Higher Spin Representations in Kac-Moody-Theory

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Contents

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Introduction

The goal of this thesis is the study of finite-dimensional representations of so-called maximal compact subalgebras $\mathfrak{k}(A)(\mathbb{R})$ of split-real Kac-Moody algebras $\mathfrak{g}(A)(\mathbb{R})$, where I mostly restrict myself to the situation that A is a simply-laced generalized Cartan matrix of indefinite type.

The structure and representation theory of finite-dimensional compact Lie algebras $\mathfrak{k}$ is well-understood for quite some time now and is treated in many standard text books both on the under-graduate and graduate level ([H72, S07, HN12]). Any finite-dimensional compact Lie algebra is reductive and any simple compact Lie algebra is the compact real form of a complex simple Lie algebra which provides a strong link between the two types of Lie algebras. Furthermore, simple split-real Lie algebras of finite dimension possess a maximal compact subalgebra that is given as the fixed-point set of the Cartan-Chevalley involution, the most standard example is probably $\mathfrak{so}(n,\mathbb{R})$ as the maximal compact subalgebra of $\mathfrak{sl}(n,\mathbb{R})$. The finite dimensional simple Lie algebras over $\mathbb{C}$ and similarly the split-real simple Lie algebras are classified by Cartan matrices A or equivalently by Dynkin diagrams $\mathcal{D}(A)$ and according to Serre's construction (cp. [S65]), a simple complex Lie algebra can be uniquely recovered from its Cartan matrix A. This way the Lie algebra is given by a presentation in form of generators and relations that are encoded in the Cartan matrix. If one relaxes the conditions on the matrix A in a certain way and performs a similar construction one obtains a Kac-Moody-algebra (cp. [K90]) denoted by $\mathfrak{g}(A)(\mathbb{K})$ as the construction can be performed over any field $\mathbb{K}$. These Lie algebras are always split in the sense that they contain a maximal abelian subalgebra whose adjoint action on $\mathfrak{g}(A)(\mathbb{K})$ is diagonalizable. As Kac-Moody-algebras possess a natural generalization of the Cartan-Chevalley involution ω, one can introduce the maximal compact subalgebra $\mathfrak{k}(A)$ of $\mathfrak{g}(A)(\mathbb{R})$ as the fixed point set of ω in analogy to the classical situation.

In contrast to the classical situation, neither maximal compact subalgebras of split-real Kac-Moody algebras nor their complexification are of Kac-Moody type if A is not a Cartan matrix. This can be seen from the fact that these Lie algebras admit finite-dimensional representations (see for instance [BHP06, DKN06, HKL15]) despite being infinite-dimensional and in a lot of cases also perfect. Kac-Moody-algebras of irreducible indefinite type on the other hand are essentially simple up to a finite-dimensional center contained in their Cartan subalgebra and therefore cannot admit finite-dimensional representations. Very little is known about the structure of these maximal compact subalgebras apart from a presentation result going back to [B89] and apart from a few examples, next to nothing is known about their representation theory.

The first nontrivial finite-dimensional representations were discovered in physics, where the finite dimensional representations of $\mathfrak{k}(E_{10})$ and $\mathfrak{k}(E_9)$ play a crucial role for certain unified theories of gravity. In total, there are four different representations known for $\mathfrak{k}(E_{10})$, labeled $\mathcal{S}_{\frac{1}{2}}$, $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$ and $\mathcal{S}_{\frac{7}{2}}$ by their "spin". Of these, $\mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}}$ were discovered first (see [BHP06, DKN06]) as they arise as hidden symmetries in the fermionic sector of 11-dimensional super gravity. The discovery of such a hidden symmetry sparked additional research concerning the representation theory of $\mathfrak{k}(E_{10})$ which among other results produced the representations $\mathcal{S}_{\frac{5}{2}}$ and $\mathcal{S}_{\frac{7}{2}}$ (cp. [KN13, KN17]).

The representation $\mathcal{S}_{\frac{1}{2}}$ has been studied mathematically in [HKL15] for the first time, where its definition was also extended to include all $\mathfrak{k}(A)$ for A a symmetrizable generalized Cartan matrix. The representations $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$ and $\mathcal{S}_{\frac{7}{2}}$ build on these so-called generalized spin representations and a coordinate-free version of $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$ was given in [LK18] that goes back to their description in [KN13]. So far, such a description of $\mathcal{S}_{\frac{7}{2}}$ was not available from a mathematical perspective.

As one my results I provide a unified description of the representations $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$ and $\mathcal{S}_{\frac{7}{2}}$ in terms of $\mathcal{S}_{\frac{1}{2}}$ and the Weyl group $W(A)$. The representation is at first only given on the level of Berman generators (named after the author of [B89]) of $\mathfrak{k}(A)$. Furthermore, I study the lift of these representations to the group level, thus linking [KN13, KN17] to [GHKW17] which might be helpful for the scientific community as the two are rather different in language. Just as $\mathfrak{k}(A)(\mathbb{R})$ is defined as the fixed-point set of the Chevalley involution on $\mathfrak{g}(A)(\mathbb{R})$ one defines the maximal compact subgroup $K(A)(\mathbb{R})$ as the fixed-point set of its lift to the Kac-Moody group $G(A)(\mathbb{R})$. Similar to the classical situation, it is a priori unclear if a given representation of $\mathfrak{k}(A)(\mathbb{R})$ lifts to $K(A)(\mathbb{R})$, as the fundamental group of $K(A)(\mathbb{R})$ generally is nontrivial ([H20, HK2x]). My result in this direction is that these representations do not lift to the maximal compact subgroup $K(A)(\mathbb{R})$ but only to its so-called spin-cover $Spin(A)$ introduced in [GHKW17] (this cover is simply connected in the irreducible simply-laced case by [H20, HK2x] and [GHKW17]). This justifies the term spin representations.

Theorem A. *Let $A \in \mathbb{Z}^{n\times n}$ be a simply-laced generalized Cartan matrix, let $\mathfrak{k}(A)(\mathbb{R})$ be the maximal compact subalgebra of type A, let $\mathfrak{h}^*$ denote the dual Cartan subalgebra of $\mathfrak{g}(A)(\mathbb{R})$ and let $W(A)$ denote the Weyl group of type A. Furthermore let $\eta_n : W(A) \to End(Sym^n(\mathfrak{h}^*))$ denote the representation that is induced by the standard representation on $\mathfrak{h}^*$ and let $\rho : \mathfrak{k}(A)(\mathbb{R}) \to End(\mathbb{C}^s)$ denote a generalized spin representation as in [HKL15]. Then the following assignment on the level of Berman generators $X_1, \dots, X_n$ of $\mathfrak{k}(A)(\mathbb{R})$ extends to a homomorphism of Lie algebras $\sigma_n : \mathfrak{k}(A)(\mathbb{R}) \to End(Sym^n(\mathfrak{h}^*) \otimes \mathbb{C}^s)$:*

$$\sigma_n(X_i) = \left(\eta_n(s_i) - \frac{1}{2}Id\right) \otimes 2\rho(X_i) \text{ for } n = 1, 2,$$

$$\sigma_n(X_i) = \left(\eta_n(s_i) - \frac{1}{2} Id + f(\alpha_i)\right) \otimes 2\rho(X_i) \ \ for\ n = 3,$$

where $f(\alpha_i)$ is a rank 1 matrix described in more detail in thm. 3.23. All these representations lift to the group $Spin(A)$ as described in [GHKW17], while they do not lift to $K(A)(\mathbb{R})$, the maximal compact subgroup of $G(A)(\mathbb{R})$.

Another important part of this work is my study of (ir-)reducibility of these representations and some of their tensor products. The properties of the image of the generalized spin representation $\mathcal{S}_{\frac{1}{2}}$ have been studied to great extent in [HKL15] and in the cases that I care about the most, these images always form a semi-simple Lie algebra so that it is always possible to choose $\mathcal{S}_{\frac{1}{2}}$ to be irreducible.

Theorem B. *Let $\mathcal{S}_{\frac{1}{2}}$ be an irreducible generalized spin representation of $\mathfrak{k}(A)$, let $\mathcal{S}_{\frac{2n+1}{2}}$ be the higher spin representations described in theorem A and let A be regular and simply-laced. Then $\mathcal{S}_{\frac{3}{2}}$ is irreducible. The module $\mathcal{S}_{\frac{5}{2}}$ always splits into two orthogonal pieces $\mathcal{S}_{\frac{5}{2}} \cong \widetilde{\mathcal{S}}_{\frac{5}{2}} \oplus \mathcal{S}_{\frac{1}{2}}$, where the properties of $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ may vary from case to case. $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible if $Sym^2(\mathfrak{h}^*)$ has exactly two $W(A)$-invariant sub-modules, one of which will always be the trivial $W(A)$-module.*

Towards the properties of $\mathrm{Sym}^2(\mathfrak{h}^*)$ as a $W(A)$-module there exist examples in both directions even in the situation of classical A. For $A = A_{n-1}$ the $W(A)$-module $\mathrm{Sym}^2(\mathfrak{h}^*)$ decomposes into more than two factors, while the above condition is satisfied for $A = E_n$ and $n = 6, 7, 8$ and one can show that this holds for E_{10} as well by direct computation.

Corollary. *The $\mathfrak{k}(E_n)$-modules $\mathcal{S}_{\frac{1}{2}}$, $\mathcal{S}_{\frac{3}{2}}$ and $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ for $n \in \{6, 7, 8, 10\}$ are irreducible.*

Towards tensor products I found the following

Theorem C. *Assume that $\mathcal{S}_{\frac{1}{2}}$, $\mathcal{S}_{\frac{3}{2}}$ and $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ are irreducible $\mathfrak{k}(A)$-modules. Then $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\widetilde{\mathcal{S}}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ are irreducible. In particular, this holds for $A = E_n$ with $n \in \{6, 7, 8, 10\}$.*

It is a curious feature of these representations that their tensor products can again be irreducible as such a behavior is rarely witnessed for semi-simple Lie algebras of finite dimension. In connection with the representations' kernels, the irreducible tensor products can provide new ideals of $\mathfrak{k}(A)$ and therefore provide a few more sporadic glimpses at its structure. For instance, I will show that the kernels of the tensor products are precisely the intersections of the individual kernels. The original plan for this project however was to find a system behind the representations $\mathcal{S}_{\frac{n}{2}}$, which I did, that allows for a construction of infinitely many independent finite-dimensional representations, which I failed at. The search for this sequence of representations is connected to the hope that the corresponding kernels become more and more faithful such that one can recover $\mathfrak{k}(A)$ from these representations (this would also show that $\mathfrak{k}(A)$ is residually finite-dimensional, as of yet it is unclear if this property holds for any $\mathfrak{k}(A)$ of indefinite type). So far, this goal could only be achieved for A of untwisted affine type (cp. [KKLN21]).

The text is structured as follows. I will start with a collection of standard results from Kac-Moody-theory that are needed throughout the remainder. In section 2 I will start with an example, the $\mathfrak{k}(E_n)$-series, and show that there exists a connection to Slodowy's theory of $\mathfrak{gim}$-Lie algebras (a shorthand for Generalized-Intersection-Matrix-Lie-algebras introduced in [S84]) that goes in the other direction than the one in [S84, B89]. I will explicitly derive a description of $\mathfrak{k}(E_n)(\mathbb{C})$ that is adapted to its natural $\mathfrak{so}(n, \mathbb{C})$-algebra (cp. prop. 2.8) and show that it can be realized as the quotient of a certain $\mathfrak{gim}$-algebra (prop. 2.7). Such a result can be expected to hold for other diagrams $\mathcal{D}(A)$ as well because the involved computations are not very specific to E_n but so far, I did not see an easy way to predict the resulting structure universally. Hence, the structure theory of $\mathfrak{k}(A)$ is connected to the structure theory of $\mathfrak{gim}$-Lie algebras which unfortunately is not well-understood either.

In section 3 I will review the representations $\mathcal{S}_{\frac{1}{2}}$, $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$ which are already known in mathematics, where I will give a more unified description for $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ in thm. 3.19 than the one in [LK18]. In section 3.3 I will translate the results from [KN13, KN17] concerning $\mathcal{S}_{\frac{7}{2}}$ into a more mathematical and in particular coordinate-free statement, where the main result is thm. 3.23.

In section 4 I connect these results to [GHKW17] by showing that all these representations lift to the spin cover $Spin(A)$ of the maximal compact subgroup $K(A) \leq G(A)$. After reviewing only the most essential parts of [GHKW17] I do this in two steps by first showing that the representations $\mathcal{S}_{\frac{3}{2}}$, $\mathcal{S}_{\frac{5}{2}}$ lift to $Spin(A)$ but not to $K(A)$ in prop. 4.9 and showing secondly that the representation $\mathcal{S}_{\frac{7}{2}}$ lifts to $Spin(A)$ as well but not to $K(A)$. I use this lift to the group level to deduce a parametrization result about the representation matrices in props. 4.16 and 4.17.

Section 5 is again devoted to the example $\mathfrak{k}\,(E_{10})$. I reproduce the decompositions of $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ w.r.t. the natural $\mathfrak{so}\,(10)$-subalgebra of $\mathfrak{k}\,(E_{10})$ that were first mentioned in [KN13]. The main results are prop. 5.5 and thm. 5.14. I translate the technical insight gained from this example into the more general statement that $\mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}}$ are $\mathfrak{k}\,(A)$-irreducible, whenever A is simply-laced, indecomposable and regular (cp. 5.8). As mentioned in theorem B, one can also see in general that $\mathcal{S}_{\frac{5}{2}}$ contains an invariant sub-module isomorphic to $\mathcal{S}_{\frac{1}{2}}$ and that under the previous conditions on A the module $\mathcal{S}_{\frac{5}{2}}$ splits into two invariant pieces $\widetilde{\mathcal{S}}_{\frac{5}{2}} \oplus \mathcal{S}_{\frac{1}{2}}$ (cp. 5.10). There, I also show by examples that the (ir-)reducibility of $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ depends on the case. I establish in thm. 5.14 that $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible for the case of $\mathfrak{k}\,(E_{10})$ (cp. also [KN13]).

In section 6 I start with the description of a computer-based analysis of the tensor products $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}} \otimes \bigwedge^2 \left(\mathcal{S}_{\frac{1}{2}}\right)$ of the $\mathfrak{k}\,(E_{10})$-modules $\mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}}$. Both these modules turn out to be irreducible and I explain this fact theoretically for $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\widetilde{\mathcal{S}}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ whenever all factors are irreducible and $\mathcal{S}_{\frac{1}{2}}$ satisfies certain properties in prop. 6.7.

I conclude with section 7 which treats the case $\mathfrak{k}\,(A)$ for A of untwisted affine type. This is the only example where an infinite series of f.d. representations is known (cp. prop. 7.20). This series of representations is in fact enough to recover $\mathfrak{k}\,(A)$ as it acts faithfully on the projective limit of these modules (cp. prop. 7.23).

Part I
Kac-Moody algebras, their involutory subalgebras and the E_n-series

If not specified otherwise, $\mathbb{K}$ always denotes the field of real or complex numbers. All results that are not specific to $\mathbb{R}$ or $\mathbb{C}$ should also be true for any field of characteristic 0.

1 Preliminaries

In this section I will fix the notation concerning Kac-Moody algebras $\mathfrak{g}$ and cite the standard results that I will need later on. I will use the same notation as [K90]. In subsection 1.2 I will review Berman's results concerning presentations of involutory subalgebras of Kac-Moody algebras where I also provide the definition of a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ and cite the explicit formulas concerning their presentation from [HKL15], as [B89] does not provide them explicitly for this particular case of involutory subalgebras. In section 1.3 I will collect some results on highest weight representations of symmetrizable Kac-Moody algebras.

1.1 Kac-Moody algebras

Kac-Moody algebras can be constructed through a generalization of Serre's construction of finite-dimensional simple split Lie algebras over $\mathbb{K}$. There one starts with a Cartan matrix A to which one associates a root system Δ, a coroot system $\Delta^\vee$ and a (split) Cartan subalgebra $\mathfrak{h}_\mathbb{K} = \mathrm{span}_\mathbb{K}\Delta^\vee$ in a natural way as the (reduced, irreducible, crystallographic) root systems are classified by Cartan matrices. Then one introduces the Chevalley generators $e_1, \dots, e_n, f_1, \dots, f_n$ and constructs the Lie algebra from generators and relations, where the relations are pairwise and depend on the entries of A. A similar approach works for Kac-Moody algebras but the resulting object can have quite different properties if A is not a Cartan matrix. Also, some parts of the construction become more complicated. I start with recalling the standard definitions from [K90]. First of all, one needs to fix how one wants to deviate from Cartan matrices.

Definition 1.1. (Generalized Cartan matrix[1])
A matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{Z}^{n\times n}$ is called a **generalized Cartan matrix** (GCM) if $\forall\, i \neq j \in \{1, \dots, n\}$

$$\begin{aligned} a_{ii} &= 2 \\ a_{ij} &\leq 0 \\ a_{ij} &= 0 \Leftrightarrow a_{ji} = 0. \end{aligned}$$

All Cartan matrices associated to the root system of a finite-dimensional semi-simple complex Lie algebra satisfy the above axioms but the converse is not true. If A is a Cartan matrix, there exists a natural choice of root system and Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In the Kac-Moody setting one works with realizations instead, where the following terminology is fairly standard (cp. [K90, sec. 1.1]).

Definition 1.2. (Realization[2])
Let $A \in \mathbb{K}^{n\times n}$ be of rank $l \leq n$ and let $\mathfrak{h}$ be a $\mathbb{K}$-vector space of dimension $2n - l$. For subsets $\Pi =$

[1] Compare [K90, sec. 1.1], especially equations (C1-3).
[2] Compare equations (1.1.1-3) in [K90, sec. 1.1].

$\{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a ($\mathbb{K}$-)**realization** of A if Π and $\Pi^\vee$ are linearly independent and such that

$$\alpha_j\left(\alpha_i^\vee\right) = a_{ij} \; \forall\, i, j \in \{1, \ldots, n\}\,.$$

One calls Π the **simple roots** and $\Pi^\vee$ the **simple coroots**. Two realizations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ are called isomorphic if there exists an isomorphism of $\mathbb{K}$-vector spaces $\varphi : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that $\varphi\left(\Pi_1^\vee\right) = \Pi_2^\vee$ and $\varphi^* : \Pi_2 \to \Pi_1$ such that $\varphi^*\left(\Pi_2\right) = \Pi_1$, where $\varphi^* : \mathfrak{h}_2^* \to \mathfrak{h}_1^*$ denotes the dual map to φ.

A matrix A admits a unique-up-to-isomorphism realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ and the realizations of two matrices A_1 and A_2 are isomorphic if and only if A_2 can be obtained from A_1 by a permutation of the index set (cp. [K90, prop. 1.1]). A GCM A and correspondingly any of its realizations is called decomposable if it can be brought into block diagonal form $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with nontrivial A_1, A_2 by a permutation of the index set.

Definition 1.3. (Root lattice, height)[3]
Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of a GCM A. One calls $Q(A) \coloneqq \mathrm{span}_{\mathbb{Z}}\Pi$ the **root lattice** and $Q^\vee(A) \coloneqq \mathrm{span}_{\mathbb{Z}}\Pi^\vee$ the **coroot lattice**. For $\alpha = \sum_{i=1}^n k_i\alpha_i \in Q(A)$ one calls $\mathrm{ht}\,(\alpha) \coloneqq \sum_{i=1}^n k_i$ the **height** of α. One imposes a partial order $\leq$ on $Q(A)$ via $\alpha \leq \beta$ if and only if $\beta - \alpha \in Q_+(A)$, where $Q_+ \coloneqq \sum_{i=1}^n \mathbb{N}\alpha_i$.

The entries of any Cartan matrix A are of the form $\frac{2(\alpha|\beta)}{(\alpha|\alpha)}$ where α and β range over the simple roots Π and $(\cdot|\cdot)$ is a positive definite bilinear form on $\mathrm{span}_{\mathbb{R}}\Pi$. For GCMs one drops the requirement of positive definiteness but even then it is not always the case that there exists a bilinear form s.t. $\alpha_j\left(\alpha_i^\vee\right) = \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$. This property is tied to the existence of a so-called symmetrization of A.

Definition 1.4. (Symmetrizability[4])
Let $A = \left(a_{ij}\right)_{i,j=1}^n \in \mathbb{Z}^{n\times n}$ be a GCM. A is called **symmetrizable** if there exists a regular diagonal matrix D and a symmetric matrix B such that $A = DB$. The pair of matrices D and B is called a **symmetrization** of A. The GCM A is called **simply-laced** if $a_{ij} \in \{0, -1\}$ for all $i \neq j$.

One can show that it is always possible to achieve a symmetrization with B rational and $D = \mathrm{diag}\left(\varepsilon_1, \ldots, \varepsilon_n\right)$ such that $\varepsilon_i > 0$ for all $i = 1, \ldots, n$. If A is indecomposable, then D is uniquely determined up to a constant factor (cp. [K90, sec. 2.3]). I will now provide the definition of a Kac-Moody algebra associated to a symmetrizable GCM A that I will use in the entire text. The definition I use is closer to [M18, def. 3.17] than the one in [K90, sec. 1.2-3] but for symmetrizable A they coincide due to the Gabber-Kac-theorem (cp. [GK81], also see [K90, thm. 9.11]).

Definition 1.5. (Kac-Moody algebra)
Let $A \in \mathbb{Z}^{n\times n}$ be a symmetrizable GCM with $\mathbb{K}$-realization $(\mathfrak{h}, \Pi, \Pi^\vee)$. Let $\mathfrak{g}\left(A\right)\left(\mathbb{K}\right)$ be the Lie algebra on generators $\mathfrak{h} \cup \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ subject to the relations

$$[h, h'] = 0,\;\; [e_i, f_j] = \delta_{ij}\alpha_i^\vee,$$

$$[h, e_i] = \alpha_i\left(h\right) e_i,\;\; [h, f_i] = -\alpha_i\left(h\right) f_i,$$

$$\mathrm{ad}\left(e_i\right)^{1-a_{ij}}\left(e_j\right) = 0 = \mathrm{ad}\left(f_i\right)^{1-a_{ij}}\left(f_j\right)$$

$\forall\, h, h' \in \mathfrak{h}$ and $\forall\; i, j \in \{1, \ldots, n\}$. One calls $\mathfrak{g}\left(A\right)\left(\mathbb{K}\right)$ the **split Kac-Moody algebra over $\mathbb{K}$ of type A**.

[3]These definitions are also as in [K90, sec. 1.1].
[4]Compare [K90, eq. (2.1.1)].

Now let $\mathfrak{n}_+$ be the subalgebra of $\mathfrak{g}(A)(\mathbb{K})$ that is generated by $e_1, \ldots, e_n$ and let $\mathfrak{n}_-$ be the subalgebra generated by $f_1, \ldots, f_n$. Then one has the following triangular decomposition as vector spaces (cp. [K90, thm. 1.2 and sec. 1.3]):

$$\mathfrak{g}(A)(\mathbb{K}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{1}$$

This triangular decomposition induces a decomposition of the universal enveloping algebra[5]

$$\mathcal{U}(\mathfrak{g}(A)(\mathbb{K})) = \mathcal{U}(\mathfrak{n}_-)\,\mathcal{U}(\mathfrak{h})\,\mathcal{U}(\mathfrak{n}_+)$$

according to the PBW-theorem (cp. [B36]).

Definition 1.6. (Root spaces)[6]
For a split Kac-Moody-algebra $\mathfrak{g} = \mathfrak{g}(A)(\mathbb{K})$ and $\alpha \in \mathfrak{h}$ set $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \,|\, [h,x] = \alpha(h)\,x \;\forall\, h \in \mathfrak{h}\}$. If $\mathfrak{g}_\alpha \neq \{0\}$ and $\alpha \neq 0$ one calls α a **root**, $\mathfrak{g}_\alpha$ a **root space** and $\operatorname{mult}(\alpha) := \dim \mathfrak{g}_\alpha$ the **multiplicity** of α. The set of roots is denoted by Δ and due to (1) it decomposes into a disjoint union $\Delta = \Delta_- \cup \Delta_+ \subset Q$ where $\Delta_\pm := \{\alpha \in \Delta \,|\, \alpha \gtrless 0\}$. The vector space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

is referred to as the **root space decomposition** of $\mathfrak{g}$.

Proposition 1.7. *(Chevalley involution)*
There exists an involutive automorphism ω of $\mathfrak{g}(A)(\mathbb{K})$ that is determined by

$$\omega(e_i) = -f_i,\; \omega(f_i) = -e_i,\; \omega(h) = -h \;\forall\, h \in \mathfrak{h}.$$

*It is called the **Chevalley involution** and it satisfies $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Delta$.*

Proof. This statement can be found at the end of [K90, sec. 1.3], where it arises as a direct consequence of [K90, thm. 1.2]. □

Definition 1.8. (Precursor of the invariant bilinear form[7])
Let $A \in \mathbb{Z}^{n\times n}$ be a symmetrizable GCM with symmetrization $A = DB$, where $D = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ is s.t. $\varepsilon_i > 0$ for all $i = 1, \ldots, n$ and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A according to def. 1.2. Set $\mathfrak{h}' := \operatorname{span}_{\mathbb{K}}\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ and let $\mathfrak{h}''$ be a complementary subspace of $\mathfrak{h}' \subset \mathfrak{h}$. Fix a symmetric $\mathbb{K}$-bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ by

$$(h|\alpha_i^\vee) = \alpha_i(h)\,\varepsilon_i \;\forall\, h \in \mathfrak{h}, \quad (h_1|h_2) = 0 \;\forall\, h_1, h_2 \in \mathfrak{h}''. \tag{2}$$

According to [K90, lem. 2.1], $(\cdot|\cdot)$ is non-degenerate on all of $\mathfrak{h}$ and its kernel on the restriction to $\mathfrak{h}'$ is equal to $\mathfrak{c} := \{h \in \mathfrak{h} \,|\, \alpha_i(h) = 0 \;\forall\, i = 1, \ldots, n\}$, which is equal to the center of $\mathfrak{g}$ (cp. [K90, prop. 1.6]). Now $(\cdot|\cdot)$ induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ which allows to induce a bilinear form on $\mathfrak{h}^*$, also denoted by $(\cdot|\cdot)$. One collects the following formulas (cp. [K90, eqs. (2.1.4-6)]):

$$\nu(h_1)(h_2) := (h_1|h_2) \;\; \forall\, h_1, h_2 \in \mathfrak{h},$$
$$\nu(\alpha_i^\vee) = \varepsilon_i \alpha_i,\; \left(\alpha_i^\vee|\alpha_j^\vee\right) = b_{ij}\varepsilon_i\varepsilon_j,\; (\alpha_i|\alpha_j) = b_{ij} \;\forall\, i,j = 1, \ldots, n,$$

where b_{ij} denote the entries of B in A's symmetrization $A = DB$.

[5] The universal enveloping algebra $(\mathcal{U}(\mathfrak{g}), \phi)$ consists of a unital associative algebra $\mathcal{U}(\mathfrak{g})$ together with a map $\phi : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ satisfying $\phi([x,y]) = \phi(x)\cdot\phi(y) - \phi(y)\cdot\phi(x)$ for all $x, y \in \mathfrak{g}$. It is universal, meaning that for all unital associative algebras A and $\psi : \mathfrak{g} \to A$ satisfying $\psi([x,y]) = \psi(x)\cdot\psi(y) - \psi(y)\cdot\psi(x)$ for all $x, y \in \mathfrak{g}$ there exists a uniqe algebra homomorphism $\hat{\psi} : \mathcal{U}(\mathfrak{g}) \to A$ such that $\psi = \hat{\psi} \circ \phi$.

[6] Compare [K90, sec. 1.3].

[7] Compare equations (2.1.2-3) of [K90].

Proposition 1.9. *(Invariant bilinear form)*
Let $\mathfrak{g} = \mathfrak{g}(A)(\mathbb{K})$ be a split Kac-Moody algebra with a symmetrizable GCM A and fix a symmetrization $A = DB$ with $D = diag(\varepsilon_1, \ldots, \varepsilon_n)$ s.t. $\varepsilon_i > 0$ for all $i = 1, \ldots, n$. Then there exists a $\mathbb{K}$-bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ s.t. $(\cdot|\cdot)|_{\mathfrak{h}}$ coincides with the bilinear form of def. 1.8 and s.t.

$$\begin{aligned}
([x, y] \,|\, z) &= (x \,|\, [y, z]) \;\; \forall\, x, y, z \in \mathfrak{g}, \\
(\mathfrak{g}_\alpha | \mathfrak{g}_\beta) &= 0 \;\forall\, \alpha, \beta \in \Delta \text{ s.t. } \alpha \neq -\beta, \\
(\cdot|\cdot)|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}} & \;\; \text{is non-degenerate } \forall\, \alpha \in \Delta \\
[x, y] &= (x|y)\, \nu^{-1}(\alpha) \;\; \forall\, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha},\; \alpha \in \Delta.
\end{aligned}$$

This form is called the ***standard invariant bilinear form*** *and it is unique w.r.t. a fixed symmetrization. Without such a reference it is unique up to scalar multiples if A is indecomposable.*

Proof. This is essentially [K90, thm. 2.2] together with the conventions on D employed in [K90, sec. 2.3]. □

In terms of the standard invariant bilinear form one has without reference to a symmetrization $A = DB$ the relations (cp. [K90, eq. 2.3.5])

$$\alpha_i^\vee = \frac{2}{(\alpha_i|\alpha_i)} \nu^{-1}(\alpha_i), \quad A = \left(\frac{2\,(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} \right)_{i,j=1}^{n} \tag{3}$$

Definition 1.10. (Integrable $\mathfrak{g}(A)(\mathbb{K})$-modules[8])
Let V be a $\mathfrak{g}(A)(\mathbb{K})$-module and set $V_\lambda := \{v \in V | h.v = \lambda(h)\, v \;\forall\, h \in \mathfrak{h}\}$ for $\lambda \in \mathfrak{h}^*$. Call V_λ the **weight space** to the **weight** $\lambda \in \mathfrak{h}^*$ if $V_\lambda \neq 0$ and call $\operatorname{mult}(\lambda, V) := \dim V_\lambda$ the **multiplicity of** λ. The module V is called **$\mathfrak{h}$-diagonalizable** if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$. It is called ***integrable*** if it is $\mathfrak{h}$-diagonalizable and the Chevalley generators $e_1, \ldots, e_n, f_1, \ldots, f_n$ are locally nilpotent on V. An element $x \in \mathfrak{g}$ is called **locally nilpotent** if for any $v \in V$ one can find $N > 0$ such that $x^N.v = 0$.

Fact 1.11. *(This is a consequence of [K90, lem 3.5]) The Kac-Moody algebra $\mathfrak{g}(A)(\mathbb{K})$ is an integrable $\mathfrak{g}(A)(\mathbb{K})$-module w.r.t. the adjoint action of $\mathfrak{g}(A)(\mathbb{K})$ on itself.*

Proposition 1.12. *(Cp. [K90, prop. 3.6])*
Let V be an integrable $\mathfrak{g}(A)(\mathbb{K})$-module, denote the set of weights of V by $P(V)$ and set $\mathfrak{g}_i := \mathbb{K}f_i \oplus \mathbb{K}\alpha_i^\vee \oplus \mathbb{K}e_i$. Then w.r.t. $\mathfrak{g}_i$ the module V decomposes into a direct sum of $\mathfrak{h}$-invariant, finite-dimensional, irreducible $\mathfrak{g}_i$-modules. For $\lambda \in P(V)$ there exist p, q either nonnegative integers or equal to infinity such that $\lambda + t\alpha_i \in P(V)$ if and only if $t \in [-p, q] \cap \mathbb{Z}$. If $mult(\lambda, V) < \infty$ then p, q are finite and in this case $p - q = \lambda(\alpha_i^\vee)$. The action of e_i defines an injective linear map from $V_{\lambda + t\alpha_i}$ to $V_{\lambda + (t+1)\alpha_i}$ whenever $-p \leq t \leq -\frac{1}{2}\lambda(\alpha_i^\vee)$ and the multiplicities $mult(\lambda + t\alpha_i, V)$ as a function of t are symmetric w.r.t. $t = -\frac{1}{2}\lambda(\alpha_i^\vee)$. Also, if $\lambda, \lambda + \alpha_i \in P(V)$ then there exists $v \in V_\lambda$ such that $e_i v \neq 0$.

Corollary 1.13. *(Cp. [K90, cor. 3.6])*
For an integrable $\mathfrak{g}(A)(\mathbb{K})$-module V and a weight $\lambda \in P(V)$ s.t. $\lambda + \alpha_i \notin P(V)$ it follows from prop. that $\lambda(\alpha_i^\vee) \geq 0$. If conversely λ is such that $\lambda - \alpha_i \notin P(V)$ one has that $\lambda(\alpha_i^\vee) \leq 0$. For $\lambda \in P(V)$ and $i \in I$ it is always true that $\lambda - \lambda(\alpha_i^\vee)\,\alpha_i \in P(V)$ and that $mult(\lambda - \lambda(\alpha_i^\vee)\,\alpha_i, V) = mult(\lambda, V)$.

The statement about the weight $\lambda - \lambda(\alpha_i^\vee)\,\alpha_i$ above is quite important towards the action of the Weyl group on the set of weights.

[8]Compare the beginning of [K90, sec. 3.6].

Definition 1.14. (Weyl group[9])
Let $\mathfrak{g}(A)(\mathbb{K})$ be a split Kac-Moody algebra associated to the GCM $A \in \mathbb{Z}^{n\times n}$ and define the **fundamental reflections** $s_i \in GL(\mathfrak{h}^*)$ for $i = 1, \ldots, n$ via

$$s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\,\alpha_i \;\forall\, \lambda \in \mathfrak{h}^*.$$

One defines the **Weyl group** $W(A)$ of $\mathfrak{g}(A)(\mathbb{K})$ as $W(A) := \langle s_1, \ldots, s_n\rangle \subset GL(\mathfrak{h}^*)$.

Another way of defining the Weyl group is by providing its presentation as a Coxeter group:

Proposition 1.15. *(This is [K90, prop. 3.13]) Let $A \in \mathbb{Z}^{n\times n}$ be a symmetrizable GCM and let*

$$m_{ij} = \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0 \\ 3 & \text{if } a_{ij}a_{ji} = 1 \\ 4 & \text{if } a_{ij}a_{ji} = 2 \\ 6 & \text{if } a_{ij}a_{ji} = 3 \\ 0 & \text{if } a_{ij}a_{ji} \geq 4. \end{cases} \tag{4}$$

Then the Weyl group $W(A)$ is given by the presentation

$$\begin{aligned} W(A) \;=\; \Big\langle s_1, \ldots, s_n \,\Big|\; & s_i^2 = e \;\forall\, i \in \{1, \ldots, n\}, \\ & \underbrace{s_i s_j s_i \cdots}_{m_{ij}\ factors} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}\ factors} \;\forall\, i \neq j \in \{1, \ldots, n\} \Big\rangle. \end{aligned}$$

Here, $m_{ij} = 0$ factors means that there exists no pairwise relation between s_i and s_j.

The Weyl group's action on the weight system of an integrable module V leaves the set of weights and their multiplicities invariant:

Proposition 1.16. *(Cp. [K90, prop. 3.7])*
Let V be an integrable $\mathfrak{g}(A)(\mathbb{K})$-module and let $\lambda \in P(V)$. Then $W(A).P(V) = P(V)$ and $mult(\omega(\lambda), V) = mult(\lambda, V)$ for all $\omega \in W(A)$. Specialized to the adjoint representation of $\mathfrak{g}(A)(\mathbb{K})$ this yields that the root system Δ is invariant under the action of $W(A)$ and that $mult(\omega(\alpha)) = mult(\alpha)$ for all $\alpha \in \Delta$, $\omega \in W(A)$.

Proposition 1.17. *(Cp. [K90, prop. 3.9])*
Let $(\cdot|\cdot)$ denote the bilinear form on $\mathfrak{h}^*$ that is induced by the standard invariant bilinear form of $\mathfrak{g}(A)(\mathbb{K})$. Then

$$(\omega(\lambda)\,|\omega(\mu)) = (\lambda|\mu) \;\;\forall\, \lambda, \mu \in \mathfrak{h}^*,\; \forall\, \omega \in W(A),$$

i.e., $(\cdot|\cdot)$ is $W(A)$-invariant.

There exist three types - the finite, affine and indefinite type - of Kac-Moody algebras which possess quite distinct features. One distinguishes and classifies them by their generalized Cartan matrices or equivalently by the associated generalized Dynkin diagrams.

[9] This definition of Weyl groups in the Kac-Moody context is standard (cp. [K90, sec. 3.7]) as it is the straightforward generalization of the definition in the classical setting of crystallographic root systems.

Definition 1.18. (Generalized Dynkin diagram[10]) To a GCM $A = (a_{ij})_{i,j=1}^{n}$ one associates a **generalized Dynkin diagram** $\mathcal{D}(A)$ with n vertices as follows. The vertices i and j are connected by an edge if $a_{ij} \neq 0$. In general one depicts the edge (i,j) with a solid line and ordered pair $(|a_{ij}|, |a_{ji}|)$ but if $a_{ij}a_{ji} \leq 4$ and such that $|a_{ij}| \geq |a_{ji}|$ one draws $|a_{ij}|$ lines instead with arrow pointing to node i if $|a_{ij}| > 1$.

Except for the case $a_{ij} = -2 = a_{ji}$ (which results in an edge $\Leftrightarrow$) these exceptional rules produce rank 2 diagrams that are classical Dynkin diagrams. Generalized Dynkin diagrams $\mathcal{D}(A)$ and GCMs A are in one-to-one correspondence and a GCM A is indecomposable if $\mathcal{D}(A)$ has only one connected component. For a vector $u \in \mathbb{R}^n$, set $u > 0$ if $u_i > 0$ for all $i = 1, \ldots, n$ and similarly $u \geq 0$, $u < 0$, $u \leq 0$. The categorization of Kac-Moody-algebras into the three types is due to the following theorem[11]

Theorem 1.19. *Let $A \in \mathbb{Z}^{n\times n}$ be an indecomposable GCM. Then exactly one of the following applies:*
(Finite) A is regular and $\exists\, u > 0$ such that $Au > 0$. If $Av \geq 0$ then either $v > 0$ or $v = 0$.
(Affine) A is of rank $n-1$ and there exists $u > 0$ s.t. $Au = 0$. If $Av \geq 0$ then $Av = 0$.
(Indefinite) There exists $u > 0$ such that $Au < 0$. If $v \geq 0$ and $Av \geq 0$ then $v = 0$.
Hence, A is of finite/affine/indefinite type if and only if there exists $u > 0$ such that $Au > 0$/ $Au = 0$/ $Au < 0$.

In [K90, sec. 4.8], all possible generalized Dynkin diagrams of finite and affine type are listed. The indefinite type does not admit such a systematic classification.

In Kac-Moody algebras that are not of finite type, a new phenomenon arises in the root system. In classical simple Lie algebras the root system is finite and every root can be written as $\omega(\alpha_i)$ for a suitable simple root α_i and an element $\omega \in W(A)$. In the affine and the indefinite case this is no longer true, which motivates the following definition:

Definition 1.20. (Real and imaginary roots[12])
Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with root system Δ. Call a root $\alpha \in \Delta$ **real** if there exists $i \in I$ and $\omega \in W(A)$ s.t. $\alpha = \omega(\alpha_i)$ and denote the set of such roots by Δ^{re} and set $\Delta_+^{re} := \Delta^{re} \cap \Delta_+$. Its complement $\Delta^{im} := \Delta \setminus \Delta^{re}$ is called the set of **imaginary roots** and one again sets $\Delta_+^{im} := \Delta^{im} \cap \Delta_+$. One associates a reflection s_α to every $\alpha \in \Delta^{re}$ via

$$s_\alpha(\mu) := \mu - \mu(\alpha^\vee)\,\alpha \;\forall\, \mu \in \mathfrak{h}^*. \tag{5}$$

For $\alpha = \sum_{i=1}^{n} k_i\alpha_i$ call $\operatorname{supp}(\alpha) := \{i \in \{1,\ldots,n\} \mid k_i \neq 0\}$ the **support** of α.

A lot of computations become easier with the following lemma:

Lemma 1.21. *(This is a consequence of [K90, lem. 1.6])*
Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with root system Δ and let $\alpha \in \Delta$. Then supp(α) is a connected subset of the generalized Dynkin diagram $\mathcal{D}(A)$.

Proposition 1.22. *(This is [K90, prop. 5.1])*
Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with root system Δ and let $\alpha = \sum_i k_i\alpha_i \in \Delta^{re}$ be a real root. Then mult$(\alpha) = 1$, $k\alpha \in \Delta$ if and only if $k = \pm 1$. For $\beta \in \Delta$ not necessarily real there exist nonnegative integers

[10] There exist different conventions concerning the edges of the diagram. I use Kac's convention as described in [K90, ch. 4].
[11] This is a specialized version of [K90, thm. 4.3], where I included the consequence of [K90, cor. 4.3] as well.
[12] This definition is standard, compare [K90, secs. 5.1-2]. Towards the well-definedness of the reflection one needs that $\alpha = \omega(\alpha_i)$ is equivalent to $\alpha^\vee = \omega(\alpha_i^\vee)$ as argued at the beginning of [K90, sec. 5.1]. This equivalence is shown in [K90, eq. 3.10.3].

p, q such that $\beta + k\alpha \in \Delta \cup \{0\}$ if and only if $k \in [-p, q] \cap \mathbb{Z}$ and one has $p - q = \beta(\alpha^\vee)$ (cp. prop. 1.12). If A is symmetrizable then

$$(\alpha|\alpha) > 0,\ \alpha^\vee = \frac{2\nu^{-1}(\alpha)}{(\alpha|\alpha)},\ k_i(\alpha_i|\alpha_i) \in (\alpha|\alpha)\mathbb{Z}\ \forall \alpha \in \Delta^{re}.$$

If $\alpha \neq \pm\alpha_i$ for all $i \in \{1, \dots, n\}$ then there exists $j \in \{1, \dots, n\}$ such that $|ht(s_j\alpha)| < |ht(\alpha)|$.

Concerning imaginary roots one has

Proposition 1.23. *(This is [K90, prop. 5.2])*
Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with root system Δ and (positive) imaginary roots Δ^{im} (resp. Δ^{im}_+). The positive and negative imaginary roots are $W(A)$-invariant independently, i.e., $W(A).\Delta^{im}_\pm \subseteq \Delta^{im}_\pm$. For every $\alpha \in \Delta^{im}_+$ there exists a root $\bar{\alpha} \in \Delta^{im}_+$ s.t. $\bar{\alpha}(\alpha_i^\vee) \leq 0\ \forall i = 1, \dots, n$ and $\omega \in W(A)$ s.t. $\alpha = \omega(\bar{\alpha})$. If A is symmetrizable then $\alpha \in \Delta^{im}$ if and only if $(\alpha|\alpha) \leq 0$.

There exist more detailed characterizations of the imaginary roots (cp. [K90, thm 5.4]) in terms of orbits of the Weyl group but here I would only like to collect a crucial statement about their existence:

Theorem 1.24. *(This is [K90, thm. 5.6])*
Let $A \in \mathbb{Z}^{n\times n}$ be an indecomposable GCM and $\mathfrak{g}(A)$ its associated Kac-Moody algebra. If A is of finite type, then the set of imaginary roots Δ^{im} is empty. If A is of affine type, there exists an isotropic[13] *root $\delta = \sum_{i=1}^n k_i\alpha_i$ such that $\Delta^{im}_\pm = \{\pm m\delta,\ m \in \mathbb{N}\}$, where the coefficients k_i are the labels of the Dynkin diagrams of affine type in [K90, sec. 4.8]. If A is of indefinite type there exists $\alpha = \sum_{i=1}^n k_i\alpha_i \in \Delta^{im}_+$ such that $k_i > 0$ and $\alpha(\alpha_i^\vee) < 0$ for all $i = 1, \dots, n$.*

Also, one has the following characterization of isotropic roots:

Proposition 1.25. *(This is [K90, prop. 5.7])*
Let A be a symmetrizable GCM. Then $\alpha \in \Delta^{im}$ is isotropic,i.e., $(\alpha|\alpha) = 0$, if and only if there exists β s.t. $supp(\beta) \subset D(A)$ is of affine type and $\omega \in W(A)$ s.t. $\omega(\beta) = \alpha$.

1.2 Some involutive subalgebras of the second kind

The involutory subalgebras of Berman studied in [B89] are fixed-point-subalgebras w.r.t. an automorphism σ of the Kac-Moody-algebra $\mathfrak{g}(A)(\mathbb{K})$ which is built from three ingredients. In this subsection I will collect a result of [B89] concerning their presentation and an adapted version from [HKL15] which will provide a presentation of $\mathfrak{k}(A)(\mathbb{R})$, the definition is given in 1.27, by generators and relations. First and most essential one defines the involutive automorphism η on the level of Chevalley generators via

$$\eta(e_i) = f_i\ ,\ \eta(f_i) = e_i, \eta(\alpha_i^\vee) = -\alpha_i^\vee\ \forall i = 1, \dots, n. \tag{6}$$

One can also include an automorphism τ (which may include a field automorphism such as complex conjugation) of order 2 via (anti-)linear extension of

$$\tau(e_i) = \rho_i e_i\ ,\ \tau(f_i) = \rho_i^{-1} f_i, \tau(\alpha_i^\vee) = \alpha_i^\vee \text{ with } \rho_i \in \{\pm 1\}\ \forall i = 1, \dots, n. \tag{7}$$

As a third component one can introduce automorphisms γ arising from diagram automorphisms π of $\mathcal{D}(A)$. For this define γ via

$$\gamma(e_i) = e_{\pi(i)},\ \gamma(f_i) = f_{\pi(i)},\ \gamma(\alpha_i^\vee) = \alpha_{\pi(i)}^\vee\ \forall i = 1, \dots, n. \tag{8}$$

[13] An isotropic root is a root α such that $(\alpha|\alpha) = 0$.

Lemma 1.26. *(Cp. [B89, rem. 1.7]) The maps defined on the level of generators in eqs. (6)-(8) extend uniquely to automorphisms of $\mathfrak{g}(A)(\mathbb{K})$. The automorphisms η and γ commute and if $\rho_i = \rho_{\pi(i)}$ then τ commutes with both η and γ. Their product $\sigma = \eta\gamma\tau$ is an order-2 automorphism of $\mathfrak{g}(A)(\mathbb{K})$ if γ is of order 2.*

Proof. One checks that each of the maps (6), (7), (8) extends uniquely to an automorphism of $\mathfrak{g}(A)(\mathbb{K})$ and that η and τ are of order 2. As $\rho_i^{-1} = \rho_i$ one has that τ and η always commute but for $\tau\gamma = \gamma\tau$ one indeed needs $\rho_i = \rho_{\pi(i)}$. It is also apparent that $\eta\gamma = \gamma\eta$. Finally,

$$\sigma^2 = \eta\gamma\tau\eta\gamma\tau = \eta^2\gamma^2\tau^2 = Id$$

because each of the involved automorphisms is of order 2, in case of γ due to the assumptions of the lemma. □

Definition 1.27. (Maximal compact subalgebras)
Denote by $\sigma = \eta\gamma\tau$ the involutive automorphism of $\mathfrak{g}(A)(\mathbb{K})$ defined by eqs. (6), (7), (8). Denote by $\mathfrak{s}_\sigma := \{x \in \mathfrak{g}(A)(\mathbb{K}) \,|\sigma(x) = x\}$ its fixed-point subalgebra. If $\mathbb{K} = \mathbb{R}$, $\gamma = Id$ and $\rho_i = -1$ for all $i = 1, \ldots, n$ I denote $\mathfrak{s}_\sigma$ by $\mathfrak{k}(A)(\mathbb{R})$ and call it the **maximal compact subalgebra**[14] of $\mathfrak{g}(A)(\mathbb{R})$.

Proposition 1.28. *(This is [B89, prop. 1.12]) The elements $x_i := e_i + \sigma(e_i)$ and $z_i := h_i + \sigma(h_i)$ for $1 \le i \le n$ generate $\mathfrak{s}_\sigma$ defined in 1.27.*

For the special case $\sigma = \eta\gamma$, Berman provides relations on the x_i, z_i such that $\mathfrak{s}_\sigma$ is isomorphic to the quotient algebra of the free Lie algebra on generators x_i, z_i by these relations.

Theorem 1.29. *(Due to [B89]) For $j \neq k \in \{1, \ldots, n\}$ define coefficients $c_{s,t}^{(j,k)} \in \mathbb{Z}$ for all $s, t \in \mathbb{Z}$ via $c_{0,0}^{(j,k)} = 1$ and*

$$c_{s,t}^{(j,k)} = \begin{cases} 0 & \text{if either } s < 0 \text{ or } t < 0 \text{ or } t > s \\ c_{s-1,t-1}^{(j,k)} + (s-1)\left[a_{jk} + (s-2)\right] c_{s-2,t}^{(j,k)} & \text{otherwise.} \end{cases}$$

Let $\sigma = \eta\gamma$ with η and γ as in eqs. (6), (8) without the field automorphism τ. Then the involutive subalgebra $\mathfrak{s}_\sigma$ from def. 1.27 admits a presentation by generators x_i, z_i for $1 \le i \le n$ and relations

$$\begin{aligned} [z_j, z_k] &= 0 = z_j + z_{\pi(j)} \\ [z_j, x_k] &= \left(a_{kj} - a_{\pi(k)j}\right) x_k \end{aligned}$$

for $j, k \in \{1, \ldots, n\}$ and

$$(ad\, x_k)^{2m+1}(x_j) + \delta_{k,\pi(k)} \sum_{t=0}^{m-1} c_{2m+1,2t+1}^{(j,k)} (ad\, x_k)^{2t+1}(x_j) + \delta_{m,0}\delta_{j,\pi(k)} z_j = 0$$

if $|a_{jk}| = 2m$ and

$$(ad\, x_k)^{2m+2}(x_j) + \delta_{k,\pi(k)} \sum_{t=0}^{m} c_{2(m+1),2t}^{(j,k)} (ad\, x_k)^{2t}(x_j) - \delta_{m,1}\delta_{j,\pi(k)} \left(2a_{kj} - a_{\pi(k)j}\right) x_j = 0$$

[14] This terminology is as in [HKL15] and the maximal compact subalgebra is not to be confused with the compact form as defined in [K90, sec. 2.7] which is also denoted by $\mathfrak{k}(A)$ there. The difference is that in [K90], the involution is antilinear and the resulting fixed-point subalgebra is a real form of $\mathfrak{g}(A)(\mathbb{C})$, whereas in [HKL15] the base field is equal to $\mathbb{R}$ to begin with. If A is of finite type, then $\mathfrak{k}(A)(\mathbb{R})$ coincide with the maximal compact subalgebra of split-real $\mathfrak{g}(A)(\mathbb{R})$ in the usual sense of compactness, i.e., negative-definiteness of the Cartan-Killing-form.

for $|a_{jk}| = 2m+1$ *for all* $j \neq k \in \{1, \ldots, n\}$. *The generators can be taken to be the same as in the previous proposition, i.e.,* $x_i = e_i + \sigma(e_i)$ *and* $z_i = h_i + \sigma(h_i)$ *for* $1 \leq i \leq n$.

Proof. This is [B89, thm. 1.31] together with [B89, prop. 1.18]. The coefficients $c_{s,t}^{(j,k)}$ are the same as in [B89, def. 1.17]. □

In principle, the same techniques that lead to theorem 1.29 can be applied to the case where τ is nontrivial but linear. One simply has to carry along the signs ρ_i. In the case of maximal compact subalgebras $\mathfrak{k}(A)$ the generators and relations look simpler. As an adaption of Berman's result [B89, thm. 1.31] one has

Theorem 1.30. *(Cp. [HKL15, thm. 1.8]) Let* $\mathfrak{g}(A)(\mathbb{R})$ *be a split-real symmetrizable Kac-Moody algebra with Chevalley generators* $e_1, \ldots, e_n, f_1, \ldots, f_n$ *and Cartan subalgebra* $\mathfrak{h}$. *Then the maximal compact subalgebra* $\mathfrak{k}(A)(\mathbb{R})$ *has a presentation by generators* $X_1, \ldots, X_n$ *and relations*

$$P_{-a_{ij}}(ad\, X_i)(X_j) = 0 \ \forall\, i \neq j \in \{1, \ldots, n\},$$

where

$$P_m(t) := \begin{cases} \prod_{k=0}^{\frac{m-1}{2}} \left(t^2 + (m-2k)^2\right) & \text{if } m \text{ is odd,} \\ t \cdot \prod_{k=0}^{\frac{m}{2}-1} \left(t^2 + (m-2k)^2\right) & \text{if } m \text{ is even.} \end{cases}$$

Concretely, one has $X_i = e_i - f_i$ *for* $i = 1, \ldots, n$ *and one calls these elements the* ***Berman generators*** *of* $\mathfrak{k}(A)(\mathbb{R})$.

For A simply-laced these relations spell out as follows:

Corollary 1.31. *(Cp. [B89, thm. 1.31], also [HKL15, thm. 1.8]) Let* $\mathfrak{g}(A)(\mathbb{R})$ *be a split-real simply-laced Kac-Moody algebra, let* $\mathfrak{k}(A)(\mathbb{R})$ *be its maximal compact subalgebra and denote by* $\mathcal{E}(A)$ *the edges of* $\mathcal{D}(A)$. *Then* $\mathfrak{k}(A)(\mathbb{R})$ *has a presentation by generators* $X_1, \ldots, X_n$ *and relations*

$$[X_i, [X_i, X_j]] = -X_j \ \forall\, (i,j) \in \mathcal{E}(A)$$

$$[X_i, X_j] = 0 \ \forall\, (i,j) \notin \mathcal{E}(A).$$

Lemma 1.32. *The maximal compact subalgebra* $\mathfrak{k}(A)(\mathbb{R})$ *as well as its complexification* $\mathfrak{k}(A)(\mathbb{C}) := \mathfrak{k}(A)(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ *is filtered by* $\Delta_+(A)$. *Explicitly, one has a decomposition*

$$\mathfrak{k}(A) = \bigoplus_{\alpha \in \Delta_+(A)} \mathfrak{k}_\alpha \tag{9}$$

as vector spaces, where $\mathfrak{k}_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$. *For* $x_\alpha \in \mathfrak{k}_\alpha$, $x_\beta \in \mathfrak{k}_\beta$ *one has*

$$[x_\alpha, x_\beta] \in \mathfrak{k}_{\alpha+\beta} \oplus \mathfrak{k}_{\pm(\alpha-\beta)}, \tag{10}$$

where the sign depends on whether or not $\alpha - \beta \in \Delta_+$ *or* $\beta - \alpha \in \Delta_+$.

Proof. The filtered structure of $\mathfrak{k}(A)$ is used both in [B89] and [HKL15] to show the main results about the presentations 1.29 and 1.30. It arises rather directly from the graded structure of $\mathfrak{g}(A)(\mathbb{K})$. If $e_\alpha \in \mathfrak{g}_\alpha$ then clearly $e_\alpha + \omega(e_\alpha) \in \mathfrak{k}$ and it is not hard to see that such elements exhaust $\mathfrak{k}_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$ because $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and $\dim \mathfrak{g}_\alpha < \infty$. Also, $\mathfrak{k} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{k}_\alpha$ because any $x \in \mathfrak{k}$ can be decomposed w.r.t. the gradation of $\mathfrak{g}$ as $x = \sum_{\alpha\in\Delta(A)} v_\alpha$. The demand $\omega(x) = x$ then implies $\omega(v_{\pm\alpha}) = v_{\mp\alpha}$ for all $\alpha \in \Delta_+(A)$. Now eq. (10) follows from

$$\begin{aligned}[x_\alpha, x_\beta] &= [e_\alpha + \omega(e_\alpha), e_\beta + \omega(e_\beta)] \\ &= [e_\alpha, e_\beta] + [\omega(e_\alpha), \omega(e_\beta)] + [e_\alpha, \omega(e_\beta)] + [\omega(e_\alpha), e_\beta] \\ &= \underbrace{[e_\alpha, e_\beta] + \omega([e_\alpha, e_\beta])}_{\in\mathfrak{k}_{\alpha+\beta}} + \underbrace{[e_\alpha, \omega(e_\beta)] + \omega([e_\alpha, \omega(e_\beta)])}_{\in\mathfrak{k}_{\alpha-\beta}}\end{aligned}$$

for $\alpha, \beta \in \Delta_+(A)$ such that $\alpha - \beta > 0$. □

1.3 Integrable highest weight modules

In this subsection I will summarize some facts about highest weight modules of split Kac-Moody-algebras.

Definition 1.33. (Category $\mathcal{O}$[15])
Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}(A)(\mathbb{K})$-module with finite-dimensional weight spaces. Let the set of weights $P(V)$ be such that there exist $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$ such that

$$P(V) \subset \bigcup_{j=1}^m D(\lambda_j), \quad D(\lambda_j) \coloneqq \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda_j\}.$$

The objects of the category $\mathcal{O}$ are $\mathfrak{g}(A)(\mathbb{K})$-modules V that satisfy the above properties. Its morphisms are homomorphisms of such $\mathfrak{g}(A)(\mathbb{K})$-modules.

Due to the properties of $\mathfrak{h}$-diagonalizable modules, the category $\mathcal{O}$ is closed under taking quotients, direct sums or tensor products. Also, any submodule $U \leq V$ of a module V in $\mathcal{O}$ is again an object of $\mathcal{O}$.

Definition 1.34. A $\mathfrak{g}(A)(\mathbb{K})$-module V is called a **highest-weight module** to the **highest weight** Λ if there exists $v_\Lambda \neq 0$ such that[16]

$$h.v_\Lambda = \Lambda(h)v_\Lambda \ \forall\, h \in \mathfrak{h},\ x.v_\Lambda = 0 \ \forall\, x \in \mathfrak{n}_+$$

and $V = \mathcal{U}(\mathfrak{n}_-)\, v_\Lambda$. One calls v_Λ a **highest weight vector**. A highest weight module $M(\Lambda)$ to the highest weight Λ is called a **Verma module** if every highest weight module to the highest weight Λ is a quotient of $M(\Lambda)$.

Verma modules are unique up to isomorphism and can be constructed by the use of induced modules. The next definition of an induced module is standard:

Definition 1.35. Let V be a $\mathfrak{g}_1$-module and let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of Lie algebras. Impose an equivalence relation $\sim$ on $\mathcal{U}(\mathfrak{g}_2) \otimes_{\mathbb{K}} V$ via bilinear extension of

$$y\phi(x) \otimes v \sim y \otimes x.v$$

and call

$$\mathcal{U}(\mathfrak{g}_2) \otimes_{\mathcal{U}(\mathfrak{g}_1)} V \coloneqq \mathcal{U}(\mathfrak{g}_2) \otimes_{\mathbb{K}} V \big/ \sim \tag{11}$$

the **induced $\mathfrak{g}_2$-module**.

[15] Compare [K90, sec. 9.1].
[16] Compare [K90, eqs. 9.2.1-3].

Proposition 1.36. *(Cp. [K90, prop. 1.2] and [K90, rem. 1.2])*
Let $\Lambda \in \mathfrak{h}^*$ and let $\mathfrak{g} = \mathfrak{g}(A)(\mathbb{K})$ be a symmetrizable split Kac-Moody algebra. Define the $(\mathfrak{n}_+ + \mathfrak{h})$-module $\mathbb{K}_\Lambda$ as the one-dimensional $\mathbb{K}$-vector space $\mathbb{K} \cdot v_\Lambda$ with action $x.v_\Lambda = 0\ \forall x \in \mathfrak{n}_+$ and $hv_\Lambda = \Lambda(h)\ \forall h \in \mathfrak{h}$. Then

$$M(\Lambda) \coloneqq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{n}_+ + \mathfrak{h})} \mathbb{K}_\Lambda \tag{12}$$

is a Verma module to the highest weight Λ. The Verma module to the highest weight Λ is unique up to isomorphism and it contains a unique maximal submodule $M'(\Lambda) \subsetneq M(\Lambda)$. Set

$$L(\Lambda) \coloneqq M(\Lambda) \diagup M'(\Lambda)\,. \tag{13}$$

For A of spherical type, I will sometimes denote $L(\omega)$ by Γ_ω for a dominant weight $\omega \in \mathfrak{h}^*$.

Definition 1.37. Let V be a $\mathfrak{g}(A)(\mathbb{K})$-module and let $v \in V_\lambda$. Then v is called **primitive**[17] if there exists a submodule $U \subset V$ such that $v \notin U$ but $x.v \in U\ \forall x \in \mathfrak{n}_+$. In this case, λ is called a **primitive weight**.

Obviously, highest weight vectors are always primitive, since then $\{0\}$ is a submodule such that the above demand is met.

Proposition 1.38. *(This is [K90, prop. 9.3])*
Let V be a $\mathfrak{g}(A)(\mathbb{K})$-module from the category $\mathcal{O}$. Then V is generated by its primitive vectors and there exists $\lambda \in P(V)$ and $0 \neq v \in V_\lambda$ such that $x.v_\lambda = 0$ for all $x \in \mathfrak{n}_+$. The following are equivalent:
(i) V is irreducible.
(ii) Any primitive vector is a highest weight vector and V is a highest weight module.
(iii) There exists $\Lambda \in \mathfrak{h}^*$ such that V is isomorphic to $L(\Lambda)$.

One has half of the Schur lemma for irreducible highest weight modules:

Lemma 1.39. *(Cp. [K90, lem. 9.3]) Let $L(\Lambda)$ be the irreducible highest weight module to the highest weight $\Lambda \in \mathfrak{h}^*$. Then the only $\mathfrak{g}(A)(\mathbb{K})$-intertwining linear maps $A : L(\Lambda) \to L(\Lambda)$ are of the form $A = c \cdot Id$ for some $c \in \mathbb{K}$.*

Definition 1.40. Consider the irreducible $\mathfrak{g}(A)(\mathbb{K})$-module $L(\Lambda)$. A nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ on $L(\Lambda)$ that satisfies

$$\langle gu, v\rangle = -\langle u, \omega(g)v\rangle\ \ \forall x \in \mathfrak{g},\ u, v \in L(\Lambda)$$

is called a **contravariant form**[18], where ω denotes the Chevalley involution.

Proposition 1.41. *(This is [K90, prop. 9.4]) Every $\mathfrak{g}(A)(\mathbb{K})$-module $L(\Lambda)$ as in (13) possesses a contravariant form $\langle\cdot,\cdot\rangle$ that is symmetric and unique up to a constant prefactor. For all $u \in V_\mu, v \in V_\lambda$ such that $\mu \neq \lambda$ it satisfies*

$$\langle u, v\rangle = 0\ . \tag{14}$$

Next, a result on complete reducibility.

Proposition 1.42. *(This is [K90, prop. 9.9])*
Let $\mathfrak{g}(A)(\mathbb{K})$ be symmetrizable and let $\rho \in \mathfrak{h}^*$ be such that $\rho(\alpha_i^\vee) = 1$ for all $i = 1, \ldots, n$. For $\Lambda \in \mathfrak{h}^*$ the Verma module $M(\Lambda)$ is irreducible if $2(\Lambda + \rho|\beta) \neq (\beta|\beta)$ for all $0 \neq \beta \in Q_+$. Let V be a $\mathfrak{g}(A)(\mathbb{K})$-module from the category $\mathcal{O}$ such that for any two primitive weights λ, μ of V with $0 < \lambda - \mu =: \beta$ one has that $2(\lambda + \rho|\beta) \neq (\beta|\beta)$. Then V is completely reducible.

[17] Compare [K90, sec. 9.3].
[18] Compare [K90, sec. 9.4].

I would like to conclude this section with an observation about highest weight modules in connection to the maximal compact subalgebra.

Proposition 1.43. *Let V be a $\mathfrak{g}(A)(\mathbb{K})$-highest weight module to the weight Λ and highest weight vector v_Λ. Then $\mathcal{U}(\mathfrak{k})\, v_\Lambda = V$.*

Proof. One has that $\mathcal{U}(\mathfrak{n}_-)\, v_\Lambda = V$ and one proves the claim by induction on the "depths" of weight spaces. Any weight $\lambda \in P(V)$ has the shape $\lambda = \Lambda - \alpha$ for $\alpha \in Q_+$ and one puts an $\mathbb{N}$-gradation on the weight spaces V_λ by setting $\deg(V_{\Lambda-\alpha}) = \mathrm{ht}(\alpha)$. Denote the corresponding decomposition of V by

$$V = \bigoplus_{n=0}^{\infty} V_n$$

and note that each V_n is finite-dimensional. As $\mathcal{U}(\mathfrak{k})$ is a unital associative algebra one has that $V_0 = \mathbb{K}v_\Lambda \subset \mathcal{U}(\mathfrak{k})\, v_\Lambda$, where the equality $V = \mathbb{K}v_\Lambda$ follows from the fact that V is a highest weight module. Now assume that $\bigoplus_{n=0}^{N} V_n \subset \mathcal{U}(\mathfrak{k})\, v_\Lambda$. As $V_{N+1} = \mathrm{span}_{\mathbb{K}}\{f_i v \mid i \in I,\ v \in V_n\}$ and $X_i v = (e_i - f_i)\, v$ one has that $f_i v = e_i v - X_i v$. But since $e_i v \in V_{n-1}$ if $v \in V_n$ there exist $y, z \in \mathcal{U}(\mathfrak{k})$ such that $e_i v = y v_\Lambda$ and $v = z v_\Lambda$ by induction. Hence,

$$f_i v = y v_\Lambda - X_i z v_\Lambda = (y - X_i z)\, v_\Lambda \in \mathcal{U}(\mathfrak{k})\, v_\Lambda$$

for all $i \in I$ and $v \in V_n$. This concludes the proof. □

2 A presentation of $\mathfrak{k}(E_n)(\mathbb{C})$ that is adapted to $\mathfrak{so}(n,\mathbb{C})$

In this section I will develop a description of the $\mathfrak{k}(E_n)(\mathbb{C})$-series that is adapted to its natural $\mathfrak{so}(n,\mathbb{C})$-subalgebra which is given in the form of a presentation result by generators and relations in prop. 2.8 together with various additional relations that are obtained along the way. I use these relations to show that $\mathfrak{k}(E_n)(\mathbb{C})$ is the quotient of a suitably chosen $\mathfrak{gim}$-algebra (see def. 2.5). It is known that $\mathfrak{gim}$-algebras admit a presentation as an involutory subalgebra of a Kac-Moody algebra (cp. [S84, B89]). The involution used is not the Chevalley involution but it is related to it by some sign- and diagram automorphisms. The converse statement that a maximal compact subalgebra of indefinite type is isomorphic to the quotient of a $\mathfrak{gim}$-algebra is new and provides another connection between the two objects.

For a split Kac-Moody algebra $\mathfrak{g}(A)(\mathbb{K})$ and a Dynkin subdiagram $\mathcal{D}(B) \subset \mathcal{D}(A)$ one always has a natural inclusion $\mathfrak{g}'(B)(\mathbb{K}) \subset \mathfrak{g}(A)(\mathbb{K})$ of subalgebras as $\mathfrak{g}'(B)(\mathbb{K}) = \langle e_i, f_i, \alpha_i^\vee \mid i \in \mathcal{V}_{\mathcal{D}(B)} \rangle$, where $\mathcal{V}_{\mathcal{D}(B)}$ denotes the set of vertices of $\mathcal{D}(B)$ and $\mathfrak{g}'(B)(\mathbb{K})$ is the derived subalgebra $\mathfrak{g}'(B)(\mathbb{K}) := [\mathfrak{g}(B)(\mathbb{K}), \mathfrak{g}(B)(\mathbb{K})]$ of $\mathfrak{g}(B)(\mathbb{K})$. If B is regular then there is no difference between the two as the realization of B will coincide with $\mathrm{span}_{\mathbb{K}}\{\alpha_i^\vee \mid i \in \mathcal{V}_{\mathcal{D}(B)}\}$. The Dynkin diagram of the E_n-series can be viewed as an A_{n-1}-diagram with an additional, exceptional, node. Therefore $\mathfrak{g}(E_n)(\mathbb{K})$ contains $\mathfrak{g}(A_{n-1})(\mathbb{K})$ naturally as a subalgebra by restriction to the sub-diagram $\mathcal{D}(A_{n-1}) \subset \mathcal{D}(E_n)$. As the Chevalley involution ω is compatible with the restriction to subdiagrams, i.e. $\omega(\mathfrak{g}'(B)(\mathbb{K})) \subseteq \mathfrak{g}'(B)(\mathbb{K})$ whenever $\mathcal{D}(B) \subset \mathcal{D}(A)$ one has that $\mathfrak{k}(B)(\mathbb{K}) \subset \mathfrak{k}(A)(\mathbb{K})$. This can also be seen from their presentations since

$$\begin{aligned} \mathfrak{k}(B)(\mathbb{K}) &\cong \left\langle X_i,\, i \in \mathcal{V}_{\mathcal{D}(B)} \mid P_{-a_{ij}}(\mathrm{ad}X_i)(X_j) = 0\ \forall\, i,j \in \mathcal{V}_{\mathcal{D}(B)} \right\rangle, \\ \mathfrak{k}(A)(\mathbb{K}) &\cong \left\langle X_i,\, i \in \mathcal{V}_{\mathcal{D}(A)} \mid P_{-a_{ij}}(\mathrm{ad}X_i)(X_j) = 0\ \forall\, i,j \in \mathcal{V}_{\mathcal{D}(A)} \right\rangle. \end{aligned}$$

Hence, $\mathfrak{k}(E_n)(\mathbb{R})$ contains $\mathfrak{k}(A_{n-1})(\mathbb{R}) \cong \mathfrak{so}(n,\mathbb{R})$ naturally as a subalgebra. However, this $\mathfrak{so}(n,\mathbb{R})$ is not in split form. After complexification one has that $\mathfrak{k}(E_n)(\mathbb{C})$ allows a natural $D_m(\mathbb{C})$-, resp. $B_m(\mathbb{C})$-subalgebra (with $m = \lfloor \frac{n}{2} \rfloor$) which is the complexification of $\mathfrak{k}(A_{n-1})(\mathbb{R}) \cong \mathfrak{so}(n, \mathbb{R})$. I will provide this description in terms of the Chevalley generators of $D_m(\mathbb{C})$, resp. $B_m(\mathbb{C})$, the exceptional Berman generator X_n and some relations among them in prop. 2.8. The Cartan subalgebra of $\mathfrak{k}(A_{n-1})(\mathbb{C})$ does not act diagonally on X_n via the adjoint action which is why I replace X_n by two elements $X_\pm$ that are diagonal. This provides a description of $\mathfrak{k}(E_n)(\mathbb{C})$ that is graded by the root lattice of $D_m(\mathbb{C})$, resp. $B_m(\mathbb{C})$. Unfortunately it is not graded by the respective root systems and it is unlikely that homogeneous components are finite-dimensional. Especially the 0-eigenspace is likely to be infinite-dimensional. Such phenomena are known to occur for so-called generalized intersection matrix algebras ($\mathfrak{gim}$-algebra for short). It turns out that there exists an epimorphism of Lie algebras from a suitably constructed $\mathfrak{gim}$-algebra to $\mathfrak{k}(E_n)(\mathbb{C})$ which I will show in prop. 2.7.

The section is structured as follows: First, an explicit realization of the Chevalley generators is given in terms of the Berman generators of $\mathfrak{k}(A_{n-1})(\mathbb{C})$ in section 2.1 where the more technical parts can be found in the appendix A.1. This explicit realization is used to obtain relations among the Chevalley generators and $X_\pm$ (resp. X_n) in section 2.3. The definition of $\mathfrak{gim}$-algebras is given in section 2.2 and the result that $\mathfrak{k}(E_n)(\mathbb{C})$ is a quotient of a $\mathfrak{gim}$-algebra is proven in section 2.3. In section 2.4 I briefly sketch the consequences that this presentation has for finite-dimensional $\mathfrak{k}(E_n)(\mathbb{C})$-modules or rather for the search of such modules.

2.1 Fixing notation and the root space decomposition

According to cor. 1.31, the maximal compact subalgebra $\mathfrak{k}(E_n)(\mathbb{R})$ of split-real E_n is generated by its n Berman generators $X_1, \ldots, X_n$ which are subject to the relations

$$\begin{aligned} [X_i, X_j] &= 0 \quad \text{if } (i,j) \notin \mathcal{E} \\ [X_i, [X_i, X_j]] &= -X_j \text{ if } (i,j) \in \mathcal{E} , \end{aligned}$$

where $\mathcal{E}$ denotes the set of edges in $\mathcal{D}(E_n)$. I use the convention that the n-th node corresponds to the one that sets apart $\mathcal{D}(E_n)$ from $\mathcal{D}(A_{n-1})$ (compare figure 1) but that the n-th node attaches to the third node. As $\mathfrak{g}(A_{n-1})(\mathbb{R})$ is split it is isomorphic to $\mathfrak{sl}(n, \mathbb{R})$ and so one deduces $\mathfrak{k}(A_{n-1})(\mathbb{R}) \cong \mathfrak{so}(n, \mathbb{R})$. Since $\mathcal{D}(A_{n-1})$ is a subdiagram of $\mathcal{D}(E_n)$ one has that $\mathfrak{k}(A_{n-1})(\mathbb{R})$ is a subalgebra of $\mathfrak{k}(E_n)(\mathbb{R})$ in such a way that it is generated by $X_1, X_2, \ldots X_{n-1}$.

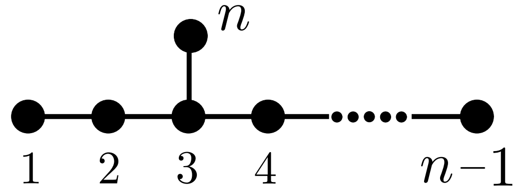

Figure 1: The Dynkin diagram of E_n. Its A_{n-1}-subdiagram is obtained upon deletion of node n.

For $n = 2m$, $\mathfrak{so}(n, \mathbb{R})$ is the real compact form of $\mathfrak{so}(2m, \mathbb{C}) \cong D_m(\mathbb{C})$ whereas for $n = 2m + 1$ it is the real compact form of $\mathfrak{so}(2m+1, \mathbb{C}) \cong B_m(\mathbb{C})$. In the following, I will determine a basis for both cases that is adjusted to the root space decomposition of B_m, D_m respectively. My approach is derived from the treatment of classical compact simple Lie algebras in [C84, app. G], although most choices that have to be made along the way are rather natural.

First, fix a Cartan subalgebra for $\mathfrak{k}(A_{n-1})(\mathbb{C})$. Since all Cartan subalgebras of finite-dimensional simple Lie algebras are conjugate one can just pick any abelian subalgebra $\mathfrak{h}$ of suitable dimension. If the adjoint action of $\mathfrak{h}$ on $\mathfrak{k}(A_{n-1})(\mathbb{C})$ can be diagonalized one then knows that $\mathfrak{h}$ is in fact a Cartan subalgebra. Choose

$$H_j := -iX_{2j-1}, \text{ for } j = 1, \ldots, m \tag{15}$$

as the generators of a distinguished abelian subalgebra

$$\mathfrak{h} := \operatorname{span}_{\mathbb{C}}\{H_1, \ldots, H_m\}. \tag{16}$$

For $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ set

$$X_{\alpha_{i_1}+\cdots+\alpha_{i_k}} := [X_{i_1}, [X_{i_2}, [\ldots, [X_{i_{k-1}}, X_{i_k}]]]] \tag{17}$$

and note that the order in the sum $\alpha_{i_1}+\cdots+\alpha_{i_k}$ matters but as long as $i_1, i_2, \ldots, i_k$ are such that $\{i_1, \ldots, i_k\}$ is a connected subdiagram of A_{n-1} it is quite unambiguous because an A_{n-1}-root is defined by its support. I will sometimes also write

$$X^{(i_1,\ldots,i_k)} := [X_{i_1}, [X_{i_2}, [\ldots, [X_{i_{k-1}}, X_{i_k}]]]] \tag{18}$$

in cases where the other definition is too unwieldy. For $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$ one also defines

$$e_\beta := [e_{i_1}, [e_{i_2}, [\ldots, [e_{i_{k-1}}, e_{i_k}]]]] \in \mathfrak{g}(E_n)_\beta \ , \ e_{-\beta} := -\omega(e_\beta) \in \mathfrak{g}(E_n)_{-\beta} \ , \ X_\beta := e_\beta - e_{-\beta}. \tag{19}$$

Here, the way β is decomposed into simple roots implicitly defines the structure constants. If the $i_1 \ldots, i_k$ are pairwise different, the nested commutators (17) and the description via (19) are related as follows:

$$\begin{aligned} X_{\alpha_{i_1}+\cdots+\alpha_{i_k}} &= [X_{i_1}, [X_{i_2}, [\ldots, [X_{i_{k-1}}, X_{i_k}]]]] \\ &= e_{\alpha_{i_1}+\cdots+\alpha_{i_k}} + (-1)^k [f_{i_1}, [f_{i_2}, [\ldots, [f_{i_{k-1}}, f_{i_k}]]]] \\ &= e_{\alpha_{i_1}+\cdots+\alpha_{i_k}} + (-1)^{2k} [\omega(e_{i_1}), [\omega(e_{i_2}), [\ldots, [\omega(e_{i_{k-1}}), \omega(e_{i_k})]]]] \\ &= e_{\alpha_{i_1}+\cdots+\alpha_{i_k}} + \omega\left(e_{\alpha_{i_1}+\cdots+\alpha_{i_k}}\right) = e_{\alpha_{i_1}+\cdots+\alpha_{i_k}} - e_{-\alpha_{i_1}-\cdots-\alpha_{i_k}}. \end{aligned}$$

In the above computation it is crucial that the $i_1 \ldots, i_k$ are pairwise different because only then one can ensure that commutators such as $[e_i, [f_{i_j}, [\ldots, [f_{i_{k-1}}, f_{i_k}]]]]$ vanish.

