On a Transcendental Equation in the Stability Analysis of a Population Growth Model

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Summary

We consider the rate equation $\dot{n} = rn$ for the density n of a single species population in a constant environment. We assume only that there is a positive constant solution n^*, that the rate of increase r depends on the history of n and that r decreases for great n. The stability properties of the solution n^* depend on the location of the eigenvalues of the linearized functional differential equation. These eigenvalues are the complex solutions λ of the equation $\lambda + \alpha \int_{-1}^{0} \exp[\lambda a]\, ds(a) = 0$ with $\alpha > 0$ and s increasing, $s(-1) = 0$, $s(0) = 1$. We give conditions on α and s which ensure that all eigenvalues have negative real part, or that there are eigenvalues with positive real part. In the case of the simplest smooth function s ($s = id + 1$), we obtain a theorem which describes the distribution of all eigenvalues in the complex plane for every $\alpha > 0$.

1. A population living in a constant environment cannot increase at constant rate. In laboratory experiments, the population density n tends to a limit n^*, or it shows undamped oscillations (see e.g. Halbach et al., 1972). A simple and general model in accordance with these observations is the functional differential equation

$$\dot{n}(t) = r(n_t)\, n(t), \quad t > 0 \tag{1}$$

for the population density $n: [-1, \infty) \to R_0^+$. Here the rate of increase is given by a real-valued mapping r defined on the set of non-negative functions on the interval $[-1, 0]$, with the only properties

$$r(n^*) = 0 \text{ for a positive constant function } n^* \tag{R 1}$$

and

$$\exists\, \varepsilon > 0: n^* - \varepsilon \leq \varphi \leq \psi \Rightarrow r(\psi) \leq r(\varphi). \tag{R 2}$$

The function n_t is defined by $n_t(a) := n(t+a)$ for $-1 \leq a \leq 0$ and $t \geq 0$. — The dependence of r on the density in the past allows oscillations (see e.g. Wright 1955), and $(R\,1)$ and $(R\,2)$ ensure that high densities result in a decay of the population size — which is a natural assumption. — The first model of this type (with $r(\varphi) = b(K - \varphi(-1))/K$, b and K positive) was proposed by G. E. Hutchinson in 1948.

We are interested in the stability of the constant solution defined by n^*. (This solution and its value are called n^*, too.) In this paper, we investigate the eigenvalues of the corresponding linearization of equation (1). Suppose in addition that

the Fréchet-derivative $Dr(n^*)$ of r in n^* exists (R 3)

(with respect to the supremum-norm on the continuous functions on $[-1,0]$). With $z:=n-n^*$ and $H(\varphi):=n^* r(\varphi+n^*)+\varphi(0)\, r(\varphi+n^*)$, equation (1) implies $\dot{z}(t)=H(z_t)$, and the linearization near n^* is $\dot{y}(t)=DH(0)(y_t)=n^* Dr(n^*)(y_t)$ since the derivative of the second term of H vanishes. By (R 1) and (R 2), $Dr(n^*)(\varphi)\le 0$ for $\varphi\ge 0$ and continuous. Hence $n^* Dr(n^*)(\varphi)=-\alpha\int_{-1}^{0}\varphi(a)\,ds(a)$ for all continuous functions $\varphi:[-1,0]\to R$, with $\alpha=n^*\,\|Dr(n^*)\|$ and $s\in S:=\{\sigma:[-1,0]\to R \mid \sigma \text{ increasing. } \sigma(-1)=0, \sigma(0)=1\}$.

The parameter α may serve as a measure of the power of the negative feedback in our system. The function s indicates how $Dr(n^*)(\varphi)$ — or $r(\varphi)$ for φ near n^* — depends on the values of φ at the different times in the past. For example, let $r(\varphi)=b-d(\varphi)=b-\alpha\cdot\int_{-1}^{0}\varphi(a)\,ds(a)$, where $b\in R^+$ stands for the birth rate and d for the death rate. Then s concave means that $r(n_t)$ is influenced more by $n\mid[-1,-1/2]$ than by $n\mid[-1/2,0]$. — One might expect that for s concave the stability of n^* is in some way less than for s convex because in the first case the system takes longer to produce a sufficient reaction to a perturbation of the equilibrium n^*. We shall see below in which way this conjecture turns out to be right.

The linearized equation becomes

$$\dot{y}(t)=-\alpha\int_{-1}^{0} y(t+a)\,ds(a). \tag{2}$$

The eigenvalues of equation (2) are the complex solutions of the transcendental equation

$$\lambda+\alpha\int_{-1}^{0}\exp[\lambda a]\,ds(a)=0$$

or, in other words, the zeros of the entire function $f(\cdot,\alpha,s):\lambda\mapsto\lambda+\alpha\int_{-1}^{0}\exp[\lambda a]\,ds(a)$.

