vortex case.

**Lemma 4.10.** *Let $K \subset B_*$ be compact. As $r \to 0$ the following asymptotics hold in $C^2(K, \mathbb{R})$:*

$$h(\psi_r(u)) + \frac{1}{2\pi} \log r = h_{B_1(0)}(u) + o(1),$$

*where $h_{B_1(0)}(u) = -\frac{1}{2\pi} \log(1 - |u|^2)$ is the Robin function of the unit disc, cf. (B.3). Moreover,*

$$\lim_{r \to 0} \partial_r \nabla(h \circ \psi_r)(u) = 0 \tag{4.5}$$

*uniformly on $K$.*

*Proof.* Recall that $-2\pi h(z) = \log \rho(z)$. Since the logarithm and its derivatives are uniformly continuous on compact subsets of $(0, \infty)$, it is enough to prove that

$$r^{-1} \rho(\psi_r(u)) \xrightarrow{r \to 0} \rho_{B_1(0)}(u) = 1 - |u|^2$$

in $C^2(K, \mathbb{R})$.

By Assumption 4.1 we can expand $\rho$ and its derivatives uniformly on a neighborhood of $C$:

$$\begin{aligned}
\rho(z) &= 2d(z) - \kappa(p(z))d(z)^2 + o(d(z)^2),\\
\nabla\rho(z) &= \big(-2 + 2\kappa(p(z))d(z)\big)\nu(p(z)) + \frac{1}{2}d(z)^2\nabla^3\rho(p(z))[\nu(p(z)), \nu(p(z))] + o(d(z)^2),\\
\nabla^2\rho(z) &= -2\kappa(p(z))\,\mathrm{id}_{\mathbb{R}^2} + d(z)\nabla^3\rho(p(z))[-\nu(p(z))] + o(d(z))
\end{aligned}$$

as $d(z) \longrightarrow 0$. Evaluating these expansions at $\psi_r(u)$ shows that

$$\begin{aligned}
\rho(\psi_r(u)) &= r(1 - |u|^2) + o(r^2),\\
\nabla\rho(\psi_r(u)) &= \big(-2 + r\kappa(\iota(u))(1 - |u|^2)\big)\,J\dot{\gamma}(\iota(u))\\
&\quad + r^2\frac{(1 - |u|^2)^2}{8}\nabla^3\rho(\gamma(\iota(u)))[J\dot{\gamma}(\iota(u)), J\dot{\gamma}(\iota(u))] + o(r^2),\\
\nabla^2\rho(\psi_r(u)) &= -2\kappa(\iota(u))\,\mathrm{id}_{\mathbb{R}^2} + r\frac{1 - |u|^2}{2}\nabla^3\rho(\gamma(\iota(u)))[-J\dot{\gamma}(\iota(u))] + o(r)
\end{aligned} \tag{4.6}$$

as $r \longrightarrow 0$ and uniformly in $u \in K$, cf. (4.2), (4.3). Therefore we can directly conclude the convergence of $r^{-1}\rho(\psi_r(u)) \longrightarrow \rho_{B_1(0)}(u)$ in $C^0(K, \mathbb{R})$.

The derivatives of the stated limit function are given by

$$\nabla\rho_{B_1(0)}(u) = -2u, \quad \nabla^2\rho_{B_1(0)}(u) = -2\,\mathrm{id}_{\mathbb{R}^2}.$$

For the convergence of the derivatives of $r^{-1}\rho \circ \psi_r$ we will need also some expansions of $\psi_r(u)$ and its derivatives. By the expression of $\alpha$ in the power series (4.3) we have

$$\begin{aligned}
f(\varepsilon, \theta) := \psi_r\left((1 + \varepsilon)e^{-J\theta}u\right) &= \gamma(\iota(u) + \theta)\\
&\quad - J\dot{\gamma}(\iota(u) + \theta)\left(r\frac{1 - (1 + \varepsilon)^2|u|^2}{2} + r^2\kappa(\iota(u) + \theta)\frac{(1 - (1 + \varepsilon)^2|u|^2)^2}{8} + O(r^3)\right),
\end{aligned}$$

with $O(r^3)$ being the reminder of the power series. Thus we obtain again uniformly in $u \in K$ as $r \longrightarrow 0$:

$$\begin{aligned}
D\psi_r(u)[u] &= \partial_\varepsilon f(0, 0) = \left(r|u|^2 + r^2\kappa(\iota(u))\frac{(1 - |u|^2)}{2}|u|^2\right)J\dot{\gamma}(\iota(u)) + o(r^2),\\
D\psi_r(u)[-Ju] &= \partial_\theta f(0, 0) = \dot{\gamma}(\iota(u)) - \left(r\frac{1 - |u|^2}{2} + r^2\kappa(\iota(u))\frac{(1 - |u|^2)^2}{8}\right)J\ddot{\gamma}(\iota(u))\\
&\quad - r^2\dot{\kappa}(\iota(u))\frac{(1 - |u|^2)^2}{8}J\dot{\gamma}(\iota(u)) + o(r^2),\\
D^2\psi_r(u)[u, u] &= \partial_\varepsilon^2 f(0, 0) = r|u|^2 J\dot{\gamma}(\iota(u)) + o(r),\\
D^2\psi_r(u)[u, -Ju] &= \partial_\theta\partial_\varepsilon f(0, 0) - D\psi_r(u)[-Ju]\\
&= -\dot{\gamma}(\iota(u)) + r\frac{1 + |u|^2}{2}J\ddot{\gamma}(\iota(u)) + o(r),\\
D^2\psi_r(u)[-Ju, -Ju] &= \partial_\theta^2 f(0, 0) + D\psi_r(u)[u]\\
&= \ddot{\gamma}(\iota(u)) - r\frac{1 - |u|^2}{2}J\gamma^{(3)}(\iota(u)) + r|u|^2 J\dot{\gamma}(\iota(u)) + o(r),
\end{aligned} \tag{4.7}$$

with $\gamma^{(3)}$ being the third derivative of the parametrization $\gamma$. Combining now (4.6), (4.7) and $|\dot\gamma| \equiv 1$ shows

$$\begin{aligned}
\langle \nabla(\rho\circ\psi_r)(u), u\rangle_{\mathbb{R}^2} &= \langle \nabla\rho(\psi_r(u)), D\psi_r(u)[u]\rangle_{\mathbb{R}^2} = -2r\,|u|^2 + o(r)\\
&= r\left\langle \nabla\rho_{B_1(0)}(u), u\right\rangle_{\mathbb{R}^2} + o(r),\\
\langle \nabla(\rho\circ\psi_r)(u), -Ju\rangle_{\mathbb{R}^2} &= \langle O(1)J\dot\gamma(\iota(u)) + o(r), \dot\gamma(\iota(u)) + O(1)J\ddot\gamma(\iota(u)) + o(r)\rangle_{\mathbb{R}^2} = o(r)\\
&= r\left\langle \nabla\rho_{B_1(0)}(u), -Ju\right\rangle_{\mathbb{R}^2} + o(r)
\end{aligned}$$

uniformly. So we can conclude the convergence in $C^1(K,\mathbb{R})$.

In the same straightforward way one can check that

$$\nabla^2(\rho\circ\psi_r)(u)[u,u] = -2r\,|u|^2 + o(r), \quad \nabla^2(\rho\circ\psi_r)(u)[u,-Ju] = o(r).$$

Only $\nabla^2(\rho\circ\psi_r)(u)[-Ju,-Ju]$ is a little trickier. Here we need the first order expansion

$$\begin{aligned}
&\nabla^2\rho(\psi_r(u))[\dot\gamma(\iota(u)), \dot\gamma(\iota(u))] + 2\kappa(\iota(u))\\
&\quad= r\frac{1-|u|^2}{2}\nabla^3\rho(\gamma(\iota(u)))[-J\dot\gamma(\iota(u)), \dot\gamma(\iota(u)), \dot\gamma(\iota(u))] + o(r)\\
&\quad= r\frac{1-|u|^2}{2}\frac{d}{d\varepsilon}_{|\varepsilon=0}\nabla^2\rho(\gamma(\iota(u)+\varepsilon))[-J\dot\gamma(\iota(u)), \dot\gamma(\iota(u))] + o(r) = o(r).
\end{aligned}$$

This together with

$$\left\langle \dot\gamma(\iota(u)), \gamma^{(3)}(\iota(u))\right\rangle_{\mathbb{R}^2} = -\,|\ddot\gamma(\iota(u))|^2 = -\kappa(\iota(u))^2,$$

which is a consequence of $|\dot\gamma| \equiv 1$, shows

$$\begin{aligned}
\nabla^2(\rho\circ\psi_r)(u)[-Ju,-Ju] &= \left\langle \nabla^2\rho(\psi_r(u))D\psi_r(u)[-Ju], D\psi_r(u)[-Ju]\right\rangle_{\mathbb{R}^2}\\
&\quad+\left\langle \nabla\rho(\psi_r(u)), D^2\psi_r(u)[-Ju,-Ju]\right\rangle_{\mathbb{R}^2}\\
&= -2\kappa(\iota(u))\left|\dot\gamma(\iota(u)) - r\frac{1-|u|^2}{2}J\ddot\gamma(\iota(u))\right|^2\\
&\quad+\left\langle \left(-2 + r\kappa(\iota(u))(1-|u|^2)\right)J\dot\gamma(\iota(u)), \ddot\gamma(\iota(u))\right\rangle_{\mathbb{R}^2}\\
&\quad+\left\langle -2J\dot\gamma(\iota(u)), -r\frac{1-|u|^2}{2}J\gamma^{(3)}(\iota(u)) + r\,|u|^2\,J\dot\gamma(\iota(u))\right\rangle_{\mathbb{R}^2} + o(r)\\
&= -2r\,|u|^2 + o(r).
\end{aligned}$$

Therefore $r^{-1}\rho\circ\psi_r \longrightarrow \rho_{B_1(0)}$ in $C^2(K,\mathbb{R})$.

It remains to prove that $\partial_r\nabla(h\circ\psi_r) = o(1)$ in $C^0(K,\mathbb{R}^n)$. By the expansion in (4.6) and since $(r,u)\mapsto \rho(\psi_r(u))$ is in $C^2([0,r_0)\times B_*,\mathbb{R})$, there holds $\partial_r\left(r^{-1}\rho(\psi_r(u))\right) = o(1)$ in $C^0(K,\mathbb{R})$. Suppose for a moment that we also know $\partial_r\left(r^{-1}\nabla(\rho\circ\psi_r)\right) = o(1)$ in $C^0(K,\mathbb{R}^2)$, then

$$\begin{aligned}
-2\pi\partial_r\nabla(h\circ\psi_r)(u) &= \partial_r\left(\frac{r}{\rho(\psi_r(u))}\frac{\nabla(\rho\circ\psi_r)(u)}{r}\right)\\
&= -\frac{r^2}{\rho(\psi_r(u))^2}\partial_r\left(\frac{\rho(\psi_r(u))}{r}\right)O(1) + O(1)\partial_r\left(\frac{\nabla(\rho\circ\psi_r)(u)}{r}\right) = o(1).
\end{aligned}$$

Thus we use the second order expansion of $D\psi_r(u)[u]$ in (4.7) to improve the previous expansion to $\langle\nabla(\rho\circ\psi_r)(u), u\rangle_{\mathbb{R}^2} = -2r\,|u|^2 + o(r^2)$. In a similar way the identity

$$\nabla^3\rho(\gamma(\iota(u)))[J\dot\gamma(\iota(u)), J\dot\gamma(\iota(u)), \dot\gamma(\iota(u))] = -2\dot\kappa(\iota(u))$$

shows that $\langle\nabla(\rho\circ\psi_r)(u), -Ju\rangle_{\mathbb{R}^2} = o(r^2)$. Since $(r,u)\mapsto\nabla(\rho\circ\psi_r)(u)$ is in $C^2([0,r_0)\times B_*, \mathbb{R}^2)$ we get the desired asymptotics $\partial_r\left(r^{-1}\nabla(\rho\circ\psi_r)\right) = o(1)$ in $C^0(K,\mathbb{R}^n)$.

This finishes the proof of the Lemma. □

So we have shown that the scaled and transformed 1-vortex system on $B_*$

$$\dot u = -J\nabla(h\circ\psi_r)(u) \tag{4.8}$$

is in the limit $r\to 0$ a perturbation of the actual 1-vortex system in the unit disc

$$\dot u = -J\nabla h_{B_1(0)}(u) = -\frac{Ju}{\pi(1-|u|^2)}, \tag{4.9}$$

which for the initial condition $u_0\in B_*$ has the solution

$$u(t) = \exp\left(-\frac{1}{\pi(1-|u_0|^2)}Jt\right)u_0.$$

From these solutions we pick out one that is $2\pi$-periodic, e.g. we fix

$$u^*(t) = \sqrt{1-\pi^{-1}}\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}. \tag{4.10}$$

**Lemma 4.11.** *The space of* $2\pi$*-periodic solutions of the linearization*

$$\dot v = -J\nabla^2 h_{B_1(0)}(u^*(t))v \tag{4.11}$$

*is* $\mathbb{R}\dot u^*$.

*Proof.* By the invariance of $h_{B_1(0)}$ under rotations we can use a rotating coordinate frame $v(t) = e^{-Jt}w(t)$, such that equation (4.11) is equivalent to

$$\dot w = Jw - J\nabla^2 h_{B_1(0)}\left(\sqrt{1-\pi^{-1}}e_1\right)w.$$

The explicit formula of the Hessian

$$\nabla^2 h_{B_1(0)}(z) = \frac{1}{\pi(1-|z|^2)}\,\mathrm{id}_{\mathbb{R}^2} + \frac{2zz^T}{\pi(1-|z|^2)^2},$$

cf. B.3, therefore gives

$$\dot w = \begin{pmatrix}0 & 0\\ 2\pi-2 & 0\end{pmatrix}w.$$

This system clearly has only the stationary points $\mathbb{R}e_2$ as $2\pi$-periodic solutions and hence any $2\pi$-periodic solution $v$ of (4.11) satisfies $v\in\mathbb{R}\dot u^*$. □

*Proof of Theorem 4.3 in the case* $N=1$. Combining Lemma 4.10, Lemma 4.11 and Theorem 2.7 we obtain a $C^1$-family $\left(u^{(r)}\right)_{r\in[0,r_1)}\subset X = H^1$ of $2\pi$-periodic solutions of (4.8) having 1 as geometrically simple Floquet multiplier and with $u^{(0)} = u^*$, $\left\langle u^{(r)}, \dot u^*\right\rangle_X = 0$. By Theorem

2.7 we also are allowed to differentiate

$$\dot{u}^{(r)}(t) = -J\nabla(h \circ \psi_r)\left(u^{(r)}(t)\right)$$

with respect to $r$ at $r = 0$, which by (4.5) leads to

$$\partial_t\left(\partial_r u^{(0)}(t)\right) = -J\nabla^2(h \circ \psi_r)(u^*(t))\left[\partial_r u^{(0)}(t)\right].$$

So $\partial_r u^{(0)}$ is a $2\pi$-periodic solution of (4.11) and therefore Lemma 4.11 implies $\partial_r u^{(0)} = \lambda \dot{u}^*$ for some $\lambda \in \mathbb{R}$. On the the other hand differentiation of $\left\langle u^{(r)}, \dot{u}^*\right\rangle_X = 0$ shows $\lambda = 0$ and thus

$$\partial_r u^{(0)} = 0. \tag{4.12}$$

By Proposition 4.9 we obtain a family $z^{(r)} = \psi_r\left(u^{(r)}(\cdot/r)\right)$, $r \in (0, r_1)$ of $2\pi r$-periodic solutions of the original 1-vortex system on $\Omega$. Moreover, $(0, r_1) \times \mathbb{R} \ni (r,t) \mapsto z^{(r)}(t) \in \Omega$ is $C^1$ and the mixed partial derivatives $\partial_r\partial_t z^{(r)}(t)$, $\partial_t\partial_r z^{(r)}(t)$ exist, are equal and continuous. If we rescale the solutions to $z^{(r)}(r\cdot) = \psi_r \circ u^{(r)}$, the same regularity holds now up to $r = 0$. So by (4.7),(4.12) and $u^*(t) = \sqrt{1-\pi^{-1}}e^{-Jt}e_1$ the following expansions hold uniformly in $t$ as $r \to 0$:

$$\begin{aligned}
z^{(r)}(rt) &= \psi_0(u^*(t)) + r\partial_r\psi_0(u^*(t)) + o(r)\\
&= \gamma(\iota(u^*(t))) - r\frac{1-|u^*(t)|^2}{2}J\dot{\gamma}(\iota(u^*(t))) + o(r)\\
&= \gamma(t) - \frac{r}{2\pi}J\dot{\gamma}(t) + o(r),\\
\frac{d}{dt}\left(z^{(r)}(rt)\right) &= D\psi_0(u^*(t))[\dot{u}^*(t)] + r\partial_r D\psi_0(u^*(t))[\dot{u}^*(t)] + o(r)\\
&= \dot{\gamma}(t) - \frac{r}{2\pi}J\ddot{\gamma}(t) + o(r)\\
&= \left(1 - \frac{r}{2\pi}\kappa(t)\right)\dot{\gamma}(t) + o(r).
\end{aligned} \tag{4.13}$$

For the distance to the boundary component $C$ we have by (4.3)

$$\begin{aligned}
d\left(z^{(r)}(rt)\right) &= \alpha\left(\iota\left(u^{(r)}(t)\right), r\frac{1-\left|u^{(r)}(t)\right|^2}{2}\right)\\
&= \frac{r}{2\pi} + \frac{r^2}{8\pi^2}\kappa(t) + o(r^2).
\end{aligned} \tag{4.14}$$

So far we have proven the local properties of Theorem 4.3 in the case of a single vortex.

