Contribution to top down portfolio modeling and systemic risk

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Katrin Anita Zilch

Faculty of Mathematics and Computer Science, Physics and
Geography of the Justus-Liebig-University Giessen

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General notation

$\vee$	$a \vee b := \max\{a, b\}$, $\mathcal{A} \vee \mathcal{B} := \sigma(A \cup B : A \in \mathcal{A}, B \in \mathcal{B})$ for σ-algebras $\mathcal{A}$ and $\mathcal{B}$
$\wedge$	$a \wedge b := \min\{a, b\}$
$\ll$	absolute continuity
$(\cdot)^+$	$a^+ := \max\{a, 0\}$
A^c	complement of the set A
$\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\overline{\mathbb{R}})$, $\mathcal{B}(\mathbb{R}^n)$	Borel-σ-algebra on $\mathbb{R}$, $\overline{\mathbb{R}}$, $\mathbb{R}^n$
$\mathbb{N}$, $\mathbb{N}_0$	natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{R}$, $\overline{\mathbb{R}}$, $\mathbb{R}_+$	real numbers, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $\mathbb{R}_+ := \{r \in \mathbb{R} \vert r \geq 0\}$

Notation in Part I

$[\cdot]$, $[\cdot, \cdot]$	quadratic variation process $[X] = ([X]_t)_{t\geq 0}$, quadratic covariation process $[X, Y] = ([X, Y]_t)_{t\geq 0}$
$\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$	right-continuous filtration generated by $\sigma(W_s : s \leq t) \vee \sigma(\kappa^1, \ldots, \kappa^n) \vee \mathcal{K}_\infty$, p. 40
$\mathscr{A}^i$	$\mathscr{A}^i_t := \int_0^t (1/Z^i_{s-}) d\mathscr{C}^i_s$, p. 51
a_i	parameter of ψ^i, p. 63
C	$(\mathbb{P}, \mathbb{G})$-compensator of N, p. 60
C^i	$(\mathbb{P}, \mathbb{G})$-compensator of N^i, p. 50
$\mathscr{C}^i$	$(\mathbb{P}, \mathbb{F}^{i-1})$-compensator of Z^i, p. 51
$\mathbb{E}^{\infty,\mathscr{N}} = (\mathcal{E}^{\infty,\mathscr{N}}_t)_{t\geq 0}$	$\mathcal{E}^{\infty,\mathscr{N}}_t := \mathcal{K}_\infty \vee \sigma(\mathscr{N}_s : s \leq t)$, p. 44
$F^{(2,i)}(\cdot)$	$F^{(2,i)}(t) := \mathbb{P}^{i-1}[S_i - S_{i-1} \leq t]$ in IIM2, p. 67
$F^{\Delta S}(\cdot, x)$	$F^{\Delta S}(t, x) := \mathbb{P}[\min_{s\leq t}(x + \sigma W_s + \mu s) \leq 0]$, $x > 0$, p. 33
$F^i(x) = (F^i_t(x))_{t\geq 0}$	$F^i_t(x) := F^{\Delta S}(G^i_{t-T_{i-1}}, x) I_{\{T_{i-1}<t\}}$, $x > 0$, p. 54
$\mathbb{F}^{i-1} = (\mathcal{F}^{i-1}_t)_{t\geq 0}$	$\mathcal{F}^{i-1}_t := \bigcap_{u>t} \mathcal{K}_u \vee \sigma(I_{\{T_k\leq s\}} : s \leq u, k \leq i-1)$, p. 39
$\mathbb{F}^{\mathscr{N}} = (\mathcal{F}^{\mathscr{N}}_t)_{t\geq 0}$	$\mathcal{F}^{\mathscr{N}}_t := \sigma(\mathscr{N}_s : s \leq t)$, p. 13
$\mathbb{F}^W = (\mathcal{F}^W_t)_{t\geq 0}$	standard Brownian filtration, p. 30
$f^{\Delta S}(\cdot, x)$	derivative of $F^{\Delta S}(\cdot, x)$, $x > 0$, p. 54
$f^i(x) = (f^i_t(x))_{t\geq 0}$	density process of $F^i(x)$, $x > 0$, p. 56
$(\cdot)^{GT}$	random time/ process/ filtration in the model of Giesecke and Tomecek (2005), p. 37

G	(overall) time change specified by $G_t := \sum_{k=1}^{i-1} G^k_{T_k - T_{k-1}} + G^i_{t-T_{i-1}}$ on $\{T_{i-1} \le t < T_i\}$ for $i \in \{1, \dots, n\}$, $G_t := \sum_{k=1}^{n-1} G^k_{T_k - T_{k-1}} + G^n_{t-T_{n-1}}$ on $\{T_n \le t\}$, pp. 35, 40
G^i	absolutely continuous time change, p. 40
$\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$	investor/ information filtration, p. 18, $\mathcal{G}_t := \bigcap_{u>t} \mathcal{K}_u \vee \sigma(I_{\{T_i \le s\}} : s \le u, i \le n)$, p. 39
$\mathbb{G}' = (\mathcal{G}'_t)_{t\ge 0}$	minimal filtration expansion of $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$, $\mathcal{G}'_t := \mathcal{F}_t \vee \sigma(\tau \wedge t)$, p. 16
$\mathbb{G}^{i-1} = (\mathcal{G}^{i-1}_t)_{t\ge 0}$	$\mathcal{G}^{i-1}_t := \mathcal{F}^{i-1}_{T_{i-1}+t}$, p. 39
$\mathbb{G}^\tau = (\mathcal{G}^\tau_t)_{t\ge 0}$	progressive filtration expansion of $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$, $\mathcal{G}^\tau_t := \{A \in \mathcal{G}_\infty \mid \exists F_t \in \mathcal{F}_t, A \cap \{\tau > t\} = F_t \cap \{\tau > t\}\}$, $\mathcal{G}_\infty := \mathcal{F}_\infty \vee \sigma(\tau)$, p. 16
g	density process of G, p. 35
g^i	density process of G^i, p. 55
H	absolutely continuous time change, p. 79
$\mathbb{H} = (\mathcal{H}_t)_{t\ge 0}$	$\mathcal{H}_t := \mathcal{K}_\infty \vee \sigma(I_{\{S_i \le s\}} : s \le t, i \le n)$, p. 40
h	density process of H, p. 79
K^i	ith default barrier, pp. 30, 36
$\mathbb{K} = (\mathcal{K}_t)_{t\ge 0}$	filtration that models the random environment, p. 38
κ^i	$\kappa^i := \log(K^{i-1}/K^i)$, p. 30, independent random variables with values in $(0, \infty)$, p. 36
λ	intensity process, pp. 12, 13
λ^i	ith intensity process, p. 51
λ^N	$(\mathbb{P}, \mathbb{G})$-intensity of N, p. 60
M	absolutely continuous time change defined by $M_t := \sum_{k=1}^{i-1} M^k_{\bar{T}_k - \bar{T}_{k-1}} + M^i_{t-\bar{T}_{i-1}}$ on $\{\bar{T}_{i-1} \le t < \bar{T}_i\}$ for $i \in \{1, \dots, n\}$, $M_t := \sum_{k=1}^{n-1} M^k_{\bar{T}_k - \bar{T}_{k-1}} + M^n_{t-\bar{T}_{n-1}}$ on $\{\bar{T}_n \le t\}$, p. 79
M^i	absolutely continuous time change, p. 79
μ	$\mu := \mu_V - \sigma_V^2/2$, pp. 30, 37
μ_V	drift of geometric Brownian motion V, pp. 29, 35
N	default counting process $N_t := \sum_{i=1}^n I_{\{T_i \le t\}}$, pp. 27, 37
N^0	counting process $N^0_t := \sum_{i=1}^n I_{\{S_i \le t\}}$, p. 37
N^i	default indicator process $N^i_t := I_{\{T_i \le t\}}$, p. 27
$N^{(k)}$	default indicator process $N^{(k)}_t := I_{\{\tau_k \le t\}}$, p. 27
$\mathcal{N}(\bar{\mu}, \bar{\sigma}^2)$	normal distribution with mean $\bar{\mu}$ and variance $\bar{\sigma}^2$
$\mathscr{N}$	general counting process with arrival times (ζ_m), p. 12
n	number of names in the underlying portfolio, pp. 27, 29, 36
ν^i	impact process defined by $\nu^0_t := g^1_t$, $\nu^i_t := (g^{i+1}_t - g^i_{t+T_i-T_{i-1}}) I_{\{T_i < \infty\}}$ for $i \in \{1, \dots, n-1\}$, p. 71
$(\Omega, \mathcal{A}, \mathbb{P})$	underlying probability space in Chapter 3, p. 35