For $1 \leq i < j \leq m$ define roots $\beta_{i,j}^{(1)}, \ldots, \beta_{i,j}^{(4)} \in \Delta(A_{n-1}) \subset \Delta(E_n)$ and an order of how they are to be constructed from simple roots by

$$\beta_{i,j}^{(1)} = \alpha_{2i} + \cdots + \alpha_{2j-1} \ , \ \beta_{i,j}^{(2)} = \alpha_{2i} + \cdots + \alpha_{2j-2} \tag{20}$$

$$\beta_{i,j}^{(3)} = \alpha_{2i-1} + \cdots + \alpha_{2j-1} \ , \ \beta_{i,j}^{(4)} = \alpha_{2i-1} + \cdots + \alpha_{2j-2}, \tag{21}$$

where the order of the summands in the above equations is not to be altered. Now for $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ set[19]

$$e_{\varepsilon_1 L_i + \varepsilon_2 L_j} := \frac{i}{2} \cdot \left(X_{\beta_{i,j}^{(1)}} - i\varepsilon_2 X_{\beta_{i,j}^{(2)}} - i\varepsilon_1 X_{\beta_{i,j}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(4)}}\right) \tag{22}$$

for $1 \leq i < j \leq m$. Furthermore, if $n = 2m+1$ set

$$e_{\pm L_j} := i \cdot \left(X_{\alpha_{2j}+\cdots+\alpha_{n-1}} \mp iX_{\alpha_{2j-1}+\cdots+\alpha_{n-1}}\right) = i \cdot \left(X_{\beta_{j,m+1}^{(2)}} \mp iX_{\beta_{j,m+1}^{(4)}}\right). \tag{23}$$

[19]My choices for the Cartan subalgebra in (15) and the Chevalley generators in (22) and (23) are close to the ones in [C84, app. G.2] but my normalization differs.

Remark. For even n the support of the $\beta_{i,j}^{(k)}$ for $k \in \{1,2,3,4\}$ and $1 \leq i < j \leq m$ ranges over the entire A_{n-1}-subdiagram whereas for odd n they never use the root α_{n-1}. This root only appears in the definition of $e_{\pm L_j}$. Effectively, the elements (22) generate the subalgebra $\mathfrak{k}(A_{n-2})$ of type D_m that one obtains from the nodes 1 to $n-2$. Adding X_{n-1} or equivalently (23) to the list of generators produces a Lie algebra of type B_m where all short root operators depend on Berman elements whose support contains α_{n-1}.

Lemma 2.1. *Consider the linear functionals $L_i : \mathfrak{h}^* \to \mathbb{C}$ defined via $L_i(H_j) = \delta_{ij}$. Then with the above definitions (15), (22) and (23) one has with $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$ that*

$$\left[h, e_{\varepsilon_1 L_i + \varepsilon_2 L_j}\right] = (\varepsilon_1 L_i + \varepsilon_2 L_j)(h)\, e_{\varepsilon_1 L_i + \varepsilon_2 L_j} \;\; \forall\, h \in \mathfrak{h} \tag{24}$$

and for $n = 2m+1$ one additionally has

$$\left[h, e_{\pm L_j}\right] = \pm L_j(h)\, e_{\pm L_j} \;\; \forall\, h \in \mathfrak{h}. \tag{25}$$

Thus, (22) and (23) provide a root space decomposition of $\mathfrak{k}(A_{n-1})(\mathbb{C})$ w.r.t. the Cartan subalgebra $\mathfrak{h}$ spanned by (15). Note that while the elements $e_{\pm L_j}$ exist in $\mathfrak{k}(A_{n-1})(\mathbb{C})$ for $n = 2m$ eq. (25) is not valid in this case.

Proof. Observe that because of $[X_i, [X_i, X_j]] = -X_j$ for $(i,j) \in \mathcal{E}$ one has

$$\begin{aligned} \left[X_{2i-1}, X_{\beta_{i,j}^{(3)}}\right] &= [X_{2i-1}, [X_{2i-1}, [X_{2i}, \ldots X_{2j-1}]]] \\ &= \left[[X_{2i-1}, [X_{2i-1}, X_{2i}]], X_{\alpha_{2i+1}+\cdots+\alpha_{2j-1}}\right] + 0 \\ &= -\left[X_{2i}, X_{\alpha_{2i+1}+\cdots+\alpha_{2j-1}}\right] = -X_{\beta_{i,j}^{(1)}} \end{aligned} \tag{26}$$

and

$$\left[X_{2i-1}, X_{\beta_{i,j}^{(4)}}\right] = -X_{\beta_{i,j}^{(2)}}. \tag{27}$$

Also,

$$\left[X_{2i-1}, X_{\beta_{i,j}^{(1)}}\right] = X_{\beta_{i,j}^{(3)}}, \;\; \left[X_{2i-1}, X_{\beta_{i,j}^{(2)}}\right] = X_{\beta_{i,j}^{(4)}}, \tag{28}$$

which leads to ($\varepsilon_i^{-1} = \varepsilon_i$)

$$\begin{aligned} \left[H_i, e_{\varepsilon_1 L_i + \varepsilon_2 L_j}\right] &= -\frac{i^2}{2}\left[X_{2i-1}, X_{\beta_{i,j}^{(1)}} - i\varepsilon_2 X_{\beta_{i,j}^{(2)}} - i\varepsilon_1 X_{\beta_{i,j}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(4)}}\right] \\ &= \frac{1}{2}\left(X_{\beta_{i,j}^{(3)}} - i\varepsilon_2 X_{\beta_{i,j}^{(4)}} + i\varepsilon_1 X_{\beta_{i,j}^{(1)}} + \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(2)}}\right) \\ &= \frac{i\varepsilon_1}{2}\left(X_{\beta_{i,j}^{(1)}} - i\varepsilon_2 X_{\beta_{i,j}^{(2)}} - i\varepsilon_1 X_{\beta_{i,j}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(4)}}\right) \\ &= \varepsilon_1 e_{\varepsilon_1 L_i + \varepsilon_2 L_j}. \end{aligned}$$

Also, one computes

$$\begin{aligned} \left[X_{2j-1}, X_{\beta_{i,j}^{(1)}}\right] &= \left[X_{2j-1}, X_{\alpha_{2i}+\cdots+\alpha_{2j-1}}\right] = [X_{2j-1}, [X_{2i}, \ldots, [X_{2j-2}, X_{2j-1}]]] \\ &= -[X_{2i}, \ldots [X_{2j-1}, [X_{2j-1}, X_{2j-2}]]] = [X_{2i}, \ldots [X_{2j-3}, X_{2j-2}]] \\ &= X_{\beta_{i,j}^{(2)}} \end{aligned}$$

and similarly

$$\left[X_{2j-1}, X_{\beta_{i,j}^{(2)}}\right] = -X_{\beta_{i,j}^{(1)}},\ \left[X_{2j-1}, X_{\beta_{i,j}^{(3)}}\right] = X_{\beta_{i,j}^{(4)}},\ \left[X_{2j-1}, X_{\beta_{i,j}^{(4)}}\right] = -X_{\beta_{i,j}^{(3)}}.$$

This then yields

$$\begin{aligned}
\left[H_j, e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right] &= -\frac{i^2}{2}\left[X_{2j-1}, X_{\beta_{i,j}^{(1)}} - i\varepsilon_2 X_{\beta_{i,j}^{(2)}} - i\varepsilon_1 X_{\beta_{i,j}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(4)}}\right] \\
&= \frac{1}{2}\left(X_{\beta_{i,j}^{(2)}} + i\varepsilon_2 X_{\beta_{i,j}^{(1)}} - i\varepsilon_1 X_{\beta_{i,j}^{(4)}} + \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(3)}}\right) \\
&= \frac{i}{2}\varepsilon_2\left(X_{\beta_{i,j}^{(1)}} - i\varepsilon_2 X_{\beta_{i,j}^{(2)}} - i\varepsilon_1 X_{\beta_{i,j}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(4)}}\right) \\
&= \varepsilon_2 e_{\varepsilon_1 L_i+\varepsilon_2 L_j}
\end{aligned}$$

The relation

$$\left[H_j, e_{\varepsilon_1 L_i+\varepsilon_2 L_k}\right] = 0\ \forall\, i \neq j \neq k$$

can be seen from the A_{n-1}-root system. In this case $\beta_{i,k}^{(l)} \pm \alpha_{2j-1} \notin \Delta(A_{n-1})$ because either a root appears with coefficient 2 if $\alpha_{2j-1} \in \operatorname{supp}\left(\beta_{i,k}^{(l)}\right)$ or $\operatorname{supp}\left(\beta_{i,k}^{(l)} \pm \alpha_{2j-1}\right)$ is disconnected. This shows with the filtered structure of $\mathfrak{k}$ (see lemma 1.32) that $\left[X_{2j-1}, X_{\beta_{i,k}^{(l)}}\right] = 0$. Also, by the same argument

$$\left[H_i, e_{\pm L_j}\right] = -i^2\left[X_{2i-1}, X_{\beta_{j,m+1}^{(2)}} \mp iX_{\beta_{j,m+1}^{(4)}}\right] = 0\ \forall\, i \neq j$$

as H_{m+1} does not exist. Finally,

$$\begin{aligned}
\left[H_j, e_{\pm L_j}\right] &= -i^2\left[X_{2j-1}, X_{\beta_{j,m+1}^{(2)}} \mp iX_{\beta_{j,m+1}^{(4)}}\right] \\
&= X_{\beta_{j,m+1}^{(4)}} \pm iX_{\beta_{j,m+1}^{(2)}} = \pm i\cdot\left(X_{\beta_{j,m+1}^{(2)}} \mp iX_{\beta_{j,m+1}^{(4)}}\right) \\
&= \pm e_{\pm L_j}.
\end{aligned}$$

□

The previous lemma both provides a link between the Berman elements and a basis of $\mathfrak{so}(n,\mathbb{C})$ that is adapted to the respective root system. Its proof shows that it can be tedious to work out all relations of interest which is why all these computations are collected in the appendix A.1. The result is:

Proposition 2.2. *Let H_i, $e_{\varepsilon_1 L_i+\varepsilon_2 L_j}$ and $e_{\pm L_i}$ (only for $n = 2m+1$) be as in (15), (22) and (23). Then one has for $1 \leq i < j < k$*

$$\begin{aligned}
\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_j}, e_{-\varepsilon_1 L_i-\varepsilon_2 L_j}\right] &= \varepsilon_1 H_i + \varepsilon_2 H_j \\
\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_{i+1}}, e_{-\varepsilon_2 L_{i+1}+\varepsilon_3 L_{i+k}}\right] &= i\cdot e_{\varepsilon_1 L_i+\varepsilon_3 L_{i+k}} \\
\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_k}, e_{\varepsilon_3 L_j-\varepsilon_2 L_k}\right] &= -ie_{\varepsilon_1 L_i+\varepsilon_3 L_j}
\end{aligned}$$

and

$$\left[e_{\varepsilon_1 L_i}, e_{\varepsilon_2 L_j}\right] = -2ie_{\varepsilon_1 L_i+\varepsilon_2 L_j},\ \left[e_{L_i}, e_{-L_i}\right] = 2H_i$$

$$\left[e_{-\varepsilon_1 L_i}, e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right] = ie_{\varepsilon_2 L_j},\quad \left[e_{-\varepsilon_2 L_j}, e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right] = -ie_{\varepsilon_1 L_i}.$$

2.2 GIM-Lie-algebras

GIM-Lie-algebras, where GIM is a shorthand for Generalized Intersection Matrix, are constructed similarly to Kac-Moody algebras. One starts from a so-called generalized intersection matrix A (see def. 2.3) where one replaces the condition $A_{ij} \leq 0$ for $i \neq j$ by $A_{ij} \leq 0 \Leftrightarrow A_{ji} \leq 0$ and $A_{ij} > 0 \Leftrightarrow A_{ji} > 0$. Due to the work [S84] of Slodowy, it is known that GIM-Lie-algebras fall into two classes. The first class consists of those GIM-Lie-algebras which are in fact isomorphic to a Kac-Moody-algebra and the second class are those which are isomorphic to an involutory subalgebra of a Kac-Moody-algebra. Such involutory subalgebras are studied in Berman's paper [B89], where a description of them is provided via generators and relations. I will collect the most essential definitions and results from [S84], but the ones relevant for this work can also be found in [B89, sec. 2] which is more accessible.

Definition 2.3. (GIM)[20] Let $A \in \mathbb{Z}^{l\times l}$ such that

$$\begin{array}{rrcl} (i) & A_{ii} & = & 2\ \forall\, i = 1, \ldots, l \\ (ii) & A_{ij} & < & 0 \ \Leftrightarrow\ A_{ji} < 0\ \forall\, i \neq j \\ (iii) & A_{ij} & > & 0 \ \Leftrightarrow\ A_{ji} > 0\ \forall\, i \neq j, \end{array}$$

then A is called a **generalized intersection matrix** (GIM). As in def. 1.4, A is called symmetrizable if there exist $D, B \in \mathbb{Q}^{l\times l}$ such that D is diagonal and B is symmetric and it holds $A = DB$. One calls A simply-laced if $|A_{ij}| \in \{0, 1\}$ for all $i \neq j$.

Note that GIMs possess a unique realization $(\mathfrak{h}, \Pi^\vee, \Pi)$ just as GCMs (compare 1.2 and the following paragraph).To a GIM one can associate a Dynkin-like diagram.

Definition 2.4. (GIM-diagram)[21] Let $A \in \mathbb{Z}^{n\times n}$ be a GIM and construct a diagram $\mathcal{C}(A)$ with n vertices and edges according to the following rules:

1. Two vertices i and j are connected by a dotted edge if $A_{ij} = \alpha_j(\alpha_i^\vee) > 0$.
2. Two vertices i and j are connected by a solid edge if $A_{ij} = \alpha_j(\alpha_i^\vee) < 0$.
3. There is no edge between two vertices i and j if $A_{ij} = 0$.
4. The edges (i, j) are equipped with the ordered pair $(|A_{ij}|, |A_{ji}|)$ but if $|A_{ij}| = 1 = |A_{ji}|$ one draws a single edge instead. Also, if $A_{ij}A_{ji} \leq 4$ one draws $\max(|A_{ij}|, |A_{ji}|)$ edges between the vertices with an arrow pointing towards i if $|A_{ij}| > 1$ and an arrow pointing towards j if $|A_{ji}| > 1$. Note that this may result in an arrow $\Leftrightarrow$ in the case that $|A_{ij}| = 2 = |A_{ji}|$.

Definition 2.5. (GIM-Lie-algebra)[22] Let A be a GIM with $\mathbb{C}$-realization $(\mathfrak{h}, \Pi^\vee, \Pi)$ and let $\mathfrak{f}$ be the free Lie algebra over $\mathbb{C}$ generated by $\mathfrak{h}$ and elements $e_\alpha, e_{-\alpha}$ for $\alpha \in \Pi$. Let $\mathfrak{I}$ be the ideal in $\mathfrak{f}$ generated by the relations (identify $h_{\pm\alpha} \equiv \pm\alpha^\vee$ in the last line)

$$\begin{array}{rcl} [h, h'] & = & 0\ \forall\, h, h' \in \mathfrak{h}, \\ [h, e_\alpha] & = & \alpha(h)e_\alpha\ \forall\, h \in \mathfrak{h},\ \alpha \in \Pi, \\ [e_\alpha, e_{-\alpha}] & = & \alpha^\vee\ \forall\, \alpha \in \Pi, \\ \mathrm{ad}\,(e_\alpha)^{\max(1,1-\beta(h_\alpha))}\,(e_\beta) & = & 0\ \forall\, \alpha, \beta \in \pm\Pi. \end{array}$$

[20] Cp. [S84, 4.1]
[21] Cp. [S84, 4.2], but the conventions concerning the edges' labeling is closer to the conventions of [K90].
[22] Cp. [S84, 4.4]

Then $\mathfrak{gim}(A) := \mathfrak{f}/\mathfrak{I}$ is called the **GIM-Lie-algebra** to A .

Starting from a GIM-Lie-algebra $\mathfrak{gim}(A)$ with a symmetrizable GIM $A \in \mathbb{Z}^{l\times l}$ construct a generalized Dynkin diagram as follows. Take the GIM-diagram $\mathcal{C}(A)$ and double all the vertices, where the labels shall be $1,\dots,l,\hat{1},\dots,\hat{l}$. If $A_{ij} < 0$ connect the vertices i and j with a line labeled with an ordered pair $(|A_{ij}|,|A_{ji}|)$ and do the same with $\hat{i}$, $\hat{j}$. If $A_{ij} > 0$ connect i and $\hat{j}$ with a line labeled with an ordered pair $(|A_{ij}|,|A_{ji}|)$ and do the same with $\hat{i}$ and j. This way one obtains a generalized Dynkin diagram $\mathcal{D}\left(\widetilde{A}\right)$ as in def. 1.18 and a Kac-Moody-algebra $\mathfrak{g}\left(\widetilde{A}\right)$ associated to $\mathfrak{gim}(A)$. If the GIM-diagram $\mathcal{C}(A)$ is connected one distinguishes two cases:

1. $\mathcal{D}\left(\widetilde{A}\right)$ is disconnected. In this case one says that $\mathcal{C}(A)$ is oriented. By [S84, prop. 4.6] $\mathcal{D}\left(\widetilde{A}\right)$ decomposes into two isomorphic connected pieces $\mathcal{D}(A_0) \oplus \mathcal{D}(A_0)$ and $\mathfrak{gim}(A)$ is isomorphic to the Kac-Moody-algebra $\mathfrak{g}(A_0)$.

2. $\mathcal{D}\left(\widetilde{A}\right)$ is connected. In this case one calls $\mathcal{C}(A)$ unoriented. Then by [S84, prop. 4.8] (but see also [B89, prop. 2.1]) $\mathfrak{gim}(A)$ is isomorphic to $\mathfrak{s}_\sigma\left(\widetilde{A}\right)$ (recall the def. 1.27 and its presentation 1.29 that is not contained in [S84]) where the automorphism is given by $\sigma = \eta\gamma$ with η as in 6 and γ (cp. eq. 8) associated to the diagram automorphism $\pi(i) = \hat{i}$,$\pi(\hat{i}) = i$ for all $i = 1,\dots,l$.

Due to the involved diagram automorphism the resulting involutory subalgebra is quite different from the case I want to consider. Recall from 1.29 that the generators of $\mathfrak{s}_\sigma$ are

$$x_i = e_i + f_{\hat{i}}\ ,\ x_{\hat{i}} = e_{\hat{i}} + f_i\ ,\ z_i = \alpha_i^\vee - \alpha_{\hat{i}}^\vee\,.$$

Denote by $\mathcal{E}$ the solid edges of A and by $\mathcal{F}$ the dotted ones. In the simply-laced case they satisfy the relations

$$[x_i, x_j] = 0 \text{ if } (i,j) \notin \mathcal{E}\ ,\ \operatorname{ad}(x_i)^2(x_j) = 0 \text{ if } (i,j) \in \mathcal{E}$$

$$\left[x_i, x_{\hat{j}}\right] = 0 \text{ if } (i,j) \notin \mathcal{F}\ ,\ \operatorname{ad}(x_i)^2\left(x_{\hat{j}}\right) = 0 \text{ if } (i,j) \in \mathcal{F}$$

$$[z_j, z_k] = 0\ ,\ [z_j, x_k] = \left(A_{kj} - A_{\pi(k)j}\right)x_k\ ,\ \left[x_i, x_{\hat{j}}\right] = \delta_{i\pi(\hat{j})} z_i\ .$$

The relations of the form $[X_i,[X_i,X_j]] = -X_j$ are completely absent which is why one has to be a little more resourceful to obtain a relation between GIM-Lie-algebras and the maximal compact subalgebra $\mathfrak{k}(E_n)(\mathbb{C})$.

2.3 The adapted presentation

In the description via Berman generators, $X_1,\dots,X_{n-1}$ generate $\mathfrak{so}(n,\mathbb{C})$ and only including X_n gives rise to full $\mathfrak{k}(E_n)(\mathbb{C})$. This Berman generator is not $\mathfrak{h}$-diagonal (with $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$ as in (16)), the following two elements however, are:

$$X_\pm := i\left(X_n \mp i\,[X_3, X_n]\right) = i\left(X_{\alpha_n} \pm iX_{\alpha_n+\alpha_3}\right) \tag{29}$$

This is verified easily as

$$[H_j, X_\pm] = 0\ \forall\, j \neq 2$$

and

$$\begin{aligned}[H_2, X_\pm] &= -i^2 [X_3, X_n \mp i [X_3, X_n]] = [X_3, X_n] \pm i X_n \\ &= \pm i (X_n \mp i [X_3, X_n]) = \pm X_\pm.\end{aligned}$$

Since $X_+ + X_- = 2iX_n$ one sees that

$$\langle X_1, \ldots, X_{n-1}, X_n \rangle_{\mathbb{C}} = \langle X_1, \ldots, X_{n-1}, X_+, X_- \rangle_{\mathbb{C}} = \mathfrak{k}(E_n)(\mathbb{C}).$$

One has the following relations among the $e_{\varepsilon_1 L_i + \varepsilon_2 L_j} \in \mathfrak{so}(n, \mathbb{C}) < \mathfrak{k}(E_n)(\mathbb{C})$ and $X_\pm$. All stated elements are nonzero unless explicitly stated and their explicit form in terms of Berman elements is computed in appendix A.2:

$$[X_+, X_-] = 2H_2, \ [H_2, X_\pm] = \pm X_\pm, \ [H_i, X_\pm] = 0 \ \forall\, i \neq 2, \tag{30}$$

$$[X_+, e_{\varepsilon L_1 - L_2}] = [X_-, e_{\varepsilon L_1 + L_2}], \quad [X_+, e_{-L_2 + \varepsilon L_3}] = -[X_-, e_{L_2 + \varepsilon L_3}], \tag{31}$$

$$\mathrm{ad}(X_\pm)^2 (e_{\varepsilon L_1 \mp L_2}) = 2 e_{\varepsilon L_1 \pm L_2}, \ \mathrm{ad}(X_\pm)^2 (e_{\mp L_2 + \varepsilon L_3}) = -2 e_{\pm L_2 + \varepsilon L_3}, \tag{32}$$

$$0 = [X_\pm, e_{\varepsilon L_1 \pm L_2}] = [X_+, e_{+L_2 + \varepsilon L_3}] = [X_-, e_{-L_2 + \varepsilon L_3}] = \left[X_\pm, e_{\varepsilon_1 L_i + \varepsilon_2 L_j}\right] \ \forall\, 2 < i < j, \tag{33}$$

$$0 = \mathrm{ad}(X_\pm)^2 \left(e_{\varepsilon_1 L_1 + \varepsilon_2 L_j}\right) = \mathrm{ad}\left(e_{\varepsilon_1 L_1 + \varepsilon_2 L_j}\right)^2 (X_\pm) \ \ \forall\, j \geq 3 \tag{34}$$

For $n = 2m+1$ there are additional relations among $X_\pm$ and $e_{\varepsilon L_i}$:

$$[X_\pm, e_{\varepsilon L_1}] \neq 0, \quad 0 = \mathrm{ad}(X_\pm)^2 (e_{\varepsilon L_1}) = \mathrm{ad}(e_{\varepsilon L_1})^2 (X_\pm), \tag{35}$$

$$\mathrm{ad}(X_\pm)^2 (e_{\mp L_2}) = -2 e_{\pm L_2}, \quad \mathrm{ad}(e_{\pm L_2})^2 (X_\mp) = -2 X_\pm, \quad [X_+, e_{-L_2}] = -[X_-, e_{+L_2}] \tag{36}$$

$$0 = [X_\pm, e_{\pm L_2}] = \mathrm{ad}(X_\pm)^3 (e_{\mp L_2}) = \mathrm{ad}(e_{\pm L_2})^3 (X_\mp) = [X_\pm, e_{\varepsilon_1 L_i}] \ \ \forall\, i > 2. \tag{37}$$

Lemma 2.6. *Consider the elements $X_\pm$ defined in (29) and for $n = 2m$ denote the Chevalley generators of $\mathfrak{so}(n, \mathbb{C}) < \mathfrak{k}(E_n)(\mathbb{C})$ by*

$$e_i := e_{L_i - L_{i+1}}, \ f_i := e_{-L_i + L_{i+1}}, \ h_i := H_i - H_{i-1} \ \ i = 1, \ldots, m-1$$

$$e_m := e_{L_{m-1} + L_m}, \ f_m := e_{-L_{m-1} + L_m}, \ h_m := H_{m-1} + H_{m+1}.$$

For $n = 2m+1$ replace e_m, f_m and h_m by $e_m := e_{L_m}$, $f_m := e_{-L_m}$, $h_m := 2H_m$. Then one has the following relations:

$$[X_+, y] = 0 \ \forall\, y \in \{f_1, e_2\} \cup \{e_3, \ldots, e_m, f_3, \ldots, f_m\} \tag{38}$$

$$ad(X_+)^3 (y) = 0 = ad(y)^2 (X_+) \ \forall\, y \in \{e_1, f_2\} \cup \{e_3, \ldots, e_m, f_3, \ldots, f_m\} \tag{39}$$

$$[X_-, y] = 0 \ \forall\, y \in \{e_1, f_2\} \cup \{e_3, \ldots, e_m, f_3, \ldots, f_m\} \tag{40}$$

$$ad(X_-)^3 (y) = 0 = ad(y)^2 (X_-) \ \forall\, y \in \{f_1, e_2\} \cup \{e_3, \ldots, e_m, f_3, \ldots, f_m\} \tag{41}$$

Proof. The relations among $X_\pm$ and the elements of $\mathfrak{so}(n, \mathbb{C})$ are computed in various lemmas in section A.2 and are stated in eqs. (30-37). ☐

Consider the following GIMs $B_m^\diamond, D_m^\diamond \in \mathbb{Z}^{(m+1)\times(m+1)}$,

$$B_m^\diamond = \begin{pmatrix} 2 & -2 & 2 & & & & \\ -1 & 2 & -1 & & & & \\ 1 & -1 & 2 & -1 & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & -1 & 0 \\ & & & & -1 & 2 & -1 \\ & & & & 0 & -2 & 2 \end{pmatrix}, \ D_m^\diamond = \begin{pmatrix} 2 & -2 & 2 & & & & \\ -1 & 2 & -1 & & & & \\ 1 & -1 & 2 & -1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix}$$

and denote the associated GIM-algebras over $\mathbb{C}$ by $\mathfrak{gim}\left(B_m^\diamond\right)(\mathbb{C})$ and $\mathfrak{gim}\left(D_m^\diamond\right)(\mathbb{C})$. The following proposition has been published without a proof in [KKLN21, appendix A] for $n = 9$ and is a result of the above computations.

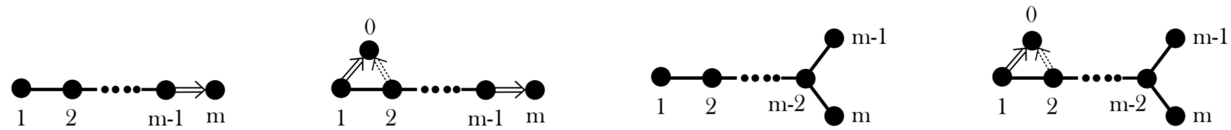

Figure 2: The (GIM-)diagrams associated to B_m, $B_m^\diamond$, D_m and $D_m^\diamond$ respectively.

Proposition 2.7. *For $n = 2m+1$ ($n = 2m$ respectively) denote the Chevalley generators of $\widetilde{\mathfrak{g}} := \mathfrak{gim}\left(B_m^\diamond\right)(\mathbb{C})$ (respectively those of $\widetilde{\mathfrak{g}} := \mathfrak{gim}\left(D_m^\diamond\right)(\mathbb{C})$) by $E_0, \ldots, E_m$, $F_0, \ldots, F_m$ and $H_{\gamma_0}, \ldots, H_{\gamma_m}$. There exists a surjective homomorphism of Lie algebras $\phi : \widetilde{\mathfrak{g}} \to \mathfrak{k}\left(E_n\right)(\mathbb{C})$ that is given on the level of generators via*

$$\phi\left(E_0\right) = X_+,\ \phi\left(F_0\right) = X_-,\ \phi\left(H_{\gamma_0}\right) = 2H_2$$

$$\phi\left(E_i\right) = e_i,\ \phi\left(F_i\right) = f_i,\ \phi\left(H_{\gamma_i}\right) = h_i\ \forall\, i = 1, \ldots, m.$$

Proof. One verifies that the defining relations between the generators from definition (2.5) are satisfied. The B_m- (resp. D_m-) relations are unproblematic and the relations

$$\mathrm{ad}\left(E_0\right)^3\left(E_1\right) = 0 = \mathrm{ad}\left(E_1\right)^2\left(E_0\right),\ \mathrm{ad}\left(E_0\right)^3\left(F_2\right) = 0 = \mathrm{ad}\left(F_2\right)^2\left(E_0\right)$$

follow from equations (38-41). One also has to check that $H_{\gamma_0} \mapsto 2H_2$ satisfies all necessary identities which is the case. Thus, ϕ is a homomorphism of Lie-algebras. Surjectivity follows from the fact that all Berman generators of $\mathfrak{k}\left(E_{2m+1}\right)(\mathbb{C})$ can be recovered from the image of the generators of $\mathfrak{gim}\left(B_m^\diamond\right)(\mathbb{C})$. For $X_1, \ldots, X_{n-1}$ this is a basis transformation within $\mathfrak{so}\left(n, \mathbb{C}\right)$ and for X_n one notes that (29) implies

$$X_+ + X_- = 2iX_n.$$

The same holds for $\mathfrak{k}\left(E_{2m+1}\right)(\mathbb{C})$ as the image of $\mathfrak{gim}\left(D_m^\diamond\right)(\mathbb{C})$. □

Equip the natural subalgebra $\mathfrak{g}\left(B_m\right) < \mathfrak{gim}\left(B_m^\diamond\right)$ with the same basis and structure coefficients as $\mathfrak{so}\left(2m+1, \mathbb{C}\right) < \mathfrak{k}\left(E_{2m+1}\right)(\mathbb{C})$ but denote it by $E_{\varepsilon_1 L_i + \varepsilon_2 L_j}$ and $E_{\pm L_i}$. Then the above result implies that $\mathfrak{k}\left(E_{2m+1}\right)(\mathbb{C})$ is a nontrivial quotient of $\mathfrak{gim}\left(B_m^\diamond\right)(\mathbb{C})$. This is because of elements like $\left[E_0, \left[E_0, E_1\right]\right] - 2E_{L_1+L_2}$ which is nonzero but because of (32), stating that

$$\mathrm{ad}\left(X_+\right)^2\left(e_{\varepsilon L_1 - L_2}\right) = 2e_{\varepsilon L_1 + L_2},$$

it is equal to 0 in the image. Nontriviality of $[E_0,[E_0,E_1]]-2E_{L_1+L_2}$ follows because $[E_0,[E_0,E_1]]$ and $E_{L_1+L_2}$ lie in different root spaces w.r.t. $\mathfrak{h}^*_{B_m^\diamond}$ and GIM-algebras are graded w.r.t. their root system. It is also useful to provide an alternative description of $\mathfrak{k}(E_n)(\mathbb{C})$ by generators and relations.

Proposition 2.8. *Let H_i, $e_{\varepsilon_1 L_i+\varepsilon_2 L_j}$ and $e_{\pm L_i}$ (only for $n=2m+1$) be a weight space basis of $\mathring{\mathfrak{g}} := \mathfrak{g}(D_m)(\mathbb{C})$ (resp. $\mathring{\mathfrak{g}} := \mathfrak{g}(B_m)(\mathbb{C})$) with structure coefficients as in prop. 2.2. Let $\mathfrak{g}$ be the Lie algebra over $\mathbb{C}$ generated by $\mathring{\mathfrak{g}}\cup\{x_+,x_-\}$ modulo the relations of $\mathring{\mathfrak{g}}$ and the relations*

$$[x_+,x_-]=2H_2,\ \ [H_2,x_\pm]=\pm x_\pm,\ \ [H_i,x_\pm]=0\ \forall\, i\neq 2,$$

$$[x_+,e_{\varepsilon L_1-L_2}]=[x_-,e_{\varepsilon L_1+L_2}]\,,\quad [x_+,e_{-L_2+\varepsilon L_3}]=-[x_-,e_{L_2+\varepsilon L_3}]\,,$$

$$0=[x_\pm,e_{\varepsilon L_1\pm L_2}]=[x_+,e_{+L_2+\varepsilon L_3}]=[x_-,e_{-L_2+\varepsilon L_3}]=\left[x_\pm,e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right]\ \ \forall\, 2<i<j,$$

together with

$$[x_\pm,e_{\varepsilon L_m}]=0$$

in the case of $n=2m+1>5$. Then $\mathfrak{g}\cong\mathfrak{k}(E_n)(\mathbb{C})$ via the following isomorphism $\phi:\mathfrak{k}(E_n)(\mathbb{C})\to\mathfrak{g}$:

$$\phi(X_{2i}) = \frac{1}{2}\left(e_{L_i+L_{i+1}}-e_{L_i-L_{i+1}}+e_{-L_i+L_{i+1}}-e_{-L_i-L_{i+1}}\right)\ \forall\, i=1,\dots,m\ (\text{resp. } m-1), \tag{42}$$

$$\phi(X_{2j-1}) = i\cdot H_j\ \forall\, j=1,\dots,m, \tag{43}$$

$$\phi(X_n) = -\frac{i}{2}(x_++x_-) \tag{44}$$

$$\phi(X_{2m}) = -\frac{i}{2}(e_{+L_m}+e_{-L_m})\quad \text{for } n=2m+1. \tag{45}$$

The inverse $\phi^{-1}:\mathfrak{g}\to\mathfrak{k}(E_n)(\mathbb{C})$ is given by identifying the H_i, $e_{\varepsilon_1 L_i+\varepsilon_2 L_j}$ and $e_{\pm L_i}$ in $\mathring{\mathfrak{g}}$ with their counterparts in $\mathfrak{k}(A_{n-1})(\mathbb{C})\cong\mathring{\mathfrak{g}}$ from (15), (22) and (23) as well as

$$\phi^{-1}(x_\pm)=i\,(X_n\mp i\,[X_3,X_n])\,.$$

Proof. One checks with eqs. (15), (22) and (23) that eqs. (42), (43) and (45) are the correct translations between $\mathfrak{so}(n,\mathbb{C})$ from its description by Berman-generators and by its root space decomposition. Then prop. 2.2 implies that this is an isomorphism. The set of relations that defines $\mathfrak{g}$ is a subset of the relations (30)-(37) that hold inside $\mathfrak{k}(E_n)(\mathbb{C})$ and therefore ϕ^{-1} extends to a surjective homomorphism of Lie algebras because all generators of $\mathfrak{k}(E_n)(\mathbb{C})$ are contained in the image.

So it suffices to study if the pairwise relations that include X_n are satisfied after application of ϕ. For $j\neq 2$ one has that $[\phi(X_{2j-1}),\phi(X_n)]=0$ follows from (30). For $j<2$ one has that $[\phi(X_{2j}),\phi(X_n)]=0$ follows from (33) which only leaves the relations $[X_3,[X_3,X_n]]=-X_n$ and $[X_n,[X_n,X_3]]=-X_3$ to be checked:

$$\begin{aligned}
[\phi(X_3),[\phi(X_3),\phi(X_n)]] &= \left[iH_2,\left[iH_2,-\frac{i}{2}(x_++x_-)\right]\right]\\
&= \frac{i}{2}[H_2,x_+-x_-]=\frac{i}{2}(x_++x_-)=-\phi(X_n)\,,\\
[\phi(X_n),[\phi(X_n),\phi(X_3)]] &= -\frac{i}{4}[x_++x_-,[x_++x_-,H_2]]=-\frac{i}{4}[x_++x_-,-x_++x_-]\\
&= -\frac{i}{4}(2H_2+2H_2)=-iH_2=-\phi(X_3)\,.
\end{aligned}$$

This shows that ϕ is a homomorphism of Lie algebras. Since $x_\pm = i\left(\phi(X_n) \mp i\phi([X_3, X_n])\right)$, ϕ is surjective and as ϕ and ϕ^{-1} are inverses on the level of generators this shows that ϕ is an isomorphism of Lie algebras. $\square$

In view of prop. 2.7, representations of $\mathfrak{gim}(B_m^\diamond)(\mathbb{C})$ could potentially be useful to find representations of $\mathfrak{k}(E_{2m+1})(\mathbb{C})$ if it is possible to check whether or not a given representation factors through the projection of proposition 2.7. Conversely, the results from section 3 provide representations of $\mathfrak{gim}(B_m^\diamond)(\mathbb{C})$ of finite dimension. The main benefit of prop. 2.8 is that the set of relations one needs to check on a potential representation is reduced compared to the set of relations given before. In particular, all relations involve single commutators.

The weight of $X_\pm$ is $\pm L_2$ and in case of $\mathfrak{so}(n,\mathbb{C}) \cong B_m(\mathbb{C})$, it appears in the root lattice of B_m. Therefore, it may be possible to check irreducible B_m-representations for conditions when they extend to representations of $\mathfrak{k}(E_n)(\mathbb{C})$. For $n = 2m$ this is not possible because there $X_\pm$ always map between D_m-irreps. as L_2 is not contained in the D_m root lattice.

2.4 An $\mathfrak{so}(n)$-adapted description of $\mathfrak{k}(E_n)$-modules

In this section I would like to outline an approach towards finite-dimensional representations of $\mathfrak{k}(E_n)$ that is based on the presentation of $\mathfrak{k}(E_n)(\mathbb{C})$ given in 2.8. Consider a B_m-module V where $m = \lfloor \frac{n}{2} \rfloor$ (for $n = 2m$ we effectively consider a $\mathfrak{k}(E_{n+1})$-representation and restrict it to $\mathfrak{k}(E_n)$). Denote the set of weights of V by $P(V)$. For $\alpha \in \Delta(B_m)$ the action of e_α as defined in eqs. (22) and (23) on the weight spaces is known from classical representation theory (cp. [GT50], also see [M00] for a modern derivation via Yangians and the references therein for other work on the subject). This means that in principle the matrix elements $D(\mu, \mu+\alpha)_{ij}$ in

$$e_\alpha v_{\mu,i} = \sum_{j=1}^{m(\mu+\alpha)} D(\mu, \mu+\alpha)_{ji} v_{\mu+\alpha,j}$$

are known, where $\{v_{\mu,i} \mid i = 1, \ldots, m(\mu)\}$ is a basis for the weight space V_μ. Now $X_\pm$ must map between the weight spaces like

$$X_\pm v_{\mu,i} = \sum_{j=1}^{m(\mu\pm L_2)} C(\mu, \mu \pm L_2)_{ji}\, v_{\mu\pm L_2,j}.$$

Now one can derive equations for the matrix elements $C(\mu, \mu \pm L_2)_{ji}$. Start with $[X_+, X_-] = 2H_2$:

$$\begin{aligned}
[X_+, X_-]\, v_{\mu,i} &= \sum_{j=1}^{m(\mu-L_2)} C(\mu, \mu-L_2)_{ji}\, X_+ v_{\mu-L_2,j} - \sum_{j=1}^{m(\mu+L_2)} C(\mu, \mu+L_2)_{ji}\, X_- v_{\mu+L_2,j} \\
&= \sum_{j=1}^{m(\mu-L_2)} \sum_{k=1}^{m(\mu)} C(\mu, \mu-L_2)_{ji}\, C(\mu-L_2, \mu)_{kj}\, v_{\mu,k} \\
&\quad - \sum_{j=1}^{m(\mu+L_2)} \sum_{k=1}^{m(\mu)} C(\mu, \mu+L_2)_{ji}\, C(\mu+L_2, \mu)_{kj}\, v_{\mu,k} \\
&= \sum_{k=1}^{m(\mu)} \left(\sum_{j=1}^{m(\mu+L_2)} C(\mu+L_2, \mu)_{kj}\, C(\mu, \mu+L_2)_{ji} - \sum_{j=1}^{m(\mu-L_2)} C(\mu-L_2, \mu)_{kj}\, C(\mu, \mu-L_2)_{ji} \right) v_{\mu,k} \\
&= 2\mu(H_2) \cdot v_{\mu,i},
\end{aligned}$$

so that

$$\sum_{j=1}^{m(\mu+L_2)} C(\mu+L_2,\mu)_{kj}\, C(\mu,\mu+L_2)_{ji} - \sum_{j=1}^{m(\mu-L_2)} C(\mu-L_2,\mu)_{kj}\, C(\mu,\mu-L_2)_{ji} = 2\mu(H_2)\,\delta_{i,k}. \tag{46}$$

Now for the other defining relations it is most useful to pick the linear relations (31) and (36). For $[X_+, e_{-L_2}] = -[X_-, e_{+L_2}]$ one obtains

$$\begin{aligned}
[X_+, e_{-L_2}]\, v_{\mu,i} &= \sum_{j=1}^{m(\mu-L_2)} D(\mu,\mu-L_2)_{ji}\, X_+ v_{\mu-L_2,j} - \sum_{j=1}^{m(\mu+L_2)} C(\mu,\mu+L_2)_{ji}\, e_{-L_2} v_{\mu+L_2,j} \\
&= \sum_{j=1}^{m(\mu-L_2)} \sum_{k=1}^{m(\mu)} D(\mu,\mu-L_2)_{ji}\, C(\mu-L_2,\mu)_{kj}\, v_{\mu,k} \\
&\quad - \sum_{j=1}^{m(\mu+L_2)} \sum_{k=1}^{m(\mu)} C(\mu,\mu+L_2)_{ji}\, D(\mu+L_2,\mu)_{kj}\, v_{\mu,k}
\end{aligned}$$

$$\begin{aligned}
[X_-, e_{L_2}]\, v_{\mu,i} &= \sum_{j=1}^{m(\mu+L_2)} D(\mu,\mu+L_2)_{ji}\, X_- v_{\mu+L_2,j} - \sum_{j=1}^{m(\mu-L_2)} C(\mu,\mu-L_2)_{ji}\, e_{+L_2} v_{\mu-L_2,j} \\
&= \sum_{j=1}^{m(\mu+L_2)} \sum_{k=1}^{m(\mu)} D(\mu,\mu+L_2)_{ji}\, C(\mu+L_2,\mu)_{kj}\, v_{\mu,k} \\
&\quad - \sum_{j=1}^{m(\mu-L_2)} \sum_{k=1}^{m(\mu)} C(\mu,\mu-L_2)_{ji}\, D(\mu-L_2,\mu)_{kj}\, v_{\mu,k},
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_{j=1}^{m(\mu-L_2)} D(\mu,\mu-L_2)_{ji}\, C(\mu-L_2,\mu)_{kj} - \sum_{j=1}^{m(\mu+L_2)} C(\mu,\mu+L_2)_{ji}\, D(\mu+L_2,\mu)_{kj} \\
= &\sum_{j=1}^{m(\mu+L_2)} D(\mu,\mu+L_2)_{ji}\, C(\mu+L_2,\mu)_{kj} - \sum_{j=1}^{m(\mu-L_2)} C(\mu,\mu-L_2)_{ji}\, D(\mu-L_2,\mu)_{kj}
\end{aligned} \tag{47}$$

for all $i,k = 1,\dots,m(\mu)$. Similar equations can be obtained from all the other relations (35) and (37). Note that all defining relations apart from $[X_+, X_-] = 2H_2$ are linear in the unknowns $X_\pm$. So one could pursue the strategy of first simplifying/solving the system of linear equations and then trying to solve eq. (46). One could also investigate if it is possible to derive conditions under which the system of linear equations (35), (47), (37) has more than just the trivial solution. Also note that among all highest weight vectors of V (as V is not assumed to be irreducible as a B_m-module) there exists a maximal one with weight Λ such that $\Lambda + L_2 \notin P(V)$. Then eqs. (46) and (47) become easier because all $C(\Lambda,\Lambda+L_2)_{ij}$ and $C(\Lambda+L_2,\Lambda)_{ij}$ are equal to 0. This could potentially open a door to deduce all other matrix elements of $X_\pm$ inductively.

Part II
Higher spin representations of simply-laced maximal compact subalgebras

3 Higher spin representations

It is a genuinely fascinating feature of maximal compact subalgebras $\mathfrak{k}(A)$ that they admit finite-dimensional representations even if A is of indefinite type. In this section I will present all presently known nontrivial finite-dimensional representations of $\mathfrak{k}(A)$ for A a simply-laced GCM of indefinite type (the results also hold for A finite or affine, but for these cases many more representations are known). All these representations were first found during the investigation of $\mathfrak{k}(E_{10})$ as a hidden symmetry of certain super gravity theories (see for instance [BHP06] and [DKN06]). The lowest-dimensional representation of $\mathfrak{k}(E_{10})$ arises as an extension of the $\mathfrak{so}(10,\mathbb{R})$-Dirac spinor, which is why this representation is called the $\frac{1}{2}$-spin representation. In more mathematical terms one takes the representation $\Gamma_\alpha \oplus \Gamma_\beta$ of $\mathfrak{so}(10,\mathbb{R})$, where α, β denote the fundamental dominant weights that describe the two elementary spin representations of $\mathfrak{g}(D_5)(\mathbb{C})$, and shows that one can establish an action of $\mathfrak{k}(E_{10})$ on this module. This extension can be described quite neatly in terms of Clifford algebras (cp. 3.5) and I will summarize the construction in section 3.1. Based on the $\frac{1}{2}$-spin representation $\mathcal{S}_{\frac{1}{2}}$ one can build further representations on the carrier space $\mathrm{Sym}^n(\mathfrak{h}^*)\otimes\mathcal{S}_{\frac{1}{2}}$ for $n = 1,2,3$ and one calls these representations the $\left(n+\frac{1}{2}\right)$-spin representations, hence "higher" spin representations. It is important to stress that $\mathfrak{h}^*$ is not a representation of $\mathfrak{k}$ and therefore the higher spin representations are not the tensor product of other representations. However, as $\dim \mathfrak{h}^* = 10$ has the same dimension as the "vector representation" Γ_{ω_1} of $\mathfrak{so}(10,\mathbb{R})$, one treats each power of $\mathfrak{h}^*$ as if it added $+1$ to the spin which would indeed be the case for the product representation $\Gamma_{\omega_1} \otimes (\Gamma_\alpha \oplus \Gamma_\beta)$ of $\mathfrak{so}(10,\mathbb{R})$. I will treat the $\frac{3}{2}$- and $\frac{5}{2}$-spin representation in section 3.2 as they allow a uniform treatment via Weyl group actions on $\mathrm{Sym}^n(\mathfrak{h}^*)$. This approach has to be augmented for the $\frac{7}{2}$-representation which I will explain in detail in section 3.3. All these representations already appeared in the physics literature: The $\frac{1}{2}$- and $\frac{3}{2}$-representations are discussed in ([BHP06], [DKN06]), the $\frac{5}{2}$- and $\frac{7}{2}$-representations are introduced in ([KN13], [KN17]). While all these sources only treat the case E_{10}, the representations are more general, as shown for $\mathcal{S}_{\frac{1}{2}}$ in a mathematical setting in [HKL15]. The way the higher spin representations $\mathcal{S}_{\frac{n}{2}}$ are constructed in ([KN13], [KN17]) is directly seen to be generalizable to the simply-laced case, and by the covering techniques of [HKL15] this allows such representations for any symmetrizable GCM A. A first mathematical treatment of the representations $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ has been given in [LK18], where the main feature is that the coordinate-dependent formulation of ([KN13], [KN17]) is compared to a coordinate-free version that puts more emphasis on how the Weyl group $W(A)$ describes the action of $\mathfrak{k}(A)$. So far, such a coordinate-free description of $\mathcal{S}_{\frac{7}{2}}$ in terms of the Weyl group was missing, which is why section 3.3 contains genuinely new material even though the representation $\mathcal{S}_{\frac{7}{2}}$ was already known. The main results of this section are theorems 3.19 and 3.23 that provide the coordinate-free and Weyl-group based description of the higher spin representations.

3.1 The $\frac{1}{2}$-spin representation $\mathcal{S}_{\frac{1}{2}}$

Let A be GCM, denote by $\mathfrak{g}(A)(\mathbb{R})$ the split-real Kac-Moody algebra of type A and by $\mathfrak{k}(A)$ its maximal compact subalgebra. If not specified otherwise I will assume A to be of simply-laced type. First, I will recall the definition of generalized $\frac{1}{2}$-spin representations which first appeared in the physics literature (see

[DKN06], [BHP06]). Their first appearance in mathematical literature was [HKL15], whose conventions I will use in the following summary. Afterwards I will provide a mathematical foundation for the calculus of generalized Γ-matrices from [KN13] which will be needed in section 4 to derive a parametrization result of representation matrices.

Definition 3.1. (Cp. [HKL15, def. 3.6]) Let $X_1, \ldots, X_n$ denote the Berman generators of $\mathfrak{k}(A)$ for simply-laced A. One calls a homomorphism $\rho : \mathfrak{k} \to \mathrm{End}\,(\mathbb{C}^s)$ a **generalized spin representation** if

$$\rho\left(X_i\right)^2 = -\frac{1}{4} Id_{s\times s}\ \forall\, i = 1, \ldots, n\,.$$

Proposition 3.2. *Let $\{A, B\} := AB + BA$ denote the anticommutator and let $\rho : \mathfrak{k} \to End\,(\mathbb{C}^s)$ be a generalized spin representation. Then one has ($\forall\, 1 \leq i \neq j \leq n$)*

$$\begin{aligned} \left[\rho\left(X_i\right)\,,\,\rho\left(X_j\right)\right] &= 0 \quad \text{if } a_{ij} = 0 \\ \left\{\rho\left(X_i\right)\,,\,\rho\left(X_j\right)\right\} &= 0 \quad \text{if } a_{ij} = -1 \quad . \end{aligned}$$

Vice versa, given matrices $A_1, \ldots, A_n \in \mathbb{C}^{s\times s}$ that satisfy

$$\begin{aligned} (i)\ A_i^2 &= -\frac{1}{4} id_s \\ (ii)\ [A_i, A_j] &= 0 \text{ if } a_{ij} = 0 \\ (iii)\ \{A_i, A_j\} &= 0 \text{ if } a_{ij} = -1\ , \end{aligned}$$

the extension of the map $X_i \mapsto A_i$ defines a generalized spin representation.

Proof. This is [HKL15, rem. 3.7]. □

Theorem 3.3. *Let A be a symmetrizable GCM, then a generalized spin representation*[23] *exists. Its image considered as a representation $\rho : \mathfrak{k} \to End\left(\mathbb{R}^{2s}\right)$ is compact, hence reductive. It is furthermore semisimple if for all $i = 1, \ldots, n$ there exists $j \in \{1, \ldots, n\}$ such that a_{ji} is odd.*

Proof. This is a merger of [HKL15, thm. 3.9] and [HKL15, thm. 3.14]. □

Note however, that the claim $\mathfrak{k} \cong \ker\rho \oplus \mathrm{im}\rho$ in case of a semisimple image is false.

Proposition 3.4. *Let A be a simply-laced, indecomposable GCM of indefinite type and let ρ be a generalized spin representation of $\mathfrak{k}(A)$ according to def. 3.1. Furthermore, any ideal in $\mathfrak{k}$ that is orthogonal to $\ker\rho$ is trivial and hence, contrary to the claim of [HKL15, thm. 3.14], $\mathfrak{k} \ncong \ker\rho \oplus \mathrm{im}\rho$.*

Proof. Set

$$\ker\rho := \{x \in \mathfrak{k} \mid \rho(x) = 0\}\,,\ \ker\rho^{\perp} := \{x \in \mathfrak{k} \mid\ (x|y) = 0\ \forall\, y \in \ker\rho\}$$

where $\ker\rho^{\perp}$ only means the algebraic object without completion. One checks that $\ker\rho^{\perp}$ is in fact an ideal:

$$\left(\left[z, x\right] | y\right) = -\left(x | \left[z, y\right]\right) = 0\ \forall\, x \in \ker\rho^{\perp}, y \in \ker\rho, z \in \mathfrak{k},$$

[23] I have not defined the notion of a generalized spin representation for diagrams which are not simply-laced. This is done in [HKL15, def. 3.13]. One obtains these representations by means of embedding $\mathfrak{g}\,(A)$ into a larger Kac-Moody algebra $\mathfrak{g}\left(\widetilde{A}\right)$, where $\widetilde{A}$ is a suitable GCM of simply-laced type, a so-called simply-laced cover of A. One then considers generalized spin representations of $\mathfrak{k}\left(\widetilde{A}\right)$ which is why the general case essentially follows from the simply-laced one.

because $\ker\rho$ is an ideal (therefore $[z,y]\in\ker\rho$) and the bilinear form is invariant. Since $(\cdot|\cdot)$ is anisotropic on $\mathfrak{k}$ (see [K90, thm. 11.7]) one knows furthermore that $\ker\rho\cap\ker\rho^{\perp}=\{0\}$. Next, I will show that $\ker\rho^{\perp}=\{0\}$.

Assume there exists a Berman generator $X_i\in\ker\rho^{\perp}$. Then, as A is indecomposable, one has that all X_i are in $\ker\rho^{\perp}$ and hence $\ker\rho^{\perp}\cong\mathfrak{k}$ which shows that $\ker\rho=\{0\}$, a contradiction because ρ is not faithful (unless A is of finite type which was excluded in the assumptions). Hence, for at least one X_i there exists $y\in\ker\rho$ s.t. $(X_i|y)\neq 0$. By invariance of the form this extends to any nested Berman element $X^{(i_1,\dots,i_n)}=[X_{i_1},[X_{i_2},[\dots,X_{i_n}]]]$ which span $\mathfrak{k}$. Now one only needs to ensure that this property also holds for linear combinations $\sum_{i=1}^n c_iX_{\beta_i}$of such elements. As to each X_{β_i} there exists y_i such that $(X_{\beta_i}|y_i)\neq 0$ there exist d_i s.t. $(\sum_{i=1}^n c_iX_{\beta_i}|\sum_{i=1}^n d_iy_i)\neq 0$. Hence, $\ker\rho^{\perp}=\{0\}$. □

As the standard invariant form of $\mathfrak{g}$ restricted to $\mathfrak{k}$ is negative definite[24], $\mathfrak{k}$ has a completion as a Hilbert space which I denote by $\widehat{\mathfrak{k}}$ and from this point of view the case is slightly more complicated. Let X and Y be Hilbert spaces, then to each operator $T:X\supset D(X)\to Y$ that is defined on a dense subset $D(X)$ there exists a formally adjoint operator $T^*:Y\supset D(T^*)\to X$ and T is bounded if and only if T^* is. As ρ is a finite-dimensional representation one has $\dim Y<\infty$ and thus, ρ^* is by default bounded. Therefore, ρ has a continuous extension $\hat\rho:\hat{\mathfrak{k}}\to End(Y)$. Since $\hat\rho$ is continuous, $\ker\hat\rho$ is closed and one obtains

$$\hat{\mathfrak{k}}=\ker\hat\rho\oplus\ker\hat\rho^{\perp}$$

as an orthogonal sum of vector spaces. Furthermore one has $\hat{\mathfrak{k}}/\ker\hat\rho\cong \mathrm{im}\hat\rho$ as vector spaces and hence $\hat{\mathfrak{k}}=\ker\hat\rho\oplus\mathrm{im}\hat\rho$ as vector spaces but that does not imply the same as ideals unless $\hat{\mathfrak{k}}$ carries the structure of a Hilbert Lie algebra. This is never the case if A is of indefinite type, as is shown in [KKLN21, appendix B].

Example 3.5. (Originally due to [DKN06] and [BHP06], this phrasing is [HKL15, prop. A.14])
Let $v_1,\dots,v_n\in\mathbb{R}^n$ be orthonormal w.r.t. the standard euclidean inner product $\langle\cdot,\cdot\rangle$ and denote their image in the Clifford algebra $Cl(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ by $v_1,\dots,v_n$ as well. A generalized spin representation of $\mathfrak{k}(E_n)(\mathbb{R})$ for $n\geq 4$ is given by

$$X_i\mapsto\frac{1}{2}v_iv_{i+1}\ \forall\, i=1,\dots,n-1,\ X_n\mapsto\frac{1}{2}v_1v_2v_3.$$

In view of the fact that $\langle X_1,\dots,X_{n-1}\rangle_{\mathbb{R}}\cong\mathfrak{so}(n,\mathbb{R})$ this shows quite explicitly that this representation is an extension of the $\mathfrak{so}(n,\mathbb{R})$-Dirac spinor as the definition via $X_i\mapsto\frac{1}{2}v_iv_{i+1}$ is a particularly easy and direct way to construct it.

Remark 3.6. Note that generalized spin representations need not be unique. Consider for instance $\widetilde{A}_3$, the affine extension of A_3. Then the constructive procedure of [HKL15, cor. 3.10] provides a generalized spin representation ρ for which there exist distinct non-adjacent nodes i,j such that $\rho(X_i)=\rho(X_j)$. However, one can check that

$$\phi(X_1):=\frac{1}{2}v_1v_2,\ \phi(X_2):=\frac{1}{2}v_2v_3,\ \phi(X_3):=\frac{1}{2}v_3v_4,\ \phi(X_0):=\frac{1}{2}v_1v_4,$$

where $v_1,\dots,v_4$ denote the orthonormal basis elements of $\mathbb{R}^4\subset Cl\left(\mathbb{R}^4\right)$, also defines a generalized spin representation. As all Berman generators have a distinct representation matrix under ϕ, the two representations can not be equivalent.

[24]Cp. [K90, thm. 11.7], the hermitian form on $\mathfrak{n}_+^{\mathbb{C}}\oplus\mathfrak{n}_-^{\mathbb{C}}\subset\mathfrak{g}(A)(\mathbb{C})$ becomes a bilinear form on $\mathfrak{n}_+^{\mathbb{R}}\oplus\mathfrak{n}_-^{\mathbb{R}}\subset\mathfrak{g}(A)(\mathbb{R})$ with the same properties.

The way prop. 3.2 describes generalized spin representations is a local one, as everything reduces to computations in rank 2 subalgebras. One can take a slightly more global perspective if one allows the root system of $\mathfrak{g}(A)$ to play a more dominant role. This can be done by introducing 2-cocycles on the root lattices and associated linear maps, so-called generalized Γ-matrices (this name stems from the origin of these maps in physics as they occurred first as generalizations of the Dirac-matrices γ_μ).

Definition 3.7. Let $Q(A)$ denote the root lattice of $\mathfrak{g}(A)$. A map $\varepsilon : Q(A) \times Q(A) \to C_2$ is called an **associated**, **normalized 2-cocycle** if

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)}\ ,\ \varepsilon(\alpha,0) = \varepsilon(0,\alpha) = 1 \tag{48}$$

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\beta+\gamma)\varepsilon(\beta,\gamma) \tag{49}$$

for all $\alpha,\beta,\gamma \in Q$.

One verifies with a short computation that these relations imply

$$\varepsilon(\alpha,\alpha) = (-1)^{\frac{1}{2}(\alpha|\alpha)} \tag{50}$$

and

$$\varepsilon(\alpha,\beta) = \begin{cases} \varepsilon(\beta,\alpha) & if\ \ (\alpha|\beta) = 0 \mod 2 \\ -\varepsilon(\beta,\alpha) & if\ \ (\alpha|\beta) = 1 \mod 2. \end{cases} \tag{51}$$

Lemma 3.8. *Let A be a symmetrizable GCM with symmetrization $A = DB$ s.t. the invariant bilinear form on $\mathfrak{h}^*$ described by $b_{ij} = (\alpha_i|\alpha_j)$ satisfies $(\alpha_i|\alpha_i) = b_{ii} \in 2\mathbb{Z}$ for all $i = 1,\dots,n$. Define a bilinear form $\underline{\varepsilon} : Q(A) \times Q(A) \to \mathbb{Z}$ on the root lattice via bilinear extension of*

$$\underline{\varepsilon}\,(\alpha_i,\alpha_j) := \begin{cases} b_{ij} & \text{if } i<j \\ \frac{1}{2}b_{ii} & \text{if } i=j \\ 0 & \text{if } i>j. \end{cases}$$

Then $\varepsilon : Q(A) \times Q(A) \to C_2 := \{-1,1\}$ given by

$$\varepsilon\,(\alpha,\beta) := (-1)^{\underline{\varepsilon}(\alpha,\beta)}$$

is an associated normalized 2-cocycle called the ***standard 2-cocycl****e to $Q(A)$.*

Proof. One has

$$\underline{\varepsilon}\,(\alpha_i,\alpha_j) + \underline{\varepsilon}\,(\alpha_j,\alpha_i) = b_{ij}$$

for all $i,j = 1,\dots,n$. For $\beta = \sum_{i=1}^n b_i\alpha_i$ and $\gamma = \sum_{i=1}^n c_i\alpha_i$ this implies

$$\begin{aligned} \underline{\varepsilon}\,(\beta,\gamma) + \underline{\varepsilon}\,(\gamma,\beta) &= \sum_{i,j} b_ic_j\underline{\varepsilon}\,(\alpha_i,\alpha_j) + \sum_{i,j} b_ic_j\underline{\varepsilon}\,(\alpha_j,\alpha_i) \\ &= \sum_{i,j} b_ic_j\left(\underline{\varepsilon}\,(\alpha_i,\alpha_j) + \underline{\varepsilon}\,(\alpha_j,\alpha_i)\right) = \sum_{i,j} b_ic_jb_{ij} \\ &= (\beta|\gamma)\ \ \forall\,\beta,\gamma \in Q(A). \end{aligned}$$

From this one has for all $\beta,\gamma \in Q(A)$ that

$$\varepsilon\,(\beta,\gamma)\,\varepsilon\,(\gamma,\beta) = (-1)^{\underline{\varepsilon}(\beta,\gamma)+\underline{\varepsilon}(\gamma,\beta)} = (-1)^{(\beta|\gamma)}$$

which implies (48) together with bilinearity of $\underline{\varepsilon}$. Towards (49) note that

$$\begin{aligned}\underline{\varepsilon}(\alpha,\beta)+\underline{\varepsilon}(\alpha+\beta,\gamma) &= \underline{\varepsilon}(\alpha,\beta)+\underline{\varepsilon}(\alpha,\gamma)+\underline{\varepsilon}(\beta,\gamma)\\ &= \underline{\varepsilon}(\alpha,\beta+\gamma)+\underline{\varepsilon}(\beta,\gamma)\end{aligned}$$

by bilinearity which implies

$$\begin{aligned}\varepsilon(\alpha,\beta)\,\varepsilon(\alpha+\beta,\gamma) &= (-1)^{\underline{\varepsilon}(\alpha,\beta)+\underline{\varepsilon}(\alpha+\beta,\gamma)} = (-1)^{\underline{\varepsilon}(\alpha,\beta+\gamma)+\underline{\varepsilon}(\beta,\gamma)}\\ &= \varepsilon(\alpha,\beta+\gamma)\,\varepsilon(\beta,\gamma)\ \ \forall\,\alpha,\beta,\gamma\in Q(A).\end{aligned}$$

□

Definition 3.9. (Cp. [KN13, eq. 4.6]) A map $\Gamma : Q(A)\to\mathbb{C}^{s\times s}$ is called a **generalized Γ-matrix** if

$$\Gamma(\alpha)\Gamma(\beta) = (-1)^{(\alpha|\beta)}\Gamma(\beta)\Gamma(\alpha) \tag{52}$$

$$\Gamma(0)=Id\ ,\ \Gamma(\alpha)^2=(-1)^{\frac{1}{2}(\alpha|\alpha)}\ ,\ \Gamma(\alpha)=\Gamma(-\alpha) \tag{53}$$

$$\Gamma(\alpha)\Gamma(\beta)=\varepsilon(\alpha,\beta)\Gamma(\alpha+\beta)\ , \tag{54}$$

for all $\alpha,\beta\in Q(A)$ and an associated normalized 2-cocycle ε.

Proposition 3.10. *A generalized Γ-matrix $\Gamma : Q(A)\to\mathbb{C}^{s\times s}$ gives rise to a generalized spin representation $\rho :\ \mathfrak{k}\to\mathbb{C}^{s\times s}$.*

Proof. One checks with eq. (52) that

$$\Gamma(\alpha_i)\Gamma(\alpha_j)=-\Gamma(\alpha_j)\Gamma(\alpha_i)$$

for adjacent simple roots and that

$$\Gamma(\alpha_i)\Gamma(\alpha_j)=\Gamma(\alpha_j)\Gamma(\alpha_i)$$

for non-adjacent simple roots. This settles the relevant (anti-)commutators. One then verifies that

$$\rho(X_i):=\frac{1}{2}\Gamma(\alpha_i)$$

provides a proper normalization such that all requirements from prop. 3.2 are satisfied. □

Proposition 3.11. *For a simply-laced GCM A , a generalized spin representation $\rho:\mathfrak{k}(A)\to\mathbb{C}^{s\times s}$ gives rise to a generalized Γ-matrix $\Gamma:Q(A)\to\mathbb{C}^{s\times s}$ via*

$$\Gamma(\pm\alpha_i):=2\rho(X_i)\,,\ \Gamma(\alpha_{i_1}+\cdots+\alpha_{i_n}):=\left(\prod_{k=1}^{n-1}\varepsilon\left(\alpha_{i_k},\alpha_{i_{k+1}}+\cdots+\alpha_{i_n}\right)\right)\Gamma(\alpha_{i_1})\cdots\Gamma(\alpha_{i_n})\,.$$

Proof. First of all one checks that equations (52), (53), (54) are satisfied for $\alpha,\beta\in Q(A)$ of height 1 and 2. Note that one can assume $\alpha=\sum_{i=1}^n k_i\alpha_i$ with $k_i\geq 0$ w.l.o.g. because $\Gamma(-\alpha_i)$ is by definition equal to $\Gamma(\alpha_i)$ and the 2-cocycles do not distinguish between $\pm\alpha_i$ because everything is counted only modulo 2, as $(-1)^n=(-1)^{-n}$ for $n\in\mathbb{Z}$. For height 2 one has to check if the definition of $\Gamma(\alpha)$ for $\alpha=\alpha_i+\alpha_j$ is unambiguous:

$$\Gamma(\alpha_i+\alpha_j)=\varepsilon(\alpha_i,\alpha_j)\,\Gamma(\alpha_i)\,\Gamma(\alpha_j)\,,$$

$$\Gamma(\alpha_j+\alpha_i)=\varepsilon(\alpha_j,\alpha_i)\,\Gamma(\alpha_j)\,\Gamma(\alpha_i)\,.$$

If $(\alpha_i,\alpha_j)=-1$ then $\Gamma(\alpha_i)\,\Gamma(\alpha_j)=-\Gamma(\alpha_j)\,\Gamma(\alpha_i)$ from the properties of a generalized spin representation and $\varepsilon(\alpha_i,\alpha_j)\,\varepsilon(\alpha_j,\alpha_i)=-1$ which shows that $\Gamma(\alpha_i+\alpha_j)=\Gamma(\alpha_j+\alpha_i)$. If $(\alpha_i,\alpha_j)=0$ one has $\varepsilon(\alpha_i,\alpha_j)=1=\varepsilon(\alpha_j,\alpha_i)$ as well as $\Gamma(\alpha_i)\,\Gamma(\alpha_j)=\Gamma(\alpha_j)\,\Gamma(\alpha_i)$ which again shows that $\Gamma(\alpha_i+\alpha_j)=\Gamma(\alpha_j+\alpha_i)$.

Now assume that equations (52), (53), (54) hold for all $\alpha,\beta\in Q(A)_+$ with $\mathrm{ht}(\alpha)+\mathrm{ht}(\beta)\le n-1$ and proceed with induction on the height n. In particular one has for all $\alpha,\beta\in Q(A)_+$ with $\mathrm{ht}(\alpha)+\mathrm{ht}(\beta)\le n-1$ that

$$\varepsilon(\alpha,\beta)\,\Gamma(\alpha)\,\Gamma(\beta)=\Gamma(\alpha+\beta)$$

is true and hence the definition of $\Gamma(\gamma)$ for $\mathrm{ht}(\gamma)\le n-1$ is unambiguous. Now take $\alpha=\alpha_{i_1}+\cdots+\alpha_{i_n}$ and $\beta,\gamma\in Q(A)_+$ s.t. $\alpha=\beta+\gamma$ and assume w.l.o.g. that β contains α_{i_1}. One needs to show $\Gamma(\alpha)=\varepsilon(\beta,\gamma)\,\Gamma(\beta)\,\Gamma(\gamma)$ and to show this one multiplies from the left with $\Gamma(\alpha_{i_1})$:

$$\varepsilon(\beta,\gamma)\,\Gamma(\alpha_{i_1})\,\Gamma(\beta)\,\Gamma(\gamma)=\varepsilon(\beta,\gamma)\,\varepsilon(\alpha_{i_1},\beta)\,\Gamma(\beta-\alpha_{i_1})\,\Gamma(\gamma)\,,$$

where one exploits that $\Gamma(\alpha_{i_1})=\Gamma(-\alpha_{i_1})$ and therefore

$$\Gamma(\beta-\alpha_{i_1})=\Gamma(-\alpha_{i_1}+\beta)=\varepsilon(-\alpha_{i_1},\beta)\,\Gamma(-\alpha_{i_1})\,\Gamma(\beta)=\varepsilon(\alpha_{i_1},\beta)\,\Gamma(\alpha_{i_1})\,\Gamma(\beta)\,.$$

All the relations used here hold, because the involved elements of the root lattice are of height less or equal than $n-1$. As $\mathrm{ht}(\beta-\alpha_{i_1})+\mathrm{ht}(\gamma)=n-1$ one can now use that

$$\Gamma(\beta-\alpha_{i_1})\,\Gamma(\gamma)=\varepsilon(\beta-\alpha_{i_1},\gamma)\,\Gamma(\beta-\alpha_{i_1}+\gamma)\,,$$

so that with $\varepsilon(\beta-\alpha_{i_1},\gamma)=\varepsilon(\beta+\alpha_{i_1},\gamma)$ and $\beta-\alpha_{i_1}+\gamma=\alpha_{i_2}+\cdots+\alpha_{i_n}$

$$\varepsilon(\beta,\gamma)\,\Gamma(\alpha_{i_1})\,\Gamma(\beta)\,\Gamma(\gamma)=\varepsilon(\beta,\gamma)\,\varepsilon(\alpha_{i_1},\beta)\,\varepsilon(\beta+\alpha_{i_1},\gamma)\,\Gamma(\alpha_{i_2}+\cdots+\alpha_{i_n})\,,$$

where now one uses cocycle property (49) to show

$$\varepsilon(\alpha_{i_1}+\beta,\gamma)=\varepsilon(\alpha_{i_1},\beta)\,\varepsilon(\alpha_{i_1},\beta+\gamma)\,\varepsilon(\beta,\gamma)$$

and therefore

$$\varepsilon(\beta,\gamma)\,\varepsilon(\alpha_{i_1},\beta)\,\varepsilon(\beta+\alpha_{i_1},\gamma)=\varepsilon(\alpha_{i_1},\beta+\gamma)\,.$$

Thus,

$$\begin{aligned}\varepsilon(\beta,\gamma)\,\Gamma(\alpha_{i_1})\,\Gamma(\beta)\,\Gamma(\gamma) &= \varepsilon(\alpha_{i_1},\beta+\gamma)\,\Gamma(\alpha_{i_2}+\cdots+\alpha_{i_n})\\ &= \varepsilon(\alpha_{i_1},\alpha_{i_1}+\cdots+\alpha_{i_n})\left(\prod_{k=2}^{n-1}\varepsilon\left(\alpha_{i_k},\alpha_{i_{k+1}}+\cdots+\alpha_{i_n}\right)\right)\Gamma(\alpha_{i_2})\cdots\Gamma(\alpha_{i_n})\end{aligned}$$

and with

$$\begin{aligned}\varepsilon(\alpha_{i_1},\alpha_{i_1}+\cdots+\alpha_{i_n}) &= \varepsilon(\alpha_{i_1},\alpha_{i_1})\,\varepsilon(2\alpha_{i_1},\alpha_{i_2}+\cdots+\alpha_{i_n})\,\varepsilon(\alpha_{i_1},\alpha_{i_2}+\cdots+\alpha_{i_n})\\ &= (-1)\cdot 1\cdot\varepsilon(\alpha_{i_1},\alpha_{i_2}+\cdots+\alpha_{i_n})\end{aligned}$$

this yields

$$\varepsilon(\beta,\gamma)\,\Gamma(\alpha_{i_1})\,\Gamma(\beta)\,\Gamma(\gamma)=-\left(\prod_{k=1}^{n-1}\varepsilon\left(\alpha_{i_k},\alpha_{i_{k+1}}+\cdots+\alpha_{i_n}\right)\right)\Gamma(\alpha_{i_2})\cdots\Gamma(\alpha_{i_n})\,.$$

Compare this to

$$\begin{aligned}\Gamma(\alpha_{i_1})\Gamma(\alpha_{i_1}+\cdots+\alpha_{i_n}) &= \left(\prod_{k=1}^{n-1}\varepsilon\left(\alpha_{i_k},\alpha_{i_{k+1}}+\cdots+\alpha_{i_n}\right)\right)\Gamma(\alpha_{i_1})^2\,\Gamma(\alpha_{i_2})\cdots\Gamma(\alpha_{i_n})\\ &= -\left(\prod_{k=1}^{n-1}\varepsilon\left(\alpha_{i_k},\alpha_{i_{k+1}}+\cdots+\alpha_{i_n}\right)\right)\Gamma(\alpha_{i_2})\cdots\Gamma(\alpha_{i_n}),\end{aligned}$$

because $\Gamma(\alpha_{i_1})^2 = -Id$. As left-multiplication with $\Gamma(\alpha_{i_1})$ is an equivalence relation, this shows

$$\varepsilon(\beta,\gamma)\Gamma(\beta)\Gamma(\gamma) = \Gamma(\alpha_{i_1}+\cdots+\alpha_{i_n})$$

for all β,γ of height $< n$ s.t. $\beta+\gamma = \alpha_{i_1}+\cdots+\alpha_{i_n}$. Now (54) can be used to show (52) via induction since $\Gamma(\alpha+\beta) = \Gamma(\beta+\alpha)$ spells out as

$$\varepsilon(\alpha,\beta)\Gamma(\alpha)\Gamma(\beta) = \varepsilon(\beta,\alpha)\Gamma(\beta)\Gamma(\alpha) \;\Leftrightarrow\; \Gamma(\alpha)\Gamma(\beta) = (-1)^{(\alpha|\beta)}\Gamma(\beta)\Gamma(\alpha)$$

via (48). One then uses this to compute

$$\begin{aligned}\Gamma(\alpha+\beta)^2 &= \varepsilon(\alpha,\beta)\Gamma(\alpha)\Gamma(\beta)\varepsilon(\beta,\alpha)\Gamma(\beta)\Gamma(\alpha)\\ &= \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)(-1)^{\frac{1}{2}(\beta|\beta)}(-1)^{\frac{1}{2}(\alpha|\alpha)}\\ &= (-1)^{(\alpha|\beta)+\frac{1}{2}(\beta|\beta)+\frac{1}{2}(\alpha|\alpha)} = (-1)^{\frac{1}{2}(\alpha+\beta|\alpha+\beta)}.\end{aligned}$$

□

Remark 3.12. Note that even though a generalized Γ-matrix provides a matrix for every root $\alpha\in\Delta(A)$, it is a priori unclear if $\rho(x) = c(x)\cdot\Gamma(\alpha)$ for all $x\in\mathfrak{k}_\alpha$ with a suitable $c(x)\in\mathbb{K}$.