The zero solution of equation (2) and n^* are asymptotically stable if all eigenvalues lie in the left half-plane $C^-:=R^-+iR$ (see [3], chapter 22, and [5]). If one eigenvalue is in $C^+:=R^++iR$ then the zero solution of equation (2) is unstable, and for $r(\varphi)=\alpha-\varphi(-1)$, $n^*\,(=\alpha)$ is unstable too (Wright 1955).

Due to [5], we have

$$0<\alpha<\pi/2\wedge s\in S\wedge f(\lambda,\alpha,s)=0\Rightarrow\lambda\in C^-. \tag{3}$$

For the minimal (convex) function s_0 in S (i.e. $s_0(a)=0$ for $a<0$), the equation for the eigenvalues reduces to $\lambda+\alpha=0$, hence $\lambda\in C^-$ for *all* $\alpha>0$. In the case of the maximal (concave) functions s_1 in S (i.e. $s_1(a)=1$ for $-1<a$) there exist $\alpha>0$ such that at least one eigenvalue is in C^+ (in fact, $\alpha>\pi/2$ is sufficient), see (Wright 1955).

Theorem 2 in section 2 shows that this property of s_0 carries over to a class of smooth convex functions in S. In Theorem 3 and Theorem 4 we present a class A of functions s in S with eigenvalues in C^+ for certain α, like s_1. The class A contains every concave function and all $s \geq id+1$ which are continuously differentiable.

The proof that n^* is unstable for $r(\varphi)=\alpha-\varphi(-1)$ and $\alpha>\pi/2$ requires — apart from the existence of an eigenvalue $\lambda=u+iv$ of the linearized equation in C^+ — the estimate $|v|<\pi$. Therefore we examine the boundedness of the branches of the eigenvalues in C^+ for given functions s (Theorem 5).

Section 3 deals with the eigenvalues for the simplest smooth function in S, that is $s=id+1$. Theorem 6 describes the location of the eigenvalues in C for all $\alpha>0$. In particular we see that $|v|>\pi$ for every eigenvalue in C^+.

2. In the following, we always assume $\alpha>0$ and $s \in S \setminus \{s_0\}$. For a subset $M \subset C$, $Z(\alpha, s, M)$ denotes the number of zeros of $f(\cdot, \alpha, s)$ in M. For $\lambda \in C$, we write $\lambda=u+iv$ with u and v real. We have

$$\begin{aligned} f(\lambda, \alpha, s)=0 \Leftrightarrow u+\alpha \int_{-1}^{0} \exp[ua] \cos va\, ds(a)=0 \wedge \\ v+\alpha \int_{-1}^{0} \exp[ua] \sin va\, ds(a)=0, \end{aligned} \tag{4}$$

$$f(\lambda, \alpha, s)=0 \Leftrightarrow f(\bar{\lambda}, \alpha, s)=0, \tag{5}$$

$$f(\lambda, \alpha, s)=0 \wedge u \geq 0 \Rightarrow |\lambda| \leq \alpha \wedge v \neq 0 \text{ (in particular } f(0, \alpha, s) \neq 0). \tag{6}$$

First, we consider real eigenvalues. Set $g(u, s) := -u / \int_{-1}^{0} \exp[ua]\, ds(a)$ for $u \leq 0$.

Theorem 1: There are real eigenvalues if and only if $\alpha \leq \max g(\cdot, s)$. Every real eigenvalue is negative.

Proof: $s \in S$, $s \neq s_0$ and $u \leq 0$ imply $\int_{-1}^{0} \exp[ua]\, ds(a) \geq \int_{-1}^{-\varepsilon} \exp[ua]\, ds(a) \geq$ $\geq \exp[-\varepsilon u]\, s(-\varepsilon)>0$ for small $\varepsilon>0$, hence $g(u, s) \to 0$ for $u \to -\infty$. By $g(0, s)=0$, $\max g(\cdot, s)$ exists. By (4), $u \in R \wedge f(u, \alpha, s)=0 \Leftrightarrow \alpha=g(u, s) \wedge u<0$. This implies Theorem 1.

Theorem 2: Let $s \in C^2[-1, 0] \cap C^3(-1, 0]$ be given with $s'(-1)=0$, $s''(-1)=0$, $s''' \geq 0$, $s'''(a^*)>0$ for a certain $a^* \in (-1, 0)$. Then for every $\alpha>0$, every eigenvalue has negative real part.