$(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$	probability space defined by $\Omega^{i-1} := \{S_{i-1} < \infty\}$, $\mathcal{A}^{i-1} := \mathcal{A} \cap \Omega^{i-1} := \{A \cap \Omega^{i-1} \vert A \in \mathcal{A}\}$ and $\mathbb{P}^{i-1}[A] := \mathbb{P}[A \vert S_{i-1} < \infty]$ for $A \in \mathcal{A}$, p. 31
$\Phi(\cdot)$	cumulative distribution function of $\mathcal{N}(0,1)$
$\Phi_2(\cdot,\cdot,\rho)$	2-dimensional normal distribution function with standard normal marginal distributions and correlation coefficient ρ
$\varphi(\cdot)$	probability density function of $\mathcal{N}(0,1)$
$\varphi(\cdot;\bar\mu,\bar\sigma)$	probability density function of $\mathcal{N}(\bar\mu,\bar\sigma^2)$
$\Psi^i(\cdot)$	cumulative distribution function of κ^i in IIM2, p. 63
$\psi^i(\cdot;a_i,v_i,\sigma_i)$	probability density function of κ^i in IIM2, p. 63
S_i	$S_i := \inf\{t \geq 0 \vert V_t \leq K^i\}$, pp. 30, 36
$\sigma = \sigma_V$	volatility of geometric Brownian motion V, pp. 29, 35
σ_i	parameter of ψ^i, p. 63
T_i	time of the ith default in the underlying portfolio, p. 27, $T_i := \inf\{t \geq 0 \vert V_{G_t} \leq K^i\}$, p. 36
$\bar T_i$	$\bar T_i := M^{-1}_{S_i}$, p. 79
τ_k	time of default of firm k, p. 27
V	asset value process in Chapter 2, pp. 18, 26, geometric Brownian motion in Chapter 3, pp. 29, 35
v_i	parameter of ψ^i, p. 63
W	standard Brownian motion, pp. 18, 29, 35
Z^i	$\mathbb{F}^{i-1}$-conditional survival probability of T_i, p. 50
(ζ_m)	general point process, p. 12

Notation in Part II

$\star$	$\bar a \star \bar X := (\bar a_1 \bar X_1, \ldots, \bar a_n \bar X_n)$ for $\bar a \in \mathbb{R}^n_+$, $\bar X \in (L^p)^n$, p. 131
$\\|\cdot\\|$, $\\|\cdot\\|_{\mathcal{X}}$	norm of a linear space $\mathcal{X}$
$\\|\cdot\\|_{\mathcal{A}^1}$, $\\|\cdot\\|_{\mathcal{A}^{1,m}}$	$\\|\cdot\\|_{\mathcal{A}^1} := \\|\cdot\\|_{\mathcal{A}^{1,1}}$, $\\|\bar\xi\\|_{\mathcal{A}^{1,m}} := \sum_{i=1}^m \mathbb{E}[\sum_{t\in\mathbb{N}_0} \vert\Delta\bar\xi^i_t\vert]$ with $\bar\xi^i_{-1} := 0$, $\Delta\bar\xi^i_t := \bar\xi^i_t - \bar\xi^i_{t-1}$ for $\bar\xi \in \mathcal{A}^{1,m}$, pp. 137, 148
$\\|\cdot\\|_{(L^\infty_{\mathcal{H}})^m}$, $\\|\cdot\\|_{(L^1_{\mathcal{H}})^m}$	$\\|\bar X\\|_{(L^\infty_{\mathcal{H}})^m} := \max_{i\in\{1,\ldots,n\}} \\|\bar X^i\\|_{L^\infty_{\mathcal{H}}}$ for $\bar X \in (L^\infty_{\mathcal{H}})^m$, $\\|\bar\Xi\\|_{(L^1_{\mathcal{H}})^m} := \sum_{i=1}^n \\|\bar\Xi^i\\|_{L^1_{\mathcal{H}}}$ for $\bar\Xi \in (L^1_{\mathcal{H}})^m$, p. 149
$\\|\cdot\\|_p$, $\\|\cdot\\|_{p,m}$	(m-dimensional) L^p-norm defined by $\\|\cdot\\|_p := \\|\cdot\\|_{p,1}$ for all $1 \leq p \leq \infty$, $\\|\bar X\\|_{p,m} := \sum_{i=1}^m (\int \vert\bar X_i\vert^p d\mathbb{P})^{1/p}$ for $\bar X \in (L^p)^m, 1 \leq p < \infty$, $\\|\bar X\\|_{\infty,m} := \sum_{i=1}^m \inf\{r \in \mathbb{R} \vert \vert\bar X_i\vert \leq r\ \mathbb{P}\text{-a.s.}\}$ for $\bar X \in (L^\infty)^m$, pp. 97, 105, 137
$\\|\cdot\\|_{\mathcal{R}^\infty}$, $\\|\cdot\\|_{\mathcal{R}^{\infty,m}}$	$\\|\cdot\\|_{\mathcal{R}^\infty} := \\|\cdot\\|_{\mathcal{R}^{\infty,1}}$, $\\|\bar X\\|_{\mathcal{R}^{\infty,m}} := \max_{i\in\{1,\ldots,n\}} \inf\{r \in \mathbb{R} \vert \sup_{t\in\mathbb{N}_0} \vert\bar X^i_t\vert \leq r\}$ for $\bar X \in \mathcal{R}^{0,m}$, pp. 137, 148
$\\|\cdot\\|_{\tau,\theta}$	$\\|X\\|_{\tau,\theta} := \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau \vert \sup_{t\in\mathbb{N}_0} \vert p^{\tau,\theta}(X_t)\vert \leq \gamma\}$ for $X \in \mathcal{R}^\infty$, p. 137