Lemma 3.13. *Let A be a symmetrizable GCM with the additional assumptions from lemma 3.8 and let ε be the associated standard 2-cocycle. Then a generalized Γ-matrix satisfies*

$$\Gamma(\alpha+\beta) = \Gamma(\alpha-\beta),\ \Gamma(\alpha+2\beta) = (-1)^{(\beta|\beta)}\Gamma(\alpha)\ \ \forall\,\alpha,\beta\in Q(A). \tag{55}$$

Proof. One has

$$\Gamma(\alpha+\beta) = \varepsilon(\alpha,\beta)\Gamma(\alpha)\Gamma(\beta),\ \Gamma(\alpha-\beta) = \varepsilon(\alpha,-\beta)\Gamma(\alpha)\Gamma(-\beta)$$

and due to (53) one has $\Gamma(-\beta) = \Gamma(\beta)$. As ε is a standard 2-cocycle one computes

$$\varepsilon(\alpha,-\beta) = (-1)^{\underline{\varepsilon}(\alpha,-\beta)} = (-1)^{-\underline{\varepsilon}(\alpha,\beta)} = (-1)^{\underline{\varepsilon}(\alpha,\beta)} = \varepsilon(\alpha,\beta)$$

so that the first part of (55) follows. Towards the second part compute

$$\varepsilon(\alpha,2\beta) = (-1)^{\underline{\varepsilon}(\alpha,2\beta)} = (-1)^{2\underline{\varepsilon}(\alpha,\beta)} = 1^{\underline{\varepsilon}(\alpha,\beta)} = 1\ \forall\,\alpha,\beta\in Q(A),$$

$$\begin{aligned}\Gamma(2\beta) &= \varepsilon(\beta,\beta)\Gamma(\beta)\Gamma(\beta) = \varepsilon(\beta,\beta)\Gamma(\beta)\Gamma(-\beta) = \varepsilon(\beta,\beta)\varepsilon(\beta,-\beta)\Gamma(0)\\ &= \varepsilon(\beta,\beta)\varepsilon(\beta,\beta) = (-1)^{(\beta|\beta)}\end{aligned}$$

and therefore

$$\Gamma(\alpha+2\beta) = \varepsilon(\alpha,2\beta)\Gamma(\alpha)\Gamma(2\beta) = (-1)^{(\beta|\beta)}\Gamma(\alpha)\ \ \forall\,\alpha,\beta\in Q(A).$$

□

Proposition 3.14. *Let A be a symmetrizable GCM with the additional assumptions from lemma 3.8 and associated standard 2-cocycle ε. Furthermore, let (ρ, V) be a generalized spin representation with corresponding generalized Γ-matrix $\Gamma : Q(A) \to End(V)$. For $x \in \mathfrak{k}_\alpha := \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ one has that*

$$\rho(x) = c(x) \cdot \Gamma(\alpha) \ \textit{for}\ c(x) \in \mathbb{K}. \tag{56}$$

Proof. For $\alpha \in \Delta_+(A)$ there exist simple roots $\beta_1, \dots, \beta_n \in \Pi(A)$ such that $\alpha = \beta_1 + \cdots + \beta_n$. For such an ordered decomposition of α set $X_{\beta_1+\cdots+\beta_n} := [X_{\beta_1}, [X_{\beta_2}, [\dots, X_{\beta_n}] \dots]]$ as in eq. (17). Then

$$\rho(X_{\beta_1+\cdots+\beta_n}) = \frac{1}{2^n}\left[\Gamma(\beta_1), \left[\Gamma(\beta_2), \left[\dots, \Gamma(\beta_n)\right] \dots\right]\right]$$

and since $\Gamma(\alpha)\Gamma(\beta) = (-1)^{(\alpha|\beta)}\Gamma(\beta)\Gamma(\alpha)$ one has

$$[\Gamma(\alpha), \Gamma(\beta)] = \Gamma(\alpha)\Gamma(\beta) - \Gamma(\beta)\Gamma(\alpha) = \begin{cases} 0 & \text{if } (\alpha|\beta) \in 2\mathbb{Z} \\ 2\Gamma(\alpha)\Gamma(\beta) & \text{if } (\alpha|\beta) \in \mathbb{Z} \setminus 2\mathbb{Z}. \end{cases}$$

This provides two possibilities for $\rho(X_{\beta_1+\cdots+\beta_n})$:

$$\rho(X_{\beta_1+\cdots+\beta_n}) = \frac{1}{2^n}\left[\Gamma(\beta_1), \left[\Gamma(\beta_2), \left[\dots, \Gamma(\beta_n)\right] \dots\right]\right] = \begin{cases} 0 \\ \Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_n) \end{cases},$$

where the first case applies if there exists i such that $(\alpha_i|\alpha_{i+1} + \cdots + \alpha_n) \in 2\mathbb{Z}$. As $\Gamma(\alpha) = \pm\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_n)$ this shows the claim for elements of $\mathfrak{k}$ of the form $X_{\beta_1+\cdots+\beta_n}$. While it is true that these elements span $\mathfrak{k}$ it does not show (56) yet because $X_{\beta_1+\cdots+\beta_n}$ is not necessarily contained in $\mathfrak{k}_\alpha$ but may have parts in (arbitrarily many) $\mathfrak{k}_\beta$ for $\beta < \alpha$. Set $\mathfrak{k}_{<\alpha} := \bigoplus_{\beta<\alpha}\mathfrak{k}_\beta$ and let $x \in \mathfrak{k}_\alpha$. Then there exist ordered decompositions $\beta_1^{(j)} + \cdots + \beta_n^{(j)} = \alpha$ for $j = 1, \dots, k$, $c_j \in \mathbb{K}$ and $r \in \mathfrak{k}_{<\alpha}$ such that

$$\sum_{j=1}^{k} c_j X_{\beta_1^{(j)}+\cdots+\beta_n^{(j)}} = x + r.$$

Recall that the Lie-bracket is filtered w.r.t. $\Delta_+(A)$ as one has for $y_1 \in \mathfrak{k}_\alpha, y_2 \in \mathfrak{k}_\beta$ that $[y_1, y_2] \in \mathfrak{k}_{\alpha+\beta} \oplus \mathfrak{k}_{\pm(\alpha-\beta)}$. Now the remainder r possesses a decomposition $r = \bigoplus_\gamma y_\gamma$ such that $y_\gamma \in \mathfrak{k}_\gamma$ is nonzero only if $\gamma < \alpha$ and $\alpha - \gamma \in 2Q(A)$. This shows with that

$$\rho(x) = \sum_{j=1}^{k} c_j \rho\left(X_{\beta_1^{(j)}+\cdots+\beta_n^{(j)}}\right) - \rho(r) = c(x) \cdot \Gamma(\alpha)$$

for all $x \in \mathfrak{k}_\alpha$ for all $\alpha \in \Delta_+(A)$ via induction on $\mathrm{ht}(\alpha)$ because $\Gamma(\alpha) = \pm\Gamma(\gamma)$ if $\alpha - \gamma \in 2Q(A)$ according to (55). □

3.2 The higher spin representations $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$

In this subsection I summarize some of the results of [KN13] in the phrasing of [LK18]. Set $\Delta^{re} \supset \widetilde{\Delta} = \{\alpha_1, \dots, \alpha_n\} \cup \{\alpha_i + \alpha_j \mid (i,j) \in \mathcal{E}(A)\}$.

Proposition 3.15. *Let $\mathfrak{k}(A)$ be simply-laced, then a map $\tau : \widetilde{\Delta} \to \mathrm{End}\,(V)$ which satisfies*

$$[\tau(\alpha), \tau(\beta)] = 0 \qquad \text{if } (\alpha|\beta) = 0 \tag{57}$$

$$\{\tau(\alpha), \tau(\beta)\} = \tau(\alpha \pm \beta) \quad \text{if } (\alpha|\beta) = \mp 1 \text{and } \alpha \pm \beta \in \widetilde{\Delta} \tag{58}$$

for all $\alpha, \beta \in \widetilde{\Delta}$ provides a finite-dimensional representation σ of $\mathfrak{k}$ via the assignment $\sigma(X_i) := \tau(\alpha_i) \otimes 2\rho(X_i) \in \mathrm{End}\,(V \otimes S)$, where $X_1, \ldots, X_n$ denote the Berman generators of $\mathfrak{k}$.

Proof. This is [KN13, eq. (5.1)], the above phrasing is as in [LK18]. □

A technical consequence of this is the following

Lemma 3.16. *Let $\tau : \widetilde{\Delta} \to \mathrm{End}\,(V)$ satisfy equations (57) and (58). Let (i,j), $(j,k) \in \mathcal{E}(A)$ but $(i,k) \notin \mathcal{E}(A)$, then*

$$\sigma([X_i, X_j]) = \tau(\alpha_i + \alpha_j) \otimes 2\rho([X_i, X_j]) \tag{59}$$

$$\sigma([X_i, [X_j, X_k]]) = \tau(\alpha_i + \alpha_j + \alpha_k) \otimes 2\rho([X_i, [X_j, X_k]]) \ . \tag{60}$$

Remark. Note that the second factor in the tensor product is proportional to a generalized Γ-matrix and that the prefactor ± 1 depends on the order of the simple roots.

Proof. One computes

$$\begin{aligned}
[\sigma(X_i), \sigma(X_j)] &= \tau(\alpha_i)\tau(\alpha_j) \otimes 4\rho(X_i)\rho(X_j) - \tau(\alpha_j)\tau(\alpha_i) \otimes 4\rho(X_j)\rho(X_i) \\
&= \tau(\alpha_i)\tau(\alpha_j) \otimes 4\rho(X_i)\rho(X_j) + \tau(\alpha_j)\tau(\alpha_i) \otimes 4\rho(X_i)\rho(X_j) \\
&\quad -\tau(\alpha_j)\tau(\alpha_i) \otimes 4\rho(X_i)\rho(X_j) - \tau(\alpha_j)\tau(\alpha_i) \otimes 4\rho(X_j)\rho(X_i) \\
&= \{\tau(\alpha_i), \tau(\alpha_j)\} \otimes 4\rho(X_i)\rho(X_j) - \tau(\alpha_j)\tau(\alpha_i) \otimes 4\{\rho(X_i)\rho(X_j)\}
\end{aligned}$$

regardless of $(i,j) \in \mathcal{E}(A)$ or not. If $(i,j) \in \mathcal{E}(A)$ as is assumed then $\{\rho(X_i), \rho(X_j)\} = 0$ and $\{\tau(\alpha_i), \tau(\alpha_j)\} = \tau(\alpha_i + \alpha_j)$ so that one has

$$[\sigma(X_i), \sigma(X_j)] = \tau(\alpha_i + \alpha_j) \otimes 4\rho(X_i)\rho(X_j)$$

and since $\rho(X_i)\rho(X_j) = \frac{1}{2}[\rho(X_i)\rho(X_j) - \rho(X_j)\rho(X_i)] = \frac{1}{2}[\rho(X_i), \rho(X_j)]$ one finds

$$[\sigma(X_i), \sigma(X_j)] = \tau(\alpha_i + \alpha_j) \otimes 2\rho([X_i, X_j])$$

as desired. In addition, if i, j, k are as assumed then

$$\begin{aligned}
\{\rho(X_i), [\rho(X_j), \rho(X_k)]\} &= \rho(X_i)\rho(X_j)\rho(X_k) - \rho(X_i)\rho(X_k)\rho(X_j) \\
&\quad +\rho(X_j)\rho(X_k)\rho(X_i) - \rho(X_k)\rho(X_j)\rho(X_i) \\
&= 2\rho(X_i)\rho(X_j)\rho(X_k) + 2\rho(X_j)\rho(X_k)\rho(X_i) \\
&= 0
\end{aligned}$$

because $\rho(X_k)\rho(X_i) = \rho(X_i)\rho(X_k)$ and $\rho(X_j)\rho(X_i) = -\rho(X_i)\rho(X_j)$. With this and the previous result one computes

$$\begin{aligned}
\sigma([X_i, [X_j, X_k]]) &= [\sigma(X_i), [\sigma(X_j), \sigma(X_k)]] \\
&= \{\tau(\alpha_i), \tau(\alpha_j + \alpha_k)\} \otimes 4\rho(X_i)\rho([X_j, X_k]) \\
&\quad -\tau(\alpha_j + \alpha_k)\tau(\alpha_i) \otimes \{2\rho(X_i), 2\rho([X_j, X_k])\} \\
&= \tau(\alpha_i + \alpha_j + \alpha_k) \otimes \frac{4}{2}[\rho(X_i), \rho([X_j, X_k])] - 0 \\
&= \tau(\alpha_i + \alpha_j + \alpha_k) \otimes 2\rho([X_i, [X_j, X_k]])
\end{aligned}$$

This completes the proof. □

There are currently 3 nontrivial maps known (compare [KN17]) which satisfy eqs. (57) and (58), two of which I have discussed in [LK18] from a mathematical perspective. Back in 2018 it was unclear how a "coordinate-free" form of the $\frac{7}{2}$-spin representation of [KN17] would look like. It turns out that the perspective proposed by Paul Levy (see [LK18, rem. 4.2]) is the most useful towards a unified coordinate-free description of these representations.

Let $\eta : W(A) \to \mathrm{End}(V)$ be a finite-dimensional representation of the Weyl group $W(A)$ for a simply-laced GCM A. Let α, β denote real roots with $(\alpha, \beta) = -1$ and let s_α denote the reflection w.r.t. α. Then $\mathfrak{S}_{\alpha,\beta} := \langle s_\alpha, s_\beta \rangle$ is a subgroup of $W(A)$ which is isomorphic to $\mathfrak{S}_3$, the symmetric group on three letters. Now $\mathfrak{S}_3$ possesses three distinct irreducible representations called the trivial, the sign and the standard representation and denoted by U, U' and E respectively. All its finite-dimensional representations are completely reducible. The three conjugacy classes of $\mathfrak{S}_3$ are given by cycles of different length:

$$\mathcal{C}_1 := [e]\,,\ \mathcal{C}_2 := [(12)]\,,\ \mathcal{C}_3 := [(123)]$$

and the character table is

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
U	1	1	1
U'	1	-1	1
E	2	0	-1

Table 1: Character table of $\mathfrak{S}_3$.

Proposition 3.17. *(This is [LK18, rem. 4.2 (iv)], which is based on a remark by Paul Levy)*
Let $\eta : W(A) \to End(V)$ be a finite-dimensional representation of the Weyl group $W(A)$ for a simply-laced GCM A. Then

$$\tau : \Delta^{re}(A) \to End(V)\,,\ \alpha \mapsto \eta(s_\alpha) - \frac{1}{2}Id \tag{61}$$

satisfies eqs. (57) and (58) if the restriction of η to any $\mathfrak{S}_{\alpha_i,\alpha_j}$ such that α_i, α_j are adjacent simple roots does not contain the sign representation of $\mathfrak{S}_3$ as an irreducible factor.

Proof. If $(\alpha|\beta) = 0$ then s_α, s_β commute and so do $\tau(\alpha)$, $\tau(\beta)$. For $(\alpha|\beta) = -1$ one has

$$\begin{aligned}\{\tau(\alpha), \tau(\beta)\} &= \left\{\eta(s_\alpha) - \frac{1}{2}Id, \eta(s_\beta) - \frac{1}{2}Id\right\} \\ &= \eta(s_\alpha)\eta(s_\beta) + \eta(s_\beta)\eta(s_\alpha) - \eta(s_\alpha) - \eta(s_\beta) + \frac{1}{2}Id\end{aligned}$$

and with $s_{\alpha+\beta} = s_\beta s_\alpha s_\beta$ one has

$$\tau(\alpha+\beta) = \eta(s_\beta s_\alpha s_\beta) - \frac{1}{2}Id$$

so that $\tau(\alpha+\beta) = \{\tau(\alpha), \tau(\beta)\}$ is equivalent to

$$0 \overset{!}{=} -\eta(s_\beta s_\alpha s_\beta) + \eta(s_\alpha)\eta(s_\beta) + \eta(s_\beta)\eta(s_\alpha) - \eta(s_\alpha) - \eta(s_\beta) + Id.$$

For the trivial representation this is easily seen to be true, whereas it is false for the sign representation as then

$$-1 = \eta\left(s_\beta s_\alpha s_\beta\right) = \eta\left(s_\alpha\right) = \eta\left(s_\beta\right).$$

One can set up the standard representation as the subspace $\mathrm{span}_{\mathbb{R}}\left\{\alpha,\beta\right\} \subset \mathfrak{h}^*$. In this basis one has

$$s_\alpha = \begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix},\ s_\beta = \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix},\ s_{\alpha+\beta} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$$

so that one computes

$$\left\{s_\alpha, s_\beta\right\} = \begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix} + \begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

and

$$\begin{aligned}-s_{\alpha+\beta} + \left\{s_\alpha, s_\beta\right\} - s_\alpha - s_\beta + Id &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= 0.\end{aligned}$$

Now $\mathfrak{S}_3$ has only these three irreducible representations and as any finite-dimensional representation of $\mathfrak{S}_3$ is completely reducible one concludes that (61) provides a representation if η contains no copies of the sign representation when restricted to $\mathfrak{S}_{\alpha_i,\alpha_j}$. □

Proposition 3.18. *Let $V_1 := \mathfrak{h}^*$ and $V_2 := Sym^2\left(\mathfrak{h}^*\right)$ be the dual Cartan subalgebra and its symmetric product with itself, respectively. Then the standard representation $W\left(A\right) \to O\left(\mathfrak{h}^*\right)$ and the induced representation on V_2 contain no copies of the sign representation when restricted to any $\mathfrak{S}_{\alpha_i,\alpha_j}$ for $(i,j) \in \mathcal{E}(A)$.*

Proof. For $\mathfrak{h}^*$ and the restriction to $\mathfrak{S}_{\alpha_i,\alpha_j}$ consider the basis $\left\{\alpha_i,\alpha_j\right\} \cup \left\{b_1,\ldots,b_{m-2}\right\}$ where the b_i are orthogonal to both α_i and α_j and $m := \dim\mathfrak{h}^*$. Now span $\left\{\alpha_i,\alpha_j\right\}$ forms a copy of the standard representation of $\mathfrak{S}_{\alpha_i,\alpha_j}$ while span $\left\{b_1,\ldots,b_{m-2}\right\}$ decomposes into $m-2$ copies of the trivial representation. For $\mathrm{Sym}^2\left(\mathfrak{h}^*\right)$ one considers symmetric products of the above basis:

$$\left\{\alpha_i\alpha_i,\ \alpha_i\alpha_j,\ \alpha_j\alpha_j\right\} \cup \left\{\alpha_i b_k, \alpha_j b_k \mid k = 1,\ldots,m-2\right\} \cup \left\{b_k b_l \mid 1 \le k \le l \le m-2\right\}.$$

Now each $b_k b_l$ spans a trivial representation, while each pair $\left\{\alpha_i b_k, \alpha_j b_k\right\}$ spans a standard representation. For the last piece it is better to use the alternative basis elements $\left\{\alpha_i\alpha_i,\ \alpha_j\alpha_j,\ \left(\alpha_i+\alpha_j\right)\left(\alpha_i+\alpha_j\right)\right\}$. Then one sees that $\alpha_i\alpha_i + \alpha_j\alpha_j + \left(\alpha_i+\alpha_j\right)\left(\alpha_i+\alpha_j\right)$ spans a trivial representation while their "trace-free" linear combinations

$$\left\{a_1\cdot\alpha_i\alpha_i + a_2\cdot\alpha_j\alpha_j + a_3\cdot\left(\alpha_i+\alpha_j\right)\left(\alpha_i+\alpha_j\right) \mid a_1+a_2+a_3 = 0\right\}$$

form a standard representation. □

Theorem 3.19. *Let $\left(\eta_1, V_1\right)$ and $\left(\eta_2, V_2\right)$ be the representations of the Weyl group from the previous proposition. Then for $n \in \{1,2\}$ the map $\tau_n : \widetilde{\Delta} \to End\left(V_n\right)$, $\tau_n(\alpha) = \eta_n\left(s_\alpha\right) - \frac{1}{2}Id$ satisfies eqs. (57) and (58). The assignment*

$$\sigma_{\frac{2n+1}{2}} : X_i \mapsto \tau_n\left(\alpha_i\right) \otimes 2\rho\left(X_i\right),$$

where $X_1,\ldots,X_m$ denote the Berman generators of $\mathfrak{k}$ and ρ denotes the $\frac{1}{2}$-spin representation from theorem 3.3, extends to a representation of $\mathfrak{k}$. This representation is called the ***$\frac{2n+1}{2}$-spin representation*** *in [KN17] and [LK18].*

Proof. This is an immediate consequence of the previous proposition because one can apply proposition 3.15 whenever the restriction of η_n to $\mathfrak{S}_{\alpha_i,\alpha_j}$ does not contain a copy of the sign representation. Note that proposition 3.15 can be applied even if A is not regular. □

The above ansatz does not work for $V_3 := \mathrm{Sym}^3(\mathfrak{h}^*)$, however. This is because for adjacent simple roots α, β one can check that $\alpha\beta(\alpha+\beta)$ spans a sign representation of $\mathfrak{S}_{\alpha,\beta}$. This sign representation is in fact the only one that appears in the decomposition of V_3, a fact that I will collect for future reference:

Lemma 3.20. *For a simply-laced GCM A the induced representation of the Weyl group $W(A)$ on $Sym^3(\mathfrak{h}^*)$ contains exactly one copy of the sign representation when restricted to $\mathfrak{S}_{\alpha,\beta}$ for adjacent simple roots $\alpha, \beta \in \Pi(A)$. It is spanned by $\alpha\beta(\alpha+\beta) \in Sym^3(\mathfrak{h}^*)$.*

Proof. Again one uses a basis for $\mathfrak{h}^*$ that uses α, β and elements $b_1, \ldots, b_{m-2}$ orthogonal to both of them. Then elements which contain at least one b_k behave like copies of V or $\mathrm{Sym}^2(V)$ and therefore span trivial or standard representations according to prop. 3.18. The only subspace left to consider is therefore elements which contain only α and/or β. It is 4-dimensional and one checks that $\alpha\beta(\alpha+\beta)$ is a copy of the sign representation. One could try to find a basis for the remaining representations or one can use the characters of $\mathfrak{S}_{\alpha,\beta}$. One has

$$s_\alpha : \alpha\alpha\alpha \mapsto -\alpha\alpha\alpha,\ \alpha\alpha\beta \mapsto \alpha\alpha\alpha + \alpha\alpha\beta,$$

$$s_\alpha : \alpha\beta\beta \mapsto -\alpha\alpha\alpha - 2\alpha\alpha\beta - \alpha\beta\beta,\ \beta\beta\beta \mapsto \alpha\alpha\alpha + 3\alpha\alpha\beta + 3\alpha\beta\beta + \beta\beta\beta,$$

$$s_\beta : \alpha\alpha\alpha \mapsto \alpha\alpha\alpha + 3\alpha\alpha\beta + 3\alpha\beta\beta + \beta\beta\beta,\ \alpha\alpha\beta \mapsto -\alpha\alpha\beta - 2\alpha\beta\beta - \beta\beta\beta,$$

$$s_\beta : \alpha\beta\beta \mapsto \alpha\beta\beta + \beta\beta\beta,\ \beta\beta\beta \mapsto -\beta\beta\beta,$$

so that in this basis

$$s_\alpha = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ s_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix},\ s_\alpha s_\beta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & -1 & 2 & -3 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

From this one computes the character of this representation (the class $\mathcal{C}_3$ is that of $s_\alpha s_\beta$ while that of s_α and s_β is $\mathcal{C}_2$):

$$\chi = (4, 0, 1)\,.$$

A comparison to the irreducible characters of $\mathfrak{S}_3$ (cp. table 1),

$$\chi_U = (1,1,1)\,,\ \chi_{U'} = (1,-1,1)\,,\ \chi_V = (2,0,-1)\,,$$

shows that

$$\chi = \chi_U + \chi_{U'} + \chi_V\,.$$

Indeed, the sign representation occurs exactly once. □

3.3 The higher spin representation $\mathcal{S}_{\frac{7}{2}}$

This section is dedicated to how to fix the Weyl-group type ansatz for extended spin representations of simply-laced $\mathfrak{k}(A) = \mathrm{Fix}_\omega(\mathfrak{g}(A))$ over the module $\mathrm{Sym}^3 V \otimes \mathcal{S}$. The main result is thm. 3.23 which is a coordinate free version of the $\frac{7}{2}$-spin representation described in [KN13]. Throughout, $V := \mathfrak{h}^*$ denotes the dual of $\mathfrak{g}(A)$'s Cartan subalgebra and $(\rho, \mathcal{S})$ denotes the $\frac{1}{2}$-spin representation of $\mathfrak{k}(A)$. I will denote the standard invariant form on $\mathfrak{h}^*$ by $Q(\cdot,\cdot)$ in this section because it increases readability in comparison with $(\cdot|\cdot)$. The full representation σ is given on the level of the Berman generators $X_1, \ldots, X_m$ via

$$\sigma(X_i) = \tau(\alpha_i) \otimes 2\rho(X_i)$$

where τ is defined in terms of the real roots of $\mathfrak{g}(A)$. According to prop. 3.15, σ provides a representation if

$$\begin{aligned} [\tau(\alpha), \tau(\beta)] &= 0 \text{ if } Q(\alpha,\beta) = 0 && (62)\\ \{\tau(\alpha), \tau(\beta)\} &= \tau(\alpha \pm \beta) \text{ if } Q(\alpha,\beta) = \mp 1 && (63) \end{aligned}$$

It therefore suffices to consider the map τ but first, I will review the structure of $\mathrm{Sym}^3 V$ in more detail. Fix a normalization on $\mathrm{Sym}^3 V$ w.r.t. $V^{\otimes 3}$ by setting

$$\mathrm{Sym}^3 V \ni v_1 \cdot v_2 \cdot v_3 := \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} \;. \quad (64)$$

As $V = \mathfrak{h}^*$ possesses a non-degenerate bilinear form Q (cp. 1.9) there exists an induced form on $V^{\otimes 3}$ and $\mathrm{Sym}^3 V$ given by

$$\begin{aligned} Q(v_1 \cdot v_2 \cdot v_3\, ,\, u_1 \cdot u_2 \cdot u_3) &= \left(\frac{1}{3!}\right)^2 \sum_{\sigma,\tau \in \mathfrak{S}_3} Q\left(v_{\sigma(1)}, u_{\tau(1)}\right) \ldots Q\left(v_{\sigma(3)}, u_{\tau(3)}\right) \\ &= \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} Q\left(v_{\sigma(1)}, u_1\right) \ldots Q\left(v_{\sigma(3)}, u_3\right) \end{aligned}$$

Let $e_1, \ldots, e_m$ be a basis of V and set

$$\omega_{ij} := Q(e_i, e_j)\,,\ \left(\omega^{ij}\right) := (\omega_{ij})^{-1} \;\Leftrightarrow\; \sum_{k,l} \omega^{kl} \omega_{ln} = \delta^k_n. \quad (65)$$

Definition 3.21. Define the **symmetric insertion** $\psi : V \to \mathrm{Sym}^3 V$ via

$$\psi(v) \;=\; \frac{1}{3!} \cdot \sum_{k,l=1}^{m} \omega^{kl} \left(v \otimes e_k \otimes e_l + e_k \otimes v \otimes e_l + e_k \otimes e_l \otimes v\right) \;.$$

Symmetric insertions play an important role in invariant theory and one can show that this map does not depend on the chosen basis (cp. [FH91, secs. 17.3 & 19.5]). The analogous element in $\mathrm{Sym}^2 V$, $\Psi := \sum_{k,l} \omega^{kl} e_k \otimes e_l$, spans the one-dimensional trivial submodule under the action of $O(V)$. Start with an ansatz

$$\tau(\alpha) = s_\alpha^3 - \frac{1}{2} Id + f(\alpha) \quad \forall\, \alpha \in \Delta^{re}(A)\,, \quad (66)$$

where s_α^3 denotes the induced Weyl reflection w.r.t. α on $\mathrm{Sym}^3 V$. As there is only one copy of the sign representation it seems plausible to assume $f(\alpha)$ to be of rank 1. I will go even one step further and associate an element $v(\alpha) \in \mathrm{Sym}^3 V$ to α such that setting

$$f(\alpha) \coloneqq v(\alpha) \cdot Q(v(\alpha) \,|\, \cdot)$$

in the ansatz (66) solves (62), (63). The more general approach would be to replace one $v(\alpha)$ by an element $w(\alpha)$ that can be adjusted independently. However, the result will be that $v(\alpha)$ needs to equal $w(\alpha)$. As the computation is lengthy, I decided to make it more tractable by assuming $v(\alpha) = w(\alpha)$ from the start. Now for $\mathrm{Sym}^3 V$ there are only two vectors which are related to α in a meaningful way: $\alpha\alpha\alpha$ and $\psi(\alpha)$. Thus, set

$$v(\alpha) = p \cdot \alpha\alpha\alpha + q \cdot \psi(\alpha) \ . \tag{67}$$

Lemma 3.22. *For $\alpha, \beta \in \Delta^{re}(A)$ and A simply-laced one has with $m \coloneqq \dim V$ that*

$$s_\alpha^3(\psi(\beta)) = \psi(s_\alpha \beta),$$
$$Q(\psi(\alpha), \psi(\beta)) = \frac{m+2}{12} Q(\alpha, \beta) \ , \ Q(\alpha\alpha\alpha, \psi(\beta)) = Q(\alpha, \beta) \ .$$

Proof. The first statement reduces to the two-dimensional case:

$$\begin{aligned} s_\alpha\left(\sum_{k,l=1}^m \omega^{kl} e_k \otimes e_l\right) &= \sum_{k,l} \omega^{kl} \left(\sum_a S(\alpha)^a_k e_a\right) \otimes \left(\sum_b S(\alpha)^b_l e_b\right) \\ &= \sum_{k,l,a,b} \omega^{kl} S(\alpha)^a_k S(\alpha)^b_l e_a \otimes e_b \\ &= \sum_{a,b} \omega^{ab} e_a \otimes e_b, \end{aligned}$$

where $S(\alpha)^a_k$ denotes the representation matrix of $s_\alpha$on $\mathfrak{h}^*$. The last line follows from the definition of what it means for a linear map s_α to leave the nondegenerate bilinear form Q invariant. Hence, s_α intertwines with ψ, i.e. $s_\alpha^3 \circ \psi = \psi \circ s_\alpha$. The other statements can be computed directly:

$$\begin{aligned} Q(\psi(\alpha), v_1 v_2 v_3) &= \frac{1}{36} \sum_{\sigma \in \mathfrak{S}_3} \sum_{k,l=1}^m \omega^{kl} Q(\alpha \otimes e_k \otimes e_l + e_k \otimes \alpha \otimes e_l + e_k \otimes e_l \otimes \alpha, \\ &\quad v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}) \\ &= \frac{1}{36} \sum_{\sigma \in \mathfrak{S}_3} \sum_{k,l=1}^m \omega^{kl} \left[Q\left(\alpha, v_{\sigma(1)}\right) Q\left(e_k, v_{\sigma(2)}\right) Q\left(e_l, v_{\sigma(3)}\right) + \right. \\ &\quad \left. Q\left(e_k, v_{\sigma(1)}\right) Q\left(\alpha, v_{\sigma(2)}\right) Q\left(e_l, v_{\sigma(3)}\right) + Q\left(e_k, v_{\sigma(1)}\right) Q\left(e_l, v_{\sigma(2)}\right) Q\left(\alpha, v_{\sigma(3)}\right)\right] \\ &= \frac{1}{36} \sum_{\sigma \in \mathfrak{S}_3} \left[Q\left(\alpha, v_{\sigma(1)}\right) Q\left(v_{\sigma(2)}, v_{\sigma(3)}\right) + Q\left(\alpha, v_{\sigma(2)}\right) Q\left(v_{\sigma(1)}, v_{\sigma(3)}\right)\right. \\ &\quad \left. + Q\left(\alpha, v_{\sigma(3)}\right) Q\left(v_{\sigma(1)}, v_{\sigma(2)}\right)\right] \\ &= \frac{1}{12} \sum_{\sigma \in \mathfrak{S}_3} Q\left(\alpha, v_{\sigma(1)}\right) Q\left(v_{\sigma(2)}, v_{\sigma(3)}\right) \\ &= \frac{1}{6} \left[Q(\alpha, v_1) Q(v_2, v_3) + Q(\alpha, v_2) Q(v_1, v_3) + Q(\alpha, v_3) Q(v_1, v_2)\right] \ . \end{aligned}$$

For $v_1v_2v_3 = \beta\beta\beta$ this specializes to

$$Q\left(\psi(\alpha), \beta\beta\beta\right) = \frac{1}{2} Q\left(\alpha, \beta\right) Q\left(\beta, \beta\right) = Q\left(\alpha, \beta\right),$$

since for $\beta \in \Delta^{re}(A)$ one has $Q(\beta, \beta) = 2$ as A is simply-laced. Also, one computes

$$\begin{aligned}
Q\left(\psi\left(\alpha\right), \psi\left(\beta\right)\right) &= \frac{1}{36}\sum_{k,l,i,j} \omega^{kl}\omega^{ij} Q\left(\alpha\otimes e_k\otimes e_l + e_k\otimes\alpha\otimes e_l + e_k\otimes e_l\otimes\alpha,\right.\\
&\left.\beta\otimes e_i\otimes e_j + e_i\otimes\beta\otimes e_j + e_i\otimes e_j\otimes\beta\right)\\
&= \frac{1}{36}\sum_{k,l,i,j}\omega^{kl}\omega^{ij}\left[Q\left(\alpha,\beta\right)\omega_{ki}\omega_{lj} + Q\left(\alpha, e_i\right) Q\left(e_k, \beta\right)\omega_{lj} + Q\left(\alpha, e_i\right)\omega_{kj} Q\left(e_l, \beta\right)\right.\\
&+ Q\left(e_k, \beta\right) Q\left(\alpha, e_i\right)\omega_{lj} + \omega_{ki} Q\left(\alpha, \beta\right)\omega_{lj} + \omega_{ki} Q\left(\alpha, e_j\right) Q\left(e_l, \beta\right)\\
&\left. Q\left(e_k, \beta\right)\omega_{li} Q\left(\alpha, e_j\right) + \omega_{ki} Q\left(e_l, \beta\right) Q\left(\alpha, e_j\right) + \omega_{ki}\omega_{lj} Q\left(\alpha, \beta\right)\right]\\
&= \frac{1}{36}\left[3Q\left(\alpha,\beta\right)\left(\sum_{k,l,i,j}\omega^{kl}\omega^{ij}\omega_{ki}\omega_{lj}\right) + 6Q\left(\alpha,\beta\right)\right]\\
&= \frac{1}{36}\left(3m+6\right) Q\left(\alpha,\beta\right) = \frac{m+2}{12} Q\left(\alpha,\beta\right),
\end{aligned}$$

where one uses

$$\sum_{k,l,i,j}\omega^{kl}\omega^{ij}\omega_{ki}\omega_{lj} = \sum_{k,i,j}\delta^k_j\omega^{ij}\omega_{ki} = \sum_{k,i}\omega^{ik}\omega_{ki} = \sum_i \delta^i_i = m.$$

□

Theorem 3.23. *Let A be a simply-laced GCM and let ρ denote the $\frac{1}{2}$-spin representation of $\mathfrak{k}(A)$ from theorem 3.3. The assignment*

$$\sigma\left(X_i\right) = \tau\left(\alpha_i\right) \otimes 2\rho\left(X_i\right)\ \ \forall\, i = 1, \ldots, m$$

on the level of Berman generators extends to a representation of $\mathfrak{k}(A)$ if τ satisfies the equations (62) and (63). The ansatz from eq. (66) is

$$\tau(\alpha) = s^3_\alpha - \frac{1}{2} Id + v\left(\alpha\right) Q\left(v\left(\alpha\right)|\cdot\right) \in End\left(Sym^3\left(\mathfrak{h}^*\right)\right)\ \ \forall\, \alpha \in \Delta^{re}\left(A\right)$$

with $v\left(\alpha\right) = p\cdot\alpha\alpha\alpha + q\cdot\psi\left(\alpha\right)$. This ansatz satisfies the equations (62) and (63) if one fixes p and q to be ($\varepsilon = \pm 1$, $m := \dim\mathfrak{h}$)

$$q_{\pm}\left(p_\varepsilon\right) = -\varepsilon\frac{12 \mp 2\sqrt{6(m+8)}}{(m+2)\sqrt{3}},\ \ p_\varepsilon = \varepsilon\frac{1}{\sqrt{3}}. \tag{68}$$

The representation is denoted by $\mathcal{S}_{\frac{7}{2}}$ in the remainder of the text and called the $\frac{7}{2}$-spin representation as in [KN13].

Proof. In the ansatz (66), eq. (62) spells out to be

$$\begin{aligned}
\left[\tau(\alpha), \tau(\beta)\right] &= \left[s^3_\alpha - \frac{1}{2}Id + f(\alpha), s^3_\beta - \frac{1}{2}Id + f(\beta)\right]\\
&= \left[s^3_\alpha, s^3_\beta\right] + \left[s^3_\alpha, f(\beta)\right] + \left[f(\alpha), s^3_\beta\right]\\
&+ \left[f(\alpha), f(\beta)\right]
\end{aligned}$$

which needs to vanish for $Q(\alpha,\beta)=0$. In the case of $Q(\alpha,\beta)=\mp 1$ one has

$$\begin{aligned}\{\tau(\alpha),\tau(\beta)\} &= \left\{s_\alpha^3-\frac{1}{2}Id+f(\alpha), s_\beta^3-\frac{1}{2}Id+f(\beta)\right\}\\ &\left\{s_\alpha^3-\frac{1}{2}Id, s_\beta^3-\frac{1}{2}Id\right\}+\left\{s_\alpha^3,f(\beta)\right\}+\left\{f(\alpha),s_\beta^3\right\}\\ &+\{f(\alpha),f(\beta)\}-f(\alpha)-f(\beta)\\ &\stackrel{!}{=} s_{\alpha\pm\beta}^3-\frac{1}{2}Id+f(\alpha\pm\beta)\ .\end{aligned}$$

This equation is satisfied if and only if

$$\left\{s_\alpha^3-\frac{1}{2}Id, s_\beta^3-\frac{1}{2}Id\right\}+T(\alpha,\beta)=s_{\alpha\pm\beta}^3-\frac{1}{2}Id, \tag{69}$$

where

$$T(\alpha,\beta)=\left\{s_\alpha^3,f(\beta)\right\}+\left\{f(\alpha),s_\beta^3\right\}+\{f(\alpha),f(\beta)\}-f(\alpha)-f(\beta)-f(\alpha\pm\beta)\ . \tag{70}$$

By lemma 3.22 one has $s_\beta(v(\alpha))=v(\alpha)$ for all β s.t. $Q(\alpha,\beta)=0$ as well as $Q(v(\alpha),v(\beta))=0$. Thus, the commutation relation (62) is satisfied because the Weyl reflections s_α and s_β commute as well. For the more interesting case of $Q(\alpha,\beta)=\mp 1$, eq. (69) needs to be satisfied. From prop. 3.17 it is known that

$$\left\{s_\alpha^3-\frac{1}{2}Id, s_\beta^3-\frac{1}{2}Id\right\}=s_{\alpha\pm\beta}^3-\frac{1}{2}Id$$

holds on all representations of $\mathfrak{S}_3=\langle s_\alpha,s_\beta\rangle$ which do not contain the sign representation. Hence, support and image of $T(\alpha,\beta)$ (defined in eq. (70)) must be the subspace of Sym^3V which is spanned by the copies of the sign representation of $\mathfrak{S}_3=\langle s_\alpha,s_\beta\rangle$. From lemma 3.20 one knows that there is only one copy and that it is spanned by $\alpha\beta(\alpha+\beta)$. One computes

$$s_\alpha^3 v(\beta)=v(\beta\pm\alpha)=\pm v(\alpha\pm\beta)=-Q(\alpha,\beta)\cdot v(\alpha\pm\beta)$$

as well as

$$\begin{aligned}v(\beta)\cdot Q\left(v(\beta),s_\alpha^3(u_1u_2u_3)\right) &= v(\beta)\cdot Q(v(s_\alpha\beta),u_1u_2u_3)=v(\beta)\cdot Q(v(\beta\pm\alpha),u_1u_2u_3)\\ &= -Q(\alpha,\beta)\cdot v(\beta)\cdot Q(v(\alpha\pm\beta),u_1u_2u_3)\ \ \forall\, u_1u_2u_3\in\mathrm{Sym}^3V\ .\end{aligned}$$

Thus,

$$\begin{aligned}\left\{s_\alpha^3,f(\beta)\right\}+\left\{f(\alpha),s_\beta^3\right\} &= v(\alpha)\cdot Q(v(\alpha\pm\beta),\cdot)+v(\alpha\pm\beta)\cdot Q(v(\alpha),\cdot)\\ &-Q(\alpha,\beta)\cdot v(\beta)\cdot Q(v(\alpha\pm\beta),\cdot)-Q(\alpha,\beta)\cdot v(\alpha\pm\beta)\cdot Q(v(\beta),\cdot)\ .\end{aligned}$$

In addition one has with $X(\alpha,\beta):=Q(v(\alpha),v(\beta))$

$$\{f(\alpha),f(\beta)\}=X(\alpha,\beta)\left[v(\alpha)\cdot Q(v(\beta),\cdot)+v(\beta)\cdot Q(v(\alpha),\cdot)\right]\ .$$

With this, one determines $T(\alpha,\beta)$ to be

$$\begin{aligned} T(\alpha,\beta) &= \left\{s_\alpha^3, f(\beta)\right\} + \left\{f(\alpha), s_\beta^3\right\} + \left\{f(\alpha), f(\beta)\right\} - f(\alpha) - f(\alpha) - f(\alpha\pm\beta) \\ &= v(\alpha)\cdot Q(v(\alpha\pm\beta),\cdot) + v(\alpha\pm\beta)\cdot Q(v(\alpha),\cdot) - Q(\alpha,\beta)\cdot v(\beta)\cdot Q(v(\alpha\pm\beta),\cdot) \\ &\quad -Q(\alpha,\beta)\cdot v(\alpha\pm\beta)\cdot Q(v(\beta),\cdot) \\ &\quad +X(\alpha,\beta)\left[v(\alpha)\cdot Q(v(\beta),\cdot) + v(\beta)\cdot Q(v(\alpha),\cdot)\right] \\ &\quad -v(\alpha)\cdot Q(v(\alpha),\cdot) - v(\beta)\cdot Q(v(\beta),\cdot) - v(\alpha\pm\beta)\cdot Q(v(\alpha\pm\beta),\cdot) \\ &= v(\alpha)\cdot Q(v(\alpha\pm\beta) + X(\alpha,\beta)\,v(\beta) - v(\alpha),\,\cdot) \\ &\quad +v(\beta)\cdot Q(X(\alpha,\beta)\,v(\alpha) - Q(\alpha,\beta)\,v(\alpha\pm\beta) - v(\beta),\,\cdot) \\ &\quad +v(\alpha\pm\beta)\cdot Q(v(\alpha) - Q(\alpha,\beta)\cdot v(\beta) - v(\alpha\pm\beta),\,\cdot)\,. \end{aligned}$$

Now the demand that $T(\alpha,\beta)$ may only be supported on the sign representation which is spanned by the vector $V_{\alpha,\beta} := \alpha\cdot\beta\cdot(\alpha\pm\beta)$ for $\mathrm{Sym}^3 V$ leads to three equations:

$$\begin{aligned} v(\alpha\pm\beta) + X(\alpha,\beta)\,v(\beta) - v(\alpha) &= k_1\cdot V_{\alpha,\beta}, \\ X(\alpha,\beta)\,v(\alpha) - Q(\alpha,\beta)\,v(\alpha\pm\beta) - v(\beta) &= k_2\cdot V_{\alpha,\beta}, \\ v(\alpha) - Q(\alpha,\beta)\cdot v(\beta) - v(\alpha\pm\beta) &= k_3\cdot V_{\alpha,\beta}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} v(\alpha\pm\beta) - v(\alpha) + X(\alpha,\beta)\,v(\beta) &= k_1\cdot V_{\alpha,\beta}, \\ -Q(\alpha,\beta)\left[v(\alpha\pm\beta) - Q(\alpha,\beta)^{-1}X(\alpha,\beta)\,v(\alpha) + Q(\alpha,\beta)^{-1}v(\beta)\right] &= k_2\cdot V_{\alpha,\beta}, \\ -\left[v(\alpha\pm\beta) - v(\alpha) + Q(\alpha,\beta)\cdot v(\beta)\right] &= k_3\cdot V_{\alpha,\beta}. \end{aligned}$$

By the definition of $v(\alpha)$ and linearity of ψ one has

$$\begin{aligned} v(\alpha\pm\beta) - v(\alpha) + \underbrace{Q(\alpha,\beta)}_{=\mp1}\cdot v(\beta) &= p\cdot\left[(\alpha\pm\beta)^3 - \alpha^3 \mp \beta^3\right] + q\cdot\left[\psi(\alpha\pm\beta) - \psi(\alpha)\mp\psi(\beta)\right] \\ &= 3p\left[\pm\alpha\alpha\beta + \alpha\beta\beta\right] = \pm3p\cdot\alpha\beta\,(\alpha\pm\beta) \\ &= -Q(\alpha,\beta)\cdot 3pV_{\alpha,\beta}. \end{aligned}$$

Therefore, one has to demand that

$$X(\alpha,\beta) = Q(\alpha,\beta)$$

in order to satisfy all three equations concerning the support of $T(\alpha,\beta)$. The following computation shows that this solves the corresponding problem towards the image as well.

$$\begin{aligned} T(\alpha,\beta) &= v(\alpha)\cdot Q(v(\alpha\pm\beta) + X(\alpha,\beta)\,v(\beta) - v(\alpha),\,\cdot) \\ &\quad +v(\beta)\cdot Q(X(\alpha,\beta)\,v(\alpha) - Q(\alpha,\beta)\,v(\alpha\pm\beta) - v(\beta),\,\cdot) \\ &\quad +v(\alpha\pm\beta)\cdot Q(v(\alpha) - Q(\alpha,\beta)\cdot v(\beta) - v(\alpha\pm\beta),\,\cdot) \\ &= -Q(\alpha,\beta)\cdot 3p\cdot v(\alpha)\cdot Q(V_{\alpha,\beta},\cdot) \\ &\quad +3p\cdot v(\beta)\cdot Q(V_{\alpha,\beta},\,\cdot) + Q(\alpha,\beta)\cdot 3p\cdot v(\alpha\pm\beta)\cdot Q(V_{\alpha,\beta},\,\cdot) \\ &= 3p\cdot(\pm v(\alpha) + v(\beta) \mp v(\alpha\pm\beta))\,Q(V_{\alpha,\beta},\,\cdot) \\ &= -9p^2\cdot V_{\alpha,\beta}\cdot Q(V_{\alpha,\beta},\,\cdot)\,. \end{aligned}$$

One computes with lemma 3.22 that

$$\begin{aligned} Q\left(v\left(\alpha\right),v\left(\beta\right)\right) &= Q\left(p\alpha\alpha\alpha+q\psi(\alpha),p\beta\beta\beta+q\psi(\beta)\right) \\ &= p^2Q\left(\alpha\alpha\alpha,\beta\beta\beta\right)+pqQ\left(\psi(\alpha),\beta\beta\beta\right)+pqQ\left(\alpha\alpha\alpha,\psi(\beta)\right) \\ &\quad +q^2Q\left(\psi(\alpha),\psi(\beta)\right) \\ &= p^2Q\left(\alpha,\beta\right)^3+2pqQ\left(\alpha,\beta\right)+q^2\frac{m+2}{12}Q\left(\alpha,\beta\right), \end{aligned} \tag{71}$$

so that

$$X\left(\alpha,\beta\right)=Q\left(v\left(\alpha\right),v\left(\beta\right)\right)=Q\left(\alpha,\beta\right)$$

is equivalent to

$$p^2+2pq+\frac{m+2}{12}q^2=1\ . \tag{72}$$

Now that $v\left(\alpha\right)$ has been determined so far as that $T\left(\alpha,\beta\right)$ has correct support and image, one needs to fix p^2 in the ansatz for $v(\alpha)$ such that it solves eq. (63). For this one needs to evaluate all maps on $V_{\alpha,\beta}$:

$$\begin{aligned} \left[\left\{s_\alpha^3-\frac{1}{2}Id,s_\beta^3-\frac{1}{2}Id\right\}+T\left(\alpha,\beta\right)\right]V_{\alpha,\beta} &= \left[s_{\alpha\pm\beta}^3-\frac{1}{2}Id\right]V_{\alpha,\beta} \\ \Leftrightarrow\ \left[2\cdot\left(-\frac{3}{2}\right)^2-9p^2Q\left(V_{\alpha,\beta},V_{\alpha,\beta}\right)\right]V_{\alpha,\beta} &= -\frac{3}{2}V_{\alpha,\beta} \\ \Leftrightarrow\ \left(\frac{9}{2}-18p^2+\frac{3}{2}\right)V_{\alpha,\beta} &= 0 \\ \Leftrightarrow\ p^2 &= \frac{1}{3} \\ \Leftrightarrow\ p_\pm &= \pm\frac{1}{\sqrt{3}}\ . \end{aligned} \tag{73}$$

Plugging this into eq. (72) and solving for q yields

$$q_\pm\left(p_\varepsilon\right)=-\varepsilon\frac{12\mp2\sqrt{6(m+8)}}{(m+2)\sqrt{3}}. \tag{74}$$

□

For later use in section 4, it is convenient to collect the behavior of powers of $f\left(\alpha\right)$:

Lemma 3.24. *Let $\left(\sigma,\mathcal{S}_{\frac{7}{2}}\right)$ be the $\frac{7}{2}$-spin representation described in thm. 3.23. For $\sigma\left(X_i\right)=\tau\left(\alpha_i\right)\otimes2\rho\left(X_i\right)$ with $\tau\left(\alpha\right):=\eta\left(s_\alpha\right)-\frac{1}{2}Id+f\left(\alpha\right)$, where η is the induced representation of the Weyl group on $Sym^3\left(\mathfrak{h}^*\right)$ and $f\left(\alpha\right)\in End\left(Sym^3\left(\mathfrak{h}^*\right)\right)$ as in (66) and (67, one has*

$$f\left(\alpha\right)^2=4f\left(\alpha\right)\ \ \forall\,\alpha\in\Delta_+^{re}. \tag{75}$$

Proof. According to the ansatz (66) one has $f\left(\alpha\right):=v\left(\alpha\right)\cdot Q\left(v\left(\alpha\right)|\cdot\right)\in End\left(\mathrm{Sym}^3\left(\mathfrak{h}^*\right)\right)$ with $v\left(\alpha\right)=p\cdot\alpha\alpha\alpha+q\cdot\psi\left(\alpha\right)$. As σ is assumed to be a representation on must only consider the values

$$p_\pm=\pm\frac{1}{\sqrt{3}},\quad q_\pm\left(p_\varepsilon\right)=-\varepsilon\frac{12\mp2\sqrt{6(m+8)}}{(m+2)\sqrt{3}}$$

according to prop. 3.23. One immediately sees that

$$f(\alpha)^2 = v(\alpha) \cdot Q(v(\alpha)|v(\alpha)) \cdot Q(v(\alpha)|\cdot) = a \cdot f(\alpha)$$

and together with (71) one has

$$\begin{aligned} a &= Q(v(\alpha), v(\alpha)) = 8p^2 + 4pq + q^2 \frac{m+2}{6} \\ &= \frac{8}{3} - 4\frac{12 \mp 2\sqrt{6(m+8)}}{3(m+2)} + \frac{\left(12 \mp 2\sqrt{6(m+8)}\right)^2}{(m+2)^2 3} \cdot \frac{m+2}{6} \\ &= \frac{8}{3} - \frac{48}{3(m+2)} \pm \frac{8\sqrt{6(m+8)}}{3(m+2)} + \frac{24 + 4(m+8) \mp 8\sqrt{6(m+8)}}{3(m+2)} \\ &= \frac{8}{3} + \frac{24 + 4m + 32 - 48}{3(m+2)} = \frac{8}{3} + \frac{4m+8}{3(m+2)} = \frac{12}{3} = 4. \end{aligned}$$

□

3.4 Further representations

So far, any attempts at extending the ansatz (66) to higher powers $\mathrm{Sym}^n(\mathfrak{h}^*)$ or other Schur modules $\mathcal{S}_\lambda(\mathfrak{h}^*)$, where λ is some partition of n, have failed. The reason is usually that the number of copies of the sign representation exceeds the number of free parameters in the ansatz. An indicator of why the representation $\mathcal{S}_{\frac{7}{2}}$ is somewhat special is that the occurrence of exactly one sign representation is universal, it does not depend on $\mathfrak{h}^*$. In any other case I studied, the number of sign representations depended on the dimension of $\mathfrak{h}^*$. Another approach could be to study representations of the Weyl group that are not some Schur-module of $\mathfrak{h}^*$ and therefore not obtained from the natural action on $\mathfrak{h}^*$. However, these approaches limit the spectrum of the Berman generators very much. In section 5 I will show that the $\mathfrak{k}(E_{10})(\mathbb{C})$-module $\mathcal{S}_{\frac{7}{2}}$ contains $\mathfrak{so}(10,\mathbb{C})$-weights such as $2\omega_1 + \alpha$ which would be impossible for any ansatz $\tau(\alpha) = \eta(s_\alpha) - \frac{1}{2}Id$ as in order to achieve a weight such as $2\omega_1 + \alpha$ the map $\tau(\alpha)$ needs to have the eigenvalue $\frac{5}{2}$ for some α_i.

4 Lift to the group level

In this section I will show that the higher spin representations lift to the spin group $Spin(A)$ for any simply-laced GCM A. I will start with reviewing the construction of $Spin(A)$ (in section 4.1) which was introduced in [GHKW17] and its relation to the "maximal compact subgroup" $K(A)$ of the minimal split-real Kac-Moody group of type A. For A indecomposable and simply-laced I will show that any finite-dimensional representation (ρ, V) lifts to $Spin(A)$ and formulate a criterion when it additionally lifts to $K(A)$ (prop. 4.8). I will apply this criterion in section 4.2 to show that the higher spin representations lift only to $Spin(A)$ in propositions 4.9 and 4.10 which hence justifies the term *spin representations.* In section 4.3 I will use this lift to deduce the form of representation matrices $\sigma(x_\alpha)$ for all $x_\alpha \in \mathfrak{k}_\alpha$ with $\alpha \in \Delta_+^{re}$ up to constant nonzero scalar multiples (see propositions 4.16 and 4.17).

4.1 Maximal compact subgroups and their spin covers

The goal is to formulate a criterion (prop. 4.8) that allows to check if a given finite-dimensional representation (ρ, V) of $\mathfrak{k}(A)$ lifts to the maximal compact subgroup $K(A)$ of the split-real Kac-Moody-group $\mathfrak{G}(A)$ or to its spin cover $Spin(A)$. In order to provide a description of these groups via amalgams and their properties I will follow [GHKW17] closely. In particular, I will use the definitions and notations of [GHKW17].

Definition 4.1. (Amalgam of groups[25]) Let I be an index set and let G_i, G_{ij} for $i \neq j \in I$ be groups with monomorphisms $\psi^i_{ij} : G_i \to G_{ij}$. The set

$$\mathcal{A} := \left\{G_i, G_{ij}, \psi^i_{ij} | i \neq j \in I\right\}$$

is called an **amalgam of groups** and the ψ^i_{ij} are called **connecting homomorphisms**. If $G_i \cong U$ for all $i \in I$ then $\mathcal{A}$ is called an U-amalgam over I. It is **continuous** if all G_i, G_{ij} are topological groups with continuous connecting homomorphisms ψ^i_{ij}.

Definition 4.2. (Universal enveloping groups[26]) Let $\mathcal{A} = \left\{G_i, G_{ij}, \psi^i_{ij} | i \neq j \in I\right\}$ be an amalgam of groups and let G be a group together with a set of homomorphisms $\tau := \{\tau_{ij} : G_{ij} \to G\}$ such that $\tau_{ij} \circ \psi^i_{ij} = \tau_{ik} \circ \psi^i_{ik}$ for all $i \neq j \neq k \in I$. Then (G, τ) is called an **enveloping group** of $\mathcal{A}$ with **enveloping homomorphisms** τ_{ij}. It is called **faithful**, if all τ_{ij} are injective. An enveloping group (G, τ) is called **universal** if there exists a unique epimorphism $\pi : G \to H$ such that $\pi \circ \tau_{ij} = \tilde{\tau}_{ij}$ for all $i \neq j \in I$, whenever $(H, \tilde{\tau})$ is an enveloping group of $\mathcal{A}$.

It is true by universality that two universal enveloping groups of an amalgam are uniquely isomorphic. One particular choice is the canonical universal enveloping group (CUEG) which is defined as in [GHKW17] as

$$G(\mathcal{A}) := \left\langle \bigcup_{i \neq j \in I} G_{ij} \,|\, \text{all relations in } G_{ij},\ \forall\, i \neq j \neq k,\, \forall\, x \in G_j : \psi^j_{ij}(x) = \psi^j_{kj}(x) \right\rangle . \tag{76}$$

According to lemma 1.3.2 of [IS02] the CUEG is indeed a universal enveloping group as the name suggests. Usually, split minimal Kac-Moody groups over a field $\mathbb{F}$ associated to a Kac-Moody algebra $\mathfrak{g}(A)(\mathbb{F})$ are defined via the constructive Tits functor (see [T87]). As I am only interested in the simply-laced situation over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ I will use a different approach. Let Π be a two-spherical generalized Dynkin diagram with GCM A, $\mathfrak{G}_i = SL(2, \mathbb{F})$ split and $\mathfrak{G}_{ij}$ the split algebraic group over $\mathbb{F}$ of type $\Pi_{\{i,j\}}$, where $\Pi_{\{i,j\}}$ denotes the subdiagram of Π corresponding to the vertices i and j. Each $\mathfrak{G}_{ij}$ is generated by its two fundamental root groups which define canonical inclusion maps $\phi^i_{ij} : \mathfrak{G}_i \hookrightarrow \mathfrak{G}_{ij}$. From the main result of [AM97] it follows that the split minimal Kac-Moody group $\mathfrak{G}$ over $\mathbb{R}$ of type Π is a universal enveloping group of the amalgam $\mathcal{A} = \left\{\mathfrak{G}_i, \mathfrak{G}_{ij}, \phi^i_{ij}\right\}$. Thus, one can identify the split minimal Kac-Moody group $\mathfrak{G}$ over $\mathbb{R}$ of type Π with this amalgam. One can now introduce involutive automorphisms on $\mathfrak{G}$. The relevant involution in this case is the Cartan-Chevalley involution θ on $\mathfrak{G}$ which is the analogue of ω on $\mathfrak{g}$. Since $\mathfrak{G}$ has a presentation by its rank 2 subgroups one can define θ on the $\mathfrak{G}_{ij}$ and extend it to $\mathfrak{G}$. One simply demands that $\theta|_{\mathfrak{G}_{ij}}$ coincides with the Cartan-Chevalley involution on the classical split-real Lie group $\mathfrak{G}_{ij}$. The fixed-point subgroups $\mathfrak{G}^\theta_{ij}$ coincide with the classical maximal-compact subgroups. Hence, it is natural to wonder whether $\mathfrak{G}^\theta$ can be described by the universal enveloping group of an amalgam of the $\mathfrak{G}^\theta_{ij}$. I will collect the relevant results from [GHKW17] concerning this question.

In the simply-laced situation the rank 1 groups $G_i := \mathfrak{G}^\theta_i$ are always isomorphic to $SO(2)$ while the rank 2 groups $G_{ij} := \mathfrak{G}^\theta_{ij}$ are isomorphic either to $SO(3)$ or $SO(2) \times SO(2)$. For a simply-laced GCM $A \in \mathbb{Z}^{n \times n}$

[25] Compare def. 3.1 and remark 3.4 of [GHKW17]
[26] Compare definitions 3.5 and 3.6 of [GHKW17]

denote by Π its generalized Dynkin diagram with edges $\mathcal{E}$. With $I = \{1, \dots, n\}$ set

$$G_{ij} := \begin{cases} SO(3) & \text{if } (i,j) \in \mathcal{E} \\ SO(2) \times SO(2) & \text{if } (i,j) \notin \mathcal{E} \end{cases} \tag{77}$$

For a general group H set

$$i_1 : H \to H \times H,\ h \mapsto (h, e),\ i_2 : H \to H \times H,\ h \mapsto (e, h). \tag{78}$$

Let $\varepsilon_{12} : SO(2) \hookrightarrow SO(3)$ describe the embedding via the upper-left $SO(2)$-subgroup and let $\varepsilon_{23} : SO(2) \hookrightarrow SO(3)$ describe the embedding via the lower-right $SO(2)$-subgroup. These maps should be intuitively clear from a presentation of the involved groups via matrices and I refer to [GHKW17, sec. 5] for a fully rigorous introduction of them.

Definition 4.3. (Standard $SO(2)$-amalgams[27])Let $A \in \mathbb{Z}^{n\times n}$ be a GCM and Π its generalized Dynkin diagram with labels $I = \{1, \dots, n\}$. An $SO(2)$**-amalgam** with respect to Π and the chosen labels is defined to be an amalgam

$$\mathcal{A} = \left\{G_i \cong SO(2), G_{ij}, \phi^i_{ij} | i \neq j \in I\right\}$$

with G_{ij} as in (77) and such that for all $i < j \in I$:

$$\phi^i_{ij}\left(SO(2)\right) = \begin{cases} \varepsilon_{12}\left(SO(2)\right) & \text{if } (i,j) \in \mathcal{E} \\ i_1\left(SO(2)\right) & \text{if } (i,j) \notin \mathcal{E} \end{cases},\ \phi^j_{ij}\left(SO(2)\right) = \begin{cases} \varepsilon_{23}\left(SO(2)\right) & \text{if } (i,j) \in \mathcal{E} \\ i_2\left(SO(2)\right) & \text{if } (i,j) \notin \mathcal{E}. \end{cases}$$

The **standard** $SO(2)$**-amalgam** with respect to Π and the chosen labels is defined as the amalgam

$$\mathcal{A}\left(\Pi, SO(2)\right) := \left\{G_i \cong SO(2), G_{ij}, \phi^i_{ij} | i \neq j \in I\right\}$$

with G_{ij} as in (77) and for all $i < j \in I$:

$$\phi^i_{ij} = \begin{cases} \varepsilon_{12} & \text{if } (i,j) \in \mathcal{E} \\ i_1 & \text{if } (i,j) \notin \mathcal{E} \end{cases},\ \phi^j_{ij} = \begin{cases} \varepsilon_{23} & \text{if } (i,j) \in \mathcal{E} \\ i_2 & \text{if } (i,j) \notin \mathcal{E}. \end{cases}$$

It is shown in [GHKW17] as consequence 9.5 that the labeling of the generalized Dynkin diagram is irrelevant for the isomorphism type of the standard $SO(2)$-amalgam and its CUEG[28]. A general $SO(2)$-amalgam and the standard $SO(2)$-amalgam do not need to be isomorphic. However, this is the case if the connecting homomorphisms are continuous (cp. [GHKW17, thm. 9.8]) and therefore, the standard $SO(2)$-amalgam $\mathcal{A}\left(\Pi, SO(2)\right)$ is the unique up to isomorphism $SO(2)$-amalgam with respect to Π with continuous connecting homomorphisms. It is pointed out in [GHKW17, rem. 9.2] that the restriction to continuous connecting homomorphisms is natural in the Kac-Moody setting with 2-spherical diagrams because split-real Kac-Moody groups and their maximal compact subgroups carry a natural topology, known as the Kac-Peterson topology, that induces the Lie topology on their spherical subgroups (cp. [KP83] and [HKM13]).

The embeddings $\varepsilon_{12}, \varepsilon_{23} : SO(2) \hookrightarrow SO(3)$ have canonical counterparts $\tilde{\varepsilon}_{12}$, $\tilde{\varepsilon}_{23} : Spin(2) \hookrightarrow Spin(3)$ (cp. [GHKW17, lem. 6.10]) and one defines the standard $Spin(2)$-amalgam w.r.t. Π accordingly:

[27]Cp. [GHKW17, def. 9.1]

[28]In fact I have dropped some subtleties concerning the diagram's labeling in the above definition to begin with because of this consequence. Also, I have not defined what it means for two amalgams to be isomorphic. For this section it suffices to understand this as equivalent to an isomorphism of the CUEGs.

Definition 4.4. (Standard $Spin(2)$-amalgam[29]) Let $A \in \mathbb{Z}^{n\times n}$ be a GCM and Π its generalized Dynkin diagram with labels $I = \{1, \dots, n\}$. Set

$$G_{ij} := \begin{cases} Spin(3) & \text{if } (i,j) \in \mathcal{E} \\ Spin(2) \times Spin(2) \diagup \langle(-1,-1)\rangle & \text{if } (i,j) \notin \mathcal{E}. \end{cases} \tag{79}$$

A $Spin(2)$**-amalgam** with respect to Π and the chosen labels is defined as the amalgam

$$\mathcal{A} = \left\{G_i \cong Spin(2), G_{ij}, \phi_{ij}^i | i \neq j \in I\right\}$$

such that for all $i < j \in I$:

$$\phi_{ij}^i\left(Spin(2)\right) = \begin{cases} \tilde{\varepsilon}_{12}\left(Spin(2)\right) & \text{if } (i,j) \in \mathcal{E} \\ i_1\left(Spin(2)\right) & \text{if } (i,j) \notin \mathcal{E} \end{cases}, \; \phi_{ij}^j\left(SO(2)\right) = \begin{cases} \tilde{\varepsilon}_{23}\left(Spin(2)\right) & \text{if } (i,j) \in \mathcal{E} \\ i_2\left(Spin(2)\right) & \text{if } (i,j) \notin \mathcal{E}. \end{cases}$$

The **standard** $Spin(2)$**-amalgam** with respect to Π and the chosen labels is defined as the amalgam

$$\mathcal{A}\left(\Pi, Spin(2)\right) := \left\{G_i \cong Spin(2), G_{ij}, \phi_{ij}^i | i \neq j \in I\right\}$$

with G_{ij} as in (79) and for all $i < j \in I$:

$$\phi_{ij}^i = \begin{cases} \tilde{\varepsilon}_{12} & \text{if } (i,j) \in \mathcal{E} \\ i_1 & \text{if } (i,j) \notin \mathcal{E} \end{cases}, \; \phi_{ij}^j = \begin{cases} \tilde{\varepsilon}_{23} & \text{if } (i,j) \in \mathcal{E} \\ i_2 & \text{if } (i,j) \notin \mathcal{E}. \end{cases}$$

Just as in the $SO(2)$-case one can show that the labeling does not matter ([GHKW17, cor. 10.7]) and that any continuous $Spin(2)$-amalgam with respect to Π is isomorphic ([GHKW17, thm. 10.9]) to $\mathcal{A}\left(\Pi, Spin(2)\right)$.

Definition 4.5. (Spin group w.r.t. Π [30])For a simply-laced GCM A with associated generalized Dynkin diagram Π, define $Spin(\Pi)$ as the CUEG of the standard $Spin(2)$-amalgam $\mathcal{A}\left(\Pi, Spin(2)\right)$.

Now two important results are

Theorem 4.6. *(Cp. [GHKW17, thm. 11.2])Let A be a simply-laced GCM with generalized Dynkin diagram Π and $\mathfrak{G}$ the minimal split-real Kac-Moody group of type A. Then the maximal compact subgroup $K(\Pi) := \mathfrak{G}^\theta$ is a faithful universal covering group of the standard $SO(2)$-amalgam $\mathcal{A}\left(\Pi, SO(2)\right)$, where θ denotes the Cartan Chevalley involution on $\mathfrak{G}$.*

Theorem 4.7. *(Cp. [GHKW17, thm. 11.17])For a simply-laced GCM A with generalized Dynkin diagram Π the group $Spin(\Pi)$ is a 2^n-fold central extension of $K(\Pi)$ where n is the number of connected components of Π.*

I would like to formulate a criterion that allows one to check the lifting properties of a given finite-dimensional representation $\rho : \mathfrak{k} \to \mathrm{End}(V)$.

[29] Cp. [GHKW17, def. 10.1]
[30] Cp. [GHKW17, def. 11.5]

$$\begin{tikzcd} \mathfrak{k}_J \arrow[r, "\phi"] \arrow[d, "\exp_0"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{End}(U) \arrow[d, "\exp"] \\ \widetilde{K}_J \arrow[r, "\Phi"'] & GL(U) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{k}_i \arrow[r, "\widetilde{\exp}_i"] \arrow[d, "(\psi^i_{ij})_*"'] \arrow[dr, phantom, "\circlearrowright"] & \widetilde{K}_i \arrow[d, "\psi^i_{ij}"] \\ \mathfrak{k}_{ij} \arrow[r, "\widetilde{\exp}_{ij}"'] & \widetilde{K}_{ij} \end{tikzcd}$$

Figure 3: The various exponential maps and homomorphisms between Lie algebras and groups form commutative diagrams due to the properties of finite-dimensional Lie-groups.

Proposition 4.8. *Let A be a simply-laced irreducible GCM with generalized Dynkin diagram Π. Let $\rho : \mathfrak{k}(A)(\mathbb{R}) \to End(V)$ be a finite-dimensional representation then one of the following two cases occurs:*

$$\exp\left(2\pi\rho\left(X_i\right)\right) = \begin{cases} -Id_V \\ Id_V \end{cases} \qquad \forall\, i \in I, \tag{80}$$

where the X_i denote the Berman-generators of $\mathfrak{k}(A)(\mathbb{R})$. The representation ρ lifts to a representation Ω of $Spin(\Pi)$ in both cases but it lifts to $K(\Pi)$ only in the second case. As only one of the two cases can occur it suffices to check the exponential on a single Berman generator.

Proof. Denote by $\mathfrak{k}_J := \langle X_j : j \in J\rangle$ the canonical subalgebras generated by the subdiagram $J \subset I$. If J is spherical and (ϕ, U) is a f.d. irreducible representation, there exists a Lie group $\widetilde{K}_J$ with Lie algebra $\mathfrak{k}_J$ and an exponential map $\exp_0 : \mathfrak{k}_J \to \widetilde{K}_J$ together with a group homomorphism $\Phi : \widetilde{K}_J \to GL(U)$ such that the first diagram in figure 3 commutes. Since every finite-dimensional representation of $\mathfrak{k}_J$ is completely reducible the same is true for ρ. As $SO(2) \cong U(1) \cong Spin(2)$ one has for any rank 1-subdiagram that ρ lifts to the desired group because one can adjust the normalization in $\exp_0$. In the rank 2 case there occur differences when $(i,j) \in \mathcal{E}$. There, ρ always lifts to $Spin(3)$ as it is the fundamental cover of $\mathfrak{k}_{\{i,j\}} \cong \mathfrak{so}(3)$. This lift is compatible with the adjoint action on $\mathfrak{k}_{\{i,j\}}$ and since X_i and X_j are conjugate via the adjoint action of $Spin(3)$, one has that $\exp\left(2\pi\rho\left(X_i\right)\right) = \pm Id_V$ implies the same for X_j. As the diagram is irreducible and simply-laced X_i can be conjugated to X_j for $j \in I$ arbitrary via successive conjugations inside rank-2 groups $\widetilde{K}_{j_1 j_2}$.

If a restricted representation $\rho_{\{i,j\}}$ also lifts to $SO(3)$ can be determined by any 1-parameter subgroup due to conjugation. The adjoint action of $\mathfrak{so}(3)$ on itself lifts to $SO(3)$ and therefore is suitable to determine the Berman generators' normalization w.r.t. exponentiation. Recall that $[X_i, [X_i, X_j]] = -X_j$ so one has

$$\begin{aligned} \exp\left(\phi \cdot \mathrm{ad}_{X_i}\right)\left(X_j\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} X_j + \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n+1}}{(2n+1)!} \left[X_i, X_j\right] \\ &= \cos\left(\phi\right) X_j + \sin\left(\phi\right)\left[X_i, X_j\right] . \end{aligned}$$

This shows that the exponential of the ad_{X_i} is 2π-periodic and hence, any representation that lifts to $SO(3)$ has to satisfy case two of (80). Conversely any representation that falls under case one of (80) does not lift to $SO(3)$ but only to $Spin(3)$. Now similar to the proof of theorem 11.14 of [GHKW17] these lifts induce enveloping homomorphisms $\tau = \left\{\tau_{ij} : \widetilde{K}_{ij} \to GL(V)\right\}$ of the amalgam[31] $\mathcal{A}\left(\Pi, Spin(2)\right)$ in case one

[31] Since the Lie algebra is called $\mathfrak{k}$, I replace the names of the G_{ij} from definitions (4.3) and (4.4) by K_{ij} and $\widetilde{K}_{ij}$ respectively.

of (80) and $\tau = \{\tau_{ij} : K_{ij} \to GL(V)\}$ of $\mathcal{A}(\Pi, SO(2))$ in case two of (80). Towards this one identifies the $\widetilde{K}_J$ for $|J| \leq 2$ with their exponential image of $\mathfrak{k}_J$ under $\widetilde{\exp}_J : \mathfrak{k}_J \to \widetilde{K}_J$ which induces canonical connecting monomorphisms $\psi^i_{ij} : \widetilde{K}_i \to \widetilde{K}_{ij}$ that are compatible with exponentiation, i.e., $\psi^i_{ij} \circ \widetilde{\exp}_i = \widetilde{\exp}_{ij} \circ \left(\psi^i_{ij}\right)_*$ (see the second diagram in figure 3). Then the $\tau_{ij} : \widetilde{K}_{ij} \to GL(V)$ can be defined via $\tau_{ij}\left(\widetilde{\exp}_{ij}(x)\right) = \exp(\rho(x))\ \forall x \in \mathfrak{k}_{\{i,j\}}$ and because ρ is globally defined on $\mathfrak{k}$ one has

$$\tau_{ij} \circ \psi^i_{ij}\left(\widetilde{\exp}_i(x)\right) = \exp(\rho(x)) = \tau_{ik} \circ \psi^i_{ik}\left(\widetilde{\exp}_i(x)\right)\ \forall x \in \mathfrak{k}_{\{i\}}.$$

□

4.2 Lift of higher spin representations

Given a higher spin representation $\sigma(X_i) := \tau(\alpha_i) \otimes \Gamma(\alpha_i)$ as in (3.19) and (3.23) set

$$\Sigma_i(\phi) := \exp(\phi \cdot \sigma(X_i)). \tag{81}$$

Proposition 4.9. *Let (σ, V) be the $\frac{3}{2}$- or $\frac{5}{2}$-spin representation of $\mathfrak{k}(A)(\mathbb{R})$ from theorem 3.19 for A simply-laced. Then one has*

$$\begin{aligned}\Sigma_i(\phi) \quad = \quad & \Big[\cos(\phi)\cos\left(\frac{\phi}{2}\right) \cdot Id \otimes Id - \cos(\phi)\sin\left(\frac{\phi}{2}\right) \cdot Id \otimes \Gamma(\alpha_i) + \\ & \sin(\phi)\sin\left(\frac{\phi}{2}\right) \cdot \eta(s_i) \otimes Id + \sin(\phi)\cos\left(\frac{\phi}{2}\right) \cdot \eta(s_i) \otimes \Gamma(\alpha_i)\Big],\end{aligned} \tag{82}$$

and (σ, V) lifts to $Spin(A)$ but not to $K(A)$.

