A differential delay equation with a planar attractor

Hans–Otto Walther
Mathematisches Institut der Universität München
D 8000 München 2, Germany

1 Planar dynamics in the attractor

Consider the equation

$$x'(t) = -\mu x(t) + f(x(t-1)) \tag{1}$$

with $\mu \geq 0$ and with a C^1–function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $f(0) = 0$ and $xf(x) < 0$ for all $x \neq 0$. Equation (1.1) is the simplest differential equation for a system governed by delayed negative feedback, with friction present in case $\mu > 0$.

Solutions are either differentiable functions $x : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfy eq. (1) everywhere, or continuous real functions x which are defined on some interval of the form $[t_0 - 1, \infty)$ and which are differentiable and satisfy eq. (1) for $t > t_0$. Every continuous "initial function" $\phi : [-1, 0] \longrightarrow \mathbb{R}$ defines a unique solution $x = x^\phi$ on $[-1, \infty)$ with $x|[-1, 0] = \phi$. The relations

$$F(t, \phi) = x_t, \quad x = x^\phi,$$

$$x_t(s) = x(t+s) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad -1 \leq s \leq 0$$

define a continuous semiflow F on the phase space $C = C([-1, 0], \mathbb{R})$ equipped with the supremum–norm.

An important class of solutions are those which have all consecutive zeros spaced at distances greater than the delay (> 1). They are called slowly oscillating. For example, every $\phi \in C$ with at most one zero defines a solution which is slowly oscillating on $[0, \infty)$. This is closely related to the fact that the set S of nonzero initial functions with at most one change of sign is positively invariant under the semiflow.

For proofs of these elementary properties, and of others mentioned below, see [7] and the references quoted therein.

Suppose from now on that f is also bounded from below, or from above. This holds true in most applications. Then one can show that the semiflow F has a global attractor $A = A(\mu, f)$ in the sense of [2], i.e. a compact set which is invariant (in the sense of [2]), contains all compact invariant sets, and attracts all bounded sets. In particular, ω–limit sets of single solutions are contained in A. The attractor can be characterized as the set of segments x_t of solutions which are defined on $\mathbb{R}$ and bounded — compare e.g. [5].

The restriction of F to A may be complicated. It is likely, but has not yet been proved, that there exist non–monotone nonlinearities f for which the attractor contains chaotic motion. In the present note we are interested in more regular behaviour, which occurs for monotone f. We assume, in addition to the preceding hypotheses, that

$$f'(x) < 0 \quad \text{for all} \quad x \in \mathbb{R}$$

and that the stationary point $\phi = 0$ is linearly unstable, i.e. that the characteristic equation

$$\lambda + \mu + \alpha e^{-\lambda} = 0$$

for the linearization

$$x'(t) = -\mu x(t) - \alpha x(t-1), \quad \alpha := -f'(0) \tag{2}$$

has solutions with positive real part.

Then there exist a complex conjugate pair $\lambda = u + iv, v > 0$, and $\overline{\lambda}$ of solutions with maximal real part $u > 0$; this leading pair is also the pair with smallest imaginary part v. We have $v \in (0, \pi)$ so that all solutions of eq. (2) of the form

$$x(t) = e^{ut}(a\cos(vt) + b\sin(vt)) \tag{3}$$

(except the zero solution) are slowly oscillating. The twodimensional space $L \subset C$ given by the segments x_t of solutions of the form (3) is invariant under the C_0–semigroup of operators

$$T_t = D_2F(t, 0), \quad t \geq 0,$$

or, the restriction of the semiflow of eq. (2) to the space L is a complete flow given by a linear planar vectorfield of spiral source type.

There exists a complementary closed subspace $Q \subset C$,

$$C = L \oplus Q,$$

which is positively invariant under the semigroup. This decomposition serves as a coordinate system for a result on the nonlinear equation (1).

In [7] we showed that there exists a Lipschitz continuous map

$$\overline{w} : \overline{L_w} \longrightarrow Q, \quad L_w \quad \text{an open subset of} \quad L \quad \text{with} \quad 0 \in L_w$$

so that $\overline{w}$ is of class C^1 on L_w and satisfies $\overline{w}(0) = 0, D\overline{w}(0) = 0$, and the graph

$$W := \{\chi + \overline{w}(\chi) : \chi \in L_w\}$$

has the following properties:

A. For every $\phi \in W \setminus \{0\}$ there exists a slowly oscillating solution $y : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1) with $y_t \in W$ for all $t \in \mathbb{R}$, $y_t \longrightarrow 0$ as $t \longrightarrow -\infty$ and $y_t \longrightarrow \overline{W} \setminus W$ as $t \longrightarrow \infty$.