For the global part we need to show that the Floquet multiplier 1 of $z^{(r)}$ is geometrically simple. Let $v$ be a Floquet solution for $z^{(r)}$ to the multiplier 1, i.e. $v$ is a $2\pi r$-periodic solution of the linearization $\dot{v} = -J\nabla^2 H\left(z^{(r)}(t)\right)v$. By Lemma A.2 $v$ corresponds to a $2\pi$-periodic solution $w(t) = D\psi_r\left(u^{(r)}(t)\right)^{-1}v(rt)$ of $\dot{w} = -J\nabla^2(h \circ \psi_r)\left(u^{(r)}(t)\right)w$. But since $u^{(r)}$ has 1 as a geometrically simple multiplier, we have $w \in \mathbb{R}\dot{u}^{(r)}$. It follows $v \in \mathbb{R}\dot{z}^{(r)}$ and Corollary 2.9 implies the existence of a global continuum of periodic solutions of the generalized 1-vortex system on $\Omega$. ◻

## 4.5 Choreographic solutions with $N \geq 2$ vortices

We now turn to the case of $N \geq 2$ vortices. Consider again a boundary component $C$ of length $2\pi$ and recall that by Proposition 4.9 it is our goal to find for $r > 0$ small $2\pi$-periodic

solutions $u(t) \in \mathcal{F}_N(B_*)$ of
$$\dot{u} = J_N \nabla(H \circ \Psi_r)(u).$$
In order to achieve this we consider as usual the action functional
$$\Phi_r(u) = \frac{1}{2}\int_0^{2\pi} \langle \dot{u}, J_N u\rangle_{\mathbb{R}^{2N}}\ dt - \int_0^{2\pi} H(\Psi_r(u))\, dt.$$

As in the previous chapter we restrict ourselves to the the subspace of choreographic functions. Let $\sigma = (1\ 2\ 3\ \dots\ N)$ be the cyclic permutation of $N$ symbols and for an element $z = (z_1, \dots, z_N) \in \mathbb{R}^{2N}$ or a function $u(t) = (u_1(t), \dots, u_N(t))$ let $\sigma * z = (z_N, z_1, z_2, \dots, z_{N-1})$, $(\sigma * u)(t) = \sigma * (u(t))$. Since all the vorticities are equal, the functional $\Phi$ satisfies
$$\Phi_r\big((\sigma * u)(\cdot + 2\pi/N)\big) = \Phi_r(u)$$
for any $r > 0$ and $u \in H^1$ with $u(t) \in \mathcal{F}_N(B_*)$. This implies
$$\nabla\Phi_r\big((\sigma * u)(\cdot + 2\pi/N)\big) = \big(\sigma * \nabla\Phi_r(u)\big)(\cdot + 2\pi/N).$$
So especially for $u \in \tilde{X} = \big\{ u \in H^1 : (\sigma * u)(\cdot + 2\pi/N) = u \big\}$ we see that $\nabla\Phi_r(u) \in \tilde{X}$. Contrary to Chapter 3 we will not work on $\tilde{X}$ but prefer to use this time the isomorphism $j : X = H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^2) \to \tilde{X}$ given by
$$j(u) = \big(u, u(\cdot + 2\pi/N), u(\cdot + 4\pi/N), \dots, u(\cdot + (N-1)2\pi/N)\big).$$
The inverse clearly is $j^{-1}(u_1, \dots, u_N) = u_1$. Let $\Lambda_* = \{ u \in X : (j(u))(\mathbb{R}) \subset \mathcal{F}_N(B_*) \}$ and observe that our solution $u^*(t)$ of the 1-vortex problem in $B_1(0)$, see (4.10), is contained in $\Lambda_*$. We abbreviate $\theta_k = k2\pi/N$ for $k = 1, \dots, N$.

**Lemma 4.12.** *Let $r_1 \in (0, r_0)$. There exist $\varepsilon > 0$, a compact intervall $K \subset (0, 1)$ and an open neighborhood $O \subset \Lambda_*$ of $u^*$, such that $(r, u) \in [0, r_1] \times O$ implies $|u(t)| \in K$ and*
$$|\psi_r(u(t + \theta_k)) - \psi_r(u(t + \theta_l))| \geq \varepsilon$$
*for every $t \in \mathbb{R}$ and $k, l \in \{ 1, \dots, N \}$ with $k \neq l$.*

*Proof.* Recall that $|u^*(t)| = \sqrt{1 - \pi^{-1}} =: b_0$. Clearly there exists a constant $c > 0$ depending only on $N$ with $\inf_{t,k\neq l} |u^*(t + \theta_k) - u^*(t + \theta_l)| \geq 2c$. Since $X$ continuously embeds into the space of $2\pi$-periodic continuous functions, we can find $\delta > 0$ – without restriction we assume $2\delta < c$ and $K := [b_0 - \delta, b_0 + \delta] \subset (0, 1)$ – and an open neighborhood $O \subset \Lambda_*$ of $u^*$ such that $u \in O$ implies $|u(t)| \in K$ and
$$\inf_{t,k\neq l} |u(t + \theta_k) - u(t + \theta_l)| \geq c.$$
For $u \in O$, $t \in \mathbb{R}$, $j, k \in \{ 1, \dots, N \}$, $k \neq j$ we therefore obtain
$$\begin{aligned} &|\psi_r(u(t + \theta_k)) - \psi_r(u(t + \theta_l))| \\ &\qquad \geq \inf\big\{ |\psi_r(a) - \psi_r(b)| : r \in [0, r_1],\ a, b \in \mathbb{R}^2,\ |a|, |b| \in K,\ |a - b| \geq c \big\} =: \varepsilon. \end{aligned}$$
If $\varepsilon$ would be 0, the continuity of $(r, x) \mapsto \psi_r(x)$ shows the existence of $a, b \in \mathbb{R}^2$, $|a|, |b| \in K$, $|a - b| \geq c$ and $r \in [0, r_1]$ with $\psi_r(a) = \psi_r(b)$. This is impossible for $r > 0$, since each $\psi_r$ is a diffeomorphism. So $\varepsilon = 0$ implies $\gamma(\iota(a)) = \psi_0(a) = \psi_0(b) = \gamma(\iota(b))$. But then
$$c \leq |a - b| = ||a| - |b|| \leq b_0 + \delta - (b_0 - \delta) = 2\delta < c$$

is a contradiction. Hence $\varepsilon > 0$. □

From now on we fix $r_1 \in (0, r_0)$ and $\mathcal{O}$, $K$, $\varepsilon$ according to Lemma 4.12.

**Lemma 4.13.** *The map* $F : [0, r_1] \times \mathcal{O} \to X$,

$$F(r,u) = \begin{cases} (j^{-1} \circ \nabla\Phi_r \circ j)(u), & r > 0, \\ (\mathrm{id} - \Delta)^{-1}\left(-J\dot{u} + \nabla h_{B_1(0)}(u)\right), & r = 0 \end{cases}$$

*is of class $C^1$ with derivatives $\partial_r F(0,u) = 0$ and*

$$D_u F(0,u)[w] = (\mathrm{id} - \Delta)^{-1}\left(-J\dot{w} + \nabla^2 h_{B_1(0)}(u)w\right).$$

*Proof.* Since $\nabla\Phi_r(j(u)) \in \tilde{X}$, the map $F$ is indeed well-defined. For positive $r$ we have

$$\begin{aligned} F(r,u) &= (\mathrm{id} - \Delta)^{-1}\left(-J\dot{u} - \nabla_1(H \circ \Psi_r)(j(u))\right) \\ &= (\mathrm{id} - \Delta)^{-1}\left(-J\dot{u} + F_0(r,u) - 2\sum_{k=1}^{N-1} F_k(r,u)\right), \end{aligned}$$

where $F_0(r,u) = \nabla(h \circ \psi_r)(u)$ and $F_k(r,u) = D\psi_r(u)^T \nabla_1 G\big(\psi_r(u), \psi_r(u(\cdot + \theta_k))\big)$ for every $k = 1, \dots, N-1$. We interpret these maps as maps between $[0, r_1] \times \mathcal{O}$ and $L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^2)$.

Now if $u \in \mathcal{O}$, $t \in \mathbb{R}$, then $u(t)$ is by Lemma 4.12 contained in the compact annulus $\{ z \in B_* : |z| \in K \}$. Thus we know by Lemma 4.10

$$F_0(r,u) \to \nabla h_{B_1(0)}(u), \quad D_u F_0(r,u) \to \nabla^2 h_{B_1(0)}(u), \quad \partial_r F_0(r,u) \to 0$$

with respect to $\|\cdot\|_{L^2}$ and uniformly in $u \in \mathcal{O}$ as $r \to 0$. It therefore remains to show that $F_k(r,\cdot) \to 0$ in $C^1(\mathcal{O}, L^2)$ and $\partial_r F_k(r,\cdot) \to 0$ in $C^0(\mathcal{O}, L^2)$, $k = 1, \dots, N-1$.

To do this we use that $u \in \mathcal{O}$ not only implies $|u(t)| \in K$, but also

$$|\psi_r(u(t)) - \psi_r(u(t + \theta_k))| \geq \varepsilon,$$

such that Assumption 4.1 gives

$$|F_k(r,u)(t)| = O\big(d(\psi_r(u(t+\theta_k)))\big) = O\big(\alpha(\iota(u(t+\theta_k)), r(1 - |u(t+\theta_k)|^2)/2)\big) = O(r)$$

uniformly in $u \in \mathcal{O}$ and $t \in \mathbb{R}$ as $r \to 0$.

For the derivative we similar have

$$\begin{aligned} D_u F_k(r,u)(t) &= O(r) + D\psi_r(u(t))^T \nabla_2 \nabla_1 G(\psi_r(u(t)), \psi_r(u(t+\theta_k))) D\psi_r(u(t+\theta_k)) \\ &= O(r) + O(1) Q_{\psi_r(u(t+\theta_k))} D\psi_r(u(t+\theta_k)) \end{aligned}$$

uniformly in $u \in \mathcal{O}$ and $t \in \mathbb{R}$. Now recall that $Q_y$ is the orthogonal projection onto the normal space $JT_{p(y)}C$. In our case $y = \psi_r(u(t+\theta_k))$ and $p(y) = \gamma(\iota(u(t+\theta_k)))$, hence $Q_{\psi_r(u(t+\theta_k))}$ is the orthogonal projection onto $\mathbb{R}J\dot{\gamma}(\iota(u(t+\theta_k)))$. On the other hand by (4.7) we have $D\psi_r(u(t+\theta_k))[w] = \lambda\dot{\gamma}(\iota(u(t+\theta_k))) + O(r)$ for $w \in \mathbb{R}^2$ and with some $\lambda = \lambda(w) \in \mathbb{R}$. This shows

$$D_u F_k(r,u)(t) = O(r)$$

uniformly in $u \in \mathcal{O}$, $t \in \mathbb{R}$.

It remains to look at the partial derivative

$$\begin{aligned}\partial_r F_k(r,u)(t) &= O(r) + D\psi_r(u(t))^T \nabla_2\nabla_1 G(\psi_r(u(t)), \psi_r(u(t+\theta_k)))\partial_r\psi_r(u(t+\theta_k)) \\ &\underset{(0.1)}{=} O(r) + \left[\nabla_2\nabla_1 G(\psi_r(u(t+\theta_k)), \psi_r(u(t)))D\psi_r(u(t))\right]^T \partial_r\psi_r(u(t+\theta_k)) \\ &= O(r),\end{aligned}$$

which holds again uniformly in $u \in \mathcal{O}$ and $t \in \mathbb{R}$. □

*Proof of Theorem 4.3 for $N \geq 2$.* We still consider a boundary component $C$ of length $2\pi$. Since $u^*$ is a solution of the 1-vortex system on $B_1(0)$, we have $F(0,u^*) = 0$. Furthermore, Kern $D_uF(0,u^*) = \mathbb{R}\dot{u}^*$ by Lemma 4.11. As in the proof of Theorem 2.7 we obtain a $C^1$-map $[0,r_2) \ni r \mapsto u^{(r)} \in \mathcal{O}$ with $\left\langle u^{(r)}, \dot{u}^*\right\rangle_X = 0$, $F\left(r, u^{(r)}\right) = 0$ and Kern $D_uF\left(r,u^{(r)}\right) = \mathbb{R}\dot{u}^{(r)}$. By the equivariance of $\nabla\Phi_r$ it follows that $\nabla\Phi_r\left(j\left(u^{(r)}\right)\right) = 0$ and hence by Proposition 4.9

$$z^{(r)}(t) = \Psi_r\left(j\left(u^{(r)}\right)\left(\frac{t}{r}\right)\right)$$

is a $2\pi r$-periodic solution of the generalized $N$-vortex system (4.1). From Theorem 2.7 we also know that $(0,r_2)\times\mathbb{R} \ni (r,t) \mapsto z^{(r)}(t) \in \mathcal{F}_N(\Omega)$ is of class $C^1$ and has the continuous mixed derivatives $\partial_r\partial_t z^{(r)}(t) = \partial_t\partial_r z^{(r)}(t)$.

The construction of $z^{(r)}$ shows

$$z_k^{(r)}(t) = \psi_r\left(u^{(r)}\left(\frac{t}{r} + \theta_{k-1}\right)\right) = \psi_r\left(u^{(r)}\left(\frac{t+(k-1)2\pi r/N}{r}\right)\right) = z_1^{(r)}\left(t + \frac{(k-1)2\pi r}{N}\right).$$

The expansions of $v^{(r)}(t) := z_1^{(r)}(rt) = \psi_r(u^{(r)}(t))$ follow in exactly the same way as in the single vortex case, since $\partial_r F(0,u^*) = 0$ from Lemma 4.13 and $\left\langle u^{(r)}, \dot{u}^*\right\rangle_X = 0$ imply $\partial_r u^{(0)} = 0$.