Proof. One has for the $\frac{3}{2}$- and $\frac{5}{2}$-spin representation with $\tau(\alpha_i) = \eta(s_i) - \frac{1}{2}$, $\tau(\alpha_i)^n = a(n) + b(n)\eta(s_i)$ and $\Gamma(\alpha_i)^{2n} = (-1)^n$ that

$$\begin{aligned}\Sigma_i(\phi) &= \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \tau(\alpha_i)^n \otimes \Gamma(\alpha_i)^n \\ &= \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \left(a(n) + b(n)\eta(s_i)\right) \otimes \Gamma(\alpha_i)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} a(2n) \cdot Id \otimes Id + \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} a(2n+1) \cdot Id \otimes \Gamma(\alpha_i) \\ &\quad + \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} b(2n) \cdot \eta(s_i) \otimes Id + \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} b(2n+1) \cdot \eta(s_i) \otimes \Gamma(\alpha_i) \\ &= A_1(\phi) \cdot Id \otimes Id + A_2(\phi) \cdot Id \otimes \Gamma(\alpha_i) + A_3(\phi) \cdot \eta(s_i) \otimes Id + A_4(\phi) \cdot \eta(s_i) \otimes \Gamma(\alpha_i),\end{aligned}$$

with

$$A_1(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} a(2n),\ A_2(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} a(2n+1),$$
$$A_3(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} b(2n),\ A_4(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} b(2n+1).$$

One determines with $\binom{n}{k} = 0$ for $k > n$ and $s_i^2 = e$ that

$$\begin{aligned}
\tau(\alpha_i)^n &= \left(\eta(s_i) - \frac{1}{2}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \eta(s_i)^{n-k} (-2)^{-k} \\
&= \sum_{k=0}^{\infty} \binom{n}{2k} \eta(s_i)^{n-2k} (-2)^{-k} + \sum_{k=0}^{\infty} \binom{n}{2k+1} \eta(s_i)^{n-2k-1} (-2)^{-2k-1} \\
&= \eta(s_i)^n \cdot \left[\sum_{k=0}^{\infty} \binom{n}{2k} (-2)^{-2k} + \eta(s_i) \sum_{k=0}^{\infty} \binom{n}{2k+1} (-2)^{-2k-1}\right].
\end{aligned}$$

Now set $f(z) := (1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{k=0}^{\infty} \binom{n}{k} z^k$ and note that[32]

$$\sum_{m=0}^{\infty} a_{2m} z^{2m} = \frac{1}{2}\left[f(z) + f(-z)\right], \quad \sum_{m=0}^{\infty} a_{2m+1} z^{2m+1} = \frac{1}{2}\left[f(z) - f(-z)\right].$$

From this one finds that

$$\sum_{k=0}^{\infty} \binom{n}{2k} (-2)^{-2k} = \frac{1}{2}\left[f(z) + f(-z)\right]_{z=-\frac{1}{2}} = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n + \frac{1}{2} \cdot \left(\frac{3}{2}\right)^n,$$

$$\sum_{k=0}^{\infty} \binom{n}{2k+1} (-2)^{-2k-1} = \frac{1}{2}\left[f(z) - f(-z)\right]_{z=-\frac{1}{2}} = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n - \frac{1}{2} \cdot \left(\frac{3}{2}\right)^n.$$

This yields

$$a(2n) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n} + \left(\frac{3}{2}\right)^{2n}\right], \quad a(2n+1) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n+1} - \left(\frac{3}{2}\right)^{2n+1}\right],$$

$$b(2n) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n} - \left(\frac{3}{2}\right)^{2n}\right], \quad b(2n+1) = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^{2n+1} + \left(\frac{3}{2}\right)^{2n+1}\right].$$

Now

$$\begin{aligned}
A_1(\phi) &= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} a(2n) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} \left[\left(\frac{1}{2}\right)^{2n} + \left(\frac{3}{2}\right)^{2n}\right] \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left[\left(\frac{\phi}{2}\right)^{2n} + \left(\frac{3}{2}\phi\right)^{2n}\right] = \frac{1}{2} \cos\left(\frac{\phi}{2}\right) + \frac{1}{2} \cos\left(\frac{3\phi}{2}\right),
\end{aligned}$$

$$\begin{aligned}
A_2(\phi) &= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} a(2n+1) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} \left[\left(\frac{1}{2}\right)^{2n+1} - \left(\frac{3}{2}\right)^{2n+1}\right] \\
&= \frac{1}{2} \sin\left(\frac{\phi}{2}\right) - \frac{1}{2} \sin\left(\frac{3\phi}{2}\right)
\end{aligned}$$

[32] For a holomorphic function f with Laurent series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and a q-th root of unity ω one has in general that $\sum_{m=0}^{\infty} a_{qm+p} z^{qm+p} = \frac{1}{q} \sum_{k=0}^{q-1} \omega^{-kp} f\left(\omega^k \cdot z\right)$. For $q = 2$, $\omega = -1$ and $p = 0, 1$ this specializes to the expressions below.

$$\begin{aligned} A_3(\phi) &= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} b(2n) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n)!} \left[\left(\frac{1}{2}\right)^{2n} - \left(\frac{3}{2}\right)^{2n} \right] \\ &= \frac{1}{2} \cos\left(\frac{\phi}{2}\right) - \frac{1}{2} \cos\left(\frac{3\phi}{2}\right) \end{aligned}$$

$$\begin{aligned} A_4(\phi) &= \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} b(2n+1) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+1)!} \left[\left(\frac{1}{2}\right)^{2n+1} + \left(\frac{3}{2}\right)^{2n+1} \right] \\ &= \frac{1}{2} \sin\left(\frac{\phi}{2}\right) + \frac{1}{2} \sin\left(\frac{3\phi}{2}\right) \end{aligned}$$

Use the trigonometric identities:

$$\sin\alpha + \sin\beta = 2 \cdot \sin\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right), \ \sin\alpha - \sin\beta = 2 \cdot \cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2 \cdot \cos\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right), \ \cos\alpha - \cos\beta = -2 \cdot \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)$$

to obtain

$$A_1(\phi) = \cos(\phi) \cdot \cos\left(\frac{\phi}{2}\right), \ A_2(\phi) = -\cos(\phi) \sin\left(\frac{\phi}{2}\right), \ A_3(\phi) = \sin(\phi) \sin\left(\frac{\phi}{2}\right), \ A_4(\phi) = \sin(\phi) \cos\left(\frac{\phi}{2}\right).$$

Now for $\phi = 2\pi$ one has $A_1 = -1$ while $A_2 = A_3 = A_4 = 0$ so that

$$\exp(2\pi \cdot \sigma(X_i)) = -Id \otimes Id$$

which in combination with prop. 4.8 shows that these representations only lift to $Spin(A)$. □

The representation $\left(\sigma, \mathcal{S}_{\frac{7}{2}}\right)$ has a slightly modified structure compared to $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$. If $\eta : W \to GL(V)$ denotes the action of the Weyl group on V for $V \in \left\{\mathfrak{h}^*, \mathrm{Sym}^2(\mathfrak{h}^*), \mathrm{Sym}^3(\mathfrak{h}^*)\right\}$ then

$$\sigma(X_i) = \tau(\alpha_i) \otimes \Gamma(\alpha_i), \quad \tau(\alpha) \coloneqq \eta(s_\alpha) - \frac{1}{2} Id + f(\alpha) \ \forall \alpha \in \Delta_+^{re}$$

according to (66), where $f(\alpha)$ is a linear rank one map for all $\alpha \in \Delta_+^{re}$ with the following properties[33]:

$$f(\alpha)^2 = a \cdot f(\alpha), \ \eta(s_\alpha) f(\alpha) = f(\alpha) \eta(s_\alpha) = -f(\alpha). \tag{83}$$

Denote $\tilde{\tau}(\alpha) \coloneqq \eta(s_\alpha) - \frac{1}{2} Id$, then one has

$$\tilde{\tau}(\alpha) f(\alpha) = f(\alpha) \tilde{\tau}(\alpha) = -\frac{3}{2} f(\alpha), \tag{84}$$

[33] One has $a = 4$ according to eq. (75) but one can do this computation for a general a and I will comment on possible other values for a later on, so I will leave it undetermined for now.

which provides

$$\begin{aligned}
\tau(\alpha)^n &= [\tilde{\tau}(\alpha)+f(\alpha)]^n = \sum_{k=0}^{n}\binom{n}{k}\tilde{\tau}(\alpha)^{n-k}f(\alpha)^k = \tilde{\tau}(\alpha)^n + \sum_{k=1}^{n}\binom{n}{k}\left(-\frac{3}{2}\right)^{n-k}a^{k-1}f(\alpha)\\
&= \tilde{\tau}(\alpha)^n + \left[a^{-1}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{3}{2}\right)^{n-k}a^k - a^{-1}\binom{n}{0}\left(-\frac{3}{2}\right)^{n}\right]f(\alpha)\\
&= \tilde{\tau}(\alpha)^n + a^{-1}\left[\left(a-\frac{3}{2}\right)^n - \left(-\frac{3}{2}\right)^n\right]f(\alpha)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\Sigma}_i(\phi) &:= \exp(\phi\sigma(X_i)) = \sum_{n=0}^{\infty}\frac{\phi^n}{n!}\tau(\alpha_i)^n\otimes\Gamma(\alpha_i)^n\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\phi^{2n}}{(2n)!}\tau(\alpha_i)^{2n}\otimes Id + \sum_{n=0}^{\infty}\frac{(-1)^n\phi^{2n+1}}{(2n+1)!}\tau(\alpha_i)^{2n+1}\otimes\Gamma(\alpha_i)\\
&= \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n\phi^{2n}}{(2n)!}\tilde{\tau}(\alpha_i)^{2n}\otimes Id + \sum_{n=0}^{\infty}\frac{(-1)^n\phi^{2n+1}}{(2n+1)!}\tilde{\tau}(\alpha_i)^{2n+1}\otimes\Gamma(\alpha_i)}_{=:\Sigma_i(\phi)}\\
&\quad + a^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n\phi^{2n}}{(2n)!}\left[\left(a-\frac{3}{2}\right)^{2n} - \left(-\frac{3}{2}\right)^{2n}\right]f(\alpha_i)\otimes Id\\
&\quad + a^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n\phi^{2n+1}}{(2n+1)!}\left[\left(a-\frac{3}{2}\right)^{2n+1} - \left(-\frac{3}{2}\right)^{2n+1}\right]f(\alpha_i)\otimes\Gamma(\alpha_i)\\
&= \Sigma_i(\phi) + a^{-1}\left[\cos\left(\left(a-\frac{3}{2}\right)\phi\right) - \cos\left(\frac{3}{2}\phi\right)\right]f(\alpha_i)\otimes Id\\
&\quad + a^{-1}\left[\sin\left(\left(a-\frac{3}{2}\right)\phi\right) + \sin\left(\frac{3}{2}\phi\right)\right]f(\alpha_i)\otimes\Gamma(\alpha_i)
\end{aligned}$$

where $\Sigma_i(\phi)$ coincides with the expression from (82) for a different $\eta: W(A)\to GL(V)$. From prop. 4.9 one knows that Σ_i is 4π-periodic as the proof only relies on the fact that η is a representation of the Weyl group. The periodicity of the remainder in the above expression depends on the eigenvalue a of $f(\alpha)^2 = af(\alpha)$. Hence, with $a=4$ from eq. (75) one obtains

$$\widetilde{\Sigma}_i(\phi) = \Sigma_i(\phi) + \frac{1}{4}\left[\cos\left(\frac{5}{2}\phi\right) - \cos\left(\frac{3}{2}\phi\right)\right]f(\alpha_i)\otimes Id + \frac{1}{4}\left[\sin\left(\frac{5}{2}\phi\right) + \sin\left(\frac{3}{2}\phi\right)\right]f(\alpha_i)\otimes\Gamma(\alpha_i),$$

which shows that $\widetilde{\Sigma}_i$ is 4π-periodic as well. Note that in comparison to the $\frac{3}{2}$- and $\frac{5}{2}$-representations the highest "frequency" that occurs is $\frac{5}{2}$ and not $\frac{3}{2}$. For the special case $\mathfrak{k}(E_{10})$ this frequency is connected to the different weight structure of the module under its $\mathfrak{so}(10)$-subalgebra. The "highest" highest weights that occur in $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ are $\omega_1+\alpha$ and $\omega_2+\alpha$ respectively, where I chose the order such that $\alpha>\beta>\omega_3>\omega_2>\omega_1$. Now while $\mathcal{S}_{\frac{7}{2}}$ exhibits the $\mathfrak{so}(10)$-highest weight $\omega_3+\alpha$, there exists a highest weight vector $v_{2\omega_1+\alpha}$ to the weight $2\omega_1+\alpha$ as well. The action of $H_1=-iX_1$ on $v_{2\omega_1+\alpha}$ is diagonal with eigenvalue $(2\omega_1+\alpha)(H_1) = 2+\frac{1}{2}=\frac{5}{2}$ which is consistent with the above occurrence of the frequency $\frac{5}{2}$ in the exponential.

The eigenvalue $a = 4$ is indirectly fixed by the demand that σ defines a representation of $\mathfrak{k}$. If one does not take this fact into account and determines the adjoint action $\widetilde{\Sigma}_i\left(\frac{\pi}{2}\right)\sigma(X_j)\widetilde{\Sigma}_i\left(-\frac{\pi}{2}\right)$ one finds (after a tedious computation) that the result is proportional to $\sigma([X_i, X_j])$ only if $a = 0 \bmod 4$. As the action realizes the action of the (spin-extended) Weyl group if $\widetilde{\Sigma}_i$ defines a representation, this constrains the possible eigenvalues of $f(\alpha)$ to $4\mathbb{Z}$. For future reference:

Proposition 4.10. *Let (σ, V) be the $\frac{7}{2}$-spin representation of $\mathfrak{k}(A)(\mathbb{R})$ from prop. 3.23 for A simply-laced. Then the lifts $\widetilde{\Sigma}_i$ to the fundamental rank 1-groups are given by*

$$\widetilde{\Sigma}_i(\phi) = \Sigma_i(\phi) + \frac{1}{4}\left[\cos\left(\frac{5}{2}\phi\right) - \cos\left(\frac{3}{2}\phi\right)\right] f(\alpha_i) \otimes Id + \frac{1}{4}\left[\sin\left(\frac{5}{2}\phi\right) + \sin\left(\frac{3}{2}\phi\right)\right] f(\alpha_i) \otimes \Gamma(\alpha_i), \quad (85)$$

where $\Sigma_i(\alpha)$ is as in (82) with the induced representation of the Weyl group $\eta : W(A) \to GL(Sym^3(\mathfrak{h}^*))$. The $\frac{7}{2}$-spin representation $\left(\sigma, \mathcal{S}_{\frac{7}{2}}\right)$ lifts to $Spin(A)$ but not to $K(A)$.

4.3 Compatibility with $W^{spin}(\Pi)$-action

In this section, the goal is to show that the higher spin representations behave well with the action of the spin-extended Weyl group introduced in [GHKW17, sec. 18]. This result is used to derive the representation matrix of $x_\alpha \in \mathfrak{k}_\alpha$ for $\alpha \in \Delta^{re}$ up to a sign.

Let A be a symmetrizable GCM with associated generalized Dynkin diagram Π, set

$$n(i,j) = \begin{cases} 0 & \text{if } A_{ij} \text{ is even} \\ 1 & \text{if } A_{ij} \text{ is odd} \end{cases},$$

and recall the m_{ij} from eq. (4). The Weyl group $W(\Pi)$ is not contained in the minimal Kac-Moody group $G(\Pi)$. Given the adjoint action or in fact any integrable representation π of $\mathfrak{g}$ one can set

$$t_i := \exp\pi(f_i)\exp(-\pi(e_i))\exp\pi(f_i)$$

which have the property that the weight spaces V_λ of the representation are conjugated like (cp. [K90, lem. 3.8])

$$t_i(V_\lambda) = V_{s_i.\lambda}, \quad (86)$$

where s_i denotes the simple Weyl reflection $s_i \in W(\Pi)$. To each integrable representation (π, V) one associates a group $G^\pi \leq GL(V)$ which is generated by the $\exp\pi(\phi f_i), \exp\pi(\phi\alpha_i^\vee)$ and $\exp\pi(\phi e_i)$ for $\phi \in \mathbb{K}$. The t_i now generate a subgroup $\widetilde{W}(\Pi) < G^\pi$ which contains an abelian normal subgroup $D^\pi = \langle t_i^2\rangle$. If $\ker\pi \subset \mathfrak{h}$ one has that $\widetilde{W}(\Pi) \diagup D^\pi \cong W(\Pi)$ (this is [K90, rem. 3.8], originally due to [KP85]).

Without reference to any representation I use the definitions of the extended Weyl group $W^{ext}(\Pi)$ and the spin extended Weyl group $W^{spin}(\Pi)$ from [GHKW17, def. 18.4]:

Definition 4.11. The **extended Weyl group** $W^{ext}(\Pi)$ to the generalized 2-spherical Dynkin diagram Π is defined by its presentation ($n := |I|$)

$$\text{(T1)}\ W^{ext}(\Pi) = \langle t_1, \ldots, t_n |\ t_i^4 = e\ \forall\, i \in I, \quad (87)$$

$$\text{(T2)}\quad t_j^{-1}t_i^2 t_j = t_i^2 t_j^{2n(i,j)}\ \forall\, i \neq j \in I, \quad (88)$$

$$\text{(T3)}\quad \underbrace{t_i t_j t_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{t_j t_i t_j \cdots}_{m_{ij}\text{ factors}}\ \forall\, i \neq j \in I \Big\rangle. \quad (89)$$

and similarly the **spin-extended Weyl group** $W^{spin}(\Pi)$ is defined by

$$\begin{aligned}
&\text{(R1) } W^{spin}(\Pi) &=& \langle r_1,\ldots,r_n |\ r_i^8 = e\ \forall\, i\in I, && (90)\\
&\text{(R2)} && r_j^{-1} r_i^2 r_j = r_i^2 r_j^{2n(i,j)}\ \forall\, i\neq j\in I, && (91)\\
&\text{(R3)} && \left.\underbrace{r_i r_j r_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{r_j r_i r_j\cdots}_{m_{ij}\text{ factors}}\ \forall\, i\neq j\in I\right\rangle. && (92)
\end{aligned}$$

From [KP85, cor. 2.4] together with a few steps explained in [GHKW17, rem. 18.5]) one has that $W^{ext}(\Pi)\cong\widetilde{W}(\Pi)$ as defined above for representations π that satisfy $\ker\pi\subset\mathfrak{h}$. Now as a matter of fact $K(\Pi)$ contains $W^{ext}(\Pi)$ and its spin cover $Spin(\Pi)$ contains $W^{spin}(\Pi)$. In order to see this one first defines the following subgroups of $Spin(\Pi)$ and $K(\Pi)$:

Definition 4.12. (Cp. [GHKW17, def. 18.3]) Let Π be a simply-laced generalized Dynkin diagram, let $\mathcal{A}(\Pi, Spin(2))$ be the standard spin-amalgam of type Π with connecting monomorphisms $\tilde{\phi}_{ij}^i:\tilde{G}_i\to\tilde{G}_{ij}$, and let $\mathcal{A}(\Pi, SO(2))$ be the standard $SO(2)$-amalgam of type Π with connecting monomorphisms $\phi_{ij}^i: G_i\to G_{ij}$. Denote the enveloping homomorphisms of the amalgams by $\tilde{\psi}_{ij}:\tilde{G}_{ij}\to Spin(\Pi)$ and $\psi_{ij}: G_{ij}\to\Pi$. As in [GHKW17, 8.1] denote by $S:\mathbb{R}\to Spin(2)$ and $D:\mathbb{R}\to SO(2)$ the 2π-periodic covering maps[34]. For $i\neq j$ set

$$\hat{r}_i := \tilde{\psi}_{ij}\circ\tilde{\phi}_{ij}^i\left(S\left(\frac{\pi}{4}\right)\right),\quad \widehat{W}(\Pi) := \langle\hat{r}_i | i\in I\rangle < Spin(\Pi),$$

$$\tilde{s}_i := \psi_{ij}\circ\phi_{ij}^i\left(D\left(\frac{\pi}{2}\right)\right),\quad \widetilde{W}(\Pi) := \langle\tilde{s}_i | i\in I\rangle < K(\Pi).$$

Note that $\widetilde{W}(\Pi)\cong W^{ext}(\Pi)$ by [KP85, cor. 2.4] together with [GHKW17, rem. 18.5] as mentioned earlier. For $\widehat{W}(\Pi)$ one has from [GHKW17, thm. 18.15] that $\widehat{W}(\Pi)\cong W^{spin}(\Pi)$, where the isomorphism is given by $\hat{r}_i\mapsto r_i$ for all $i\in I$. Now the spin representations of $\mathfrak{k}$ lift only to $Spin(\Pi)$ which is a central extension of $K(\Pi)$.

Proposition 4.13. *The adjoint action of $Spin(\Pi)$ on $\mathfrak{k}$ factors through the natural projection $\varphi: Spin(\Pi)\to K(\Pi)$. Furthermore one has an action on $\mathfrak{g}$ via*

$$Ad_g(x) := Ad_{\varphi(g)}(x)\ \ \forall\, g\in Spin(\Pi),\ x\in\mathfrak{g}. \tag{93}$$

This action satisfies for $r_i\in W^{spin}(\Pi)$

$$Ad_{r_i}(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i.\alpha},\ \ \forall\,\alpha\in\Delta \tag{94}$$

and for all $\omega\in W(\Pi)$ and $\alpha\in\Delta$ there exists $\hat{\omega}\in W^{spin}(\Pi)$ such that $Ad_{\hat{\omega}}(\mathfrak{g}_\alpha)=\mathfrak{g}_{\omega.\alpha}$.

Proof. The adjoint action of $Spin(\Pi)$ on $\mathfrak{k}$ factors through φ because $\ker\varphi$ is central. Therefore, the adjoint action of $W^{spin}(\Pi)$ on $\mathfrak{g}$ factors through the projection to $W^{ext}(\Pi)$ (the observation that the center $Z=\ker\varphi$

[34] Explicitly one has $D(\alpha)=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}$ and $S(\alpha)=\cos\alpha+\sin\alpha e_1e_2$, where $e_1,e_2\in Cl(\mathbb{R}^2)$ are orthonormal w.r.t. the euclidean form.

lies inside $W^{spin}(\Pi)$ is [GHKW17, 18.11]) and that $W^{ext}(\Pi)$ acts like (94) has been mentioned earlier in (86). According to [KP85, cor. 2.3 b)] there exists a unique map from $W(\Pi)$ to $W^{ext}(\Pi)$ such that

$$\begin{aligned} e &\mapsto e \\ s_i &\mapsto t_i \\ \omega\omega' &\mapsto \widetilde{\omega}\widetilde{\omega}' \text{ if } l(\omega\omega') = l(\omega) + l(\omega'). \end{aligned}$$

This way it is always possible to find a word $\widetilde{w}$ in W^{ext} to a reduced word $w \in W$. Since any ambiguity in $W^{spin}(\Pi)$ consists of central terms and therefore is removed by the projection to $W^{ext}(\Pi)$ one can just fix the following translation for a reduced word $w \in W(\Pi)$:

$$w = s_{i_1} \cdots s_{i_k} \mapsto \widehat{w} = r_{i_1} \cdots r_{i_k} \in W^{spin}(\Pi).$$

□

This implies that one can perform all $W(\Pi)$-conjugations of root spaces by the action with $W^{spin}(\Pi)$ as well. In the following I will show that such conjugations on $\mathfrak{k}$ behave well with the $\frac{1}{2}$-spin representation $\left(\mathcal{S}_{\frac{1}{2}}, \rho\right)$.

Lemma 4.14. *Let Π (resp. A) be simply-laced*[35] *and let $\rho : \mathfrak{k}(A) \to End(V)$ be a finite-dimensional representation that lifts to $K(\Pi)$ or $Spin(\Pi)$, where the lift is denoted by Ω. Then one has*

$$\rho\left(Ad_g(x)\right) = \Omega(g)\,\rho(x)\,\Omega(g)^{-1} \quad \forall g \in Spin(\Pi), \ \forall x \in \mathfrak{k}(A).$$

Proof. According to [K90, (3.8.1)] one has

$$\exp(\rho(a))\,\rho(x)\exp(-\rho(a)) = \rho(\exp(\text{ad } a)(x)) \tag{95}$$

for all $a, x \in \mathfrak{k}$ such that $\rho(a)$, $\text{ad}(a)$ and $\rho(x)$ are locally nilpotent. As pointed out later in [K90, sec. 3.8] the above formula is also correct if $\rho(a)$ is locally finite and the span of the $\text{ad}(a)^n(x)$ for $n \in \mathbb{N}$ is finite-dimensional, which is in particular the case if $\text{ad}(a)$ is locally finite or locally nilpotent. This is true because in the derivation of (95) one uses the binomial formula of associative algebras,

$$\text{ad}(a)^n x = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n-k} a x^k,$$

in combination with the exponential of linear maps $exp(a) = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$. The necessary rearrangements of the two infinite sums is easy for locally nilpotent maps because then only finitely many terms are relevant. If the maps are only locally finite, one can still rearrange the terms because for any $v \in V$ or $x \in \mathfrak{k}$, the evaluation of the exponential can be done on a finite-dimensional vector space where it exists unconditionally. Therefore, eq. (95) is satisfied for all ad-locally finite elements $a \in \mathfrak{k}$ because V is finite-dimensional. The elements of $\mathfrak{k}(A)$ which are ad-locally finite include the Berman-elements $x_\alpha := e_\alpha - \omega(e_\alpha)$ for $\alpha \in \Delta_+^{re}$ because the e_α are locally nilpotent for $\alpha \in \Delta_+^{re}$. In particular, all $a \in \mathfrak{k}_J$ for J spherical are ad-locally finite, where $\mathfrak{k}_J := \langle X_j \,|\, j \in J \subset I \rangle$ and J is called spherical if the corresponding sub-diagram of Π is a spherical Dynkin diagram. For J spherical, there exists a well-defined exponential map $exp_J : \mathfrak{k}_J \to K_J$, where K_J

[35]The proof also works for 2-spherical and symmetrizable but as I have not introduced the groups $K(\Pi)$ and $Spin(\Pi)$ for diagrams that are not simply-laced, I will only discuss the simply-laced situation.

$$\begin{array}{ccc} \mathfrak{k}_J & \xrightarrow{\rho_J} & \mathrm{End}(U) \\ \exp_J \downarrow & \circlearrowright & \downarrow \exp \\ \widetilde{K}_J & \xrightarrow[\Omega_J]{} & GL(U) \end{array}$$

Figure 4: The above diagram commutes because the involved Lie groups and Lie lagebras as well as the representation are finite-dimensional.

can denote either the maximal compact subgroup $K_J < G_J$ or its spin cover. For connected compact Lie groups the exponential map is onto, so that any $g \in K_J$ can be written as[36] $g = \exp_J(a)$ for $a \in \mathfrak{k}_J$. Denote the restriction of Ω to K_J by Ω_J (the restriction of ρ to $\mathfrak{k}_J$ will be denoted by ρ_J if necessary) and note that it satisfies (in other words, diagram 4 commutes)

$$\Omega(g) = \Omega_J(\exp_J(a)) = \exp(\rho_J(a)) = \exp(\rho(a)) \;\; \forall\, g = \exp_J a \in K_J.$$

With this compute for all $a \in \mathfrak{k}_J, x \in \mathfrak{k}$

$$\begin{aligned} \Omega_J(\exp_J(a))\,\rho(x)\Omega_J(\exp_J(a))^{-1} &= \exp(\rho(a))\,\rho(x)\exp(-\rho(a)) \\ &\overset{(95)}{=} \rho(\exp(\mathrm{ad}\ a)(x)) \\ &= \rho\left(Ad_{\exp_J a}(x)\right) = \rho(Ad_g(x)). \end{aligned}$$

The penultimate equality holds because for finite-dimensional Lie groups the exponential map intertwines with finite-dimensional representations of the group and the Lie-algebra and hence, *Ad*, *exp* and *ad* form a commutative diagram here. As any $g \in K(\Pi)$ or $Spin(\Pi)$ is a finite product of elements in the fundamental rank 1 subgroups this shows the claim of the lemma by applying the above equation finitely many times. ☐

Lemma 4.15. *Let $\left(\mathcal{S}_{\frac{1}{2}}, \rho\right)$ be a generalized spin representation of $\mathfrak{k}(A)$ as in def. (3.1) with generalized Γ-matrices according to prop. 3.11 for A simply-laced. Set*

$$\widehat{r}_i(\phi) := \exp(\phi \cdot \rho(X_i)), \tag{96}$$

then one has for all $\alpha \in \Delta$

$$\widehat{r}_i(\phi)\,\Gamma(\alpha)\,\widehat{r}_i(\phi)^{-1} = \begin{cases} \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z} \\ \cos\phi \cdot \Gamma(\alpha) + \sin\phi \cdot \Gamma(\alpha_i)\,\Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1 \end{cases}$$

$$r_i\Gamma(\alpha)\,r_i^{-1} = \begin{cases} \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z} \\ \varepsilon(\alpha_i, \alpha)\,\Gamma(s_i.\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1 \end{cases}.$$

where $\varepsilon : Q(A) \times Q(A) \to \{\pm 1\}$ is the standard normalized 2-cocycle from lemma 3.8 and $r_i := \widehat{r}_i\left(\frac{\pi}{2}\right)$ are the generators of $W^{spin}(A)$ on the representation side.

[36] One does not really need this fact in such a strong version, it suffices to note that for a connected finite-dimensional Lie group any element can be written as the product of finitely many exponentials.

Proof. One has $\rho(X_i) = \frac{1}{2}\Gamma(\alpha_i)$ and so the exponential is given by

$$\widehat{r}_i(2\phi) := \exp(2\phi \cdot \rho(X_i)) = \cos\phi \cdot Id + \sin\phi\Gamma(\alpha_i).$$

Then

$$\begin{aligned}
\widehat{r}_i(2\phi)\Gamma(\alpha)\widehat{r}_i(2\phi)^{-1} &= \exp(2\phi\cdot\rho(X_i))\Gamma(\alpha)\exp(-2\phi\cdot\rho(X_i)) \\
&= \cos^2\phi\cdot\Gamma(\alpha) - \sin^2\phi\Gamma(\alpha_i)\Gamma(\alpha)\Gamma(\alpha_i) \\
&\quad + \sin\phi\cos\phi\left(\Gamma(\alpha_i)\Gamma(\alpha) - \Gamma(\alpha)\Gamma(\alpha_i)\right) \\
&= \cos^2\phi\cdot\Gamma(\alpha) - \sin^2\phi\Gamma(\alpha_i)^2\Gamma(\alpha) \\
&\quad - \sin^2\phi\Gamma(\alpha_i)\left[\Gamma(\alpha),\Gamma(\alpha_i)\right] \\
&\quad + \sin\phi\cos\phi\left[\Gamma(\alpha_i),\Gamma(\alpha)\right] \\
&= \Gamma(\alpha) - \left(\sin^2\phi\Gamma(\alpha_i) + \sin\phi\cos\phi\right)\left[\Gamma(\alpha),\Gamma(\alpha_i)\right].
\end{aligned}$$

From (52) one has that

$$\left[\Gamma(\alpha),\Gamma(\alpha_i)\right] = \begin{cases} 0 & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z} \\ -2\Gamma(\alpha_i)\Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1 \end{cases}$$

and therefore with $2\Gamma(\alpha_i)\Gamma(\alpha_i)\Gamma(\alpha) = -2\Gamma(\alpha)$ one has

$$\widehat{r}_i(2\phi)\Gamma(\alpha)\widehat{r}_i(2\phi)^{-1} = \begin{cases} \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z} \\ \left(1-2\sin^2\phi\right)\Gamma(\alpha) + 2\sin\phi\cos\phi\Gamma(\alpha_i)\Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1. \end{cases}$$

With $\cos^2\phi - \sin^2\phi = \cos 2\phi$, $2\sin\phi\cos\phi = \sin 2\phi$ this simplifies to

$$\widehat{r}_i(2\phi)\Gamma(\alpha)\widehat{r}_i(2\phi)^{-1} = \begin{cases} \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z} \\ \cos 2\phi\Gamma(\alpha) + \sin 2\phi\Gamma(\alpha_i)\Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1 \end{cases}$$

Furthermore, with (54), (55) and $(\alpha|\alpha_i) \in 2\mathbb{Z}+1$ and one has

$$\begin{aligned}
\Gamma(\alpha_i)\Gamma(\alpha) &= \varepsilon(\alpha_i,\alpha)\Gamma(\alpha_i+\alpha) = \varepsilon(\alpha_i,\alpha)\Gamma(\alpha-\alpha_i) \\
&= \varepsilon(\alpha_i,\alpha)\Gamma\left(\alpha - \frac{2(\alpha|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i\right) \\
&= \varepsilon(\alpha_i,\alpha)\Gamma(s_i.\alpha)
\end{aligned}$$

while for $(\alpha|\alpha_i) \in 2\mathbb{Z}$

$$\Gamma(\alpha) = \Gamma\left(\alpha - \frac{2(\alpha|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i\right) = \Gamma(s_i.\alpha),$$

so that in total

$$\widehat{r}_i(2\phi)\Gamma(\alpha)\widehat{r}_i(2\phi)^{-1} = \begin{cases} \Gamma(s_i.\alpha) = \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z} \\ \cos 2\phi\Gamma(\alpha) + \sin 2\phi\cdot\varepsilon(\alpha_i,\alpha)\Gamma(s_i.\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1 \end{cases}.$$

Now $r_i := \widehat{r}_i\left(\frac{\pi}{2}\right)$ generate W^{spin} on the image side, so that one obtains the desired Weyl-group-like conjugation

$$r_i\Gamma(\alpha)r_i^{-1} = \begin{cases} \Gamma(\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z} \\ \varepsilon(\alpha_i,\alpha)\Gamma(s_i.\alpha) & \text{if } (\alpha|\alpha_i) \in 2\mathbb{Z}+1 \end{cases}.$$

□

By the Γ-matrix calculus from lemma 3.14 one knows that $\rho(x_\alpha)$ for $x_\alpha \in \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ is given by a generalized Γ-matrix, i.e.,

$$\rho(x_\alpha) = c(x_\alpha)\Gamma(\alpha),$$

with $c(x_\alpha) \in \mathbb{C}$. However, one does not know when $c(x_\alpha) \neq 0$. The next proposition answers this question for some cases. The previous lemma basically implies that $c(x_\alpha) \neq 0$ depends on the orbit of the Weyl group[37].

Proposition 4.16. *(Cp. [KN13]) Let $\left(\mathcal{S}_{\frac{1}{2}}, \rho\right)$ be a generalized spin representation of $\mathfrak{k}(A)$ for A simply-laced and indecomposable. Let $0 \neq x \in \mathfrak{k}_\alpha$ then*

$$\rho(x) = c \cdot \Gamma(\alpha) \text{ s.t. } c \neq 0 \text{ if } \alpha \in \Delta^{re}(A), \quad \rho(x) = 0 \text{ if } \alpha \text{ is an isotropic root.}$$

For $\alpha, \beta \in \Delta^{re}$ s.t. $\alpha - \beta \in 2Q(A)$ and $0 \neq x_\alpha \in \mathfrak{k}_\alpha$ and $0 \neq x_\beta \in \mathfrak{k}_\beta$ there exists $c \in \mathbb{K} \setminus \{0\}$ s.t.

$$\rho(x_\alpha) = c \cdot \rho(x_\beta).$$

Proof. Let Γ and ε be as in the previous lemma then one obtains the representation matrices of $\mathfrak{k}_{\omega.\alpha}$ with lemma 4.14 as

$$\rho(Ad_{\widetilde{\omega}}(x_\alpha)) \quad = \quad \rho(Ad_{\widehat{\omega}}(x_\alpha)) = \Omega(\widehat{\omega})\rho(x_\alpha)\Omega(\widehat{\omega})^{-1},$$

where $\widetilde{\omega} \in W^{ext}(A)$ and $\hat{\omega} \in W^{spin}(A)$ are the corresponding elements to $\omega \in W(A)$ from prop. 4.13. Since $\Delta^{re} = W(A) \cdot \{\alpha_1, \ldots, \alpha_n\}$, all $\mathfrak{k}_\alpha$ for $\alpha \in \Delta^{re}$ are conjugate to $\mathfrak{k}_{\alpha_i}$ for any $i \in I$ (A is indecomposable and simply-laced). As $\mathfrak{k}_{\alpha_i} = \mathbb{K} \cdot X_i$ and the $\rho(X_i) \neq 0$ because of $\rho(X_i)^2 = -\frac{1}{4}Id$ one has $\rho(x) \neq 0$ for all $0 \neq x \in \mathfrak{k}_\alpha$ for all $\alpha \in \Delta^{re}$.

If $\alpha, \beta \in \Delta^{re}$ such that $2\gamma := \alpha - \beta \in 2Q(A)$ then

$$\Gamma(\alpha) = \Gamma(\beta + 2\gamma) = (-1)^{(\gamma|\gamma)}\Gamma(\beta)$$

by (55) and since $\rho(x_\alpha) = c(x_\alpha)\Gamma(\alpha)$, $\rho(x_\beta) = c(x_\beta)\Gamma(\beta)$ with $c(x_\alpha) \neq 0 \neq c(x_\beta)$ one knows that the two are proportional.

For an affine null root δ the space $\mathfrak{k}_\delta$ is spanned by all $[x_{\alpha_i}, x_{\delta-\alpha_i}]$ for $i \in I$. Now $(\alpha_i|\delta - \alpha_i) = -2$ and therefore $[\Gamma(\alpha_i), \Gamma(\delta - \alpha_i)] = 0$ which shows $\mathfrak{k}_\delta \subset \ker\rho$. In fact the same argument works for any affine null root $n \cdot \delta$. According to [K90, prop. 5.7] any isotropic root is $W(A)$-equivalent to an imaginary root whose support is a sub-diagram of affine type, hence any isotropic is $W(A)$-conjugate to an affine null root $n \cdot \delta$. By prop. 4.13 one then has again that

$$\rho(Ad_{\widetilde{\omega}}(x_{n\delta})) = \rho(Ad_{\widehat{\omega}}(x_{n\delta})) = \Omega(\widehat{\omega})\rho(x_{n\delta})\Omega(\widehat{\omega})^{-1} = 0.$$

□

Proposition 4.17. *(Cp.[38] [KN13]) Let $\sigma : \mathfrak{k}(A) \to End(V) \otimes End\left(\mathcal{S}_{\frac{1}{2}}\right)$ denote the $\frac{3}{2}$- , the $\frac{5}{2}$-, or the $\frac{7}{2}$-spin representation (cp. thms. 3.19 and 3.23). Let $0 \neq x_\alpha \in \mathfrak{k}_\alpha$ for $\alpha \in \Delta^{re}$, then there exists $c(x_\alpha) \neq 0$ s.t.*

$$\sigma(x_\alpha) = c(x_\alpha) \cdot \tau(\alpha) \otimes \Gamma(\alpha).$$

Furthermore, if x_α is conjugate to X_i for some $i \in I$, then $c(x_\alpha) \in \{-1, +1\}$.

[37] Most of these statements can be found in [KN13] in one way or another. The focus here is more on the Weyl group than in [KN13] and I do not need the assumption that it is possible to write any positive real root γ in the form $\gamma = \alpha + \beta$, where α, β are positive real roots.

[38] The idea to use the root lattice of E_{10} as a way of parametrizing the $\frac{n}{2}$-spin representations of $\mathfrak{k}(E_{10})$ first appeared in [KN13]. The authors checked that the root-dependent formula is correct for all suitably normalized Berman-elements $x \in \mathfrak{k}_\alpha$

Proof. Denote the lift of σ to the group $Spin(A)$ by Σ for $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ and denote the lifts to the fundamental rank 1 subgroups of $Spin(A)$ by Σ_i. For $\mathcal{S}_{\frac{7}{2}}$ denote the lift of σ by $\widetilde{\Sigma}$, because the formulas for $\mathcal{S}_{\frac{7}{2}}$ will split into a piece that is identical to that of $\mathcal{S}_{\frac{3}{2}}$ and an additional term. Then prop. 4.9 provides the formula

$$\begin{aligned}\Sigma_i(\phi) \quad = \quad & \left[\cos(\phi)\cos\left(\frac{\phi}{2}\right)\cdot Id\otimes Id - \cos(\phi)\sin\left(\frac{\phi}{2}\right)\cdot Id\otimes\Gamma(\alpha_i) + \right.\\ & \left.\sin(\phi)\sin\left(\frac{\phi}{2}\right)\cdot\eta(s_i)\otimes Id + \sin(\phi)\cos\left(\frac{\phi}{2}\right)\cdot\eta(s_i)\otimes\Gamma(\alpha_i)\right]\end{aligned}$$

for $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ which specializes to

$$\Sigma_i\left(\pm\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\cdot\eta(s_i)\otimes(Id\pm\Gamma(\alpha_i)).$$

For $\mathcal{S}_{\frac{7}{2}}$ prop. 4.10 provides the following formula for the lift of σ:

$$\widetilde{\Sigma}_i(\phi) = \Sigma_i(\phi) + \frac{1}{4}\left[\cos\left(\frac{5}{2}\phi\right) - \cos\left(\frac{3}{2}\phi\right)\right] f(\alpha_i)\otimes Id + \frac{1}{4}\left[\sin\left(\frac{5}{2}\phi\right) + \sin\left(\frac{3}{2}\phi\right)\right] f(\alpha_i)\otimes\Gamma(\alpha_i),$$

where $\Sigma_i(\phi)$ is the same expression as above. This expression specializes to

$$\widetilde{\Sigma}_i\left(\pm\frac{\pi}{2}\right) = \Sigma_i\left(\pm\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\cdot\eta(s_i)\otimes(Id\pm\Gamma(\alpha_i))$$

because $\cos\left(\pm\frac{5\pi}{4}\right) = \cos\left(\pm\frac{3\pi}{4}\right)$ and $\sin\left(\pm\frac{5\pi}{4}\right) = -\sin\left(\pm\frac{3\pi}{4}\right)$. As Σ_i (resp. $\widetilde{\Sigma}_i$) is 4π-periodic, the $\Sigma_i\left(\frac{\pi}{2}\right)$ (resp. $\widetilde{\Sigma}_i\left(\frac{\pi}{2}\right)$) generate the spin-extended Weyl group $W^{spin}(A)$ on the representation side. As in the previous proposition one uses that one can obtain the representation matrices of W-conjugates via

$$\sigma\left(Ad_{\widetilde{\omega}}(x_\alpha)\right) = \sigma\left(Ad_{\widehat{\omega}}(x_\alpha)\right) = \Sigma(\widehat{\omega})\,\sigma(x_\alpha)\,\Sigma(\widehat{\omega})^{-1},$$

where $\widetilde{\omega}\in W^{ext}(A)$ is the image of $\widehat{\omega}\in W^{spin}(A)$ under projection. Now one computes

$$\begin{aligned}\Sigma_i\left(\frac{\pi}{2}\right)\sigma(X_j)\,\Sigma_i\left(-\frac{\pi}{2}\right) \quad &= \quad \frac{1}{2}\eta(s_i)\otimes(Id+\Gamma(\alpha_i))\cdot\tau(\alpha_j)\otimes\Gamma(\alpha_j)\cdot\eta(s_i)\otimes(Id-\Gamma(\alpha_i))\\ &= \quad \frac{1}{2}\left(\eta(s_i)\,\tau(\alpha_j)\,\eta(s_i)\right)\otimes(Id+\Gamma(\alpha_i))\,\Gamma(\alpha_j)\,(Id-\Gamma(\alpha_i))\end{aligned}$$

and

$$\begin{aligned}(Id+\Gamma(\alpha_i))\,\Gamma(\alpha_j)\,(Id-\Gamma(\alpha_i)) \quad &= \quad \Gamma(\alpha_j)+\Gamma(\alpha_i)\,\Gamma(\alpha_j)-\Gamma(\alpha_j)\,\Gamma(\alpha_i)-\Gamma(\alpha_i)\,\Gamma(\alpha_j)\,\Gamma(\alpha_i)\\ &= \quad \begin{cases}2\Gamma(\alpha_j) & \text{if } (\alpha_i|\alpha_j)=0\\ 2\;\underbrace{\Gamma(\alpha_i)\,\Gamma(\alpha_j)}_{=\varepsilon(\alpha_i,\alpha_j)\Gamma(\alpha_i+\alpha_j)} & \text{if } (\alpha_i|\alpha_j)=-1.\end{cases}\end{aligned}$$

with $\alpha\in\Delta_+^{re}$ and $\mathrm{ht}(\alpha)\leq 100$. Their argument for an extension to all real roots is to decompose "a given (positive) real root α into two other (positive) real roots β and γ by $\alpha=\beta+\gamma$" ([KN13, p. 18]). When I started investigating these representations I could not find a proof of such a decomposition, although its existence appears very reasonable in the simply-laced situation. During a discussion with the first author of [KN13] he came up with the idea of using Weyl group conjugation to show that this parametrization holds for all real roots. This proposition fills in all technical details that one needs to do so. A key ingredient is that one can achieve the action of the Weyl group via the action of $W^{spin}(A)$ and that this action is compatible with the representation. Hence, one needs to build the bridge to [GHKW17] in order to have a firm grip on $W^{spin}(A)$ and $Spin(A)$.

For $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ one has $\tau(\alpha_j) = \eta(s_j) - \frac{1}{2}Id$ and therefore

$$\begin{aligned} \eta(s_i)\,\tau(\alpha_j)\,\eta(s_i) &= \eta(s_i)\left(\eta(s_j) - \frac{1}{2}Id\right)\eta(s_i) = \eta(s_i s_j s_i) - \frac{1}{2}Id \\ &= \eta\left(s_{s_i.\alpha_j}\right) - \frac{1}{2}Id = \tau(s_i.\alpha_j)\,. \end{aligned}$$

For $\mathcal{S}_{\frac{7}{2}}$ one has $\tau(\alpha_j) = \eta(s_j) - \frac{1}{2}Id + f(\alpha_j)$ with $f(\alpha_j) = v(\alpha_j)\,Q(v(\alpha_j)\,|\cdot)$ and $v(\alpha_j)$ as in eq. 67. From lemma 3.22 it follows that $\eta(s_i)\,v(\alpha_j) = v(s_i.\alpha_j)$ and so one computes

$$\begin{aligned} \eta(s_i)\,f(\alpha_j)\,\eta(s_i) &= \eta(s_i)\,v(\alpha_j)\,Q(v(\alpha_j)\,|\cdot)\,\eta(s_i) \\ &= v(s_i.\alpha_j)\,Q(v(s_i.\alpha_j)\,|\cdot) = f(s_i.\alpha_j) \end{aligned}$$

and

$$\begin{aligned} \eta(s_i)\,\tau(\alpha_j)\,\eta(s_i) &= \eta(s_i)\left[\eta(s_j) - \frac{1}{2}Id + f(\alpha_j)\right]\eta(s_i) \\ &= \eta\left(s_{s_i.\alpha_j}\right) - \frac{1}{2}Id + f(s_i.\alpha_j) = \tau(s_i.\alpha_j)\,. \end{aligned}$$

In total this provides

$$\Sigma_i\left(\frac{\pi}{2}\right)\sigma(X_j)\,\Sigma_i\left(-\frac{\pi}{2}\right) = \begin{cases} \tau(s_i.\alpha_j)\otimes\Gamma(\alpha_j) & \text{if } (\alpha_i|\alpha_j) = 0 \\ \varepsilon(\alpha_i,\alpha_j)\,\tau(s_i.\alpha_j)\otimes\Gamma(\alpha_i+\alpha_j) & \text{if } (\alpha_i|\alpha_j) = -1 \end{cases}$$

for $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ as well as for $\mathcal{S}_{\frac{7}{2}}$ because $\widetilde{\Sigma}_i\left(\pm\frac{\pi}{2}\right) = \Sigma_i\left(\pm\frac{\pi}{2}\right)$ and so

$$\Sigma_i\left(\frac{\pi}{2}\right)\sigma(X_j)\,\Sigma_i\left(-\frac{\pi}{2}\right) = c\cdot\tau(s_i.\alpha_j)\otimes\Gamma(s_i.\alpha_j)\,,$$

where c is either 1 or $\varepsilon(\alpha_i,\alpha_j)$, hence $c = \pm 1$. Now this implies

$$\Sigma(\widehat{\omega})\,\sigma(X_j)\,\Sigma(\widehat{\omega})^{-1} = c\cdot\tau(\omega.\alpha_j)\otimes\Gamma(\omega.\alpha_j)\,, \tag{97}$$

where Σ now denotes the lift of any $\mathcal{S}_{\frac{n}{2}}$ for $n = 3,5,7$ and $\omega \in W$ is the projection of $\widehat{\omega}$. As any real root space $\mathfrak{g}_\alpha$ is W^{ext}-conjugate and hence, W^{spin}-conjugate to a simple root space $\mathfrak{g}_{\alpha_i}$ and the same is true for $\mathfrak{k}_\alpha$, eq. (97) implies that all $0 \neq x_\alpha \in \mathfrak{k}_\alpha$ for $\alpha \in \Delta_+^{re}$ have a nontrivial image because the multiplicity of real roots is equal to 1. □

5 Decompositions of the $\mathfrak{k}(E_{10})(\mathbb{C})$-module $\mathcal{S}_{\frac{n}{2}}$ under $\mathfrak{so}(10,\mathbb{C})$

In this section, I will alternate between studying the higher spin representations $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ for general $\mathfrak{k}(A)$ and for the example of $\mathfrak{k}(E_{10})$. I will analyze how the higher spin representations $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ of $\mathfrak{k}(E_{10})(\mathbb{C})$ decompose under restriction to $\mathfrak{so}(10,\mathbb{C})$ which will reproduce some results of [KN17]. The representation $\mathcal{S}_{\frac{1}{2}}$ of $\mathfrak{k}(E_{10})(\mathbb{C})$ is not irreducible under $\mathfrak{so}(10,\mathbb{C})$ but splits into two irreducible parts $\Gamma_\alpha \oplus \Gamma_\beta$ denoted by their respective highest weights α and β. Under $\mathfrak{k}(E_{10})(\mathbb{C})$ and $\mathfrak{k}(E_{10})(\mathbb{R})$ however, it is irreducible. The same holds for $\mathcal{S}_{\frac{3}{2}}$ whose decomposition into irreducible $\mathfrak{so}(10,\mathbb{C})$-modules is given in prop. 5.5. In prop.

5.8 I furthermore provide a general criterion on the GCM A that ensures irreducibility of $\mathcal{S}_{\frac{3}{2}}$. The module $\mathcal{S}_{\frac{5}{2}}$ splits into an invariant copy of $\mathcal{S}_{\frac{1}{2}}$ and its orthogonal complement $\widetilde{\mathcal{S}}_{\frac{5}{2}}$, called the trace-free part, is also invariant if the GCM A is regular (see prop. 5.10). The trace-free part $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ can be irreducible or not even for A of classical type as is shown in the remark of prop. 5.10, so here no general statement is possible but the question can be reduced to the question of $W(A)$-irreducible submodules of $\mathrm{Sym}^2(\mathfrak{h}^*)$. Theorem 5.14 shows that for $\mathfrak{k}(E_{10})$, the trace-free spin representation $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible and furthermore provides its decomposition into irreducible $\mathfrak{so}(10,\mathbb{C})$-modules.

5.1 The $\frac{3}{2}$-spin representation of $\mathfrak{k}(E_{10})$

As a vector space the $\mathfrak{k}(E_{10})$-module is $\mathcal{S}_{\frac{3}{2}} \cong \mathfrak{h}^* \otimes S$, where $S \cong \mathbb{C}^{32}$ is the module of the $\frac{1}{2}$-spin representation. On $\mathfrak{h}^*$, the action basically works via Weyl-reflections and elements in $\mathfrak{k}(A_9)(\mathbb{C})$ only include the reflections $s_1, \ldots, s_9$. Hence there should exist an $W(A_9)$-invariant subspace of $\mathfrak{h}^*$:

Lemma 5.1. *There exists a vector $v \in \mathfrak{h}^*$ such that $s_i(v) = v$ for all $i = 1, \ldots, 9$. This vector is unique up to scalar multiples and in the basis $\{\alpha_1, \ldots, \alpha_{10}\}$ of $\mathfrak{h}^*$ it is given by*

$$v = c \cdot \left(\frac{7}{3}, \frac{14}{3}, 7, 6, 5, 4, 3, 2, 1, \frac{10}{3}\right) \tag{98}$$

Proof. Note that $s_i(v) = v$ is equivalent to $(v|\alpha_i) = 0$. This makes v any solution of

$$(v|\alpha_i) = 0 \text{ for } i = 1, \ldots, 9$$

and since the $\alpha_1, \ldots, \alpha_9$ are linearly independent and $(\cdot|\cdot)$ is non-degenerate these equations define a $\dim(\mathfrak{h}^*) - 9 = 1$ dimensional subspace. Let $v = \sum_{i=1}^{10} a_i \alpha_i$ then

$$(\alpha_i|v) = 0 \Leftrightarrow -a_{i-1} + 2a_i - a_{i+1} = 0 \text{ for } i = 4, \ldots, 8$$

and

$$\begin{gathered} 2a_9 - a_8 = 0\ ,\ 2a_3 - a_2 - a_4 - a_{10} = 0 \\ 2a_2 - a_1 - a_3 = 0\ ,\ 2a_1 - a_2 = 0 \end{gathered}$$

This implies

$$a_8 = 2a_9,\ a_7 = 2a_8 - a_9 = 3a_9,\ a_6 = 2a_7 - a_8 = 4a_9,$$

$$a_{9-k} = (k+1)a_9 \text{ for } k = 0, \ldots, 6\ ,\ a_2 = 2a_1\ .$$

Using the equation $2a_2 - a_1 - a_3 = 0$ then yields $a_1 = \frac{7}{3}a_9$ and ultimately one finds

$$a_{10} = 2a_3 - a_2 - a_4 = \frac{10}{3}a_9$$

so that

$$v = c \cdot \left(\frac{7}{3}, \frac{14}{3}, 7, 6, 5, 4, 3, 2, 1, \frac{10}{3}\right)$$

for $c \in \mathbb{R}, \mathbb{C}$. $\square$

Remark. Note that $(v|\alpha_{10}) = c \cdot \left(-7 + \frac{20}{3}\right) = -\frac{1}{3}c$ so that for $c = 3$ one has

$$v_{c=3} = (7, 14, 21, 18, 15, 12, 9, 6, 3, 10) = -\omega_{10} \ , \tag{99}$$

where ω_{10} is the 10-th fundamental weight[39] of E_{10}.

With v as above one has for $i = 1, \ldots, 9$ that

$$\sigma(X_i)\, v \otimes s = \tau(\alpha_i)\, v \otimes 2\rho(X_i)\, s = v \otimes \rho(X_i)\, s \ \forall\, s \in S.$$

As ρ was initially defined such that its restriction to $\mathfrak{k}(A_9)(\mathbb{C})$ coincides with the classical spin representation, one recovers just that.

Proposition 5.2. *Upon restriction to $\mathfrak{so}(10, \mathbb{C})$ the generalized $\frac{3}{2}$-spin representation of $\mathfrak{k}(E_{10})(\mathbb{C})$ over the module $\mathcal{S}_{\frac{3}{2}} := V \otimes S$ contains a copy of the $\frac{1}{2}$-spin representation $\Gamma_\alpha \oplus \Gamma_\beta$. The corresponding highest weight vectors in the $\mathfrak{so}(10, \mathbb{C})$-module are given by $v \otimes s_\alpha$ and $v \otimes s_\beta$, where $v \perp span_{\mathbb{C}} \{\alpha_1, \alpha_2, \ldots, \alpha_9\}$ and s_α, s_β denote the highest weight vectors in S.*

There are still other $\mathfrak{so}(10, \mathbb{C})$-irreducible parts contained in the $\frac{3}{2}$spin representation. Consider the set of vectors $\alpha_1 \otimes s_\lambda$ where s_λ is a weight vector to the weight $\lambda \in \Delta_W(\Gamma_\alpha), \Delta_W(\Gamma_\beta)$. Then (recall $H_j = -iX_{2j-1}$) one has

$$\begin{aligned} \sigma(H_j)\, \alpha_1 \otimes s_\lambda &= \tau(\alpha_{2j-1})\, \alpha_1 \otimes 2\rho(H_j)\, s_\lambda = 2\lambda(H_j) \cdot \tau(\alpha_{2j-1})\, \alpha_1 \otimes s_\lambda \\ &= \begin{cases} -3\lambda(H_j) \cdot \alpha_1 \otimes s_\lambda & j = 1 \\ \lambda(H_j) \cdot \alpha_1 \otimes s_\lambda & j = 2, \ldots, 5 \ . \end{cases} \end{aligned} \tag{100}$$

Hence, these vectors are $\mathfrak{h}_{D_5}$-diagonal. In order to be a highest weight vector it is also required that $\sigma\left(e_{\gamma_j}\right) \alpha_1 \otimes s_\lambda = 0$ for $j = 1, \ldots, 5$, where γ_j denote the positive simple roots of D_5. So one could determine the action of the Chevalley generators $e_{\gamma_1}, \ldots, e_{\gamma_5}$ and see if there is a vector $\alpha_1 \otimes s_\lambda$ with these properties. However, one can proceed more generally and find the highest weight vectors with less computational effort.

First, spell out $V \otimes S$ in a basis that is diagonal w.r.t. the action of $\mathfrak{h}_{D_5}$. Let $t_1, \ldots, t_5 \in V$ be pairwise orthogonal and orthogonal to $\alpha_1, \alpha_3, \ldots, \alpha_9$. Then

$$\mathcal{B} = \{\alpha_1, \alpha_3, \ldots, \alpha_9\} \cup \{t_1, \ldots, t_5\} \tag{101}$$

is an orthogonal basis of V.

Lemma 5.3. *For $V = \mathfrak{h}^*$ and S the generalized $\frac{1}{2}$-spin representation's module, the action of $\mathfrak{h}_{D_5}$ is diagonal on elements of the form $w \otimes s_\lambda$, where $w \in \mathcal{B}$ and $s_\lambda \in S$ is a weight vector to the weight $\lambda \in \Delta(\Gamma_\alpha \oplus \Gamma_\beta)$. The D_5-weight system of the $\mathfrak{k}(E_{10})$-module $\mathcal{S}_{\frac{3}{2}}$ is*

$$\Delta_{\frac{3}{2}} := \left\{ \frac{1}{2} \sum_{i=1}^{5} a_i L_i \mid \textit{at most one } a_i = \pm 3, \ \textit{the others } \pm 1 \right\}, \tag{102}$$

where as usual $L_i \in \mathfrak{h}^*_{D_5}$ are defined via $L_i(H_j) = \delta_{ij}$.

[39] In general, the fundamental weights are defined by $\omega_i\left(\alpha_j^\vee\right) = \delta_{ij}$ but for the simply-laced case this is equivalent to $(\omega_i|\alpha_j) = \delta_{ij}$.

Proof. As
$$\sigma(H_j)\, w \otimes s_\lambda = \tau(\alpha_{2j-1})\, w \otimes 2\rho(H_j)\, s_\lambda = 2\lambda(H_j) \cdot \tau(\alpha_{2j-1})\, w \otimes s_\lambda$$
and $\tau(\alpha_{2j-1}) = s_{\alpha_{2j-1}} - \frac{1}{2} Id$ one has

$$\tau(\alpha_{2j-1})\, w = \begin{cases} -\frac{3}{2} w & \text{if } w = \alpha_{2j-1} \\ \frac{1}{2} w & \text{if } w \in \mathcal{B} \setminus \{\alpha_{2j-1}\} \end{cases} .$$

Since $\mathcal{B}$ is a basis of V and S decomposes into weight spaces this clearly provides an $\mathfrak{h}_{D_5}$-diagonal basis for $V \otimes S$. Now
$$\sigma(H_j)\, \alpha_{2i-1} \otimes s_\lambda = (1 - 4\delta_{ij})\, \lambda(H_j) \cdot \alpha_{2i-1} \otimes s_\lambda$$
and so possible modifications to the weights of $\Delta(\Gamma_\alpha \oplus \Gamma_\beta)$ consist of multiplying the prefactor of up to one L_i by -3. Since (cp. [FH91, ch. 20])

$$\Delta(\Gamma_\alpha \oplus \Gamma_\beta) = \left\{ \frac{1}{2} \sum_{i=1}^{5} a_i L_i \mid a_1, \ldots, a_5 \in \{\pm 1\} \right\}$$

this provides all the weights of $V \otimes S$ to be as in eq. 102. □

Now that a decomposition of $V \otimes S$ into weight spaces has been obtained, determining its decomposition into irreducible $\mathfrak{so}(10, \mathbb{C})$-modules is equivalent to finding all highest weight vectors in $V \otimes S$.

Proposition 5.4. *The vectors $\alpha_1 \otimes s_{\lambda_1}$ and $\alpha_1 \otimes s_{\lambda_2}$, where $\lambda_1 = \beta - \omega_1$, $\lambda_2 = \alpha - \omega_1$, are highest weight vectors to the weights $\beta + \omega_1, \alpha + \omega_1$ respectively.*

Proof. Highest weights Λ of D_5 have the shape (cp. [FH91, chps. 18 & 20])

$$\Lambda = n_1 L_1 + n_2 (L_1 + L_2) + n_3 (L_1 + L_2 + L_3) + \frac{n_4}{2} (L_1 + \ldots L_4 - L_5) + \frac{n_5}{2} (L_1 + \ldots L_4 + L_5)$$

and the only weights in $\Delta_{\frac{3}{2}}$ for which the coefficients of $L_1, \ldots, L_4$ are positive are of the shape $\alpha + aL_i$ or $\beta + aL_i$ for some $i \in \{1, \ldots, 5\}$ and $a \in \{0, 1\}$. The only highest weights that can appear this way are $L_1 + \alpha = \omega_1 + \alpha$, $\omega_1 + \beta$, α and β. So given a weight vector $v_\Lambda = w \otimes s_\lambda$ to the weight $\omega_1 + \alpha$ or $\omega_1 + \beta$ one can immediately conclude that this vector is a highest weight vector. Now observe that

$$\begin{aligned} \lambda_1 &= -\frac{1}{2} L_1 + \frac{1}{2}(L_2 + L_3 + L_4) - \frac{1}{2} L_5 = \beta - L_1 = \beta - \omega_1 \in \Delta(\Gamma_\alpha) \\ \lambda_2 &= -\frac{1}{2} L_1 + \frac{1}{2}(L_2 + L_3 + L_4 + L_5) = \alpha - L_1 = \alpha - \omega_1 \in \Delta(\Gamma_\beta) \end{aligned}$$

have the property that multiplying the coefficient of L_1 by -3 yields the weights $\beta + \omega_1$ and $\alpha + \omega_1$ respectively[40]. Therefore $\alpha_1 \otimes s_{\lambda_1}$ and $\alpha_1 \otimes s_{\lambda_2}$ are HWVs to the weights $\beta + \omega_1$ and $\alpha + \omega_1$ respectively. □

Proposition 5.5. *The $\frac{3}{2}$-spin representation $\mathcal{S}_{\frac{3}{2}}$ of $\mathfrak{k}(E_{10})(\mathbb{C})$ is irreducible. Upon restriction to its $\mathfrak{so}(10, \mathbb{C})$-subalgebra it decomposes as*
$$\mathcal{S}_{\frac{3}{2}} \cong_{\mathfrak{so}(10,\mathbb{C})} \Gamma_\alpha \oplus \Gamma_\beta \oplus \Gamma_{\alpha+\omega_1} \oplus \Gamma_{\beta+\omega_1} .$$

[40] Since Γ_α and Γ_β are conjugate to each other, $\lambda_2 \in \Delta(\Gamma_\beta)$ if $-\lambda_2 \in \Delta(\Gamma_\alpha)$ and so one sees that $-\lambda_2 = -\alpha + L_1$ is located at depth 6 whereas λ_1 is located at depth 4 in the weight diagram (cp. figure 5).

$\mathfrak{so}(10, \mathbb{C})$-weight diagram of Γ_α

$$\lambda = \frac{1}{2}(a_1 L_1 + \cdots + a_5 L_5) \longleftrightarrow (a_1\ a_2\ a_3\ a_4\ a_5)$$

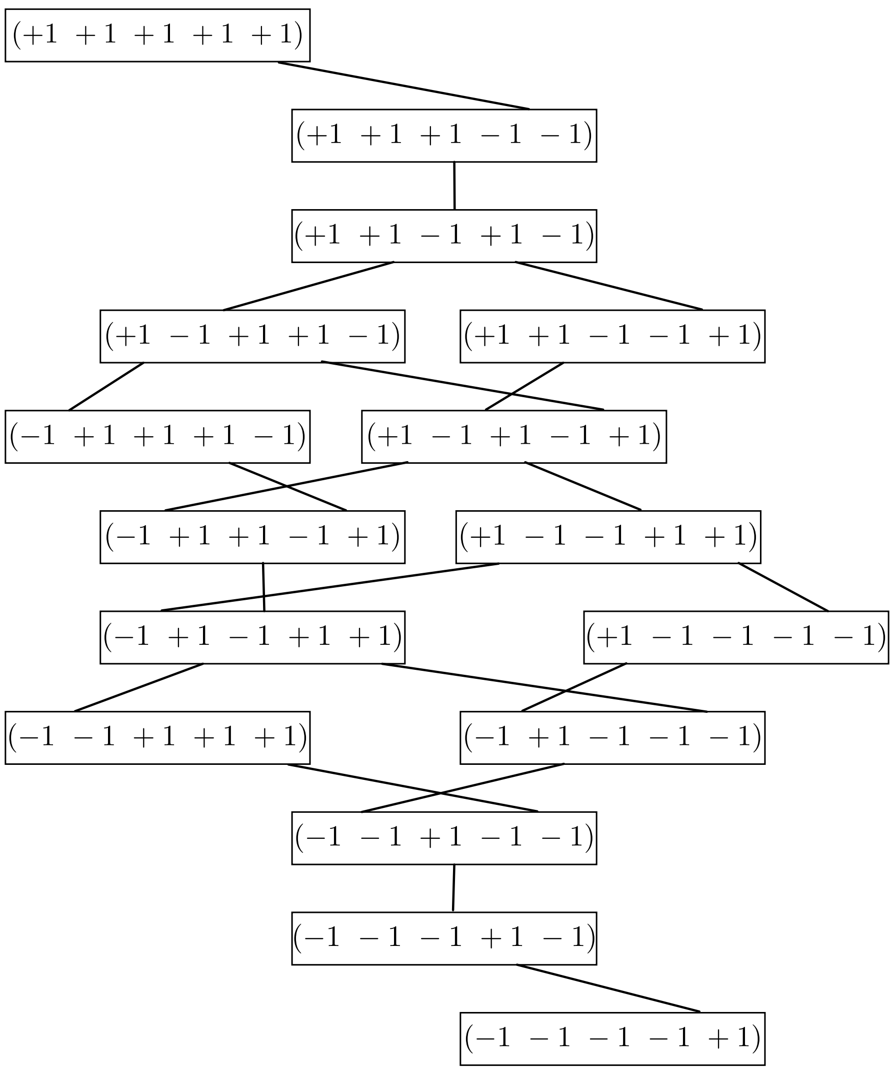

Figure 5: The weight diagram of the irreducible representation Γ_α of $\mathfrak{so}(10, \mathbb{C})$. The lines that are drawn indicate which simple root has to be subtracted to descend to the lower weights. For example the very first line starts at position 5 to indicate that $\gamma_5 = L_4 + L_5$ needs to be substracted. Recall that the other simple roots are given by $\gamma_i = L_i - L_{i+1}$ for $i = 1, \ldots, 4$.

Proof. There are two complementary subspaces of $\mathfrak{h}^* \otimes S$ that are invariant under $\mathfrak{so}(10,\mathbb{C})$. The first is spanned by elements of the form $V_1 := \{v \otimes s \mid s \in S\}$ where v is the vector from lemma 5.1 that is orthogonal to $\alpha_1, \ldots, \alpha_9$. The subspace $V_2 := \{\alpha_i \otimes s \mid s \in S,\ i = 1, \ldots, 9\}$ is also invariant under $\mathfrak{so}(10,\mathbb{C})$, because $s_i \alpha_j \in \operatorname{span}\{\alpha_1, \ldots, \alpha_9\}$ for all $i, j \in \{1, \ldots, 9\}$. Since $\dim V_1 + \dim V_2 = 1 \cdot 32 + 9 \cdot 32 = 320 = \dim \mathcal{S}_{\frac{3}{2}}$ one concludes that $\mathcal{S}_{\frac{3}{2}} = V_1 \oplus V_2$ as an orthogonal direct sum of $\mathfrak{so}(10,\mathbb{C})$-modules. Note that orthogonality is defined w.r.t. the standard inner product on S and the invariant bilinear form on $\mathfrak{h}^*$. The invariant bilinear form on $\mathfrak{h}^*$ is indefinite but one has that its restrictions to $\mathbb{K} \cdot \omega_{10}$ and $\mathbb{K}\{\alpha_1, \ldots, \alpha_9\}$ are non-degenerate so that $\mathfrak{h}^*$ splits into orthogonal complements

One had already seen $V_1 \cong \mathcal{S}_{\frac{1}{2}} \cong \Gamma_\alpha \oplus \Gamma_\beta$ as $\mathfrak{so}(10,\mathbb{C})$-modules in proposition (5.2) and from proposition (5.4) it follows that $V_2 \cong \Gamma_{\alpha+\omega_1} \oplus \Gamma_{\beta+\omega_1}$. The last equality follows from $\dim(\Gamma_{\alpha+\omega_1} \oplus \Gamma_{\beta+\omega_1}) = 2 \cdot 144 = 288 = \dim V_2$. Thus,

$$\mathcal{S}_{\frac{3}{2}} \cong_{\mathfrak{so}(10,\mathbb{C})} \Gamma_\alpha \oplus \Gamma_\beta \oplus \Gamma_{\alpha+\omega_1} \oplus \Gamma_{\beta+\omega_1} .$$

Given the vector $v = -\omega_{10}$ from lemma 5.1 eq. (98) one notes that

$$\tau(\alpha_{10})(v) = \frac{1}{2}v - \alpha_{10}(\alpha_{10}|v) = \frac{1}{2}v + \alpha_{10}$$

and therefore

$$\sigma(X_{10})\, v \otimes s = \underbrace{v \otimes \rho(X_{10})\, s}_{\in \Gamma_\alpha \oplus \Gamma_\beta} + \underbrace{2\alpha_{10} \otimes \rho(X_{10})\, s}_{\notin \Gamma_\alpha \oplus \Gamma_\beta} .$$

Since the $\frac{1}{2}$-spin representation is known to be irreducible[41] it has to mix Γ_α and Γ_β and therefore there needs to exist $s \in S$ such that $\rho(X_{10})\, s \neq 0$. Displaying $\sigma(X_{10})$ as a block matrix

$$\sigma(X_{10}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Hom}\ \left((\Gamma_\alpha \oplus \Gamma_\beta) \oplus (\Gamma_{\alpha+\omega_1} \oplus \Gamma_{\beta+\omega_1})\right) \tag{103}$$

this shows that B is non-zero and A can not be decomposed, meaning it is irreducible. Observing that $\tau(\alpha_{10})(\alpha_3) = \frac{1}{2}\alpha_3 + \alpha_{10}$ shows that $(v|\tau(\alpha_{10})(\alpha_3)) = 1$ and therefore

$$\sigma(X_{10})\, \alpha_3 \otimes s = a \otimes \rho(X_{10})\, s + 2v \otimes \rho(X_{10})\, s$$

where a is the part of $2\tau(\alpha_{10})(\alpha_3)$ that is orthogonal to v. Now $\alpha_3 \otimes s \in \Gamma_{\alpha+\omega_1} \oplus \Gamma_{\beta+\omega_1}$ for all $s \in S$ and by the properties of ρ one has $\rho(X_{10})\, s \neq 0$ for all $s \neq 0$. This implies that C in eq. (103) is nonzero. Since the weight vectors to $\Gamma_{\alpha+\omega_1}$ and $\Gamma_{\beta+\omega_1}$ are of the form $\alpha_i \otimes s_\lambda$, $\alpha_i \otimes s_\kappa$ with $i = 1, \ldots, 9$, λ a weight of Γ_β and κ a weight of Γ_α one can again rely on ρ to mix between $\Gamma_{\alpha+\omega_1}$ and $\Gamma_{\beta+\omega_1}$ so that D is also irreducible. This shows that the $\frac{3}{2}$-spin representation is irreducible as a $\mathfrak{k}(E_{10})(\mathbb{C})$ −representation. □

There exists an alternative way of showing irreducibility that relies certain polynomial identities of the representation matrices and I will use variants of this trick again in section 6.2.

[41] This follows for instance from [HKL15, thm. A] which says that the image of $\mathfrak{k}(E_{10})(\mathbb{R})$ under ρ is isomorphic to $\mathfrak{so}(32)$. After complexification this acts irreducibly on $S \cong \mathbb{C}^{32}$. One can also use prop. 2.7 to show this. The commutation relations satisfied by $X_\pm$ are not compatible with the weight system of a single highest weight module w.r.t. D_5 because $\pm L_2 \notin Q(D_5)$. Hence, $X_{10} \propto X_+ + X_-$ has to mix the highest weight modules Γ_α and Γ_β of $\mathfrak{so}(10,\mathbb{C})$ in $S \cong \Gamma_\alpha \oplus \Gamma_\beta$.

Lemma 5.6. *Let $(\sigma, V\otimes S)$ denote the $\frac{3}{2}$- or $\frac{5}{2}$-spin representation of $\mathfrak{k}(A)(\mathbb{K})$ for A simply-laced. Then one has for $\alpha\in\Delta_+^{re}$ and $X_\alpha\in\mathfrak{k}_\alpha$ s.t. $(X_\alpha|X_\alpha)=(X_i|X_i)$ for some $i=1,\dots,n$ that*

$$\sigma(X_\alpha)^2=\left(\eta(s_\alpha)-\frac{5}{4}\right)\otimes Id,\ \sigma(X_\alpha)^3=-\frac{7}{4}\sigma(X_\alpha)+\frac{3}{4}Id\otimes 2\rho(X_\alpha),\ \sigma(X_\alpha)^4=-\frac{5}{2}\left(\eta(s_\alpha)-\frac{41}{40}\right)\otimes Id$$

$$Id\otimes\rho(X_\alpha)=\frac{2}{3}\sigma(X_\alpha)^3+\frac{7}{6}\sigma(X_\alpha),\ \eta(s_\alpha)\otimes Id=-\frac{20}{9}\sigma(X_\alpha)^4-\frac{41}{9}\sigma(X_\alpha)^2 \tag{104}$$

$$Id\otimes Id=\frac{16}{9}\sigma(X_\alpha)^4+\frac{40}{9}\sigma(X_\alpha)^2 \tag{105}$$

where η denotes the representation of the Weyl group $W(A)$ on V. Hence, for all $w\in W(A)$ there exists $y_1\in\mathcal{U}(\mathfrak{k})$ s.t. $\sigma(y_1)=\eta(w)\otimes Id$ and for all $x\in\mathfrak{k}$ there exists $y_2\in\mathcal{U}(\mathfrak{k})$ s.t. $\sigma(y_2)=Id\otimes\rho(x)$.

Proof. One has from prop. 4.17 that

$$\sigma(X_\alpha)=\pm\left(\eta(s_\alpha)-\frac{1}{2}\right)\otimes\Gamma(\alpha)=\left(\eta(s_\alpha)-\frac{1}{2}\right)\otimes(2\rho(X_\alpha)),$$

where the prefactor is just a sign, because X_α has the same norm as X_i. One now computes

$$\sigma(X_\alpha)^2=\left(\eta(s_\alpha)-\frac{1}{2}\right)^2\otimes(2\rho(X_\alpha))^2=\left(\eta(s_\alpha)-\frac{5}{4}\right)\otimes Id,$$

$$\begin{aligned}\sigma(X_\alpha)^3 &= \left(\eta(s_\alpha)-\frac{1}{2}\right)\left(\eta(s_\alpha)-\frac{5}{4}\right)\otimes 2\rho(X_\alpha)=\left(\frac{13}{8}-\frac{7}{4}\eta(s_\alpha)\right)\otimes 2\rho(X_\alpha)\\ &= -\frac{7}{4}\left(\eta(s_\alpha)-\frac{13}{14}\right)\otimes 2\rho(X_\alpha)=-\frac{7}{4}\sigma(X_\alpha)+\frac{3}{4}Id\otimes 2\rho(X_\alpha),\end{aligned}$$

$$\begin{aligned}\sigma(X_\alpha)^4 &= \left(\eta(s_\alpha)-\frac{5}{4}\right)^2\otimes Id=\left(-\frac{5}{2}\eta(s_\alpha)+1+\frac{25}{16}\right)\otimes Id\\ &= -\frac{5}{2}\left(\eta(s_\alpha)-\frac{41}{40}\right)\otimes Id\end{aligned}$$

and from there (104) and (105) follow. As the s_i generate $W(A)$ and the X_i generate $\mathfrak{k}$ one can always find $y_1,y_2\in\mathcal{U}(\mathfrak{k})$ s.t. $\sigma(y_1)=\eta(w)\otimes Id$ and $\sigma(y_2)=Id\otimes\rho(x)$ which completes the proof. □

Lemma 5.7. *Let $A\in\mathbb{Z}^{n\times n}$ be a simply-laced generalized Cartan matrix of full rank and let $\mathfrak{h}^*$ be the dual Cartan subalgebra to a realization of A. Then $\mathfrak{h}^*$ is an irreducible $W(A)$-module.*

Proof. Assume that $\mathfrak{h}^*$ contains an invariant submodule V that is not all of $\mathfrak{h}^*$. Because the invariant bilinear form is nondegenerate for any $0\neq v\in V$ there needs to exist $i\in\{1,\dots,n\}$ such that $(v|\alpha_i)\neq 0$. Hence $s_iv=v-(v|\alpha_i)\alpha_i\in V$ by invariance but then also $\alpha_i\in V$ because $(v|\alpha_i)\neq 0$. But $W(A).\alpha_i=\Delta^{re}(A)$ and so $V=\mathfrak{h}^*$in contradiction to the original assumption. Hence, $\mathfrak{h}^*$ is an irreducible $W(A)$-module. □

Remark. Note that the assumption about full rank is necessary. In the affine situation one has that $\mathbb{K}\cdot\delta$, where δ denotes the null root, is an invariant subspace of $\mathfrak{h}^*$ as a $W(A)$-module.

Proposition 5.8. *Let $A \in \mathbb{Z}^{n\times n}$ be a simply-laced indecomposable GCM of full rank greater than two. Then there exists an irreducible generalized spin representation $\mathcal{S}_{\frac{1}{2}}$ and the $\mathfrak{k}(A)$-module $\mathcal{S}_{\frac{3}{2}}$ is irreducible if built on this irreducible $\mathcal{S}_{\frac{1}{2}}$.*

Proof. First of all, why can $\mathcal{S}_{\frac{1}{2}}$ be assumed to be irreducible? According to thm. 3.3 (originally [HKL15, thm. 3.14]) the image of $\rho : \mathfrak{k}(A) \to \mathrm{End}\left(\mathcal{S}_{\frac{1}{2}}\right)$ is semi-simple if A is such that to each i there exists j with a_{ij} odd. This is the case for A simply-laced, indecomposable and of rank greater than two. Hence, $\mathcal{S}_{\frac{1}{2}}$ is a finite-dimensional imρ-module and therefore completely reducible. One now restricts to an arbitrary decomposition factor in order to obtain an irreducible generalized spin representation.

In view of lemmas 5.6 and 5.7 the irreducibility of $\mathcal{S}_{\frac{3}{2}}$ follows from the fact that $\mathfrak{h}^*$ is an irreducible $W(A)$-module and that $\mathcal{S}_{\frac{1}{2}}$ is irreducible as well as long as one can show that any nontrivial submodule needs to contain an elementary tensor $\alpha \otimes s \in \mathfrak{h}^* \otimes \mathcal{S}_{\frac{1}{2}}$. This is because then one has

$$\mathcal{U}(\mathfrak{k})\, U \subseteq \mathcal{U}(\mathfrak{k})\,(\alpha \otimes s) = \mathcal{U}(W(A))\,\{\alpha\} \otimes \mathcal{U}(\mathfrak{k})\,\{s\}\,,$$

where the last equality follows from lemma 5.6. As $\mathcal{U}(W(A))\,\{\alpha\}$ is a $W(A)$-invariant submodule of $\mathfrak{h}^*$ one deduces from lemma 5.7 that $\mathcal{U}(W(A))\,\{\alpha\} = \mathfrak{h}^*$ and the same argument shows $\mathcal{U}(\mathfrak{k})\,\{s\}$ as $\mathcal{S}_{\frac{1}{2}}$ was assumed to be irreducible.