$\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_m$	general pairing, p. 198, $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_1$, pp. 105, 137, $\langle\cdot,\cdot\rangle_m : (L^p)^m \times (L^q)^m \to \mathbb{R}$ defined by $\langle\bar{X},\bar{\xi}\rangle_m := \sum_{i=1}^m \mathbb{E}[\bar{X}^i\bar{\xi}^i]$, p. 105, $\langle\cdot,\cdot\rangle_m : \mathcal{R}^{\infty,m} \times \mathcal{A}^{1,m} \to \mathbb{R}$ defined by $\langle\bar{X},\bar{\xi}\rangle_m := \sum_{i=1}^m \mathbb{E}[\sum_{t\in\mathbb{N}_0} \bar{X}_t^i \Delta\bar{\xi}_t^i]$, p. 148
$\langle\cdot,\cdot\rangle^{\tau,\theta}$, $\langle\cdot,\cdot\rangle_m^{\tau,\theta}$	$\langle\cdot,\cdot\rangle^{\tau,\theta} := \langle\cdot,\cdot\rangle_1^{\tau,\theta}$, $\langle\bar{X},\bar{\xi}\rangle_m^{\tau,\theta} := \sum_{i=1}^m \mathbb{E}[\sum_{t\in[\tau,\theta]\cap\mathbb{N}_0} \bar{X}_t^i \Delta\bar{\xi}_t^i \mid \mathcal{F}_\tau]$ for $\bar{X} \in \mathcal{R}^{\infty,m}$, $\bar{\xi} \in \mathcal{A}^{1,m}$, pp. 140, 166
1_m, 0_m	$1_m := (1,\dots,1)$, $0_m := (0,\dots,0)$ (m-times), p. 106
AVaR_λ	Average Value at Risk at level λ, p. 102
$\mathcal{A}^1$, $\mathcal{A}^{1,m}$	$\mathcal{A}^1 := \mathcal{A}^{1,m}$, $\mathcal{A}^{1,m} := \{\bar{\xi} \in \mathcal{R}^{0,m} \mid \|\bar{\xi}\|_{\mathcal{A}^{1,m}} < \infty\}$, pp. 137, 148
$\mathcal{A}_+^1$, $\mathcal{A}_+^{1,m}$	$\mathcal{A}_+^1 := \mathcal{A}_+^{1,1}$, $\mathcal{A}_+^{1,m} := \{\bar{\xi} \in \mathcal{A}^{1,m} \mid \Delta\bar{\xi}_t^i \geq 0$ for all $t \in \mathbb{N}_0$, $i \in \{1,\dots,m\}\}$, pp. 140, 166
$\mathcal{A}_{\tau,\theta}^1$, $\mathcal{A}_{\tau,\theta}^{1,m}$	$\mathcal{A}_{\tau,\theta}^1 := \mathcal{A}_{\tau,\theta}^{1,1}$, $\mathcal{A}_{\tau,\theta}^{1,m} := p_m^{\tau,\theta}\mathcal{A}^1$, pp. 137, 150
$(\mathcal{A}_{\tau,\theta}^1)_+$, $(\mathcal{A}_{\tau,\theta}^{1,m})_+$	$(\mathcal{A}_{\tau,\theta}^1)_+ := (\mathcal{A}_{\tau,\theta}^{1,1})_+$, $(\mathcal{A}_{\tau,\theta}^{1,m})_+ := p_m^{\tau,\theta}\mathcal{A}_+^{1,m}$, pp. 140, 166
$\mathcal{A}_\Lambda$	acceptance set $\mathcal{A}_\Lambda := \{(Y,\bar{Z}) \in L^p \times (L^p)^n \mid Y \geq \Lambda(\bar{Z})\}$, p. 119
$\mathcal{A}_\Lambda^*$	$\mathcal{A}_\Lambda^* := \{(\xi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^* \mid \mathbb{E}[V\xi] - \mathbb{E}[\bar{Z}^t\bar{\xi}] \geq 0$ for all $(V,\bar{Z}) \in \mathcal{A}_\Lambda\}$, p. 128
$\mathcal{A}_\rho$	acceptance set $\mathcal{A}_\rho := \{X \in \mathcal{X}^{\mathrm{fp}} \mid \rho(X) \leq 0\}$, p. 94
$\mathcal{A}_{\rho_0}$	acceptance set $\mathcal{A}_{\rho_0} = \{(r,X) \in \mathbb{R} \times L^p \mid r \geq \rho_0(X)\}$, p. 119
$\mathcal{A}_{\rho_0}^*$	$\mathcal{A}_{\rho_0}^* := \{(x,\psi) \in \mathbb{R} \times (L^p)^* \mid rx - \mathbb{E}[Y\psi] \geq 0$ for all $(r,Y) \in \mathcal{A}_{\rho_0}\}$, p. 128
α	penalty function on $\mathcal{M}_{1,f}$, p. 96
α_{min}	minimal penalty function on $\mathcal{M}_{1,f}$ (or $\mathcal{M}_1(\mathbb{P})$), pp. 96, 98
α_n	$\alpha_n : (L^p)^* \times ((L^p)^n)^* \to \mathbb{R} \cup \{+\infty\}$ given by $\alpha_n(\xi,\bar{\xi}) = \sup_{(r,Y)\in\mathcal{A}_{\rho_0},(V,\bar{Z})\in\mathcal{A}_\Lambda}\{-r + \mathbb{E}[(Y-V)\xi] + \sum_{i=1}^n \mathbb{E}[\bar{Z}_i\bar{\xi}_i]\}$, p. 123
$\alpha^{\tau,\theta}$	penalty function on $\mathcal{D}_{\tau,\theta}$, p. 140
$\alpha_{min}^{\tau,\theta}$	minimal penalty function on $\mathcal{D}_{\tau,\theta}$, p. 141
$\mathcal{B}_\Lambda$	acceptance set $\mathcal{B}_\Lambda := \{(Y,\bar{X}) \in \mathcal{R}_{\tau,\theta}^\infty \times \mathcal{R}_{\tau,\theta}^{\infty,n} \mid Y \geq \Lambda(\bar{X})\}$, p. 157
$\mathcal{B}_{\infty,m}^r$	$\mathcal{B}_{\infty,m}^r := \{\bar{X} \in (L_{\mathcal{H}}^\infty)^m \mid \|\bar{X}\|_{(L_{\mathcal{H}}^\infty)^m} \leq r\}$ for $r > 0$, p. 159
$\mathcal{B}_\rho$	acceptance set $\mathcal{B}_\rho := \{X \in \mathcal{R}_{\tau,\theta}^\infty \mid \rho(X) \leq 0\}$, p. 138
$\mathcal{B}_{\rho_0}$	acceptance set $\mathcal{B}_{\rho_0} := \{(X,Z) \in \mathcal{R}_{\tau,\theta}^\infty \times \mathcal{R}_{\tau,\theta}^\infty \mid X = \gamma I_{[\tau,\infty)}$ for $\gamma \in L_\tau^\infty, \gamma \geq \rho_0(Z)\}$, p. 158
$\tilde{\mathcal{B}}_{\rho_0}$	acceptance set $\tilde{\mathcal{B}}_{\rho_0} := \{(\gamma,X) \in L_\tau^\infty \times \mathcal{R}_{\tau,\theta}^\infty \mid \gamma \geq \rho_0(X)\}$, p. 157
ba	space of all bounded, finitely additive functions μ on $(\Omega,\mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$, p. 105

$\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda}$	$\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda} := \{(V,X,Z,\bar{Z}) \in \mathcal{R}^{\infty n+3}_{\tau,\theta} \| V = \phi I_{[\tau,\infty)}$ for some $\phi \in L^\infty_\tau$ and $\mathbb{E}[\rho_0(\Lambda(\bar{V}-\bar{Z})-X-Z)]-\mathbb{E}[\phi] \leq d\}$ for $d \in \mathbb{R}$, $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$, p. 163
$\mathrm{cl}\, f$	closure of $f : \mathcal{X} \to \overline{\mathbb{R}}$, p. 199
$\mathrm{conv}\, f$	convex hull of $f : \mathcal{X} \to \overline{\mathbb{R}}$, p. 199
$D_G\Upsilon(x_0)$	Gâteaux derivative of $\Upsilon : \mathcal{X} \to \mathcal{Y}$ at $x_0 \in \mathcal{X}$, p. 201
$\tilde{\mathcal{D}}_T$, $\tilde{\mathcal{D}}^{rel}_T$	$\tilde{\mathcal{D}}_T := \{\varsigma \in L^1_T \| \varsigma \geq 0, \mathbb{E}[\varsigma] = 1\}$, $\tilde{\mathcal{D}}^{rel}_T := \{\varsigma \in \tilde{\mathcal{D}}_T \| \varsigma > 0\}$, p. 144
$\mathcal{D}_{\tau,\theta}$	$\mathcal{D}_{\tau,\theta} := \{\xi \in (\mathcal{A}^1_{\tau,\theta})_+ \| \langle 1,\xi\rangle^{\tau,\theta} = 1\}$, p. 140
$\mathcal{D}^{rel}_{\tau,\theta}$	$\mathcal{D}^{rel}_{\tau,\theta} := \{\xi \in \mathcal{D}_{\tau,\theta} \| \mathbb{P}[\sum_{j\geq t\wedge\theta} \Delta\xi_j > 0] = 1$ for all $t \in \mathbb{N}_0\}$, p. 143
$d^+\upsilon(x_0)$	directional derivative of $\upsilon : \mathcal{X} \to \overline{\mathbb{R}}$ at $x_0 \in \mathcal{X}$ in the direction $x \in \mathcal{X}$, p. 201
$\mathrm{dom}\, f$	effective domain of $f : \mathcal{X} \to \overline{\mathbb{R}}$, p. 199
$\mathcal{E}_{\tau,\theta}$	$\mathcal{E}_{\tau,\theta} := \{\xi \in (\mathcal{A}^1_{\tau,\theta})_+ \| \langle 1,\xi\rangle^{\tau,\theta} \leq 1\}$, p. 166
f^*	convex conjugate of $f : \mathcal{X} \to \overline{\mathbb{R}}$, p. 199
f_Λ	function in f_Λ-constancy property, p. 108
f_ρ	function in f_ρ-constancy property, pp. 107, 151
$I_{[\tau,\theta)}$	$I_{[\tau,\theta)} : \mathbb{N}_0 \times \Omega \to \mathbb{R}$, $I_{[\tau,\theta)}(t,\omega) := I_{\{\tau\leq t<\theta\}}(\omega)$, pp. 138, 150
$\iota_\mathcal{E}$	$\iota_\mathcal{E}(x) := 0$ if $x \in \mathcal{E}$, $\iota_\mathcal{E}(x) := +\infty$ otherwise, p. 122
K	Lagrangian function, p. 200
$k(\bar{X})$	systemic risk attribution of $\bar{X} \in \mathrm{dom}\,\rho$, p. 130
L^0	space of all random variables on $(\Omega,\mathcal{F},\mathbb{P})$, p. 102
$L^0(\overline{\mathbb{R}})$	space of all extended random variables, p. 136
$L^0_\tau(\overline{\mathbb{R}})$	$L^0_\tau(\overline{\mathbb{R}}) := \{\gamma \in L^0(\overline{\mathbb{R}}) \| \gamma$ is $\mathcal{F}_\tau$-measurable$\}$, p. 140
$L^0_\tau(\overline{\mathbb{R}}_+)$	$L^0_\tau(\overline{\mathbb{R}}_+) := \{\gamma \in L^0_\tau(\overline{\mathbb{R}}) \| \gamma \geq 0\}$, p. 140
L^∞_τ	$L^\infty_\tau := L^\infty(\Omega,\mathcal{F}_\tau,\mathbb{P})$, p. 137
$(L^\infty_\tau)_+$	$(L^\infty_\tau)_+ := \{\gamma \in L^\infty_\tau \| \gamma \geq 0\}$, p. 138
$L^\infty_\mathcal{H}$, $L^1_\mathcal{H}$	$L^p_\mathcal{H} := L^p(\mathbb{N}_0 \times \Omega, \mathcal{H}, \eta)$ for $p \in \{1,\infty\}$, p. 148
L^p	$L^p := \{X \in L^0 \| \left\|X\right\|_p < \infty\}$, pp. 97, 105, 137
L^p_+	$L^p_+ := \{X \in L^p \| X \geq 0\}$, p. 108
$\mathcal{L}(\mathcal{X},\mathcal{Y})$	set of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, p. 202
Λ	aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$, p. 109
$\Lambda_{t,T}$	aggregation function at time t, $\Lambda_{t,T} : \mathbb{R}^n \to \mathbb{R}$, p. 178
$\mathrm{lsc}\, f$	lower semicontinuous hull of $f : \mathcal{X} \to \overline{\mathbb{R}}$, p. 199
$\mathcal{M}_1$	space of all probability measures on $(\Omega,\mathcal{F})$, p. 96
$\mathcal{M}_1(\mathbb{P})$	space of all probability measures on $(\Omega,\mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$, p. 97
$\mathcal{M}^q_1(\mathbb{P})$	$\mathcal{M}^q_1(\mathbb{P}) := \{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \| d\mathbb{Q}/d\mathbb{P} \in L^q\}$, p. 101
$\mathcal{M}_{1,f}$	space of all finitely additive set functions $Q : \mathcal{F} \to [0,1]$ with $Q[\Omega] = 1$, p. 96
$\mathcal{M}_{1,f}(\mathbb{P})$	space of all finitely additive set functions $Q : \mathcal{F} \to [0,1]$ with $Q[\Omega] = 1$ which are absolutely continuous with respect to $\mathbb{P}$, p. 97