Finally by Corollary 2.9 and especially Remark 2.14 for the choreographic version we need to show that $\mathbb{R}\dot{z}^{(r)}$ are the only $2\pi r$-periodic solutions of $\dot{v} = J_N\nabla^2 H\left(z^{(r)}(t)\right)v$, which satisfy $(\sigma * v)(t + 2\pi r/N) = v(t)$. And indeed if $v$ is a $2\pi r$-periodic function with these properties, the $r$-symplectic transformation $\Psi_r$ shows that

$$D\Psi_r\left(ju^{(r)}(t)\right)^{-1}v(rt) = j\left(D\psi_r\left(u^{(r)}(t)\right)^{-1}v_1(rt)\right) \in \text{Kern}\,\nabla^2\Phi_r\left(ju^{(r)}\right).$$

Hence $D\psi_r\left(u^{(r)}(t)\right)^{-1}v_1(rt)$ is an element of the kernel Kern $D_uF\left(r,u^{(r)}\right) = \mathbb{R}\dot{u}^{(r)}$. It follows $v \in \mathbb{R}\dot{z}^{(r)}$ and hence we obtain a global continuum of choreographic solutions. □

# Chapter 5

# Conclusion and open questions

We have seen that we can combine existing solutions, i.e. stationary solutions of an $m$-vortex problem on a domain $\Omega$ and relative equilibria of the whole plane system, to get new periodic solutions on $\Omega$. The easiest example is given in 3.2 and illustrated in Figure 1.2. We have also seen that choreographic solutions with an arbitrary number of identical vortices can be found near the boundary of the domain, and that both types of solutions (at least if $m = 1$) give rise to a global connected set of periodic solutions. In this last chapter we investigate the $N$-Gon family in the unit disc as a concrete example for a global set of solutions, and we also discuss some open questions.

## 5.1 The $N$-Gon in the unit disc

Let $\Omega = B_1(0)$ and $g(x,y) = g_{B_1(0)}(x,y) = -\frac{1}{4\pi}\log\left(|x|^2|y|^2 - 2\langle x,y\rangle_{\mathbb{R}^2} + 1\right)$. The Hamiltonian for $N$ identical vortices of unit strength then reads

$$H_{B_1(0)}(z) = -\frac{1}{2\pi}\sum_{k\neq j}\log|z_k - z_j| + \frac{1}{4\pi}\sum_{k,j}\log\left(|z_k|^2|z_j|^2 - 2\langle z_k, z_j\rangle_{\mathbb{R}^2} + 1\right).$$

It is known that the point vortex system $\dot{z} = J_N\nabla H_{B_1(0)}(z)$ has a family of choreographic solutions $z^{N,s}(t) = (z_1(t),\ldots,z_N(t))$ given by the points

$$z_k(t) = s\begin{pmatrix}\cos\left(\omega_N(s)t + \frac{2\pi(k-1)}{N}\right)\\ \sin\left(\omega_N(s)t + \frac{2\pi(k-1)}{N}\right)\end{pmatrix}$$

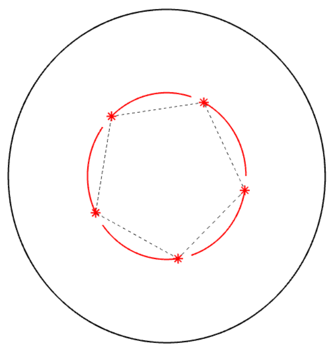

Figure 5.1: Rigidly rotating pentagon in the disc

at the radius $s \in (0, 1)$ and with uniform angular velocity

$$\omega_N(s) = \frac{1}{\pi s^2}\left(\frac{N}{1-s^{2N}} - \frac{N+1}{2}\right),$$

see Figure 5.1. The detailled calculation can be found for example in the thesis by Qianhui Dai, [26]. Note that the Hamiltonian there differs by a factor 2 with the one used here.

### 5.1.1 Local aspects

The solutions can be seen in the following two ways: They are emanating from the boundary where $s = 1$, i.e. they serve as an example of Theorem 4.3, and they are emanating from the critical point of the Robin function $h_{B_1(0)}$ at the origin where $s = 0$, which gives an example of Theorem 3.9 with $m = 1$.

In the first point of view we consider $s \approx 1$ and observe that $z^{N,s}$ is $2\pi r$-periodic if $r\omega_N(s) = 1$, i.e. if

$$r = \frac{2\pi s^2(1-s^{2N})}{N-1+(N+1)s^{2N}}.$$

Close to $s = 1$ and $r = 0$ this equation can be inverted to write $s = s(r)$, such that the functions $z^{(r)} := z^{N,s(r)}$ form the local $r$-dependent family of Theorem 4.3. An expansion of the implicitly defined function $s(r)$ shows that

$$d\big(z_1^{(r)}\big) = 1 - s(r) = \frac{r}{2\pi} + \frac{r^2}{8\pi^2} + o(r^2),$$

as $r \longrightarrow 0$. This coincides with property (3) of Theorem 4.3.

Similar we can view the family in the context of Theorem 3.9. Consider $s \approx 0$ and $\tau = 2\pi N$. The solution $z^{N,s}$ is $\tau r^2$-periodic if $r^2 N\omega_N(s) = 1$, i.e. if

$$r^2 = \frac{2\pi s^2(1-s^{2N})}{N(N-1+(N+1)s^{2N})}.$$

Again we can locally solve this equation to write $s = s(r)$. Then $z^{(r)} := z^{N,s(r)}$ is the $\tau r^2$-periodic solution of Theorem 3.9. In complex notation we get for the rescaled $\tau$-periodic function $u^{(r)} = r^{-1}z^{(r)}(r^2\cdot)$ the following expansion

$$u_k^{(r)}(t) = \frac{s(r)}{r}e^{i\omega_N(s(r))r^2 t}e^{i2\pi(k-1)/N} = \sqrt{\frac{N(N-1)}{2\pi}}e^{i(t+2\pi(k-1))/N} + o(1).$$

One can then check that the limiting function $u^{(0)}(t)$ is indeed the $2\pi N$-periodic Thomson $N$-Gon configuration of the whole plane system.

### 5.1.2 Global aspects

To discuss the global extension of the solutions we normalize the period of the solutions to $2\pi$ by introducing an additional parameter as in section 2.2.3, i.e. we consider for $r > 0$ the family

$$\dot{z} = rJ_N\nabla H_{B_1(0)}(z). \tag{5.1}$$

If $u$ is a $2\pi$ periodic solution of this equation, then $u(\cdot/r)$ is a $2\pi r$-periodic solution of the original $N$-vortex system on the unit disc. Let $\Lambda = \left\{u \in H^1 : u(t) \in \mathcal{F}_N(B_1(0)) \text{ for all } t \in \mathbb{R}\right\}$

and $\mathcal{S} = \{ (r, u) \in \mathbb{R}^+ \times \Lambda : u \text{ solves } (5.1) \}$. We denote by

$$C_{N\text{-Gon}} = \left\{ \left( \frac{1}{\omega_N(s)}, z^{N,s}(\cdot/\omega_N(s)) \right) : s \in (0,1) \right\} \subset \mathcal{S}$$

the known regular $N$-Gon family, by $C_N^{\text{chor}}$ the connected component of $C_{N\text{-Gon}}$ with respect to choreographic solutions $\mathcal{S} \cap \{ (r, u) : u_k = u_1(\cdot + 2\pi(k-1)/N) \}$ and by $C_N$ the full connected component of $C_{N\text{-Gon}}$ in $\mathcal{S}$. In general we have

$$C_{N\text{-Gon}} \subset C_N^{\text{chor}} \subset C_N. \tag{5.2}$$

In the easiest case of a single vortex we know all possible solutions of the system on $B_1(0)$. As the radius $s$ tends to 0 the periodic solutions $z^{1,s}$ merge at the parameter value $\lim_{s\to 0} \omega_1(s)^{-1} = \pi$, which corresponds to a limiting period of $2\pi^2$, into the stationary solution at the origin, see Figure 5.2. We can therefore conclude $C_1 = C_{1\text{-Gon}} \cup \mathbb{R}^+ \times \{0\}$.

Figure 5.3 shows the set $C_{N\text{-Gon}}$ in terms of $\|\cdot\|_{H^1}$ for some $N > 1$. In that case a full characterization of $C_N$ is not available, especially it is not clear if for some $N$ the inclusions in (5.2) can be replaced by equalities. In fact Bolsinov, Borisov and Mamaev show in [19] that bifurcations from the equilateral triangle into other relative equilibrium configurations occur. This means $C_{3\text{-Gon}} \subset C_3^{\text{chor}} \neq C_3$. In general it would be interesting to know, if bifurcations from $C_{N\text{-Gon}}$ into other choreographic solutions can be detected with the help of the equivariant degree $S^1\text{-deg}^\nabla$. For this purpose one would need to actually compute the degree with the formula of Theorem 2.24 or by other means. So far we have only used this formula to conclude that a nondegenerate solution has a nontrivial degree.

## 5.2 The general case

In the case of a general domain $\Omega \subset \mathbb{R}^2$ the structure of the set of periodic solutions is of course even less clear. Suppose that $\Omega$ is bounded, simply connected and has a smooth boundary of length $2\pi$. Suppose further that the Robin function $h_\Omega$ has $l$ nondegenerate critical points $a_1, \ldots, a_l$ and recall that $l \geq 1$ for a generic bounded domain. For the vorticities we assume $\Gamma_1 = \ldots = \Gamma_N = 1$ and denote by $Z(t)$ the Thomson $N$-Gon configuration of the whole plane system. Then for each critical point $a_i$ Theorem 3.9 provides the existence of a global connected set of choreographies $C(a_i, Z)$. Another continuum $C(\partial\Omega)$ emanates from the boundary by Theorem 4.3.

If we try to illustrate these continua in terms of a $r$ vs. $\|u\|_{H^1}$ plot as in Figures 5.2, 5.3, we only can say something for the local parts that are given as a graph. For example for the local part of $C(\partial\Omega)$ corresponding to the solutions close to $\partial\Omega$ we know by Remark 4.4 c) that

$$\|u\|_{H^1}^2 = N\left( \|\gamma\|_{H^1}^2 - \frac{2r}{\pi}\left(\text{vol}_2(\Omega) + \pi\right)\right) + o(r).$$

This provides the initial height and the initial slope of the curve in terms of the domain $\Omega$. A similar first order expansion can be obtained for the local parts of $C(a_i, Z)$. This time in terms of $a_i$ and $Z$.

Beyond these local expansions much more is so far not known. For example in the unit disc we have $C(\partial B_1(0)) = C(0, Z)$, but even in a different convex domain this is not clear anymore. The general open question is what happens at the "other end(s)" of the continua? When do some of the sets $C(\partial\Omega)$, $C(a_i, Z)$, or even $C(a_i, \tilde{Z})$ for some other relative equilibrium $\tilde{Z}$, actually coincide? When do they merge into stationary or heteroclinic solutions?

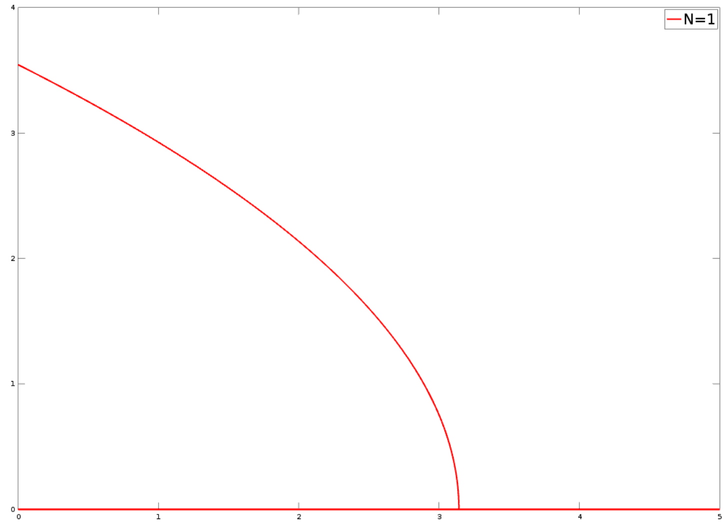


FIGURE 5.2: This shows the structure of all periodic solutions $(r, u) \in C_{1\text{-Gon}}$ in terms of $r$ at the $x$-axis vs. the norm $\|u\|_{H^1}$ at the $y$-axis. The solutions start at the boundary of the disc with arbitrarily small period and $H^1$-norm $2\sqrt{\pi}$. With shrinking radius they merge at $r = \omega_1(0)^{-1} = \pi$ into the stationary solution. In terms of Remark 2.10 we see that options a) and c) are valid for the global continuum induced by the solutions near $\partial B_1(0)$.

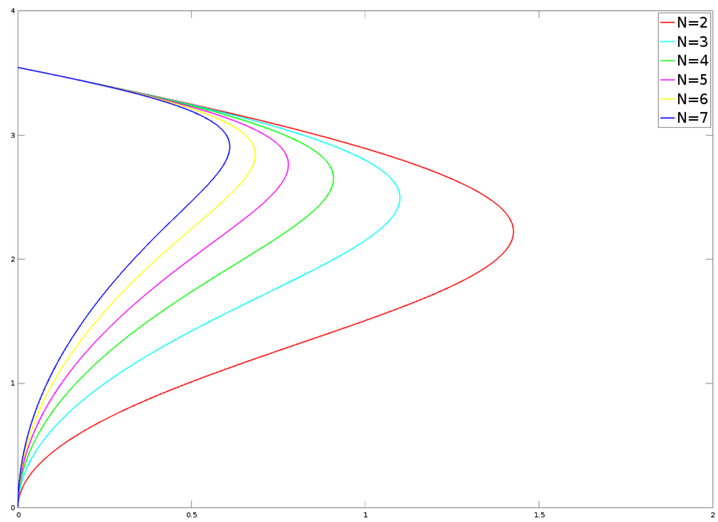


FIGURE 5.3: Similar to the $N = 1$ case we illustrate the set $C_{N\text{-Gon}}$ via the plot of $r$ vs. $N^{-1} \|u\|_{H^1}$. One sees that the periodic solutions emanating from the boundary (upper end of the lines) are connected with the solutions emanating from the critical point of the Robin function (lower end of the lines). On both ends the solutions approach $\partial\mathcal{F}_N(B_1(0))$ with periods going to 0. So for $C_N \supset C_{N\text{-Gon}}$ at least options b) and c) of Remark 2.10 are valid.

## 5.3 Further remarks

As already mentioned before, an actual computation of the degree $S^1\text{-deg}^\nabla$ in the case of a nondegenerate periodic solution has not been carried out. It would also be nice to know, how the degree behaves under a symplectic transformation of the Hamiltonian system.

It seems likely that the solutions of Theorem 3.8 consisting of $m \geq 2$ clusters also give rise to global continua, but this has not been verified for a concrete example, cf. Section 3.4.1.

Concerning solutions near the boundary of a domain, one could try to scale more than one vortex towards the same (time dependent) boundary point in order to find more complicated configurations chasing along the boundary.

As explained in section 1.1 the $N$-vortex system arises as some sort of singular limit of more sophisticated models given by partial differential equations. A natural and interesting question is therefore what kind of conclusions one can draw for these PDEs from solutions of the point vortex system.

By constructing appropriate stream functions it is for example possible to desingularize stationary solutions of the $N$-vortex problem to stationary solutions of the 2D Euler equations (1.2), see [20] and references therein. A similar result for the Euler equations and periodic solutions is so far not available. Concerning the Gross-Pitaevskii equation (1.4) Venkatraman has shown in [76] that rigidly rotating solutions of (1.3) in the unit disc $\Omega = B_1(0)$ give rise to corresponding periodic solutions of (1.4). The same is true for rigidly rotating configurations on the sphere $S^2$, see [39]. Apart from that the desingularization of general periodic solutions like the ones discussed in this thesis is also for the Gross-Pitaevskii equation an open problem.

# Appendix A

# Hamiltonian systems and their variational structure

We collect here some facts about first order Hamiltonian systems and the associated action functional. The elaboration is getting only to a point sufficient for this thesis. For further properties we refer to the books [43, 60, 61].

## A.1 Hamiltonian systems on $\mathbb{R}^{2N}$ and symplectic transformations

A symplectic form on the vectorspace $\mathbb{R}^{2N}$ is a bilinear map $\omega : \mathbb{R}^{2N} \times \mathbb{R}^{2N} \to \mathbb{R}$, which is nondegenerate and skew-symmetric, i.e. $\omega(v, w) = 0$ for every $w \in \mathbb{R}^{2N}$ implies $v = 0$ and $\omega(v, w) = -\omega(w, v)$ for all $v, w \in \mathbb{R}^{2N}$.