Proof: a) No eigenvalue on iR: By (4), (5) and $u=0$, we only have to show $\int_{-1}^{0} \cos va\, ds(a)>0$ for all $v>0$. Let $v>0$. With $s'(-1)=0$, $\int_{-1}^{0} \cos va\, ds(a)=$ $=-(1/v) \int_{-1}^{0} s''(a) \sin va\, da$. $-\int_{-1}^{0} s''(a) \sin va\, da = \lim_{\varepsilon \to 0} - \int_{\varepsilon-1}^{0} s''(a) \sin va\, da = (1/v)$ $\lim_{\varepsilon \to 0} (s''(0)-s''(\varepsilon-1) \cos(v\varepsilon-v) - \int_{\varepsilon-1}^{0} s'''(a) \cos va\, da)$.

By $s''(-1)=0$ and $s''(0)=\int_{\varepsilon-1}^{0} s'''(a)\,da+s''(\varepsilon-1)$, the last term equals $(1/v)$ $\lim \int_{\varepsilon-1}^{0} s'''(a)(1-\cos va)\,da \geq (1/v)\int_{a^*}^{0} s'''(a)(1-\cos va)\,da>0$.

b) No eigenvalue in C^+: $Z(\alpha, s, C^+)>0$ for $\alpha>0$ implies $\tilde{\alpha}:=\inf\{\alpha>0 \mid Z(\alpha, s, C^+)>0\}\in[\pi/2, \infty)$, see (3). Then there are sequences $\alpha_n \to \tilde{\alpha}$, λ_n in C^+ with $f(\lambda_n, \alpha_n, s)=0$. By (6), there exist subsequences $\alpha'_n \to \tilde{\alpha}$, $\lambda'_n \to \tilde{\lambda}\in \overline{C^+}$ with $f(\tilde{\lambda}, \tilde{\alpha}, s)=0$. a) gives $\tilde{\lambda}\in C^+$. Then $Z(\tilde{\alpha}, s, \partial D)=0$ for a compact disk D in C^+ with $\tilde{\lambda}\in \mathring{D}$, and Theorem 9.17.4 of [1] guarantees the existence of $\varepsilon>0$ with $Z(\tilde{\alpha}-\varepsilon, s, D)=$ $=Z(\tilde{\alpha}, s, D)>0$, contradiction.

Examples: Theorem 2 holds for $s: a \mapsto (a+1)^\beta$ with $\beta>2$. The case $\beta=2$ shows that the theorem is optimal in a certain sense: The function $s: a \mapsto (a+1)^2$ fulfills the hypotheses except of $s'''(a^*)>0$, and one verifies easily that $f(2\pi k i, (2\pi k)^2/2, s)$ is zero for every integer $k \neq 0$.

A class of discontinuous functions s with $Z(\alpha, s, \overline{C^+})=0$ for all $\alpha>0$ is defined by $\lim_{a\to 0} s(a)<1/2$.

Proof: Set $s^*(0):=\lim_{a\to 0} s(a)$ and $s^*(a):=s(a)$ for $a<0$. Assume $f(\lambda, \alpha, s)=0$ and $u\geq 0$. By (4), $u=-\alpha\int_{-1}^{0}\exp[ua]\cos va\,ds(a)=-\alpha\left(1-s^*(0)+\int_{-1}^{0}\exp[ua]\cos va\,ds^*(a)\right)$. By $\left|\int_{-1}^{0}\exp[ua]\cos va\,ds^*(a)\right|\leq s^*(0)$, we obtain $u\leq -(1-2s^*(0))$, contradiction to $u\geq 0$.

Theorem 3: For $s\in A:=\{\sigma\in S \mid \exists\, v\in[0,\pi]\ \exists\, \alpha>0: f(iv, \alpha, \sigma)=0\}$, there exist eigenvalues with positive real part for certain $\alpha>0$.

Proof: a) For $s\in S$, define a map $F=\begin{pmatrix}F_1\\F_2\end{pmatrix}: R^2\times R^+ \to R^2$ by $F_1(u, v, \alpha):=\operatorname{Re} f(u+iv, \alpha, s)$, $F_2(u, v, \alpha):=\operatorname{Im} f(u+iv, \alpha, s)$. If $F(0, v, \alpha)=0$ (which is equivalent to $f(iv, \alpha, s)=0$) and if

$$0<d:=\det\begin{pmatrix}\dfrac{\partial F_1}{\partial u} & \dfrac{\partial F_1}{\partial v}\\[2ex] \dfrac{\partial F_2}{\partial u} & \dfrac{\partial F_2}{\partial v}\end{pmatrix}(0, v, \alpha)=\left(1+\alpha\int_{-1}^{0} a\cos va\,ds(a)\right)^2+\alpha^2\left(\int_{-1}^{0} a\sin va\,ds(a)\right)^2,$$