B. The boundary $\overline{W} \setminus W$ of the graph W is the orbit in C of a slowly oscillating periodic solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1).

Note that W is tangent to L at $\phi = 0$. In general, W is a proper subset of the unstable set $W^u(0)$ formed by the segments y_t of solutions $y : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1) such that $y_t \longrightarrow 0$ as $t \longrightarrow -\infty$. If all solutions $z \notin \{\lambda, \overline{\lambda}\}$ of the characteristic equation satisfy $\Re z < 0$ then

$$W = W^u(0).$$

The restriction of F to W is a complete flow on a set homeomorphic to a closed disk in the plane.

We have

$$\overline{W} \subset A$$

since $\overline{W}$ consists of segments of bounded solutions defined on $\mathbb{R}$. In general, $\overline{W}$ is a proper subset of A. The purpose of the present note is to provide examples where equality holds.

2 An example

Let a C^1–function $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given which is odd,

$$f(x) = -f(-x) \quad \text{for all} \quad x \in \mathbb{R},$$

and bounded, and which has the following properties:

$$f'(x) < 0 \quad \text{for all} \quad x \in \mathbb{R},$$

$$f'(0) < -\pi/2 \tag{4}$$

$$-2 < f'(0) \tag{5}$$

$$f' \quad \text{is strictly increasing on} \quad [0, \infty) \tag{6}$$

Consider the equation

$$x'(t) = f(x(t-1)) \tag{7}$$

which is a special case of eq. (1) with $\mu = 0$. Property (4) implies that the stationary point $\phi = 0$ is linearly unstable. According to [7] there exists a graph W as described in Section 1. A first consequence of property (5) is that all solutions $z \notin \{\lambda, \overline{\lambda}\}$ of the characteristic equation satisfy $\Re z < 0$ so that

$$W = W^u(0).$$

Theorem. $\overline{W} = A(0, f)$

The proof combines several nontrivial results on the dynamics of eq. (7) which we first collect. According to the uniqueness theorem of R.D. Nussbaum [6], eq. (7) has exactly one slowly oscillating periodic solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ so that

(U) there are zeros $z_0 = -1, z_1 > 0, z_2 > z_1 + 1$; the solution is positive on $(-1, z_1)$, negative on (z_1, z_2), and has period $z_2 - z_1$.

J.A. Kaplan and J.A. Yorke [3] proved that (for more general monotone f) there exist two slowly oscillating periodic solutions p and r, both with property (U), so that the simple closed curves in $\mathbb{R}^2$

$$P : t \longrightarrow (p(t), -p(t-1)), \quad R : t \longrightarrow (r(t), -r(t-1))$$

contain the origin in their interior, and

$$P(\mathbb{R}) \subset \text{int}(R) \cup R(\mathbb{R});$$

furthermore, for every slowly oscillating solution y of eq. (7) the curve

$$Y : t \longrightarrow (y(t), -y(t-1))$$

converges in $\mathbb{R}^2$ to the closed annulus

$$(P(\mathbb{R}) \cup \operatorname{ext}(P)) \cap (R(\mathbb{R}) \cup \operatorname{int}(R))$$

between the traces of P and R, as $t \longrightarrow \infty$. For the situation considered here we conclude that $p = r = x$;
for every slowly oscillating solution y of eq. (7) we have

$$\operatorname{dist}(Y(t), X(\mathbb{R})) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \tag{8}$$