$(\mathbb{N}_0 \times \Omega, \mathcal{H}, \eta)$	probability space, $\mathcal{H}$ generated by the sets $\{t\} \times B$, $t \in \mathbb{N}_0$, $B \in \mathcal{F}_t$ and $\eta(\{t\} \times B) := 2^{-(t+1)}\mathbb{P}[B]$, p. 148
n	number of nodes in the underlying financial system, pp. 104, 147
$(\Omega, \mathcal{F}, \mathbb{P})$	underlying probability space, pp. 97, 104
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	underlying filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{N}_0}$, $\mathcal{F}_0 := \{\emptyset, \Omega\}$, pp. 136, 147
$\mathfrak{P}(\Omega)$	set of all subsets of Ω, p. 95
$p^{\tau,\theta}$, $p^{\tau,\theta}_m$	$p^{\tau,\theta} := p^{\tau,\theta}_1$, $p^{\tau,\theta}_m : \mathcal{R}^{0,m} \to \mathcal{R}^{0,m}$, $p^{\tau,\theta}_m(\bar{X})_t := I_{\{\tau\leq t\}}\bar{X}_{t\wedge\theta}$, pp. 137, 150
$\mathcal{R}^0$, $\mathcal{R}^{0,m}$	$\mathcal{R}^0 := \mathcal{R}^{0,1}$, space of $\mathbb{F}$-adapted m-dimensional stochastic processes $\bar{X} = (\bar{X}_t)_{t\in\mathbb{N}_0}$, pp. 137, 148
$\mathcal{R}^\infty$, $\mathcal{R}^{\infty,m}$	$\mathcal{R}^\infty := \mathcal{R}^{\infty,1}$, $\mathcal{R}^{\infty,m} := \{\bar{X} \in \mathcal{R}^{0,m} \| \|\bar{X}\|_{\mathcal{R}^{\infty,m}} < \infty\}$, pp. 137, 148
$\mathcal{R}^\infty_{\tau,\theta}$, $\mathcal{R}^{\infty,m}_{\tau,\theta}$	$\mathcal{R}^\infty_{\tau,\theta} := \mathcal{R}^{\infty,1}_{\tau,\theta}$, $\mathcal{R}^{\infty,m}_{\tau,\theta} := p^m_{\tau,\theta}\mathcal{R}^\infty$, pp. 137, 150
$(\mathcal{R}^\infty_{\tau,\theta})_+$	$(\mathcal{R}^\infty_{\tau,\theta})_+ := \{X \in \mathcal{R}^\infty_{\tau,\theta} \| X \geq 0\}$, p. 150
$\partial\upsilon(\bar{X})$	subdifferential of the proper convex function $\upsilon : (L^p)^m \to \mathbb{R} \cup \{+\infty\}$ at $\bar{X} \in \operatorname{dom}\upsilon$, p. 129
ρ	risk measure $\rho : \mathcal{X}^{\mathrm{fp}} \to \mathbb{R}$, p. 93, risk measure $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$, p. 99, systemic risk measure $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$, p. 107, conditional risk measure $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$, p. 138, conditional systemic risk measure $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_\tau$, p. 152
$\rho_{t,T}$	conditional risk measure $\rho_{t,T} : \mathcal{R}^\infty_{t,T} \to L^\infty_t$, p. 142, conditional systemic risk measure $\rho_{t,T} : \mathcal{R}^{\infty,n}_{t,T} \to L^\infty_t$, p. 178
$\rho_{\tau,\theta}$	risk measure $\rho_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ defined by $\rho_{\tau,\theta}(X) := \sum_{t\in[S,T]\cap\mathbb{N}_0} \rho_{t,T}(I_{\{\tau=t\}}X)$ for $S \leq \tau \leq \theta \leq T$, p. 142
ρ_0	single-firm risk measure $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$, p. 106, conditional single-firm risk measure $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$, p. 151
$\rho^0_{t,T}$	single-firm risk measure $\rho^0_{t,T} : \mathcal{R}^\infty_{t,T} \to L^\infty_t$, p. 178
$s_{\mathcal{B}_\Lambda}$	$s_{\mathcal{B}_\Lambda} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \to L^0_\tau(\overline{\mathbb{R}})$, $s_{\mathcal{B}_\Lambda}(\phi, \bar{\xi}) := \operatorname{ess\,sup}_{(Y,\bar{Z})\in\mathcal{B}_\Lambda}\{\langle Y, \phi\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n\}$, p. 169
$\tilde{s}_{\mathcal{B}_\Lambda}$	$\tilde{s}_{\mathcal{B}_\Lambda} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \to \mathbb{R} \cup \{+\infty\}$, $\tilde{s}_{\mathcal{B}_\Lambda}(\phi, \bar{\xi}) := \mathbb{E}[s_{\mathcal{B}_\Lambda}(\phi, \bar{\xi})]$, p. 169
$s_{\mathcal{B}_{\rho_0}}$	$s_{\mathcal{B}_{\rho_0}} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta} \to L^0_\tau(\overline{\mathbb{R}})$, $s_{\mathcal{B}_{\rho_0}}(\psi, \xi) := \operatorname{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0}}\{\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y, \xi\rangle^{\tau,\theta}\}$, p. 168
$\tilde{s}_{\mathcal{B}_{\rho_0}}$	$\tilde{s}_{\mathcal{B}_{\rho_0}} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta} \to \mathbb{R} \cup \{+\infty\}$, $\tilde{s}_{\mathcal{B}_{\rho_0}}(\psi, \xi) := \mathbb{E}[s_{\mathcal{B}_{\rho_0}}(\psi, \xi)]$, p. 168
$\sigma(\mathcal{X}, \mathcal{V})$	weakest topology on $\mathcal{X}$ such that for every $v \in \mathcal{V}$, the linear functional $\mathcal{X} \ni x \mapsto \langle x, v\rangle$ is continuous, pp. 137, 150, 198
$\mathfrak{T}^m_{\tau,\theta}$	$\mathfrak{T}^m_{\tau,\theta} := \{\mathcal{U} \cap \mathcal{R}^{\infty,m}_{\tau,\theta} \| \mathcal{U} \in \sigma(\mathcal{R}^{\infty,m}, \mathcal{A}^{1,m})\}$, p. 165