then there are neighborhoods U of α and W of $(0, v)$ and a differentiable map $G=(G_1, G_2): U\to W$ with $G(\alpha)=(0, v)$ and $F\circ(G, id)=0$. With $G(\alpha)=(0, v)$, we obtain $G'_1(\alpha)=-d^{-1}\left(\dfrac{\partial F_1}{\partial \alpha}\cdot\dfrac{\partial F_2}{\partial v}-\dfrac{\partial F_1}{\partial v}\cdot\dfrac{\partial F_2}{\partial \alpha}\right)(0, v, \alpha)=d^{-1}\int_{-1}^{0} va\sin va\,ds(a)$. Therefore, Theorem 3 will follow from the existence of $\alpha>0$ and $v>0$ with $f(iv, \alpha, s)=0$ and

$$\int_{-1}^{0} a\sin va\,ds(a)>0. \tag{7}$$

b) $s\in A$, $f(iv, \alpha, s)=0$ and $v\in[0,\pi]$ imply $v>0$, by (6). Hence $a\sin va>0$ for $-1<a<0$. If $v<\pi$, we have in addition $-\sin(-v)>0$, and $s\neq s_0$ gives (7). The case $v=\pi$ and $\int_{-1}^{0} a \sin va\, ds(a)=0$ is impossible because $\int_{-1}^{0} a \sin \pi a\, ds(a)=0$ implies $s\,|\,(-1,0)$ constant, hence by (4) $-\pi/\alpha=\int_{-1}^{0} \sin \pi a\, ds(a)=s(1/2)\sin(-\pi)+$ $+(1-s(1/2))\sin 0=0$, contradiction.

Theorem 4: For s in $C^1[-1,0]$, $s\in A$ is equivalent with $-\int_{-1}^{0} \pi s(a)\sin \pi a\, da \geq 1$. In particular, every function s in $C^1[-1,0]$ with $s\geq id+1$ is in A. A contains every concave function in S.

Proof: a) Let $s\in C^1[-1,0]$, set $h(t)=\int_{-1}^{0} \cos ta\, ds(a)$ for $t\geq 0$. We have $h'(t)\leq 0$ for $0\leq t\leq \pi$ and $h(0)=1$, hence $h(\pi)\leq 0 \Leftrightarrow \exists\, v\in(0,\pi]: h(v)=0$. For every $v\in(0,\pi]$ with $h(v)=0$, $\int_{-1}^{0} \sin va\, ds(a)<0$ by $s\in C^1$. Then $f\left(iv, -v/\int_{-1}^{0} \sin va\, ds(a), s\right)=0$, by (4). We infer $(s\in A \Leftrightarrow h(\pi)\leq 0)$. Obviously, $0\geq h(\pi)=\int_{-1}^{0} s'(a)\cos \pi a\, da=1+\int_{-1}^{0} \pi s(a)$ $\sin \pi a\, da$.

b) $s\geq id+1 \Rightarrow -\int_{-1}^{0} \pi s(a)\sin \pi a\, da \geq -\int_{-1}^{0} \pi(a+1)\sin \pi a\, da=1$.

c) Let s be concave. For $n\in N$, set $a_v:=-1+v/2n$ for $v=0,1,\dots,2n$. Then $\cos \pi a_{v-1}=-\cos \pi a_{2n-(v-1)}$ and $s(a_v)-s(a_{v-1})\geq s(a_{n+\mu})-s(a_{n+\mu-1})$ for $1\leq v,\mu\leq n$. We infer $0\geq \sum_{\kappa 1}^{2n}(s(a_\kappa)-s(a_{\kappa-1}))\cos a_{\kappa-1}\pi$, hence $h(\pi)\leq 0$ and $h(v)=0$ for a certain $v\in[0,\pi]$. Obviously, $\int_{-1}^{0} \sin va\, ds(a)\leq 0$. $\int_{-1}^{0} \sin va\, ds(a)=0$ implies $v=\pi$ (since $s\geq id+1$) and $s\,|\,(-1,0)$ constant, hence $s\,|\,(-1,0]=1$ and $\int_{-1}^{0} \cos va\, ds(a)=-1$, contradiction. Now $s\in A$ follows as in a).

For fixed s, let P denote the set of eigenvalues with positive real part, that is $\{\lambda\in C^+ \,|\, \exists\, \alpha>0: f(\lambda,\alpha,s)=0\}$. In general, P is unbounded, see Theorem 6 below. But we have

Theorem 5: For $s\in C^3[-1,0]$ with $s'(-1)>0$ and $s'(0)>0$, every connected subset of P is bounded.