Let now $\omega$ be such a symplectic form, $U \subset \mathbb{R}^{2N}$ open and $H : U \to \mathbb{R}$ be a $C^2$ function. By the nondegenerateness of $\omega$ there exists a unique $C^1$ vectorfield $X_H : U \to \mathbb{R}^{2N}$ satisfying for all $z \in U$, $w \in \mathbb{R}^{2N}$

$$\omega(X_H(z), w) = DH(z)w.$$

The Hamiltonian system associated to $\omega$ and $H$ is then the first order differential equation

$$\dot{z} = X_H(z). \tag{A.1}$$

**Lemma A.1.** *There exists a skew-symmetric, regular matrix $A$ depending only on $\omega$, such that* (A.1) *is equivalent to*

$$A\dot{z} = \nabla H(z).$$

*Proof.* Let $e_1, \ldots, e_{2N}$ be the usual basis of $\mathbb{R}^{2N}$ consisting of unit vectors and $S$ be the $2N \times 2N$ matrix with entries $s_{ij} = \omega(e_i, e_j)$. Since $\omega$ is nondegenerate, $S$ is regular and the skew-symmetry of $\omega$ implies $S^T = -S$. Moreover, for $v = \sum_i v_i e_i$, $w = \sum_j w_j e_j$ there holds

$$\omega(v, w) = \sum_{i,j} v_i w_j \omega(e_i, e_j) = \langle v, Sw \rangle_{\mathbb{R}^{2N}} .$$

Thus with $A := S^T = -S$ follows

$$\langle AX_H(z), w \rangle_{\mathbb{R}^{2N}} = \omega(X_H(z), w) = DH(z)w$$

for any $z \in U$, $w \in \mathbb{R}^{2N}$ and hence $AX_H(z) = \nabla H(z)$. □

Let now $\tilde{U} \subset \mathbb{R}^{2N}$ be another open subset of $\mathbb{R}^{2N}$ and assume that we have a $C^2$ diffeomorphism $\varphi : U \to \tilde{U}$. It is called symplectic with multiplier $r \neq 0$ or $r$-symplectic, if

$$\omega(D\varphi(z)v, D\varphi(z)w) = r\omega(v, w)$$

for any $z \in U$, $v, w \in \mathbb{R}^{2N}$. If $r = 1$, then $\varphi$ is just called symplectic.

**Lemma A.2.** *Let $\varphi \in C^2(U, \tilde{U})$ be a $r$-symplectic diffeomorphism, $r \neq 0$ and $H \in C^2(\tilde{U}, \mathbb{R})$. Then $z(t)$ is a solution of $\dot{z} = X_H(z)$ on $\tilde{U}$, if and only if $u(t) := \varphi^{-1}(z(rt))$ solves $\dot{u} = X_{H\circ\varphi}(u)$ on $U$. Moreover, we have a similar equivalence between the linearizations, i.e. $v(t)$ solves $\dot{v}(t) = DX_H(z(t))v(t)$, if and only if $y(t) := D\varphi(u(t))^{-1}v(rt)$ solves the corresponding equation $\dot{y}(t) = DX_{H\circ\varphi}(u(t))y(t)$.*

*Proof.* The function $z(t) = \varphi(u(t/r))$ solves $\dot{z} = X_H(z)$, if and only if for every $w \in \mathbb{R}^{2N}$

$$\frac{1}{r}\omega\big(D\varphi(u(t))\dot{u}(t), w\big) = \omega(\dot{z}(rt), w) = DH(z(rt))w = DH(\varphi(u(t)))w.$$

Since every $D\varphi(u(t))$ is an isomorphism, we may replace $w$ in this equation by $D\varphi(u(t))\tilde{w}$ with $\tilde{w} \in \mathbb{R}^{2N}$ and obtain this way $\omega(\dot{u}(t), \tilde{w}) = D(H \circ \varphi)(u(t))\tilde{w}$ for every $\tilde{w} \in \mathbb{R}^{2N}$, i.e. $\dot{u} = X_{H\circ\varphi}(u)$. This shows the first part.

For the linearized equations we have that $v(t) = D\varphi(u(t/r))y(t/r)$ solves $\dot{v} = DX_H(z(t))v$, if and only if

$$\begin{aligned}&\frac{1}{r}\omega\big(D^2\varphi(u(t))[\dot{u}(t), y(t)] + D\varphi(u(t))\dot{y}(t), w\big)\\ &\qquad = \omega(\dot{v}(rt), w) = D^2H(z(rt))[v(rt), w] = D^2H(\varphi(u(t)))[D\varphi(u(t))y(t), w]\end{aligned}$$

for every $w \in \mathbb{R}^{2N}$. Replacing again $w$ by $D\varphi(u(t))\tilde{w}$, $\tilde{w} \in \mathbb{R}^{2N}$ we see that

$$\omega(\dot{y}(t), \tilde{w}) = D^2(H \circ \varphi)(u(t))[y(t), \tilde{w}]$$

holds true provided

$$\frac{1}{r}\omega\big(D^2\varphi(u(t))[\dot{u}(t), y(t)], D\varphi(u(t))\tilde{w}\big) = -DH(\varphi(u(t))D^2\varphi(u(t))[y(t), \tilde{w}],$$

but this identity is a consequence of

$$\begin{aligned}0 &= \frac{d}{d\varepsilon}_{|\varepsilon=0}\omega(\dot{u}(t), \tilde{w}) = \frac{1}{r}\frac{d}{d\varepsilon}_{|\varepsilon=0}\omega\big(D\varphi(u(t) + \varepsilon y(t))\dot{u}(t), D\varphi(u(t) + \varepsilon y(t))\tilde{w}\big)\\ &= \frac{1}{r}\omega\big(D^2\varphi(u(t))[\dot{u}(t), y(t)], D\varphi(u(t))\tilde{w}\big) + \omega\big(\dot{z}(rt), D^2\varphi(u(t))[y(t), \tilde{w}]\big).\end{aligned}$$

□

## A.2 The Sobolev spaces $H^s$

This summary of facts is taken from Section 3.3 of [43].

Let $u : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^{2N}$ be a square-integrable function with $L^2$-Fourier-series representation

$$u(t) = \sum_{k\in\mathbb{Z}} e^{-J_N kt}\alpha_k.$$

For $s \in [0, \infty)$ the Sobolev space $H^s$ is defined by saying that $u \in H^s$, if and only if the Fourier-coefficients $\alpha_k \in \mathbb{R}^{2N}$ satisfy

$$\sum_{k\in\mathbb{Z}} |k|^{2s}\,|\alpha_k|^2 < \infty.$$

The vector space $H^s$ equipped with the inner product

$$\langle u, v \rangle_s = \langle \alpha_0, \beta_0 \rangle_{\mathbb{R}^{2N}} + 2\pi \sum_{k\in\mathbb{Z}} |k|^{2s} \langle \alpha_k, \beta_k \rangle_{\mathbb{R}^{2N}}$$

for $u = \sum_k e^{-J_N kt}\alpha_k$, $v = \sum_k e^{-J_N kt}\beta_k$ and induced norm $\|u\|_s^2 = \langle u, u \rangle_s$ is a Hilbert space. We will especially use the space $H^1$ and like to point out that we use instead of $\langle u, v \rangle_1$ the inner product

$$\langle u, v \rangle_{H^1} = \int_0^{2\pi} \langle u(t), v(t) \rangle_{\mathbb{R}^{2N}} + \langle \dot{u}(t), \dot{v}(t) \rangle_{\mathbb{R}^{2N}} \; dt = 2\pi \sum_{k\in\mathbb{Z}} (1 + k^2) \langle \alpha_k, \beta_k \rangle_{\mathbb{R}^{2N}} ,$$

but this causes no problems, since the induced norm $\|\cdot\|_{H^1}^2 = \langle \cdot, \cdot \rangle_{H^1}$ is equivalent to $\|\cdot\|_1$.

We have the following continuous embeddings (Prop. 3 and 4 of [43]):

- $H^t \hookrightarrow H^s$ compactly for $t > s$,
- $H^s \hookrightarrow C^k(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^{2N})$ for $s > k + \frac{1}{2}$, $k \in \mathbb{N}_0$.

In particular for $H^1$ we can find a constant $c$, such that

$$\|u\|_{C^0} = \sup_{t\in[0,2\pi]} |u(t)| \le c \, \|u\|_{H^1}$$

for every $u \in H^1$. In fact one can prove that $H^1$-functions are $\frac{1}{2}$-Hölder continuous and that the embedding $H^1 \hookrightarrow C^0$ is compact and as a consequence completely continuous, cf. [60].

## A.3 The action functional on $H^1$

On the space $H^1$ we will now set up the action functional $\Phi$ associated to a Hamiltonian system (A.1), prove its regularity and the correspondence between critical points and periodic solutions.

Since we need it in our application, we directly consider a family of Hamiltonian systems on the symplectic space $(\mathbb{R}^{2N}, \omega)$. I.e. let $D \subset \mathbb{R} \times \mathbb{R}^{2N}$ open, $H : D \to \mathbb{R}$, $(r, z) \mapsto H_r(z)$ with each $H_r$ being of class $C^2$ and $H$, $D_z H$, $D_z^2 H$ continuous.

Let $\mathcal{D} = \{ (r, u) \in \mathbb{R} \times H^1 : (r, u(t)) \in D \text{ for all } t \in \mathbb{R} \}$ and $\mathcal{D}_r = \{ u \in H^1 : (r, u) \in \mathcal{D} \}$ and observe that these sets are open subsets of $\mathbb{R} \times H^1$ and $H^1$ by the embedding $H^1 \hookrightarrow C^0$.

The action functional associated to the family of Hamiltonian systems

$$\dot{z} = X_{H_r}(z) \tag{A.2}$$

is defined by $\Phi : \mathcal{D} \to \mathbb{R}$,

$$\Phi_r(u) = \frac{1}{2} \int_0^{2\pi} \langle A\dot{u}, u \rangle_{\mathbb{R}^{2N}} \; dt - \int_0^{2\pi} H_r(u) \, dt,$$

where $A$ is the skew-symmetric matrix of Lemma A.1.

Let $(\mathrm{id} - \Delta) : H^{s+2} \to H^s$, $s \ge 0$ denote the isomorphism

$$u = \sum_{k\in\mathbb{Z}} B_k \alpha_k \mapsto u - \ddot{u} = \sum_{k\in\mathbb{Z}} (1 + k^2) B_k \alpha_k,$$

such that for $u \in H^1$, $v \in L^2$ there holds

$$\left\langle u, (\mathrm{id} - \Delta)^{-1} v \right\rangle_{H^1} = \int_0^{2\pi} \langle u, v \rangle_{\mathbb{R}^{2N}} \; dt = \langle u, v \rangle_{L^2} .$$

Thus if we define $L : H^1 \to H^1$, $Lu = (\mathrm{id} - \Delta)^{-1} A\dot{u}$, then $L$ is self-adjoint and we can write the quadratic form of $\Phi_r$ as

$$Q(u) = \frac{1}{2} \int_0^{2\pi} \langle A\dot{u}, u \rangle_{\mathbb{R}^{2N}} \; dt = \frac{1}{2} \langle Lu, u \rangle_{H^1} .$$

Clearly $Q$ is defined on all of $H^1$ and smooth with $\nabla Q(u) = Lu$.

We now turn to the nonlinear part. Let $\mathfrak{K} : \mathcal{D} \to \mathbb{R}$,

$$\mathfrak{K}_r(u) = \int_0^{2\pi} H_r(u) \, dt.$$

**Lemma A.3.** *a) The functional $\mathfrak{K}$ is as smooth as $H$, i.e. each $\mathfrak{K}_r \in C^2(\mathcal{D}_r, \mathbb{R})$ with gradient $\nabla \mathfrak{K}_r(u) = (\mathrm{id} - \Delta)^{-1} \nabla H_r(u)$, $\nabla^2 \mathfrak{K}_r(u) = (\mathrm{id} - \Delta)^{-1} \nabla^2 H_r(u)$ and the maps $\mathfrak{K}$, $\nabla \mathfrak{K}$, $\nabla^2 \mathfrak{K}$ are continuous.*

*b) For each $r$ we even have $\nabla \mathfrak{K}_r \in C^1(\mathcal{D}_r, H^2)$ and if $\mathcal{B} \subset \mathcal{D}$ is closed in $\mathbb{R} \times H^1$ and bounded, then $\nabla \mathfrak{K}(\mathcal{B})$ is relatively compact in $H^2$.*

*c) More generally, if $H \in C^k(D, \mathbb{R})$ for $k \geq 1$, then $\mathfrak{K} \in C^k(\mathcal{D}, \mathbb{R})$ and*

$$\partial_r^j D_u^l \mathfrak{K}_r(u)[v^1, \ldots, v^l] = \int_0^{2\pi} \partial_r^j D_u^l H_r(u(t))[v^1(t), \ldots, v^l(t)] \, dt$$

*for all $0 \leq j, l$ with $j + l \leq k$ and $v^1, \ldots, v^l \in H^1$.*

*Proof.* a) The proof of these properties will rely mainly on the fact that if a sequence $u_n \to u$ in $H^1$, then $u_n \to u$ in $C^0$. Hence $u_n(t)$, $t \in \mathbb{R}$ is for $n$ large enough in a compact neighborhood of the orbit $u(\mathbb{R})$, on which $H_r$ and its derivatives are bounded and therefore serve as an integrable majorant. Alternatively one can also argue that $H_r$ restricted to the orbit $u(\mathbb{R})$ is uniformly continuous and hence $H_r \circ u_n \to H_r \circ u$ uniformly.

To get started consider $(r, u) \in \mathcal{D}$ and sequences $r_n \to r$ in $\mathbb{R}$, $u_n \to u$ in $H^1$. Then

$$\left| \mathfrak{K}_r(u) - \mathfrak{K}_{r_n}(u_n) \right| \leq \int_0^{2\pi} \left| H_r(u(t)) - H_{r_n}(u_n(t)) \right| \; dt = o(1),$$

since the integrand converges pointwise to 0 and $H$ is bounded on a compact neighborhood of $\{r\} \times u(\mathbb{R})$. Thus $\mathfrak{K}$ is continuous.

Next we have for $\|v\|_{H^1} \to 0$:

$$\begin{aligned} &\left| \mathfrak{K}_r(u + v) - \mathfrak{K}_r(u) - \int_0^{2\pi} \langle \nabla H_r(u), v \rangle_{\mathbb{R}^{2N}} \; dt \right| \\ &\quad = \left| \int_0^{2\pi} \left\langle \int_0^1 \nabla H_r(u + \lambda v) \, d\lambda - \nabla H_r(u), v \right\rangle_{\mathbb{R}^{2N}} \; dt \right| \leq o(1) \cdot \|v\|_{L^2} \leq o(\|v\|_{H^1}). \end{aligned}$$

Hence each $\mathfrak{K}_r$ is differentiable with $\nabla \mathfrak{K}_r(u) = (\mathrm{id} - \Delta)^{-1} \nabla H_r(u)$.

Since $(\mathrm{id} - \Delta)^{-1} : L^2 \to H^2$ is an isomorphism, it is sufficient to prove that the map $\mathcal{D} \ni (r, u) \mapsto \nabla H_r(u) \in L^2$ is continuous in order to conclude the continuity of $\nabla \mathfrak{K}$. But the continuity of this map follows in the same way as shown before for $\mathfrak{K}$ itself.

In a similar way as above we obtain

$$\begin{aligned}\left\|\nabla\mathfrak{K}_r(u+v)-\nabla\mathfrak{K}_r(u)-(\mathrm{id}-\Delta)^{-1}\nabla^2 H_r(u)v\right\|_{H^1} &\le \left\|\nabla H_r(u+v)-\nabla H_r(u)-\nabla^2 H_r(u)v\right\|_{L^2}\\ &= o(\|v\|_{H^1}),\end{aligned}$$

which shows that each $\mathfrak{K}_r$ is twice-differentiable. Again as before we get the continuity of $\mathcal{D}\ni(r,u)\mapsto\nabla^2\mathfrak{K}_r(u)=(\mathrm{id}-\Delta)^{-1}\nabla^2 H_r(u)\in\mathcal{L}(H^1)$. This shows $\mathfrak{K}_r\in C^2(\mathcal{D}_r,\mathbb{R})$ for every $r$ and $\mathfrak{K}$, $\nabla\mathfrak{K}$, $\nabla^2\mathfrak{K}$ are continuous.

b) That $\nabla\mathfrak{K}_r\in C^1(\mathcal{D}_r,H^2)$ is a consequence of $\nabla H_r\in C^1(\mathcal{D}_r,L^2)$ and that the map $(\mathrm{id}-\Delta)^{-1}:L^2\to H^2$ is an isomorphism. Next we will show that $\nabla H$ actually maps continuously into $H^1$ and that the image of a bounded closed subset $\mathcal{B}\subset\mathcal{D}$ under $\nabla H$ is bounded in $H^1$.