Now let U be an invariant submodule of $\mathcal{S}_{\frac{3}{2}}$ then for now one can only assume the most general form for $u \in U$, where $m := \dim\left(\mathcal{S}_{\frac{1}{2}}\right)$ and $\{b_1, \dots, b_m\}$ is some basis of $\mathcal{S}_{\frac{1}{2}}$:

$$u = \sum_{i=1}^{n}\sum_{j=1}^{m} d_i c_j \alpha_i \otimes b_j \in U. \tag{106}$$

Under the proposition's assumptions one knows from thm. 3.3 that the image of $\rho : \mathfrak{k}(A) \to \mathrm{End}\left(\mathcal{S}_{\frac{1}{2}}\right)$ is semi-simple. As $\mathcal{S}_{\frac{1}{2}}$ is an irreducible finite-dimensional imρ -module it is a highest weight module w.r.t. $\mathring{\mathfrak{h}}$, where $\mathring{\mathfrak{h}}$ is a Cartan subalgebra of imρ. Denote the triangular decomposition of imρ by $\mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$. Then the basis $\{b_1, \dots, b_m\}$ of $\mathcal{S}_{\frac{1}{2}}$ can be assumed to be a weight space basis and the decomposition (106) can be rewritten as

$$u = \sum_{i=1}^{n}\sum_{\lambda \in P}\sum_{j=1}^{m(\lambda)} d_i c_j^{(\lambda)} \alpha_i \otimes b_j^{(\lambda)} \in U,$$

where P denotes the weights of $\mathcal{S}_{\frac{1}{2}}$ w.r.t. $\mathring{\mathfrak{h}}$. Since any weight λ can be written as $\lambda = \Lambda - \sum_{i=1}^{l} k_i \gamma_i$ with $k_i \in \mathbb{N}_0$ and $\gamma_1, \dots, \gamma_l$ the simple roots of imρ, the depth $|\lambda| := \sum_{i=1}^{l} k_i$ provides a partial order on the weights. The above decomposition contains one or more $b_j^{(\lambda)}$ of maximal depth, where different weights λ of the same depth may occur. To each of these $b_j^{(\lambda)}$, there exists an element $e_+^{(\lambda,j)} \in \mathring{\mathfrak{n}}_+$ s.t. $e_+^{(\lambda,j)} b_i^{(\lambda)} = \kappa_{i,j}^{(\lambda)} b_\Lambda$ with $\kappa_{j,j}^{(\lambda)} = 1$ is equal to the highest weight vector, because b_Λ is up to prefactors the only nontrivial primitive (w.r.t. imρ) vector of $\mathcal{S}_{\frac{1}{2}}$. Furthermore, one has that $e_+^{(\lambda,j)} b_i^{(\mu,i)} = 0$ if $\lambda \neq \mu$ and $|\mu| \leq |\lambda|$, which is again a consequence of b_Λ's uniqueness. From lemma 5.6 one knows that there exists $y \in \mathcal{U}(\mathfrak{k})$ s.t.

$\sigma(y) = Id \otimes \rho\left(e_+^{(\lambda,k)}\right)$ and therefore

$$\begin{aligned}\sigma(y)u &= \sum_{i=1}^{n}\sum_{\lambda\in P}\sum_{j=1}^{m(\lambda)} d_i c_j^{(\lambda)} \alpha_i \otimes \rho\left(e_+^{(\lambda,k)}\right) b_j^{(\lambda)} = \sum_{i=1}^{n}\sum_{j=1}^{m(\lambda)} d_i c_j^{(\lambda)} \kappa_{j,k}^{(\lambda)} \alpha_i \otimes b_\Lambda \\ &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m(\lambda)} d_i c_j^{(\lambda)} \kappa_{j,k}^{(\lambda)} \alpha_i\right) \otimes b_\Lambda =: v \otimes b_\Lambda \in U.\end{aligned}$$

Note that one can find $e_+^{(\lambda,k)}$ s.t. $v \neq 0$ because otherwise there exists $e_\sim \in \mathcal{U}(\mathring{\mathfrak{n}}_+)$ s.t. $e_\sim w$ with $w = \sum_{j=1}^{m(\lambda)} c_j^{(\lambda)} b_j^{(\lambda)}$ is a nonzero primitive vector that is not proportional to b_Λ which is a contradiction. Hence, U contains an elementary tensor and by the argument given at the beginning of this proof this shows that U must be all of $\mathcal{S}_{\frac{3}{2}}$. □

Lemma 5.9. *Let A be a simply-laced indecomposable GCM. Then the generalized spin representations $\left(\sigma, \mathcal{S}_{\frac{n}{2}}\right)$ of $\mathfrak{k}(A)(\mathbb{R})$ for $n \in \{3,5,7\}$ admit a contravariant bilinear form, i.e., a nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ with respect to which the representation matrices $\sigma(X_i)$ of the Berman generators are skew-adjoint.*

Proof. Denote the module $\mathcal{S}_{\frac{n}{2}} = V \otimes S$, where V is some symmetric power of $\mathfrak{h}^*$ and S is the $\frac{1}{2}$-spin representation of $\mathfrak{k}(A)(\mathbb{K})$. Since A has no isolated nodes, the $\frac{1}{2}$-spin representation is compact according to prop. 3.2. Hence, the $\rho(X_i)$ are skew-adjoint w.r.t. an inner product $(\cdot|\cdot)_S$ on S. Now $\mathfrak{h}^*$ carries the invariant bilinear form $(\cdot|\cdot)$ which induces a nondegenerate bilinear form $(\cdot|\cdot)_V$ on V such that both the Weyl reflections s_α and projections are symmetric w.r.t. $(\cdot|\cdot)_V$. Define a bilinear form on $V \otimes S$ via their product:

$$\langle\cdot,\cdot\rangle := (\cdot|\cdot)_V \otimes (\cdot|\cdot)_S\,,\ \langle a\otimes s, b\otimes t\rangle = (a|b)_V \cdot (s|t)_S \ \ \forall\, a,b \in V,\ s,t \in S.$$

This bilinear form is nondegenerate and the $\sigma(X_i) = \tau(\alpha_i) \otimes 2\rho(X_i)$ are skew-adjoint because $\tau(\alpha_i)$ consists of (induced) Weyl reflections and projections (hence symmetric w.r.t. $(\cdot|\cdot)_V$), while the $\rho(X_i)$ are skew-adjoint w.r.t. $(\cdot|\cdot)_S$. □

5.2 The $\frac{5}{2}$-spin representation of $\mathfrak{k}(E_{10})$

The $\frac{5}{2}$-spin representation is not irreducible but contains an invariant submodule isomorphic to $\mathcal{S}_{\frac{1}{2}}$ such that its orthogonal complement is also invariant. This works for any indecomposable simply-laced A:

Proposition 5.10. *Let A be an indecomposable simply-laced GCM and let $\mathcal{S}_{\frac{5}{2}}$ be the representation from thm. 3.19. With respect to the bilinear form of lemma 5.9 $\mathcal{S}_{\frac{5}{2}}$ decomposes into a direct sum of invariant submodules as*

$$\mathcal{S}_{\frac{5}{2}} \cong \widetilde{\mathcal{S}}_{\frac{5}{2}} \oplus \mathcal{S}_{\frac{1}{2}}.$$

The module $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is called the "trace-free" part of $\mathcal{S}_{\frac{5}{2}}$. It is irreducible, if the $W(A)$-module Sym$^2(\mathfrak{h}^*)$ decomposes into two irreducible factors, where one of them is the trivial representation.

Remark. In the proof below it will become clear that the factor $\mathcal{S}_{\frac{1}{2}}$ above is due to a $W(A)$-invariant element $\Psi \in \mathrm{Sym}^2(\mathfrak{h}^*)$ that exists for arbitrary types A as it is constructed from the invariant bilinear form on $\mathfrak{h}^*$. The behavior of the remainder of $\mathrm{Sym}^2(\mathfrak{h}^*)$ however can not be predicted universally and can be irreducible or not; an example for the latter case is $A = A_{n-1}$. The Weyl group $W(A_{n-1})$ is isomorphic to the symmetric group on n letters $\mathfrak{S}_n$ and $\mathfrak{h}^*$ is isomorphic to its standard representation denoted by V as in [FH91, ch 4]. According to [FH91, ex. 4.19], $\mathrm{Sym}^2 V \cong U \oplus V \oplus V_{(n-2,2)}$ where U denotes the trivial representation and $V_{(n-2,2)}$ the irreducible representation associated to the partition $(n-2,2)$ of n. Hence, the remainder is not irreducible but consists of two irreducible submodules. For $W(A_{n-1})$ the above decomposition would be $\mathcal{S}_{\frac{5}{2}} \cong \hat{\mathcal{S}}_{\frac{5}{2}} \oplus \mathcal{S}_{\frac{3}{2}} \oplus \mathcal{S}_{\frac{1}{2}}$. For E_6, E_7 and E_8 the picture is different. According to [GP00, tbls. C.4-6], $W(E_n)$ for $n = 6,7,8$ admits an irreducible character of degree $\binom{n+1}{2} - 1$ that occurs[42] in $\mathrm{Sym}^2(V)$, where V denotes the standard representation of $W(E_n)$.

Proof. According to lemma 5.6, the action of $\mathfrak{k}(A)$ on $V \otimes S$ can be split into an action on V and S respectively, i.e. for all $w \in W(A)$ there exists $y_1 \in \mathcal{U}(\mathfrak{k})$ s.t. $\sigma(y_1) = \eta(w) \otimes Id$ and for all $x \in \mathfrak{k}$ there exists $y_2 \in \mathcal{U}(\mathfrak{k})$ s.t. $\sigma(y_2) = Id \otimes \rho(x)$. The action on $V = \mathrm{Sym}^2(\mathfrak{h}^*)$ is fully determined by the action of $W(A)$. The symmetric element

$$\Psi := \sum_{k,l} \omega^{kl} e_k \otimes e_l \tag{107}$$

with ω from eq. (65) is invariant under any $A \in End(V)$ that is induced by an orthogonal transformation $g \in O(\mathfrak{h}^*)$ (cp. [FH91, secs. 17.3 & 19.5]). As $W(A) < O(\mathfrak{h}^*)$ this immediately implies that Ψ is invariant under the action of $W(A)$. Also, $(\Psi|\Psi) = \dim \mathfrak{h}^*$ and as the image of $\rho_{\frac{1}{2}} : \mathfrak{k}(A) \to \mathrm{End}(S)$ is compact, the bilinear form on S can be chosen to be positive-definite. Hence, $\mathbb{K} \cdot \Psi \otimes S$ is anisotropic w.r.t. the form of lemma 5.9 and therefore its orthogonal complement is an invariant submodule, too. If $\Psi^\perp$ is irreducible w.r.t. the action of $W(A)$, then $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible by the same argument as in the proof of prop. 5.8. □

I will work out the example for $\mathfrak{k}(E_{10})$ and show explicitly that $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible as no general statements about the representation theory of indefinite Weyl groups seems to be known (and even the classical cases differ from case to case as apparent from the above remark).

Proposition 5.11. *Let $v_\alpha, v_\beta \in S \cong \mathbb{C}^{32}$ be highest weight vectors to the representations $\Gamma_\alpha, \Gamma_\beta$ of $\mathfrak{so}(10,\mathbb{C})$ and denote by Ψ the symmetric element in $Sym^2(\mathfrak{h}^*)$. Let $v = -\omega_{10} \in \mathfrak{h}^*$ be the vector perpendicular to $\{\alpha_1, \dots, \alpha_9\}$ and denote by $v_{32} := v \cdot v + \Psi$ the projection of $v \cdot v$ to $(span_{\mathbb{C}}\Psi)^\perp \subset Sym^2(\mathfrak{h}^*)$. Then $\Psi \otimes v_\alpha$, $\Psi \otimes v_\beta$ and $v_{32} \otimes v_\alpha$, $v_{32} \otimes v_\beta$ are highest weight vectors to the weights α, β respectively. Under all of $\mathfrak{k}(E_{10})(\mathbb{C})$, $(span_{\mathbb{C}}\Psi) \otimes S$ is the irreducible submodule isomorphic to $\mathcal{S}_{\frac{1}{2}}$ from prop. 5.10.*

Proof. One has

$$\tau(\alpha_i)\Psi = \left(\eta(s_{\alpha_i}) - \frac{1}{2}\right)\Psi = \frac{1}{2}\Psi \;\forall\, i = 1, \dots, 10,$$
$$\tau(\alpha_i) v_{32} = \left(\eta(s_{\alpha_i}) - \frac{1}{2}\right) v_{32} = \frac{1}{2} v_{32} \;\forall\, i = 1, \dots, 9.$$

Thus, for all $x \in \mathfrak{so}(10,\mathbb{C})$, the action of $\sigma(x)$ on $\Psi \otimes s$ and $v_{32} \otimes s$ reduces to that of $\rho(x)$:

$$\sigma(x)\Psi \otimes s = \Psi \otimes \rho(x) s,\;\; \sigma(x) v_{32} \otimes s = v_{32} \otimes \rho(x) s \;\forall\, s \in S,\; \forall\, x \in \mathfrak{so}(10,\mathbb{C}).$$

For Ψ, this holds for all Berman generators and therefore also under $\mathfrak{k}(E_{10})(\mathbb{K})$ which shows that $(\mathrm{span}_{\mathbb{C}}\Psi) \otimes S$ is irreducible. Towards the concrete form of the projection $v_{32} = v \cdot v + \Psi$ note that

$$(\Psi|\Psi) = \sum_{i,j,k,l} \omega^{ij}\omega^{lk} (e_i \otimes e_j | e_l \otimes e_k) = \sum_{i,j,k,l} \omega^{ij}\omega^{lk}\omega_{il}\omega_{jk} = \sum_{j,l} \delta^j{}_l \delta^l{}_j = \delta^j{}_j = 10\ .$$

[42] This can be seen from the value of b_χ in this table. For an irreducible character χ a value of $b_\chi = d$ means that $\mathrm{Sym}^d(V)$ is the smallest symmetric product of V that affords χ as an irreducible component.

Now $(u \cdot w|\Psi) = (u|w)\ \forall u, w \in \mathfrak{h}^*$ and $v = -\omega_{10}$ one has

$$(v \otimes v|\Psi) = (\omega_{10}|\omega_{10}) = -(v|\omega_{10}) = -10,$$

because $k_{10} = 10$ in $v = \sum_{i=1}^{10} k_i \alpha_i$ according to eq. (99).

With this and $v \cdot v = \frac{2}{2!} v \otimes v$ one finds that

$$v \otimes v - \frac{(v \otimes v|\Psi)}{(\Psi|\Psi)} \Psi = v \otimes v + \Psi$$

is perpendicular to Ψ. □

Lemma 5.12. *An $\mathfrak{h}_{D_5}$-diagonal basis of $\mathcal{S}_{\frac{5}{2}}$ is given by the set $\{b_i \cdot b_j \otimes s_\lambda \mid b_i, b_j \in \mathcal{B}\ \forall i \leq j, \lambda \in \Delta(\Gamma_\alpha \oplus \Gamma_\beta)\}$, where $\mathcal{B} = \{\alpha_1, \alpha_3, \dots, \alpha_9, t_1, \dots, t_4, t_5 \coloneqq v\}$ with $v = -\omega_{10}$ from (99) is the orthogonal basis of $\mathfrak{h}^*$ (cp. eq. 101) and $\{s_\lambda \mid \lambda \in \Delta(\Gamma_\alpha \oplus \Gamma_\beta)\}$ is an $\mathfrak{h}_{D_5}$-diagonal basis of S. The weights of $\mathcal{S}_{\frac{5}{2}}$ as a $\mathfrak{so}(10, \mathbb{C})$-module are*

$$\Delta_{\frac{5}{2}} \coloneqq \left\{ \frac{1}{2} \sum_{i=1}^{5} a_i L_i \mid \textit{up to two } a_i = \pm 3, \textit{ the others } \pm 1 \right\} . \tag{108}$$

Proof. By using the orthogonal basis $\mathcal{B} = \{\alpha_1, \alpha_3, \dots, \alpha_9, t_1, \dots, t_4, t_5\}$ of eq. (101), one immediately obtains such a basis for $\mathrm{Sym}^2 V$. One finds

$$\begin{aligned}
\sigma(H_j)(\alpha_i \cdot \alpha_k \otimes s_\lambda) &= [2(1 - 2\delta_{i,2j-1})(1 - 2\delta_{2j-1,k}) - 1]\, \lambda(H_j)\ \alpha_i \cdot \alpha_k \otimes s_\lambda \\
\sigma(H_j)(t_i \cdot \alpha_k \otimes s_\lambda) &= [2(1 - 2\delta_{2j-1,k}) - 1]\, \lambda(H_j)\ t_i \cdot \alpha_k \otimes s_\lambda \\
\sigma(H_j)(t_i \cdot t_k \otimes s_\lambda) &= \lambda(H_j)\ t_i \cdot t_k \otimes s_\lambda
\end{aligned}$$

which shows (108). □

According to the previous lemma the potential highest weights that can appear are $\alpha, \beta, \omega_1 + \alpha, \omega_1 + \beta, \omega_2 + \alpha$ and $\omega_2 + \beta$ and only the respective multiplicities are unknown. In order to describe the HWVs to the weights $\omega_1 + \alpha$ and $\omega_1 + \beta$ one needs to specify the orthogonal basis $\mathcal{B}$ further. So far, only $t_5 \coloneqq v = -\omega_{10}$ was set. It is possible to take $t_2, t_3, t_4 \in \mathrm{span}\{\alpha_3, \dots, \alpha_9\}$ so that t_1 and t_5 are the only elements of $\mathcal{B}$ that involve α_2 and α_{10} in their decomposition into simple roots. However, as $(t_5|t_1) = -(\omega_{10}|t_1) = 0$ must be satisfied it follows that $t_1 \in \mathrm{span}_{\mathbb{C}}\{\alpha_1, \dots, \alpha_9\}$.

Proposition 5.13. *In $\mathcal{S}_{\frac{5}{2}}$ restricted to $\mathfrak{so}(10, \mathbb{C})$ there is one highest weight vector each to the weights $\omega_2 + \alpha$ and $\omega_2 + \beta$. The multiplicity of the weights $\omega_1 + \alpha, \omega_1 + \beta$ in $\mathcal{S}_{\frac{5}{2}}$ is 5 each and to each of them there exist two highest weight vectors. The highest weight vectors are*

$$v^{(1)}_{\omega_1+\alpha} = \alpha_1 \cdot t_1 \otimes s_{\lambda_1},\ v^{(2)}_{\omega_1+\alpha} = \alpha_1 \cdot t_5 \otimes s_{\lambda_1},\ v^{(1)}_{\omega_1+\beta} = \alpha_1 \cdot t_1 \otimes s_{\lambda_2},\ v^{(2)}_{\omega_1+\beta} = \alpha_1 \cdot t_5 \otimes s_{\lambda_2},$$

$$v_{\omega_2+\alpha} = \alpha_1 \cdot \alpha_3 \otimes s_{\lambda_3},\ v_{\omega_2+\beta} = \alpha_1 \cdot \alpha_3 \otimes s_{\lambda_4},$$

with $\lambda_1 = \alpha - \omega_1 \in \Delta(\Gamma_\beta)$, $\lambda_2 = \beta - \omega_1 \in \Delta(\Gamma_\alpha)$, $\lambda_3 = \alpha - \omega_2 \in \Delta(\Gamma_\alpha)$, $\lambda_4 = \beta - \omega_2 \in \Delta(\Gamma_\beta)$ and $0 \neq s_{\lambda_i}$ vectors in the corresponding weight space. The vectors t_1, t_5 are elements of the orthogonal basis $\mathcal{B}$ of $\mathfrak{h}^*$ described in eq. (101) such that $t_2, t_3, t_4 \in \mathit{span}\{\alpha_3, \dots, \alpha_9\}$, $t_5 = -\omega_{10}$ and $t_1 \in \mathit{span}\{\alpha_1, \dots, \alpha_9\}$.

Proof. Expressing the weights $\omega_2+\alpha$ and $\omega_2+\beta$ in terms of $L_1,\ldots,L_5$ one has

$$\begin{aligned}\omega_2+\alpha &= L_1+L_2+\frac{1}{2}(L_1+\cdots+L_5)=\frac{1}{2}\begin{pmatrix}3&3&1&1&1\end{pmatrix}\\ \omega_2+\beta &= L_1+L_2+\frac{1}{2}(L_1+\cdots-L_5)=\frac{1}{2}\begin{pmatrix}3&3&1&1&-1\end{pmatrix}\end{aligned}$$

and one therefore needs a vector $r\otimes s_\lambda$ that satisfies

$$\sigma(H_j)(r\otimes s_\lambda)=\begin{cases}-3\lambda(H_j)\,r\otimes s_\lambda & j=1,\,2\\ \lambda(H_j)\,r\otimes s_\lambda & j=3,\,4,\,5\end{cases}$$

to obtain these weights. A vector that has the above properties is $r=\alpha_1\cdot\alpha_3$ and the appropriate weights λ_3 and λ_4 are given by

$$\Delta(\Gamma_\alpha)\ \ni\ \lambda_3=\alpha-\omega_2=\frac{1}{2}\begin{pmatrix}-1&-1&1&1&1\end{pmatrix} \tag{109}$$

$$\Delta(\Gamma_\beta)\ \ni\ \lambda_4=\beta-\omega_2=\frac{1}{2}\begin{pmatrix}-1&-1&1&1&-1\end{pmatrix}\ . \tag{110}$$

From lemma 5.12 one has that there are no higher weights than $\omega_2+\alpha$ and $\omega_2+\beta$ in $\mathcal{S}_{\frac{5}{2}}$ and since neither is contained in the weight system of the other both must be highest weights. Recall the weights

$$\begin{aligned}\lambda_1 &= \alpha-L_1=\alpha-\omega_1\in\Delta(\Gamma_\beta)\\ \lambda_2 &= \beta-L_1=\beta-\omega_1\in\Delta(\Gamma_\alpha)\end{aligned}$$

from prop. 5.4, then for $i=1,\ldots,5$ the vectors $\alpha_1\cdot t_i\otimes s_{\lambda_1}$ and $\alpha_1\cdot t_i\otimes s_{\lambda_2}$ are weight vectors to the weights $\omega_1+\alpha$ and $\omega_1+\beta$ respectively. Not all of them are highest weight vectors to the weights $\omega_1+\alpha$ or $\omega_1+\beta$ however, as[43]

$$\operatorname{mult}(\omega_1+\alpha;\omega_2+\beta)=3,\ \operatorname{mult}(\omega_1+\beta;\omega_2+\alpha)=3.$$

One can also see this from an argument concerning the dimension of $\mathcal{S}_{\frac{5}{2}}$: $\mathfrak{h}^*$ has dimension 10 and S has dimension 32. Hence, $\mathcal{S}_{\frac{5}{2}}=\mathrm{Sym}^2V\otimes S$ has dimension $55\cdot 32=1760$. The sum of highest weight modules $\Gamma_{\omega_2+\alpha}\oplus\Gamma_{\omega_2+\beta}$ is of dimension $2\cdot 560=1120$. Since $1760-1120=640$ and $\dim(\Gamma_{\omega_1+\alpha}\oplus\Gamma_{\omega_1+\beta})=288$ there can be at most two vectors each to the weights $\omega_1+\alpha$ and $\omega_1+\beta$ which are also highest weight vectors. Observe that

$$\omega_1+\alpha=\omega_2+\beta-(L_2-L_5)=\omega_2+\beta-(\gamma_2+\gamma_3+\gamma_4)\,,\ \omega_1+\beta=\omega_2+\alpha-(\gamma_2+\gamma_3+\gamma_5)$$

and that the HWVs $v_{\omega_2+\alpha},v_{\omega_2+\beta}$ are of the shape $(\alpha_1\cdot\alpha_3)\otimes s_\lambda$ for suitable λ. The t_2,t_3,t_4 were chosen such that they are contained in $\mathrm{span}_{\mathbb{C}}\{\alpha_3,\alpha_4,\ldots,\alpha_9\}$. Now the $e_{-\gamma_j}$ involve commutators of the Berman generators X_{2j-1},X_{2j},X_{2j+1} (see eq. 22) and so the action of $\sigma\left(e_{-\gamma_j}\right)$ for[44] $j=2,3,4$ involves several of the reflections $\tau(\alpha_{2j-1}+\alpha_{2j})$, $\tau(\alpha_{2j})$, $\tau(\alpha_{2j-1}+\alpha_{2j}+\alpha_{2j+1})$, $\tau(\alpha_{2j}+\alpha_{2j+1})$ on $\alpha_1\cdot\alpha_3\in\mathrm{Sym}^2V$. The action on α_1 is always trivial, So both t_1 and $t_5:=v$ cannot be obtained by descent from $(\alpha_1\cdot\alpha_3)\otimes s_\lambda$ because they contain $\alpha_1\cdot\alpha_2$ and $\alpha_1\cdot\alpha_{10}$ in a decomposition w.r.t. the basis $\{\alpha_i\alpha_j|1\le i\le j\le 10\}$, which makes $v^{(1)}_{\omega_1+\alpha}$, $v^{(2)}_{\omega_1+\alpha}$, $v^{(1)}_{\omega_1+\beta}$ and $v^{(2)}_{\omega_1+\beta}$ highest weight vectors. □

[43] A direct computation of this would be possible by application of the Weyl character formula. Alternatively, one can use Software solutions such as Sagemath [SAGE], and there in particular the routines connected to the WeylCharacterRing.

[44] Note that in the case of studying $V_{\omega_1+\alpha}\subset\Gamma_{\omega_2+\beta}$ one has $j=2,3,5$ but $e_{-\gamma_5}$ uses the same commutators as $e_{-\gamma_4}$ and so the argument holds here as well.

Theorem 5.14. *(cp. [KN13, sec. 5.1]) The $\mathfrak{k}(E_{10})(\mathbb{C})$-module $\mathcal{S}_{\frac{5}{2}}$ splits upon restriction to its $\mathfrak{so}(10,\mathbb{C})$-subalgebra as*

$$\mathcal{S}_{\frac{5}{2}} \cong_{\mathfrak{so}(10,\mathbb{C})} \Gamma_{\omega_2+\alpha} \oplus \Gamma_{\omega_2+\beta} \oplus 2 \times (\Gamma_{\omega_1+\alpha} \oplus \Gamma_{\omega_1+\beta}) \oplus 2 \times (\Gamma_\alpha \oplus \Gamma_\beta) \tag{111}$$

with highest weight vectors

$$\begin{aligned} (\alpha_1 \cdot \alpha_3) \otimes s_{\lambda_3} \ , \ (\alpha_1 \cdot \alpha_3) \otimes s_{\lambda_4} \quad &\leftrightarrow \quad \Gamma_{\omega_2+\alpha} \oplus \Gamma_{\omega_2+\beta} \\ (\alpha_1 \cdot t_j) \otimes s_{\lambda_2} \ , \ (\alpha_1 \cdot t_j) \otimes s_{\lambda_1} \ , j = 1,5 \quad &\leftrightarrow \quad \Gamma_{\omega_1+\alpha} \oplus \Gamma_{\omega_1+\beta} \\ v_{32} \otimes s_\alpha \ , \ v_{32} \otimes s_\beta \quad &\leftrightarrow \quad \Gamma_\alpha \oplus \Gamma_\beta \\ \Psi \otimes s_\alpha \ , \ \Psi \otimes s_\beta \quad &\leftrightarrow \quad \Gamma_\alpha \oplus \Gamma_\beta \end{aligned}$$

where $\lambda_1 = \alpha - \omega_1$, $\lambda_2 = \beta - \omega_1$, $\lambda_3 = \alpha - \omega_2$, $\lambda_4 = \beta - \omega_2$, Ψ is the symmetric element in $Sym^2(\mathfrak{h}^*)$ (cp. 107), v_{32} from prop. 5.11 and t_1, t_5 as in prop. 5.13.
As a $\mathfrak{k}(E_{10})$-module $\mathcal{S}_{\frac{5}{2}}$ is not irreducible, it splits as

$$\mathcal{S}_{\frac{5}{2}} \cong \widetilde{\mathcal{S}}_{\frac{5}{2}} \oplus \mathcal{S}_{\frac{1}{2}},$$

where $\mathcal{S}_{\frac{1}{2}} = \{\Psi \otimes s \,|\, s \in S\}$ and orthogonality is defined w.r.t. the contravariant form on $\mathcal{S}_{\frac{5}{2}}$ described in lemma 5.9. The trace-free part $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible.

Proof. Adding up the dimensions of the highest weight modules provided in propositions 5.11 and 5.13 yields

$$\begin{aligned} & \dim(\Gamma_{\omega_2+\alpha} \oplus \Gamma_{\omega_2+\beta}) + 2 \cdot \dim(\Gamma_{\omega_1+\alpha} \oplus \Gamma_{\omega_1+\beta}) + 2 \cdot \dim(\Gamma_\alpha \oplus \Gamma_\beta) \\ = \ & 1120 + 2 \cdot 288 + 2 \cdot 32 \\ = \ & 1120 + 640 = 1760 \\ = \ & \dim \mathcal{S}_{\frac{5}{2}} \ . \end{aligned}$$

Hence, these $\mathfrak{so}(10,\mathbb{C})$-modules exhaust $\mathcal{S}_{\frac{5}{2}}$. The split of $\mathcal{S}_{\frac{5}{2}}$ into a copy of $\mathcal{S}_{\frac{1}{2}}$ given by $\{\Psi \otimes s \,|\, s \in S\}$ and an invariant complement was shown in prop. 5.13. In order to show irreducibility of $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ I will pursue the strategy of showing that the $\mathfrak{so}(10,\mathbb{C})$-modules are mixed under the full $\mathfrak{k}(E_{10})$-action. If $\langle \cdot, \cdot \rangle$ denotes the $\mathfrak{so}(10,\mathbb{C})$-contravariant bilinear form on $\mathcal{S}_{\frac{5}{2}}$ w.r.t. which the decomposition in (111) is orthogonal, then a $\mathfrak{so}(10,\mathbb{C})$-module U_1 mixes with U_2 under the action of $\mathfrak{k}(E_{10})(\mathbb{C})$ if there exists $u_1 \in \mathcal{U}(\mathfrak{k}(E_{10})(\mathbb{C}))U_1$ and $u_2 \in U_2$ s.t. $\langle u_1, u_2 \rangle \neq 0$. Since $\langle \cdot, \cdot \rangle$ is not necessarily contravariant w.r.t. $\mathfrak{k}(E_{10})(\mathbb{C})$ this does not need to be a symmetric relation. However, I will show that this question can be reduced to a question of orthogonality in $\mathrm{Sym}^2(\mathfrak{h}^*)$ w.r.t. the induced invariant bilinear form $(\cdot|\cdot)$.

According to lemma 5.6, there exists $y_1 \in \mathcal{U}(\mathfrak{k})$ s.t. $\sigma(y_1) = \eta(w) \otimes Id$ for all $w \in W(A)$ and there exists $y_2 \in \mathcal{U}(\mathfrak{k})$ s.t. $\sigma(y_2) = Id \otimes \rho(x)$ for all $x \in \mathfrak{k}$. Since S is an irreducible $\mathrm{im}(\rho)$-module and $\mathrm{im}(\rho)$ is simple for E_{10}, one can figure out if modules are mixed by just looking at the $\mathrm{Sym}^2(\mathfrak{h}^*)$-component, because the highest weight vectors of all $\mathfrak{so}(10,\mathbb{C})$-modules are elementary tensors. The highest weight vectors have the $\mathrm{Sym}^2(\mathfrak{h}^*)$-factors Ψ, $v_{32} = t_5 \cdot t_5 + \Psi$, $\alpha_1 \cdot t_1$, $\alpha_1 \cdot t_5$ and $\alpha_1 \cdot \alpha_3$. One observes that all these vectors are perpendicular w.r.t. the standard invariant form $(\cdot|\cdot)$ on $\mathrm{Sym}^2(\mathfrak{h}^*)$. As a consequence, the $\mathfrak{so}(10,\mathbb{C})$-modules are orthogonal w.r.t. the contravariant bilinear form described in lemma 5.9. This form restricted to any $\mathfrak{so}(10,\mathbb{C})$-module is therefore proportional to the $\mathfrak{so}(10,\mathbb{C})$-contravariant form as this is unique up to constant multiples. For some this multiple should be negative because $(\cdot|\cdot)$ has mixed signature and indeed this is the case for $\alpha_1 \cdot t_5$. Hence, two $\mathfrak{so}(10,\mathbb{C})$-modules U_1 and U_2 are mixed under the action of $\mathfrak{k}(E_{10})(\mathbb{C})$ if there exist $w_1, w_2 \in W(E_{10})$ s.t. $(w_1 u_1 | w_2 u_2) \neq 0$, where u_1 and u_2 denote the $\mathrm{Sym}^2(\mathfrak{h}^*)$-factors of the

HWVs of U_1 and U_2 . Any mixing that is found this way is symmetric because $(\cdot|\cdot)$ is $W(E_{10})$-invariant. Now $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible if any $\mathfrak{so}(10,\mathbb{C})$-module in $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ mixes with every other $\mathfrak{so}(10,\mathbb{C})$-module in $\widetilde{\mathcal{S}}_{\frac{5}{2}}$. Note that "mixing" is a transitive relation, because the action of $\mathcal{U}(\mathfrak{so}(10,\mathbb{C}))$ on an irreducible $\mathfrak{so}(10,\mathbb{C})$-module is transitive. The rest is a case-by-case computation.

First, show that v_{32} and $\alpha_1 t_5$ mix (recall $t_5 = -\omega_{10}$). One has $s_{10}v_{32} = (\omega_{10}-\alpha_{10})^2 + \Psi$ and also that there exists $w \in W(A_9) < W(E_{10})$ s.t. $w(\alpha_1\omega_{10}) = \alpha_7\omega_{10}$. With $(\alpha_7\omega_{10}|\Psi) = (\alpha_7|\omega_{10}) = \omega_{10}(\alpha_7^\vee) = 0$ and $(\omega_{10}|\omega_{10}) = -10$ one computes

$$\begin{aligned}
\left(\alpha_7\omega_{10}|\,(\omega_{10}-\alpha_{10})^2+\Psi\right) &= (\alpha_7\omega_{10}|\omega_{10}^2 - 2\omega_{10}\alpha_{10} + \alpha_{10}^2) \\
&= 0 - \frac{2}{2!}\left[(\alpha_7|\omega_{10})(\omega_{10}|\alpha_{10}) + (\alpha_7|\alpha_{10})(\omega_{10}|\omega_{10})\right] + (\alpha_7|\alpha_{10})(\omega_{10}|\alpha_{10}) \\
&= -10-1 = -11
\end{aligned}$$

and concludes that v_{32} and $\alpha_1 t_5$ mix in both directions.

Next, show that $\alpha_1\omega_{10}$ and $\alpha_1\alpha_3$ mix. There exists $w \in W(A_9)$ s.t. $w(\alpha_1\alpha_3) = \alpha_1\alpha_7$ and then $s_{10}(\alpha_1\alpha_7) = \alpha_1\cdot(\alpha_7+\alpha_{10})$ is not perpendicular to $\alpha_1\omega_{10}$. One concludes that $v_{32}, \alpha_1 t_5$ and $\alpha_1\alpha_3$ all mix among each other.

The last case is to show that $\alpha_1 t_1$ mixes with any of the others symmetrically. As $t_1 \perp \{\alpha_1,\alpha_3,\dots,\alpha_9\}$ there needs to exist $i = 1,\dots,5$ s.t. $(t_1|\alpha_{2i}) \neq 0$ and for convenience set $(t_1|\alpha_{2i}) = -c$ with $c \neq 0$. Continue by case distinction and start with $i=1$, then

$$\begin{aligned}
s_2(\alpha_1 t_1) &= (\alpha_1+\alpha_2)(t_1+c\alpha_2) = \alpha_1 t_1 + c\alpha_1\alpha_2 + \alpha_2 t_1 + c\alpha_2\alpha_2 \\
(s_2(\alpha_1 t_1)\,|\alpha_1\alpha_3) &= (\alpha_1 t_1 + c\alpha_1\alpha_2 + \alpha_2 t_1 + c\alpha_2\alpha_2\,|\,\alpha_1\alpha_3) \\
&= 0 + \frac{c}{2}(-2+1) + 0 + c = \frac{c}{2} \neq 0.
\end{aligned}$$

In the other case $i \geq 2$ one has that there exists $w \in W(A_9)$ s.t. $w(\alpha_1\alpha_3) = \alpha_1\alpha_{2i}$ and therefore

$$\begin{aligned}
(\alpha_1 t_1|w(\alpha_1\alpha_3)) &= (\alpha_1 t_1\,|\,\alpha_1\alpha_{2i}) \\
&= \frac{1}{2}(-2c+0) = -c \neq 0
\end{aligned}$$

shows that $\alpha_1 t_1$ and $\alpha_1\alpha_3$ mix symmetrically. □

6 Tensor products of higher spin representations

This section is dedicated to the study of tensor products of some of the higher spin representations described in section 3. The first subsection is dedicated to higher spin representations of $\mathfrak{k}(E_{10})$ in a computer-based approach, where the main goal is to deduce the decomposition into $\mathfrak{so}(10,\mathbb{C})$-modules and determine if the tensor products are (ir-)reducible and in case of reducibility if they are completely reducible. Apart from the reproduction of some of the results of section 5 the main result is that $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}} \otimes \bigwedge^2\left(\mathcal{S}_{\frac{1}{2}}\right)$ are irreducible $\mathfrak{k}(E_{10})$-modules. In section 6.2 I will approach this intriguing result from a more general perspective as I will show in proposition 6.7 for arbitrary indecomposable simply-laced types that the tensor products $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\widetilde{\mathcal{S}}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ are irreducible if each of the factors is.

6.1 Computational point of view

In this section I will lay out the algorithms and findings which underlie my computer-based analysis of tensor products of generalized spin representations of $\mathfrak{k}(E_{10})$. My analysis has been conducted using Sage 9.0 ([SAGE]). The results are obtained for the complexification $\mathfrak{k}(E_{10})(\mathbb{C})$ of $\mathfrak{k}(E_{10})(\mathbb{R})$ but results concerning irreducibility also hold for $\mathfrak{k}(E_{10})(\mathbb{R})$. As some of the computations can be performed on a regular computer but others require the use of a computer cluster, the details of which result was obtained how and in particular how these results can be reproduced[45] is treated in full detail in appendix B. I will start by describing the algorithm that is used to obtain the $\mathfrak{so}(10,\mathbb{C})$-decomposition of the $\mathfrak{k}(E_{10})(\mathbb{C})$-modules and how one can use it to test for irreducibility. Afterwards I will discuss each of my findings for the analytically known test cases $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$ as well as the tensor products $\bigwedge^2\left(\mathcal{S}_{\frac{1}{2}}\right)$, $\mathrm{Sym}^2\left(\mathcal{S}_{\frac{1}{2}}\right)$, $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}}\otimes\left(\bigwedge^2\mathcal{S}_{\frac{1}{2}}\right)$. The last two are the the first nontrivial examples of lowest dimension and therefore their analysis is still feasible with computer-based methods.

6.1.1 The algorithm

The results of section 3 provide the representation matrices for the Berman generators of $\mathfrak{k}(E_{10})$. This involves fixing the matrices for the generalized spin representation $\mathcal{S}_{\frac{1}{2}}$ from theorem 3.3 as well as providing the (induced) Weyl reflections on $\mathfrak{h}^*$ and $\mathrm{Sym}^2\mathfrak{h}^*$. This provides the Berman generators' representation matrices via the tensor product. Together with eqs. (15) and (22) the Weyl canonical form of $\mathfrak{so}(10,\mathbb{C})$ can be computed, which is a particular choice of root space basis (cp. [C84, app. G]). If one chooses $\sigma(X_1),\ldots,\sigma(X_9)$ to be skew-hermitian, then the standard inner product of $\mathbb{C}^n$ defines a contravariant form w.r.t. $\mathfrak{so}(10,\mathbb{C})$ which is used to define orthogonality (details can be found in sec. B.1).

I used two approaches to obtain $\mathfrak{so}(10,\mathbb{C})$-decompositions of the given representations. One of them scales rather poorly but still provides an interesting example of how one can end up with a decomposition s.t. the representation matrices of $\mathfrak{so}(10,\mathbb{C})$ are not block-diagonal despite the representation being completely reducible.

The first approach is to determine the $\mathfrak{so}(10,\mathbb{C})$-highest weight vectors rather straightforwardly in two steps. First, determine the vector space of primitive vectors, i.e., the intersection of the kernels of $E_1,\ldots,E_5$. Afterwards one determines a basis of weight vectors for this vector space which will automatically yield highest weight vectors. I used this approach for preliminary computations with $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{1}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$, the only occurrence of this method in the final version is $\mathcal{S}_{\frac{5}{2}}$ since the major issue with this approach is that computation of the kernels becomes quite expensive. Hence, for most of the time I used the following approach which I will refer to as the orbit-method (which has nothing to do with Kirillov's orbit method).

Start with a reasonably unique vector of the module. My choice was to pick the highest weight vector of an $\mathfrak{so}(10,\mathbb{C})$-module that occurs only with multiplicity one in the decomposition and therefore is unique up to scalar multiples. From this highest weight vector one constructs an explicit basis of the corresponding $\mathfrak{so}(10,\mathbb{C})$-module which I denote by V_1. Now one can apply X_{10} to each basis vector of V_1 and the result will never[46] be in V_1 unless it is trivial. For $v_\lambda\in V_1$ pick the part of $X_{10}v_\lambda$ that is orthogonal to V_1, denoted by $(X_{10}v_\lambda)_{\perp,1}$. Given some vector v successively apply E_is for $i\in\{1,\ldots,5\}$ until one reaches a vector w with the property that $E_iw=0$ for all $i=1,\ldots,5$. In all performed computations these primitive vectors were

[45] All the Sagemath-notebooks and scripts are available online here: http://dx.doi.org/10.22029/jlupub-533.

[46] This is a consequence of the weight system of f.d. irreducible $\mathfrak{so}(10,\mathbb{C})$-modules. If λ is a weight then $\lambda\pm L_2$ is not because any weight of an highest weight module is of the form $\Lambda-\sum_{i=1}^5 k_i\gamma_i$, where $k_i\in\mathbb{N}_0$ and γ_i are the simple D_5-roots. Since $\pm L_2$ is not part of the D_5-root lattice, $\lambda\pm L_2$ cannot be a weight if λ is as long as the module is f.d. and irreducible because then it is a highest weight module. But now $X_{10}=-\frac{i}{2}(X_++X_-)$ and $X_\pm$ are $\mathfrak{h}_{D_5}$-diagonal with weight $\pm L_2$. Hence $X_{10}v_\lambda$ cannot lie inside V_1.

also highest weight vectors even though exceptions can in principle occur (see sec. B.2 for details). To this new highest weight vector one now constructs the corresponding $\mathfrak{so}(10,\mathbb{C})$-module V_2. Denote the projection to V_i by π_i, and write

$$(X_{10}v_\lambda)_{\perp,n} = X_{10}v_\lambda - \sum_{i=1}^{n} \pi_i (X_{10}v_\lambda),$$

where n is the number of $\mathfrak{so}(10,\mathbb{C})$-modules that one has already found[47]. Now go back to $X_{10}v_\lambda$ and pick the part orthogonal to $V_1 \oplus V_2$, i.e. $(X_{10}v_\lambda)_{\perp,2}$, and repeat the procedure until the orthogonal piece is zero. Note that this may take several iterations. Consider the case where $(X_{10}v_\lambda)_{\perp,1}$ can be decomposed into the sum $(X_{10}v_\lambda)_{\perp,1} = v_\mu^{(2)} + v_\mu^{(3)} + v_\mu^{(4)}$ where $v_\mu^{(i)} \in V_i$ such that μ is of depth $k_2 > k_3 > k_4$ w.r.t. the highest weights $\Lambda_2, \Lambda_3, \Lambda_4$ of V_2, V_3, V_4 respectively. Then the first iteration will[48] yield $v_\mu^{(2)}$ as highest weight vector because the others are annihilated after at most k_3, k_4 steps up. In the second iteration $(X_{10}v_\lambda)_{\perp,2}$ will only consist of $v_\mu^{(3)} + v_\mu^{(4)}$ because $v_\mu^{(2)}$ can be expressed in terms of the basis of the previously added module V_2.

If one has computed the n modules $V_1, \dots, V_n$ s.t. $(X_{10}v_\lambda)_{\perp,n} = 0$ one proceeds with the next basis vector of V_1. Once all elements of V_1 have been checked this way, one continues with V_2 and so on.

This way one obtains the $\mathfrak{k}(E_{10})$-orbit of the first $\mathfrak{so}(10,\mathbb{C})$-module $V_1 = L(\Lambda_1)$. The orbit now requires additional analysis because one cannot a priori conclude that such an orbit is an irreducible submodule. This is because it may be reducible but not completely reducible. For this one needs to compute if an $\mathfrak{so}(10)$-module $L(\Lambda_i)$ can be reached from $L(\Lambda_j)$ via some intermediate modules $L(\Lambda_{i_1}), \dots, L(\Lambda_{i_k})$. Towards this one computes an adjacency matrix A_{adj} that encodes which modules are connected by a single application of X_{10}. Irreducibility of the $\mathfrak{k}(E_{10})$-orbit is then equivalent to the question if any two points in the directed graph described by this adjacency matrix A_{adj} are connected by a directed path. This is equivalent to saying that the directed graph defined by A_{adj} is strongly connected. If the orbits of a module $L(\Lambda_i)$ do not exhaust the entire $\mathfrak{k}(E_{10})$-module one has to find a vector that is not contained in the orbit. There are several ways to do this, one being the construction of a random vector. But one can also use some analytical insight into the module for this as I will do in the analysis of $\mathcal{S}_{\frac{5}{2}}$.

6.1.2 Decomposition and orbits of $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$

These are the first test cases, as the results are known analytically (cp. prop. 5.5 and thm. 5.14). For $\mathcal{S}_{\frac{3}{2}}$ one implements the weight vector $s_\lambda \in \mathcal{S}_{\frac{1}{2}}$ of the weight $-\frac{1}{2}L_1 + \frac{1}{2}(L_1 + L_2 + L_3 + L_4)$ manually and then builds the tensor product $w_1 \coloneqq \alpha_1 \otimes s_\lambda$, where $\alpha_1 \in \mathfrak{h}^*$ denotes the first simple root of E_{10}. The vector w_1 is a highest weight vector of weight $\frac{3}{2}L_1 + \frac{1}{2}(L_1 + L_2 + L_3 + L_4) = \omega_1 + \alpha$ as is also shown in prop. 5.5. Now the $\mathfrak{k}(E_{10})$-orbit of its highest weight module is computed according to the algorithm described in the previous section. The orbit includes the highest weight modules (in this order) to the following weights:

$$\Lambda_1 = \omega_1 + \alpha\ ,\ \Lambda_2 = \omega_1 + \beta\ ,\ \Lambda_3 = \alpha\,,\ \Lambda_4 = \beta.$$

[47] One would generally expect these modules to be orthogonal because one started with a vector that is orthogonal. I explicitly show this in lemma B.2 but it is simply a consequence of properties of contravariant forms.

[48] To be fully precise: There exists a choice of successively applying E_is to $(X_{10}v_\lambda)_{\perp,1}$ s.t. this claim is true. It can always happen that $E_1 v_\mu^{(4)} = 0$ but $E_1 v_\mu^{(3)} \neq 0$ so that if the algorithm starts with application of E_1 the result will be different.

The adjacency matrix of X_{10} displays which modules $L(\Lambda_i)$ are mixed under X_{10}:

$$A_{adj}(X_{10}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The graph corresponding to $A_{adj}(X_{10})$ is shown to be strongly connected which shows that $\mathcal{S}_{\frac{3}{2}}$ is irreducible as a $\mathfrak{k}(E_{10})(\mathbb{C})$-module, because the orbit contains the entire module. Hence, it is so as a $\mathfrak{k}(E_{10})(\mathbb{R})$-module[49], although there one might not have the same $\mathfrak{so}(10)$-irreducible pieces in the decomposition (for instance $\Gamma_{\omega_1+\alpha} \oplus \Gamma_{\omega_1+\beta}$ may be one irreducible piece if one restricts to $\mathbb{R}$-linear combinations of $X_1, \dots, X_9$). Note that the adjacency matrix is symmetric, i.e., if a module $L(\Lambda_i)$ is mapped to $L(\Lambda_j)$ under X_{10} the converse is also true.

For $\mathcal{S}_{\frac{5}{2}}$ I begin with a different approach and compute the $\mathfrak{so}(10,\mathbb{C})$-highest weight vectors directly from the intersection of the kernels of $E_1, \dots, E_5$ which yields the subspace of primitive vectors. Afterwards, diagonalize a random linear combination $H := \sum_{i=1}^5 \lambda_i H_i$ restricted to the subspace of primitive vectors. This yields the following $\mathfrak{so}(10,\mathbb{C})$-decomposition of $\mathcal{S}_{\frac{5}{2}}$:

$$\Lambda_0 = \beta \ , \ \Lambda_1 = \beta \ , \ \Lambda_2 = \omega_1 + \beta \ , \ \Lambda_3 = \omega_1 + \beta$$

$$\Lambda_4 = \alpha \ , \ \Lambda_5 = \alpha \ , \ \Lambda_6 = \omega_1 + \alpha \ , \ \Lambda_7 = \omega_1 + \alpha$$

$$\Lambda_8 = \omega_2 + \beta \ , \ \Lambda_9 = \omega_2 + \alpha$$

Checking which $L(\Lambda_i)$ are mapped to which $L(\Lambda_j)$ yields the adjacency matrix[50]:

$$A_{adj} = \begin{pmatrix} & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \end{pmatrix}$$

One can already see that it is not symmetric and that something interesting is going on in lines 2 and 6 which correspond to the weight $\Lambda_1 = \beta$ and $\Lambda_5 = \alpha$. Via $\mathcal{U}(\mathfrak{k}(E_{10})(\mathbb{C}))$ one can reach every module $L(\Lambda_j)$ if one starts in module $L(\Lambda_i)$ for $i \in \{0, \dots, 9\} \setminus \{1, 5\}$. For $L(\Lambda_1)$ the only other module that can be reached is $L(\Lambda_5)$ and vice versa. Hence the module is reducible under $\mathfrak{k}(E_{10})(\mathbb{C})$ but is it also completely reducible?

If one looks at the $\mathfrak{so}(10,\mathbb{C})$-decomposition one realizes that the highest weights α and β both occur with multiplicity 2 inside the decomposition. The question therefore is if there exists a linear combination of the highest weight vectors $v_\alpha^{(1)}, v_\alpha^{(2)}$ and $v_\beta^{(1)}, v_\beta^{(2)}$ such that the representation matrices become block diagonal. One could of course determine this by solving a linear system of equations but I would like to display how the other approach via $\mathfrak{k}(E_{10})$-orbits avoids this ambiguity of mixing isomorphic representations to begin with.

[49] If U is any nontrivial invariant submodule w.r.t. the action of $\mathfrak{k}(E_{10})(\mathbb{C})$ then it is also invariant w.r.t. the action of $\mathfrak{k}(E_{10})(\mathbb{R})$ as it is a real form of $\mathfrak{k}(E_{10})(\mathbb{C})$.

[50] I left out the diagonal because it will always be empty. This is because if λ is a weight of $L(\Lambda)$ then both $\lambda \pm L_2$ are not.

Application of the orbit method to the module $L(\Lambda_9)$ with highest weight $\omega_2+\alpha$, which is of multiplicity 1 inside the decomposition, determines the orbit, denoted by $\widetilde{\mathcal{S}}_{\frac{5}{2}}$, to contain the following modules

$$\Lambda_0^{(1)} = \omega_2+\alpha\ ,\ \Lambda_1^{(1)} = \omega_2+\beta\ ,\ \Lambda_2^{(1)} = \omega_1+\alpha\ ,\ \Lambda_3^{(1)} = \omega_1+\alpha$$

$$\Lambda_4^{(1)} = \beta\ ,\ \Lambda_5^{(1)} = \omega_1+\beta\ ,\ \Lambda_6^{(1)} = \omega_1+\beta\ ,\ \Lambda_7^{(1)} = \alpha.$$

The adjacency matrix to this orbit has the shape

$$A_{adj}^{(1)} = \begin{pmatrix} & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & \end{pmatrix}$$

which is symmetric. Furthermore, the corresponding graph is connected so that this orbit is irreducible. This shows that $\mathcal{S}_{\frac{5}{2}}$ splits into the above orbit $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ and $\Gamma_\alpha \oplus \Gamma_\beta$ such that $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible. By the analytical study from thm. 5.14 one knows that there exists a copy of $\mathcal{S}_{\frac{1}{2}} \cong \Gamma_\alpha \oplus \Gamma_\beta$ that is invariant under the full $\mathfrak{k}(E_{10})$. As $\Gamma_\alpha \oplus \Gamma_\beta = L\left(\Lambda_7^{(1)}\right) \oplus L\left(\Lambda_4^{(1)}\right) \subset \widetilde{\mathcal{S}}_{\frac{5}{2}}$ are not invariant this shows that the remainder has to be isomorphic to $\mathcal{S}_{\frac{1}{2}}$. Thus, the split is given as a split of $\mathfrak{k}(E_{10})$-invariant modules.

The above analysis shows that it is important how one sets up the $\mathfrak{so}(10)$-modules which appear multiple times as this may result in a decomposition that is not block-diagonal. If one computes a weight space basis of the space of primitive vectors one will almost always be in the situation, where the action is reducible but not in block-diagonal form. This is because no choice of basis for the highest weight vectors to the same weight is preferred in this case in contrast to the other approach. Starting with a $\mathfrak{so}(10)$-module that occurs only once yields a unique $\mathfrak{so}(10)$-decomposition of the $\mathfrak{k}(E_{10})$-orbit of that module and therefore a reducible structure is not the outcome of an unlucky choice of highest weight vectors but inherent to the module's $\mathfrak{k}(E_{10})$-structure.

6.1.3 Decomposition and orbits of $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$, $\bigwedge^2 \mathcal{S}_{\frac{1}{2}}$ and Sym$^2\left(\mathcal{S}_{\frac{1}{2}}\right)$

In this subsection I will decompose the $\mathfrak{k}(E_{10})$-invariant submodules of the tensor representation $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ with the orbit method before applying it to $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{3}{2}}$ in the next subsection. This is the largest test case, since thanks to the fact that $\mathrm{im}\rho_{\frac{1}{2}} \cong \mathfrak{so}(32)$ (cp. [HKL15, thm. A]) one knows that the irreducible pieces of $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ are $\bigwedge^2 \mathcal{S}_{\frac{1}{2}}, S_{[2]}\left(\mathcal{S}_{\frac{1}{2}}\right)$ and 1 where $S_{[2]}\left(\mathcal{S}_{\frac{1}{2}}\right)$ denotes the traceless symmetric part of the tensor product and 1 is the trivial representation. One can apply the method to $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ but in order to increase performance, I decided to work directly with $\bigwedge^2 \mathcal{S}_{\frac{1}{2}}$ and $\mathrm{Sym}^2\mathcal{S}_{\frac{1}{2}}$, as the first one is needed later anyways. For the exterior product one knows theoretically[51] that under $\mathfrak{so}(10)$

$$\bigwedge^2 \mathcal{S}_{\frac{1}{2}} \cong 1 \oplus \Gamma_{\omega_2} \oplus 2 \times \Gamma_{\omega_3} \oplus \Gamma_{\alpha+\beta}\,.$$

[51] For [SAGE] this is a simple computation in the character ring of D_5.

Since the weight $\alpha+\beta$ does not appear in the other representations, the orbit method will work best if one starts from this highest weight module. A vector to this weight is

$$v_{\alpha+\beta} \coloneqq s_\alpha \wedge s_\beta = \frac{1}{\sqrt{2}} \left(s_\alpha \otimes s_\beta - s_\beta \otimes s_\alpha \right)$$

and the orbit method applied to this module finds the following modules with highest weights

$$\Lambda_1 = \alpha+\beta \ , \ \Lambda_2 = \omega_3 \ , \ \Lambda_3 = \omega_3 \ , \ \Lambda_4 = 0 \ , \ \Lambda_5 = \omega_2$$

and one concludes that this exhausts $\bigwedge^2 \mathcal{S}_{\frac{1}{2}}$. The adjacency matrix of X_{10} w.r.t. this decomposition is computed to be

$$A^{(ext)}_{adj} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} .$$

Note that this adjacency matrix is symmetric. Under the full action of $\mathfrak{k}\left(E_{10}\right)(\mathbb{C})$ this orbit turns out to be irreducible as expected.

For $\mathrm{Sym}^2\left(\mathcal{S}_{\frac{1}{2}}\right)$ one has the following $\mathfrak{so}(10)$-decompositions on abstract grounds:

$$\mathrm{Sym}^2\left(\mathcal{S}\right) \cong 1 \oplus 2 \times \Gamma_{\omega_1} \oplus \Gamma_{\omega_2} \oplus \Gamma_{\alpha+\beta} \oplus \Gamma_{2\alpha} \oplus \Gamma_{2\beta} \, .$$

The vector

$$v_{\alpha+\beta} \coloneqq s_\alpha \cdot s_\beta = \frac{1}{\sqrt{2}} \left(s_\alpha \otimes s_\beta + s_\beta \otimes s_\alpha \right)$$

is a vector to the weight $\alpha+\beta$ and one obtains the following highest weights in the orbit connected to $\Gamma_{\alpha+\beta}$:

$$\Lambda_1 = \alpha+\beta \ , \ \Lambda_2 = 2\beta \ , \ \Lambda_3 = 2\alpha \ , \ \Lambda_4 = \omega_1 \ , \ \Lambda_5 = \omega_2 \ , \ \Lambda_6 = \omega_1 \, .$$

With respect to this decomposition, the adjacency matrix of X_{10} is equal to

$$A^{(sym)}_{adj} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and as the corresponding graph is undirected and connected one concludes that the orbit is irreducible under $\mathfrak{k}\left(E_{10}\right)(\mathbb{C})$ and thus under $\mathfrak{k}\left(E_{10}\right)(\mathbb{R})$. A comparison to the theoretically predicted decomposition shows that the orbit lacks the trivial representation. The trivial representation can be computed by direct intersection of the kernels of all Berman generators, as this is still feasible in this dimension. The resulting vector is then shown to be orthogonal to all the other $\mathfrak{so}(10)$-modules. Since all Berman generators act trivially on this vector, there is no mixing between the trivial representation and the above orbit. This shows reducibility of $\mathrm{Sym}^2\left(\mathcal{S}_{\frac{1}{2}}\right)$ into two $\mathfrak{k}\left(E_{10}\right)$-irreducible pieces:

$$\mathrm{Sym}^2\left(\mathcal{S}_{\frac{1}{2}}\right) \cong S_{[2]}\left(\mathcal{S}_{\frac{1}{2}}\right) \oplus 1,$$

where

$$S_{[2]}\left(\mathcal{S}_{\frac{1}{2}}\right) \cong 2\times\Gamma_{\omega_1}\oplus\Gamma_{\omega_2}\oplus\Gamma_{\alpha+\beta}\oplus\Gamma_{2\alpha}\oplus\Gamma_{2\beta}\,.$$

Since $\mathcal{S}_{\frac{1}{2}}\otimes S_{\frac{1}{2}} \cong \mathrm{Sym}^2\left(\mathcal{S}_{\frac{1}{2}}\right)\oplus\bigwedge^2\mathcal{S}_{\frac{1}{2}}$ this completes the analysis of $\mathcal{S}_{\frac{1}{2}}\otimes S_{\frac{1}{2}}$. In agreement with the representation theory of $\mathfrak{so}(32)$ one has seen that $\mathcal{S}_{\frac{1}{2}}\otimes S_{\frac{1}{2}}$ splits into three irreducible pieces.

6.1.4 Decomposition and orbits of $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}}\otimes\left(\bigwedge^2\mathcal{S}_{\frac{1}{2}}\right)$

The goal is to decompose $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}}\otimes\left(\bigwedge^2\mathcal{S}_{\frac{1}{2}}\right)$ via the orbit method. The result will be that both modules are irreducible under the action of $\mathfrak{k}\left(E_{10}\right)(\mathbb{C})$. This computation may not work on any PC as it is quite memory intensive. The computation of $\mathcal{S}_{\frac{3}{2}}\otimes\left(\bigwedge^2\mathcal{S}_{\frac{1}{2}}\right)$ is modified such that it needs less memory in comparison to the approach used for $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ but trades this for a longer (relative) run time.

Under the restriction to $\mathfrak{so}(10,\mathbb{C})$ the module splits as follows (plug the $\mathfrak{so}\,(10,\mathbb{C})$-decompositions of $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{1}{2}}$ into [SAGE] and multiply the characters):

$$\begin{aligned}\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}} \quad\cong\quad & \Gamma_{\omega_1+2\alpha}\oplus\Gamma_{\omega_1+2\beta}\oplus 2\times\left(\Gamma_{\omega_1+\alpha+\beta}\oplus\Gamma_{\omega_1+\omega_3}\oplus\Gamma_{\omega_1+\omega_2}\oplus\Gamma_{2\omega_1}\oplus\Gamma_{2\alpha}\oplus\Gamma_{2\beta}\right)\\ & \oplus 4\times\left(\Gamma_{\alpha+\beta}\oplus\Gamma_{\omega_3}\oplus\Gamma_{\omega_2}\oplus\Gamma_{\omega_1}\right)\oplus 2\times\Gamma_0.\end{aligned}$$

As before one first constructs with a little guesswork the weight vector $s_\lambda\in\mathcal{S}_{\frac{1}{2}}$ with weight $\lambda=-\frac{1}{2}L_1+\frac{1}{2}\left(L_2+L_3+L_4+L_5\right)$. The tensor product $w:=\alpha_1\otimes s_\lambda$ is then a highest weight vector of $\mathcal{S}_{\frac{3}{2}}$ to the weight $\frac{3}{2}L_1+\frac{1}{2}\left(L_2+L_3+L_4+L_5\right)=\omega_1+\alpha$. The tensor product of w with s_α, the highest weight vector of $\Gamma_\alpha\subset\mathcal{S}_{\frac{1}{2}}$ then provides a vector to the weight $\omega_1+2\alpha$ which is confirmed as a highest weight vector by checking that $E_i\left(w\otimes s_\alpha\right)=0$ holds for all $i=1,\ldots,5$. This mus be the case as the weight $\omega_1+2\alpha$ is not contained in any of the other $\mathfrak{so}\,(10,\mathbb{C})$-representations. Afterwards the full $\mathfrak{so}(10,\mathbb{C})$-module associated to this highest weight vector is constructed. Therefore, one can be assured that the computed $\mathfrak{k}(E_{10})$-orbit gives some meaningful answer because $\Gamma_{\omega_1+2\alpha}$ occurs with multiplicity one in this decomposition. It turns out that the orbit contains each of the above $\mathfrak{so}(10)$-modules and so one knows that it is the entire module. In order to investigate reducibility one computes the adjacency matrix A_{adj} of X_{10} which is a 32×32 matrix for $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$. The corresponding graph is analyzed and shown to be strongly connected (the adjacency matrix is not symmetric in this case). Hence, the $\mathfrak{k}\left(E_{10}\right)(\mathbb{C})$-module $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ is irreducible which came as a slight surprise; I will take this point up again in the next section. This implies that $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ is irreducible as a $\mathfrak{k}\left(E_{10}\right)(\mathbb{R})$-module as well.

In order to do a similar analysis for $\mathcal{S}_{\frac{3}{2}}\otimes\left(\bigwedge^2\mathcal{S}_{\frac{1}{2}}\right)$ one definitely should work on a computer cluster as the computations take time and are still memory-intensive despite some optimizations in comparison to the previous computation. Under restriction to $\mathfrak{so}(10,\mathbb{C})$ the module splits into a total of 116 modules as follows (again, plug the D_5-characters of $\mathcal{S}_{\frac{3}{2}}$ and $\bigwedge^2\mathcal{S}_{\frac{1}{2}}$ into [SAGE] and multiply them):

$$\begin{aligned}\mathcal{S}_{\frac{3}{2}}\otimes\left(\bigwedge^2\mathcal{S}_{\frac{1}{2}}\right) \quad\cong\quad & \Gamma_{\omega_1+2\alpha+\beta}\oplus\Gamma_{\omega_1+\alpha+2\beta}\oplus 3\times\left(\Gamma_{\omega_1+\omega_3+\alpha}\oplus\Gamma_{\omega_1+\omega_3+\beta}\right)\\ & \oplus 4\times\left(\Gamma_{\omega_1+\omega_2+\alpha}\oplus\Gamma_{\omega_1+\omega_2+\beta}\oplus\Gamma_{2\omega_1+\alpha}\oplus\Gamma_{\omega_1+\beta}\oplus\Gamma_{2\alpha+\beta}\oplus\Gamma_{\alpha+2\beta}\right)\\ & \oplus\Gamma_{3\alpha}\oplus\Gamma_{3\beta}\oplus 8\times\left(\Gamma_{\omega_3+\alpha}\oplus\Gamma_{\omega_3+\beta}\right)\oplus 11\times\left(\Gamma_{\omega_2+\alpha}\oplus\Gamma_{\omega_2+\beta}\right)\\ & \oplus 13\times\left(\Gamma_{\omega_1+\alpha}\oplus\Gamma_{\omega_1+\beta}\right)\oplus 9\times\left(\Gamma_\alpha\oplus\Gamma_\beta\right).\end{aligned}$$

Again one starts by producing a unique-up-to-prefactor highest weight vector, in this case of $\Gamma_{\omega_1+2\alpha+\beta}$, and computes the first module. To this module one then again applies the orbit method which yields that the orbit contains all 116 $\mathfrak{so}(10,\mathbb{C})$-modules. A subsequent analysis of these modules and how they mix under X_{10} showed that the representation is irreducible, as the graph created from the adjacency matrix $A_{adj}(X_{10})$ is strongly connected.

The above two examples exhibit the curious feature that the tensor product of two irreducible representations is again irreducible, which is something one rarely observes for semisimple, finite-dimensional, Lie algebras. As I will explain in the next section, these representations have the same kernel because $\bigwedge^2 \mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{1}{2}}$ do. Hence, the only additional information about $\mathfrak{k}(E_{10})$ would come from the tensor products $\mathcal{S}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$,$\mathcal{S}_{\frac{7}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$,$\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{5}{2}}$, ... Without further optimization, these representations cannot be computed in reasonable time.

The main caveat of my implementation is that it is not parallelized. The main issue here is that the only substantial acceleration would come from parallelization of matrix multiplication. My implementation via Sage uses symbolic implementations of certain fields (which Sage provides plenty of) which allows for an exact evaluation instead of an approximate one in comparison to the use of floats. My attempt at parallelization of these matrix multiplications turned out to be slower. Parallelized matrix multiplication over arbitrary fields is (at least in version 9.0) nontrivial in Sage and the amount of time to set this up successfully didn't seem worth the time, as the code still runs in reasonable time and the goal was to obtain any information about how the tensor products behave. If one wants to analyze more and higher tensor products, this is something one needs to address if one still wants to compute exactly. If not, one can just switch to SciPy or NumPy and use their implementations of parallelized matrix multiplication. The reasons why $\mathcal{S}_{\frac{5}{2}}$ and $S_{[2]}\left(\mathcal{S}_{\frac{1}{2}}\right)$ are not among the analyzed tensor products and $\mathcal{S}_{\frac{7}{2}}$ is not analyzed at all is that their matrices contain certain normalization factors that are not rational. For the computations of $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}} \otimes \left(\bigwedge^2 \mathcal{S}_{\frac{1}{2}}\right)$ one can actually use Sage's implementation of the rational numbers which is quite fast. For $S_{[2]}\left(\mathcal{S}_{\frac{1}{2}}\right)$ one encounters the additional problem that the involved matrices are rather dense in the basis that is induced from the one used for $\mathcal{S}_{\frac{1}{2}}$.

6.2 Irreducibility and ideals

In this section I would like to collect some theoretical considerations concerning the tensor product representations. I will show that the kernels of most of these representations are given by the intersection of the individual kernels, which is of interest as the kernels are usually not contained in each other and therefore the intersection provides a smaller ideal. Also, I provide a result on the action of $\mathfrak{k}(A)$ on tensor product representations $\mathcal{S}_{\frac{n}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$. There one observes that the action factors, meaning that one can act on the factors individually. This is essentially a consequence of the $\mathcal{S}_{\frac{1}{2}}$-representation matrices squaring to $-\frac{1}{4}Id$.

Let $\rho_i : \mathfrak{k}(A)(\mathbb{C}) \to \mathrm{End}(V_i)$ be f.d. representations for $i = 1, 2$ and $x \in \mathfrak{k}$ such that $x \in \ker \rho_1$ and $x \in \ker \rho_2$. Then one has $x \in \ker(\rho_1 \otimes \rho_2)$ because of

$$\begin{aligned} (\rho_1 \otimes \rho_2)(x)(v \otimes w) &= (\rho_1(x)v) \otimes w + v \otimes (\rho_2(x)w) \\ &= 0 \otimes w + v \otimes 0 = 0 \;\forall v \in V_1, w \in V_2. \end{aligned}$$

Hence, in general $\ker \rho_1 \cap \ker \rho_2 \subset \ker(\rho_1 \otimes \rho_2)$. But what does the nontrivial condition $x \in \ker(\rho_1 \otimes \rho_2)$ but $x \notin \ker(\rho_1)$ imply?

Towards this consider bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ of V_1, resp. V_2. Then one computes

$$\begin{aligned} \rho_1(x)e_i \otimes f_j + e_i \otimes \rho_2(x) f_j &= 0 \\ \Leftrightarrow \sum_{k \neq i}^{n} \rho_1(x)_{ki} e_k \otimes f_j + \rho_1(x)_{ii} e_i \otimes f_j & \\ + \sum_{l \neq i}^{m} \rho_2(x)_{lj} e_i \otimes f_l + \rho_2(x)_{jj} e_i \otimes f_j &= 0 \; \forall \, 1 \leq i \leq n, \, 1 \leq j \leq m. \end{aligned}$$

Since $e_k \otimes f_j$ and $e_i \otimes f_l$ are linearly independent for $k \neq i$ and $l \neq j$ this holds if and only if $\rho_1(x)$ and $\rho_2(x)$ are diagonal matrices such that $\rho_1(x)_{ii} = -\rho_2(x)_{jj} \neq 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$ or in other terms if the matrices are proportional to the identity. The prefactor needs to be different from 0 as otherwise $\rho_1(x) = 0 = \rho_2(x)$ and therefore $x \in \ker(\rho_1)$ in contradiction to the assumption. In short one has the following

Lemma 6.1. *Let $\rho_i : \mathfrak{k}(A)(\mathbb{C}) \to End(V_i)$ for $i = 1, 2$ be f.d. representations and let $x \in \ker(\rho_1 \otimes \rho_2)$. Then either $x \in \ker(\rho_1) \cap \ker(\rho_2)$ or there exists $0 \neq \lambda \in \mathbb{C}$ such that*

$$\rho_1(x) = \lambda \cdot Id_{V_1} \text{ und } \rho_2(x) = -\lambda \cdot Id_{V_2}. \tag{112}$$

Another important piece of information is that powers (regular, symmetric or antisymmetric) of finite-dimensional representations don't produce larger kernels:

Lemma 6.2. *Let $\rho : \mathfrak{k}(A)(\mathbb{C}) \to End(V)$ be a finite-dimensional representation, then*

$$\ker(\rho \otimes \rho) = \ker(\rho), \;\; \ker(Sym^n(\rho)) = \ker(\rho), \;\; \ker(\wedge^n \rho) = \ker \rho, \;\; \textit{as long as } n < \dim(V).$$

Proof. For $\rho \otimes \rho$ one can apply the previous lemma and for $\mathrm{Sym}^n(\rho)$ one observes the following, where $\{b_1, \ldots, b_m\}$ is any basis of V:

$$x.b_i \cdot b_i \cdot \cdots \cdot b_i = n \cdot \sum_{j=1}^{m} \rho(x)_{ji} \, b_j \cdot b_i \cdots b_i = 0 \;\Leftrightarrow\; \rho(x)_{ji} = 0 \; \forall \, i, j = 1, \ldots, m.$$

For $\bigwedge^n V$ it is instructional to look at $\bigwedge^2 V$ first:

$$\begin{aligned} x.b_i \wedge b_j &= \sum_{k=1}^{m} \rho(x)_{ki} \, b_k \wedge b_j + \sum_{k=1}^{m} \rho(x)_{kj} \, b_i \wedge b_k \\ &= \left(\rho(x)_{ii} + \rho(x)_{jj}\right) b_i \wedge b_j + \sum_{k \neq i}^{m} \rho(x)_{ki} \, b_k \wedge b_j + \sum_{k \neq j}^{m} \rho(x)_{kj} \, b_i \wedge b_k. \end{aligned}$$

As $b_k \wedge b_j$ is linearly independent from all $b_l \wedge b_i$ if $k \neq i$ and $l \neq j$ this is equal to 0 if and only if

$$\rho(x)_{ki} = 0 \; \forall \, k \neq i \text{ and } \rho(x)_{ii} + \rho(x)_{jj} = 0 \; \forall \, i \neq j.$$

The second condition yields $\rho(x)_{ii} = 0$ for all $i = 1, \ldots, m$ if $m > 2$. In the same manner one has in

$$x.b_{i_1} \wedge \cdots \wedge b_{i_n} = \left(\sum_{k=1}^{n} \rho(x)_{i_k i_k}\right) b_{i_1} \wedge \cdots \wedge b_{i_n} + \sum_{k=1}^{n} \sum_{j \neq i_k}^{m} \rho(x)_{i_k j} \, b_{i_1} \wedge \cdots \wedge b_j \wedge \cdots \wedge b_{i_n}$$

that $x.b_{i_1} \wedge \cdots \wedge b_{i_n} = 0$ is equivalent to $\sum_{k=1}^{n} \rho(x)_{i_k i_k} = 0$ and $\rho(x)_{ij} = 0$ for all $j \neq i$ because the terms in the second sum are linearly independent. Again, if $n < m$ then $\sum_{k=1}^{n} \rho(x)_{i_k i_k} = 0$ implies $\rho(x)_{i_k i_k} = 0$ for all $i_k = 1, \ldots, m$. □

Proposition 6.3. *Let A be simply-laced and indecomposable, then for all the higher spin representations $\left(\rho_{\frac{n}{2}}, \mathcal{S}_{\frac{n}{2}}\right)$ of $\mathfrak{k}(A)$ for $n = 1, 3, 5, 7$ one has* $\ker \rho_{\frac{n_1}{2}} \otimes \rho_{\frac{n_2}{2}} \cong \ker \rho_{\frac{n_1}{2}} \cap \ker \rho_{\frac{n_2}{2}}$.

Proof. All these representations satisfy that there exists a non-degenerate bilinear form on the module w.r.t. which the action of $\mathfrak{k}(A)$ is skew (cp. lemma 5.9). Hence, the representation matrices must be traceless which excludes the second case of lemma 6.1. □

Note that the above proposition also holds in the case of $\mathfrak{k}(E_9)$ (or more generally, if A is not of full rank) but its implications are rather trivial as the kernels form a chain of inclusions

$$\ker \rho_{\frac{7}{2}} \subsetneq \ker \rho_{\frac{5}{2}} \subsetneq \ker \rho_{\frac{3}{2}} \subsetneq \ker \rho_{\frac{1}{2}}$$

as is shown in [KN21]. The next proposition shows for the example $\mathfrak{k}(E_{10})$ that this condition can provide truly smaller ideals.

Proposition 6.4. *(Due to a personal discussion with Axel Kleinschmidt) For $\mathfrak{k}(E_{10})$ one has* $\ker\left(\rho_{\frac{3}{2}}\right) \cap \ker\left(\rho_{\frac{1}{2}}\right) \subsetneq \ker\left(\rho_{\frac{n}{2}}\right)$ *for $n = 1, 3$.*

Proof. Show first that $\ker\left(\rho_{\frac{1}{2}}\right) \not\subseteq \ker\left(\rho_{\frac{3}{2}}\right)$. Towards this choose suitably normalized $x_\alpha \in \mathfrak{k}_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$, $x_\beta \in \mathfrak{k}_\beta$ with $\alpha, \beta \in \Delta_+^{re}$ such that

$$\rho_{\frac{1}{2}}(x_\alpha) = \frac{1}{2}\Gamma(\alpha), \ \rho_{\frac{1}{2}}(x_\beta) = \frac{1}{2}\Gamma(\beta)$$

$$\rho_{\frac{3}{2}}(x_\alpha) = \left(s_\alpha - \frac{1}{2}Id\right) \otimes \Gamma(\alpha), \ \rho_{\frac{3}{2}}(x_\beta) = \left(s_\beta - \frac{1}{2}Id\right) \otimes \Gamma(\beta),$$

according to 4.16 and 4.17. If now $\alpha - \beta =: \gamma \in 2Q(E_{10})$ then $\Gamma(\alpha) = \Gamma(\beta)$ and therefore $x_\alpha - x_\beta \in \ker\left(\rho_{\frac{1}{2}}\right)$. On the other hand,

$$\begin{aligned} s_\alpha - s_\beta &= Id - \alpha(\alpha|\cdot) - Id + \beta(\beta|\cdot) \\ &= -(\gamma + \beta)(\gamma + \beta|\cdot) + \beta(\beta|\cdot) \\ &= -(\gamma)(\gamma|\cdot) - \beta(\gamma|\cdot) - (\gamma)(\beta|\cdot) \end{aligned}$$

$$\Rightarrow \ \rho_{\frac{3}{2}}(x_\alpha) - \rho_{\frac{3}{2}}(x_\beta) = (s_\alpha - s_\beta) \otimes \Gamma(\alpha) \neq 0.$$

As $\mathrm{im}\rho_{\frac{1}{2}} \cong \mathfrak{so}(32)$ (cp. [KN13, sec. 4.4], also [HKL15, thm. A]) and $\mathrm{im}\rho_{\frac{3}{2}} \cong \mathfrak{so}(32, 288)$ (cp. [KN13, sec. 4.5]) one knows that $\ker\left(\rho_{\frac{3}{2}}\right) \not\subseteq \ker\left(\rho_{\frac{1}{2}}\right)$because otherwise $\mathfrak{so}(32, 288)$ would also be able to act on $\mathcal{S}_{\frac{1}{2}}$ via factoring through $\mathfrak{so}(32)$ but $\mathfrak{so}(32, 288)$ does not admit an irreducible module of dimension 32. □

A similar argument as above should work for $\mathcal{S}_{\frac{5}{2}}$ and $\mathcal{S}_{\frac{1}{2}}$ as computation of im $\rho_{\frac{5}{2}}$ should show that $\ker\left(\rho_{\frac{5}{2}}\right) \not\subseteq \ker\left(\rho_{\frac{1}{2}}\right)$. Similarly this could show $\ker\left(\rho_{\frac{5}{2}}\right) \not\subseteq \ker\left(\rho_{\frac{3}{2}}\right)$ and conversely one needs to investigate whether there exist linear combinations $\sum_j s_{\beta_j}$ with $\beta_j - \beta_i \in 2Q(E_{10})$ such that $\sum_j s_{\beta_j} = 0 \in Gl(\mathfrak{h}^*)$ but

$\sum_j s_{\beta_j} \neq 0 \in Gl\left(\mathrm{Sym}^2\left(\mathfrak{h}^*\right)\right)$. Therefore, investigation of $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{1}{2}} \otimes \widetilde{\mathcal{S}}_{\frac{5}{2}}$ are interesting as they are irreducible (for the second one this result is shown soon). Also, their triple product $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{3}{2}} \otimes \widetilde{\mathcal{S}}_{\frac{5}{2}}$ could be of interest if one can show that the pairwise intersections of their kernels are not contained in each other.