Proof: a) $f(\lambda,\alpha,s)=0$ implies $\lambda\neq 0$. Let $s\in C^3$. Integration by parts yields

$$(\lambda^2)/\alpha+s'(0)=s'(-1)\exp[-\lambda]+\int_{-1}^{0} \exp[\lambda a]\, s''(a)\, da \tag{8}$$

and

$$(\lambda^3)/\alpha+\lambda s'(0)=s'(-1)\exp[-\lambda]+s''(0)-\exp[-\lambda]\, s''(-1)-\int_{-1}^{0} \exp[\lambda a]\, s'''(a)\, da. \tag{9}$$

b) Assume $P\neq\emptyset$, $\operatorname{Re} P$ unbounded and $\operatorname{Im} P$ bounded. Then (8) holds for sequences $\lambda_n=u_n+iv_n$ in P and α_n in R^+, with $u_n\to\infty$. (8) gives

$$(u_n^2-v_n^2)/\alpha_n+s'(0)=\operatorname{Re}(\lambda_n^2/\alpha_n+s'(0))\to 0. \tag{10}$$

By (6) and $u_n\to\infty$, $\alpha_n\to\infty$. Hence $v_n^2/\alpha_n\to 0$, $0<s'(0)\leq s'(0)+u_n^2/\alpha_n\to 0$, contradiction.

b) Now let Q be a connected subset of P. By (5) and (6), we may assume $\operatorname{Im} Q \subset R^+$. For $\operatorname{Im} Q$ unbounded, there exist sequences α_n, u_n in R^+ and an integer n_0 with $f(u_n + i(2n\pi + \pi/2), \alpha_n, s) = 0$ for $n \geq n_0$. Set $\lambda_n := u_n + i(2n\pi + \pi/2)$. We have $\alpha_n \to \infty$ and $\int_{-1}^{0} \exp[\lambda_n a]\, s''(a)\, da \to 0$, $\int_{-1}^{0} \exp[\lambda_n a]\, s'''(a)\, da \to 0$. (Proof: It is sufficient to show the assertion in the cases $u_n \to \infty$ and $u_n \to u^* \geq 0$. The first case is trivial. In the second case, Theorem 4.6 of [7] gives, for example, $\int_{-1}^{0} \exp[u^* a]\, s''(a) \cos v_n a\, da \to 0$. In addition, $\left| \int_{-1}^{0} \exp[u^* a]\, s''(a) \cos v_n a\, da - \int_{-1}^{0} \exp[u_n a]\, s''(a) \cos v_n a\, da \right| \leq \max |s''|\, |\exp[-u_n] - \exp[-u^*]| \to 0$.) — Assume u_n bounded. By (8) and $\cos v_n = 0$, (10) holds for λ_n and α_n, if $n \geq n_0$. Hence $v_n^2/\alpha_n \to s'(0)$. Then $v_n/\alpha_n \to 0$ and $2u_n v_n/\alpha_n \to 0$. On the other hand, (8) yields $2u_n v_n/\alpha_n - s'(-1)\exp[-u_n] = \operatorname{Im}(\lambda_n^2/\alpha_n + s'(0) - s'(-1)\exp[-\lambda_n]) \to 0$. We infer $\exp[-u_n] \to 0$, contradiction.

Next, suppose $u_{n_k} \to \infty$ for a subsequence. Set $g_k := u_{n_k}$, $h_k := v_{n_k}$, $\beta_k := \alpha_{n_k}$. Taking real and imaginary parts in (9), we obtain

$$g_k\left((g_k^2 - 3h_k^2)/\beta_k + s'(0)\right) - s'(-1)\exp[-g_k]\, h_k - s''(0) \to 0 \tag{11}$$

and $h_k\left((3g_k^2 - h_k^2)/\beta_k + s'(0)\right) \to 0$, therefore

$$(3g_k^2 - h_k^2)/\beta_k + s'(0) \to 0. \tag{12}$$

We have $-s'(-1)\exp[-g_k]\, h_k \to -\infty$. (Proof: For a subsequence $g'_k + ih'_k$ with $s'(-1)\exp[-g'_k]\, h'_k$ bounded, (11) and $g'_k \to \infty$ imply $({g'_k}^2 - 3{h'_k}^2)/\beta'_k + s'(0) \to 0$. By (12), we obtain $({g'_k}^2 + {h'_k}^2)/\beta'_k \to 0$. By (12) again, $s'(0) = 0$, contradiction.)

Now (11) yields $0 < (g_k^2 - 3h_k^2)/\beta_k + s'(0)$ (for k large) $\leq (3g_k^2 - h_k^2)/\beta_k + s'(0) \to 0$, therefore $s'(0) = 0$, contradiction.