Let $(r,u)\in\mathcal{D}$ and fix a compact neighborhood $\mathcal{O}\subset D$ of the orbit $\{(r,u(t)):t\in\mathbb{R}\}$, on which $\nabla H$ and $\nabla^2 H$ are bounded by a constant $c>0$. Then $\left\|\nabla^2 H_r(u)[\dot u]\right\|_{L^2}\le c\,\|\dot u\|_{L^2}$ and by an approximation of $u$ with $2\pi$-periodic $C^1$ functions we can conclude that $\nabla^2 H_r(u)[\dot u]$ is indeed the weak derivative of $\nabla H_r(u)$. Thus $\nabla H_r(u)\in H^1$.

For the continuity of $\nabla H:\mathcal{D}\to H^1$ it remains to show that $\nabla^2 H_{r_n}(u_n)[\dot u_n]\to\nabla^2 H_r(u)[\dot u]$ in $L^2$, when $u_n\to u$ in $H^1$, $r_n\to r$ in $\mathbb{R}$. Checking this gives us

$$\begin{aligned}\int_0^{2\pi}\left|\nabla^2 H_{r_n}(u_n)[\dot u_n]-\nabla^2 H_r(u)[\dot u]\right|^2\,dt &\le \tilde c\,\|\dot u_n-\dot u\|_{L^2}^2\\ &\quad+\int_0^{2\pi}\left\|\nabla^2 H_{r_n}(u_n)-\nabla^2 H_r(u)\right\|_{\mathcal{L}(\mathbb{R}^{2N})}^2|\dot u(t)|^2\,dt\to 0.\end{aligned}$$

Let now $\mathcal{B}\subset\mathcal{D}$ be bounded and closed in $\mathbb{R}\times H^1$. As a consequence of the compact embedding $H^1\hookrightarrow C^0$, the set $B=\{(r,u(t)):(r,u)\in\mathcal{B}\}$ is a compact subset of $\mathbb{R}\times\mathbb{R}^{2N}$. Therefore

$$\sup_{(r,u)\in\mathcal{B}}\|\nabla H_r(u)\|_{H^1}^2\le 2\pi\sup_B|\nabla H|^2+\sup_B\|\nabla^2 H\|_{\mathcal{L}(\mathbb{R}^{2N})}^2\cdot\sup_{(r,u)\in\mathcal{B}}\|\dot u\|_{L^2}^2<\infty.$$

So $\nabla H(\mathcal{B})$ is bounded in $H^1$. Applying the isomorphism $(\mathrm{id}-\Delta)^{-1}:H^1\to H^3$ and the compact embedding $H^3\hookrightarrow H^2$ we finally see that $\nabla\mathfrak{K}(\mathcal{B})$ is relatively compact in $H^2$.

c) For $k=1$ we have already seen that $D_u\mathfrak{K}_r(u)[v]=\int_0^{2\pi}D_u H_r(u(t))[v(t)]\,dt$ and that $\nabla\mathfrak{K}$, hence also $D_u\mathfrak{K}$, is continuous. For the partial derivative with respect to $r$ one easily gets

$$\mathfrak{K}_{r_0+r}(u)-\mathfrak{K}_{r_0}(u)-r\int_0^{2\pi}\partial_r H_{r_0}(u)\,dt=o(r),$$

as well as the continuity of $\partial_r\mathfrak{K}_r(u)=\int_0^{2\pi}\partial_r H_r(u)\,dt$.

For $k>1$ the statement follows by induction. □

**Lemma A.4.** *A $2\pi$-periodic function $u$ is a solution of (A.2), if and only if $u\in\mathcal{D}_r$ is a critical point of $\Phi_r$. Similar $v\in$ Kern $\nabla^2\Phi_r(u)$, if and only if $v$ is a $2\pi$-periodic solution of the linearization $\dot v=DX_{H_r}(u(t))v$.*

*Proof.* If $u\in\mathcal{D}_r$ is a critical point of $\Phi_r$, then

$$\int_0^{2\pi}\langle A\dot u-\nabla H_r(u),v\rangle_{\mathbb{R}^{2N}}\,dt=0$$

for any $v\in H^1$. The fundamental lemma of calculus of variations implies $A\dot u-\nabla H_r(u)=0$ almost everywhere. But since $\nabla H_r(u)\in H^1$, this means that $\dot u$ has a continuous representation and hence $A\dot u-\nabla H_r(u)=0$ holds everywhere.

In a similar way $v \in \operatorname{Kern} \nabla^2\Phi_r(u)$ implies $A\dot{v} - \nabla^2 H_r(u)v = 0$, which is equivalent to $\dot{v} = DX_{H_r}(u)v$.

The other directions are obvious. $\square$

# Appendix B

# Green's and Robin function

The $N$-vortex Hamiltonian arising in fluid dynamics is classically determined by the Dirichlet - or more generally a hydrodynamic - Green's function of the domain. We present here some properties of the Dirichlet Green's function, that are needed for our existence results on periodic solutions. For some words on the more general hydrodynamic version skip to section B.4.

## B.1 Basic properties

First we recall some basic facts of the Green's function for the Dirichlet-Laplace operator, which can be found for example in [32, 40]. Let $\Omega \subset \mathbb{R}^2$ be a bounded or unbounded domain with non empty boundary. A Dirichlet Green's function $G_\Omega$ for $\Omega$ is a real-valued function, defined on $(\overline{\Omega} \times \Omega) \setminus \{ (z,z) : z \in \Omega \}$ and satisfying for every $y \in \Omega$

$$\begin{cases} -\Delta G_\Omega(\cdot, y) = \delta_y, & \text{in } \Omega, \\ G_\Omega(\cdot, y) = 0, & \text{on } \partial\Omega \end{cases} \tag{B.1}$$

in the sense that $\Delta_x G_\Omega(x,y) = 0$ for $x \in \Omega \setminus \{y\}$ and

$$\int_\Omega G_\Omega(x,y)\Delta\varphi(x)\,dx = -\varphi(y)$$

for every real valued $C^\infty$ function $\varphi$ compactly supported in $\Omega$.

In the case $\Omega = \mathbb{R}^2$ one takes the fundamental solution of $-\Delta$ as the Green's function

$$G_{\mathbb{R}^2}(x,y) = -\frac{1}{2\pi}\log|x-y|\,.$$

Since $-\Delta G_{\mathbb{R}^2}(\cdot, y) = \delta_y$ for any $y \in \mathbb{R}^2$, the ansatz

$$G_\Omega(x,y) = G_{\mathbb{R}^2}(x,y) - g_\Omega(x,y)$$

shows that a Green's function for $\Omega$ exists, provided one can solve for every $y \in \Omega$ the boundary value problem

$$\begin{cases} \Delta g_\Omega(\cdot, y) = 0, & \text{in } \Omega, \\ g_\Omega(\cdot, y) = -\frac{1}{2\pi}\log|\cdot - y|\,, & \text{on } \partial\Omega. \end{cases}$$

Hence Perron's method, section 2.8 in [40], guarantees the existence of a Green's function for a bounded domain, if all way-components of the complement of the domain consist of more than a single point. Moreover, in the case of a bounded domain, the maximum principle implies the uniqueness of $G_\Omega$ and also the positivity $G_\Omega(x,y) > 0$ for any $x, y \in \Omega$, $x \neq y$. In general $G_\Omega$ is symmetric, which means $G_\Omega(x,y) = G_\Omega(y,x)$, whenever $G_\Omega(x,y)$

is defined. The function $g_\Omega$ is therefore symmetric as well. It is called the regular part of $G_\Omega$ and smooth on all of $\Omega \times \Omega$. The evaluation of $g_\Omega$ at the same point defines the Robin function $h_\Omega : \Omega \longrightarrow \mathbb{R}$,

$$h_\Omega(z) = g_\Omega(z, z),$$

and $\rho_\Omega : \Omega \longrightarrow \mathbb{R}$ satisfying $h(z) = -\frac{1}{2\pi} \log \rho_\Omega(z)$ is called harmonic radius.

Before we state some explicit examples, we mention a consequence of the positivity of $G_\Omega$ and Hopf's Lemma, see [32]. Let $\Omega$ be bounded, $y \in \Omega$, $p \in \partial\Omega$, such that the interior ball condition at $p$ is satisfied with some ball $B$, then

$$\partial_\nu G_\Omega(p, y) < 0, \tag{B.2}$$

where $\nu$ is the exterior unit normal of $B$ at $p$. If $\partial\Omega \in C^2$, then the interior ball condition is satisfied at every boundary point and $\nu$ is just the exterior unit normal for $\partial\Omega$.

## B.2 Explicit cases

This section collects formulas of three important cases, in which the Green's function is explicitly known. These are the whole plane $\mathbb{R}^2$, the upper halfplane $\mathbb{R}^2_+$ and the unit disc $B_1(0)$.

### B.2.1 The wohle plane

On $\mathbb{R}^2$ the Green's function is just the fundamental solution of $-\Delta$, thus

$$\begin{gathered}
G_{\mathbb{R}^2}(x, y) = -\frac{1}{2\pi} \log|x - y|, \quad g_{\mathbb{R}^2}(x, y) = 0, \quad h_{\mathbb{R}^2}(z) = 0, \\
\nabla_1 G_{\mathbb{R}^2}(x, y) = -\frac{1}{2\pi} \frac{x - y}{|x - y|^2}, \quad \nabla_2 G_{\mathbb{R}^2}(x, y) = \nabla_1 G_{\mathbb{R}^2}(y, x) = -\nabla_1 G_{\mathbb{R}^2}(x, y), \\
\nabla_1^2 G_{\mathbb{R}^2}(x, y) = -\frac{1}{2\pi} \left( \frac{1}{|x - y|^2} \cdot \mathrm{id}_{\mathbb{R}^2} - 2\frac{(x - y)(x - y)^T}{|x - y|^4} \right), \\
\nabla_2 \nabla_1 G_{\mathbb{R}^2}(x, y) = -\nabla_1^2 G_{\mathbb{R}^2}(x, y) = -\nabla_1^2 G_{\mathbb{R}^2}(y, x).
\end{gathered}$$

### B.2.2 The upper halfplane

For the upper halfplane $\mathbb{R}^2_+ = \left\{ (z_1, z_2) \in \mathbb{R}^2 : z_2 > 0 \right\}$ the Green's function can be constructed by the method of images, i.e. with the reflection at the $x$-axis $\tau : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $z = (z_1, z_2) \mapsto (z_1, -z_2) = \bar{z}$ holds

$$\begin{aligned}
g_{\mathbb{R}^2_+}(x, y) = -\frac{1}{2\pi} \log|x - \bar{y}| = G_{\mathbb{R}^2}(x, \bar{y}), &\quad h_{\mathbb{R}^2_+}(z) = -\frac{1}{2\pi} \log(2z_2), \\
\nabla_1^k G_{\mathbb{R}^2_+}(x, y) = \nabla_1^k G_{\mathbb{R}^2}(x, y) - \nabla_1^k G_{\mathbb{R}^2}(x, \bar{y}), &\quad k = 0, 1, 2 \\
\nabla_2 \nabla_1 G_{\mathbb{R}^2_+}(x, y) &= \nabla_2 \nabla_1 G_{\mathbb{R}^2}(x, y) - \nabla_2 \nabla_1 G_{\mathbb{R}^2}(x, \bar{y}) \circ \tau \\
&= -\nabla_1^2 G_{\mathbb{R}^2_+}(x, y) - \nabla_1^2 G_{\mathbb{R}^2}(x, \bar{y}) \circ (\mathrm{id}_{\mathbb{R}^2} - \tau), \\
\nabla h_{\mathbb{R}^2_+}(z) = -\frac{1}{2\pi z_2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, &\quad \nabla^2 h_{\mathbb{R}^2_+}(z) = \frac{1}{2\pi z_2^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Note that the Green's function for the upper halfplane is not unique. For example $\tilde{G}(x, y) = G_{\mathbb{R}^2_+}(x, y) + x_2 y_2$ satisfies (B.1) as well. But $G_{\mathbb{R}^2_+}$ induces a velocity field that tends to zero for points far away from the source.

### B.2.3 The unit disc

Denote by $R : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$, $R(z) = \frac{z}{|z|^2}$ the reflection at the unit circle. For $y, z \neq 0$, the Green's and Robin function of the disc $B_1(0)$ and their derivatives can then be written as

$$\begin{aligned}
g_{B_1(0)}(x,y) &= -\frac{1}{4\pi}\log\left(|x|^2|y|^2 - 2\langle x,y\rangle_{\mathbb{R}^2} + 1\right) = G_{\mathbb{R}^2}(x,R(y)) - \frac{1}{2\pi}\log|y|,\\
G_{B_1(0)}(x,y) &= G_{\mathbb{R}^2}(x,y) - G_{\mathbb{R}^2}(x,R(y)) + \frac{1}{2\pi}\log|y|,\\
\nabla_1^k G_{B_1(0)}(x,y) &= \nabla_1^k G_{\mathbb{R}^2}(x,y) - \nabla_1^k G_{\mathbb{R}^2}(x,R(y)), \quad k=1,2,\\
\nabla_2\nabla_1 G_{B_1(0)}(x,y) &= \nabla_2\nabla_1 G_{\mathbb{R}^2}(x,y) - \nabla_2\nabla_1 G_{\mathbb{R}^2}(x,R(y))DR(y)\\
&= -\nabla_1^2 G_{B_1(0)}(x,y) - \nabla_1^2 G_{\mathbb{R}^2}(x,R(y))(\mathrm{id}_{\mathbb{R}^2} - DR(y)),\\
h_{B_1(0)}(z) &= -\frac{1}{2\pi}\log\left(1-|z|^2\right), \quad \nabla h_{B_1(0)}(z) = \frac{z}{\pi(1-|z|^2)},\\
\nabla^2 h_{B_1(0)}(z) &= \frac{1}{\pi(1-|z|^2)}\mathrm{id}_{\mathbb{R}^2} + \frac{2zz^T}{\pi(1-|z|^2)^2}.
\end{aligned} \tag{B.3}$$

## B.3 Boundary behaviour

For the periodic solutions of Chapter 4 emanating from the boundary of a domain we need a sound asymptotic behaviour of the Green's and Robin function, when pushing several vortices towards the boundary. Starting with the Robin function, recall that $h_\Omega$ is determined by the harmonic radius $\rho_\Omega$ via $h_\Omega(z) = -\frac{1}{2\pi}\log\rho_\Omega(z)$. The maximum principle implies a monotonicity property for the harmonic radius, $\rho_{\Omega'} \le \rho_\Omega$ if $\Omega' \subset \Omega$. Hence if $p \in \partial\Omega$ is a boundary point at which the double sided ball condition is satisfied, a comparison of $\rho_\Omega$ with the harmonic radius of the interior ball and of the complement of the exterior ball leads to the expansion

$$\rho_\Omega(p - d\nu) = 2d + o(d), \tag{B.4}$$

where $\nu$ denotes the exterior unit normal of the interior ball at $p$. For details see [8] or [34]. It turns out that this first order expansion is not sufficient for our application. Luckily it can be improved in the simply connected case.

### B.3.1 The harmonic radius in simply connected domains

Let now $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain, such that a conformal equivalence in terms of a Riemann mapping $f : \Omega \to B_1(0)$ exists. Since the Green's function (in dimension 2) is invariant under conformal transformations, see e.g. [34], we can write $G_\Omega$ in terms of $f$ and the explicitly known Green's function for the unit disc:

$$G_\Omega(x,y) = G_{B_1(0)}(f(x), f(y)). \tag{B.5}$$

Also the harmonic radius – in the simply connected case called conformal radius – can be expressed in terms of the Riemann map, which allows an improvement of expansion (B.4). We interpret here $f$ as a holomorphic map defined on $\Omega \subset \mathbb{C}$ and write $f'$ and $f^{(k)}$ for the first and $k$th complex derivative of $f$.