The polynomial identities from lemma 5.6 imply a factorization of the action of $\mathfrak{k}(A)$ on the tensor products $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{1}{2}} \otimes \mathcal{S}_{\frac{5}{2}}$.

Lemma 6.5. *Let A be a simply-laced GCM and let $\left(\rho, \mathcal{S}_{\frac{1}{2}}\right)$ denote a generalized spin representation according to def. 3.1 and let $\left(\sigma, \mathcal{S}_{\frac{n}{2}}\right)$ for $n = 3$ or 5 denote a higher spin representation from theorem 3.19. Denote the tensor product of σ and ρ by μ then one has*

$$Id \otimes \rho(x) = \frac{2}{9}\left[\mu(x)^5 + 5\mu(x)^3 + \frac{23}{8}\mu(x)\right], \quad \sigma(x) \otimes Id = -\frac{2}{9}\left[\mu(x)^5 + 5\mu(x)^3 - \frac{13}{8}\mu(x)\right] \tag{113}$$

for all $x \in \mathfrak{k}_\alpha$, $\alpha \in \Delta^{re}$ that have the same norm as a Berman generator. This implies that for each $x \in \mathcal{U}(\mathfrak{k})$, there exist $y_1, y_2 \in \mathcal{U}(\mathfrak{k})$ such that $\mu(y_1) = Id \otimes \rho(x)$ and $\mu(y_2) = \sigma(x) \otimes Id$.

Proof. Use from lemma 5.6 that

$$\frac{16}{9}\sigma(X_i)^4 + \frac{40}{9}\sigma(X_i)^2 = Id \in End\left(\mathcal{S}_{\frac{n}{2}}\right) \text{ for } n = 3, 5$$

for all Berman generators X_i and suitably normalized Berman elements $x \in \mathfrak{k}_\alpha$ for $\alpha \in \Delta^{re}$. Recall that for such x one has $\rho(x)^2 = -\frac{1}{4}Id \in End\left(\mathcal{S}_{\frac{1}{2}}\right)$. Denote by μ the Lie-tensor product of σ and ρ. One now computes

$$\mu(x) = \sigma(x) \otimes Id + Id \otimes \rho(x)$$

$$\mu(x)^2 = \sigma(x)^2 \otimes Id + 2\sigma(x) \otimes \rho(x) - \frac{1}{4}Id \otimes Id$$

$$\mu(x)^3 = \sigma(x)^3 \otimes Id + 3\sigma(x)^2 \otimes \rho(x) - \frac{3}{4}\sigma(x) \otimes Id - \frac{1}{4}Id \otimes \rho(x)$$

$$\mu(x)^3 + \frac{1}{4}\mu(x) = \sigma(x)^3 \otimes Id + 3\sigma(x)^2 \otimes \rho(x) - \frac{1}{2}\sigma(x) \otimes Id.$$

Furthermore, with

$$\sigma(x)^4 = \frac{9}{16}Id - \frac{5}{2}\sigma(x)^2$$

one simplifies

$$\begin{aligned} \mu(x)^4 + \frac{1}{4}\mu(x)^2 &= \sigma(x)^4 \otimes Id + \sigma(x)^3 \otimes \rho(x) + 3\sigma(x)^2 \otimes \rho(x) - \frac{3}{4}\sigma(x)^2 \otimes Id \\ &\quad -\frac{1}{2}\sigma(x)^2 \otimes Id - \frac{1}{2}\sigma(x) \otimes \rho(x) \\ &= \frac{9}{16}Id \otimes Id + 4\sigma(x)^3 \otimes \rho(x) - \frac{15}{4}\sigma(x)^2 \otimes Id - \frac{1}{2}\sigma(x) \otimes \rho(x). \end{aligned}$$

$$\begin{aligned}
\mu(x)^5 + \frac{1}{4}\mu(x)^3 &= 4\sigma(x)^4 \otimes \rho(x) - \sigma(x)^3 \otimes Id - \frac{15}{4}\sigma(x)^3 \otimes Id - \frac{15}{4}\sigma(x)^2 \otimes \rho(x) \\
&\quad - \frac{1}{2}\sigma(x)^2 \otimes \rho(x) + \frac{1}{8}\sigma(x) \otimes Id + \frac{9}{16}\sigma(x) \otimes Id + \frac{9}{16} Id \otimes \rho(x) \\
&= \frac{9}{4} Id \otimes \rho(x) - 10\sigma(x)^2 \otimes \rho(x) - \frac{19}{4}\sigma(x)^3 \otimes Id - \frac{17}{4}\sigma(x)^2 \otimes \rho(x) \\
&\quad + \frac{11}{16}\sigma(x) \otimes Id + \frac{9}{16} Id \otimes \rho(x) \\
&= -\frac{19}{4}\sigma(x)^3 \otimes Id - \frac{57}{4}\sigma(x)^2 \otimes \rho(x) + \frac{11}{16}\sigma(x) \otimes Id + \frac{45}{16} Id \otimes \rho(x)
\end{aligned}$$

$$\begin{aligned}
\mu(x)^5 + \left(\frac{1}{4} + \frac{19}{4}\right)\mu(x)^3 &= \frac{11}{16}\sigma(x) \otimes Id + \frac{45}{16} Id \otimes \rho(x) - \frac{19}{4}\left(\frac{3}{4}\sigma(x) \otimes Id + \frac{1}{4} Id \otimes \rho(x)\right) \\
\mu(x)^5 + 5\mu(x)^3 &= -\frac{23}{8}\sigma(x) \otimes Id + \frac{13}{8} Id \otimes \rho(x)
\end{aligned}$$

$$\begin{aligned}
\mu(x)^5 + 5\mu(x)^3 + \frac{23}{8}\mu(x) &= \frac{9}{2} Id \otimes \rho(x) \\
\mu(x)^5 + 5\mu(x)^3 - \frac{13}{8}\mu(x) &= -\frac{9}{2}\sigma(x) \otimes Id
\end{aligned}$$

which shows eq. (113). Now for each $x \in \mathcal{U}(\mathfrak{k})$, there exist $y_1, y_2 \in \mathcal{U}(\mathfrak{k})$ such that $\mu(y_1) = Id \otimes \rho(x)$ and $\mu(y_2) = \sigma(x) \otimes Id$ because the X_i generate $\mathfrak{k}$. □

Lemma 6.6. *Let $\mathfrak{g}$ be a semi-simple finite-dimensional Lie algebra with Cartan subalgebra $\mathfrak{h}$ and let U be a finite-dimensional $\mathfrak{g}$-module with weight space decomposition $U = \bigoplus_{\lambda \in P(U)} U_\lambda$. Then projection to U_λ can be achieved within $\mathcal{U}(\mathfrak{h})$, i.e. for each $\lambda \in P(U)$ there exists an element $\Pi_\lambda \in \mathcal{U}(\mathfrak{h})$ s.t. $\Pi_\lambda U_\mu = \{0\}$ for all $\mu \neq \lambda$ and $\Pi_\lambda U_\lambda = U_\lambda$.*

Proof. Let $\{H_1, \ldots, H_l\}$ be an orthonormal basis of $\mathfrak{h}$ and set

$$P_{i,\lambda}(U) := \{\mu \in P(U) \mid \mu(H_i) \neq \lambda(H_i)\}.$$

Then

$$\pi_{i,\lambda} := \prod_{\mu \in P_{i,\lambda}(U)} \frac{H_i - \mu(H_i)}{\lambda(H_i) - \mu(H_i)}$$

has the property that with $u_\nu \in U_\nu$ and $\nu \in P_{i,\lambda}(U)$

$$\pi_{i,\lambda}(u_\nu) = 0 \;\forall\, \nu \in P_{i,\lambda}(U), \quad \pi_{i,\lambda}(u_\lambda) = u_\lambda \;\forall\, u_\lambda \in U_\lambda.$$

But then

$$\left(\prod_{i=1}^{l} \pi_{i,\lambda}\right)(u_\nu) = 0 \;\forall\, u_\nu \in U_\nu \; s.t. \; \nu \neq \lambda, \; \left(\prod_{i=1}^{l} \pi_{i,\lambda}\right)(u_\lambda) \;\; \forall\, u_\lambda \in U_\lambda,$$

as any $\nu \neq \lambda$ is contained in at least one $P_{i,\lambda}(U)$. □

Proposition 6.7. *Let $A \in \mathbb{Z}^{n\times n}$ be a simply laced, indecomposable, regular GCM s.t. $n \geq 2$. Let $\left(\rho, \mathcal{S}_{\frac{1}{2}}\right)$ denote an irreducible generalized spin representation according to def. 3.1 and let $\left(\sigma, \mathcal{S}_{\frac{3}{2}}\right)$, $\left(\sigma, \widetilde{\mathcal{S}}_{\frac{5}{2}}\right)$ denote a higher spin representation from theorem 3.19 (respectively the trace-free part of $\mathcal{S}_{\frac{5}{2}}$ from prop. 5.10). Then $\left(\mu, \mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}\right)$ is irreducible and if $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ is irreducible then the same holds for $\left(\mu, \widetilde{\mathcal{S}}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{1}{2}}\right)$.*

Remark. It may be possible to drop the assumption about A being indecomposable, as long as the diagram is not totally disconnected. In the proof one uses that imρ is semisimple (which follows from thm. 3.3 under the assumption that A is simply-laced and indecomposable). If one allows isolated nodes for instance then imρ is just compact, hence reductive but if its semisimple part $\mathfrak{s} \subset$ imρ is nonzero, one would still have that $\mathcal{S}_{\frac{1}{2}}$ is a highest weight module w.r.t. $\mathfrak{s}$ as the abelian part of imρ acts diagonally anyways. Therefore one needs to make sure that there exists a sub-diagram of $\Pi(A)$ that meets the demands of the proposition.

Proof. Let U be an invariant submodule of $V \otimes W$, where $V \in \left\{\mathcal{S}_{\frac{3}{2}}, \widetilde{\mathcal{S}}_{\frac{5}{2}}\right\}$ and $W = \mathcal{S}_{\frac{1}{2}}$. According to lemma 6.5 one can act on W separately, i.e. for each $x \in \mathcal{U}(\mathfrak{k})$, there exist $y_1, y_2 \in \mathcal{U}(\mathfrak{k})$ such that $\mu(y_1) = Id \otimes \rho(x)$ and $\mu(y_2) = \sigma(x) \otimes Id$. Under the assumptions of the proposition one obtains from thm. 3.3 that imρ is semisimple. Hence, over $\mathbb{C}$ there exists a weight space decomposition of $W = \bigoplus_{\lambda \in P(W)} W_\lambda$ w.r.t. a Cartan subalgebra $\mathring{\mathfrak{h}}$ of imρ. Then any element in U can be written as

$$u = \sum_{i=1}^{\dim V} \sum_{\lambda \in P(W)} \sum_{j=1}^{\dim W_\lambda} c_i d_\lambda^{(i,j)} b_i \otimes w_\lambda^{(j)}, \tag{114}$$

where $\{b_i \,|\, i = 1, \ldots, \dim V\}$ is any basis of V and $\left\{w_\lambda^{(j)} \,|\, \lambda \in P(W),\, j = 1, \ldots, \dim W_\lambda\right\}$ is a weight space basis of W. Let π_λ be the projector to the weight space W_λ, i.e. the linear map $\pi_\lambda : W \to W_\lambda$ s.t. $\pi_\lambda w = 0$ $\forall\, w \in W_\mu$ with $\mu \neq \lambda$ and $\pi_\lambda|_{W_\lambda} = Id$. From lemma 6.6 one knows that there exists an element $\tilde{\pi}_{\lambda,W}$ in $\mathcal{U}\left(\mathring{\mathfrak{h}}\right)$ s.t. $\tilde{\pi}_{\lambda,W} = \pi_\lambda$. Then there exists $x \in \mathcal{U}(\mathfrak{k}(A)(\mathbb{C}))$ s.t.

$$\begin{aligned} u' &= \mu(x)\, u = 1 \otimes \tilde{\pi}_{\lambda,W} \left[\sum_{i=1}^{\dim V} \sum_{\lambda \in P(W)} \sum_{j=1}^{\dim W_\lambda} c_i d_\lambda^{(j)} b_i \otimes w_\lambda^{(j)}\right] = \sum_{i=1}^{\dim V} \sum_{j=1}^{\dim W_\lambda} c_i d_\lambda^{(i,j)} b_i \otimes w_\lambda^{(j)} \\ &= \sum_{i=1}^{\dim V} c_i b_i \otimes \left(\sum_{j=1}^{\dim W_\lambda} d_\lambda^{(i,j)} w_\lambda^{(j)}\right) = \sum_{i=1}^{\dim V} c_i b_i \otimes w_i \end{aligned}$$

for a weight $\lambda \in P(W)$ that occurs in the decomposition (114) of u and $w_i := \sum_{j=1}^{\dim W_\lambda} d_\lambda^{(i,j)} w_\lambda^{(j)} \in W_\lambda$. As im$\rho$ is semisimple its complexification (if one started over $\mathbb{R}$) admits a triangular decomposition $\mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$and as W is irreducible, there exists $e_+ \in \mathcal{U}(\mathring{\mathfrak{n}}_+)$ s.t.

$$e_+ w_i = k_i w_\Lambda,\ k_i \in \mathbb{C},$$

where w_Λ is the (up to prefactors) unique highest weight vector of W.

Then by lemma 6.5 there again exists an element $x \in \mathcal{U}(\mathfrak{k}(A)(\mathbb{C}))$ s.t. $\mu(x) = 1 \otimes e_+$ and therefore

$$(1 \otimes e_+)\, u' = \sum_{i=1}^{\dim V} c_i b_i \otimes k_i w_\Lambda = \left(\sum_{i=1}^{\dim V} c_i k_i b_i\right) \otimes w_\Lambda = v' \otimes w_\Lambda \in U$$

is an elementary tensor that is contained in U, where $v' \neq 0$ because the b_i are linearly independent and at least one k_i can be assumed to be nonzero. One now applies lemma 6.5 one last time together with irreducibility of V, W (for $\mathcal{S}_{\frac{3}{2}}$ this follows from prop. 5.8 and for $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ this was an assumption) to obtain

$$\mathcal{U}(\mathfrak{k}(A))\, U = (\mathcal{U}(\mathfrak{k}(A))\, v') \otimes \mathcal{U}(\mathfrak{k}(A))\, w_\Lambda = V \otimes W$$

which shows irreducibility of $V \otimes W$. □

Corollary 6.8. *Let $\mathcal{S}_{\frac{1}{2}}$ denote the generalized spin representation of $\mathfrak{k}(E_{10})$ from example 3.5 and let $\mathcal{S}_{\frac{3}{2}}$ and $\widetilde{\mathcal{S}}_{\frac{5}{2}}$ denote the higher spin representations of $\mathfrak{k}(E_{10})$ from thm. 3.19 and prop. 5.10. Then $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\widetilde{\mathcal{S}}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ are irreducible.*

For now it remains an open question how to adapt the above strategy to the case of $\mathcal{S}_{\frac{3}{2}} \otimes \widetilde{\mathcal{S}}_{\frac{5}{2}}$, where one does not have the result on the image on one of the factors. A closer look on the images could be worthwhile as there always exists a nondegenerate bilinear form w.r.t. which the representation matrices are skew-adjoint and I have the feeling that this fact could be exploited more. Also, one would have to take lemma 6.5 to the next level. It should be possible to derive polynomials that show how the action on the two pieces factors but the computation will probably be much longer than the above one. Another approach is to weaken the assumptions in that regard that one assumes the existence of a spherical subalgebra $\mathfrak{k}\left(\mathring{A}\right)$ such that V or W admit a multiplicity-free decomposition w.r.t. $\mathfrak{k}\left(\mathring{A}\right)(\mathbb{C})$. Together with a factorized action the above proof then works with slight modifications.

7 An infinite series of representations of maximal compact subalgebras of affine Kac-Moody-algebras

In this section I will present results that were obtained in joint work with A. Kleinschmidt, R. Köhl and H. Nicolai and that are currently in the process of being published, which is why I will refer to the preprint [KKLN21]. I will use the same terminology as in [KKLN21]. The proofs are essentially the ones in [KKLN21] although I occasionally decided to restructure or expand them.

Untwisted affine Kac-Moody algebras $\mathfrak{g}(A)$ admit a realization as the central extension of the loop algebra $\mathfrak{L}(\mathring{\mathfrak{g}}) \cong \mathfrak{L} \otimes_{\mathbb{K}} \mathring{\mathfrak{g}}$, where $\mathring{\mathfrak{g}}$ denotes the unique classical subalgebra of $\mathfrak{g}(A)$ (see def. 7.2 for details) and $\mathfrak{L}$ are the Laurent polynomials over $\mathbb{K}$. One can show without much effort, that $\mathfrak{k}(A)$ is contained in $\mathfrak{L}(\mathring{\mathfrak{g}})$ and furthermore has a vector space decomposition $\mathfrak{k}(A) \cong \mathfrak{L}^+ \otimes \mathring{\mathfrak{k}} \oplus \mathfrak{L}^- \otimes \mathring{\mathfrak{p}}$, where $\mathring{\mathfrak{g}} \cong \mathring{\mathfrak{k}} \oplus \mathring{\mathfrak{p}}$ is the Cartan decomposition of $\mathring{\mathfrak{g}}$. The composition is such that $\mathfrak{L}^+$ is a subring of $\mathfrak{L}$ and $\mathfrak{L}^-$ is a $\mathfrak{L}^+$-module, just as $\mathring{\mathfrak{k}}$ is a subalgebra of $\mathring{\mathfrak{g}}$ and $\mathring{\mathfrak{p}}$ is a $\mathring{\mathfrak{k}}$-module. Any homomorphism $\phi : \mathfrak{L} \to R$ of rings with involution will provide a homomorphism of Lie algebras $\rho : \mathfrak{L}^+ \otimes \mathring{\mathfrak{k}} \oplus \mathfrak{L}^- \otimes \mathring{\mathfrak{p}} \to R^+ \otimes \mathring{\mathfrak{k}} \oplus R^- \otimes \mathring{\mathfrak{p}}$. I will reproduce a result of [KKLN21] that there exist such homomorphism for $R = \mathbb{K}[[u]]$ and certain quotients of it (power series which are truncated at degree N). Afterwards, I will study the induced representation of $R^+ \otimes \mathring{\mathfrak{k}} \oplus R^- \otimes \mathring{\mathfrak{p}}$ of a $\mathring{\mathfrak{k}}$-module V. It will be crucial, that multiplication in R and its quotients behaves differently w.r.t. the degree than in $\mathfrak{L}$ which makes it possible analyze the action on the induced module and describe some invariant submodules. The main result (in [KKLN21] and in this section) is that there exist infinitely many reducible but not completely reducible representations of $\mathfrak{k}(A)$. The projective limit of this series of representations is shown to provide a faithful representation of $\mathfrak{k}(A)$.

7.1 Maximal compact subalgebras of untwisted affine Kac-Moody-algebras

Let A be a generalized Cartan matrix of affine type and $\Pi(A)$ its associated generalized Dynkin diagram. According to [K90, prop. 4.7], A is of affine type if and only if $\det A = 0$ and all proper principal minors of A are positive. This implies that any proper sub-diagram of $\Pi(A)$ is a union of generalized Dynkin diagrams of finite type but the converse is generally not true (a major source of counterexamples are rank 2 diagrams with non-spherical edges). The generalized Dynkin diagrams of affine type are classified in [K90, sec. 4.8] and a subclass, the untwisted affine ones, are given in table Aff 1 of [K90, sec. 4.8]. All of them can be obtained by adding an additional node to an existing generalized Dynkin diagram of finite type, which is often called affine extension.

Definition 7.1. (Current algebra[52]) Let R be a commutative, unital $\mathbb{K}$-algebra and let $\mathring{\mathfrak{g}}$ be a semi-simple finite dimensional Lie algebra over $\mathbb{K}$. Then $R \otimes_{\mathbb{K}} \mathring{\mathfrak{g}}$ with the Lie bracket

$$[a \otimes x, b \otimes y] := (ab) \otimes [x, y]_0$$

is called the **current algebra** of $\mathring{\mathfrak{g}}$ over R, where $[\cdot,\cdot]_0$ denotes the Lie bracket of $\mathring{\mathfrak{g}}$. Let $\mathfrak{L} := \mathbb{K}\left[t, t^{-1}\right]$ denote the Laurent polynomials over $\mathbb{K}$, then the current algebra of $\mathring{\mathfrak{g}}$ over $\mathfrak{L}$ is called the **loop algebra** of $\mathring{\mathfrak{g}}$ which will be denoted by $\mathfrak{L}(\mathring{\mathfrak{g}})$. For $\mathring{\mathfrak{g}}$ and any of its subalgebras, $\mathring{\mathfrak{g}} \subset \mathfrak{L}(\mathring{\mathfrak{g}})$ always refers to the canonical inclusion $1 \otimes \mathring{\mathfrak{g}} \subset \mathfrak{L}(\mathring{\mathfrak{g}})$.

Typically, I will denote the loop parameter by t, i.e., I consider $\mathfrak{L}$ as the ring of Laurent polynomials in the variable t, which is rather common (cp. for instance [K90, sec. 7]). The untwisted affine Kac-Moody algebras admit a rather explicit description in terms of the loops algebra over a distinguished finite-dimensional simple subalgebra $\mathring{\mathfrak{g}}$ together with an enlargement of the Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$ by elements that are usually denoted by K and d.

Definition 7.2. (Affine extension) Let $\mathring{\mathfrak{g}}$ be a simple finite dimensional Lie algebra over $\mathbb{K}$ and $\mathfrak{L}(\mathring{\mathfrak{g}})$ its loop algebra. Let ψ be a $\mathbb{K}$-valued 2-cocycle on $\mathfrak{L}(\mathring{\mathfrak{g}})$ and let $\mathfrak{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{K} \cdot K$ denote the universal central extension of $\mathfrak{L}(\mathring{\mathfrak{g}})$ by a one dimensional center, which is spanned by K, w.r.t. the 2-cocycle ψ. Now let d denote the derivation on $\mathfrak{L}(\mathring{\mathfrak{g}})$ given by $t \cdot \frac{d}{dt}$ and set

$$\widehat{\mathfrak{L}}(\mathring{\mathfrak{g}}) := \mathfrak{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{K} \cdot K \oplus \mathbb{K} \cdot d,$$

where the bracket is defined as

$$[x + a_1 K + b_1 d, y + a_2 K + b_2 d] = [x, y] + b_1 d(y) - b_2 d(x) + \psi(x, y) K \ \forall\, x, y \in \mathring{\mathfrak{g}},\ a_1, a_2, b_1, b_2 \in \mathbb{K}. \tag{115}$$

Call $\widehat{\mathfrak{L}}(\mathring{\mathfrak{g}})$ the **affine extension** of $\mathring{\mathfrak{g}}$.

While the above description is rather explicit as soon as one fixes a 2-cocycle ψ, the question remains how this description relates to the constructive definition of KM-algebras given in def. 1.5.

Proposition 7.3. *(This is [K90, thm. 7.4] together with [K90, sec. 7.6])*
Let $\mathring{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra over $\mathbb{K}$. To its Cartan matrix $\mathring{A} \in \mathbb{Z}^{n \times n}$ associate the so-called extended Cartan matrix A by setting $A_{ij} := \beta_j\left(\beta_i^\vee\right)$ for all $i, j = 0, \ldots, n$ with $\beta_0 := -\theta$, $\beta_0^\vee := -\theta^\vee$, where θ denotes the highest root of the root system of $\mathring{\mathfrak{g}}$ and $\beta_i := \alpha_i$, $\beta_i^\vee := \alpha_i^\vee$ $\forall\, i = 1, \ldots, n$. Then the affine

[52] The term seems to originate from particle physics in the 1960s and has made its way into mathematics since, where it is usually understood in the way of this definition.

extension $\widehat{\mathfrak{L}}(\mathring{\mathfrak{g}})$ from def. 7.2 is isomorphic to the untwisted affine Kac-Moody algebra $\mathfrak{g}(A)(\mathbb{K})$ constructed as in def. 1.5.

The Cartan subalgebra of $\mathfrak{g}(A)(\mathbb{K})$ is given by[53] $\mathfrak{h} = \mathring{\mathfrak{h}} + \mathbb{K}\cdot K + \mathbb{K}\cdot d$ *and the additional simple (co-)root is $\alpha_0 := \delta - \theta$ (resp. $\alpha_0^\vee := \frac{2}{(\theta|\theta)}K - 1\otimes\theta^\vee$), where δ is defined via $\delta(d) = 1$ and $\delta(h) = 0\ \forall\, h \in \mathring{\mathfrak{h}}\cup\mathbb{K}\cdot K$. Let E_i, F_i for $i = 1, \ldots, n$ denote the Chevalley generators of $\mathring{\mathfrak{g}}$ and let $F_0 \in \mathring{\mathfrak{g}}_\theta$ be normalized s.t. $(F_0|\mathring{\omega}(F_0)) = -\frac{2}{(\theta|\theta)}$, where $\mathring{\omega}$ denotes the Chevalley involution of $\mathring{\mathfrak{g}}$. Set $E_0 := -\mathring{\omega}(F_0)$, then the Chevalley generators of $\mathfrak{g}(A)(\mathbb{K})$ are given by*

$$e_0 = t\otimes E_0,\ e_i = 1\otimes E_i,\ f_0 = t^{-1}\otimes F_0,\ f_i = 1\otimes F_i\ \forall\, i = 1, \ldots, n.$$

The Chevalley involution ω on $\mathfrak{g}(A)(\mathbb{K}) \cong \widehat{\mathfrak{L}}(\mathring{\mathfrak{g}})$ is given by

$$\omega\left(q(t)\otimes x + a\cdot K + b\cdot d\right) = q\left(t^{-1}\right)\otimes\mathring{\omega}(x) - a\cdot K - b\cdot d\ \forall\, q\in\mathfrak{L}, x\in\mathring{\mathfrak{g}},\ a, b\in\mathbb{K}. \tag{116}$$

Corollary 7.4. *(Cp. [KKLN21, eq. (2.6)]) If A is a GCM of untwisted affine type, the maximal compact subalgebra $\mathfrak{k}(A)(\mathbb{K})$ is contained in the loop algebra $\mathfrak{L}(\mathring{\mathfrak{g}}) \subset \widehat{\mathfrak{L}}(\mathring{\mathfrak{g}})$. Denote by $\eta : \mathfrak{L}\to\mathfrak{L}$ the involution determined by $t^n \mapsto (-1)^n t^n\ \forall\, n\in\mathbb{N}_0$ and denote its ± 1 eigenspaces by $\mathfrak{L}_\pm$. Then*

$$\mathfrak{k}(A)(\mathbb{K}) \cong \mathfrak{L}_+\otimes\mathring{\mathfrak{k}} \oplus \mathfrak{L}_-\otimes\mathring{\mathfrak{p}}$$

as vector spaces, where $\mathring{\mathfrak{g}} = \mathring{\mathfrak{k}}\oplus\mathring{\mathfrak{p}}$ denotes the Cartan decomposition of $\mathring{\mathfrak{g}}$.

Proof. From prop. 7.3 and in particular eq. (116) it follows that $K, d \notin \mathfrak{k}(A)(\mathbb{K})$. Recall that the Chevalley involution on $\mathring{\mathfrak{g}}$ was denoted by $\mathring{\omega}$ and check that $\omega_{|\mathfrak{L}(\mathring{\mathfrak{g}})} = \eta\otimes\mathring{\omega}$ which immediately implies that the $+1$ eigenspace of ω on $\mathfrak{L}\otimes\mathring{\mathfrak{g}}$ is equal to $(\mathfrak{L}_+\otimes\mathring{\mathfrak{g}}_+)\oplus(\mathfrak{L}_-\otimes\mathring{\mathfrak{g}}_-)$, where the $\pm$ denote the respective ± 1 eigenspaces w.r.t. η and $\mathring{\omega}$. But the ± 1 eigenspaces of $\mathring{\mathfrak{g}}$ w.r.t. $\mathring{\omega}$ are exactly $\mathring{\mathfrak{k}}$ and $\mathring{\mathfrak{p}}$ from the Cartan decomposition of $\mathring{\mathfrak{g}}$. □

7.2 R-models of maximal compact subalgebras

If one wants to view $\mathfrak{k}(A)(\mathbb{K}) \cong \mathfrak{L}_+\otimes\mathring{\mathfrak{k}}\oplus\mathfrak{L}_-\otimes\mathring{\mathfrak{p}}$ from a more abstract perspective, one can replace $\mathfrak{L}$ by any ring with involution. In [KKLN21] the ring of formal power series in a single variable u, the ring of polynomials in u and quotients thereof are considered and they are referred to as so-called parabolic models $\mathfrak{N}(\mathbb{K}[[u]])$, $\mathfrak{N}(\mathbb{K}[u])$ and $\mathfrak{N}(\mathbb{P}_N)$ of $\mathfrak{k}(A)(\mathbb{K})$. Here, I use a slightly different notation and name for this object, as I would like to display the ring in question as well as the finite-dimensional root datum from $\mathring{A}$ more explicitly.

Definition 7.5. (Cp. [KKLN21, eq. 3.1, rem. 13]) Let R be a unital, associative $\mathbb{K}$-algebra with involution η, whose ± 1 eigenspaces are denoted by $R_\pm$ and let $\mathring{\mathfrak{g}} = \mathring{\mathfrak{k}}\oplus\mathring{\mathfrak{p}}$ be the Cartan decomposition of a finite-dimensional, simple split Lie algebra $\mathring{\mathfrak{g}} = \mathfrak{g}\left(\mathring{A}\right)(\mathbb{K})$. Denote the extended Cartan matrix of $\mathring{A}$ by A and define a subalgebra

$$\mathfrak{N}\left(R, \mathring{A}\right) := R_+\otimes\mathring{\mathfrak{k}}\oplus R_-\otimes\mathring{\mathfrak{p}}$$

of the current algebra of $\mathring{\mathfrak{g}}$ over R. Call $\mathfrak{N}\left(R, \mathring{A}\right)$ the R-model of $\mathfrak{k}(A)(\mathbb{K})$.

[53] One does not write this as a direct sum as in def. 7.2 because there it is meant as a direct sum of vector spaces without respect to any bilinear form but if one puts a standard invariant form on $\mathfrak{g}(A)$ one sees that w.r.t. this form K and d are not orthogonal to each other, although both of them are orthogonal to $\mathfrak{L}(\mathring{\mathfrak{g}})$.

In the sequel R-models of $\mathbb{K}$ with R such that there exist homomorphisms $\mathfrak{L} \to R$ that respect the involution will become important.

Lemma 7.6. *Let R_1, R_2 be unital commutative $\mathbb{K}$-algebras with involution such that there exists a homomorphism $\phi : R_1 \to R_2$. Then ϕ defines a homomorphism of Lie algebras*

$$\rho : \mathfrak{N}\left(R_1, \mathring{A}\right) \to \mathfrak{N}\left(R_2, \mathring{A}\right),\ \rho\left(p \otimes x\right) = \phi(p) \otimes x \ \forall p \in R_1,\ x \in \mathring{\mathfrak{g}}.$$

Remark. Since $\mathfrak{k}\left(A\right)\left(\mathbb{K}\right)$ is isomorphic to its $\mathfrak{L}$-model this result applies whenever $R_1 = \mathfrak{L}$, where $\mathfrak{L}$ denotes the Laurent polynomials over $\mathbb{K}$.

Proof. Let $p_1, p_2 \in R_1^+$, $q_1, q_2 \in R_1^-$, $x_1, x_2 \in \mathring{\mathfrak{k}}$, $y_1, y_2 \in \mathring{\mathfrak{p}}$, then

$$\left[p_1 \otimes x_1 + q_1 \otimes y_1, p_2 \otimes x_2 + q_2 \otimes y_2\right] = p_1p_2 \otimes \left[x_1, x_2\right] + q_1q_2 \otimes \left[y_1, y_2\right] + p_1q_2 \otimes \left[x_1, y_2\right] - q_1p_2 \otimes \left[x_2, y_1\right].$$

Now

$$\rho\left(r.h.s.\right) \;=\; \phi\left(p_1p_2\right) \otimes \left[x_1, x_2\right] + \phi\left(q_1q_2\right) \otimes \left[y_1, y_2\right] + \phi\left(p_1q_2\right) \otimes \left[x_1, y_2\right] - \phi\left(q_1p_2\right) \otimes \left[x_2, y_1\right]$$

and

$$\begin{aligned}\rho\left(l.h.s.\right) &= \left[\rho\left(p_1 \otimes x_1 + q_1 \otimes y_1\right), \rho\left(p_2 \otimes x_2 + q_2 \otimes y_2\right)\right]\\ &= \phi\left(p_1\right)\phi\left(p_2\right) \otimes \left[x_1, x_2\right] + \phi\left(q_1\right)\phi\left(q_2\right) \otimes \left[y_1, y_2\right]\\ &\quad + \phi\left(p_1\right)\phi\left(q_2\right) \otimes \left[x_1, y_2\right] - \phi\left(q_1\right)\phi\left(p_2\right) \otimes \left[x_2, y_1\right]\\ &= \rho\left(r.h.s.\right)\end{aligned}$$

because ϕ is a homomorphism of commutative $\mathbb{K}$-algebras. □

In order to determine if a given $\mathbb{K}$-linear map $\phi : \mathfrak{L} \to R$ is a homomorphism it is useful to spell out the multiplication of the basis elements of $\mathfrak{L} = \mathfrak{L}_+ \oplus \mathfrak{L}_-$. Since $\mathfrak{L}_\pm = \mathrm{span}_{\mathbb{K}}\left\{t^n \pm t^{-n} \,|\, n \in \mathbb{N}\right\}$ one has

$$\begin{aligned}\left(t^n + t^{-n}\right)\left(t^m + t^{-m}\right) &= t^{n+m} + t^{-(n+m)} + t^{n-m} + t^{m-n}\\ \left(t^n + t^{-n}\right)\left(t^m - t^{-m}\right) &= t^{n+m} - t^{-(n+m)} - \mathrm{sgn}\left(n-m\right)\left(t^{|n-m|} - t^{-|n-m|}\right)\\ \left(t^n - t^{-n}\right)\left(t^m - t^{-m}\right) &= t^{n+m} + t^{-(n+m)} - \left(t^{n-m} + t^{-(n-m)}\right).\end{aligned}$$

Lemma 7.7. *(Cp. [KKLN21, lem. 3]) Let $\mathbb{P} := \mathbb{K}\left[\left[u\right]\right]$ denote the algebra of formal power series with coefficients in $\mathbb{K}$. The coefficients $\left(\pm 1\right)^n a_{2N}^{(n)}$ and $\left(\pm 1\right)^n a_{2N+1}^{(n)}$ with*

$$a_{2N}^{(n)} = 2 \cdot \sum_{k=0}^{n} \binom{2n}{2k}\binom{N-k+n-1}{N-k}, \quad a_{2N+1}^{(n)} = -2 \cdot \sum_{k=0}^{n-1} \binom{2n}{2k+1}\binom{N-k+n-1}{N-k} \tag{117}$$

define two homomorphisms of $\mathbb{K}$-algebras $\phi : \mathfrak{L} \to \mathbb{P}$ by linear extension of

$$\left(t^n + t^{-n}\right) \mapsto \left(\pm 1\right)^n \sum_{k=0}^{\infty} a_{2k}^{(n)} u^{2k}, \quad \left(t^n - t^{-n}\right) \mapsto \left(\pm 1\right)^n \sum_{k=0}^{\infty} a_{2k+1}^{(n)} u^{2k+1}. \tag{118}$$

The map $\sum_{k=0}^{\infty} c_k u^k \mapsto \sum_{k=0}^{\infty} (-1)^k c_k u^k$ defines an involution on $\mathbb{P}$ such that $\phi\left(\mathfrak{L}_\pm\right) \subset \mathbb{P}_\pm$, where $\mathbb{P}_\pm$ are the ± 1 eigenspaces of $\mathbb{P}$, which are the even and odd formal power series respectively.

Proof. The above coefficients arise as the coefficients of Taylor series of the meromorphic functions $f_\pm^{(n)}$: $z \mapsto z^n \pm z^{-n}$ after application of a Möbius transformation. The Möbius transformations map the Riemann sphere $\mathbb{C}^\infty$ onto itself and define automorphisms of the algebra of meromorphic functions $\mathcal{M}(\mathbb{C}^\infty)$. Now the Taylor series $T\left(\widetilde{f}_\pm^{(n)} \cdot \widetilde{f}_\pm^{(m)}\right)$ of the transformed functions $\widetilde{f}_\pm^{(n)}, \widetilde{f}_\pm^{(m)}$ coincides with the product of the individual Taylor series $T\left(\widetilde{f}_\pm^{(n)}\right) \cdot T\left(\widetilde{f}_\pm^{(m)}\right)$ if the radius of convergence for all involved series is nonzero. As the product on power series is given by convolution, this provides a homomorphism of algebras. So the only thing to show is that the coefficients in (117) indeed arise as the Taylor coefficients of the transformed functions $\widetilde{f}_\pm^{(n)}$.

I will use the Möbius transformations

$$m_1 : z \mapsto \frac{1-z}{1+z}, \quad m_1^{-1} = m_1$$

and

$$m_2 : z \mapsto \frac{1+z}{1-z}, \quad m_2^{-1} : z \mapsto -\frac{1-z}{1+z}.$$

One has

$$\begin{aligned}
f_\pm^{(n)} \circ m_1^{-1}(z) &= \left(\frac{1-z}{1+z}\right)^n \pm \left(\frac{1+z}{1-z}\right)^n = \frac{(1-z)^{2n} \pm (1+z)^{2n}}{(1-z^2)^n} \\
&= \left(1-z^2\right)^{-n} \cdot \sum_{k=0}^{2n} \binom{2n}{k} \left[(-z)^k \pm z^k\right] \\
&= \left(1-z^2\right)^{-n} \cdot \begin{cases} 2\sum_{k=0}^{n} \binom{2n}{2k} z^{2k} & \text{for } f_+^{(n)} \circ m_1^{-1} \\ -2\sum_{k=0}^{n-1} \binom{2n}{2k+1} z^{2k+1} & \text{for } f_-^{(n)} \circ m_1^{-1} \end{cases}
\end{aligned}$$

Expand $\left(1-z^2\right)^{-n}$ into a power series which is convergent for $|z| < 1$:

$$(1-x)^{-n} = \sum_{k=n-1}^{\infty} \binom{k}{k-n+1} x^{k-n+1} = \sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k$$

$$\Rightarrow \left(1-z^2\right)^{-n} = \sum_{k=0}^{\infty} \binom{k+n-1}{k} z^{2k}.$$

Therefore

$$\begin{aligned}
f_+^{(n)} \circ m_1^{-1}(z) &= 2\sum_{l=0}^{\infty}\binom{l+n-1}{l}\sum_{k=0}^{n}\binom{2n}{2k}z^{2k+2l} \\
&= 2\sum_{l=0}^{\infty}\sum_{k=0}^{n}\binom{l+n-1}{l}\binom{2n}{2k}z^{2k+2l} \\
&\overset{N=l+k}{=} 2\sum_{N=0}^{\infty}\sum_{k=0}^{\min\{n,N\}}\binom{N-k+n-1}{N-k}\binom{2n}{2k}z^{2N} \\
&= 2\sum_{N=0}^{\infty}\sum_{k=0}^{n}\binom{N-k+n-1}{N-k}\binom{2n}{2k}z^{2N} \\
&= \sum_{N=0}^{\infty}a_{2N}^{(n)}z^{2N} \;\forall\,|z|<1
\end{aligned}$$

with $\binom{n}{k}=0$ for $k<0$ in the penultimate step and

$$\begin{aligned}
f_-^{(n)} \circ m_1^{-1}(z) &= -2\sum_{l=0}^{\infty}\binom{l+n-1}{l}\sum_{k=0}^{n-1}\binom{2n}{2k+1}z^{2k+2l+1} \\
&= -2\sum_{l=0}^{\infty}\sum_{k=0}^{n-1}\binom{l+n-1}{l}\binom{2n}{2k+1}z^{2k+2l+1} \\
&\overset{N=l+k}{=} -2\sum_{N=0}^{\infty}\sum_{k=0}^{\min\{n-1,N\}}\binom{N-k+n-1}{N-k}\binom{2n}{2k+1}z^{2N+1} \\
&= -2\sum_{N=0}^{\infty}\sum_{k=0}^{\min\{n-1,N\}}\binom{N-k+n-1}{N-k}\binom{2n}{2k+1}z^{2N+1} \\
&= \sum_{N=0}^{\infty}a_{2N+1}^{(n)}z^{2N+1} \;\forall\,|z|<1.
\end{aligned}$$

The same computation goes through with m_2^{-1} instead of m_1^{-1}. From

$$f_\pm^{(n)} \circ m_2^{-1}(z) = (-1)^n\left(\frac{1-z}{1+z}\right)^n \pm (-1)^n\left(\frac{1+z}{1-z}\right)^n = (-1)^n\, f_\pm^{(n)} \circ m_1^{-1}(z)$$

one deduces that $(-1)^n a_{2N}^{(n)}$ and $(-1)^n a_{2N+1}^{(n)}$ are the coefficients corresponding to m_2. $\square$

Proposition 7.8. *(Cp. [KKLN21, prop. 5]) Let A be of untwisted affine type, denote by $\mathring{A}$ the unique Cartan matrix whose extended Cartan matrix is A and set $\mathbb{P} := \mathbb{K}[[u]]$. Let $a_{2k}^{(n)}$, $a_{2k+1}^{(n)}$ be as in eq. (117) and define $\rho_\pm : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}, \mathring{A}\right)$ as the linear extension of*

$$\left(t^n + t^{-n}\right)\otimes x \mapsto (\pm 1)^n \sum_{k=0}^{\infty} a_{2k}^{(n)}u^{2k}\otimes x \quad \forall\, x\in\mathring{\mathfrak{k}}, \tag{119}$$

$$\left(t^n - t^{-n}\right) \otimes y \mapsto (\pm 1)^n \sum_{k=0}^{\infty} a_{2k+1}^{(n)} u^{2k+1} \otimes y \quad \forall\, y \in \mathring{\mathfrak{p}}. \tag{120}$$

Then $\rho_\pm$ is a homomorphism of Lie algebras.

Proof. This follows immediately from lemmas 7.6 and 7.7. ☐

Proposition 7.9. *(Cp. [KKLN21, cor. 7]) Let A and $\mathbb{P}$ be as in prop. 7.8, let $\mathfrak{i}_N$ denote the ideal in $\mathbb{P}$ that is generated by the element $u^{N+1} \equiv \sum_{n=0}^{\infty} \delta_{n,N+1} u^n$ and set $\mathbb{P}_N := \mathbb{K}\,[[u]] \diagup \mathfrak{i}_N$. Then the quotient map $\pi_N : \mathbb{P} \mapsto \mathbb{P}_N$ induces homomorphisms $\rho_\pm^{(N)} : \mathfrak{k}\,(A)\,(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ via $\rho_\pm^{(N)} := \pi_N \circ \rho_\pm$, where $\rho_\pm$ is given in prop. 7.8.*

Proof. The quotient map $\pi_N : \mathbb{P} \to \mathbb{P}_N$ is a homomorphism of $\mathbb{K}$-algebras which by lemma 7.6 implies a homomorphism of the R-models $\mathfrak{N}\left(\mathbb{P}, \mathring{A}\right) \to \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$. By prop. 7.8 one has a homomorphism $\rho_\pm$: $\mathfrak{k}\,(A)\,(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}, \mathring{A}\right)$, so set $\rho_\pm^{(N)} := \pi_N \circ \rho_\pm$. ☐

The next lemma will be needed to show that $\rho_\pm$ is injective but not surjective.

Lemma 7.10. *(This is [KKLN21, lem. 8]) To each $m \in \mathbb{N}$ and linearly independent $q_1, \ldots, q_m \in \mathbb{K}[u]$ there exist $N_1, \ldots, N_m \in \mathbb{N}$ such that the "evaluation matrix" $\mathcal{E}\,(N_1, \ldots, N_m) := (q_i\,(N_j))_{i,j=1}^m$ is of full rank and therefore invertible.*

Proof. Use induction on n. For $n = 1$ the matrix $\mathcal{E} = (q_1\,(N_1))$ is regular iff $q_1\,(N_1) \neq 0$. Since $q_1(u) = 0$ only for finitely many $u \in \mathbb{K}$ there exist infinitely many $N \in \mathbb{N}$ with that property. Now assume that $N_1, \ldots, N_n$ are chosen such that $\mathcal{E}^{(n)}\,(N_1, \ldots, N_n)$ is regular. Consider the $(n+1) \times (n+1)$-matrix

$$\mathcal{E}^{(n+1)}(u) = \begin{pmatrix} q_1\,(N_1) & q_1\,(N_2) & \cdots & q_1\,(N_n) & q_1\,(u) \\ q_2\,(N_1) & \ddots & & & \\ \vdots & & \ddots & & \\ q_n\,(N_1) & & & q_n\,(N_n) & q_n\,(u) \\ q_{n+1}\,(N_1) & \cdots & & q_{n+1}\,(N_n) & q_{n+1}\,(u) \end{pmatrix}$$

and denote by $\mathcal{E}_{i,j}^{(n+1)}$ its minors with the i-th row and j-th column removed. Then

$$\begin{aligned} \det \mathcal{E}^{(n+1)}(u) \quad &= \quad \left|\mathcal{E}^{(n)}\,(N_1, \ldots, N_n)\right| q_{n+1}\,(u) - q_n\,(u) \cdot \left|\mathcal{E}_{n,n+1}^{(n+1)}\,(N_1, \ldots, N_n)\right| \\ &\qquad + \cdots + (-1)^n q_1\,(u) \cdot \left|\mathcal{E}_{1,n+1}^{(n+1)}\,(N_1, \ldots, N_n)\right|, \end{aligned}$$

where I use the notation $\mathcal{E}_{i,n+1}^{(n+1)}\,(N_1, \ldots, N_n)$ to indicate that these minors do not depend on u. It follows that

$$\det \mathcal{E}^{(n+1)}(u) = \sum_{j=1}^{n+1} c_j q_j(u)$$

with $c_{n+1} \neq 0$ because by the induction hypothesis $\left|\mathcal{E}^{(n)}\,(N_1, \ldots, N_n)\right| \neq 0$. As $q_1, \ldots, q_{n+1}$ are linearly independent the above polynomial is nonzero and therefore there exist only finitely many $u \in \mathbb{K}$ such that $\det \mathcal{E}^{(n+1)}(u) = 0$. Thus, there exist infinitely many $u \in \mathbb{N}$ such that $\mathcal{E}^{(n+1)}(u)$ is regular. This proves the claim by induction on n. ☐

Proposition 7.11. *(Cp. [KKLN21, prop. 9]) Let A and $\mathbb{P}$ be as in prop. 7.8, then $\rho_\pm : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}, \mathring{A}\right)$ from prop. 7.8 is injective. Furthermore,*

$$\left\{ \sum_{N=0}^{\infty} c_{2N} u^{2n} \otimes x + \sum_{N=0}^{\infty} c_{2N+1} u^{2n+1} \otimes y \mid |\{N \mid c_N \neq 0\}| < \infty,\ x \in \mathring{\mathfrak{k}},\ y \in \mathring{\mathfrak{p}} \right\} \cap im\ \rho_\pm = \{1\} \otimes \mathring{\mathfrak{k}}.$$

Proof. Note that $\mathfrak{L}^+ \otimes \mathring{\mathfrak{k}}$ and $\mathfrak{L}^- \otimes \mathring{\mathfrak{p}}$ are mapped to different subspaces of $\mathfrak{N}\left(\mathbb{P}, \mathring{A}\right)$ under $\rho_\pm$ and therefore one can analyze their images separately. Consider elements of the form

$$\chi := \sum_{i=1}^{K} \left(t^{n_i} + t^{-n_i}\right) \otimes x_i,\ \ \gamma := \sum_{i=1}^{K} \left(t^{n_i} - t^{-n_i}\right) \otimes y_i$$

for $x_i \in \mathring{\mathfrak{k}}$ and $y_i \in \mathring{\mathfrak{p}}$. One has

$$\begin{aligned} \rho_\pm(\chi) &= \sum_{i=1}^{K} \sum_{N=0}^{\infty} (\pm 1)^{n_i} a_{2N}^{(n_i)} u^{2N} \otimes x_i = \sum_{N=0}^{\infty} \sum_{i=1}^{K} (\pm 1)^{n_i} a_{2N}^{(n_i)} u^{2N} \otimes x_i = 0 \\ \Leftrightarrow\ 0 &= \sum_{i=1}^{K} (\pm 1)^{n_i} a_{2N}^{(n_i)} x_i\ \forall N \in \mathbb{N}_0. \end{aligned}$$

Spelling this out in a basis of $\mathring{\mathfrak{k}}$ shows that the above equation has nontrivial solutions if and only if

$$\sum_{i=1}^{K} (\pm 1)^{n_i} a_{2N}^{(n_i)} z_i = 0\ \forall N \geq 0 \tag{121}$$

does, where now z_i is a $\mathbb{K}$-valued indeterminate. It is (cp. eq. 117)

$$a_{2N}^{(n)} = 2 \sum_{k=0}^{n} \binom{2n}{2k} \binom{n+N-k-1}{N-k}$$

and so $a_{2N}^{(n)}$ is given by the evaluation of $p_n \in \mathbb{K}[x]$ at $x = N$ s.t. $\deg(p_n) = n-1$. If the $n_1, \ldots, n_K$ are pairwise distinct, the $p_{n_1}, \ldots, p_{n_K}$ are each of different degree and therefore linearly independent. For $N_1, \ldots, N_K \in \mathbb{N}$ one has a finite subsystem of linear equations of (121)

$$\begin{aligned} \sum_{i=1}^{K} a_{2k}^{(n_i)} z_i &= 0\ \forall k \in \{N_1, \ldots, N_K\} \\ \Leftrightarrow \sum_{i=1}^{K} p_{n_i}(k) z_i &= 0\ \forall k \in \{N_1, \ldots, N_K\} \end{aligned} \tag{122}$$

and according to lemma (7.10) there exist $N_1, \ldots, N_K$ such that there exists an inverse to $\left(p_{n_i}(N_j)\right)_{i,j=1}^{K}$. This shows that (122), hence also (121), only admits the trivial solution $z_i = 0$ for all i. As the $a_{2N+1}^{(n)}$ are also polynomials in N of degree $n-1$, exactly the same computation shows that $\rho_\pm(\gamma) = 0$ for $\gamma \in \mathfrak{L}_- \otimes \mathring{\mathfrak{p}}$ if and only if $\gamma = 0$ and therefore $\rho_\pm$ is injective.

This only leaves to determine which elements in $\mathfrak{N}\left(\mathbb{P}, \mathring{A}\right)$ whose power series factor is a finite power series are contained in the image of $\rho_\pm$ and which are not. As $\mathfrak{N}\left(\mathbb{P}, \mathring{A}\right) \cong \left(\mathbb{P}_+ \otimes \mathring{\mathfrak{k}}\right) \oplus (\mathbb{P}_- \otimes \mathring{\mathfrak{p}})$ as vector spaces, this decomposition holds for $\mathrm{im}\rho_\pm$ as well and one can restrict the analysis to the case of $\mathbb{P}_+ \otimes \mathring{\mathfrak{k}}$. Towards this, let $X := \sum_j \sum_{N\geq 0} b_{2N} u^{2N} \otimes x_j \in \mathbb{P}_+ \otimes \mathring{\mathfrak{k}}$ be s.t. $b_{2N} \neq 0$ for only finitely many $N \in \mathbb{N}$. If $X \in \mathrm{im}\rho_\pm$ there exists $\chi := \sum_{i=1}^K c_i \left(t^{n_i} + t^{-n_i}\right) \otimes x_i$ s.t.

$$\rho_\pm(\chi) = \sum_{i=1}^K \sum_{N=0}^\infty c_i (\pm 1)^{n_i} a_{2N}^{(n_i)} u^{2N} \otimes x_i = \sum_{N=0}^\infty \sum_{i=1}^K c_i (\pm 1)^{n_i} a_{2N}^{(n_i)} u^{2N} \otimes x_i = \sum_{N\geq 0} \sum_j b_{2N} u^{2N} \otimes x_j,$$

where the n_i can be chosen such that they are pairwise distinct. Let $K_0 := \max_N \{N \mid b_{2N} \neq 0\}$and assume $K_0 \geq 1$, then

$$\sum_{i=1}^K c_i (\pm 1)^{n_i} a_{2N}^{(n_i)} = 0 \; \forall N > K_0.$$

But as $a_{2N}^{(n_i)} = p_i(N)$ with p_i of degree $n_i - 1$ this implies

$$\sum_{i=1}^K c_i (\pm 1)^{n_i} p_i(N) = 0 \; \forall N > K_0$$

which is a contradiction to the fact that $\sum_{j=1}^N c_j (\pm 1)^{n_i} p_i$ can be equal to 0 only at finitely many points (the polynomial is nonzero, because the p_i have different degree). If however $K_0 = 0$, then $\sum_j \sum_{N\geq 0} b_{2N} u^{2N} \otimes x_j = b_0 \cdot 1 \otimes x_0 \in \mathfrak{N}\left(\mathbb{P}, \mathring{A}\right)$ for $x_0 \in \mathring{\mathfrak{k}}$ and since $a_{2N}^{(0)} = 2\delta_{N,0}$ one has that $1 \otimes \mathring{\mathfrak{k}} \subset \mathrm{im}\rho_\pm$ as one can pick $\chi = \frac{b_0}{2} \cdot 1 \otimes x \in \mathfrak{k}(A)(\mathbb{K})$ to obtain $b_0 \cdot 1 \otimes x_0 = \rho_\pm(\chi) = b_0 \cdot 1 \otimes x_0$. $\square$

The Lie algebra $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ is graded by $\mathbb{Z}/(N+1)\mathbb{Z}$ as it inherits the grading of $\mathbb{P}_N$. One has the graded decomposition

$$\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right) \cong \bigoplus_{k=0}^{\lfloor N/2 \rfloor} \mathring{\mathfrak{k}}_{(2k)} \oplus \bigoplus_{k=0}^{\lfloor (N-1)/2 \rfloor} \mathring{\mathfrak{p}}_{(2k+1)} \tag{123}$$

as in [KKLN21, eq. 3.19], where

$$\mathring{\mathfrak{k}}_{(2k)} := \mathrm{span}_{\mathbb{K}} \left\{ \left(u^{2k} + \mathfrak{i}_N\right) \otimes x \mid x \in \mathring{\mathfrak{k}} \right\}, \; \mathring{\mathfrak{p}}_{(2k+1)} := \mathrm{span}_{\mathbb{K}} \left\{ \left(u^{2k+1} + \mathfrak{i}_N\right) \otimes y \mid y \in \mathring{\mathfrak{p}} \right\} \tag{124}$$

with $\mathfrak{i}_N := \left(u^{N+1}\right)$.

Proposition 7.12. *(Cp. [KKLN21, prop. 11])Let A , $\mathbb{P}_N$ and $\rho_\pm^{(N)} : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ be as in prop. 7.9, then $\rho_\pm^{(N)}$ is surjective.*

Proof. Since I assume $\mathring{\mathfrak{g}}$ to be simple and non-compact, one can apply [HN12, 13.1.10] which yields that $\mathring{\mathfrak{k}} = [\mathring{\mathfrak{p}}, \mathring{\mathfrak{p}}]$ and that $\mathring{\mathfrak{p}}$ is a simple $\mathring{\mathfrak{k}}$-module. Thus, for any $x \in \mathring{\mathfrak{k}}$ or $y \in \mathring{\mathfrak{p}}$ there exist $y_1, \ldots, y_L \in \mathring{\mathfrak{p}}$

s.t. $x = [y_1, [y_2, \ldots, [\ldots, y_L]]]$ and $z_1, \ldots, z_M \in \mathring{\mathfrak{p}}$ s.t. $y = [z_1, [z_2, \ldots, [\ldots, z_M]]]$. If $L = \lfloor N/2 \rfloor$ and $M = \lfloor (N-1)/2 \rfloor$ one has

$$\prod_{j=1}^{L,M} \left(\sum_{k=0}^{\lfloor (N-1)/2 \rfloor} a_{2k+1}^{(n)} \left(u^{2k+1} + \mathfrak{i}_N \right) \right) = \left(\prod_{j=1}^{N} a_1^{(n)} \right) \cdot u^N + \mathfrak{i}_N .$$

As $y_l^{(n)} := (t^n - t^{-n}) \otimes y_l$ for $y_l \in \mathring{\mathfrak{p}}$ is mapped to $\bigoplus_{k=0}^{\lfloor (N-1)/2 \rfloor} \mathring{\mathfrak{p}}_{(2k+1)}$ under $\rho_{\pm}^{(N)}$ one obtains

$$\begin{aligned} \rho_{\pm}^{(N)} \left(\left[y_1^{(1)}, \left[y_2^{(1)}, \ldots, \left[\ldots, y_L^{(1)} \right] \right] \right] \right) &= \left(\prod_{j=1}^{N} a_1^{(1)} \right) \left(u^N + \mathfrak{i}_N \right) \otimes x \in \mathring{\mathfrak{k}}_{(L)}, \\ \rho_{\pm}^{(N)} \left(\left[z_1^{(1)}, \left[z_2^{(1)}, \ldots, \left[\ldots, z_M^{(1)} \right] \right] \right] \right) &= \left(\prod_{j=1}^{N} a_1^{(1)} \right) \left(u^N + \mathfrak{i}_N \right) \otimes y \in \mathring{\mathfrak{p}}_{(M)}, \end{aligned}$$

so that the highest homogeneous components $\mathring{\mathfrak{k}}_{(L)}$ and $\mathring{\mathfrak{p}}_{(M)}$ (cp. eqs. 123 and 124) are contained in $\mathrm{im}\rho_{\pm}^{(N)}$. One can now peel off the remaining homogeneous components successively which shows the claim by induction. □

Proposition 7.13. *(Cp. [KKLN21, prop. 12]) Let A , $\mathbb{P}_N$ and $\rho_{\pm}^{(N)} : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ be as in prop. 7.9 and $\overset{\circ}{\mathfrak{k}}_{(2k)}$, $\overset{\circ}{\mathfrak{p}}_{(2k+1)}$ as in eq. (124). Then the radical of $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ is given by*

$$\mathfrak{J}_{(N)} := \mathfrak{z}\left(\overset{\circ}{\mathfrak{k}}_{(0)}\right) \oplus \bigoplus_{k=1}^{\lfloor N/2 \rfloor} \overset{\circ}{\mathfrak{k}}_{(2k)} \oplus \bigoplus_{k=0}^{\lfloor (N-1)/2 \rfloor} \overset{\circ}{\mathfrak{p}}_{(2k+1)} \subset \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right),$$

where $\mathfrak{z}\left(\overset{\circ}{\mathfrak{k}}_{(0)}\right)$ is the center[54] of $\mathring{\mathfrak{k}}$. This provides the following Levi decomposition for $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$:

$$\mathfrak{N}_N(\mathbb{K}) \cong \left[\overset{\circ}{\mathfrak{k}}_{(0)}, \overset{\circ}{\mathfrak{k}}_{(0)}\right] \ltimes \mathfrak{J}_{(N)}.$$

Proof. The graded decomposition (123) of $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ implies that

$$\left[\overset{\circ}{\mathfrak{p}}_{(2k-1)}, \overset{\circ}{\mathfrak{p}}_{(2l-1)}\right] \subseteq \overset{\circ}{\mathfrak{k}}_{(2k+2l-2)}, \quad \left[\overset{\circ}{\mathfrak{k}}_{(2k)}, \overset{\circ}{\mathfrak{p}}_{(2l-1)}\right] \subseteq \overset{\circ}{\mathfrak{p}}_{(2k+2l-1)}, \quad \left[\overset{\circ}{\mathfrak{k}}_{(2k)}, \overset{\circ}{\mathfrak{k}}_{(2l)}\right] \subseteq \overset{\circ}{\mathfrak{k}}_{(2k+2l)},$$

with $\overset{\circ}{\mathfrak{k}}_{(2k)} = \{0\} = \overset{\circ}{\mathfrak{p}}_{(2l+1)}$ for $k > \lfloor N/2 \rfloor$ and $l > \lfloor (N-1)/2 \rfloor$. This shows that $\mathfrak{J}_{(N)}$ is an ideal, so analyze the derived series $\mathfrak{J}_{(N)}^{(n)}$. The lowest degree that occurs in $\mathfrak{J}_{(N)}^{(1)} = \left[\mathfrak{J}_{(N)}, \mathfrak{J}_{(N)}\right]$ is equal to 1, because the factors of degree 0 commute with each other. Thus, $\mathfrak{J}_{(N)}$ is solvable because after $N+2$ steps the derived Lie algebra $\mathfrak{J}_{(N)}^{(N+2)}$ only contains elements of degree $N+1$ or higher, which are all equal to 0. Now take any

[54] Since A is a generalized Dynkin diagram of untwisted affine type, $\mathfrak{z}\left(\overset{\circ}{\mathfrak{k}}_{(0)}\right)$ is nontrivial only if $A = C_l^{(1)}$ or $A = A_1^{(1)}$. In the latter case, $\mathfrak{k}(A_1) \cong \mathbb{K}$ so this is rather a special case.

$x \in \mathring{\mathfrak{k}}_{(0)} \setminus \mathfrak{z}\left(\mathring{\mathfrak{k}}_{(0)}\right) = \left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)}\right]$ as those are the only elements left to add to $\mathfrak{J}_{(N)}$. The derived subalgebra $\left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)}\right]$ is semisimple and so is the ideal $\mathfrak{j}_0$ in $\left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)}\right]$ generated by x and therefore $\mathfrak{j}_0$ is not solvable. In consequence, the ideal generated by $\mathfrak{J}_{(N)} + x$ is not solvable which shows that $\mathfrak{J}_{(N)}$ is a maximal solvable ideal, hence the radical of $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ as $\dim \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right) < \infty$. □

Proposition 7.14. *(Cp. [KKLN21, rem. 14]) Let A , $\mathbb{P}_N$ and $\rho_{\pm}^{(N)} : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ be as in the previous proposition but now consider only $\mathbb{K} = \mathbb{C}$. The representation is irreducible if and only if it is the tensor product of an irreducible $\left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)}\right]$-module with a 1-dimensional $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$-module.*

Proof. As a consequence of Lie's theorem (cp. for instance [HN12, thm 5.4.8]) one has that every irreducible f.d. representation V of a complex and f.d. Lie algebra $\mathfrak{g}$ is of the form $V = V_0 \otimes L$, where V_0 is an irreducible representation of the semisimple part $\mathfrak{g}_{ss} := \mathfrak{g}/rad(\mathfrak{g})$ and L is a one-dimensional representation (cp. [FH91, prop. 9.17]). As $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ is finite-dimensional and the image of $\rho_{\pm}^{(N)}$ is surjective the claim follows from the observation that the semisimple part of $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ is $\left[\mathring{\mathfrak{k}}_{(0)}, \mathring{\mathfrak{k}}_{(0)}\right]$. □

Proposition 7.15. *(Cp. [KKLN21, prop. 15]) Let A , $\mathbb{P}_N$ and $\rho_{\pm}^{(N)} : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ be as in prop. 7.9, then the kernels satisfy*

$$\ker \rho_{\pm}^{(N)} \supset \ker \rho_{\pm}^{(N+1)} \ \forall N \in \mathbb{N}.$$

Proof. Set $x_{(m)} := (t^m + t^{-m}) \otimes x$ for $x \in \mathring{\mathfrak{k}}$ and $y_{(m)} := (t^m - t^{-m}) \otimes x$ for $y \in \mathring{\mathfrak{p}}$. Then one spells out the truncated version of eqs. (119) and (120):

$$\rho_{\pm}^{(N)}\left(x_{(m)}\right) = (\pm 1)^m \sum_{n=0}^{\lfloor N/2 \rfloor} a_{2n}^{(m)} u^{2n} \otimes x, \quad \rho_{\pm}^{(N)}\left(y_{(m)}\right) = (\pm 1)^m \sum_{n=0}^{\lfloor (N-1)/2 \rfloor} a_{2n+1}^{(m)} u^{2n+1} \otimes y.$$

Now obtain the kernel conditions as linear systems of equations:

$$\begin{aligned} \rho_{\pm}^{(N)}\left(\sum_{i=1}^{M} b_i x_{(m_i)}\right) &= \sum_{i=1}^{M} (\pm 1)^{m_i} b_i \sum_{n=0}^{\lfloor N/2 \rfloor} a_{2n}^{(m_i)} u^{2n} \otimes x \\ &= \sum_{n=0}^{\lfloor N/2 \rfloor} \left(\sum_{i=1}^{M} (\pm 1)^{m_i} b_i a_{2n}^{(m_i)}\right) u^{2n} \otimes x = 0 \\ \Leftrightarrow \sum_{i=1}^{M} (\pm 1)^{m_i} b_i a_{2n}^{(m_i)} &= 0 \ \forall n = 0, \ldots, \lfloor N/2 \rfloor . \end{aligned}$$

In the same way one finds that

$$\rho_{\pm}^{(N)}\left(\sum_{i=1}^{M} b_i y_{(m_i)}\right) = 0 \ \Leftrightarrow \ \sum_{i=1}^{M} (\pm 1)^{m_i} b_i a_{2n+1}^{(m_i)} = 0 \ \forall n = 0, \ldots, \lfloor (N-1)/2 \rfloor .$$

If one goes from N to $N+1$, then the above equations will not change. Instead, one additional equation will appear and so one concludes that

$$\ker \rho_{\pm}^{(N)} \supset \ker \rho_{\pm}^{(N+1)}.$$

□

Remark 7.16. In [KN21, sec. 4] it is shown that the kernels of the higher spin representations $\sigma_{\frac{2n+1}{2}}$: $\mathfrak{k}(E_9)(\mathbb{K}) \to \operatorname{End}\left(\mathcal{S}_{\frac{2n+1}{2}}\right)$ for $n = 0, 1, 2, 3$ coincide with the intersections of the above kernels like this:

$$\ker \sigma_{\frac{2n+1}{2}} = \ker \rho_{+}^{(n)} \cap \ker \rho_{-}^{(n)}.$$

7.3 Induced representations

Given a $\mathring{\mathfrak{k}}$-module V the induced $\mathfrak{k}(A)$-module $\mathfrak{V}$ is in general infinite-dimensional and hard to analyze. In this section, I will study the induced representations of the $\mathbb{K}[u]$-model $\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right)$ of $\mathfrak{k}(A)(\mathbb{K})$ and show how they admit quotients that are finite-dimensional representations of $\mathfrak{k}(A)(\mathbb{K})$. Throughout this subsection, let A be a GCM of untwisted affine type and $\mathring{A}$ its unique sub-GCM of finite type whose extended Cartan matrix is A.

Lemma 7.17. *(Cp. [KKLN21, eqs. 4.1-3]) The Lie algebra $\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right)$ is $\mathbb{N}$-graded via*

$$\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right) = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n, \quad \mathfrak{N}_n := \begin{cases} \mathit{span}_{\mathbb{K}} \left\{ u^n \otimes x \mid x \in \mathring{\mathfrak{k}} \right\} & \text{if } n \text{ is even,} \\ \mathit{span}_{\mathbb{K}} \left\{ u^n \otimes y \mid y \in \mathring{\mathfrak{p}} \right\} & \text{if } n \text{ is odd,} \end{cases}$$

and contains the maximal proper ideal ($\mathfrak{z}\left(\mathring{\mathfrak{k}}\right)$ denotes the center of $\mathring{\mathfrak{k}}$ as before)

$$\mathfrak{I} := \mathfrak{z}\left(\mathring{\mathfrak{k}}\right) \oplus \bigoplus_{n=1}^{\infty} \mathfrak{N}_n.$$

Its universal enveloping algebra decomposes as

$$\mathcal{U}\left(\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right)\right) = \mathcal{U}\left(\left[\mathring{\mathfrak{k}}, \mathring{\mathfrak{k}}\right]\right) \cdot \mathcal{U}(\mathfrak{I}). \tag{125}$$

Proof. The gradation is inherited from the degree of the monomials u^n that generate $\mathbb{K}[u]$. The argument that $\mathfrak{I}$ is a maximal proper ideal is a slight variation of the argument used in prop. 7.13: $\mathfrak{I}$ is certainly an ideal and as $\left[\mathring{\mathfrak{k}}, \mathring{\mathfrak{k}}\right]$ is not only semisimple but in fact simple because of the assumptions on A and $\mathring{A}$, the ideal generated by any $x \in \mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right) \setminus \mathfrak{I} = \left[\mathring{\mathfrak{k}}, \mathring{\mathfrak{k}}\right]$ contains $\left[\mathring{\mathfrak{k}}, \mathring{\mathfrak{k}}\right]$. Therefore, $\mathfrak{I}$ is a maximal proper ideal. The decomposition of the UEA follows from the vector space decomposition

$$\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right) = \left[\mathring{\mathfrak{k}}, \mathring{\mathfrak{k}}\right] \oplus \mathfrak{z}\left(\mathring{\mathfrak{k}}\right) \oplus \bigoplus_{n=1}^{\infty} \mathfrak{N}_n$$

and application of the PBW-theorem, as it is possible to establish an order which places degree 0-elements on the left in the PBW-basis of $\mathcal{U}\left(\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right)\right)$. □

Remark. Note that $\mathfrak{J}$ is no longer solvable in contrast to $\mathfrak{J}_{(N)}$ in prop. 7.13.

Definition 7.18. (Induced module) Let V be a f.d. $\mathring{\mathfrak{k}}$-module and $\tilde{\mathfrak{k}} := \mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right), \mathcal{U}\left(\tilde{\mathfrak{k}}\right) = \mathcal{U}\left(\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right)\right)$ as in the previous lemma. Then the induced $\tilde{\mathfrak{k}}$-module is defined as the $\mathcal{U}\left(\tilde{\mathfrak{k}}\right)$-left-module

$$Ind_{\mathring{\mathfrak{k}}}^{\tilde{\mathfrak{k}}}(V) := \mathcal{U}\left(\tilde{\mathfrak{k}}\right) \otimes_{\mathcal{U}(\mathring{\mathfrak{k}})} V.$$

The tensor product $\otimes_{\mathcal{U}(\mathring{\mathfrak{k}})}$ is defined by viewing $\mathcal{U}\left(\tilde{\mathfrak{k}}\right)$ as a $\mathcal{U}\left(\mathring{\mathfrak{k}}\right)$-right-module:

$$a \otimes x \cdot v = (a \cdot x) \otimes v,\ a \cdot (b \otimes v) = (a \cdot b) \otimes v \quad \forall a, b \in \mathcal{U},\ x \in \mathcal{U}\left(\mathring{\mathfrak{k}}\right),\ v \in V.$$

The grading of $\mathbb{K}[u]$ and the decomposition (125) of $\mathcal{U}\left(\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right)\right)$ have consequences for the structure of the induced representation:

Lemma 7.19. *(Cp. [KKLN21, lem. 17]) Let V be a f.d. $\mathring{\mathfrak{k}}$-module, $\tilde{\mathfrak{k}} := \mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right), \mathcal{U}\left(\tilde{\mathfrak{k}}\right) = \mathcal{U}\left(\mathfrak{N}\left(\mathbb{K}[u], \mathring{A}\right)\right)$, and consider the induced module $Ind_{\mathring{\mathfrak{k}}}^{\tilde{\mathfrak{k}}}(V)$. The $\mathbb{N}$-grading of $\mathcal{U}\left(\tilde{\mathfrak{k}}\right)$ extends to an $\mathbb{N}$-grading of $Ind_{\mathring{\mathfrak{k}}}^{\tilde{\mathfrak{k}}}(V)$. If $\mathbb{K} = \mathbb{C}$, then this grading can be extended to an $\mathbb{N} \times \mathring{\mathfrak{h}}$-grading[55], where $\mathring{\mathfrak{h}}$ denotes the Cartan subalgebra of $\mathring{\mathfrak{k}}$. As a $\mathring{\mathfrak{k}}$-module one has*

$$Ind_{\mathring{\mathfrak{k}}}^{\tilde{\mathfrak{k}}}(V) \cong \mathcal{U}(\mathfrak{J}) \otimes_{\mathbb{K}} V \tag{126}$$

with $\mathfrak{J}$ the maximal proper ideal from lemma 7.17 if $\mathring{\mathfrak{k}}$ is semisimple. If $\mathring{\mathfrak{k}}$ is not semisimple than exclude $\mathfrak{z}\left(\mathring{\mathfrak{k}}\right)$ from the definition of $\mathfrak{J}$.

Proof. From lemma 7.17 one has that $\tilde{\mathfrak{k}}$ admits an $\mathbb{N}$-grading which extends to $\mathcal{U}\left(\tilde{\mathfrak{k}}\right)$. One then sets

$$\deg(x \otimes v) = \deg(x)\ \forall\ x \in \mathcal{U}\left(\tilde{\mathfrak{k}}\right), v \in V$$

which is compatible with $\otimes_{\mathcal{U}(\mathring{\mathfrak{k}})}$ because $\deg(x) = 0$ for all $x \in \mathcal{U}\left(\mathring{\mathfrak{k}}\right)$. One has

$$x \cdot (y \otimes v) = xy \otimes v = (yx + [x, y]) \otimes v = y \otimes x \cdot v + [x, y] \otimes v\ \forall x \in \mathcal{U}\left(\mathring{\mathfrak{k}}\right), y \in \mathcal{U}\left(\tilde{\mathfrak{k}}\right), v \in V$$

which shows that left multiplication can be written in terms of the action on a tensor product of $\mathring{\mathfrak{k}}$-modules. One now needs to show that (126) holds as $\mathbb{K}$-vector spaces. For $x \in \mathcal{U}\left(\mathring{\mathfrak{k}}\right)$ and $y \in \mathcal{U}(\mathfrak{J})$ one has that $[x, y] \in \mathcal{U}(\mathfrak{J})\mathcal{U}\left(\mathring{\mathfrak{k}}\right)$ (this can be seen by successive application of the ideal property of $\mathfrak{J}$) and therefore

$$\left(\mathcal{U}\left(\mathring{\mathfrak{k}}\right) \cdot \mathcal{U}(\mathfrak{J})\right) \otimes_{\mathcal{U}(\mathring{\mathfrak{k}})} V = \mathcal{U}(\mathfrak{J}) \otimes_{\mathcal{U}(\mathring{\mathfrak{k}})} V.$$

Since $\mathfrak{J} \cap \mathring{\mathfrak{k}} = \{0\}$ one has $\mathcal{U}\left(\mathring{\mathfrak{k}}\right) \cap \mathcal{U}(\mathfrak{J}) = \mathbb{K} \cdot 1$ and therefore $\mathcal{U}(\mathfrak{J}) \otimes_{\mathcal{U}(\mathring{\mathfrak{k}})} V \cong \mathcal{U}(\mathfrak{J}) \otimes_{\mathbb{K}} V$ as $\mathbb{K}$-vector spaces which shows (126). □

[55] For $\mathbb{K} = \mathbb{R}$ this does not work because then $\mathring{\mathfrak{k}}$ is not split.

Proposition 7.20. *(Cp. [KKLN21, prop. 18]) Let V be a $\mathring{\mathfrak{k}}$-module and denote the graded decomposition of its induced $\mathfrak{N}\left(\mathbb{K}\left[u\right], \mathring{A}\right)$-module $\mathfrak{V} := Ind_{\mathring{\mathfrak{k}}}^{\mathfrak{N}\left(\mathbb{K}[u],\mathring{A}\right)}(V)$ by*

$$\mathfrak{V} = \bigoplus_{n=0}^{\infty} \mathfrak{V}_n.$$

Then $\mathfrak{V}_{(N)} := \bigoplus_{n=N}^{\infty} \mathfrak{V}_n$ is an invariant submodule and the quotient $\mathfrak{V}/\mathfrak{V}_{(N)}$ admits an action of $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$ and thus, one of $\mathfrak{k}(A)$. The module $\mathfrak{V}/\mathfrak{V}_{(N)}$ is finite-dimensional if V is.