3. *Location of the eigenvalues for* $s = id + 1$. We set $f_\alpha := f(\cdot, \alpha, id+1)$ and $Z(\alpha, M) := Z(\alpha, id+1, M)$ for $\alpha > 0$, $v_k := 2k\pi + \pi$ and $\alpha_k := v_k^2/2$ for $k \in N_0$, $G_k := R + iI_k$ with $I_0 := (-2\pi, 2\pi)$ and $I_k := (2k\pi, 2k\pi + 2\pi)$ for $k \in N$, $g := g(\cdot, id+1)$, $\alpha^* := \max g$. u^* is defined by $g(u^*) = \alpha^*$. Because of (5), we only consider eigenvalues with $v \geq 0$.

Theorem 6:

i) Every zero $\lambda = u + iv$ of f_α, $\alpha > 0$, with $v \geq 0$ lies in the set $\bigcup_{k \in N_0} G_k$.

ii) For $k \in N$ and $\alpha > 0$, f_α has exactly one zero $\lambda_k(\alpha)$ in G_k. We have $\lambda_k(\alpha_k) = i\, v_k$, $\lambda_k(\alpha) \in R^- + i(2k\pi, 2k\pi + \pi)$ for $\alpha < \alpha_k$ and $\lambda_k(\alpha) \in R^+ + i(2k\pi + \pi, 2k\pi + 2\pi)$ for $\alpha > \alpha_k$.

iii) For every $\alpha > 0$, f_α has exactly two zeros in G_0. These zeros are real and simple, if $\alpha < \alpha^*$. If we denote them by $u_1(\alpha) < u_2(\alpha)$, then $u_1(\alpha) < u^* < u_2(\alpha)$ and $u_1(\alpha) \to -\infty$, $u_2(\alpha) \to 0$ for $\alpha \to 0$. For $\alpha = \alpha^*$, u^* is a double zero. For $\alpha^* < \alpha$, f_α has one zero $\lambda_0(\alpha)$ in G_0 with positive imaginary part. We have $\lambda_0(\alpha) \in R^- + i(0, \pi)$, if $\alpha^* < \alpha < \alpha_0$, $\lambda_0(\alpha_0) = iv_0$ and $\lambda_0(\alpha) \in R^+ + i(\pi, 2\pi)$, if $\alpha > \alpha_0$.

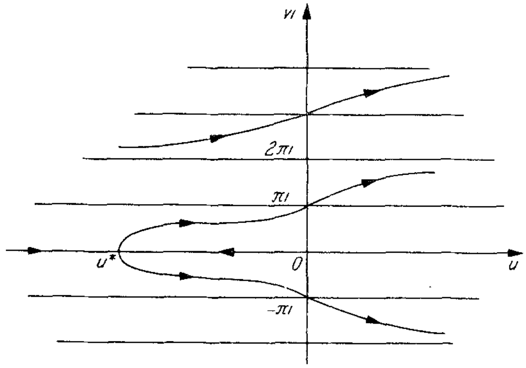

Fig. 1. The arrows indicate the direction of increasing α

Proof of Theorem 6: We have $f_\alpha(\lambda)=0 \Leftrightarrow \lambda \neq 0 \wedge$

$$(\lambda^2+\alpha)\exp[\lambda]=\alpha. \tag{13}$$

Assertion i) is a consequence of

a) Let $f_\alpha(\lambda)=0$, $v \geq 0$. Then $v \notin 2\pi N$, and we have

$$k \in N_0 \wedge \lambda \in G_k \wedge \begin{cases} u<0 \Rightarrow v<2k\pi+\pi \\ u>0 \Rightarrow v>2k\pi+\pi. \end{cases}$$

Proof: By (13), $\lambda^2+\alpha=\alpha\exp[-\lambda]$. Hence

$$2uv=\alpha\exp[-u]\sin(-v). \tag{14}$$

Therefore, $u \neq 0$ and $v \in 2\pi N$ would imply $v=0$. $u=0$ yields $v \in \pi N_0$. By (13), $u=0$ and $v \in 2\pi N_0$ would imply $\lambda^2=0$, contradiction to (6). Together, we obtain $v \notin 2\pi N$. The implication in a) is obvious from (14).

b) $f_\alpha(iv)=0 \wedge v \geq 0 \Leftrightarrow \exists\, k \in N_0 : \alpha=\alpha_k \wedge v=v_k$.

Proof: Clearly $f_{\alpha_k}(iv_k)=0$ for $k \in N_0$. As in the preceding proof, we have $(f_\alpha(iv)=0 \wedge v \geq 0 \Rightarrow \exists\, k \in N_0 : v=2k\pi+\pi=v_k)$. Then (13) gives $\alpha=v^2/2=v_k^2/2=\alpha_k$.

c) $\alpha \leq \alpha_k \Rightarrow Z(\alpha, G_k \cap C^+)=0$.