τ, θ	$\mathbb{F}$-stopping times with $\tau < \infty$ and $0 \leq \tau \leq \theta \leq \infty$, pp. 137, 150
VaR_λ	Value at Risk at level λ, p. 102
$\mathcal{X}, \mathcal{Y}, \mathcal{V}$	linear spaces
$(\mathcal{X}, \mathcal{V})$	paired spaces, p. 198
$\mathcal{X}^*$	dual space of $\mathcal{X}$
$\mathcal{X}^{\mathrm{fp}}$	space of all financial positions, p. 93
$(\xi^o, \bar{\xi}^o)$	optimal solution to optimization problem in dual representation of ρ, pp. 129, 130, 131
$\mathcal{Z}$	$\mathcal{Z} := \{(\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} \mid (\gamma I_{[\tau,\infty)}, Y) \in \mathcal{B}_{\rho_0}, (V, \bar{Z}) \in \mathcal{B}_\Lambda\}$, p. 176
$\mathcal{Z}^\#$	$\mathcal{Z}^\# := \{(\xi, \bar{\xi}) \in (L^p)^* \times ((L^p)^n)^* \mid (1, \xi) \in \mathcal{A}^*_{\rho_0},\ (\xi, \bar{\xi}) \in \mathcal{A}^*_\Lambda\}$, p. 132, $\mathcal{Z}^\# := \{(\xi, \bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \mid -\gamma + \langle Y - V, \xi\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n \leq 0$ for all $(\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{Z}\}$, p. 176
$\mathcal{Z}^\#(\bar{X})$	$\mathcal{Z}^\#(\bar{X}) := \{(\xi, \bar{\xi}) \in \mathcal{Z}^\# \mid \rho(\bar{X}) = \sum_{i=1}^n \mathbb{E}[\bar{\xi}_i \bar{X}_i]\}$ for $\bar{X} \in (L^p)^n$, p. 132

Abbreviations

a.e.	almost every
a.s.	almost surely
càdlàg	continue à droite, limites à gauche (right-continuous with left limits)
càglàd	continue à gauche, limites à droite (left-continuous with right limits)
IIM	incomplete information model
l.s.c.	lower semicontinuous
u.s.c.	upper semicontinuous

Introduction

Risk is omnipresent in existing financial systems. There exist various categories of financial risk addressing different aspects and perspectives. This thesis is devoted to two specific types of risk: *portfolio credit risk*, which originates in the possibility that borrowers are not able to repay their debts as previously agreed upon, and *systemic risk* covering the risk of an entire financial system. Throughout this thesis we go beyond the perspective of a single firm and study the interaction between multiple entities.

The recent financial crisis shed light on modeling approaches concerning both types of risk. First of all, it has revealed several problems of existing models for credit derivatives and portfolio credit derivatives in particular. Consequently, valuation and risk management of these instruments and especially the development of accurate and still tractable models is an important field of research. In addition to that, we have seen the consequences arising out of a loss of trust between market participants and the questioning of their own models. On the one hand, they adapt a more risk averse behavior, which results in counterparty contagion. For instance, contagion could spread from lender to borrower. While this behavior may still be rational, in the recent financial crisis panic led to much more drastic contagion effects and finally resulted in a collapse of financial markets. The topic of systemic risk is closely related to these observations, and the events of this crisis illustrate the importance of identification, measuring and controlling this specific type of risk.

Focusing on portfolio credit risk in the first part of this thesis, we consider a credit portfolio with n counterparties. More precisely, we look at securities issued by these firms. In this framework, important products are basket credit derivatives which offer protection against the ith default in the underlying portfolio.

The existence of various counterparties leads to the question of how to deal with the complex dependencies between these different entities. We adopt the so called top down perspective, which means that we do not explicitly study the relationship between the considered firms but focus on the portfolio as a whole. Consequently, we are interested in the so called default counting process, which simply counts the defaults in the underlying portfolio without telling us anything about the identity of the defaulted name.

Starting with a structural definition of default in the sense of Black and Cox (1976), we develop the first top down first-passage model for portfolio credit risk. Structural variables in our model are the portfolio value process of the underlying portfolio and different time independent default barriers. We model the portfolio value process by a time changed geometric Brownian motion and suppose that a default occurs if this process hits a specific barrier. Because we have to model several defaults, our model consists of n possibly stochastic sequential default barriers.

In order to obtain a tractable model, we additionally introduce an incomplete information framework which uses some of the ideas in Giesecke (2006). We study different incomplete information models with different assumptions on the availability of information concerning the portfolio value process and the default barriers. As a consequence of this, we obtain reduced form formulas for prices of credit sensitive securities and an algorithm to simulate the default times. Nonetheless, the most important feature in our top town first-passage default model is the specific time change which itself depends on the default times. This type of time change has also been studied in Giesecke and Tomecek (2005). Due to the dependence on prior defaults, the time change determines how sequential defaults depend on each other and the resulting default counting process is self-affecting. Therefore, our model has the flexibility to incorporate feedback of default events to future events. This allows for the possibility that a default increases the likelihood of the next default. As a result of this, we are able to model the contagion effects discussed above.

While in portfolio credit risk modeling we deal with a rather limited number of counterparties, in the context of systemic risk the situation is quite different because we consider an entire financial system with much more entities. The second part of this thesis is devoted to systemic risk measurement. Here, we change the perspective from a modeling point of view towards the view of a financial regulator or a central bank. These entities are interested in measuring and managing the risk in order to maintain the stability of the financial system.

In the context of single-firm risk measurement, Artzner et al. (1999) introduced and Delbaen (2000, 2002) generalized an axiomatic approach. They studied so called coherent risk measures which assign risk to random payments and satisfy four economically desirable properties. Later, Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) extended these approaches and introduced convex risk measures. Coherent risk measures, as well as convex risk measures, can be characterized by their corresponding acceptance sets, and both risk measures admit a specific dual representation which clarifies the relationship between a risk measure and the largest expected loss with respect to a family of probability measures. A further generalization includes dynamic aspects of risk measurement. For instance, taking into account the availability of additional information leads to the theory of conditional risk measures. Another possibility is to define risk measures on stochastic processes which represent the evolution of firm specific financial values; see Artzner et al. (2007), Cheridito et al. (2006) and others.

In line with these approaches for single-firm risk measurement, we generalize the approach in Chen et al. (2013) who studied positively homogeneous systemic risk measures on a finite probability space. Throughout the second part of this thesis we work on a general probability space and consider a network of n firms. The main objects of interest are convex systemic risk measures which are defined by dropping the axiom of positive homogeneity in the approach of Chen et al. (2013). Our convex systemic risk measures can be decomposed into a convex single-firm risk measure and a so called aggregation function. The latter function determines how to pool the losses of the individual firms contained in the underlying financial system and provides, in contrast to the classical portfolio approach, more flexible possibilities to do so. Based on this fundamental decomposition result, we are able to characterize our convex systemic risk measure in terms of acceptance sets. In addition, we obtain a dual representation result, which is applied to another important subject in the context of systemic risk measurement. We consider the question of what fraction each firm contributes to the systemic risk of the whole financial system. More precisely, based on our dual representation result, we study an appropriate risk attribution method.

Beyond the scope of these static convex systemic risk measures, we develop in Chapter 7 the first dynamic approach to systemic risk and study conditional and dynamic convex systemic risk measures. In more detail, we study systemic risk measures for multi-dimensional bounded discrete time stochastic processes and incorporate the availability of new information over time. We are able to extend our decomposition and primal representation result directly to this setting and with some of the techniques from Cheridito et al. (2006) we also obtain a dual representation of conditional systemic risk measures.

Another important aspect in conjunction with dynamic risk measurement is the question of how risk measures at different points in time depend on each other. In literature, there exist several different suggestions for this issue of time-consistency. We study the notion of so called strong time-consistency in line with the idea from Riedel (2004) or Artzner et al. (2007) to use the Bellman principle in conjunction with the axiomatic risk measurement approach. Because our dynamic systemic risk measures can be decomposed into a dynamic single-firm risk measure and a dynamic aggregation function, time-consistency is studied for each of these objects. In particular, we discuss the relation between these properties and focus on the question of what time-consistency means for the underlying aggregation function.

Part I.

Portfolio credit risk modeling: a top down first-passage default model

Credit risk originates in the possibility that a person or an organization, the borrowing entity, is not able to repay its debt as previously arranged in a contractual agreement. To manage this risk, the question of how to model the risk of a single security or even a whole portfolio of securities is essential.

Among the standard credit risk modeling approaches there are two types of models: *structural* and *reduced form models*.

The basis of all structural models, starting with the approaches of Merton (1974) and Black and Cox (1976), is the definition of the default event. This default event is defined by using the firm's structural variables. We can, for example, define the time of default as the first time the value of the firm's assets falls below a specific barrier, and this barrier may depend on the value of the firm's debt. In this case, relevant structural variables are the firm's assets and the value of the firm's debt. Models of this kind are models in which a default is triggered by the first hitting of a specific barrier. These are called first-passage models. Because of this explicit definition of the default event, we know the reason for this default: The firm asset value is too low and too close to the value of the firm's debt. Another important feature of most structural models is that all the information which is needed to specify the default time of a specific firm is always publicly available for everyone, which in reality is generally not the case. We discuss this aspect of structural models in more detail in Subsection 2.1.3.