Proof. $\mathfrak{V}_{(N)}$ is an invariant submodule of $\mathfrak{V}$ because the degree of any element in $\mathfrak{N}\left(\mathbb{K}\left[u\right], \mathring{A}\right)$ is greater or equal than 0. Now any monomial element $u^n \otimes x \in \mathfrak{N}\left(\mathbb{K}\left[u\right], \mathring{A}\right)$ with $n > N$ acts trivially on $\mathfrak{V}/\mathfrak{V}_{(N)}$ because the only nontrivial homogeneous components of $\mathfrak{V}/\mathfrak{V}_{(N)}$ have degree $k \in \{0, \ldots, N\}$. Denote by $\mathbb{I}_N$ the ideal in $\mathbb{K}[u]$ generated by u^{N+1} then the action of $\mathfrak{N}\left(\mathbb{K}\left[u\right], \mathring{A}\right)$ factors through $\mathfrak{N}\left(\mathbb{K}\left[u\right]/\mathbb{I}_N, \mathring{A}\right)$. As $\mathbb{K}[u]/\mathbb{I}_N \cong \mathbb{P}_N$ this shows that $\mathfrak{V}/\mathfrak{V}_{(N)}$ is a $\mathfrak{N}\left(\mathbb{P}_N, \mathring{A}\right)$-module and by prop. 7.9 it is a $\mathfrak{k}(A)$-module. Each graded component of $\mathcal{U}\left(\mathfrak{J}\right)$ is finite-dimensional because every element of $\mathfrak{J}$ has degree 1 or higher (recall that one excludes $\mathfrak{z}\left(\mathring{\mathfrak{k}}\right)$ from $\mathfrak{J}$ here if it is nontrivial). Hence, $\mathfrak{V}/\mathfrak{V}_{(N)}$ is finite-dimensional if V is according to eq. (126). □

Definition 7.21. (Projective limit) Let I be a directed set and $\left(G_i\right)_{i\in I}$ a family of objects in a category $\mathcal{C}$ together with a family of morphisms $\pi_{ij} : G_j \to G_i$ for all $i \le j$ such that

$$\pi_{ii} = Id_{G_i} \ \forall\, i \in I, \quad \pi_{ij} \circ \pi_{jk} = \pi_{ik} \ \forall\, i \le j \le k \in I.$$

Then $\left(\left(G_i\right)_{i\in I}, \left(\pi_{ij}\right)_{i\le j\in I}\right)$ is called a **projective system**. The **projective limit** $(G, (\pi_i))$ of $\left(\left(G_i\right)_{i\in I}, \left(\pi_{ij}\right)_{i\le j\in I}\right)$ is the universal object $G \in \mathcal{C}$ s.t. there exist morphisms $\pi_i : G \to G_i$ for $i \in I$ s.t.

$$\pi_i = \pi_{ij} \circ \pi_j \ \forall\, i \le j \in I.$$

The π_{ij} are called **bonding maps** and the π_i are referred to as the **limit maps**.

Example 7.22. In categories where projective limits exist one typically constructs the projective limit as follows (cp. [HM07, prop. 1.18] in the context of topological groups):

$$\lim_{\substack{\leftarrow\\ i\in I}} G_i = \left\{ \left(g_i\right)_{i\in I} \in \prod_{i\in I} G_i \mid \pi_{ij}\left(g_j\right) = g_i \ \forall\, i \le j \in I \right\} \tag{127}$$

Another example is that the ring (resp. commutative $\mathbb{K}$-algebra) of formal power series $\mathbb{K}\left[[u]\right]$ is the projective limit of the $\mathbb{P}_N$ in the category of rings over $\mathbb{K}$ (resp. commutative $\mathbb{K}$-algebras).

Proposition 7.23. *(Cp. [KKLN21, prop. 20]) Let V be a finite-dimensional $\mathring{\mathfrak{k}}$-module and $\mathfrak{V} := Ind_{\mathring{\mathfrak{k}}}^{\mathfrak{N}\left(\mathbb{K}[u],\mathring{A}\right)}(V)$ its induced $\mathfrak{N}\left(\mathbb{K}\left[u\right], \mathring{A}\right)$-module with graded decomposition $\mathfrak{V} = \bigoplus_{n=0}^{\infty} \mathfrak{V}_n$. Denote its formal completion by*

$$\overline{\mathfrak{V}} := \left\{ \left(v_n\right) \mid v_n \in \mathfrak{V}_n \right\},$$

then $\overline{\mathfrak{V}}$ admits a faithful action of $\mathfrak{N}\left(\mathbb{K}\left[[u]\right], \mathring{A}\right)$. Furthermore, $\overline{\mathfrak{V}}$ is the projective limit of the $\mathfrak{V}/\mathfrak{V}_{(N)}$ in the category of $\mathfrak{N}\left(\mathbb{K}\left[[u]\right], \mathring{A}\right)$-modules.

Proof. Consider the following action of $x = (x_n)_{n\in\mathbb{N}} \in \mathfrak{N}\left(\mathbb{K}\left[[u]\right], \mathring{A}\right)$ on $v = (v_n)_{n\in\mathbb{N}} \in \overline{\mathfrak{V}}$:

$$(x \cdot v)_n = \sum_{k=0}^{n} x_k v_{n-k} \ \forall\, n \in \mathbb{N}. \tag{128}$$

For $x \in \mathfrak{N}\left(\mathbb{K}\left[[u]\right], \mathring{A}\right)$ s.t. $\exists\, n > 0$ with $x_n \neq 0$ pick $v = (v_0, 0, 0, 0, \dots) \in \overline{\mathfrak{V}}$ with $v_0 \neq 0$, then

$$x \cdot v = (x_0 \cdot v_0, x_1 \cdot v_0, \dots)$$

is such that $x_n \cdot v_0 \neq 0$ because of eq. (126). If $x = (x_0, 0, \dots)$ one uses eq. (126) again because $\mathcal{U}(\mathfrak{J})$ is a faithful $\mathring{\mathfrak{k}}$-module (and therefore the action is faithful even if V is the trivial representation).

For $\mathfrak{V}/\mathfrak{V}_{(N)}$ and $\mathfrak{V}/\mathfrak{V}_{(M)}$ with $N \leq M$ one has that $\mathfrak{V}_{(N)}/\mathfrak{V}_{(M)}$ is a submodule of $\mathfrak{V}/\mathfrak{V}_{(M)}$ such that

$$\left(\mathfrak{V}/\mathfrak{V}_{(M)}\right) / \left(\mathfrak{V}_{(N)}/\mathfrak{V}_{(M)}\right) \cong \mathfrak{V}/\mathfrak{V}_{(N)}.$$

Denote by

$$\pi_{NM} : \mathfrak{V}/\mathfrak{V}_{(M)} \to \mathfrak{V}/\mathfrak{V}_{(N)},\ (v_0, \dots, v_M) \mapsto (v_0, \dots, v_N)$$

the resulting bonding map. Also,

$$\overline{\mathfrak{V}}_{(N)} := \left\{(v_n)_{n\in\mathbb{N}} \in \overline{\mathfrak{V}} \,|\, v_n = 0 \ \forall\, n \leq N\right\}$$

is a submodule of $\overline{\mathfrak{V}}$ because of eq. (128) and is such that

$$\overline{\mathfrak{V}}/\overline{\mathfrak{V}}_{(N)} \cong \mathfrak{V}/\mathfrak{V}_{(N)}.$$

Denote the corresponding projection which will be the limit map by

$$\pi_N : \overline{\mathfrak{V}} \to \overline{\mathfrak{V}}/\overline{\mathfrak{V}}_{(N)} \cong \mathfrak{V}/\mathfrak{V}_{(N)}.$$

Note that all maps π_{NM} and π_N are compatible with the action of $\mathfrak{N}\left(\mathbb{K}\left[[u]\right], \mathring{A}\right)$ because of eq. (128). In order to show universality, I will show that there exists a map from $\overline{\mathfrak{V}}$ to the projective limit as constructed in eq. (127):

$$\lim_{\substack{\leftarrow \\ N\in\mathbb{N}}} \mathfrak{V}/\mathfrak{V}_{(N)} = \left\{(v_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \left(\mathfrak{V}/\mathfrak{V}_{(N)}\right) \ |\ \pi_{ij}(v_j) = v_i \ \forall\, i \leq j \in \mathbb{N}\right\}.$$

This map is given by

$$\phi := \prod_{N\in\mathbb{N}} \pi_N : \overline{\mathfrak{V}} \to \lim_{\substack{\leftarrow \\ N\in\mathbb{N}}} \mathfrak{V}/\mathfrak{V}_{(N)},\ (v_i) \mapsto \left(\bigoplus_{i=0}^{N} v_i\right)_{N\in\mathbb{N}}$$

which shows that $\overline{\mathfrak{V}}$ is isomorphic to the projective limit of the $\mathfrak{V}/\mathfrak{V}_{(N)}$. □

Corollary 7.24. *(Cp. [KKLN21, porp. 20]) If A is of untwisted affine type, then $\mathfrak{k}(A)(\mathbb{K})$ is residually finite-dimensional in the sense that to each $x \in \mathfrak{k}(A)(\mathbb{K})$ there exists a f.d. representation ρ s.t. $\rho(x) \neq 0$.*

Proof. Let $x \in \mathfrak{k}(A)(\mathbb{K})$ be nontrivial and denote by $\rho_\pm : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{K}[[u]], \mathring{A}\right)$ the monomorphism from prop. 7.11. As $\overline{\mathfrak{V}}$ is a faithful $\mathfrak{N}\left(\mathbb{K}[[u]], \mathring{A}\right)$-module there exists $v = (v_i) \in \overline{\mathfrak{V}}$ s.t.

$$\rho(x)v \neq 0.$$

In view of eq. (128) this implies that there exist $N \leq M \in \mathbb{N}$ s.t.

$$\rho(x)\left(\sum_{i=0}^{N} v_i\right) = \sum_{i=0}^{M} u_i \neq 0$$

and hence $\rho_\pm^{(M)}(x) \neq 0$, where $\rho_\pm^{(M)} : \mathfrak{k}(A)(\mathbb{K}) \to \mathfrak{N}\left(\mathbb{P}_M, \mathring{A}\right)$ denotes the epimorphism from prop. 7.9. □

8 Open questions and further research

As could have been expected, answering some of the questions concerning the higher spin representations of $\mathfrak{k}(A)$ raised several new ones. For the particular case of $\mathfrak{k}(E_{10})$ one of the most pressing questions is, under which conditions a finite-dimensional $\mathring{\mathfrak{k}}$-module admits an action of $\mathfrak{k}(E_{10})$, where is a $\mathring{\mathfrak{k}}$ natural subalgebra of $\mathfrak{k}(E_{10})$. There exist several phrasings of this problems, a rather technical one that works over $\mathring{\mathfrak{k}} := \mathfrak{so}(10, \mathbb{C})$ has been outlined in sec. 2.4. Other phrasing work with $\mathring{\mathfrak{k}} := \mathfrak{k}(E_8) \oplus \mathfrak{so}(2)$ or $\mathring{\mathfrak{k}} := \mathfrak{k}(E_9)$ (cp. [KN21]), where in particular the case of $\mathfrak{k}(E_9)$ yields interesting features. As explained in [KN21], all known $\mathfrak{k}(E_{10})$-modules split into two $\mathfrak{k}(E_9)$-modules that are related by "chirality", or in other terms they factor through the homomorphisms $\rho_\pm^{(N)}$ from cor. 7.9, where the signs between the two modules differ. This could be an important clue but as the $\mathcal{S}_{\frac{n}{2}}$ are all built on $\mathcal{S}_{\frac{1}{2}}$ it could also just be a remnant of $\mathcal{S}_{\frac{1}{2}}$. Another approach for $\mathring{\mathfrak{k}} := \mathfrak{so}(10, \mathbb{C})$ stems from the observation that $\mathcal{U}\left(\mathring{\mathfrak{k}}\right).\{X_{10}\}$ w.r.t. the adjoint action is isomorphic to Γ_{ω_3} (see 13) as a $\mathring{\mathfrak{k}}$-module. Therefore the question can be rephrased to "Which f.d. $\mathfrak{so}(10, \mathbb{C})$-modules U admit a linear operator $X \in End(U)$ that transforms[56] in Γ_{ω_3}". While the properties of such operators are known by the Wigner-Eckert-theorem (cp. [C97, sec. 5.4]) I am not aware of results on their existence, although this seems like a question that representation theory of semi-simple Lie algebras could be able to answer.

Another open question concerns the (ir-)reducibility of $\mathcal{S}_{\frac{7}{2}}$ and other tensor products such as $\mathcal{S}_{\frac{7}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$, $\widetilde{\mathcal{S}}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{3}{2}}$, $\widetilde{\mathcal{S}}_{\frac{5}{2}} \otimes \mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}, \ldots$ both in general as well as in the particular case $\mathfrak{k}(E_{10})$. Here, I would be interested in the question whether or not one can use the contravariant form on the representation side to deduce semi-simplicity of the image, as this property of $\left(\rho, \mathcal{S}_{\frac{1}{2}}\right)$ proved highly useful. In addition this could open up a path that allows the characterization of the images in concrete cases, such as the E_n-series, similar to [HKL15].

A more abstract question is if finite-dimensional $\mathfrak{k}(A)$-modules are completely reducible if A is indefinite but regular. One knows that this statement is false for A of affine type, but the modules I studied in section

[56] Assume there exist $X_1, \ldots, X_n \in End(U)$, where (ρ, U) denotes a $\mathfrak{g}$-module s.t. $[\rho(y), X_i] \subset \operatorname{span}\{X_1, \ldots, X_n\}$ for all $y \in \mathfrak{g}$. Then $\operatorname{span}\{X_1, \ldots, X_n\}$ is a finite-dimensional representation Γ_ω of $\mathfrak{g}$ and an operator $X \in End(U)$ is said to transform in Γ_ω if $X \in \operatorname{span}\{X_1, \ldots, X_n\}$.

5 satisfy this property. Also, the connection between $\mathfrak{k}(A)$ and $\mathfrak{gim}$-Lie algebras has been barely touched in this thesis.

Finally, the affine situation appears to allow for further investigation, as the R-models from def. 7.5 appear like a blend of Lie-triples with generalized current algebras. The representation theory of both has been studied individually (cp. for instance [HP02] and [FL07]) and maybe one can combine the methods of both worlds to find out more about the representation theory of Lie algebras of type $(R_+ \otimes \mathfrak{k}) \oplus (R_- \otimes \mathfrak{p})$.

A Computations for section 2

In this appendix I collect the rather technical computations of section 2. Throughout, $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$ are placeholders for signs. Also, the Berman elements $X_{\beta^{(1)}_{i,j}}, \ldots, X_{\beta^{(4)}_{i,j}}$ are defined as in eq. (17) where $\beta^{(1)}_{i,j}, \ldots, \beta^{(4)}_{i,j}$ are defined in eqs. (20) and (21).

A.1 Computation of the $\mathfrak{so}(n, \mathbb{C})$-structure coefficients

First, collect the pairwise commutation relations of the $X_{\beta^{(1)}_{i,j}}, \ldots, X_{\beta^{(4)}_{i,j}}$:

Lemma A.1. *With $\beta^{(1)}_{i,j}, \ldots, \beta^{(4)}_{i,j}$ as in eqs. (20), (21) and $X_{\beta^{(k)}_{i,j}}$ as in (17) one has*

$$\left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(2)}_{i,j}}\right] = \left[X_{\beta^{(3)}_{i,j}}, X_{\beta^{(4)}_{i,j}}\right] = X_{2j-1}, \quad \left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(3)}_{i,j}}\right] = \left[X_{\beta^{(2)}_{i,j}}, X_{\beta^{(4)}_{i,j}}\right] = X_{2i-1},$$
$$\left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(4)}_{i,j}}\right] = 0 = \left[X_{\beta^{(2)}_{i,j}}, X_{\beta^{(3)}_{i,j}}\right] .$$

Proof. Initially one finds

$$\left(\beta^{(1)}_{i,j}|\beta^{(2)}_{i,j}\right) = (\alpha_{2i} + \cdots + \alpha_{2j-1}|\alpha_{2i} + \cdots + \alpha_{2j-2}) = \left(\beta^{(2)}_{i,j} + \alpha_{2j}|\beta^{(2)}_{i,j}\right) = 2 - 1 = 1$$

and similarly one computes all others as well. One obtains

$$\left(\beta^{(1)}_{i,j}|\beta^{(2)}_{i,j}\right) = 1 \ , \ \left(\beta^{(1)}_{i,j}|\beta^{(3)}_{i,j}\right) = 1 \ , \ \left(\beta^{(1)}_{i,j}|\beta^{(4)}_{i,j}\right) = 0,$$
$$\left(\beta^{(2)}_{i,j}|\beta^{(3)}_{i,j}\right) = 0 \ , \ \left(\beta^{(2)}_{i,j}|\beta^{(4)}_{i,j}\right) = 1 \ , \ \left(\beta^{(3)}_{i,j}|\beta^{(4)}_{i,j}\right) = 1.$$

This leads to (cp. eq. 19)

$$\left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(2)}_{i,j}}\right] = \left[e_{\beta^{(1)}_{i,j}} - e_{-\beta^{(1)}_{i,j}}, e_{\beta^{(2)}_{i,j}} - e_{-\beta^{(2)}_{i,j}}\right] = -\left[e_{\beta^{(1)}_{i,j}}, e_{-\beta^{(2)}_{i,j}}\right] - \left[e_{-\beta^{(1)}_{i,j}}, e_{\beta^{(2)}_{i,j}}\right] .$$

With $[e_{2j-1}, e_i] = 0 \ \forall \, (i, 2j-1) \notin \mathcal{E}$ one has

$$\begin{aligned}
e_{\beta^{(1)}_{i,j}} &= [e_{2i}, [\ldots, [e_{2j-2}, e_{2j-1}]]] = -[e_{2i}, [\ldots, [e_{2j-1}, e_{2j-2}]]] \\
&= -\mathrm{ad}\,(e_{2j-1})\,([[e_{2i}, [\ldots, [e_{2j-3}, e_{2j-2}]]]]) = -\left[e_{2j-1}, e_{\beta^{(2)}_{i,j}}\right] \\
e_{-\beta^{(1)}_{i,j}} &= -\omega\left(e_{\beta^{(1)}_{i,j}}\right) = \left[f_{2j-1}, e_{-\beta^{(2)}_{i,j}}\right]
\end{aligned}$$

and with $\pm\left(\beta^{(2)}_{i,j} - \alpha_{2j-1}\right) \notin \Delta\,(A_9)$ one finds

$$\begin{aligned}
\left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(2)}_{i,j}}\right] &= (-1)^2\left[\left[e_{2j-1}, e_{\beta^{(2)}_{i,j}}\right], e_{-\beta^{(2)}_{i,j}}\right] - \left[\left[f_{2j-1}, e_{-\beta^{(2)}_{i,j}}\right], e_{\beta^{(2)}_{i,j}}\right] \\
&= \left[e_{2j-1}, \left[e_{\beta^{(2)}_{i,j}}, e_{-\beta^{(2)}_{i,j}}\right]\right] - \left[f_{2j-1}, \left[e_{-\beta^{(2)}_{i,j}}, e_{\beta^{(2)}_{i,j}}\right]\right] \\
&= \left[e_{2j-1}, \left(\beta^{(2)}_{i,j}\right)^\vee\right] + \left[f_{2j-1}, \left(\beta^{(2)}_{i,j}\right)^\vee\right] \\
&= -\left(\beta^{(2)}_{i,j}|\alpha_{2j-1}\right) e_{2j-1} + \left(\beta^{(2)}_{i,j}|\alpha_{2j-1}\right) f_{2j-1} \\
&= e_{2j-1} - f_{2j-1} = X_{2j-1} \ .
\end{aligned}$$

Similarly, with $e_{\beta^{(3)}_{i,j}} = -\left[e_{2j-1}, e_{\beta^{(4)}_{i,j}}\right]$ and $e_{-\beta^{(3)}_{i,j}} = \left[f_{2j-1}, e_{-\beta^{(4)}_{i,j}}\right]$ one finds

$$\begin{aligned}
\left[X_{\beta^{(3)}_{i,j}}, X_{\beta^{(4)}_{i,j}}\right] &= -\left[e_{\beta^{(3)}_{i,j}}, e_{-\beta^{(4)}_{i,j}}\right] - \left[e_{-\beta^{(3)}_{i,j}}, e_{\beta^{(4)}_{i,j}}\right] \\
&= \left[e_{2j-1}, \left[e_{\beta^{(4)}_{i,j}}, e_{-\beta^{(4)}_{i,j}}\right]\right] - \left[f_{2j-1}, \left[e_{-\beta^{(4)}_{i,j}}, e_{\beta^{(4)}_{i,j}}\right]\right] \\
&= \left[e_{2j-1}, \left(\beta^{(4)}_{i,j}\right)^{\vee}\right] + \left[f_{2j-1}, \left(\beta^{(4)}_{i,j}\right)^{\vee}\right] \\
&= -\left(\beta^{(4)}_{i,j}|\alpha_{2j-1}\right) e_{2j-1} + \left(\beta^{(4)}_{i,j}|\alpha_{2j-1}\right) f_{2j-1} \\
&= e_{2j-1} - f_{2j-1} = X_{2j-1}\,.
\end{aligned}$$

From $e_{\beta^{(3)}_{i,j}} = \left[e_{2i-1}, e_{\beta^{(1)}_{i,j}}\right]$ and $e_{-\beta^{(3)}_{i,j}} = -\left[f_{2i-1}, e_{-\beta^{(1)}_{i,j}}\right]$ however, one deduces

$$\begin{aligned}
\left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(3)}_{i,j}}\right] &= -\left[e_{\beta^{(1)}_{i,j}}, e_{-\beta^{(3)}_{i,j}}\right] - \left[e_{-\beta^{(1)}_{i,j}}, e_{\beta^{(3)}_{i,j}}\right] \\
&= \left[f_{2i-1}, \left[e_{\beta^{(1)}_{i,j}}, e_{-\beta^{(1)}_{i,j}}\right]\right] - \left[e_{2i-1}, \left[e_{-\beta^{(1)}_{i,j}}, e_{\beta^{(1)}_{i,j}}\right]\right] \\
&= -\left[\left(\beta^{(1)}_{i,j}\right)^{\vee}, f_{2i-1}\right] - \left[\left(\beta^{(1)}_{i,j}\right)^{\vee}, e_{2i-1}\right] \\
&= \left(\beta^{(1)}_{i,j}|\alpha_{2i-1}\right) f_{2i-1} - \left(\beta^{(1)}_{i,j}|\alpha_{2i-1}\right) e_{2i-1} \\
&= -f_{2i-1} + e_{2i-1} = X_{2i-1}\,.
\end{aligned}$$

Since $e_{\beta^{(4)}_{i,j}} = \left[e_{2i-1}, e_{\beta^{(2)}_{i,j}}\right]$ the computation for $\left[X_{\beta^{(2)}_{i,j}}, X_{\beta^{(4)}_{i,j}}\right] = X_{2i-1}$ works exactly the same way. One has

$$\begin{aligned}
\left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(4)}_{i,j}}\right] &= \left[e_{\beta^{(1)}_{i,j}}, e_{\beta^{(4)}_{i,j}}\right] - \left[e_{\beta^{(1)}_{i,j}}, e_{-\beta^{(4)}_{i,j}}\right] - \left[e_{-\beta^{(1)}_{i,j}}, e_{\beta^{(4)}_{i,j}}\right] + \left[e_{-\beta^{(1)}_{i,j}}, e_{-\beta^{(4)}_{i,j}}\right] \\
&= 0 - 0 - 0 + 0 = 0
\end{aligned}$$

since $\beta^{(1)}_{i,j} \pm \beta^{(4)}_{i,j} \notin \Delta\left(E_{10}\right)$. Similarly one finds $\left[X_{\beta^{(2)}_{i,j}}, X_{\beta^{(3)}_{i,j}}\right] = 0$. □

Lemma A.2. *With $e_{\varepsilon_1 L_i + \varepsilon_2 L_j}$ and H_i as in eqs. (22) and (22) one has*

$$\begin{aligned}
\left[e_{\varepsilon_1 L_i + \varepsilon_2 L_j}, e_{-\varepsilon_1 L_i - \varepsilon_2 L_j}\right] &= \varepsilon_1 H_i + \varepsilon_2 H_j \\
\left[e_{\varepsilon_1 L_i + \varepsilon_2 L_{i+1}}, e_{-\varepsilon_2 L_{i+1} + \varepsilon_3 L_{i+k}}\right] &= i \cdot e_{\varepsilon_1 L_i + \varepsilon_3 L_{i+k}} \text{ for } k \geq 2.
\end{aligned}$$

Proof. Start with the first relation. As only the roots $\beta^{(1)}_{i,j}, \ldots, \beta^{(4)}_{i,j}$ appear one has to realize that only the combinations $\beta^{(1)}_{i,j} - \beta^{(2)}_{i,j}$, $\beta^{(1)}_{i,j} - \beta^{(3)}_{i,j}$, $\beta^{(2)}_{i,j} - \beta^{(4)}_{i,j}$ and $\beta^{(3)}_{i,j} - \beta^{(4)}_{i,j}$ are positive roots. This is best seen from the fact that A_9-roots are characterized uniquely by their support and sign, as their support has to be connected (cp. [K90, lem. 1.6]) and each root $\alpha_1, \ldots, \alpha_{n-1}$ can appear at most once. Thus, the only combinations among the $\beta_{i,j}^{(1)},\dots,\beta_{i,j}^{(4)}$ are differences of these roots which result in a connected support. This yields

$$\begin{aligned}
\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_j}, e_{-\varepsilon_1 L_i-\varepsilon_2 L_j}\right] &= -\frac{1}{4}\Big[X_{\beta_{i,j}^{(1)}} - i\varepsilon_2 X_{\beta_{i,j}^{(2)}} - i\varepsilon_1 X_{\beta_{i,j}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(4)}},\\
&\quad X_{\beta_{i,j}^{(1)}} + i\varepsilon_2 X_{\beta_{i,j}^{(2)}} + i\varepsilon_1 X_{\beta_{i,j}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,j}^{(4)}}\Big]\\
&= -\frac{i}{2}\Big(\varepsilon_2\left[X_{\beta_{i,j}^{(1)}}, X_{\beta_{i,j}^{(2)}}\right] + \varepsilon_1\left[X_{\beta_{i,j}^{(1)}}, X_{\beta_{i,j}^{(3)}}\right]\\
&\quad +\varepsilon_1\left[X_{\beta_{i,j}^{(2)}}, X_{\beta_{i,j}^{(4)}}\right] + \varepsilon_2\left[X_{\beta_{i,j}^{(3)}}, X_{\beta_{i,j}^{(4)}}\right]\Big)
\end{aligned}$$

and from lemma A.1 one then has that

$$\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_j}, e_{-\varepsilon_1 L_i-\varepsilon_2 L_j}\right] = -\frac{i}{2}\left[2\varepsilon_1 X_{2i-1} + 2\varepsilon_2 X_{2j-1}\right]$$

and with $H_j = -iX_{2j-1}$ one concludes

$$\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_j}, e_{-\varepsilon_1 L_i-\varepsilon_2 L_j}\right] = \varepsilon_1 H_i + \varepsilon_2 H_j\,.$$

Towards the second relation consider that only the following roots are nonzero:

$$\begin{gathered}
\beta_{i,i+1}^{(1)} + \beta_{i+1,i+k}^{(1)} = \beta_{i,i+k}^{(1)} \quad,\quad \beta_{i,i+1}^{(1)} + \beta_{i+1,i+k}^{(2)} = \beta_{i,i+k}^{(2)} \quad,\quad \beta_{i,i+1}^{(2)} + \beta_{i+1,i+k}^{(3)} = \beta_{i,i+k}^{(1)}\\
\beta_{i,i+1}^{(2)} + \beta_{i+1,i+k}^{(4)} = \beta_{i,i+k}^{(2)} \quad,\quad \beta_{i,i+1}^{(3)} + \beta_{i+1,i+k}^{(1)} = \beta_{i,i+k}^{(3)} \quad,\quad \beta_{i,i+1}^{(3)} + \beta_{i+1,i+k}^{(2)} = \beta_{i,i+k}^{(4)}\\
\beta_{i,i+1}^{(4)} + \beta_{i+1,i+k}^{(3)} = \beta_{i,i+k}^{(3)} \quad,\quad \beta_{i,i+1}^{(4)} + \beta_{i+1,i+k}^{(4)} = \beta_{i,i+k}^{(4)}\,.
\end{gathered}$$

With this one computes

$$\begin{aligned}
-4\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_{i+1}}, e_{-\varepsilon_2 L_{i+1}+\varepsilon_3 L_{i+k}}\right] &= \Big[X_{\beta_{i,i+1}^{(1)}} - i\varepsilon_2 X_{\beta_{i,i+1}^{(2)}} - i\varepsilon_1 X_{\beta_{i,i+1}^{(3)}} - \varepsilon_1\varepsilon_2 X_{\beta_{i,i+1}^{(4)}}\,,\\
&\quad X_{\beta_{i+1,i+k}^{(1)}} - i\varepsilon_3 X_{\beta_{i+1,i+k}^{(2)}} + i\varepsilon_2 X_{\beta_{i+1,i+k}^{(3)}} + \varepsilon_2\varepsilon_3 X_{\beta_{i+1,i+k}^{(4)}}\Big]\\
&= \left[X_{\beta_{i,i+1}^{(1)}}, X_{\beta_{i+1,i+k}^{(1)}}\right] - i\varepsilon_3\left[X_{\beta_{i,i+1}^{(1)}}, X_{\beta_{i+1,i+k}^{(2)}}\right] + \varepsilon_2^2\left[X_{\beta_{i,i+1}^{(2)}}, X_{\beta_{i+1,i+k}^{(3)}}\right]\\
&\quad -i\varepsilon_2^2\varepsilon_3\left[X_{\beta_{i,i+1}^{(2)}}, X_{\beta_{i+1,i+k}^{(4)}}\right] - i\varepsilon_1\left[X_{\beta_{i,i+1}^{(3)}}, X_{\beta_{i+1,i+k}^{(1)}}\right]\\
&\quad -\varepsilon_1\varepsilon_3\left[X_{\beta_{i,i+1}^{(3)}}, X_{\beta_{i+1,i+k}^{(2)}}\right] - i\varepsilon_1\varepsilon_2^2\left[X_{\beta_{i,i+1}^{(4)}}, X_{\beta_{i+1,i+k}^{(3)}}\right]\\
&\quad -\varepsilon_1\varepsilon_2^2\varepsilon_3\left[X_{\beta_{i,i+1}^{(4)}}, X_{\beta_{i+1,i+k}^{(4)}}\right]
\end{aligned}$$

All commutators in the above equation yield an element $\pm X_{\beta_{i,i+k}^{(j)}}$ by the way these are defined. The sign is determined as follows ($1 \le i < j < l < n$):

$$\begin{aligned}
\left[X^{(i,i+1,\dots,j)}, X^{(j+1,\dots,l)}\right] &= \left[-\mathrm{ad}\left(X_j\right)\left(X^{(i,\dots,j-1)}\right), X^{(j+1,\dots,l)}\right]\\
&= -\mathrm{ad}\left(X_j\right)\left(\left[X^{(i,\dots,j-1)}, X^{(j+1,\dots,l)}\right]\right)\\
&\quad +\left[X^{(i,\dots,j-1)}, \mathrm{ad}\left(X_j\right)\left(X^{(j+1,\dots,l)}\right)\right]\\
&= 0 + \left[X^{(i,\dots,j-1)}, X^{(j,\dots,l)}\right]
\end{aligned}$$

and by induction $\left[X^{(i,i+1,\ldots,j)}, X^{(j+1,\ldots,l)}\right] = X^{(i,\ldots,l)}$ which fixes all signs to be

$$\left[X_{\beta_{i,i+1}^{(m_1)}}, X_{\beta_{i+1,i+k}^{(m_2)}}\right] = +X_{\beta_{i,i+k}^{(m_3)}} \ \forall\, \beta_{i,i+1}^{(m_1)} + \beta_{i+1,i+k}^{(m_2)} = \beta_{i,i+k}^{(m_3)}. \tag{129}$$

With this:

$$\begin{aligned} -4\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_{i+1}}, e_{-\varepsilon_2 L_{i+1}+\varepsilon_3 L_{i+k}}\right] &= X_{\beta_{i,i+k}^{(1)}} - i\varepsilon_3 X_{\beta_{i,i+k}^{(2)}} + X_{\beta_{i,i+k}^{(1)}} - i\varepsilon_3 X_{\beta_{i,i+k}^{(2)}} - i\varepsilon_1 X_{\beta_{i,i+k}^{(3)}} \\ &\quad -\varepsilon_1\varepsilon_3 X_{\beta_{i,i+k}^{(4)}} - i\varepsilon_1 X_{\beta_{i,i+k}^{(3)}} - \varepsilon_1\varepsilon_3 X_{\beta_{i,i+k}^{(4)}} \\ &= 2\cdot\left(X_{\beta_{i,i+k}^{(1)}} - i\varepsilon_3 X_{\beta_{i,i+k}^{(2)}} - i\varepsilon_1 X_{\beta_{i,i+k}^{(3)}} - \varepsilon_1\varepsilon_3 X_{\beta_{i,i+k}^{(4)}}\right) \\ &= -4i\cdot e_{\varepsilon_1 L_i+\varepsilon_3 L_{i+k}}. \end{aligned}$$

□

Lemma A.3. *Let $\gamma, \gamma_1, \gamma_2$ be positiveA_{n-1}-roots such that $\gamma = \gamma_1+\gamma_2$ and $e_\gamma = [e_{\gamma_1}, e_{\gamma_2}]$ then*

$$[X_\gamma, X_{\gamma_1}] = X_{\gamma_2},\ \ [X_\gamma, X_{\gamma_2}] = -X_{\gamma_1}.$$

Proof. If $\gamma = \gamma_1+\gamma_2$ then $\gamma+\gamma_i \notin \Delta(A_{n-1})$ but $\gamma-\gamma_i$ for $i=1,2$ is. Thus, $[e_\gamma, e_{\gamma_i}] = 0$. Also, if $\gamma_1+\gamma_2 \in \Delta(A_{n-1})$ one knows that $\gamma_1-\gamma_2 \notin \Delta(A_{n-1})$ and hence, $[e_{\gamma_1}, e_{-\gamma_2}] = 0$. One computes

$$\begin{aligned} [X_\gamma, X_{\gamma_1}] &= -[e_\gamma, e_{-\gamma_1}] - [e_{-\gamma}, e_{\gamma_1}] \\ &= -[[e_{\gamma_1}, e_{\gamma_2}], e_{-\gamma_1}] + [[e_{-\gamma_1}, e_{-\gamma_2}], e_{\gamma_1}] \\ &= [[e_{-\gamma_1}, e_{\gamma_1}], e_{\gamma_2}] + 0 - [[e_{\gamma_1}, e_{-\gamma_1}], e_{-\gamma_2}] + 0 \\ &= [-\gamma_1^\vee, e_{\gamma_2}] - [\gamma_1^\vee, e_{-\gamma_2}] \\ &= -(\gamma_2|\gamma_1)\cdot(e_{\gamma_2} - e_{-\gamma_2}) = X_{\gamma_2} \end{aligned}$$

and

$$\begin{aligned} [X_\gamma, X_{\gamma_2}] &= -[e_\gamma, e_{-\gamma_2}] - [e_{-\gamma}, e_{\gamma_2}] \\ &= -[[e_{\gamma_1}, e_{\gamma_2}], e_{-\gamma_2}] + [[e_{-\gamma_1}, e_{-\gamma_2}], e_{\gamma_2}] \\ &= -\left[e_{\gamma_1}, (\gamma_2)^\vee\right] - \left[e_{-\gamma_1}, (\gamma_2)^\vee\right] \\ &= (\gamma_2|\gamma_1)\cdot(e_{\gamma_1} - e_{-\gamma_1}) = -X_{\gamma_1}. \end{aligned}$$

□

Lemma A.4. *For $1 \le i < j \le m$ one has*

$$\left[X_{\beta_{i,m+1}^{(2)}}, X_{\beta_{i,j}^{(k)}}\right] = 0 \text{ for } k=3,4,\ \ \left[X_{\beta_{i,m+1}^{(4)}}, X_{\beta_{i,j}^{(k)}}\right] = 0 \text{ for } k=1,2$$

$$\left[X_{\beta_{i,m+1}^{(2)}}, X_{\beta_{i,j}^{(1)}}\right] = X_{\beta_{j,m+1}^{(2)}},\ \ \left[X_{\beta_{i,m+1}^{(2)}}, X_{\beta_{i,j}^{(2)}}\right] = X_{\beta_{j,m+1}^{(4)}}$$

$$\left[X_{\beta_{i,m+1}^{(4)}}, X_{\beta_{i,j}^{(3)}}\right] = X_{\beta_{j,m+1}^{(2)}},\ \ \left[X_{\beta_{i,m+1}^{(4)}}, X_{\beta_{i,j}^{(4)}}\right] = X_{\beta_{j,m+1}^{(4)}}$$

and

$$\left[X_{\beta_{j,m+1}^{(2)}}, X_{\beta_{i,j}^{(k)}}\right] = 0 \text{ for } k=2,4,\ \ \left[X_{\beta_{j,m+1}^{(4)}}, X_{\beta_{i,j}^{(k)}}\right] = 0 \text{ for } k=1,3$$

$$\left[X_{\beta^{(2)}_{j,m+1}}, X_{\beta^{(1)}_{i,j}}\right] = -X_{\beta^{(2)}_{i,m+1}},\ \left[X_{\beta^{(2)}_{j,m+1}}, X_{\beta^{(3)}_{i,j}}\right] = -X_{\beta^{(4)}_{i,m+1}}$$
$$\left[X_{\beta^{(4)}_{j,m+1}}, X_{\beta^{(2)}_{i,j}}\right] = -X_{\beta^{(2)}_{i,m+1}},\ \left[X_{\beta^{(4)}_{j,m+1}}, X_{\beta^{(4)}_{i,j}}\right] = -X_{\beta^{(4)}_{i,m+1}}$$

Proof. One has from the definition of the $\beta^{(k)}_{i,j}$ that

$$\beta^{(2)}_{i,m+1} \pm \beta^{(k)}_{i,j} \notin \Delta\left(A_{n-1}\right) \text{ for } k=3,4,$$

$$\beta^{(2)}_{i,m+1} = \beta^{(1)}_{i,j} + \beta^{(2)}_{j,m+1},\ \beta^{(2)}_{i,m+1} = \beta^{(2)}_{i,j} + \beta^{(4)}_{j,m+1}$$

and

$$\beta^{(4)}_{i,m+1} \pm \beta^{(k)}_{i,j} \notin \Delta\left(A_{n-1}\right) \text{ for } k=1,2,$$

$$\beta^{(4)}_{i,m+1} = \beta^{(3)}_{i,j} + \beta^{(2)}_{j,m+1},\ \beta^{(4)}_{i,m+1} = \beta^{(4)}_{i,j} + \beta^{(4)}_{j,m+1}.$$

Therefore

$$\left[X_{\beta^{(2)}_{i,m+1}}, X_{\beta^{(k)}_{i,j}}\right] = 0 \text{ for } k=3,4,\ \left[X_{\beta^{(4)}_{i,m+1}}, X_{\beta^{(k)}_{i,j}}\right] = 0 \text{ for } k=1,2$$

and one computes with lemma A.3 that

$$\left[X_{\beta^{(2)}_{i,m+1}}, X_{\beta^{(1)}_{i,j}}\right] = \left[X_{\beta^{(1)}_{i,j}+\beta^{(2)}_{j,m+1}}, X_{\beta^{(1)}_{i,j}}\right] = X_{\beta^{(2)}_{j,m+1}},$$

$$\left[X_{\beta^{(2)}_{i,m+1}}, X_{\beta^{(2)}_{i,j}}\right] = \left[X_{\beta^{(2)}_{i,j}+\beta^{(4)}_{j,m+1}}, X_{\beta^{(2)}_{i,j}}\right] = X_{\beta^{(4)}_{j,m+1}}.$$

Note that the assumptions on the ordering of the Berman generators in lemma A.3 are met because $i<j$. For example,

$$\beta^{(2)}_{i,m+1} = \alpha_{2i} + \cdots + \alpha_{2m} = \underbrace{\alpha_{2i} + \cdots + \alpha_{2j-1}}_{\beta^{(1)}_{i,j}} + \underbrace{\alpha_{2j} + \ldots \alpha_{2m}}_{\beta^{(2)}_{j,m+1}}.$$

Similarly one computes that

$$\left[X_{\beta^{(4)}_{i,m+1}}, X_{\beta^{(3)}_{i,j}}\right] = \left[X_{\beta^{(3)}_{i,j}+\beta^{(2)}_{j,m+1}}, X_{\beta^{(3)}_{i,j}}\right] = X_{\beta^{(2)}_{j,m+1}}$$

$$\left[X_{\beta^{(4)}_{i,m+1}}, X_{\beta^{(4)}_{i,j}}\right] = \left[X_{\beta^{(4)}_{i,j}+\beta^{(4)}_{j,m+1}}, X_{\beta^{(4)}_{i,j}}\right] = X_{\beta^{(4)}_{j,m+1}}.$$

For the others one checks that

$$\beta^{(1)}_{i,j} + \beta^{(2)}_{j,m+1} = \beta^{(2)}_{i,m+1},\ \beta^{(3)}_{i,j} + \beta^{(2)}_{j,m+1} = \beta^{(4)}_{i,m+1},$$

$$\beta^{(2)}_{i,j} + \beta^{(4)}_{j,m+1} = \beta^{(2)}_{i,m+1},\ \beta^{(4)}_{i,j} + \beta^{(4)}_{j,m+1} = \beta^{(4)}_{i,m+1}$$

are the only nonzero combinations of the involved roots which implies with eq. (129) that

$$\left[X_{\beta^{(2)}_{j,m+1}}, X_{\beta^{(k)}_{i,j}}\right] = 0 \text{ for } k=2,4,\ \left[X_{\beta^{(4)}_{j,m+1}}, X_{\beta^{(k)}_{i,j}}\right] = 0 \text{ for } k=1,3$$

$$\left[X_{\beta^{(2)}_{j,m+1}}, X_{\beta^{(1)}_{i,j}}\right] = -\left[X_{\beta^{(1)}_{i,j}}, X_{\beta^{(2)}_{j,m+1}}\right] = -X_{\beta^{(2)}_{i,m+1}},$$

$$\left[X_{\beta^{(2)}_{j,m+1}}, X_{\beta^{(3)}_{i,j}}\right] = -\left[X_{\beta^{(3)}_{i,j}}, X_{\beta^{(2)}_{j,m+1}}\right] = -X_{\beta^{(4)}_{i,m+1}},$$
$$\left[X_{\beta^{(4)}_{j,m+1}}, X_{\beta^{(2)}_{i,j}}\right] = -\left[X_{\beta^{(2)}_{i,j}}, X_{\beta^{(4)}_{j,m+1}}\right] = -X_{\beta^{(2)}_{i,m+1}},$$
$$\left[X_{\beta^{(4)}_{j,m+1}}, X_{\beta^{(4)}_{i,j}}\right] = -\left[X_{\beta^{(4)}_{i,j}}, X_{\beta^{(4)}_{j,m+1}}\right] = -X_{\beta^{(4)}_{i,m+1}}.$$

□

Lemma A.5. *For $i<j<k$ the only nonzero commutators $\left[X_{\beta^{(m_1)}_{i,k}}, X_{\beta^{(m_2)}_{j,k}}\right]$ are*

$$\left[X_{\beta^{(1)}_{i,k}}, X_{\beta^{(1)}_{j,k}}\right] = -X_{\beta^{(1)}_{i,j}}\ ,\ \left[X_{\beta^{(1)}_{i,k}}, X_{\beta^{(3)}_{j,k}}\right] = -X_{\beta^{(2)}_{i,j}},\ \left[X_{\beta^{(2)}_{i,k}}, X_{\beta^{(2)}_{j,k}}\right] = -X_{\beta^{(1)}_{i,j}}\ ,\ \left[X_{\beta^{(2)}_{i,k}}, X_{\beta^{(4)}_{j,k}}\right] = -X_{\beta^{(2)}_{i,j}},$$
$$\left[X_{\beta^{(3)}_{i,k}}, X_{\beta^{(1)}_{j,k}}\right] = -X_{\beta^{(3)}_{i,j}}\ ,\ \left[X_{\beta^{(3)}_{i,k}}, X_{\beta^{(3)}_{j,k}}\right] = -X_{\beta^{(4)}_{i,j}},\ \left[X_{\beta^{(4)}_{i,k}}, X_{\beta^{(2)}_{j,k}}\right] = -X_{\beta^{(3)}_{i,j}}\ ,\ \left[X_{\beta^{(4)}_{i,k}}, X_{\beta^{(4)}_{j,k}}\right] = -X_{\beta^{(4)}_{i,j}}.$$

Proof. Apply lemma (A.3) and eq. (129) together with definitions (20) and (21) to obtain

$$\left[X_{\beta^{(1)}_{i,k}}, X_{\beta^{(1)}_{j,k}}\right] = \left[X_{\beta^{(1)}_{i,j}+\beta^{(1)}_{j,k}}, X_{\beta^{(1)}_{j,k}}\right] = -X_{\beta^{(1)}_{i,j}}\ ,\ \left[X_{\beta^{(1)}_{i,k}}, X_{\beta^{(3)}_{j,k}}\right] = \left[X_{\beta^{(2)}_{i,j}+\beta^{(3)}_{j,k}}, X_{\beta^{(3)}_{j,k}}\right] = -X_{\beta^{(2)}_{i,j}},$$
$$\left[X_{\beta^{(2)}_{i,k}}, X_{\beta^{(2)}_{j,k}}\right] = \left[X_{\beta^{(1)}_{i,j}+\beta^{(2)}_{j,k}}, X_{\beta^{(2)}_{j,k}}\right] = -X_{\beta^{(1)}_{i,j}}\ ,\ \left[X_{\beta^{(2)}_{i,k}}, X_{\beta^{(4)}_{j,k}}\right] = \left[X_{\beta^{(2)}_{i,j}+\beta^{(4)}_{j,k}}, X_{\beta^{(4)}_{j,k}}\right] = -X_{\beta^{(2)}_{i,j}},$$
$$\left[X_{\beta^{(3)}_{i,k}}, X_{\beta^{(1)}_{j,k}}\right] = \left[X_{\beta^{(3)}_{i,j}+\beta^{(1)}_{j,k}}, X_{\beta^{(1)}_{j,k}}\right] = -X_{\beta^{(3)}_{i,j}}\ ,\ \left[X_{\beta^{(3)}_{i,k}}, X_{\beta^{(3)}_{j,k}}\right] = \left[X_{\beta^{(4)}_{i,j}+\beta^{(3)}_{j,k}}, X_{\beta^{(3)}_{j,k}}\right] = -X_{\beta^{(4)}_{i,j}},$$
$$\left[X_{\beta^{(4)}_{i,k}}, X_{\beta^{(2)}_{j,k}}\right] = \left[X_{\beta^{(3)}_{i,j}+\beta^{(2)}_{j,k}}, X_{\beta^{(2)}_{j,k}}\right] = -X_{\beta^{(3)}_{i,j}}\ ,\ \left[X_{\beta^{(4)}_{i,k}}, X_{\beta^{(4)}_{j,k}}\right] = \left[X_{\beta^{(4)}_{i,j}+\beta^{(4)}_{j,k}}, X_{\beta^{(4)}_{j,k}}\right] = -X_{\beta^{(4)}_{i,j}}.$$

□

Lemma A.6. *With $e_{\varepsilon_1 L_i+\varepsilon_2 L_j}$ as in eq. (22) one has for $i<j<k$*

$$\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_k}, e_{\varepsilon_3 L_j-\varepsilon_2 L_k}\right] = -i e_{\varepsilon_1 L_i+\varepsilon_3 L_j}.$$

Proof. Start with

$$\begin{aligned}\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_k}, e_{\varepsilon_3 L_j-\varepsilon_2 L_k}\right] &= -\frac{1}{4}\Big[X_{\beta^{(1)}_{i,k}} - i\varepsilon_2 X_{\beta^{(2)}_{i,k}} - i\varepsilon_1 X_{\beta^{(3)}_{i,k}} - \varepsilon_1\varepsilon_2 X_{\beta^{(4)}_{i,k}} \\ &\quad X_{\beta^{(1)}_{j,k}} + i\varepsilon_2 X_{\beta^{(2)}_{j,k}} - i\varepsilon_3 X_{\beta^{(3)}_{j,k}} + \varepsilon_3\varepsilon_2 X_{\beta^{(4)}_{j,k}}\Big]\end{aligned}$$

and apply the previous lemma to find

$$\begin{aligned}-4\cdot\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_k}, e_{\varepsilon_3 L_j-\varepsilon_2 L_k}\right] &= \Big[X_{\beta^{(1)}_{i,k}} - i\varepsilon_2 X_{\beta^{(2)}_{i,k}} - i\varepsilon_1 X_{\beta^{(3)}_{i,k}} - \varepsilon_1\varepsilon_2 X_{\beta^{(4)}_{i,k}} \\ &\quad X_{\beta^{(1)}_{j,k}} + i\varepsilon_2 X_{\beta^{(2)}_{j,k}} - i\varepsilon_3 X_{\beta^{(3)}_{j,k}} + \varepsilon_3\varepsilon_2 X_{\beta^{(4)}_{j,k}}\Big] \\ &= -X_{\beta^{(1)}_{i,j}} + i\varepsilon_3 X_{\beta^{(2)}_{i,j}} - \varepsilon_2^2 X_{\beta^{(1)}_{i,j}} + i\varepsilon_2^2\varepsilon_3 X_{\beta^{(2)}_{i,j}} \\ &\quad + i\varepsilon_1 X_{\beta^{(3)}_{i,j}} + \varepsilon_1\varepsilon_3 X_{\beta^{(4)}_{i,j}} + i\varepsilon_1\varepsilon_2^2 X_{\beta^{(3)}_{i,j}} + \varepsilon_1\varepsilon_2^2\varepsilon_3 X_{\beta^{(4)}_{i,j}} \\ &= -2\left(X_{\beta^{(1)}_{i,j}} - i\varepsilon_3 X_{\beta^{(2)}_{i,j}} - i\varepsilon_1 X_{\beta^{(3)}_{i,j}} - \varepsilon_1\varepsilon_3 X_{\beta^{(4)}_{i,j}}\right) \\ &= 4i\cdot e_{\varepsilon_1 L_i+\varepsilon_3 L_j}\end{aligned}$$

so that $\left[e_{\varepsilon_1 L_i+\varepsilon_2 L_k}, e_{\varepsilon_3 L_j-\varepsilon_2 L_k}\right] = -i\cdot e_{\varepsilon_1 L_i+\varepsilon_3 L_j}$. □

Lemma A.7. *For $1 \leq i < j \leq m$ and $e_{\varepsilon_1 L_i}$ as in eq. (22) one has*

$$\left[e_{\varepsilon_1 L_i}, e_{\varepsilon_2 L_j}\right] = -2ie_{\varepsilon_1 L_i+\varepsilon_2 L_j}, \quad \left[e_{L_i}, e_{-L_i}\right] = 2H_i$$

$$\left[e_{-\varepsilon_1 L_i}, e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right] = ie_{\varepsilon_2 L_j}, \quad \left[e_{-\varepsilon_2 L_j}, e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right] = -ie_{\varepsilon_1 L_i}.$$

Proof. Compute with lemma A.5 that

$$\begin{aligned}
\left[e_{\varepsilon_1 L_i}, e_{\varepsilon_2 L_j}\right] &= -\left[X_{\beta^{(2)}_{i,m+1}} - i\varepsilon_1 X_{\beta^{(4)}_{i.m+1}}, X_{\beta^{(2)}_{j,m+1}} - i\varepsilon_2 X_{\beta^{(4)}_{j.m+1}}\right] \\
&= X_{\beta^{(1)}_{i,j}} - i\varepsilon_2 X_{\beta^{(2)}_{i.j}} - i\varepsilon_1 X_{\beta^{(3)}_{i.j}} - \varepsilon_1\varepsilon_2 X_{\beta^{(4)}_{i.j}} \\
&= -2ie_{\varepsilon_1 L_i+\varepsilon_2 L_j}.
\end{aligned}$$

Also,

$$\begin{aligned}
\left[e_{L_i}, e_{-L_i}\right] &= -\left[X_{\beta^{(2)}_{i,m+1}} - iX_{\beta^{(4)}_{i.m+1}}, X_{\beta^{(2)}_{i,m+1}} + iX_{\beta^{(4)}_{i.m+1}}\right] \\
&= -2i\left[X_{\beta^{(2)}_{i,m+1}}, X_{\beta^{(4)}_{i.m+1}}\right] = -2iX_{\alpha_{2i+1}} = 2H_i
\end{aligned}$$

according to lemma A.3. With lemma A.4 one computes that

$$\begin{aligned}
\left[e_{-\varepsilon_1 L_i}, e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right] &= -\frac{1}{2}\left[X_{\beta^{(2)}_{i,m+1}} + i\varepsilon_1 X_{\beta^{(4)}_{i.m+1}}, X_{\beta^{(1)}_{i,j}} - i\varepsilon_2 X_{\beta^{(2)}_{i.j}} - i\varepsilon_1 X_{\beta^{(3)}_{i.j}} - \varepsilon_1\varepsilon_2 X_{\beta^{(4)}_{i.j}}\right] \\
&= -\frac{1}{2}\left(X_{\beta^{(2)}_{j,m+1}} - i\varepsilon_2 X_{\beta^{(4)}_{j,m+1}} + \varepsilon_1^2 X_{\beta^{(2)}_{j,m+1}} - i\varepsilon_1^2\varepsilon_2 X_{\beta^{(4)}_{j,m+1}}\right) \\
&= -\left(X_{\beta^{(2)}_{j,m+1}} - i\varepsilon_2 X_{\beta^{(4)}_{j,m+1}}\right) = ie_{\varepsilon_2 L_j}
\end{aligned}$$

and

$$\begin{aligned}
\left[e_{-\varepsilon_2 L_j}, e_{\varepsilon_1 L_i+\varepsilon_2 L_j}\right] &= -\frac{1}{2}\left[X_{\beta^{(2)}_{j,m+1}} + i\varepsilon_2 X_{\beta^{(4)}_{j.m+1}}, X_{\beta^{(1)}_{i,j}} - i\varepsilon_2 X_{\beta^{(2)}_{i.j}} - i\varepsilon_1 X_{\beta^{(3)}_{i.j}} - \varepsilon_1\varepsilon_2 X_{\beta^{(4)}_{i.j}}\right] \\
&= -\frac{1}{2}\left(-X_{\beta^{(2)}_{i,m+1}} + i\varepsilon_1 X_{\beta^{(4)}_{i,m+1}} - \varepsilon_2^2 X_{\beta^{(2)}_{i,m+1}} + i\varepsilon_1\varepsilon_2^2 X_{\beta^{(4)}_{i,m+1}}\right) \\
&= X_{\beta^{(2)}_{i,m+1}} - i\varepsilon_1 X_{\beta^{(4)}_{i,m+1}} = -ie_{\varepsilon_1 L_i}.
\end{aligned}$$

□

A.2 Computation of relations

In this subsection, all relations stated in eqs. (30)-(37) are shown.

Lemma A.8. *One has with $X_\pm \coloneqq i\left(X_{\alpha_n} \pm iX_{\alpha_n+\alpha_3}\right)$ from eq. (29):*

$$\left[X_+, e_{\varepsilon L_1-L_2}\right] = X_{\alpha_2+\alpha_3+\alpha_n} - i\varepsilon X_{\alpha_1+\alpha_2+\alpha_3+\alpha_n} = \left[X_-, e_{\varepsilon L_1+L_2}\right]$$

$$\operatorname{ad}\left(X_+\right)^2\left(e_{\varepsilon L_1-L_2}\right) = 2e_{\varepsilon L_1+L_2}, \ \operatorname{ad}\left(X_-\right)^2\left(e_{\varepsilon L_1+L_2}\right) = 2e_{\varepsilon L_1-L_2}$$

$$\operatorname{ad}\left(X_+\right)^3\left(e_{\varepsilon L_1-L_2}\right) = 0 = \operatorname{ad}\left(X_-\right)^3\left(e_{\varepsilon L_1+L_2}\right), \ \operatorname{ad}\left(e_{\varepsilon_1 L_1-L_2}\right)^2\left(X_+\right) = 0 = \operatorname{ad}\left(e_{\varepsilon_1 L_1+L_2}\right)^2\left(X_-\right).$$

Proof. Starting from

$$e_{\varepsilon_1 L_1+\varepsilon_2 L_2} = \frac{i}{2}\cdot\left(X_{\beta^{(1)}_{1,2}} - i\varepsilon_2 X_{\beta^{(2)}_{1,2}} - i\varepsilon_1 X_{\beta^{(3)}_{1,2}} - \varepsilon_1\varepsilon_2 X_{\beta^{(4)}_{1,2}}\right)$$

one computes

$$\begin{aligned}
[X_\pm, e_{\varepsilon_1 L_1+\varepsilon_2 L_2}] &= -\frac{1}{2}\left[X_{\alpha_n}, X_{\alpha_2+\alpha_3} - i\varepsilon_2 X_{\alpha_2} - i\varepsilon_1 X_{\alpha_1+\alpha_2+\alpha_3} - \varepsilon_1\varepsilon_2 X_{\alpha_1+\alpha_2}\right] \\
&\quad \pm\frac{i}{2}\left[X_{\alpha_3+\alpha_n}, X_{\alpha_2+\alpha_3} - i\varepsilon_2 X_{\alpha_2} - i\varepsilon_1 X_{\alpha_1+\alpha_2+\alpha_3} - \varepsilon_1\varepsilon_2 X_{\alpha_1+\alpha_2}\right] \\
&= +\frac{1}{2}X_{\alpha_2+\alpha_3+\alpha_n} + 0 - \frac{i}{2}\varepsilon_1 X_{\alpha_1+\alpha_2+\alpha_3+\alpha_n} + 0 \\
&\quad +0 \pm i^2\frac{\varepsilon_2}{2}X_{\alpha_2+\alpha_3+\alpha_n} + 0 \pm \frac{i}{2}\varepsilon_1\varepsilon_2 X_{\alpha_1+\alpha_2+\alpha_3+\alpha_n} \\
&= \frac{1}{2}\left(1\mp\varepsilon_2\right)X_{\alpha_2+\alpha_3+\alpha_n} - \frac{i}{2}\varepsilon_1\left(1\mp\varepsilon_2\right)X_{\alpha_1+\alpha_2+\alpha_3+\alpha_n} \\
&= \frac{1\mp\varepsilon_2}{2}\left(X_{\alpha_2+\alpha_3+\alpha_n} - i\varepsilon_1 X_{\alpha_1+\alpha_2+\alpha_3+\alpha_n}\right)
\end{aligned}$$

which implies that

$$[X_+, e_{\varepsilon_1 L_1+L_2}] = 0 = [X_-, e_{\varepsilon_1 L_1-L_2}]\,.$$

Furthermore, one immediately deduces

$$[X_+, e_{\varepsilon_1 L_1-L_2}] = X_{\alpha_2+\alpha_3+\alpha_n} - i\varepsilon_1 X_{\alpha_1+\alpha_2+\alpha_3+\alpha_n} = [X_-, e_{\varepsilon_1 L_1+L_2}]\,.$$

With $[X_+, X_-] = 2H_2$ one computes further that

$$\begin{aligned}
\operatorname{ad}(X_+)^2(e_{\varepsilon_1 L_1-L_2}) &= [X_+,[X_+,e_{\varepsilon_1 L_1-L_2}]] = [X_+,[X_-,e_{\varepsilon_1 L_1+L_2}]] \\
&= [2H_2, e_{\varepsilon_1 L_1+L_2}] + [X_-, 0] = 2e_{\varepsilon_1 L_1+L_2} \\
\operatorname{ad}(X_+)^3(e_{\varepsilon_1 L_1-L_2}) &= 2\,[X_+, e_{\varepsilon_1 L_1+L_2}] = 0 \\
\operatorname{ad}(X_-)^2(e_{\varepsilon_1 L_1+L_2}) &= [X_-,[X_-,e_{\varepsilon_1 L_1+L_2}]] = [X_-,[X_+,e_{\varepsilon_1 L_1-L_2}]] \\
&= [-2H_2, e_{\varepsilon_1 L_1-L_2}] + [X_+, 0] = 2e_{\varepsilon_1 L_1-L_2} \\
\operatorname{ad}(X_-)^2(e_{\varepsilon_1 L_1+L_2}) &= 2\,[X_-, e_{\varepsilon_1 L_1-L_2}] = 0.
\end{aligned}$$

In order to obtain the other Serre-type relations one has

$$\begin{aligned}
\operatorname{ad}(e_{\varepsilon_1 L_1-L_2})^2(X_+) &= [e_{\varepsilon_1 L_1-L_2},[e_{\varepsilon_1 L_1-L_2},X_+]] = [e_{\varepsilon_1 L_1-L_2},[e_{\varepsilon_1 L_1+L_2},X_-]] \\
&= [[e_{\varepsilon_1 L_1-L_2}, e_{\varepsilon_1 L_1+L_2}], X_-] + [e_{\varepsilon_1 L_1+L_2},[e_{\varepsilon_1 L_1-L_2},X_-]] \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\operatorname{ad}(e_{\varepsilon_1 L_1+L_2})^2(X_-) &= [e_{\varepsilon_1 L_1+L_2},[e_{\varepsilon_1 L_1+L_2},X_-]] = [e_{\varepsilon_1 L_1+L_2},[e_{\varepsilon_1 L_1-L_2},X_+]] \\
&= [[e_{\varepsilon_1 L_1+L_2}, e_{\varepsilon_1 L_1-L_2}], X_+] + [e_{\varepsilon_1 L_1-L_2},[e_{\varepsilon_1 L_1+L_2},X_+]] \\
&= 0.
\end{aligned}$$

□

Lemma A.9. *One has with $X_\pm := i\left(X_{\alpha_n} \pm iX_{\alpha_n+\alpha_3}\right)$ from eq. (29):*

$$\begin{gathered}\left[X_+, e_{-L_2+\varepsilon L_j}\right] = -iX_{\alpha_n+\beta_{2,j}^{(3)}} - \varepsilon X_{\alpha_n+\beta_{2,j}^{(4)}} = -\left[X_-, e_{L_2+\varepsilon L_j}\right] \\ \mathrm{ad}\left(X_+\right)^2\left(e_{-L_2+\varepsilon L_j}\right) = -2e_{L_2+\varepsilon L_j},\ \mathrm{ad}\left(X_-\right)^2\left(e_{L_2+\varepsilon L_j}\right) = -2e_{-L_2+\varepsilon L_j} \\ \mathrm{ad}\left(X_+\right)^3\left(e_{-L_2+\varepsilon L_j}\right) = 0 = \mathrm{ad}\left(X_-\right)^3\left(e_{L_2+\varepsilon L_j}\right),\ \mathrm{ad}\left(e_{-L_2+\varepsilon L_j}\right)^2\left(X_+\right) = 0 = \mathrm{ad}\left(e_{L_2+\varepsilon L_j}\right)^2\left(X_-\right).\end{gathered}$$

Proof. Note that the support of $\beta_{2,j}^{(1)}$ and $\beta_{2,j}^{(2)}$ starts at α_4 whereas the support of $\beta_{2,j}^{(3)}$ and $\beta_{2,j}^{(4)}$ starts at α_3. Hence,

$$\alpha_n + \beta_{2,j}^{(1)},\ \alpha_n + \beta_{2,j}^{(2)},\ \alpha_3 + \alpha_n + \beta_{2,j}^{(3)},\ \alpha_3 + \alpha_n + \beta_{2,j}^{(4)} \notin \Delta\left(E_n\right),$$

because $\left(\alpha_3 + \alpha_n \mid \beta_{2,j}^{(3)}\right) = 0 = \left(\alpha_3 + \alpha_n \mid \beta_{2,j}^{(4)}\right)$. From this one computes with $\alpha_3 + \beta_{2,j}^{(1)} = \beta_{2,j}^{(3)}$, $\alpha_3 + \beta_{2,j}^{(2)} = \beta_{2,j}^{(4)}$ and lemma A.5 that

$$\begin{aligned}\left[X_\pm, e_{\varepsilon_1 L_2+\varepsilon L_j}\right] &= -\frac{1}{2}\left[X_{\alpha_n} \pm iX_{\alpha_n+\alpha_3}, X_{\beta_{2,j}^{(1)}} - i\varepsilon X_{\beta_{2,j}^{(2)}} - i\varepsilon_1 X_{\beta_{2,j}^{(3)}} - \varepsilon_1\varepsilon X_{\beta_{2,j}^{(4)}}\right] \\ &= -\frac{1}{2}\left(0 + 0 - i\varepsilon_1 X_{\alpha_n+\beta_{2,j}^{(3)}} - \varepsilon_1\varepsilon X_{\alpha_n+\beta_{2,j}^{(4)}}\right) \\ &\quad \mp\frac{i}{2}\left(X_{\alpha_n+\alpha_3+\beta_{2,j}^{(1)}} - i\varepsilon X_{\alpha_n+\alpha_3+\beta_{2,j}^{(2)}} + 0 + 0\right) \\ &= \frac{i\varepsilon_1}{2}X_{\alpha_n+\beta_{2,j}^{(3)}} + \frac{1}{2}\varepsilon_1\varepsilon X_{\alpha_n+\beta_{2,j}^{(4)}} \mp \frac{i}{2}X_{\alpha_n+\beta_{2,j}^{(3)}} \mp \frac{\varepsilon}{2}X_{\alpha_n+\beta_{2,j}^{(4)}} \\ &= \frac{i}{2}\left(\varepsilon_1 \mp 1\right)X_{\alpha_n+\beta_{2,j}^{(3)}} + \frac{\varepsilon}{2}\left(\varepsilon_1 \mp 1\right)X_{\alpha_n+\beta_{2,j}^{(4)}}\end{aligned}$$

which shows

$$\begin{gathered}\left[X_+, e_{L_2+\varepsilon L_j}\right] = 0 = \left[X_-, e_{-L_2+\varepsilon L_j}\right] \\ \left[X_+, e_{-L_2+\varepsilon L_j}\right] = -iX_{\alpha_n+\beta_{2,j}^{(3)}} - \varepsilon X_{\alpha_n+\beta_{2,j}^{(4)}} = -\left[X_-, e_{L_2+\varepsilon L_j}\right]\end{gathered}$$

From this one goes on to deduce

$$\begin{aligned}\mathrm{ad}\left(e_{-L_2+\varepsilon L_j}\right)^2\left(X_+\right) &= +\left[e_{-L_2+\varepsilon L_j}, \left[X_-, e_{+L_2+\varepsilon L_j}\right]\right] \\ &= 0 + \left[X_-, 0\right] = 0 \\ \mathrm{ad}\left(e_{L_2+\varepsilon L_j}\right)^2\left(X_-\right) &= +\left[e_{L_2+\varepsilon L_j}, \left[X_+, e_{-L_2+\varepsilon L_j}\right]\right] \\ &= 0 + \left[X_+, 0\right] = 0\end{aligned}$$

$$\begin{aligned}\mathrm{ad}\left(X_+\right)^2\left(e_{-L_2+\varepsilon L_j}\right) &= -\mathrm{ad}\left(X_+\right)\left(\left[X_-, e_{L_2+\varepsilon L_j}\right]\right) = -2\left[H_2, e_{L_2+\varepsilon L_j}\right] + 0 \\ &= -2e_{L_2+\varepsilon L_j} \\ \Rightarrow\ \mathrm{ad}\left(X_+\right)^3\left(e_{-L_2+\varepsilon L_j}\right) &= 0 \\ \mathrm{ad}\left(X_-\right)^2\left(e_{L_2+\varepsilon L_j}\right) &= -\mathrm{ad}\left(X_-\right)\left(\left[X_+, e_{-L_2+\varepsilon L_j}\right]\right) = 2\left[H_2, e_{-L_2+\varepsilon L_j}\right] + 0 \\ &= -2e_{-L_2+\varepsilon L_j} \\ \Rightarrow\ \mathrm{ad}\left(X_-\right)^3\left(e_{L_2+\varepsilon L_j}\right) &= 0.\end{aligned}$$

□

Lemma A.10. *One has for $i > 2$ that*

$$\left[X_\pm, e_{\varepsilon_1 L_i + \varepsilon_2 L_j}\right] = 0 = \left[X_\pm, e_{\varepsilon_1 L_i}\right].$$

Proof. This is seen directly from the fact that for $i > 2$

$$\alpha_n \pm \beta_{i,j}^{(k)}, \alpha_3 + \alpha_n \pm \beta_{i,j}^{(k)} \notin \Delta\left(E_n\right) \ \forall, k = 1, \dots, 4$$

and therefore all commutators vanish. $\square$

Together, lemmas A.8, A.9 and A.10 show eqs. (31)-(33). The next lemma shows eq. (36) and the nontrivial part of (37).

Lemma A.11. *One has the following relations among $X_\pm$ and $e_{\varepsilon L_2}$:*

$$[X_\pm, e_{\pm L_2}] = 0,\ [X_+, e_{-L_2}] = -2iX_{\alpha_n + \beta_{2,m+1}^{(4)}} = -\left[X_-, e_{+L_2}\right]$$

$$ad\left(X_+\right)^2\left(e_{-L_2}\right) = -2e_{L_2},\ ad\left(X_+\right)^3\left(e_{-L_2}\right) = 0 = ad\left(X_-\right)^3\left(e_{+L_2}\right),\ ad\left(X_-\right)^2\left(e_{+L_2}\right) = -2e_{-L_2}$$

$$ad\left(e_{\pm L_2}\right)^2\left(X_\mp\right) = -2X_\pm,\ ad\left(e_{\pm L_2}\right)^3\left(X_\mp\right) = 0.$$

Proof. One computes

$$\begin{aligned}
[X_\pm, e_{\varepsilon L_2}] &= i^2\left[X_{\alpha_n} \pm iX_{\alpha_n+\alpha_3}, X_{\beta_{2,m+1}^{(2)}} - i\varepsilon X_{\beta_{2,m+1}^{(4)}}\right] \\
&= 0 + i\varepsilon X_{\alpha_n + \beta_{2,m+1}^{(4)}} \mp iX_{(\alpha_n+\alpha_3)+\beta_{2,m+1}^{(2)}} + 0 \\
&= i\left(\varepsilon \mp 1\right)X_{\alpha_n + \beta_{2,m+1}^{(4)}}
\end{aligned}$$

because $X_{(\alpha_n+\alpha_3)+\beta_{2,m+1}^{(2)}} = X_{\alpha_n+(\alpha_3+\beta_{2,m+1}^{(2)})} = X_{\alpha_n+\beta_{2,m+1}^{(4)}}$. This implies the first line of relations. From this one goes on to compute

$$\begin{aligned}
\mathrm{ad}\left(X_+\right)^2\left(e_{-L_2}\right) &= -\left[X_+, \left[X_-, e_{+L_2}\right]\right] = -\left[2H_2, e_{+L_2}\right] + 0 \\
&= -2e_{L_2} \\
\Rightarrow\ \mathrm{ad}\left(X_+\right)^3\left(e_{-L_2}\right) &= 0 \\
\mathrm{ad}\left(X_-\right)^2\left(e_{+L_2}\right) &= -\left[X_-, \left[X_+, e_{-L_2}\right]\right] = \left[2H_2, e_{-L_2}\right] + 0 \\
&= -2e_{-L_2} \\
\Rightarrow\ \mathrm{ad}\left(X_-\right)^3\left(e_{+L_2}\right) &= 0\,.
\end{aligned}$$

For the last line one checks that

$$\begin{aligned}
\mathrm{ad}\left(e_{\pm L_2}\right)^2\left(X_\mp\right) &= \mathrm{ad}\left(e_{\pm L_2}\right)\left(\left[e_{\pm L_2}, X_\mp\right]\right) = -\mathrm{ad}\left(e_{\pm L_2}\right)\left(\left[e_{\mp L_2}, X_\pm\right]\right) \\
&= -\left[\pm 2H_2, X_\pm\right] - 0 = -2X_\pm \\
\Rightarrow\ \mathrm{ad}\left(e_{\pm L_2}\right)^3\left(X_\mp\right) &= 0\,.
\end{aligned}$$

$\square$

The next lemma shows eq. (130):

Lemma A.12. *One has the following relations among $X_\pm$ and $e_{\varepsilon L_1}$:*

$$0 \neq [X_\pm, e_{\varepsilon L_1}] = -X_{\alpha_n+\beta^{(2)}_{1,m+1}} + i\varepsilon X_{\alpha_n+\beta^{(4)}_{1,m+1}} \mp i\left[X_{\alpha_n+\alpha_3}, X_{\beta^{(2)}_{1,m+1}}\right] \mp \varepsilon \left[X_{\alpha_n+\alpha_3}, X_{\beta^{(4)}_{1,m+1}}\right] \tag{130}$$

$$ad\,(X_\pm)^2\,(e_{\varepsilon L_1}) = 0 = ad\,(e_{\varepsilon L_1})^2\,(X_\pm)\,.$$

Proof. The first result (130) follows from expansion of

$$\begin{aligned}
[X_\pm, e_{\varepsilon L_1}] &= i^2\left[X_{\alpha_n} \pm iX_{\alpha_n+\alpha_3}, X_{\beta^{(2)}_{1,m+1}} - i\varepsilon X_{\beta^{(4)}_{1,m+1}}\right] \\
&= -X_{\alpha_n+\beta^{(2)}_{1,m+1}} + i\varepsilon X_{\alpha_n+\beta^{(4)}_{1,m+1}} \mp i\left[X_{\alpha_n+\alpha_3}, X_{\beta^{(2)}_{1,m+1}}\right] \mp \varepsilon\left[X_{\alpha_n+\alpha_3}, X_{\beta^{(4)}_{1,m+1}}\right],
\end{aligned}$$

where this is nonzero because $\left(\beta^{(k)}_{1,m+1}|\alpha_n\right) = -1 = \left(\beta^{(k)}_{1,m+1}|\alpha_n+\alpha_3\right)$ for $k = 2, 4$. Since all the pairwise sums of roots yield different E_n-roots, the above expression is nonzero. One computes further that

$$\begin{aligned}
\mathrm{ad}\,(e_{\varepsilon L_1})^2\,(X_\pm) &= i^2\left\{\mathrm{ad}\left(X_{\beta^{(2)}_{1,m+1}}\right) - i\varepsilon\mathrm{ad}\left(X_{\beta^{(4)}_{1,m+1}}\right)\right\}^2 (iX_{\alpha_n} \mp X_{\alpha_n+\alpha_3}) \\
&= -i\left\{\mathrm{ad}\left(X_{\beta^{(2)}_{1,m+1}}\right)^2 - i\varepsilon\mathrm{ad}\left(X_{\beta^{(2)}_{1,m+1}}\right)\mathrm{ad}\left(X_{\beta^{(4)}_{1,m+1}}\right) - i\varepsilon\mathrm{ad}\left(X_{\beta^{(4)}_{1,m+1}}\right)\mathrm{ad}\left(X_{\beta^{(2)}_{1,m+1}}\right)\right. \\
&\quad \left. -\varepsilon^2\mathrm{ad}\left(X_{\beta^{(4)}_{1,m+1}}\right)^2\right\}(X_{\alpha_n}) \\
&\quad \pm\left\{\mathrm{ad}\left(X_{\beta^{(2)}_{1,m+1}}\right)^2 - i\varepsilon\mathrm{ad}\left(X_{\beta^{(2)}_{1,m+1}}\right)\mathrm{ad}\left(X_{\beta^{(4)}_{1,m+1}}\right) - i\varepsilon\mathrm{ad}\left(X_{\beta^{(4)}_{1,m+1}}\right)\mathrm{ad}\left(X_{\beta^{(2)}_{1,m+1}}\right)\right. \\
&\quad \left. -\varepsilon^2\mathrm{ad}\left(X_{\beta^{(4)}_{1,m+1}}\right)^2\right\}(X_{\alpha_n+\alpha_3})\,.
\end{aligned}$$

In general, let $\alpha, \gamma \in \Delta(A_{n-1})$ and $\delta \in \Delta(E_n)$ such that $\delta \pm \alpha \notin \Delta(E_n)$ and $(\gamma|\delta) = -1$ with $\delta - \gamma \notin \Delta(E_n)$. Then one computes

$$\begin{aligned}
[X_\gamma, [X_{\alpha+\gamma}, X_\delta]] &= \left[X_\gamma, [X_\alpha, [X_\gamma, X_\delta]] - \left[X_\gamma, \underbrace{[X_\alpha, X_\delta]}_{=0}\right]\right] \\
&= \left[X_\alpha, \mathrm{ad}\,(X_\gamma)^2\,(X_\delta)\right] - \left[\underbrace{[X_\alpha, X_\gamma]}_{=X_{\alpha+\gamma}}, [X_\gamma, X_\delta]\right] \\
&= \mathrm{ad}\,(X_\alpha)\,\mathrm{ad}\,(X_\gamma)^2\,(X_\delta) - \mathrm{ad}\,(X_{\alpha+\gamma})\,\mathrm{ad}\,(X_\gamma)\,(X_\delta)\,.
\end{aligned}$$

This is equivalent to

$$\{\mathrm{ad}\,(X_{\alpha+\gamma})\,\mathrm{ad}\,(X_\gamma) + \mathrm{ad}\,(X_\gamma)\,\mathrm{ad}\,(X_{\alpha+\gamma})\}\,(X_\delta) = \mathrm{ad}\,(X_\alpha)\,\mathrm{ad}\,(X_\gamma)^2\,(X_\delta) = 0, \tag{131}$$

where the last equality follows from the $\Delta(E_n)$-root system: Neither $\delta - \gamma$ nor $\delta + 2\gamma$ are roots and therefore $\mathrm{ad}\,(X_\gamma)^2\,(X_\delta) = c \cdot X_\delta$ with $c \neq 0$. But $\delta \pm \alpha \notin \Delta(E_n)$ and thus, the application of $\mathrm{ad}\,(X_\alpha)$ in the end renders the expression zero. This applies in the above cases with $\delta \in \{\alpha_n, \alpha_n + \alpha_3\}$, $\gamma = \beta^{(2)}_{1,m+1}$, $\alpha = \alpha_1$.