**Lemma B.1** (Bandle, Flucher [8, 34]). *The conformal radius can be written as*

$$\rho_\Omega = \frac{1-|f|^2}{|f'|}. \tag{B.6}$$

*Furthermore, at every boundary point $p \in \partial\Omega$ and for $d \longrightarrow 0$,*

$$\rho_\Omega(p - d\nu(p)) = 2d - \kappa(p)d^2 + o(d^2), \tag{B.7}$$

*provided $\partial\Omega$ is of class $C^{2,\alpha}$, $0 < \alpha < 1$.*

Note that the approximation is exact for the unit disc and the upper halfplane, since

$$\rho_{B_1(0)}(z) = 1 - |z|^2 = 2d(z) - d(z)^2, \quad \rho_{\mathbb{R}^2_+}(z) = z - \overline{z} = 2d(z).$$

**Corollary B.2.** *If $\partial\Omega \in C^{k,\alpha}$, $k \geq 2$, $\alpha \in (0,1)$, the conformal radius $\rho = \rho_\Omega$ extends $C^k$ to $\overline{\Omega}$ with*

$$\rho(p) = 0, \quad \nabla\rho(p) = -2\nu(p), \quad \nabla^2\rho(p) = -2\kappa(p) \cdot \mathrm{id}_{\mathbb{R}^2}, \quad p \in \partial\Omega.$$

*Proof.* Since $\partial\Omega \in C^{k,\alpha}$, $f$ extends to $f \in C^{k,\alpha}(\overline{\Omega})$ by the Kellogg-Warschawski Theorem, cf. [65]. As a consequence of (B.2) and (B.5) the derivative $f'$ can not be 0 on the boundary $\partial\Omega$. In the interior this is clear, since $f$ is a biholomorphic map. For the moment we therefore obtain $\rho_\Omega \in C^{k-1}(\overline{\Omega})$ by (B.6).

In order to see $\rho \in C^k(\overline{\Omega})$, derive (B.6) $k$ times in the interior of $\Omega$ and observe that the $(k+1)$st derivative of $f$ only appears in one term containing the product $\rho f^{(k+1)}$. But for $z \longrightarrow \partial\Omega$ holds

$$\left|\rho(z) f^{(k+1)}(z)\right| = \frac{\rho(z)}{d(z)} \cdot \left|\frac{d(z)}{2\pi i} \int_{|w-z|=\frac{d(z)}{2}} \frac{f^{(k)}(w) - f^{(k)}(z)}{(w-z)^2}\, dw\right| = O(d(z)^\alpha),$$

due to (B.4).

Now the expansion $\rho(p - d\nu(p)) = 2d - \kappa(p)d^2 + o(d^2)$ of Lemma B.1 shows $\rho = 0$, $\langle\nabla\rho, \nu\rangle = -2$ and $\left\langle\nabla^2\rho\nu, \nu\right\rangle = -2\kappa$ onto the boundary. Then clearly $\langle\nabla\rho, J\nu\rangle = 0$, which gives $\nabla\rho = -2\nu$ on $\partial\Omega$. Finally, since $D\nu(p) : T_p\partial\Omega \longrightarrow T_p\partial\Omega$ is given by multiplication with $\kappa(p)$, $\nabla^2\rho J\nu = D(-2\nu)[J\nu] = -2\kappa J\nu$ and therefore $\nabla^2\rho = -2\kappa \cdot \mathrm{id}_{\mathbb{R}^2}$ holds. □

### B.3.2 The harmonic radius of an annulus

The following Lemma shows that the second order expansion of $\rho_\Omega$ needed for Assumption 4.1 not only holds in simply connected domains.

**Lemma B.3.** *Let $\rho_A$ be the harmonic radius of the annulus $A = \left\{x \in \mathbb{R}^2 : a < |x| < b\right\}$ with $0 < a < b$. Then $\rho_A \in C^3(\overline{A})$ and*

$$\rho_A(p) = 0, \quad \nabla\rho_A(p) = -2\nu(p), \quad \nabla^2\rho_A(p) = -2\kappa(p) \cdot \mathrm{id}_{\mathbb{R}^2}, \quad p \in \partial A.$$

*Proof.* We prove this for $b = 1$, i.e. $A = \left\{x \in \mathbb{R}^2 : a < |x| < 1\right\}$. A formula for the Robin function $h_A$ is given by the following series

$$h_A(x) = -\frac{1}{2\pi}\left(\frac{(\log|x|)^2}{\log a} - \sum_{m=1}^{\infty} \frac{1}{m} \frac{|x|^{2m} - 2a^{2m} + a^{2m}|x|^{-2m}}{1 - a^{2m}}\right),$$

see [42], Corollary 2.1. Setting $h_A(x) = -\frac{1}{2\pi}\Big(\log(2d(x)) + \varphi(x)\Big)$ we have $\rho_A(x) = 2d(x)e^{\varphi(x)}$. A direct calculation shows that it suffices to prove $\varphi(x) \longrightarrow 0$, $\langle\nabla\varphi(x), J\nu(p(x))\rangle_{\mathbb{R}^2} \longrightarrow 0$, $\langle\nabla\varphi(x), \nu(p(x))\rangle_{\mathbb{R}^2} \longrightarrow \frac{1}{2}\kappa(p(x))$, $\left|\nabla^2\varphi\right|$ is bounded as $x \longrightarrow \partial A$ and $\lim_{x\to\partial A} D^3\varphi(x)$ exists.

We just consider the outer boundary, i.e. $|x| \to 1$. In this case we have:

$$\begin{aligned}\varphi(x) &= \frac{(\log|x|)^2}{\log a} - \log\left(2(1-|x|)\right) - \sum_{m=1}^{\infty} \frac{1}{m} \frac{|x|^{2m} - 2a^{2m} + a^{2m}|x|^{-2m}}{1-a^{2m}} \\ &= \frac{(\log|x|)^2}{\log a} - \log\left(2(1-|x|)\right) - \sum_{m=1}^{\infty} \frac{1}{m} \left(|x|^{2m} + \frac{a^{2m}(|x|^m - |x|^{-m})^2}{1-a^{2m}}\right) \\ &= \frac{(\log|x|)^2}{\log a} + \log\frac{1+|x|}{2} - \sum_{m=1}^{\infty} \left(\frac{a}{|x|}\right)^{2m} \frac{(|x|^{2m}-1)^2}{m(1-a^{2m})}.\end{aligned}$$

Now $\lim_{|x|\to 1} \varphi(x) = 0$ is equivalent to

$$\lim_{|x|\to 1} \sum_{m=1}^{\infty} \left(\frac{a}{|x|}\right)^{2m} \frac{(|x|^{2m}-1)^2}{m(1-a^{2m})} = 0.$$

The latter is true since $0 < \frac{a}{|x|} < 1$, $\frac{(|x|^{2m}-1)^2}{m(1-a^{2m})}$ is uniformly bounded, hence the series is uniformly convergent as $|x| \to 1$. Furthermore,

$$\nabla\varphi(x) = \left(\frac{1}{1+|x|} + \frac{2\log|x|}{|x|\log a} + 2\sum_{m=1}^{\infty} \left(\frac{a}{|x|}\right)^{2m} \frac{1-|x|^{4m}}{(1-a^{2m})\,|x|}\right) \nu(p(x)),$$

so $\langle \nabla\varphi(x), J\nu(p(x))\rangle_{\mathbb{R}^2} = 0$ and

$$\lim_{|x|\to 1} \langle \nabla\varphi(x), \nu(p(x))\rangle_{\mathbb{R}^2} = \frac{1}{2} + 2\lim_{|x|\to 1} \sum_{m=1}^{\infty} \left(\frac{a}{|x|}\right)^{2m} \frac{1-|x|^{4m}}{(1-a^{2m})\,|x|} = \frac{1}{2} = \frac{1}{2}\kappa(p(x)).$$

Continuing differentiating, in the same manner one can derive that $\left|\nabla^2\varphi(x)\right|$ is bounded and that $\lim_{|x|\to 1} D^3\varphi(x)$ exists. The situation when $x$ approaches the inner circle is analogous. □

### B.3.3 Green's function

Now we turn to the part of Assumption 4.1 concerning the Green's function $G_\Omega$. Recall that $Q_y : \mathbb{R}^2 \to \mathbb{R}\nu(p(y))$ is the orthogonal projection onto the normal space $N_{p(y)}\partial\Omega$ whenever this is defined for $y \in \Omega$.

**Lemma B.4.** *Let $\Omega \subset \mathbb{R}^2$ be a domain (not necessarily simply connected) and $C \subset \partial\Omega$ a compact connected component of class $C^{3,\alpha}$. Choose a bounded neighborhood $\Omega_0 \subset \Omega$ of $C$ with $\partial\Omega_0 \in C^{3,\alpha}$, such that the orthogonal projection $p$ onto $\partial\Omega$ is welldefined on $\Omega_0$. For every $\varepsilon > 0$ the function $G_\Omega$ satisfies*

$$|\nabla_1 G_\Omega(x,y)| + \left|\nabla_1^2 G_\Omega(x,y)\right| = O(d(y)), \quad \nabla_2\nabla_1 G_\Omega(x,y) = O(1)Q_y + O(d(y))$$

*as $d(y) \to 0$ uniformly on the set $A_\varepsilon := \left\{(x,y) \in \overline{\Omega} \times \Omega_0 : |x-y| \geq \varepsilon\right\}$.*

*Proof.* It is well known that $G_\Omega \in C^{3,\alpha}\left(\overline{A_\varepsilon}\right)$, see [40] Thm. 6.19. Since $G_\Omega(x,y) = 0$ for $x \in \Omega$ and $y \in C \subset \partial\Omega$, we have $\nabla_1 G_\Omega(x,y) = 0$ and $\nabla_1^2 G_\Omega(x,y) = 0$ for $x \in \Omega$ and $y \in C$. The estimate

$$|\nabla_1 G_\Omega(x,y)| + \left|\nabla_1^2 G_\Omega(x,y)\right| = O(d(y))$$

as $d(y) \to 0$, uniformly on the set $\overline{A_\varepsilon}$ now follows because $G_\Omega$ is of class $C^3$ and $\overline{A_\varepsilon}$ is compact.

Next observe that $\nabla_1 G_\Omega(x,y) = 0$, $x \in \Omega$, $y \in C$ implies that $D_y\big(\nabla_1 G_\Omega(x,y)\big)\big|_{T_yC} = 0$, i.e. $\nabla_2\nabla_1 G_\Omega(x,y) = \alpha(x,y)Q_y$ for $x \in \Omega$, $y \in C$ and with some $\alpha(x,y) \in \mathbb{R}^{2\times 2}$. Again the compactness of $\overline{A}_\varepsilon$ and $G_\Omega$ being of class $C^3$ imply

$$\nabla_2\nabla_1 G_\Omega(x,y) = O(1)Q_y + O(d(y))$$

as $d(y) \to 0$ uniformly on $\overline{A}_\varepsilon$. □

## B.4 Hydrodynamic Green's function

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega = \bigcup_{i=0}^{l} C_i$ consisting of smooth closed curves. Usually $C_0$ denotes the exterior curve. For a smooth vectorfield $u : \overline{\Omega} \to \mathbb{R}^2$ the circulation around $C_i$ is defined by

$$c_i(u) = \oint_{C_i} u \cdot ds.$$

Let $\omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ denote the rotation of $u$. Stokes' Theorem implies

$$\int_\Omega \omega\, dx = c_0(u) - \sum_{i=1}^{l} c_i(u). \tag{B.8}$$

**Theorem B.5** (Chap. 1, Thm. 2.2 of [57]). *Given $\omega : \Omega \to \mathbb{R}$ smooth and $c_1, \dots, c_l \in \mathbb{R}$. There exists a unique vectorfield $u : \overline{\Omega} \to \mathbb{R}^2$, tangent to $\partial\Omega$ and with $\operatorname{curl} u = \omega$, $\operatorname{div} u = 0$, $c_i(u) = c_i$ for $i = 1, \dots, l$. (The circulation around $C_0$ is determined by (B.8).)*

If $\Omega$ is simply connected and $u : \overline{\Omega} \to \mathbb{R}^2$ a divergence-free vectorfield, tangent to $\partial\Omega$ with compactly supported vorticity $\omega = \operatorname{curl} u$, then $u$ can be expressed with the help of the Dirichlet Green's function via

$$u(x) = \int_\Omega J\nabla_1 G_\Omega(x,y)\omega(y)\, dy =: (\nabla^\perp G_\Omega * \omega)(x). \tag{B.9}$$

Indeed the right-hand side has divergence 0 and $\operatorname{curl}(\nabla^\perp G_\Omega * \omega) = (-\Delta G_\Omega) * \omega = \omega$. Moreover, the tangential derivative of $\nabla^\perp G_\Omega * \omega$ at a boundary point is vanishing, since $G_\Omega(\cdot,y)_{|\partial\Omega} = 0$. Therefore Theorem B.5 implies $u = \nabla^\perp G_\Omega * \omega$.

In the simply connected case equation (B.9) provides an equivalence between a vectorfield and its vorticity. The notion of a hydrodynamic Green's function generalizes this equivalence to multiply connected domains. The following definition and properties can be found in [34, 35].

Let $\Omega$ be a multiply connected domain with boundary curves $C_0, \dots, C_l$.

**Definition B.6.** *The hydrodynamic Green's function with periods $\gamma_0, \dots, \gamma_l \in \mathbb{R}$, such that $\sum_i \gamma_i = -1$, is defined as the solution $G(\cdot,y)$ of the problem*

$$\begin{aligned}
-\Delta G(\cdot,y) &= \delta_y \text{ in } \Omega,\\
\partial_{J\nu} G(\cdot,y) &= 0 \text{ on } \partial\Omega,\\
\int_{C_i} \partial_\nu G(\cdot,y) &= \gamma_i \text{ for every } i = 0, \dots, l,\\
\int_{\partial\Omega} G(\cdot,y)\partial_\nu G(\cdot,z) &= 0 \text{ for every } y, z \in \Omega.
\end{aligned}$$

The second condition says that $G(\cdot, y)$ is constant on each component $C_i$ and the third can be rephrased in terms of circulations, i.e. $c_0(\nabla^\perp G(\cdot, y)) = -\gamma_0$ and $c_i(\nabla^\perp G(\cdot, y)) = \gamma_i$ for $i = 1, \ldots, l$. By (B.8) the hydrodynamic Green's function only has a chance to exist, if $\sum_{i=0}^{l} \gamma_i = -1$. The last condition in B.6 is a normalization.