Proof: Suppose $\alpha' \leq \alpha_k$ and $Z(\alpha', G_k \cap C^+)>0$. We may assume $\alpha'<\alpha_k$ ($Z(\alpha_k, G_k \cap C^+)>0$ implies $Z(\alpha', G_k \cap C^+)>0$ for certain $\alpha'<\alpha_k$, compare part b) in the proof of Theorem 2). By (3), $\alpha' \geq \pi/2$. Set $B:=(0, \alpha'+1)+iI_k$. We have $Z(\alpha', B)>0$, by (6), and $Z(\alpha, \partial B)=0$ for $1 \leq \alpha \leq \alpha'$. This follows from a) together with (6), $\alpha<\alpha_k$ and with

$$f_\alpha(iv)=0 \wedge iv \in G_k \Rightarrow \alpha=\alpha_k. \tag{15}$$

With the aid of Theorem 9.17.4 of [1], we derive $Z(1, B)=Z(\alpha', B)>0$ which contradicts (3).

d) Let $k \in N_0$ and $\alpha>0$. There exists a negative constant $T(\alpha, k)$ with

$$\alpha' \geq \alpha \wedge \lambda \in G_k \cap C^- \wedge f_{\alpha'}(\lambda)=0 \Rightarrow T(\alpha, k)<u.$$

Proof: By (13), we have

$$1+(u^2+v^2)/\alpha \geq |1+\lambda^2/\alpha'| = |\exp[-\lambda]| = \exp[-u],$$

therefore $u^2/\alpha \geq \exp[-u]-1-(2k\pi+2\pi)^2/\alpha$. This estimate and $u<0$ yield the proposition.

e) $\forall k \in N_0 \; \exists \alpha_k^* > \alpha_k : \alpha \geq \alpha_k^* \wedge f_\alpha(\lambda)=0 \wedge \lambda \in G_k \Rightarrow u>0$.

Proof: Suppose there are sequences $\alpha(n)$, λ_n with $f_{\alpha(n)}(\lambda_n)=0$, $\alpha_k < \alpha(n) \to \infty$, $\lambda_n \in G_k$, $u_n \leq 0$. By (15) and $\alpha_k < \alpha(n)$, $u_n<0$. Proposition d), (13), $u_n<0$ and $\alpha(n) > \alpha_k > 1$ give

$$((2k\pi+2\pi)^2+T(1,k)^2)/\alpha(n) \geq |\lambda_n|^2/\alpha(n) \geq \exp[-u_n]-1>0.$$

Therefore $\exp[-u_n]\to 1$, $u_n \to 0$. Together with $v_n \to 0$, this would imply $\int_{-1}^{0} \exp[\lambda_n a]\, da \to 1$, hence $|f_{\alpha(n)}(\lambda_n)| \to \infty$, contradiction. — We obtain $|v_n| \geq \delta$ for a certain $\delta>0$ and for $n \in N^*$ with $N^* \subset N$ unbounded. By (13) and $u_n<0$, $\alpha(n)^2 \leq |\lambda_n^2+\alpha(n)|^2 = \alpha(n)^2+2\alpha(n)(u_n^2-v_n^2)+|\lambda_n|^4$. Because of $u_n \to 0$, $\alpha(n) \leq |\lambda_n|^4/2(v_n^2-u_n^2)$ for large $n \in N^*$, in contradiction to $\alpha(n)\to\infty$.

f) $Z(\alpha_0, G_0)=2$. $\forall k \in N: Z(\alpha_k, G_k)=1$.

Proof: Let $k \in N_0$. The zeros iv_k, $-iv_k$ of f_{α_k} are simple $(\lambda \neq 0 \Rightarrow f_\alpha(\lambda) = \lambda+\alpha(1-\exp[-\lambda])/\lambda \Rightarrow (df_\alpha/d\lambda)(\lambda) = 1+\alpha\exp[-\lambda]/\lambda - \alpha/\lambda^2+\alpha\exp[-\lambda]/\lambda^2 \Rightarrow (df_{\alpha_k}/d\lambda)(\pm iv_k) = 2 \pm iv_k/2)$. Therefore, i) and b) and c) imply $Z(\alpha_k, \overline{G_k \cap C^+})=1$, if $k \in N$, and $\ldots = 2$ for $k=0$. Suppose $Z(\alpha_k, G_k \cap C^-)>0$. Then $Z(\alpha', G_k \cap C^-)>0$ for certain $\alpha' \in (\alpha_k, \alpha_k^*)$, compare the proof of c). Set $B := (T(\alpha_k, k), 0)+iI_k$. We have $Z(\alpha', B)>0$ and $Z(\alpha, \partial B)=0$ for $\alpha' \leq \alpha \leq \alpha_k^*$, by d), a) and (15). Hence $Z(\alpha_k^*, B) = Z(\alpha', B)>0$, in contradiction to e).

g) For every $k \in N_0$ and every $\alpha>0$, $Z(\alpha, G_k) = Z(\alpha_k, G_k)$.