Again, let $\alpha,\gamma\in\Delta(A_{n-1})$ and $\delta\in\Delta(E_n)$ such that $\delta\pm\alpha\notin\Delta(E_n)$ and $(\gamma|\delta)=-1$ with $\delta-\gamma\notin\Delta(E_n)$, then

$$\begin{aligned}\operatorname{ad}(X_{\alpha+\gamma})^2(X_\delta) &= [X_{\alpha+\gamma},[[X_\alpha,X_\gamma],X_\delta]]\\ &= [X_{\alpha+\gamma}[X_\alpha,[X_\gamma,X_\delta]]-0]\\ &= \operatorname{ad}(X_\alpha)([X_{\alpha+\gamma},[X_\gamma,X_\delta]])-[[X_\alpha,X_{\alpha+\gamma}],[X_\gamma,X_\delta]]\\ &= \operatorname{ad}(X_\alpha)\operatorname{ad}(X_{\alpha+\gamma})\operatorname{ad}(X_\gamma)(X_\delta)+\operatorname{ad}(X_\gamma)^2(X_\delta)\\ &= \operatorname{ad}(X_\gamma)^2(X_\delta),\end{aligned}\tag{132}$$

where the last equality again follows from the $\Delta(E_n)$-root system: $\gamma+\delta\in\Delta(E_n)$ but $(\gamma+\alpha|\gamma+\delta)=0$ and $(\gamma+\delta)-(\gamma+\alpha)=-\alpha+\delta\notin\Delta(E_n)$ implies that $\operatorname{ad}(X_{\alpha+\gamma})\operatorname{ad}(X_\gamma)(X_\delta)=0$. Applying (131) and (132) to $\operatorname{ad}(e_{\varepsilon L_1})^2(X_\pm)$ shows that it needs to vanish. For the last relation $\operatorname{ad}(X_\pm)^2(e_{\varepsilon L_1})=0$ one could in principle do a similar computation but it is easier to rewrite $e_{\varepsilon L_1}=c\cdot[e_{\varepsilon L_1\pm L_2},e_{\mp L_2}]$ and use that $[X_\pm,e_{\varepsilon L_1\pm L_2}]=0$ and $\operatorname{ad}(X_\pm)^2(e_{\mp L_2})=-2e_{\pm L_2}$ according to lemmas A.11 and A.8 together with $[e_{\varepsilon L_1\pm L_2},e_{\pm L_2}]=0$. □

Lemma A.13. *One has for $j\geq 3$ that*

$$\left[X_\pm,e_{\varepsilon_1L_1+\varepsilon_2L_j}\right]\neq 0,\ \ ad(X_\pm)^2\left(e_{\varepsilon_1L_1+\varepsilon_2L_j}\right)=0=ad\left(e_{\varepsilon_1L_1+\varepsilon_2L_j}\right)^2(X_\pm)$$

Proof. First observe that

$$e_{\varepsilon_1L_1+\varepsilon_2L_j}=-i\left[e_{\varepsilon_1L_1-L_2},e_{L_2+\varepsilon_2L_j}\right]=-i\left[e_{\varepsilon_1L_1+L_2},e_{-L_2+\varepsilon_2L_j}\right]$$

which implies that

$$\begin{aligned}\operatorname{ad}(X_+)^2\left(e_{\varepsilon_1L_1+\varepsilon_2L_j}\right) &= -i\left[\operatorname{ad}(X_+)^2(e_{\varepsilon_1L_1-L_2}),e_{L_2+\varepsilon_2L_j}\right]+0\\ &= -2i\left[e_{\varepsilon_1L_1+L_2},e_{L_2+\varepsilon_2L_j}\right]=0\end{aligned}$$

$$\begin{aligned}\operatorname{ad}(X_-)^2\left(e_{\varepsilon_1L_1+\varepsilon_2L_j}\right) &= -i\left[\operatorname{ad}(X_-)^2(e_{\varepsilon_1L_1+L_2}),e_{-L_2+\varepsilon_2L_j}\right]+0\\ &= -2i\left[e_{\varepsilon_1L_1-L_2},e_{-L_2+\varepsilon_2L_j}\right]=0\,.\end{aligned}$$

The second relation is shown similarly to lemma A.12. With the shorthand $\operatorname{ad}(x)=:\widetilde{x}$ one has that

$$\begin{aligned}\operatorname{ad}\left(e_{\varepsilon_1L_1+\varepsilon_2L_j}\right)^2(X_\pm) &= \frac{i^3}{4}\left\{\widetilde{X}_{\beta^{(1)}_{1,j}}-i\varepsilon_2\widetilde{X}_{\beta^{(2)}_{1,j}}-i\varepsilon_1\widetilde{X}_{\beta^{(3)}_{1,j}}-\varepsilon_1\varepsilon_2\widetilde{X}_{\beta^{(4)}_{1,j}}\right\}^2(X_{\alpha_n}\pm iX_{\alpha_n+\alpha_3})\\ &= -\frac{i}{4}\left\{\sum_{k=1}^4 d_k\widetilde{X}^2_{\beta^{(k)}_{1,j}}+\sum_{k<l}c_{kl}\left(\widetilde{X}_{\beta^{(k)}_{1,j}}\widetilde{X}_{\beta^{(l)}_{1,j}}+\widetilde{X}_{\beta^{(l)}_{1,j}}\widetilde{X}_{\beta^{(k)}_{1,j}}\right)\right\}(X_{\alpha_n})\\ &\quad\pm\frac{1}{4}\left\{\sum_{k=1}^4 d_k\widetilde{X}^2_{\beta^{(k)}_{1,j}}+\sum_{k<l}c_{kl}\left(\widetilde{X}_{\beta^{(k)}_{1,j}}\widetilde{X}_{\beta^{(l)}_{1,j}}+\widetilde{X}_{\beta^{(l)}_{1,j}}\widetilde{X}_{\beta^{(k)}_{1,j}}\right)\right\}(X_{\alpha_n+\alpha_3}),\end{aligned}$$

with $d_1=1$, $d_2=-\varepsilon_2^2=-1$, $d_3=-\varepsilon_1^2=-1$, $d_4=\varepsilon_1^2\varepsilon_2^2=1$ and $c_{23}=-\varepsilon_1\varepsilon_2=c_{14}$ (the others won't matter). The involved roots $\beta^{(k)}_{1,j}$ are all of the shape $\alpha_1+\beta^{(2)}_{1,j}+\alpha_{2j-1}$ where one or both roots α_1, α_{2j-1} may be absent. Since $\alpha_1, \alpha_{2j-1} \perp \alpha_n, \alpha_3$ one can apply the result (132) from the proof of the previous lemma. Note that signs from the order $X_{\gamma+\beta} = \pm X_{\beta+\gamma}$ do not matter in the square terms $\widetilde{X}^2_{\beta^{(k)}_{1,j}}$ and so one obtains

$$\widetilde{X}^2_{\beta^{(k)}_{1,j}}\left(X_{\alpha_n}\right) = \widetilde{X}^2_{\beta^{(2)}_{1,j}}\left(X_{\alpha_n}\right),\ \widetilde{X}^2_{\beta^{(k)}_{1,j}}\left(X_{\alpha_n+\alpha_3}\right) = \widetilde{X}^2_{\beta^{(2)}_{1,j}}\left(X_{\alpha_n+\alpha_3}\right)\ \forall k = 1,2,3,4.$$

This yields (as $\sum_{k=1}^4 d_k = 0$)

$$\sum_{k=1}^4 d_k \widetilde{X}^2_{\beta^{(k)}_{1,j}}\left(X_{\alpha_n}\right) = 0 = \sum_{k=1}^4 d_k \widetilde{X}^2_{\beta^{(k)}_{1,j}}\left(X_{\alpha_n+\alpha_3}\right).$$

For the pairs $(1,2)$, $(1,3)$, $(2,4)$ and $(3,4)$ one can apply (131) as in the previous lemma because each anticommutator yields a term proportional to $\widetilde{X}_\alpha\left(X_\delta\right)$ with $\alpha \in \{\alpha_1, \alpha_{2j-1}\}$ and $\delta \in \{\alpha_n, \alpha_n + \alpha_3\}$ which vanishes. For $(2,3)$ and $(1,4)$ however, this strategy does not work. Instead one has that because $\left(\beta^{(2)}_{1,j}|\beta^{(3)}_{1,j}\right) = 0 = \left(\beta^{(1)}_{1,j}|\beta^{(4)}_{1,j}\right)$ the corresponding $X_{\beta^{(k)}_{1,j}}$ commute. Now

$$\delta + \beta^{(2)}_{1,j} + \beta^{(3)}_{1,j} = \delta + \beta^{(1)}_{1,j} + \beta^{(4)}_{1,j} \in \Delta\left(E_n\right) \text{ for } \delta \in \{\alpha_n, \alpha_n + \alpha_3\}.$$

One then computes with $\delta + \beta^{(1)}_{1,j} + \beta^{(2)}_{1,j} \notin \Delta\left(E_n\right)$ and $\left[X_{\alpha_1}, X_\delta\right] = 0$ that

$$\begin{aligned}
\widetilde{X}_{\beta^{(3)}_{1,j}}\widetilde{X}_{\beta^{(2)}_{1,j}}\left(X_\delta\right) &= \widetilde{X}_{\alpha_1}\widetilde{X}_{\beta^{(1)}_{1,j}}\widetilde{X}_{\beta^{(2)}_{1,j}}\left(X_\delta\right) - \widetilde{X}_{\beta^{(1)}_{1,j}}\widetilde{X}_{\alpha_1}\widetilde{X}_{\beta^{(2)}_{1,j}}\left(X_\delta\right) \\
&= 0 - \widetilde{X}_{\beta^{(1)}_{1,j}}\left(\widetilde{X}_{\alpha_1}\widetilde{X}_{\beta^{(2)}_{1,j}} - \widetilde{X}_{\beta^{(2)}_{1,j}}\widetilde{X}_{\alpha_1}\right)\left(X_\delta\right) + 0 \\
&= -\widetilde{X}_{\beta^{(1)}_{1,j}}\widetilde{X}_{\beta^{(4)}_{1,j}}\left(X_\delta\right).
\end{aligned} \tag{133}$$

Note that $\alpha_1 + \beta^{(2)}_{1,j} = \beta^{(4)}_{1,j}$ implies $\left[X_{\alpha_1}, X_{\beta^{(2)}_{1,j}}\right] = X_{\beta^{(4)}_{1,j}}$ in my sign convention because A_{n-1}-Berman elements are built from left to right. With $c_{23} = c_{14}$ this implies that

$$\begin{aligned}
&\left\{c_{23}\left(\widetilde{X}_{\beta^{(2)}_{1,j}}\widetilde{X}_{\beta^{(3)}_{1,j}} + \widetilde{X}_{\beta^{(3)}_{1,j}}\widetilde{X}_{\beta^{(2)}_{1,j}}\right) + c_{14}\left(\widetilde{X}_{\beta^{(1)}_{1,j}}\widetilde{X}_{\beta^{(4)}_{1,j}} + \widetilde{X}_{\beta^{(4)}_{1,j}}\widetilde{X}_{\beta^{(1)}_{1,j}}\right)\right\}\left(X_\delta\right) \\
= \ & 2c_{23}\widetilde{X}_{\beta^{(3)}_{1,j}}\widetilde{X}_{\beta^{(2)}_{1,j}}\left(X_\delta\right) + 2c_{14}\widetilde{X}_{\beta^{(1)}_{1,j}}\widetilde{X}_{\beta^{(4)}_{1,j}}\left(X_\delta\right) = 0.
\end{aligned}$$

□

This shows eq. (34).

B Documentation of tensor products and reproducability

In this section I will explain in detail how the computer-based analysis of $\mathfrak{k}\left(E_{10}\right)$-modules from section 6 works and how to reproduce the results. Section B.1 explains how the representation matrices are implemented and which codes reproduce them[57] whereas section B.2 deals with the decomposition into $\mathfrak{so}(10, \mathbb{C})$-modules. Section B.3 explains in detail how the $\frac{1}{2}$-spin representation $\mathcal{S}_{\frac{1}{2}}$ is set up analytically in a weight space basis w.r.t. $\mathfrak{so}(10, \mathbb{C})$ such that the representation matrices are as sparse as possible. In section C I provide a technical documentation for most of the functions that are used in my scripts and notebooks. One important thing to note is that in all of the codes the exceptional Berman generator is called X_2 while throughout this document it is called X_{10}.

[57] All the Sagemath-notebooks and scripts are available online here: http://dx.doi.org/10.22029/jlupub-533.

B.1 Generating the representation matrices

In order to perform and analyze the $\mathfrak{so}(10,\mathbb{C})$-decompositions of the $\mathfrak{k}(E_{10})$-representations it is necessary to spell out how $\mathfrak{so}(10,\mathbb{C})$ is generated in terms of the Berman generators of $\mathfrak{k}(E_{10})$. Based on section 2 and more precisely eq. (22) one has that for $j=1,2,3,4$ a Weyl canonical basis for $\mathfrak{so}(10,\mathbb{C})$ is given via

$$\begin{aligned} E_j &= \frac{1}{2}\cdot\left([X_{2j},X_{2j+1}]+i\cdot X_{2j}-i\cdot[X_{2j-1},[X_{2j},X_{2j+1}]]+[X_{2j-1},X_{2j}]\right)\\ E_5 &= \frac{1}{2}\cdot\left([X_8,X_9]-i\cdot X_8-i\cdot[X_7,[X_8,X_9]]-[X_7,X_8]\right)\end{aligned}$$

$$\begin{aligned} F_j &= \frac{1}{2}\cdot\left([X_{2j},X_{2j+1}]-i\cdot X_{2j}+i\cdot[X_{2j-1},[X_{2j},X_{2j+1}]]+[X_{2j-1},X_{2j}]\right)\\ F_5 &= \frac{1}{2}\cdot\left([X_8,X_9]+i\cdot X_8+i\cdot[X_7,[X_8,X_9]]-[X_7,X_8]\right)\end{aligned}$$

$$h_j=-i\cdot(X_{2j-1}-X_{2j+1})\ ,\ h_5=-i\cdot(X_7+X_9)\,,\ H_j=-i\cdot X_{2j-1},\ \mathfrak{h}_{\mathbb{R}}:=\operatorname{span}_{\mathbb{R}}\{X_1,X_3,\ldots,X_9\}\,.$$

The major difference between the Weyl-canonical basis and the Chevalley basis is that $[E_i,F_j]=-\delta_{ij}h_i$ instead of $+\delta_{ij}h_i$. Also, in my implementation of the Berman generators, the E_i and F_i will be real and in some cases even rational which allows for faster exact computations in Sagemath because one can use the rational number field instead of the symbolic ring. Assume that the representation matrices of the Berman generators of $\mathfrak{k}(A_9)(\mathbb{R})$ are chosen to be skew-hermitian, i.e. $X_i^\dagger=-X_i$ for $i\in\{1,\ldots,9\}$. Together with the additional i in the definition of $h_1,\ldots,h_5$ and $H_1,\ldots,H_5$ this provides that $\mathbb{R}$-linear combinations of those are hermitian and that E- and F-type Weyl-operators are skew-conjugate to each other:

$$h_\alpha^\dagger=h_\alpha\ \forall\, h_\alpha\in i\cdot\mathfrak{h}_{\mathbb{R}},\ E_i^\dagger=-F_i\ .$$

Furthermore If one manages to realize these relations in a representation of $\mathfrak{k}(E_{10})$ over $\mathbb{C}^n$ one has the advantage that the standard hermitian inner product of $\mathbb{C}^n$ is proportional to the hermitian form that is induced by the $\mathfrak{so}(10,\mathbb{C})$-contravariant bilinear form. With respect to this form, the different $\mathfrak{so}(10,\mathbb{C})$-modules are orthogonal, which I will exploit in decompositions. Note that the additional Berman generator X_{10} is excluded from this. Its representation matrix will in most cases not be skew-hermitian. The only case where this will happen is for powers of $\left(\mathcal{S}_{\frac{1}{2}},\rho_{\frac{1}{2}}\right)$, since im $\left(\rho_{\frac{1}{2}}\right)$ is coincidentally isomorphic to $\mathfrak{so}(32)$ on the level of matrix algebras (cp. [HKL15]).

The above definitions of the $\mathfrak{so}(10,\mathbb{C})$-generators in terms of Berman-generators are implemented in the functions `get_E_ladder`, `get_F_ladder`, `get_H_weyl` and `get_H_orth` (see section C.1.1 for a technical documentation) which give back the matrices as a list with 5 entries. A test of the Weyl-relations is implemented for dense and sparse matrices separately, called `ladder_check` and `ladder_check_sparse` respectively (see C.1.2). The sparse check is considerably faster then the dense one in most cases studied as it checks relations $R(A,B)=0$ for sparse matrices A,B by computing the matrix norm of the sparse matrix $R(A,B)$ which is quite fast.

Additional tests are available for the Berman relations of $\mathfrak{k}(E_{10})$ as spelled out in section C.1.3 via the functions `Berman_check` and `Berman_check_sparse` for dense and sparse matrices respectively.

As in section 5 I spell out D_5-weights w.r.t. the orthonormal basis $H_1,\ldots,H_5$ of $\mathfrak{h}_{\mathbb{R}}(D_5)$. A weight λ can be written as $\lambda=\sum_{i=1}^5 a_iL_i$ with $L_i\in\mathfrak{h}_{\mathbb{R}}^*(D_5)$ such that $L_i(H_j)=\delta_{ij}$ and $a_i\in\frac{1}{2}\mathbb{Z}$. The fundamental weights of D_5 are

$$\omega_1=L_1\ ,\ \omega_2=L_1+L_2\ ,\ \omega_3=L_1+L_2+L_3$$

$$\beta = \frac{1}{2}\left(L_1 + L_2 + L_3 + L_4 - L_5\right)\ ,\ \alpha = \frac{1}{2}\left(L_1 + L_2 + L_3 + L_4 + L_5\right)$$

and for a dominant weight Λ I will denote the associated highest weight module by Γ_Λ or $L(\Lambda)$.

The generation of the representation matrices is implemented in the Sagemath 9.0-notebook "`Generating_representation_matrices`" and the therein called script "`generation routines.sage`". The notebook consists of several very similar blocks. In each block one first creates the Berman generators and performs a test of the Berman relations. Afterwards one computes the Weyl canonical form and performs the corresponding tests as well which includes testing the skewness properties described above. Afterwards the matrices are stored in a suitably named folder in a sparse format. For the tensor products of representation matrices one can skip the tests and I have done so for larger dimensions as these tests are rather expensive. Running the entire notebook can take between 3 and 4 days as the last block (creation of the $\mathcal{S}_{\frac{3}{2}} \otimes \bigwedge^2 \mathcal{S}_{\frac{1}{2}}$-representation matrices) is time-consuming. In the following I will describe some details about the individual construction of the representation matrices.

B.1.1 Representation matrices for $\mathcal{S}_{\frac{1}{2}}$

The representation matrices of the Berman generators of $\mathfrak{k}(E_{10})$ are set up w.r.t. the basis developed in section (B.3). The only result that is essentially needed from this section is the following lemma.

Lemma B.1. *With respect to a weight basis*

$$\left\{ s_\lambda \mid \lambda \in \Delta(\Gamma_\alpha) \cup \Delta(\Gamma_\beta) = \left\{ \sum_{i=1}^{5} a_i L_i \mid a_i = \pm\frac{1}{2} \right\} \right\}$$

of the $\mathfrak{so}(10,\mathbb{C})$-module $\Gamma_\alpha \oplus \Gamma_\beta$, define matrices $\rho(X_i)$ for $i = 1,\ldots,10$ as follows:

$$\rho(X_{2j-1})s_\lambda \quad = \quad i\cdot\rho(H_j)\,s_\lambda = i\cdot\lambda(H_j)\,s_\lambda\ \forall\, j = 1,\ldots,5,$$

$$\rho(X_{2j})\,s_\lambda = -\frac{i}{2}s_{\lambda-2\lambda(H_j)L_j-2\lambda(H_{j+1})L_{j+1}}\ \forall\, j = 1,\ldots,4,\ \ \rho(X_{10})\,s_\lambda = -\frac{i}{2}s_{\lambda-2\lambda(H_2)L_2},$$

where $L_i(H_j) = \delta_{ij}$. Then the matrices $\rho(X_1),\ldots,\rho(X_{10})$ form a generalized spin representation of $\mathfrak{k}(E_{10})$ as in def. 3.1.

The representation matrices for all Berman generators are skew-hermitian in the above initialization. Since every weight in $\Gamma_\alpha \oplus \Gamma_\beta$ has multiplicity one, one simply identifies each weight with one of the euclidean basis vectors of $\mathbb{C}^{32}$. I will also collect a result on orthogonality of highest weight modules.

Lemma B.2. *Let $L(\Lambda_1) \subset V$ and $L(\Lambda_2) \subset V$ be concrete realizations of highest weight modules inside a larger vector space $V \cong \mathbb{C}^n$. Assume that $E_1,\ldots,E_5,F_1,\ldots,F_5,h_1,\ldots,h_5$ are representation matrices that satisfy the relations of the Weyl canonical form of $\mathfrak{so}(10,\mathbb{C})$ and in addition are such that $E_i^\dagger = -F_i$. Then if the highest weight vectors v_{Λ_1} and v_{Λ_2} are orthogonal w.r.t. the standard hermitian product on $\mathbb{C}^n$, the modules $L(\Lambda_1)$ and $L(\Lambda_2)$ are orthogonal w.r.t. each other. Note that the highest weight vectors are always orthogonal to each other if their highest weights are different.*

Proof. A weight vector of shape $v_\lambda = \left(\prod_{i=1}^{k} F_{j_i}\right) v_\Lambda$ is said to be of depth k. Assume orthogonality to hold for vectors of depth k. Then for $v_\lambda \in L(\Lambda_1)$ and $v_\mu \in L(\Lambda_2)$ such that $F_i v_\lambda$ and $F_j v_\mu$ are vectors to the same weight one computes with $[E_i, F_j] = h_{ij} \in \mathfrak{h}$

$$\begin{aligned}(F_i v_\lambda | F_j v_\mu) \quad &= \quad -(v_\lambda|E_iF_jv_\mu) = -(v_\lambda|F_jE_iv_\mu) - (v_\lambda|h_{ij}v_\mu) \\ &= \quad (E_jv_\lambda|E_iv_\mu) - \mu(h_{ij})(v_\lambda|v_\mu) = 0.\end{aligned}$$

Since the weight spaces of depth $k+1$ are spanned by such vectors, it follows that these weight spaces are orthogonal. The claim follows by induction as it is assumed to be true for the highest weight vectors. Towards the criterion about different highest weights: If $\Lambda_1 \neq \Lambda_2$ there exists $i = 1, \ldots, 5$ such that $\Lambda_1(h_i) \neq \Lambda_2(h_i)$ but then

$$\Lambda_1(h_i)(v_{\Lambda_1}|v_{\Lambda_2}) = (h_i v_{\Lambda_1}|v_{\Lambda_2}) = (v_{\Lambda_1}|h_i v_{\Lambda_2}) = \Lambda_2(h_i)(v_{\Lambda_1}|v_{\Lambda_2})$$

and thus, $(v_{\Lambda_1}|v_{\Lambda_2}) = 0$. □

B.1.2 Representation matrices for $\mathcal{S}_{\frac{3}{2}}$ and $\mathcal{S}_{\frac{5}{2}}$

For $\mathcal{S}_{\frac{3}{2}}$ one first needs an implementation of $\mathfrak{h}^*$ of E_{10}. Towards this I use the Wall-basis $\mathcal{B} = \{e_1, \ldots, e_{10}\}$ for $\mathfrak{h}^*(E_{10})(\mathbb{R})$ from [KN13] (adjusted to the enumeration of E_{10} in figure 1) in which the simple roots have the shape

$$\alpha_j := e_j - e_{j+1} \; \forall j = 1, \ldots, 9, \;\; \alpha_{10} := -e_1 - e_2 - e_3.$$

In terms of $\mathcal{B}$ the invariant bilinear form Q on $\mathfrak{h}^*$ is given by the matrix

$$G_{ij} = \delta_{ij} - \frac{1}{9} \; \forall i, j = 1, \ldots 10,$$

so that $\mathcal{B}$ is not orthonormal w.r.t. Q but it is so w.r.t. the inner product $(\cdot|\cdot)$ of $\mathbb{C}^{10}$ by definition. The Weyl reflection

$$s_\alpha(v) = v - \frac{2Q(v,\alpha)}{Q(\alpha,\alpha)}\alpha \; \forall v \in \mathfrak{h}^*$$

w.r.t. a real E_{10}-root α is therefore given explicitly in terms of $\mathcal{B}$ by

$$(s_\alpha)_{\mathcal{B}} = (e_i|s_\alpha(e_j))_{i,j=1}^{10} \; .$$

In the code this is facilitated by the routine `weyl_orth` (see C.1.5) and as it turns out the matrices for the simple Weyl reflections $s_{\alpha_1}, \ldots, s_{\alpha_{10}}$ are rather sparse (they have a density of about $10-15\,\%$ which is quite good for an invertible 10×10-matrix). According to theorem 3.19

$$\sigma_{\frac{3}{2}} : \; X_i \mapsto \left(s_{\alpha_i} - \frac{1}{2}Id\right) \otimes 2\rho(X_i) \;\; \forall i = 1, \ldots, 10$$

defines a representation of $\mathfrak{k}(E_{10})$ known as the $\frac{3}{2}$-spin representation $\mathcal{S}_{\frac{3}{2}}$. Thus, in addition to implementing the maps $s_{\alpha_i} - \frac{1}{2}Id$ one has to obtain the tensor product matrix, which is simply the Kronecker product of the two matrices. With the functions `tensor_homemade` (see C.1.4) and `weyl_orth` (see C.1.5) the generation of the $\mathcal{S}_{\frac{3}{2}}$-matrices is straightforward. Together with the above matrices for s_α one produces the representation matrices for $\mathcal{S}_{\frac{5}{2}}$ via

$$\sigma_{\frac{5}{2}} : \; X_i \mapsto \left(\eta(s_{\alpha_i}) - \frac{1}{2}\right) \otimes 2\rho(X_i) \;\; \forall i = 1, \ldots, 10\,,$$

where η denotes the induced representation of the Weyl group on $\mathrm{Sym}^2(V)$. In order to set this up, one needs to fix certain normalizations and relate linear indexation to indexation by multi-indices. All this is implemented in the functions `normalizers` (which facilitates normalization and index conversion for $Sym^d(V)$ in general) and `induced_map_sparse_sym2` which is specialized to $Sym^2 V$. The technical documentation of these functions is given in sections C.1.6 and C.1.7. Note that I did not make the effort to work over the

$\mathfrak{so}(1,9)$-irreducible Schur-module $S_{[2]}(V)$ which excludes the so-called trace part. In a best case scenario, this would decrease the dimension of the $\mathcal{S}_{\frac{5}{2}}$-representation by 32 which is not negligible if one wants to consider tensor product representations of $\mathcal{S}_{\frac{5}{2}}$ with other representations. However, in the current basis a subtraction of the trace part will lead to a significant increase in density of the involved matrices which will mitigate the benefits of a slightly smaller dimension.

B.1.3 Representation matrices for tensor products

In order to implement the representation matrices of $\mathrm{Sym}^2\left(\mathcal{S}_{\frac{1}{2}}\right)$ one needs the restriction of maps in $\mathrm{End}\,(V\otimes V)$ to $\mathrm{Sym}^2\,(V)$, in this case one needs to restrict the map

$$\rho\,(X_i)\otimes Id + Id\otimes\rho\,(X_i)$$

to $\mathrm{Sym}^2\left(\mathcal{S}_{\frac{1}{2}}\right)$. This is facilitated by the function `Lie_sym2` (see C.1.9) together with the already mentioned function `normalizers` (see C.1.6).

Similar to the symmetric case one needs a grip on the basis and the restriction of the Lie algebra tensor product to $\bigwedge^2\mathcal{S}_{\frac{1}{2}}$. The first part is dealt with via the function `normalizers_ext` (see C.1.10) while the second problem is attended to by `Lie_ext2` (see C.1.11).

For regular tensor product representations such as $\mathcal{S}_{\frac{1}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ one simply needs to use the representation matrices of $\mathcal{S}_{\frac{1}{2}}$ and implement the Lie algebra tensor product

$$(x,y)\mapsto x\otimes Id + Id\otimes y$$

which is done in the function `Lie_tensor` (see C.1.8). For $\mathcal{S}_{\frac{3}{2}}\otimes\mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}}\otimes\bigwedge^2\mathcal{S}_{\frac{1}{2}}$ one also uses the function `Lie_tensor` (see section C.1.8) with the previously computed representation matrices. Note that it is not necessary to compute the Weyl canonical form from the Berman generators as one can just take the Lie-tensor product of the individual factors. A direct computation is very time-consuming and even the seemingly simple approach via Lie-tensor products took almost 3 days to terminate for $\mathcal{S}_{\frac{3}{2}}\otimes\bigwedge^2\mathcal{S}_{\frac{1}{2}}$.

B.2 Decomposition into $\mathfrak{so}(10)$-modules in the notebooks

The decomposition into $\mathfrak{so}(10)$-modules is performed in the notebook "`Decompositions`" which again consists out of very similar blocks where each block treats a particular $\mathfrak{k}\,(E_{10})$-module. First, the representation matrices are read in from file, where two options exist: `read_in_SR` and `read_in_QQ`. The last option works only for some representation matrices because it assumes all matrix entries can be coerced to a rational number (this does not work for $\mathcal{S}_{\frac{5}{2}}$ for instance because normalization of the symmetric matrices includes $\frac{1}{\sqrt{2}}$). Hence, the first option which uses the symbolic ring in Sage is the default, although it is substantially slower. Afterwards one constructs an $\mathfrak{so}\,(10)$-highest weight vector that serves as the starting point of the investigation via the orbit method described in section 6.1.1. Usually, this builds on analytic insight into the module and one mostly needs the functions `vec_tensor` (see C.2.6), `induced_vector_sparse_ext2` (see C.2.8) or `induced_vector_sparse_sym2` to perform tensor products of vectors and `det_weight_v2` (see C.2.4) to determine the weight that the constructed vector has (this function produces an error if one picks an inhomogeneous vector).

Then one needs one of the most essential functions of my code: The function that constructs a weight space basis for a highest weight module named `weights_and_vectors` (described in detail in section C.2.5). It builds heavily on the theory of Kashiwara crystals and their implementation in Sage which originally was the reason because of which I decided to perform this analysis in Sage. Now one goes on to compute the orbit of the initial $\mathfrak{so}(10)$-module w.r.t. the full action of $\mathfrak{k}\left(E_{10}\right)$.

Starting from the highest weight vector v_Λ one applies X_{10} to each weight vector $v_\lambda \in L(\Lambda)$ to the weight λ. The image $X_{10}v_\lambda$ is then projected away from the already known representation $L(\Lambda)$ via orthogonal projection, facilitated by the function `project_away` (see C.2.9). As one knows that X_{10} maps the weight space $L(\Lambda)_\lambda$ into weight spaces to the weights $\lambda \pm L_2$ one only needs to consider the basis vectors for these weight spaces. One collects them via the function `retrieve_known_weight_vectors` (see C.2.10) while the possible weight spaces are determined by the function `possible_weights` (see C.2.11). Afterwards this set of vectors is orthogonalized by the function `orthogonalize_v2` (see C.2.12). Later on the function `retrieve_known_weight_vectors` will be replaced by `retrieve_known_weight_vectors_from_file` which reads the weight vectors from disk (this is necessary at a certain point because one cannot store all $\mathfrak{so}\,(10)$-modules in working memory). One then finds the associated highest weight vector(s) to $X_{10}v_\lambda$. This is implemented in `find_primitive` (see C.2.3), determination of the weight space is again `det_weight_v2` (see C.2.4). Afterwards one saves the modules to disk.

Towards the analysis of the orbit one first checks if an $\mathfrak{so}(10)$-module $L(\Lambda_i)$ can be reached from $L(\Lambda_j)$ via X_{10}. This is done by the function `check_modules` (see C.2.13) and the results are stored in an adjacency matrix A_{adj}. The graph associated to this matrix is directed and its nodes are $\mathfrak{so}(10)$-modules. The function `orbit` (see C.2.14) determines all modules $L(\Lambda_j)$ that can be reached from the module $L(\Lambda_i)$ via $\mathfrak{k}\left(E_{10}\right)$-action. It is used to check if the orbits of all $L(\Lambda_1), \ldots, L(\Lambda_n)$ are equal. Out of paranoia I also double checked for some modules if they are indeed orthogonal as they should be.

An alternative approach to obtain the $\mathfrak{so}(10)$-decompositions of the modules that was described in section 6 is to compute the $\mathfrak{so}(10)$-highest weight vectors directly. Determination of the vector space of primitive vectors, i.e., the intersection of the kernels of $E_1, \ldots, E_5$ is done by the function `get_primitives` (see section C.2.1). A basis of weight vectors for this vector space is obtained by diagonalizing a random linear combination $\sum_{i=1}^5 \lambda_i H_i$ of the orthonormal basis of $\mathfrak{h}^*\left(D_5\right)$ restricted to the space of primitive vectors. This is implemented in the function `get_HWVs` (see section C.2.2). However, this approach scales very badly and therefore is only used once for $\mathcal{S}_{\frac{5}{2}}$ to demonstrate how one can end up with a reducible structure such that it remains unclear whether or not the representation is completely reducible.

Towards the setup of this computation on a computer cluster there exist scripts called `orbit_serial` and `mixing_parallel`. As the names suggest, the first script does not support parallelization, whereas the other one does. The computations in `orbit_serial` are hard to parallelize as this script creates the orbit associated to an initial $\mathfrak{so}\,(10)$-module and one always needs the information about the modules that one has already found. Once the orbit is found however, its analysis is easily parallelized. Hence the decision to split the analysis into two pieces because on most computer cluster there exist different nodes for parallel and serial computations.

In `orbit_serial` all the steps to create the orbit are wrapped in a function called `analyze_module` that can be found in the script `distributed`. All relevant information such as where to find the representation matrices and the $\mathfrak{so}\,(10)$-modules that were already computed are read in from the initialization file `configuration_serial`. In this file, two important pieces of information are how many $\mathfrak{so}\,(10)$-modules were already found (keyword: number_of_known_modules) and at which of these modules the analysis shall start (keyword: module_to_start_at). This setup has the advantage that the computation can be paused and continued if for instance cluster resources are scarce. Note that one can track how much working memory is used[58] and that one can abort the computation if a certain threshold is exceeded. Only a single $\mathfrak{so}\,(10)$-module is stored in working memory at once during the computation as the necessary basis vectors are read

[58] Note however that this may not always give the correct result on a computer cluster as it is possible that the result is the amount of memory currently used on the entire node and not just the amount of memory that is used by this code.

in from disk via the function `retrieve_known_weight_vectors_from_file`. Due to this the computation does not use that much memory but it is time-consuming.

The analysis of which $\mathfrak{so}(10)$-modules mix under the full $\mathfrak{k}(E_{10})$-action takes place in the script `mixing_parallel`. This script uses functions from two other scripts, `decomposition_routines` and `mixing_routines`, as well as the python multiprocessing package. Again, the precise job information is read in from the file `configuration_mixing`. The core of this script is the function `analyze_mixing` which basically receives as input the identification number of an $\mathfrak{so}(10)$-module V_i and then determines to which other $\mathfrak{so}(10)$-modules V_j there exist $v \in V_i$ and $w \in V_j$ s.t. $(I \cdot X_{10} v \,|\, w) \neq 0$. As this computation has to be done for all $\mathfrak{so}(10)$-modules anyways this is parallelized easily. In addition one saves a lot of time by excluding some of the modules based on analytic considerations. One can check that $\mathcal{U}(\mathfrak{so}(10, \mathbb{C})) .X_{10}$ w.r.t. the adjoint action forms a highest weight module to the weight ω_3. Hence, all highest weights that can appear in an analysis of $X_{10} v_\lambda$ for $v_\lambda \in L(\Lambda)$ are the highest weights that appear in $L(\omega_3) \otimes L(\Lambda)$. If one sets the option `enforce_Kostant_rule` to True, only the modules which appear in this tensor product are checked for an overlap, which reduces the amount of modules that need to be checked significantly.

One needs to create an initial module in order to run `orbit_serial`, and for $\mathcal{S}_{\frac{3}{2}} \otimes \mathcal{S}_{\frac{1}{2}}$ and $\mathcal{S}_{\frac{3}{2}} \otimes \left(\bigwedge^2 \mathcal{S}_{\frac{1}{2}}\right)$ this is provided in the script `initial_modules`, as these computations may be too expensive in the notebook version.

B.3 Analytical setup of the representation $\mathcal{S}_{\frac{1}{2}}$

The core of all higher spin representations of $\mathfrak{k}(E_{10})$ is the generalized spin representation $\mathcal{S}_{\frac{1}{2}}$. I will provide its description by starting with the classical spin representation of $\mathfrak{so}(10, \mathbb{C})$ and work out how the remaining Berman generator of $\mathfrak{k}(E_{10})$ acts on it. I mostly use the conventions and some of the results of [FH91, ch. 20] concerning spin representations but at some point my normalizations differ slightly.

For $V = \mathbb{C}^{10}$ consider a split $V = W \oplus W^*$ such that W and W^* are isotropic with respect to a nondegenerate bilinear form Q, i.e. $Q(w_1|w_2) = 0\ \forall\, w_1, w_2 \in W$ and $\forall\, w_1, w_2 \in W^*$. Pick bases $\{a_1, \ldots, a_5\}$ and $\{b_1, \ldots, b_5\}$ for W and W^* such that $Q(a_i|b_j) = \delta_{ij}$. For $A \in End(W)$ one has that $A^T \in End(W^*)$ as W and W^* are duals. It holds $Q(Aw, w^*) = Q\left(w, A^T w^*\right)$ for all $w \in W$, $w^* \in W^*$ and from this one deduces that the map

$$\rho_A := \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \in End(V)$$

is skew:

$$Q(\rho_A w|w^*) = Q(Aw|w^*) = Q\left(w|A^T w^*\right) = -Q(w|\rho_A w^*) .$$

Now use the diagonal matrices $diag(d_1, \ldots, d_5, -d_1, \ldots, -d_5)$ as a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g} := \mathfrak{so}(V, Q)$. The other skew endomorphisms can be parametrized by exploiting an isomorphism to the exterior algebra of V. Define $\varphi_{x \wedge y} : V \to V$ via (cp. [FH91, eq. 20.4])

$$\varphi_{x \wedge y}(v) = 2\left[Q(y, v)\, x - Q(x, v)\, y\right]\ \forall\, x, y, v \in V.$$

With respect to the above bases one has $\varphi_{a_i \wedge b_i} = 2E_{ii} - 2E_{i+5,i+5}$, where E_{ij} denotes the matrix which has a 1 at position (i, j) and 0 everywhere else. Denote by L_i the linear functional that sends $E_{ii} - E_{i+5,i+5}$ to 1 and is 0 on the other diagonal matrices. Then w.r.t. these functionals the root spaces are given as follows:

$$\varphi_{a_i \wedge a_j} \in \mathfrak{g}_{L_i + L_j}\ ,\ \varphi_{a_i \wedge b_j} \in \mathfrak{g}_{L_i - L_j}\ ,\ \varphi_{b_i \wedge b_j} \in \mathfrak{g}_{-L_i - L_j} .$$

Now consider the exterior algebra $\mathcal{S} = \bigwedge^{\bullet} W$ together with the following action of V:

$$W \oplus W^* \ni v = w + w^* \ : \ v \cdot \psi = \sqrt{2} \cdot \left(w \wedge \psi + i_{w^*} (\psi) \right),$$

where $i_{w^*} (v_1 \wedge \cdots \wedge v_k) \coloneqq \sum_{i=1}^{k} Q (w^*, v_i) (-1)^{i+1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k$. It is a fact that this map respects Clifford multiplication[59], i.e.,

$$(v \cdot w + w \cdot v) \cdot \psi = 2Q(v, w) \cdot \psi \ \forall v, w \in V, \ \psi \in \mathcal{S} .$$

Thus, the action of V extends to an action of $Cl\,(V, Q)$ turning $\mathcal{S}$ into a Clifford module. Since $\mathfrak{so}\,(V, Q)$ can be embedded into $Cl(V, Q)$ via a Lie-algebra homomorphism, $\mathcal{S}$ becomes a $\mathfrak{so}\,(V, Q)$-module. The homomorphism makes extensive use of the parametrization by the exterior product $\bigwedge^2 V$ (which is an isomorphism of vector spaces and therefore injective):

$$\bigwedge^{2} V \ni x \wedge y \mapsto \varphi_{x \wedge y} \mapsto \frac{1}{4} [x, y] = \frac{1}{4} (x \cdot y - y \cdot x) \in Cl\,(V, Q)$$

An $\mathfrak{h}$-diagonal basis of $\mathcal{S}$ is given by $\{a_{i_1} \wedge \cdots \wedge a_{i_k} \mid i_1, \ldots, i_k \in \{1, \ldots, 5\}\}$ since one checks:

$$a_i \wedge b_i \mapsto \frac{1}{4} (a_i b_i - b_i a_i)$$

$$\begin{aligned}
\varphi_{a_i \wedge b_i} . a_{i_1} \wedge \cdots \wedge a_{i_k} &= \frac{1}{4} (a_i b_i - b_i a_i) \cdot a_{i_1} \wedge \cdots \wedge a_{i_k} \\
&= \frac{\sqrt{2} \cdot \sqrt{2}}{4} \left[a_i \wedge i_{b_i} (a_{i_1} \wedge \cdots \wedge a_{i_k}) - i_{b_i} (a_i \wedge a_{i_1} \wedge \cdots \wedge a_{i_k}) \right] \\
&= \frac{1}{2} a_i \wedge i_{b_i} (a_{i_1} \wedge \cdots \wedge a_{i_k}) - \frac{1}{2} i_{b_i} (a_i \wedge a_{i_1} \wedge \cdots \wedge a_{i_k}) \\
&= \begin{cases} \frac{1}{2} a_{i_1} \wedge \cdots \wedge a_{i_k} & \text{if } i \in \{i_1, \ldots, i_k\} \\ -\frac{1}{2} a_{i_1} \wedge \cdots \wedge a_{i_k} & \text{if } i \notin \{i_1, \ldots, i_k\} \end{cases}
\end{aligned}$$

How is this related to the spin representation that is used to construct $\mathcal{S}_{\frac{1}{2}}$? Consider the following basis transformation:

$$a_j = \frac{1}{\sqrt{2}} (v_j + i v_{j+5}) \ , \ b_j = \frac{1}{\sqrt{2}} (v_j - i v_{j+5})$$

$$v_j = \frac{1}{\sqrt{2}} (a_j + b_j) \ , \ v_{j+5} = \frac{-i}{\sqrt{2}} (a_j - b_j) \ \forall 1 \leq j \leq 5$$

then

$$\begin{aligned}
Q (v_j, v_k) &= \frac{1}{2} Q (a_j + b_j, a_k + b_k) = \delta_{jk} \ \forall 1 \leq j, k \leq 5 \\
Q (v_j, v_{k+5}) &= -\frac{i}{2} Q (a_j + b_j, a_k - b_k) = 0 \ \forall 1 \leq j, k \leq 5 \\
Q (v_{j+5}, v_{k+5}) &= -\frac{1}{2} Q (a_j - b_j, a_k - b_k) = \delta_{jk} \ \forall 1 \leq j, k \leq 5
\end{aligned}$$

[59]This is [FH91, lem. 20.9] but note that my normalization differs by a factor of $\sqrt{2}$ which is compensated for later by a different normalizeation of the map $\bigwedge^2 V \to Cl\,(V, Q)$, where [FH91, eq. 20.6] uses a factor of $\frac{1}{2}$ but I use $\frac{1}{4}$.

shows that $\{v_1,\dots,v_{10}\}$ is a standard basis of (V,Q). With respect to a standard basis $\{e_1,\dots,e_{10}\}$ the representation of $\mathfrak{k}(E_{10})(\mathbb{R})$ is given by[60]

$$\rho(X_i)=\frac{1}{2}e_ie_{i+1}\ \forall\, 1\le i\le 9,\ \rho(X_{10})=\frac{1}{2}e_1e_2e_3\ ,$$

where the Berman generators are labeled as in figure 1. The bases $\{v_1,\dots,v_{10}\}$ and $\{e_1,\dots,e_{10}\}$ are now related by renumeration:

$$e_{2i-1}=v_i\ ,\ e_{2i}=v_{i+5}\ .$$

Recall that $\mathfrak{h}_{\mathbb{C}}$ was spanned by $\varphi_{a_j\wedge b_j}$ which maps to $\frac{1}{4}[a_j,b_j]=-\frac{i}{2}v_jv_{j+5}=-\frac{i}{2}e_{2j-1}e_{2j}$ and therefore

$$\mathfrak{h}_{\mathbb{C}}=\operatorname{span}_{\mathbb{C}}\{-i\rho(X_i)\ \mid i\in\{1,3,5,7,9\}\}$$

My preferred convention is that the orthonormal basis $\{H_1,\dots,H_5\}$ is equal to the above generating set and the weights are spelled out w.r.t. these orthonormal basis elements. Explicitly, fix $L_1,\dots,L_5$ by demanding

$$L_i(H_j)=\delta_{ij}\ .$$

With this,

$$\varphi_{a_i\wedge b_i}.a_{i_1}\wedge\cdots\wedge a_{i_k}=\begin{cases}\frac{1}{2}a_{i_1}\wedge\cdots\wedge a_{i_k} & \text{if } i\in\{i_1,\dots,i_k\}\\ -\frac{1}{2}a_{i_1}\wedge\cdots\wedge a_{i_k} & \text{if } i\notin\{i_1,\dots,i_k\}\end{cases}$$

implies

$$H_i\cdot a_{i_1}\wedge\cdots\wedge a_{i_k}=\begin{cases}\frac{1}{2}a_{i_1}\wedge\cdots\wedge a_{i_k} & \text{if } i\in\{i_1,\dots,i_k\}\\ -\frac{1}{2}a_{i_1}\wedge\cdots\wedge a_{i_k} & \text{if } i\notin\{i_1,\dots,i_k\}\end{cases}$$

and therefore the weight structure of $\mathcal{S}$ can be understood from this basis for $\bigwedge^\bullet W$. The two $\frac{1}{2}$-spin representations of $\mathfrak{so}(10)$, denoted by Γ_α, Γ_β with highest weights $\alpha=\frac{1}{2}(L_1+L_2+\cdots+L_5)$ and $\beta=\frac{1}{2}(L_1+\cdots+L_4-L_5)$, have the following weights, all of multiplicity one:

$$\Delta(\Gamma_\alpha)=\left\{\sum_{i=1}^{5}c_iL_i\mid c_i=\pm\frac{1}{2}\ ,|\{i:c_i>0\}|\in\{1,3,5\}\right\},$$

$$\Delta(\Gamma_\beta)=\left\{\sum_{i=1}^{5}c_iL_i\mid c_i=\pm\frac{1}{2}\ ,|\{i:c_i>0\}|\in\{0,2,4\}\right\}.$$

The even and odd number of signs above is reflected in the split of $\bigwedge^\bullet W$ into $\bigwedge^{even}W\oplus\bigwedge^{odd}W$ where $\bigwedge^{odd}W$ corresponds to Γ_α. All of this can also be found in [FH91, prop.20.15]. Now analyze how X_{10} acts on $\mathcal{S}_{\frac{1}{2}}$:

$$\rho(X_{10})=\frac{1}{2}e_1e_2e_3=iH_1e_3=iH_1\frac{1}{\sqrt{2}}(a_2+b_2)$$

Set $I=\{i_1,\dots,i_k\}$ and impose $i_1<i_2<\cdots<i_k$ then the action of a_2 and b_2 on the $\mathfrak{h}_{\mathbb{C}}$-diagonal basis of $\mathcal{S}$ is given by

$$\begin{aligned}a_2\cdot a_{i_1}\wedge\cdots\wedge a_{i_k} &= \sqrt{2}a_2\wedge a_{i_1}\wedge\cdots\wedge a_{i_k}\\ &= \begin{cases}0 & \text{if } 2\in I\\ -\sqrt{2}a_{i_1}\wedge a_2\wedge\cdots\wedge a_{i_k} & \text{if } i_1=1,2\notin I\\ \sqrt{2}a_2\wedge a_{i_1}\wedge\cdots\wedge a_{i_k} & \text{if } i_1>2\end{cases}\end{aligned}$$

[60] See [HKL15, example 3.2] for this particular phrasing, the representation is originally due to [BHP06] and also [DKN06].

and

$$\begin{aligned} b_2 \cdot a_{i_1} \wedge \cdots \wedge a_{i_k} &= \sqrt{2}\sum_{j=1}^{k}(-1)^{j+1}Q\left(\alpha_2, a_{i_j}\right) a_{i_1} \wedge \dots \hat{v}_{i_j} \wedge \cdots \wedge a_{i_k} \\ &= \begin{cases} 0 & \text{if } 2 \notin I \\ -\sqrt{2}a_{i_1} \wedge a_{i_3} \wedge \cdots \wedge a_{i_k} & \text{if } i_2 = 2 \\ \sqrt{2}a_{i_2} \wedge \cdots \wedge a_{i_k} & \text{if } i_1 = 2 \end{cases} \end{aligned}$$

which combines to ($A_I \coloneqq \bigwedge_{k\in I} a_k$)

$$\frac{1}{\sqrt{2}}v_3 \cdot A_I = \begin{cases} A_{I\cup\{2\}} & \text{if } 1,2 \notin I \\ -A_{I\cup\{2\}} & \text{if } i_1 = 1, 2 \notin I \\ -A_{I\setminus\{2\}} & \text{if } i_2 = 2 \\ A_{I\setminus\{2\}} & \text{if } i_1 = 2. \end{cases}$$

This can be rephrased more compactly with the weight component $\lambda(H_1)$. Towards this consider first

$$H_1 A_I = \begin{cases} +\frac{1}{2}A_I & \text{if } 1 \in I \\ -\frac{1}{2}A_I & \text{if } 1 \notin I \end{cases}$$

Note that either adding or removing a_2 in A_I can always be phrased in terms of weights as

$$s_\lambda \mapsto c \cdot s_{\lambda - 2\lambda(H_2)L_2}$$

To make this more precise: For $\lambda \in \Delta(\Gamma_\alpha) \cup \Delta(\Gamma_\beta)$ set $s_\lambda = A_I$ where $I = \left\{i \mid \lambda(H_i) = \frac{1}{2}\right\}$. If now $2 \notin I$ then

$$\frac{1}{\sqrt{2}}v_3 \cdot s_\lambda = \frac{1}{\sqrt{2}}v_3 \cdot A_I = \begin{cases} A_{I\cup\{2\}} & \text{if } 1 \notin I \\ -A_{I\cup\{2\}} & \text{if } i_1 = 1 \end{cases} = -2\lambda(H_1)s_{\lambda-2\lambda(H_2)L_2}$$

as $\lambda(H_1) = \mp\frac{1}{2}$. For $2 \in I$ one computes

$$\frac{1}{\sqrt{2}}v_3 \cdot s_\lambda = \frac{1}{\sqrt{2}}v_3 \cdot A_I = \begin{cases} -A_{I\setminus\{2\}} & \text{if } i_2 = 2 \\ A_{I\setminus\{2\}} & \text{if } i_1 = 2 \end{cases} = -2\lambda(H_1)s_{\lambda-2\lambda(H_2)L_2}$$

which results in

$$\begin{aligned} \rho\left(X_{10}\right) s_\lambda &= -2\lambda(H_1)\rho(X_1) \cdot s_{\lambda-2\lambda(H_2)L_2} \\ &= -2\lambda(H_1) \cdot i \cdot H_1 \cdot s_{\lambda-2\lambda(H_2)L_2} \\ &= -2i\lambda(H_1)^2 s_{\lambda-2\lambda(H_2)L_2} \end{aligned}$$

$$\rho\left(X_{10}\right) s_\lambda = -\frac{i}{2}s_{\lambda-2\lambda(H_2)L_2} \ \forall\, s_\lambda \in \mathcal{S}_{\frac{1}{2}} \ . \tag{134}$$

The only missing piece is the action of the Berman generators $X_2, X_4, \dots, X_8$:

Lemma B.3. *One has for $s_\lambda \in \mathcal{S}_{\frac{1}{2}}$ as above that*

$$\rho\left(X_{2j}\right) s_\lambda = -\frac{i}{2}s_{\lambda-2\lambda(H_j)L_j-2\lambda(H_{j+1})L_{j+1}} \ .$$

Proof. First of all, note that

$$\rho\,(X_{2j}) = \frac{1}{2}e_{2j}e_{2j+1} = -\frac{i}{4}(a_j - b_j)(a_{j+1} + b_{j+1})\ \forall\, j = 1, \ldots, 4.$$

Perform this computation via case distinction. Case 1: $j, j+1 \in I$

$$\begin{aligned}\rho\,(X_{2j})\,A_I &= -\frac{i}{4}(a_j - b_j)(a_{j+1} + b_{j+1})A_I = \frac{i}{4}b_j b_{j+1} A_I \\ &= \frac{i}{4}\sqrt{2}^2(-1)^{1+k+1}(-1)^{1+k}A_{I\setminus\{j,j+1\}} = -\frac{i}{2}A_{I\setminus\{j,j+1\}}\end{aligned}$$

or in terms of the weight vectors

$$\rho\,(X_{2j})\,s_\lambda = -\frac{i}{2}s_{\lambda-2\lambda(H_j)L_j-2\lambda(H_{j+1})L_{j+1}}\ .$$

Case 2: $j \in I, j+1 \notin I$.

$$\begin{aligned}\rho\,(X_{2j})\,A_I &= -\frac{i}{4}(a_j - b_j)(a_{j+1} + b_{j+1})A_I = \frac{i}{4}b_j a_{j+1} A_I \\ &= \frac{i}{4}\sqrt{2}^2(-1)^{k}(-1)^{1+k}A_{I\cup\{j+1\}\setminus\{j\}} = -\frac{i}{2}A_{I\cup\{j+1\}\setminus\{j\}}\end{aligned}$$

or in terms of the weight vectors

$$\rho\,(X_{2j})\,s_\lambda = -\frac{i}{2}s_{\lambda-L_j+L_{j+1}} = -\frac{i}{2}s_{\lambda-2\lambda(H_j)L_j-2\lambda(H_{j+1})L_{j+1}}\ .$$

Case 3: $j \notin I, j+1 \in I$.

$$\begin{aligned}\rho\,(X_{2j})\,A_I &= -\frac{i}{4}(a_j - b_j)(a_{j+1} + b_{j+1})A_I = -\frac{i}{4}a_j b_{j+1} A_I \\ &= -\frac{i}{4}\sqrt{2}^2(-1)^{1+k}\underbrace{a_j \wedge A_{I\setminus\{j+1\}}}_{\text{needs } k-1 \text{ swaps}} = -\frac{i}{2}A_{I\cup\{j\}\setminus\{j+1\}}\end{aligned}$$

or in terms of the weight vectors

$$\rho\,(X_{2j})\,s_\lambda = -\frac{i}{2}s_{\lambda+L_j-L_{j+1}} = -\frac{i}{2}s_{\lambda-2\lambda(H_j)L_j-2\lambda(H_{j+1})L_{j+1}}\ .$$

Case 4: $j, j+1 \notin I$.

$$\begin{aligned}\rho\,(X_{2j})\,A_I &= -\frac{i}{4}(a_j - b_j)(a_{j+1} + b_{j+1})A_I = -\frac{i}{4}a_j a_{j+1} A_I \\ &= -\frac{i}{4}\sqrt{2}^2 a_j \wedge a_{j+1} \wedge A_I = -\frac{i}{2}A_{I\cup\{j,j+1\}}\end{aligned}$$

or in terms of the weight vectors

$$\rho\,(X_{2j})\,s_\lambda = -\frac{i}{2}s_{\lambda+L_j+L_{j+1}} = -\frac{i}{2}s_{\lambda-2\lambda(H_j)L_j-2\lambda(H_{j+1})L_{j+1}}\ .$$

□

C Technical documentation

Here, I gather technical details and documentation of the functions that I use in the notebooks.

C.1 Documentation of functions that are used to generate matrices and vectors

Most of these functions are located in the script "`generation_routines.sage`".

C.1.1 `get_E_ladder` and `relatives`

The functions `get_E_ladder(Bermans)`, `get_F_ladder(Bermans)`, `get_H_weyl(Bermans)` and `get_H_orth(Bermans)` compute the elements of the Weyl-canonical form of $\mathfrak{so}(10, \mathbb{C})$ as described in section B.1. All of them expect as input the Berman generators $X_1, \ldots, X_{10}$ as a list of 10 matrices (dense or sparse). Their output is always a list of 5 matrices corresponding to the E_i, F_i, h_i or H_i. The commutator is computed with the function `Com(A,B)` which is just the regular definition of the commutator $AB - BA$. The parameters a and b inside the functions were set to experiment with different normalizations, since the normalization of [C84, eq. G.19-20] which I originally used did not behave the way I expected it to. I find it likely that the normalization of [C84] differs because the author uses explicit matrices and therefore the explicit Killing form defined by the trace.

C.1.2 `ladder_check` and `ladder_check_sparse`

Both functions take the same arguments `(E_ladder, F_ladder, H_weyl, Cartan_matrix, silent=False)` and check if the following relations for the Weyl canonical form are satisfied where $\mathcal{E}$ denotes the edges in the simply-laced Dynkin diagram of type A, where A is a simply-laced Cartan matrix:

$$[E_i, E_j] = 0 \text{ if } (i,j) \notin \mathcal{E} \ , \ [E_i, [E_i, E_j]] = 0 \text{ if } (i,j) \in \mathcal{E}$$

$$[F_i, F_j] = 0 \text{ if } (i,j) \notin \mathcal{E} \ , \ [F_i, [F_i, F_j]] = 0 \text{ if } (i,j) \in \mathcal{E}$$

$$[E_i, F_j] = -\delta_{ij} h_i \ , \ [h_i, E_j] = A_{ij} E_j \ , \ [h_i, F_j] = -A_{ij} F_j$$

The arguments `E_ladder`, `F_ladder` and `H_weyl` correspond to the E_i, F_i and h_i respectively and need to be a list of $\dim(Cartan\_matrix)$ matrices each. The argument `Cartan_matrix` is the Cartan_matrix of the type to be checked and needs to be a square matrix of a classical, simply-laced Cartan type (everything else will produce an error message). The optional argument `silent` gives the opportunity to chose between intermediate output to the console of the test routine or silence. The first case can be of interest for large matrices as the tests then take some time. The equalities are tested by rephrasing each relation in the form $R(A, B) = 0$ and then computing the matrix norm of the left-hand side. There the two versions differ because `check_ladder` uses the function `norm()` that is provided by Sage whereas `check_ladder_sparse` uses my own function `mat_norm2_sparse` which is adapted to sparse matrices and computes the square of the matrix 2-norm $\|A\|^2 = \sum_{i,j} a_{ij}^2$. The output of the functions is a list of 4 matrices, where the first matrix encodes if the relations between E_i and E_j hold. The second one is for F_i and F_j, then the relations between E_i and F_j and ultimately the relations among h_i and E_j, F_j.

C.1.3 `Berman_check` and `Berman_check_sparse`

`Berman_check_sparse(Bermans,Cartan_matrix,silent=False)` works similar to `ladder_check_sparse` above, just that the relations to be checked now are ($\mathcal{E}$ denotes the set of edges in the generalized Dynkin diagram of type `Cartan_matrix`)

$$[X_i, X_j] = 0 \text{ if } (i,j) \notin \mathcal{E} \ , \ [X_i, [X_i, X_j]] = -X_j \text{ if } (i,j) \in \mathcal{E},$$

where the X_i are the Berman generators `Bermans` handed over as a list of n sparse matrices, where n is the dimension of `Cartan_matrix`, which now is allowed to be a generalized simply-laced Cartan matrix. The output is a $n \times n$ matrix which encodes if the relation between X_i and X_j holds or not.

C.1.4 `tensor_homemade`

The function `tensor_homemade(A,B)` expects two sparse matrices as input and hands back the Kronecker product of the two matrices A and B as a sparse matrix. The advantage towards the Sage routine `A.tensor_product(B)` is that it exploits the sparse structure of the matrices by iterating only over the nonzero elements of both matrices (I am not sure if this issue was fixed from Sage 8.7 to 9.0, so this fix may be unnecessary).

C.1.5 `weyl_orth`

The function `weyl_orth(root,G)` expects a real root for `root` and the matrix corresponding to the invariant bilinear form of $\mathfrak{h}^*$ for `G`, both spelled out in the standard orthonormal basis of $\mathbb{C}^n$ where $n = \dim \mathfrak{h}^*$. It returns the matrix representation of the Weyl reflection s_α w.r.t. the standard basis of $\mathbb{C}^n$ as a sparse matrix.

C.1.6 `normalizers`

The function `normalizers(m,dim)` computes a normalized basis of $\mathrm{Sym}^m V$ where V has dimension `dim`. The output are two lists in the form `[list1,list2]`. The first list consists of pairs $[(i_1, \dots, i_m), n]$ where $i_1 \leq \cdots \leq i_m \in \{0, \dots, m-1\}$ are the multi-indices corresponding to the symmetrized basis vector ($e_0, \dots, e_{m-1}$ is the standard orthonormal basis of V)

$$e_{i_1 \dots i_m} := \sum_{\sigma \in \mathfrak{S}_m} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(m)}}$$

and n is such that $(n \cdot e_{i_1 \dots i_m} | n \cdot e_{i_1 \dots i_m}) = 1$. The second output is just the list of tuples $(i_1, \dots, i_m)$ for $i_1 \leq \cdots \leq i_m \in \{0, \dots, m-1\}$ and is meant to serve as a translation between multi-indices and linear indexation. The value n is computed by normalizing

$$(e_{i_1 \dots i_m} | e_{j_1 \dots j_m}) = \sum_{\sigma, \rho \in \mathfrak{S}_m} \left(e_{i_{\rho(1)}} | e_{j_{\sigma(1)}}\right) \cdots \left(e_{i_{\rho(m)}} | e_{j_{\sigma(m)}}\right)$$

to 1. The summation over the correct multi-indices is facilitated by the Sage routine `SemistandardTableaux` where one only has to take care of shifting the indices back by 1 (Python uses indices from 0 to $dim - 1$ but the entries in the tableaux start at 1, probably to ensure compatibility with actions of the symmetric group).

C.1.7 `induced_map_sparse_sym2`

The function `induced_map_sparse_sym2(A,basis,linear)` expects a $n \times n$ matrix A in sparse representation together with a normalized basis of $\mathrm{Sym}^2 V$, where $V \cong \mathbb{K}^n$, for `basis` as in the first output of `normalizers`. The argument `linear` is the list of multi-indices given by the second output of `normalizers`. The output is a sparse matrix that equals $A \otimes A$ restricted to $\mathrm{Sym}^2 V$ in the normalized basis. It is important that A is given in the standard basis of $V \cong \mathbb{K}^n$. Since the normalized basis is orthonormal the entries of $A \otimes A$ can be computed via

$$\left(n\left(i_1,i_2\right)e_{i_1 i_2}\right|n\left(j_1,j_2\right)A\otimes Ae_{j_1 j_2}) = n\left(i_1,i_2\right)n\left(j_1,j_2\right)\sum_{\rho,\sigma\in\mathfrak{S}_2} A_{i_{\rho(1)}j_{\sigma(1)}}A_{i_{\rho(2)}j_{\sigma(2)}}$$

which is done in the main body of the function. Before that it calls the function `create_indices_sym2(A)` which determines from the nonzero entries of A the only elements of the induced matrix that can be nonzero.

C.1.8 `Lie_tensor`

The function `Lie_tensor(A1,A2)` expects two sparse square matrices A_1 and A_2 of dimension n_1 and n_2 as input and returns the sparse matrix $A_1 \otimes Id_{n_2\times n_2} + Id_{n_1\times n_1} \otimes A_2$. The knowledge that the other matrix is always the identity matrix can help to save some time in comparison to calling the tensor product of A_1 with $Id_{n_2\times n_2}$ which is why it has its own implementation here.

C.1.9 `Lie_sym2`

The function `Lie_sym2(A,basis,linear,A_dim)` expects a sparse matrix A together with its dimension as `A_dim`. The arguments `basis` and `linear` are the outputs of normalizers. The output is the restriction A_{ind} of $A \otimes Id + Id \otimes A$ to $\mathrm{Sym}^2 V$ in a sparse format (here, one need the dimension of A because the matrix might otherwise be too small). As the basis is orthonormal one again computes the matrix elements of A_{ind} via the induced scalar product. The function `create_indices_Lie_sym2` that is called within `Lie_sym2` tells us which indices are potentially nonzero.

C.1.10 `normalizers_ext`

The function `normalizers_ext` works almost exactly as the function normalizers with the only difference that one calls `SemistandardTableaux` of Sage with a differently shaped tableaux that corresponds to the exterior product of rank m instead of the symmetric product.

C.1.11 `Lie_ext2`

Does the same thing as `Lie_sym2` but for the exterior product.

C.1.12 `skew_properties`

The function `skew_properties(Bermans,E_ladder,F_ladder,silent=False)` checks for each sparse matrix in the list `Bermans` if it is skew hermitian or not. Afterwards it checks if

$$E_i^\dagger = -F_i \;\forall\, i = 1,\ldots,5,$$

where E_i is the i-th entry of `E_ladder` and F_i is the i-th entry of `F_ladder`, which both are expected to be lists of the same length with sparse matrices of matching dimensions as entries. The test is conducted by computation of the matrix norm's square of $X_i + X_i^\dagger$ and $E_i^\dagger + F_i$ with `matrix_norm2_sparse`. The conjugation transpose is performed with the function `my_sparse_dagger`, since it is faster then the Sage routine `A.conjugate_transpose()` for sparse matrices (at least in version 8.7, maybe this issue no longer exists).

C.2 Documentation of functions that are used for $\mathfrak{so}(10)$-decompositions

C.2.1 `get_primitives`

The function `get_primitives(E)` expects a list $[E_1, \ldots, E_k]$ of matrices $E_i \in \mathbb{K}^{n\times n}$. It determines the intersection $K = \cap_{i=1}^{k} \ker(E_i)$ of the kernels of E_i and returns a matrix, where each column is a basis vector for this subspace of $\mathbb{K}^n$. It accepts sparse matrices but it calls the Sage routine `E.right_kernel()` which I suspect to perform a conversion to dense format internally.

C.2.2 `get_HWVs`

The function `get_HWVs(H,primitives,silent=True)` expects a list of commuting matrices that can be diagonalized over the vector space generated by the vectors in the matrix `primitives`. If successful it returns a basis of simultaneous eigenvectors for the space spanned by `primitives`. The simultaneous diagonalization is performed by diagonalizing a random linear combination of the matrices in `H` which works in most cases. As I didn't have any trouble with this function, I did not implement real error handling in case it doesn't work, like trying again with a different linear combination. The optional argument `silent` provides printed outputs of intermediate computations to the terminal if set to `False`.

C.2.3 `find_primitive`

The function `find_primitive(E_ladder,v)` takes a list of ladder operators of type E as its first input. The second input is a vector v to which one wants to find an associated primitive vector. All matrices and vectors should be sparse. The idea is the following: v can be expressed as a linear combination of weight vectors v_λ and each λ can be written as $\lambda = \Lambda - \sum_{i=1}^{d} k_i\alpha_i$ with $k_i \in \mathbb{N}_0$. Applying E_i to v_λ maps it to the weight space $\lambda + \alpha_i$. One can only go up a finite number of steps this way because at some point one would exceed the highest weight Λ and thus applying E_i yields 0 for all $i \in \{1, \ldots, d\}$ at this point. This is exactly the definition of a primitive vector.

The function `step_up` tries to find the first $i \in \{1, \ldots, d\}$ such that $E_i v \neq 0$ where it tries in the same order as the E_i are sorted in `E_ladder` and gives back the result $E_i v$ as its second output if $E_i v \neq 0$. It returns v if all $E_i v = 0$ for all $i = 1, \ldots, d$. The first output of `step_up` is a Boolean indicating if the output is $E_i v$ or v. One now simply applies `step_up` until a primitive vector is found and returns this. In order to be efficient, `step_up` uses my sparse inner product `dot_sparse` which is why the inputs need to be sparse as well.

C.2.4 `det_weight_v2` (`det_weight` is the old one for non-sparse, v2 is for sparse types)

The function `det_weight_v2(v,H)` computes the weight of a simultaneous sparse eigenvector v of the sparse matrices in the list `H` by solving the equation $h \cdot v - x \cdot v = 0$ for $x \in \mathbb{C}$ where h ranges over the list `H`. It returns the weight in form of a tuple of rational numbers. The length of the tuple is equal to the length of `H`. If the vector that was handed over is not a pure weight vector, an error message is returned.

C.2.5 `weights_and_vectors`

The function `weights_and_vectors(highest_weight, highest_vector, F_ladder, silent_inner=True, no_check=False, orth=False)` expects as input a highest weight Λ of type D_5 in form of a tuple of length 5 whose entries are $\Lambda(H_i)$, where the H_i are the orthonormal basis of $\mathfrak{h}_{D_5}$. The corresponding highest vector is handed over as `highest_vector` which can be a dense or sparse vector. The representation matrices of $F_1, \ldots, F_5$ of the Weyl-canonical form are handed over as a list of 5 matrices as the argument `F_ladder`. The optional argument `silent_inner` enables printing the currently evaluated level (of descent) during the process for longer computations. The output of `weights_and_vectors` is a dictionary. The keys are weights λ, in form of a tuple of length 5 spelling out $\lambda = \sum_{i=1}^{5} a_i L_i$, of the irreducible D_5-representation to the highest weight `highest_weight`. The entry to a weight λ is a list of vectors that form a basis of the weight space V_λ inside the irreducible representation.

How it works: Call a weight $\lambda = \Lambda - \sum_{i=1}^{5} k_i \alpha_i$ of level $\sum_i k_i$. The function loops over the levels of the highest weight module starting with the `highest weight` at level 0. It builds a temporary basis called `B` which is a list where the entry at position i will contain the dictionary of weight spaces of level-i weights. Each of these weight spaces consists of a list of `pairs`, where each pair consists of a point in the Kashiwara crystal of type D_5 and shape `highest_weight` and the actual vector in terms of the explicit representation (w.r.t. which the `F_ladder` are representations matrices of $F_1, \ldots, F_5$). This basis is initialized at level 0 with the highest weight vector. Iteratively one adds level by level where the function `new_vectors(elem, points,F)` is used. This function expects a pair `[crystal_point, vector]` as `elem`, a list of crystal points as `points`, and the list of the five representation matrices for $F_1, \ldots, F_5$ as `F`. It returns the list of pairs that can be reached from `elem` by descent with one of the F_i and an updated list of crystal points that are all crystal points on the current level that one already has. For instance, as f_1 and f_3 commute, the vectors $f_1 f_3 v_\Lambda$ and $f_3 f_1 v_\Lambda$ are equal and for this example assume they are nonzero. Then at level 1 the first element that will be handled is $f_1 v_\Lambda$ and one of the new pairs is $f_3 f_1 v_\Lambda$. Once the loop reaches the element $f_3 v_\Lambda$ it will realize that $f_1 f_3 v_\Lambda$ is nonzero but it will not add it to the list of `new_pairs` because the crystal point corresponding to $f_3 f_1 v_\Lambda$ has already been added to the list of known `points` and equals that of $f_1 f_3 v_\Lambda$. This recognition of equalities happens internally by the Sage implementation of Kashiwara crystals. This way one obtains a full list of weight spaces, sorted by level, where the weight vectors are given by pairs of points in the Kashiwara crystal and actual vectors. From this, one extracts a dictionary where each weight space basis is accessed by the weight as a key and the crystal points are dropped. One additional remark: Once a new level is computed it is checked for consistency because in the past this turned out to be an issue as bases were linearly dependent though having the correct number of vectors. I am not sure, if this still an issue because I redid the entire function at some point. This part can be turned off by `no_check=True`. Also, one has the option to normalize the bases via `orth=True`.

C.2.6 `vec_tensor`

The function `vec_tensor(a,b)` is an implementation for the tensor product of two sparse vectors a and b which returns their tensor product in a sparse format.

C.2.7 `induced_vector_sparse_sym2`

The function `induced_vector_sparse_sym2(v,w,basis,linear)` takes two sparse vectors v and w as input together with the two lists from `normalizers(m=2,dim)` as arguments for `basis` and `linear`. It determines the list of entries that could be nonzero via the function `create_vector_indices_sym2(v,w)` and then computes the entry for all these multi-indices according to the rule

$$(n\,(i_1,i_2)\,e_{i_1}e_{i_2}\,|\,v_1\cdot v_2)=\frac{n\,(i_1,i_2)}{2!}\sum_{\rho,\sigma\in\mathfrak{S}_2}v_{\sigma(1)}^{i_{\rho(1)}}v_{\sigma(2)}^{i_{\rho(2)}}$$

where by definition $v\cdot w=\frac{1}{2!}\,(v\otimes w-w\otimes v)$. The output of `induced_vector_sparse_sym2` is a sparse vector with entries in the symbolic ring and the dimension `dim`.

C.2.8 `induced_vector_sparse_ext2`

The function `induced_vector_sparse_ext2(v,w,basis,linear)` takes two sparse vectors v and w as input together with the two lists from `normalizers_ext(m=2,dim)` as arguments for `basis` and `linear` (in that order). It determines the list of entries that could be nonzero via the function `create_vector_indices_ext2(v,w)` and then computes the entry for all these multi-indices according to the rule

$$(e_{i_1}\wedge e_{i_2}\,|\,u\wedge v)=u_{i_1}v_{i_2}-u_{i_2}v_{i_1}$$

where by definition $v\wedge w=\frac{1}{\sqrt{2}}\,(v\otimes w-w\otimes v)$. The output of `induced_vector_sparse_ext2` is is a sparse vector with entries in the symbolic ring and the dimension `dim`.

C.2.9 `project_away`

The function `project_away(v,basis)` takes a sparse vector v as input and subtracts the projection to the subspace that is spanned by the orthogonal vectors `basis` which are handed over as a list of sparse vectors. This function does not work if the vectors in `basis` are not orthogonal.

C.2.10 `retrieve_known_weight_vectors`

The function `retrieve_known_weight_vectors(weights,modules)` takes as input a list of weights together with a list of $\mathfrak{so}(10)$-modules for the argument `modules`. It returns a list of all vectors within these modules that are weight vectors for one of the weights in `weights`. More precisely:

The input `weights` is a list of tuples representing weights. The structure of `modules` has to be such that each entry is a list `[highest_weight,weight_spaces]` where `highest_weight` is a tuple that is the highest weight of the module while `weight_spaces` is a dictionary where the keys are tuples corresponding to weights and the entry is a list of vectors that form a basis for the weight space.

C.2.11 `possible_weights`

The function `possible_weights(weight)` takes as input a tuple $\begin{pmatrix}a_1 & \dots & a_5\end{pmatrix}$ that corresponds to a D_5-weight $\lambda=\sum_{i=1}^{5}a_iL_i$ and returns the weights $\lambda-L_2$ and $\lambda+L_2$ as a list of two tuples.

C.2.12 `orthogonalize_v2`

The function `orthogonalize_v2(basis)` takes as input a list of sparse vectors and returns a list of sparse vectors that are orthogonal and span the same subspace as `basis`. It uses the sparse hermitian inner product `dot_sparse`.

C.2.13 check_modules

The function check_modules(module1,module2,A_mat) expects two D_5-modules module1 and module2 together with a matrix A_mat as arguments. The modules are of the shape [highest_weight,weight_spaces] where highest_weight is the highest weight of the module as a tuple (although this part is irrelevant for the function) and weight_spaces is a dictionary. The keys of this dictionary are D_5-weights as tuples, the entry of the weight λ is a list of vectors that form a basis for the weight space V_λ inside the module. The function checks if there exists any v_λ in module1 and v_μ in module2 such that $(v_\mu|Av_\lambda) \neq 0$ where $(\cdot|\cdot)$ denotes the standard hermitian product. If such a pair exists, the function returns 1 otherwise it returns 0.

C.2.14 orbit

The function orbit(incidence,L,start) expects an incidence matrix A as incidence, i.e. a matrix with entries 0 and 1 that is a square $L \times L$ matrix. The argument start is an integer i between 0 and $L-1$. The function determines for which j in $\{0, \ldots, L-1\}$ there exists a chain of integers $i_1, \ldots, i_k$ such that $A_{ii_1} A_{i_1 i_2} \cdots A_{i_k j} = 1$ and returns all the j for which this is possible as a list.

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