**Lemma B.7** (Lem. 15.3 of [34]). *Let $\Omega$ and $\gamma_i$, $i = 0, \ldots, l$ be as before. Then we have:*

*a) The hydrodynamic Green's function exists if and only if $\sum_{i=0}^{l} \gamma_i = -1$.*

*b) $G$ is unique, symmetric and*

$$G(x, y) = G_\Omega(x, y) + \sum_{i,j=0}^{l} g^{ij} u_i(x) u_j(y),$$

*where $G_\Omega$ is the Dirichlet Green's function, $[g^{ij}]_{i,j=0,\ldots,l}$ is a symmetric, positive semi definite matrix with one-dimensional kernel spanned by $(\gamma_0, \ldots, \gamma_l)$, and $u_j$ is the unique harmonic function with values $u_j = 1$ on $C_j$, $u_j = 0$ on $C_k$ for $k \neq j$.*

*c) The hydrodynamic Robin function $h(z) := g_\Omega(z, z) - \sum_{i,j} g^{ij} u_i(z) u_j(z)$ satisfies*

$$h(z) = -\frac{1}{2\pi} \log(d(z)) + O(1)$$

*uniformly as $z \longrightarrow \partial\Omega$.*

In particular if $\Omega$ is simply connected, we see that $G$ (with $\gamma_0 = -1$) is nothing but the Dirichlet Green's function $G_\Omega$. For the possibly multiply connected case where $\partial\Omega = \bigcup_i C_i$ we conclude:

**Corollary B.8.** *If $u : \overline{\Omega} \longrightarrow \mathbb{R}^2$ is a smooth divergence-free vectorfield, tangent to $\partial\Omega$ and with compactly supported vorticity $\omega = \operatorname{curl} u$ satisfying $\Gamma := \int_\Omega \omega \, dx \neq 0$. Then there exists a unique hydrodynamic Green's function $G$, such that $u = \nabla^\perp G * \omega$. The choice of $G$ depends only on $\Gamma$ and the circulations $c_i(u)$.*

*Proof.* It is easy to see that the convolution $\tilde{u} = \nabla^\perp \tilde{G} * \omega$ with an arbitrary hydrodynamic Green's function $\tilde{G}$ satisfies $\operatorname{div} \tilde{u} = 0$, $\operatorname{curl} \tilde{u} = \omega$ in $\Omega$ and $\langle \tilde{u}, \nu \rangle_{\mathbb{R}^2} = 0$ on the boundary. Since

$$c_i(\tilde{u}) = \int_\Omega c_i(\nabla^\perp \tilde{G}(\cdot, y)) \omega(y) \, dy = \Gamma \cdot \begin{cases} -\gamma_0, & i = 0, \\ \gamma_i, & i > 0, \end{cases} \tag{B.10}$$

we define $\gamma_0 = -\frac{c_0(u)}{\Gamma}$ and $\gamma_i = \frac{c_i(u)}{\Gamma}$ for $i = 1, \ldots, l$. By (B.8) the numbers $\gamma_i$ satisfy the consistence relation $\sum_{i=0}^{l} \gamma_i = -1$, and hence the corresponding hydrodynamic Green's function $G$ exists. By the definition of the periods $\gamma_i$ the vectorfields $u$ and $\nabla^\perp G * \omega$ have the same circulations. Thus Theorem B.5 implies $u = \nabla^\perp G * \omega$. □

If $\Gamma = \int_\Omega \omega \, dx = 0$, equation (B.10) shows that a representation of $u$ in terms of a convolution is only possible if all circulations $c_i(u)$ are vanishing. If this is the case, then $u = \nabla^\perp G * \omega$ for any hydrodynamic Green's function $G$.

# Appendix C

# A derivation of the $N$-vortex problem

Here we will see how the 2D-Euler equation gives rise to the $N$-vortex problem in terms of a localization result. Roughly speaking we will see that highly concentrated vortex blobs remain concentrated while their centers follow the corresponding point vortex solution. This has been shown in [59] by Marchioro and Pulvirenti. The discussion in the first section is based on the books [34] and [57].

## C.1 Two representations of the Euler equation

We consider an incompressible nonviscous fluid contained in a two-dimensional bounded domain $\Omega$ surrounded by smooth closed boundary curves $C_i$, $i = 0, \dots, l$, which are impenetrable for the fluid. The velocity field $u : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^2$, $(x,t) \mapsto u(x,t)$ and the pressure $p : \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto p(x,t)$ of the fluid satisfy the Euler equations

$$\begin{gathered} \partial_t u + (u \cdot \nabla) u = -\nabla p \text{ in } \Omega \times \mathbb{R}, \quad \operatorname{div} u = 0 \text{ in } \Omega \times \mathbb{R}, \\ u \cdot \nu = 0 \text{ on } \partial\Omega \times \mathbb{R}, \quad u(\cdot, 0) = u_0 \text{ in } \overline{\Omega}. \end{gathered} \tag{C.1}$$

Here $\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 = 0$, where $\partial_i = \partial_{x_i}$, corresponds to the incompressibility condition, the scalar product with the exterior unit normal $u \cdot \nu = 0$ models the impenetrability of $\partial\Omega$, and $u_0$ is the initial velocity field.

Suppose that $u, p$ are smooth solutions of the 2D-Euler equation (C.1). The vorticity associated to $u$ is given by $\omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ and its time evolution is determined by the scalar equation

$$\partial_t \omega = \operatorname{curl}(\partial_t u) = -\operatorname{curl}((u \cdot \nabla) u + \nabla p) = -u \cdot \nabla \omega. \tag{C.2}$$

For $x \in \overline{\Omega}$ we denote by $\Phi_{t,t_0}(x)$ the solution of the initial value problem

$$\dot{\phi}(t) = u(\phi(t), t), \quad \phi(t_0) = x,$$

i.e. $t \mapsto \Phi_{t,t_0}(x)$ describes the trajectory of a particle moving with the fluid starting in $x$ at time $t_0$. Note that if $x$ is initially located in a boundary curve $C_i$, then $\Phi_{t,t_0}(x) \in C_i$. This follows from $u \cdot \nu = 0$ and can for example be seen by locally writing $C_i \cap B_\varepsilon(x)$ as a regular level set $h^{-1}(0)$ and considering the Hamiltonian system $\dot{y} = J\nabla h(y)$. Since $u$ is tangent to $\partial\Omega$, the solutions of this Hamiltonian system are a reparameterization of the particle trajectories $\Phi_{t,t_0}(x)$.

In other words $\partial\Omega$ consists of invariant flow lines and as a consequence $\Phi_{t,t_0}(x)$ is globally defined for all $x \in \overline{\Omega}$, $t, t_0 \in \mathbb{R}$. Moreover, $\Phi_{t_2,t_1} \circ \Phi_{t_1,t_0} = \Phi_{t_2,t_0}$ for any triple $t_0, t_1, t_2 \in \mathbb{R}$.

This implies that each $\Phi_{t,t_0} : \overline{\Omega} \longrightarrow \overline{\Omega}$ is a diffeomorphism with inverse $\Phi_{t,t_0}^{-1} = \Phi_{t_0,t}$. Since $\operatorname{div} u = 0$, we also know that $\det D\Phi_{t,t_0}(x) = 1$ for any $t, t_0 \in \mathbb{R}$, $x \in \Omega$.

The properties of $\Phi_{t,t_0}(x)$ discussed so far are also valid for an arbitrary divergence-free vectorfield $\tilde{u}(x,t)$, tangent to $\partial\Omega$. For a vector field satisfying the Euler equation and therefore (C.2), we can in addition conclude $\omega(\Phi_{t,t_0}(x), t) = \omega(x, t_0)$, and especially

$$\omega(x,t) = \omega(\Phi_{0,t}(x), 0). \tag{C.3}$$

As a consequence we see that the total vorticity is conserved, indeed $\Phi_{t,0}$ area-preserving implies

$$\int_\Omega \omega(x,t)\, dx = \int_{\Phi_{t,0}(\Omega)} \omega(\Phi_{0,t}(x), 0)\, dx = \int_\Omega \omega(x,0)\, dx.$$

By Kelvin's Theorem the same holds for the circulations: We rewrite $(u\cdot\nabla)u = \frac{1}{2}\nabla|u|^2 - \omega J u$, and get

$$\frac{d}{dt} c_i(u(\cdot,t)) = \oint_{C_i} \partial_t u \cdot ds = -\oint_{C_i} \nabla\left(\frac{1}{2}|u|^2 + p\right) \cdot\, ds + \oint_{C_i} \omega J u \cdot\, ds = 0.$$

Suppose that the initially velocity field $u_0$ is such that $\omega(\cdot,0) \in C_c^\infty(\Omega)$ with nonzero total vorticity $\Gamma = \int_\Omega \omega(\cdot,0)\, dx$, then Corollary B.8 provides a unique hydrodynamic Green's function $G$, such that

$$u(\cdot,t) = \int_\Omega J\nabla G(\cdot,y)\omega(y,t)\, dy = \nabla^\perp G * \omega(\cdot,t).$$

Hence the vorticity satisfies in $\Omega\times\mathbb{R}$ the equation

$$\partial_t\omega + (\nabla^\perp G * \omega)\cdot\nabla\omega = 0. \tag{C.4}$$

Conversely if we suppose that $\omega(x,t)$ satisfies (C.4) with a fixed hydrodynamic Green's function $G$ and initial vorticity $\omega(\cdot,0) \in C_c^\infty(\Omega)$, then $u := \nabla^\perp G * \omega$ and $p$ defined as the (up to a constant unique) solution of the Neumann boundary value problem

$$\Delta p = \omega^2 - \frac{1}{2}\Delta|u|^2 \;\text{ in } \Omega\times\mathbb{R}, \quad \partial_\nu p = \left(\omega J u - \frac{1}{2}\nabla|u|^2\right)\cdot\nu \text{ on } \partial\Omega\times\mathbb{R},$$

solve the $2D$-Euler equation (C.1). Indeed $u$ is divergence free, tangent to $\partial\Omega$ and $\operatorname{curl} u = \omega$. Thus it remains to show that

$$\tilde{u} := \partial_t u + (u\cdot\nabla)u + \nabla p = \partial_t u + \frac{1}{2}\nabla|u|^2 - \omega J u + \nabla p = 0.$$

We have $\operatorname{curl}\tilde{u} = 0$ by (C.4), as well as $\operatorname{div}\tilde{u} = 0$, $\tilde{u}\cdot\nu = 0$ by the choice of $p$. In order to apply Theorem B.5 we thus need to show that the circulations $c_i(\tilde{u}(\cdot,t))$ are vanishing. We have

$$c_i(\tilde{u}(\cdot,t)) = c_i(\partial_t u(\cdot,t)) = \frac{d}{dt}c_i(u(\cdot,t)) = \pm\gamma_i\frac{d}{dt}\int_\Omega \omega(x,t)\, dx,$$

where $\gamma_i \in \mathbb{R}$, $i = 0,\dots,l$ are the periods of the Green's function $G$, cf. (B.10). To see that the total vorticity is constant we can argue in the same way as before, i.e. we denote by $\Phi_{t,t_0}(x)$ the particle-trajectory map associated to $u = \nabla^\perp G * \omega$, observe that it is an area preserving diffeomorphism and that (C.4) implies $\omega(\Phi_{t,t_0}(x),t) = \omega(x,t_0)$. It follows $c_i(\tilde{u}(\cdot,t)) = 0$.

Our discussion shows that the equations (C.1) and (C.4) are equivalent – at least under the condition of nonzero total vorticity. Note also that $\Psi = G * \omega$ defines a stream function for $u$, which has been used in the introduction in equation (1.2).

## C.2 The localization of vortex blobs

Following [59] we will now present the localization result. As a step in between we consider a single vortex blob on the whole plane under the influence of an external field. For $\varepsilon > 0$, $a \in \mathbb{R}^{2N}$ and $\Gamma \neq 0$ we define a set $\mathcal{B}_\varepsilon(a,\Gamma) \subset C_c^\infty(\mathbb{R}^2,\mathbb{R})$ by saying that $\omega \in \mathcal{B}_\varepsilon(a,\Gamma)$ if and only if

$$\operatorname{supp}\omega \subset B_\varepsilon(a),\ \omega\Gamma \geq 0 \text{ on all of } \mathbb{R}^2 \text{ and } \int_{\mathbb{R}^2} \omega\, dx = \Gamma.$$

So $\mathcal{B}_\varepsilon(a,\Gamma)$ contains smooth vortex blobs concentrated in the $\varepsilon$-ball around $a$ and with total vorticity $\Gamma$.

For the external field we consider a collection $(F_\varepsilon)_{\varepsilon\in(0,\varepsilon_1)} \subset C^\infty(\mathbb{R}^2\times\mathbb{R},\mathbb{R}^2)$ of smooth vectorfields, such that there exists a constant $L > 0$ independent of $\varepsilon$ with

$$\operatorname{div} F_\varepsilon(x,t) = 0,\ \operatorname{supp} F_\varepsilon(\cdot,t) \subset B_L(0),\ |F_\varepsilon(x,t)| \leq L \text{ and } |F_\varepsilon(x,t) - F_\varepsilon(y,t)| \leq L\,|x-y|$$

for all $\varepsilon \in (0,\varepsilon_1)$, $x, y \in \mathbb{R}^2$, $t \in \mathbb{R}$.

For $\varepsilon \in (0,\varepsilon_1)$ we suppose that $\omega_\varepsilon \in C^\infty(\mathbb{R}^2\times\mathbb{R},\mathbb{R})$ satisfies

$$\partial_t\omega_\varepsilon + (\nabla^\perp G_{\mathbb{R}^2} * \omega_\varepsilon + F_\varepsilon)\cdot\nabla\omega_\varepsilon = 0 \tag{C.5}$$

with initial vorticity profiles $\omega_\varepsilon(\cdot,0) \in \mathcal{B}_\varepsilon(a,\Gamma)$ such that $|\omega_\varepsilon(x,t)| = O(\varepsilon^{-\eta})$ for some $\eta < \frac{8}{3}$.

We define the center of vorticity

$$c_\varepsilon(t) = \frac{1}{\Gamma}\int_{\mathbb{R}^2} x\omega_\varepsilon(x,t)\,dx$$

and $e_\varepsilon(t)$ as the solution of the initial value problem

$$\dot e_\varepsilon(t) = F_\varepsilon(e_\varepsilon(t),t), \quad e_\varepsilon(0) = a.$$

**Theorem C.1** (cf. Thm. 3.1 of [59])**.** *For $T > 0$ and $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(T,\delta) > 0$, such that for all $\varepsilon < \varepsilon_0$ and every $t \in [0,T]$ there holds*

$$\operatorname{supp}\omega_\varepsilon(\cdot,t) \subset B_\delta(c_\varepsilon(t)).$$

*Moreover, $|c_\varepsilon(t) - e_\varepsilon(t)| \to 0$ uniformly on $[0,T]$ as $\varepsilon \to 0$.*

**Remark C.2.** *The Theorem stated above differs slightly from the original version in [59]. First of all we have formulated everything for smooth $\omega_\varepsilon$, $F_\varepsilon$ with compactly supported fields $F_\varepsilon(\cdot,t)$. This is of course way too much. The paper [59] deals with $\omega(\cdot,t) \in L^1 \cap L^\infty$, the weak form of the Euler equation and a uniformly Lipschitz continuous, uniformly bounded external field.*

*In fact the original formulation considers a single external field $F$ instead of a family $(F_\varepsilon)_\varepsilon$, but the proof only uses the uniform Lipschitz continuity and the uniform boundedness of $F$.*

*A third difference is that Thm. 3.1 of [59] considers a blob with vorticity $\Gamma = 1$, but the general case can be reduced to this case by rescaling $\tilde\omega_\varepsilon(x,t) = \Gamma^{-1}\omega_\varepsilon(\sigma x, t/|\Gamma|)$, $\sigma = \operatorname{sign}(\Gamma)$ and $\tilde F_\varepsilon(x,t) = \Gamma^{-1}F(\sigma x, t/|\Gamma|)$.*

Next we look at the Euler equation in vorticity form on a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, i.e. we fix a hydrodynamic Green's function $G(x,y) = G_{\mathbb{R}^2}(x,y) - g(x,y)$ and consider

$$\partial_t\omega + (\nabla^\perp G * \omega)\cdot\nabla\omega = 0 \quad \text{in } \Omega\times\mathbb{R}. \tag{C.6}$$

In Section C.1 we have already seen that a solution of this equation implies that the vector field $u = \nabla^\perp G * \omega$ and a suitable pressure function $p$ solve the Euler equation (C.1). Let

$(a_1, \dots, a_N) \in \mathcal{F}_N(\Omega)$, $\Gamma_1, \dots, \Gamma_N \in \mathbb{R} \setminus \{0\}$ and denote by $\omega_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ the solution of (C.6) with initial condition

$$\omega_\varepsilon(x,0) = \sum_{i=1}^{N} \omega_\varepsilon^i(x,0),$$

where $\omega_\varepsilon^i(\cdot,0) \in \mathcal{B}_\varepsilon(a_i,\Gamma_i)$ and $|\omega_\varepsilon(x,t)| = O(\varepsilon^{-\eta})$ for some $\eta < \frac{8}{3}$. Concerning global existence and uniqueness of solutions of (C.6), the proof with $G$ being the Dirichlet Green's function can be found in [57].