Reduced form models and intensity models do not establish a direct connection between the default event and the firm's structural variables. The first models of this type were introduced by Jarrow and Turnbull (1992), Artzner and Delbaen (1995) and Duffie and Singleton (1999). In reduced form models a default occurs if the so called default indicator process jumps. But in contrast to structural models, this process is exogenously given. This implies that we do not know the reason for the default, we only know that a default has occurred. Nevertheless, an advantage of the definition of the default time in reduced form models is that much less information is required: We do not necessarily need to know the value of the firm's assets and debt to obtain its time of default. Indeed, "reduced form models were originally constructed to be consistent with the information that is available to the market" (Jarrow and Protter (2004), p. 5). As a consequence, an important advantage of reduced form models is that pricing formulas are often tractable, and hence applicable in practice.

As discussed so far, structural and reduced form models distinguish between the assumptions on the information that is available to the modeler. Jarrow and Protter (2004) and Elizalde (2006) discuss this information based distinction of structural and reduced form models in detail. Jarrow and Protter (2004) emphasize that a structural model can be converted into a reduced form model if we transform the set of available information. This means that we have to relax the assumption that all information is publicly available. Therefore, a link between these two approaches is given by so called *incomplete information models*. For a more detailed discussion we refer again to Subsection 2.1.3. Since these models based on incomplete information can be arranged between structural and reduced form models, they have the potential to combine the advantages of both approaches. This means they have the advantage of an economically explainable default event as in structural models and provide applicable pricing formulas as in reduced form models; see also Giesecke and Goldberg (2004a,b).

In this part of the thesis we are interested in credit risk at the portfolio level. Therefore, we consider a reference portfolio consisting of n names. Giesecke (2008) points out that every model of portfolio credit risk has three main objects of interest. These are the available information, the default times (and in conjunction with these, the corresponding process counting the defaults in the portfolio) and finally the loss distribution at a default event. For simplicity we concentrate on the available information and the default times and consider a constant loss at a default event. The available information is modeled by a filtration which is denoted by $\mathbb{G}$ throughout this thesis. Moreover, we distinguish between two types of default times. On the one hand, we consider the default time of a specific firm $i \in \{1, \ldots, n\}$; on the other hand, we study the time of the ith default in the underlying portfolio. In the latter case, we do not necessarily know which firm belongs to the ith default.

With this terminology we are able to distinguish between *top down* and *bottom up approaches* in credit risk. Brigo et al. (2010) propose a very intuitive distinction. In bottom up models we try "to model dependence by specifying dependence across single default times" (Brigo et al. (2010), p. 2). In the top down framework we "could completely give up single-name default modelling and focus on the pool loss and default-counting process" (Brigo et al. (2010), p. 8). By using a more technical distinction, we can concentrate on the content of the filtration $\mathbb{G}$; see also Giesecke (2008) and Bielecki et al. (2010) who provide a detailed overview focused on reduced form models. In top down and bottom up approaches the filtration $\mathbb{G}$ contains enough information to identify the default events, but in bottom up models an investor or a modeler is also informed about the identity of a defaulted name. Hence, in comparison to top down models, the information filtration contains additional information.

We can find reduced form approaches with a top down perspective as well as reduced form approaches with a bottom up perspective. But to our knowledge, in the context of structural models, there exist only bottom up models in the literature so far. For an overview see the table below. Examples of these different approaches are discussed in Section 2.2.

model type	*top down*	*bottom up*
structural	×	✓
reduced form	✓	✓

We develop the first *top down first-passage model* for portfolio credit risk. More precisely, we start with a structural definition of default and incorporate the idea that only partial information about the structural variables is available to investors in the market. Structural variables in our model are the portfolio value process of the underlying portfolio that consists of n names (corresponding to the firm's asset value process in case of a single entity) and different time independent default barriers K^i with $K^1 > K^2 > \ldots > K^n$. The time of the ith default in the underlying portfolio is denoted by T_i and defined as the first time the portfolio value process hits the ith default barrier K^i. Since our model is an incomplete information model, it does not only have the advantage of an economically explainable default event, but it also incorporates advantages of reduced form models. Indeed, we obtain explicit solutions for the so called default trends of the different default times T_i for $i \in \{1, \ldots, n\}$. The specific form of these default trends allows us to determine tractable solutions for prices of credit sensitive securities. In detail, we consider prices of contingent claims that pay a specific amount at time T only if the ith default in the underlying pool of names did not occur up to time T, i.e., if $T < T_i$; otherwise, the payout is equal to zero. Moreover, we will see that our default times are totally inaccessible, which is typical for reduced form models and a desired property. As Jarrow and Protter (2004) point out, totally inaccessible default times imply that default is not predictable, i.e., it occurs by surprise. Finally, based on the previous results for default trends, we are able to determine the compensator of the default counting process of the underlying portfolio, and we obtain an algorithm to simulate our default times in a similar way to Giesecke and Goldberg (2004b). This algorithm is typical for reduced form models.

To implement the idea of partially available information, we use the ideas from the incomplete information approach of Giesecke (2006). This approach has also been considered in the bottom up first-passage structural model in Giesecke and Goldberg (2004b). Here, the authors model for each firm an individual firm value process and suppose that these are correlated with each other in order to model the dependence on common market factors. The default time is equal to the first time this process falls below a firm specific stochastic barrier, and these barriers are supposed to be dependent. This is justified by the observation that debt levels of different firms depend on each other. Moreover, neither the asset values nor the default barriers are available to investors. As a consequence of this modeling approach, the model covers contagion effects such as jumps in prices of credit sensitive securities after a default event.

There also exist top down reduced form models with a similar property. To be more precisely, the conditional portfolio default rate, i.e., the portfolio intensity λ, is sensitive to the occurrence of a default. Technically, this means that after each event, the portfolio intensity possibly changes and not simply because there are less potential defaulters in the portfolio; see Giesecke (2008). Examples of such approaches are Giesecke and Tomecek (2005), Arnsdorf and Halperin (2009), Ding et al. (2009) and Cont and Minca (2013) among many others. We refer to Giesecke (2008) for a more detailed overview and more references.

In addition, we can find other models that focus on such contagion effects and which are closely linked to the previously introduced incomplete information approaches; see, for instance, Jiao (2009), Chapter 2 in Kchia (2011) and El Karoui et al. (2013). All previously mentioned approaches start with a default-free reference filtration and consider enlargements of this filtration with respect to ordered or unordered default times. The model in El Karoui et al. (2013) is based on the approach in El Karoui et al. (2010) and mainly studies the case of ordered default times. A key assumption in El Karoui et al. (2013) is the existence of conditional densities for the default times. Thus, these kinds of models are referred to as conditional density approaches. As a result of the model definition in El Karoui et al. (2013), "conditional densities given the global information depend explicitly on the past defaults" (El Karoui et al. (2013), p. 8). In particular, the timing of these defaults matters. Kchia (2011) analyzes credit contagion in the conditional density approach with the focus on unordered default times. Jiao (2009), who also concentrates on unordered default times, studies a model framework which allows to remove the density hypothesis. In our model we do not claim a corresponding assumption to the density hypothesis in El Karoui et al. (2013), but due to our incomplete information framework, we also work with successively enlarged filtrations. Moreover, we will see that in our model the timing of past defaults is also relevant for future default events.

Since we want to construct a top down first-passage default model, the question is how to incorporate feedback of default events (see Giesecke (2008)) in such a model framework. Our solution to this problem is the usage of *time change techniques*. A time change can be seen as a map that transforms calendar time in financial time in the sense that the financial time is strongly connected with the financial activities that occurred up to calender time t. If, for example, the volatility in the market is very high in a given period, then financial time will run faster than the corresponding calender time. See, for instance, Albanese et al. (2003) or Carr and Wu (2004) for this interpretation.

In general, we can distinguish between two types of time changes. A first possibility is to use nondecreasing Lévy processes; see Cont and Tankov (2004). These time changes are also called Lévy subordinators and the first application in finance of such time changes can be found in Clark (1973). Second, we can focus on absolutely continuous time changes which are defined as integrals of stochastic processes.

Time change techniques are well known and often applied in equity modeling. Geman et al. (2001) consider models for financial market price processes that are given by purely discontinuous time changed Brownian motions. Important examples of time changed Brownian motions with a specific Lévy subordinator are the well known Variance Gamma process, see Madan and Seneta (1990) and Madan et al. (1998), or the Normal Inverse Gaussian model of Barndorff-Nielsen (1998). There are also many applications of absolutely continuous time changes in equity modeling; see, for instance, the stochastic volatility models of Heston (1993) and Carr et al. (2003).

Time changes are also increasingly popular in credit risk modeling. Albanese et al. (2003) consider, for example, the so called credit quality process of a specific firm, which takes values in $[0, 1]$. This credit quality process is given by a diffusion process that is time changed by a gamma process. Therefore, the resulting process includes jumps such that migration rates can be correctly fitted by the model.