where $X(t) = (x(t), -x(t-1))$. Set

$$\xi := \{x_t : t \in \mathbb{R}\}.$$

Proposition 1. For every slowly oscillating solution y of eq. (7), $\operatorname{dist}(y_t, \xi) \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof. (Compare [3]) The segments y_t converge to the ω–limit set $\omega(y)$ as $t \longrightarrow \infty$. Consider a point $\phi \in \omega(y)$. There is a solution $w : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (7) so that $w_0 = \phi$ and $w_t \in \omega(y)$ for all $t \in \mathbb{R}$. Using (8) we infer that $(w(t), -w(t-1)) \in X(\mathbb{R})$ for all $t \in \mathbb{R}$. By $0 \notin X(\mathbb{R})$, $w_t \neq 0$ for all $t \in \mathbb{R}$. Note that $\omega(y) \subset \overline{S}$ since all y_t belong to S. From $0 \neq w_t \in \overline{S}$ for all $t \in \mathbb{R}$ we conclude, using elementary arguments as in [7], that all zeros of w are simple and spaced at distances greater than 1. Lemma 3.2 from [3] implies that there exists $c \in \mathbb{R}$ such that $w = x(\cdot + c)$. In particular, $\phi = w_0 \in \xi$. We have shown $\omega(y) \subset \xi$, which yields the assertion. **QED.**

According to [3] the trace $X(\mathbb{R})$ is also orbitally stable with respect to curves Y given by slowly oscillating solutions. This carries over to a stability property of the orbit ξ in C. We prefer to quote a stronger result from [1] which guarantees that the periodic orbit ξ is stable and locally exponentially attractive for the full semiflow F on C. In particular,

(S) given $\epsilon > 0$ there exists $\delta > 0$ such that $\operatorname{dist}(\phi, \xi) < \delta$ implies $\operatorname{dist}(F(t, \phi), \xi) < \epsilon$ for all $t \geq 0$.

Proposition 2. A bounded solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (7) is either slowly oscillating or identically zero.

Proof. Let a bounded solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ be given which is not slowly oscillating. It follows that there exists $t_0 \in \mathbb{R}$ such that for every $t \leq t_0$, the interval $[t-1, t]$ contains at least two different zeros of x. Set $a := \sup\{|x(t)| : t \leq t_0 - 1\}$. Assume $a > 0$. There exists $\epsilon > 0$ such that

$$|f(x)| \leq (2-\epsilon)|x| \quad \text{for all} \quad x \in \mathbb{R}.$$

Choose $\delta \in (0, a)$ so small that

$$\tau := \frac{1}{2-\epsilon}\left(1 - \frac{\delta}{a}\right) \geq \frac{1}{2},$$

and pick a point $m \leq t_0 - 1$ so that

$$a - \delta < |x(m)|.$$

Eq. (7) yields $|x'(t)| \leq (2-\epsilon)a$ for $t \leq t_0$. In case $x(m) > 0$, we find that

$$x(t) \geq x(m) + (2-\epsilon)a(t-m) > a - \delta + (2-\epsilon)a(t-m) \quad \text{for} \quad t \leq m,$$

$$x(t) \geq x(m) - (2-\epsilon)a(t-m) > a - \delta - (2-\epsilon)a(t-m) \quad \text{for} \quad m \leq t \leq t_0.$$

In particular,

$$x(t) > 0 \quad \text{for} \quad m - \frac{1}{2} \leq t \leq m + \frac{1}{2},$$

which is a contradiction to the property of t_0 stated above. The case $x(m) < 0$ is analogous. **QED.**

Proof of the Theorem. We have to show $A \subset \overline{W}$. Let $\phi \in A$ be given. There is a solution $y : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (7) with $y_0 = \phi$ and $y_t \in A$ for all $t \in \mathbb{R}$; y is bounded. Its α–limit set consists of segments w_t of bounded solutions $w : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (7).
Case 1: $\alpha(y) = \{0\}$. Then $\phi \in W^u(0) = W$.
Case 2: $\alpha(y) \neq \{0\}$. Proposition 2 implies that there exists a slowly oscillating solution $w : \mathbb{R} \longrightarrow \mathbb{R}$ with $w_t \in \alpha(y)$ for all $t \in \mathbb{R}$. By Proposition 1, $w_t \longrightarrow \xi$ as $t \longrightarrow \infty$. We find a sequence $t_n \longrightarrow -\infty$ such that $y_{t_n} \longrightarrow \xi$ as $n \longrightarrow \infty$. From this we infer, using (S), that $y_t \in \xi$ for all $t \in \mathbb{R}$.

Proposition 1 and Nussbaum's uniqueness result imply that up to translations in time, there is no other slowly oscillating periodic solution than x, so that we have

$$\xi = \overline{W} \setminus W.$$

Hence $\phi = y_0 \in \xi \subset \overline{W}$. **QED.**

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