For our initial value problem we consider only $\varepsilon > 0$ satisfying $3\varepsilon < \min_{i\neq j} \left|a_i - a_j\right|$ and $2\varepsilon < \min_i \operatorname{dist}(a_i, \partial\Omega)$, such that $\operatorname{supp} \omega_\varepsilon^i(\cdot,0) \cap \operatorname{supp} \omega_\varepsilon^j(\cdot,0) = \emptyset$ and $\operatorname{supp}(\omega_\varepsilon^i(\cdot,0)) \subset \Omega$ for $i \neq j$. In view of (C.3) we define

$$\omega_\varepsilon^i(x,t) := \omega_\varepsilon^i(\Phi_{0,t}^\varepsilon(x), 0),$$

where $\Phi_{t,t_0}^\varepsilon(x)$ is the particle-trajectory map associated to the vectorfield $u_\varepsilon := \nabla^\perp G * \omega_\varepsilon$. In other words $\omega_\varepsilon^i(\cdot,t)$ tracks the evolution of the vortex blob $\omega_\varepsilon^i(\cdot,0)$ initially located around $a_i$. Clearly $\omega_\varepsilon(x,t) = \sum_i \omega_\varepsilon^i(x,t)$ and

$$\int_\Omega \omega_\varepsilon^i(x,t)\, dx = \Gamma_i.$$

Next we set up the $N$-vortex Hamiltonian[1] $H : \mathcal{F}_N(\Omega) \to \mathbb{R}$ induced by the hydrodynamic Green's function $G$, i.e.

$$H(z_1, \dots, z_N) = \frac{1}{2} \left( \sum_{i \neq j} \Gamma_i \Gamma_j G(z_i, z_j) - \sum_{i,j} \Gamma_i \Gamma_j g(z_i, z_j) \right),$$

and denote by $z(t) = (z_1(t), \dots, z_N(t))$ the solution of the initial value problem

$$\Gamma_i \dot{z}_i = J \nabla_{z_i} H(z), \quad z_i(0) = a_i, \quad i = 1, \dots, N.$$

Let $0 < T^+ \leq \infty$ be the upper bound of the existence interval of the solution $z(t)$.

**Theorem C.3** (cf. Thm. 2.1 of [59])**.** *For $T \in (0, T+)$ and $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(T,\delta) > 0$, such that for all $\varepsilon < \varepsilon_0$, $t \in [0,T]$ and $i = 1, \dots, N$ there holds*

$$\operatorname{supp} \omega_\varepsilon^i(\cdot,t) \subset B_\delta(z_i(t)).$$

*Moreover, for any $f \in C^0(\Omega, \mathbb{R})$ we have*

$$\int_\Omega \omega_\varepsilon(x,t) f(x)\, dx \to \sum_{i=1}^{N} \Gamma_i f(z_i(t))$$

*uniformly on $[0,T]$ as $\varepsilon \to 0$. That is $\omega_\varepsilon(\cdot,t) \to \sum_i \Gamma_i \delta_{z_i(t)}$ weakly in the sense of measures.*

*Proof.* First of all we observe that the identity $\Phi_{t,0}^\varepsilon(\Phi_{0,t}^\varepsilon(x)) = x$ implies

$$\frac{d}{dt} \Phi_{0,t}^\varepsilon(x) = -D\Phi_{0,t}^\varepsilon(x) u_\varepsilon(x,t)$$

[1] in the other parts of the thesis we neglect the factor $\frac{1}{2}$

and therefore

$$\begin{aligned}\partial_t \omega_\varepsilon^i(x,t) &= \frac{d}{dt}\omega_\varepsilon^i(\Phi_{0,t}^\varepsilon(x),0) = -\nabla\omega_\varepsilon^i(\Phi_{0,t}^\varepsilon(x),0)\cdot D\Phi_{0,t}^\varepsilon(x)u_\varepsilon(x,t)\\ &= -\nabla\omega_\varepsilon^i(x,t)\cdot u_\varepsilon(x,t).\end{aligned}$$

Thus if we write

$$\begin{aligned}u_\varepsilon(x,t) = (\nabla^\perp G * \omega_\varepsilon)(x,t) &= \int_\Omega \nabla^\perp G_{\mathbb{R}^2}(x,y)\omega_\varepsilon^i(y,t)\,dy\\ &\quad + \sum_{j\neq i}\int_\Omega \nabla^\perp G(x,y)\omega_\varepsilon^j(y,t)\,dy - \int_\Omega \nabla^\perp g(x,y)\omega_\varepsilon^i(y,t)\,dy\\ &=: (\nabla^\perp G_{\mathbb{R}^2} * \omega_\varepsilon^i)(x,t) + F^i\left(\omega_\varepsilon^1,\dots,\omega_\varepsilon^N\right)(x,t),\end{aligned}$$

we see that the functions $\omega_\varepsilon^i$ satisfy the following system of Euler equations

$$\partial_t\omega_\varepsilon^i + \left(\nabla^\perp G_{\mathbb{R}^2} * \omega_\varepsilon^i + F^i\left(\omega_\varepsilon^1,\dots,\omega_\varepsilon^N\right)\right)\cdot\nabla\omega_\varepsilon^i = 0, \quad \text{in } \Omega\times\mathbb{R},\ i=1,\dots,N. \tag{C.7}$$

Next we regularize the interaction between different vortex blobs, as well as the interaction with the boundary by a modification of the convolutions in $F^i$. Let $T\in(0,T^+)$ and choose $b>0$ such that $b\leq \left|z_i(t)-z_j(t)\right|$ and $b\leq \operatorname{dist}(z_i(t),\partial\Omega)$ for all $1\leq i<j\leq N$, $t\in[0,T]$. Take smooth cutoff functions $\xi_1,\xi_2:\mathbb{R}^2\to\mathbb{R}$ with

$$\xi_1(x)=\xi_1(|x|)=\begin{cases}1, & |x|\geq \frac{b}{4}\\ 0, & |x|\leq\frac{b}{8}\end{cases} \qquad \xi_2(x)=\begin{cases}1, & x\in\Omega \text{ and } \operatorname{dist}(x,\partial\Omega)\geq\frac{b}{4}\\ 0, & x\notin\Omega \text{ or } \operatorname{dist}(x,\partial\Omega)\leq\frac{b}{8}\end{cases}$$

and define $\tilde g,\tilde G:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$,

$$\tilde g(x,y)=\xi_2(x)\xi_2(y)g(x,y), \qquad \tilde G(x,y)=\xi_2(x)\xi_2(y)\xi_1(x-y)G_{\mathbb{R}^2}(x,y)-\tilde g(x,y).$$

Let $\tilde\omega_\varepsilon^i$ denote the solution of the following system of regularized Euler equations

$$\partial_t\tilde\omega_\varepsilon^i + \left(\nabla^\perp G_{\mathbb{R}^2} * \tilde\omega_\varepsilon^i + \tilde F^i\left(\tilde\omega_\varepsilon^1,\dots,\tilde\omega_\varepsilon^N\right)\right)\cdot\nabla\tilde\omega_\varepsilon^i = 0, \quad \text{in } \mathbb{R}^2\times\mathbb{R},\ i=1,\dots,N, \tag{C.8}$$

where $\tilde\omega_\varepsilon^i(\cdot,0)=\omega_\varepsilon(\cdot,0)$ and

$$\tilde F^i\left(\tilde\omega_\varepsilon^1,\dots,\tilde\omega_\varepsilon^N\right)(x,t)=\sum_{j\neq i}\int_\Omega\nabla^\perp\tilde G(x,y)\tilde\omega_\varepsilon^j(y,t)\,dy-\int_\Omega\nabla^\perp\tilde g(x,y)\tilde\omega_\varepsilon^i(y,t)\,dy.$$

This system coincides with (C.7) as long as the distance between different vortex blobs and towards the boundary are sufficiently large.

The vector fields $f_\varepsilon^i(x,t)=\tilde F^i\left(\tilde\omega_\varepsilon^1,\dots,\tilde\omega_\varepsilon^N\right)(x,t)$ are smooth, uniformly bounded, uniformly Lipschitz continuous and for each $\varepsilon,t$ the support $\operatorname{supp} f_\varepsilon^i(\cdot,t)$ is contained in $\Omega$. Thus we can apply Theorem C.1, which provides for every $\delta>0$ a number $\varepsilon_0(T,\delta)>0$, such that

$$\operatorname{supp}\tilde\omega_\varepsilon^i(\cdot,t)\subset B_\delta(c_\varepsilon^i(t)) \quad\text{with}\quad c_\varepsilon^i(t):=\frac{1}{\Gamma_i}\int_\Omega x\tilde\omega_\varepsilon^i(x,t)\,dx, \tag{C.9}$$

whenever $t\in[0,T]$, $i=1,\dots,N$ and $\varepsilon<\varepsilon_0$. Note that also for the regularized system there holds $\int_{\mathbb{R}^2}\tilde\omega_\varepsilon^i(y,t)\,dy=\Gamma_i$ for any $t$. We will now show that $c_\varepsilon^i(t)\to z_i(t)$ uniformly on $[0,T]$

as $\varepsilon \longrightarrow 0$. Let $\delta > 0$ and $\varepsilon < \min(\delta, \varepsilon_0(T, \delta))$. At time $t = 0$ we have

$$\begin{aligned} \left|z_i(0) - c_\varepsilon^i(0)\right| &= \left|a_i - \frac{1}{\Gamma_i}\int_{\mathbb{R}^2}(a_i + y)\tilde{\omega}_\varepsilon^i(a_i + y, 0)\, dy\right| \\ &\leq \varepsilon \int_{B_\varepsilon(0)} \frac{\tilde{\omega}_\varepsilon^i(a_i + y, 0)}{\Gamma_i}\, dy \leq \varepsilon \leq \delta. \end{aligned}$$

For the development in time observe that $G(z_i, z_j) = \tilde{G}(z_i, z_j)$, $g(z_i, z_i) = \tilde{g}(z_i, z_i)$ on $[0, T]$ and

$$\int_{\mathbb{R}^2}(v(x,t)\cdot \nabla\tilde{\omega}_\varepsilon^i(x,t))x\, dx = -\int_{\mathbb{R}^2} v(x,t)\tilde{\omega}_\varepsilon^i(x,t)\, dx$$

for any smooth divergence-free vector field $v$. Hence

$$\begin{aligned} \left|\dot{z}_i(t) - \dot{c}_\varepsilon^i(t)\right| &= \left|\sum_{j\neq i}\Gamma_j\nabla^\perp G(z_i, z_j) - \Gamma_i\nabla^\perp g(z_i, z_i) - \frac{1}{\Gamma_i}\int_{\mathbb{R}^2}\left(\nabla^\perp G_{\mathbb{R}^2} * \tilde{\omega}_\varepsilon^i + f_\varepsilon^i\right)\tilde{\omega}_\varepsilon^i\, dx\right| \\ &\leq \sum_{j\neq i}\left|\Gamma_j\nabla^\perp\tilde{G}(z_i(t), z_j(t)) - \frac{1}{\Gamma_i}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla^\perp\tilde{G}(x,y)\tilde{\omega}_\varepsilon^j(x,t)\tilde{\omega}_\varepsilon^i(y,t)\, dx\, dy\right| \\ &\quad + \left|\Gamma_i\nabla^\perp\tilde{g}(z_i(t), z_i(t)) - \frac{1}{\Gamma_i}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla^\perp\tilde{g}(x,y)\tilde{\omega}_\varepsilon^i(x,t)\tilde{\omega}_\varepsilon^i(y,t)\, dx\, dy\right| \\ &\quad + \left|\frac{1}{\Gamma_i}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla^\perp G_{\mathbb{R}^2}(x,y)\tilde{\omega}_\varepsilon^i(x,t)\tilde{\omega}_\varepsilon^i(y,t)\, dx\, dy\right|. \end{aligned}$$

The last term actually is 0 due to the symmetry of $G_{\mathbb{R}^2}$. For the first term we estimate

$$\left|\Gamma_j\nabla^\perp\tilde{G}(z_i(t), z_j(t)) - \Gamma_j\nabla^\perp\tilde{G}(c_\varepsilon^i(t), c_\varepsilon^j(t))\right| \leq M_1\left(\left|z_i(t) - c_\varepsilon^i(t)\right| + \left|z_j(t) - c_\varepsilon^j(t)\right|\right)$$

with a constant $M_1 > 0$ depending on $\tilde{G}$, $\tilde{g}$ and $\Gamma_1, \dots, \Gamma_N$. On the other hand (with a similar constant $M_2$)

$$\begin{aligned} &\left|\Gamma_j\nabla^\perp\tilde{G}(c_\varepsilon^i(t), c_\varepsilon^j(t)) - \frac{1}{\Gamma_i}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\nabla^\perp\tilde{G}(x,y)\tilde{\omega}_\varepsilon^j(x,t)\tilde{\omega}_\varepsilon^i(y,t)\, dx\, dy\right| \\ &\quad = \left|\frac{1}{\Gamma_i}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\nabla^\perp\tilde{G}(c_\varepsilon^i(t), c_\varepsilon^j(t)) - \nabla^\perp\tilde{G}(x,y)\right)\tilde{\omega}_\varepsilon^j(x,t)\tilde{\omega}_\varepsilon^i(y,t)\, dx\, dy\right| \leq M_2\delta, \end{aligned}$$

since $\operatorname{supp}\tilde{\omega}_\varepsilon^i(\cdot, t) \subset B_\delta(c_\varepsilon^i(t))$ for $i = 1, \dots, N$. The same procedure applied to the term involving $\nabla^\perp\tilde{g}(z_i(t), z_i(t))$ finally gives us

$$\left|\dot{z}_i(t) - \dot{c}_\varepsilon^i(t)\right| \leq M_3\sum_{j=1}^N\left|z_j(t) - c_\varepsilon^j(t)\right| + M_3\delta.$$

Thus

$$\sum_{i=1}^N\left|z_i(t) - c_\varepsilon^i(t)\right| \leq M_4\delta + M_4\int_0^t\sum_{i=1}^N\left|z_i(s) - c_\varepsilon^i(s)\right|\, ds$$

and we can apply Gronwall's lemma to conclude $\left|z_i(t) - c_\varepsilon^i(t)\right| \leq M_4\delta e^{M_4T}$ for $t \in [0, T]$. This shows $\left|z_i(t) - c_\varepsilon^i(t)\right| \longrightarrow 0$ uniformly as $\varepsilon \longrightarrow 0$.

By (C.9) we therefore can find to every $\delta > 0$ a number $\varepsilon_0(T, \delta) > 0$, such that for every $\varepsilon < \varepsilon_0$, $t \in [0, T]$ and $i = 1, \ldots, N$ there holds

$$\operatorname{supp} \tilde{\omega}_\varepsilon^i(\cdot, t) \subset B_\delta(z_i(t)).$$

In particular if $\delta < \frac{b}{4}$, we can conclude that as long as $t \in [0, T]$ the collection $\tilde{\omega}_\varepsilon^i$, $i = 1, \ldots, N$ not only is a solution of the regularized system (C.8), but also of the original system (C.7). Hence $\tilde{\omega}_\varepsilon^i(\cdot, t) = \omega_\varepsilon^i(\cdot, t)$ for $t \in [0, T]$, $i = 1, \ldots, N$. This shows that the vortex blobs remain localized around the point vortex solution.

It remains to prove the uniform convergence

$$\int_\Omega f(x)\omega_\varepsilon(x, t)\, dx \to \sum_{i=1}^N \Gamma_i z_i(t)$$

for $f \in C^0(\Omega, \mathbb{R})$ as $\varepsilon \to 0$. Let $\tilde{\delta} > 0$ and take $\delta > 0$ independent of $t \in [0, T]$, such that $|f(x) - f(z_i(t))| \leq \tilde{\delta}$ whenever $|x - z_i(t)| \leq \delta$. For $\varepsilon < \varepsilon_0(T, \delta)$ we have

$$\left| \int_\Omega f(x)\omega_\varepsilon^i(x, t)\, dx - \Gamma_i f(z_i(t)) \right| \leq \int_{B_\delta(z_i(t))} |f(x) - f(z_i(t))| \left| \omega_\varepsilon^i(x, t) \right| dx \leq \tilde{\delta}\, |\Gamma_i|$$

and the statement follows. □

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