Proof: Let $k \in N_0$, $\alpha > \alpha_k$ (the proof for $\alpha<\alpha_k$ is similar). There exists $T>0$ with

$$\lambda \in G_k \wedge \alpha_k \leq \alpha' \leq \alpha \wedge f_{\alpha'}(\lambda)=0 \Rightarrow |u|<T. \tag{16}$$

(Proof: The zeros with $u>0$ and $u<0$ are bounded by (6) and by d) respectively.) Set $B := (-T, T)+iI_k$. For $\alpha_k \leq \alpha' \leq \alpha$, $Z(\alpha', \partial B)=0$. Hence $Z(\alpha, B) = Z(\alpha_k, B)$, and (16) implies g).

h) Let $k \in N_0$ and $\alpha<\alpha_k$. By c), $u \leq 0$ for the zeros of f_α in G_k. (15) yields $u<0$. By a), $2k\pi < v < 2k\pi+\pi$ for $k \in N$ and $|v|<\pi$ for $k=0$.

j) For $k \in N_0$ and $\alpha>\alpha_k$, set $B := (T(\alpha, k), 0)+iI_k$. e) implies the existence of $\alpha_k^{**} \geq \alpha$ with $Z(\alpha_k^{**}, B)=0$. By d) and (15), $Z(\alpha', \partial B)=0$ for $\alpha \leq \alpha' \leq \alpha_k^{**}$. Hence $Z(\alpha, B) = Z(\alpha_k^{**}, B)=0$. d) gives $(f_\alpha(\lambda)=0 \wedge \lambda \in G_k \cap C^- \Rightarrow \lambda \in B)$, therefore $Z(\alpha, G_k \cap C^-)=0$. $Z(\alpha, G_k \cap iR)=0$ because of (15) and $\alpha>\alpha_k$. With a), we obtain $u>0$ and $2k\pi+\pi < v < 2k\pi+2\pi$ for the zero of f_α in G_k, $k \in N$, and $u>0$ and $\pi<|v|<2\pi$ for the two zeros of f_α in G_0.

k) By Theorem 1, f_α has real zeros if and only if $\alpha \leq \alpha^*$, and the real zeros are negative. The assertion concerning the real zeros for $\alpha<\alpha^*$ follows from

$Z(\alpha, G_0)=2$ and from the relations $u \in R \wedge f_\alpha(u)=0 \Leftrightarrow \alpha=g(u)$, $u<u^* \Rightarrow g'(u)>0$, $u>u^* \Rightarrow g'(u)<0$, $g(0)=0$, $g(u) \to 0$ for $u \to -\infty$, $g \mid R^- > 0$. u^* must be a double zero of f_{α^*} since by $g(u^*)=\alpha^*=\max g$, there is no other real zero of f_{α^*}, and the existence of a zero of f_{α^*} in $G_0 \setminus R$ would imply the existence of a third zero in G_0, in contradiction to $Z(\alpha^*, G_0)=2$.

Notation:

N, R, R^+, R^- and C denote the natural, real, positive real, negative real and the complex numbers respectively. We set $N_0 := N \cup \{0\}$, $R_0^+ := R \setminus R^-$, $R_0^- := R \setminus R^+$. For $A \subset C$, $B \subset C$ and $z \in C$, we define

$$A+zB := \{\lambda \in C \mid \exists\, a \in A\ \exists\, b \in B: \lambda = a+zb\}.$$

Re z (Im z) is the real part (imaginary part) of z,

$$\operatorname{Re} A := \{u \in R \mid \exists\, z \in A: u = \operatorname{Re} z\},\ \operatorname{Im} A := \{u \in R \mid \exists\, z \in A: u = \operatorname{Im} z\}.$$

∂A, $\mathring{A}$ and $\bar{A}$ are the boundary, the interior and the closure of the set $A \subset C$. A dot — like in $\dot{n}$ — and a prime — like in s' — indicate differentiation. $C^k[-1,0]$ and $C^k(-1,0]$ are the sets of functions on $[-1,0]$ which have continuous derivatives up to order k in $[-1,0]$ and $(-1,0]$ respectively. $f \mid A$ denotes the restriction of a given mapping $f: B \to D$ to a subset $A \subset B$.

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