Examples of time change models in multi-name credit risk are the bottom up structural models in Overbeck and Schmidt (2005), Luciano and Schoutens (2006) and Hurd (2009). Overbeck and Schmidt (2005) study the case of two underlying names and assume that the ability-to-pay processes are given by correlated Brownian motions which are transformed by deterministic time changes. In this approach the time change differs for each firm. In contrast to this, Luciano and Schoutens (2006) use the same gamma time change for all names in the portfolio. Hurd (2009) models the so called log-leverage ratio process as a time changed Brownian motion. Here, each time change is the weighted sum of a firm specific time change and a common time change shared by all names.

In the context of top down reduced form approaches, examples of models using time change techniques are the models in Giesecke and Tomecek (2005) and Ding et al. (2009). The first approach models the default counting process of the underlying portfolio as a time changed Poisson process and the second approach assumes that this counting process is given by a time changed birth process. Both approaches have a self-exiting default counting process in common, which means that prior defaults have an influence on the arrival of future default events. This is exactly the property we want to incorporate in our model. But while in the model of Ding et al. (2009) the self-affecting default counting process is a direct consequence of the definition of the underlying birth process, Giesecke and Tomecek (2005) obtain this result by the specific time change which depends at time t on all default times that occurred up to this point in time.

We will use this property and define our portfolio value process as a time changed geometric Brownian motion, and our time change is similar to the one introduced in Giesecke and Tomecek (2005). In Chapter 3 we will see that the resulting default counting process is also self-affecting such that the timing of past defaults is relevant for future default events.

The outline of the first part of this thesis is the following. In Chapter 1 we repeat important mathematical preliminaries. The aim of Chapter 2 is to give an overview of the different perspectives in modeling (portfolio) credit risk. Therefore, the differences between structural and reduced form models and bottom up and top down models are discussed in detail. Chapter 3 contains the main results of this part of the thesis. Here, we introduce and study our top down first-passage default model. To be more precisely, we discuss two different models based on different assumptions concerning the availability of information. At the end of this chapter we analyze a specific time change and determine the implications to our model.

1. Mathematical preliminaries

In this chapter we introduce the terminology needed in the subsequent study. In the first part of this thesis we focus on the jump process that counts the defaults in a given portfolio. These processes are specific types of so called point processes. We discuss in Section 1.1 important facts about point processes in general. Section 1.2 is dedicated to compensator processes and their computation. Throughout this chapter let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration on this space.

1.1. Point processes

The following definitions and statements are based on Brémaud (1981).

Definition 1.1.1. *A* point process *over* $[0, \infty)$ *is a sequence* (ζ_m) *of random variables with values in* $[0, \infty]$ *such that* $\zeta_0 = 0$ *and* $\zeta_m < \infty$ *implies* $\zeta_m < \zeta_{m+1}$. *If we define* $\zeta_\infty := \lim_{m\to\infty} \zeta_m$, *then the associated process* $\mathscr{N}$ *defined by*

$$\mathscr{N}_t := \begin{cases} m & \text{if } t \in [\zeta_m, \zeta_{m+1}), m \in \mathbb{N}_0 \\ +\infty & \text{if } t \geq \zeta_\infty \end{cases}$$

is called counting process. *The point process* (ζ_m) *is called* nonexplosive *if and only if* $\zeta_\infty = \infty$ *or equivalently* $\mathscr{N}_t < \infty$ $\mathbb{P}$*-almost surely (a.s.) for all* $t \geq 0$. *We say that* $\mathscr{N}$ *is* integrable *if and only if* $\mathbb{E}[\mathscr{N}_t] < \infty$ *for all* $t \geq 0$.

Remark 1.1.2. Since there exists a one-to-one correspondence between the sequence (ζ_m) and the corresponding process $\mathscr{N}$, the process $\mathscr{N}$ is also called *point process.*

The following definition introduces a specific point process.

Definition 1.1.3. *Consider a point process* $\mathscr{N}$ *that is adapted to the filtration* $\mathbb{F}$ *and let* λ *be a nonnegative, measurable process such that*

1. λ_t *is* $\mathcal{F}_0$*-measurable for all* $t \geq 0$,
2. $\int_0^t \lambda_s ds < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$.

We call $\mathscr{N}$ *an* $\mathbb{F}$-doubly stochastic Poisson process *with* (stochastic) intensity λ *if and only if*

$$\mathbb{E}[\exp(iu(\mathscr{N}_t - \mathscr{N}_s))|\mathcal{F}_s] = \exp\left((e^{iu} - 1)\int_s^t \lambda_v dv\right) \tag{1.1}$$

for all $0 \leq s \leq t$ *and all* $u \in \mathbb{R}$. *If* λ *is deterministic, then* $\mathscr{N}$ *is called* $\mathbb{F}$-Poisson process *and if* $\lambda \equiv 1$, *then we say that* $\mathscr{N}$ *is a standard* $\mathbb{F}$-Poisson process. *In case* *of* $\mathbb{F} = \mathbb{F}^{\mathscr{N}} = (\mathcal{F}_t^{\mathscr{N}})_{t \geq 0}$ *with* $\mathcal{F}_t^{\mathscr{N}} := \sigma(\mathscr{N}_s : s \leq t)$*, we call* $\mathscr{N}$ *a* Poisson process*. If* $\mathbb{F} = \mathbb{F}^{\mathscr{N}}$ *and* $\lambda \equiv 1$*, then* $\mathscr{N}$ *is called* standard Poisson process.

Remark 1.1.4. Brémaud (1981) points out that the first property and Equation (1.1) in Definition 1.1.3 enable us to condition on $\mathcal{F}_0$ in (1.1) such that

$$\mathbb{E}[\exp(iu(\mathscr{N}_t - \mathscr{N}_s))|\mathcal{F}_0] = \exp\left((e^{iu} - 1)\int_s^t \lambda_v dv\right) = \mathbb{E}[\exp(iu(\mathscr{N}_t - \mathscr{N}_s))|\mathcal{F}_s \vee \mathcal{F}_0].$$

Hence, $\mathscr{N}_t - \mathscr{N}_s$ is $\mathbb{P}$-independent of $\mathcal{F}_s$ given $\mathcal{F}_0$. Moreover, (1.1) implies the well known property

$$\mathbb{P}[\mathscr{N}_t - \mathscr{N}_s = k|\mathcal{F}_s] = \exp\left(-\int_s^t \lambda_v dv\right) \frac{\left(\int_s^t \lambda_v dv\right)^k}{k!} \quad \text{for } k \in \mathbb{N}_0 \text{ and } 0 \leq s \leq t. \tag{1.2}$$

In case of a standard $\mathbb{F}$-Poisson process $\mathscr{N}$ starting in 0, it follows from Equation (1.2) that

$$\mathbb{P}[\mathscr{N}_t - \mathscr{N}_s = k|\mathcal{F}_s] = \exp\left(-(t-s)\right) \frac{(t-s)^k}{k!} \quad \text{for } k \in \mathbb{N}_0 \text{ and } 0 \leq s \leq t.$$

The previous equation implies that in this special case of a standard $\mathbb{F}$-Poisson process, the waiting times $(\zeta_m - \zeta_{m-1})_{m \in \mathbb{N}}$ are $\mathbb{P}$-independent and additionally satisfy

$$\mathbb{P}[\zeta_m - \zeta_{m-1} > t] = \exp(-t) \quad \text{for } m \in \mathbb{N} \text{ and } t \geq 0.$$

This means that the waiting times are exponentially distributed with parameter $\lambda = 1$. Furthermore, it follows that $\mathbb{P}[\zeta_m < \infty] = 1$ for all $m \in \mathbb{N}_0$. We refer to Section 10.2 in Meintrup and Schäffler (2005) for more details.

So far, we have defined the intensity of a specific point process, the so called Poisson process. The following definition clarifies the term for general point processes.

Definition 1.1.5. *Consider an* $\mathbb{F}$*-adapted point process* $\mathscr{N}$ *and a nonnegative,* $\mathbb{F}$*-progressive process* λ *that satisfies* $\int_0^t \lambda_s ds < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*. If for all nonnegative,* $\mathbb{F}$*-predictable processes* Y *the equation*

$$\mathbb{E}\left[\int_0^\infty Y_s d\mathscr{N}_s\right] = \mathbb{E}\left[\int_0^\infty Y_s \lambda_s ds\right]$$

holds, then we say that $\mathscr{N}$ *admits the* ($\mathbb{F}$-)intensity λ.

Theorem 1.1.6 (See Theorem T8, Chapter II in Brémaud (1981))**.** *Suppose that the point process* $\mathscr{N}$ *admits the* $\mathbb{F}$*-intensity* λ*. Then* $\mathscr{N}$ *is nonexplosive and the following statements hold:*

1. *M defined by $M_t := \mathscr{N}_t - \int_0^t \lambda_s ds$ is an $\mathbb{F}$-local martingale.*
2. *If X is an $\mathbb{F}$-predictable process with $\mathbb{E}[\int_0^t |X_s|\lambda_s ds] < \infty$ for all $t \geq 0$, then the process Y defined by $Y_t := \int_0^t X_s dM_s$ is an $\mathbb{F}$-martingale.*

3. *If X is an $\mathbb{F}$-predictable process with $\int_0^t |X_s|\lambda_s ds < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then the process Y defined by $Y_t := \int_0^t X_s dM_s$ is an $\mathbb{F}$-local martingale.*

The next result characterizes the intensity in terms of martingale properties.

Theorem 1.1.7 (See Theorem T9, Chapter II in Brémaud (1981))**.** *Consider a nonexplosive, $\mathbb{F}$-adapted point process $\mathscr{N}$ and let λ be a nonnegative, $\mathbb{F}$-progressive process such that the process $(\mathscr{N}_{t\wedge\zeta_m} - \int_0^{t\wedge\zeta_m} \lambda_s ds)_{t\geq 0}$ is an $\mathbb{F}$-martingale for all $m \in \mathbb{N}$. Then λ is the $\mathbb{F}$-intensity of $\mathscr{N}$.*

According to Theorem T12, Chapter II in Brémaud (1981), $\mathbb{F}$-predictable intensities are unique, and Theorem T13, Chapter II in Brémaud (1981) states that for every point process $\mathscr{N}$ with $\mathbb{F}$-intensity λ, there always exists an $\mathbb{F}$-intensity $\bar{\lambda}$ that is $\mathbb{F}$-predictable.

The following results are special cases from the results in Chapter A.2 in Brémaud (1981) and will be applied in Chapter 3.

Theorem 1.1.8 (See Theorem T23, Chapter A2 in Brémaud (1981))**.** *Consider a point process $\mathscr{N}$ with corresponding sequence (ζ_m). Then the following statements hold:*

1. *ζ_m is an $\mathbb{F}^{\mathscr{N}}$-stopping time for all $m \in \mathbb{N}_0$.*
2. *$\mathcal{F}_t^{\mathscr{N}} = \sigma(I_{\{\zeta_m \leq s\}} : s \leq t, m \in \mathbb{N})$.*

Theorem 1.1.9 (See Theorem T26 and Theorem T28, Chapter A2 in Brémaud (1981))**.** *Consider an $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$-valued process Y such that there exists for all $t \geq 0$ and all $\omega \in \Omega$ an $\varepsilon(t, \omega) > 0$ with*

$$Y_{t+s}(\omega) = Y_t(\omega) \qquad \textit{on } [t, t + \varepsilon(t, \omega)).$$

Then the natural filtration $\mathbb{F}^Y$ is right-continuous and for all $\mathbb{F}^Y$-stopping times τ, we have

$$\mathcal{F}_\tau^Y = \sigma(Y_{s\wedge\tau} : s \geq 0).$$

Theorem 1.1.10 (See Theorem T30, Chapter A2 in Brémaud (1981))**.** *Consider a point process $\mathscr{N}$ with corresponding sequence (ζ_m). Then the following equations are satisfied:*

1. *$\mathcal{F}_{\zeta_m}^{\mathscr{N}} = \sigma(\zeta_i : 0 \leq i \leq m)$ for all $m \in \mathbb{N}$.*
2. *$\mathcal{F}_{\zeta_\infty}^{\mathscr{N}} = \mathcal{F}_\infty^{\mathscr{N}} = \sigma(\zeta_i : i \in \mathbb{N}_0)$.*

1.2. The compensator of a stochastic process

The following theorem is well known and enables us to define the so called compensator of a stochastic process. In what follows, we assume that $\mathbb{F}$ satisfies the usual conditions.

We call a càdlàg process (a process which is right-continuous with left limits) $C = (C_t)_{t\geq 0}$ an *increasing process* if the paths of $C : t \mapsto C_t(\omega)$ are nondecreasing for almost every (a.e.) $\omega \in \Omega$.

Theorem 1.2.1 (Doob-Meyer decomposition; see, e.g., Theorem 11, Chapter III in Protter (2005))**.** *Consider a càdlàg $\mathbb{F}$-supermartingale X with $X_0 = 0$ such that the family*

$$\{X_\tau | \tau \text{ is a finite stopping time}\}$$

is uniformly integrable. Then there exists a unique, increasing, $\mathbb{F}$-predictable process C with $C_0 = 0$ such that the process M given by $M_t = X_t + C_t$ is a uniformly integrable $\mathbb{F}$-martingale.

Definition 1.2.2. *We call the process C the* $(\mathbb{P}, \mathbb{F})$-compensator *of X. If there is no confusion about the probability measure, we also use the term* $\mathbb{F}$-compensator *or simply* compensator.

In the subsequent study we are interested in point processes $\mathscr{N}$ which have a finite number of jumps, i.e., there exists $n \in \mathbb{N}$ such that $\zeta_m = \infty$ for $m > n$. Since $\mathscr{N}$ is an increasing and bounded process, we know from the Doob-Meyer decomposition theorem that $\mathscr{N}$ admits the unique compensator C such that $\mathscr{N} - C$ is an $\mathbb{F}$-martingale. If we additionally assume that there exists a nonnegative, $\mathbb{F}$-progressive process λ such that $C_t = \int_0^t \lambda_s ds$ $\mathbb{P}$-a.s., then it follows from Theorem 1.1.7 that λ is the intensity of $\mathscr{N}$. Thus, the compensator of $\mathscr{N}$ is strongly connected to the intensity. Nevertheless, this does not imply the existence of such an intensity.

For the following remark and the corresponding proof see also Chapter II in Brémaud (1981).

Remark 1.2.3. Let us again consider a nonexplosive, $\mathbb{F}$-adapted point process $\mathscr{N}$ and a nonnegative, $\mathbb{F}$-progressive process λ such that

$$\left(\mathscr{N}_{t\wedge\zeta_m} - \int_0^{t\wedge\zeta_m} \lambda_s ds\right)_{t\geq 0} \quad \text{is an } \mathbb{F}\text{-martingale for all } m \in \mathbb{N}. \tag{1.3}$$

If we additionally assume that λ is right-continuous and bounded, then we obtain

$$\lambda_s = \lim_{t\downarrow s} \frac{1}{t-s} \mathbb{E}\left[\mathscr{N}_t - \mathscr{N}_s \middle| \mathcal{F}_s\right] \quad \mathbb{P}-\text{a.s.}$$

This equation can be verified as follows: Fix $0 \leq s \leq t$. The martingale property in (1.3) yields $\mathbb{E}[\mathscr{N}_{t\wedge\zeta_m} - \mathscr{N}_{s\wedge\zeta_m} | \mathcal{F}_s] = \mathbb{E}[\int_{s\wedge\zeta_m}^{t\wedge\zeta_m} \lambda_u du | \mathcal{F}_s]$ $\mathbb{P}$-a.s. for all $m \in \mathbb{N}$. For $m \to \infty$ we obtain

$$\mathbb{E}\left[\mathscr{N}_t - \mathscr{N}_s \middle| \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t \lambda_u du \middle| \mathcal{F}_s\right] \quad \mathbb{P}-\text{a.s.}$$

If λ is right-continuous and bounded, it follows from Lebesgue's averaging theorem and Lebesgue's dominated convergence theorem that

$$\lim_{t\downarrow s} \frac{1}{t-s} \mathbb{E}\left[\mathscr{N}_t - \mathscr{N}_s \middle| \mathcal{F}_s\right] = \lim_{t\downarrow s} \frac{1}{t-s} \mathbb{E}\left[\int_s^t \lambda_u du \middle| \mathcal{F}_s\right] = \mathbb{E}\left[\lambda_s \middle| \mathcal{F}_s\right] = \lambda_s \quad \mathbb{P}-\text{a.s.}$$

Finally, the following lemma addresses important properties of the so called compensated process $\mathscr{N} - C$.

Lemma 1.2.4 (See Lemma 3.1 in Pang et al. (2007))**.** *Let $\mathscr{N}$ be a nonexplosive, integrable, $\mathbb{F}$-adapted point process with compensator C. If C is continuous, then the $\mathbb{F}$-martingale $M := \mathscr{N} - C$ is square-integrable and the corresponding quadratic variation process satisfies $[M]=\mathscr{N}$.*

Computing the compensator via the Jeulin-Yor theorem

Let us consider a nonnegative random variable τ. If τ is not an $\mathbb{F}$-stopping time, we can expand $\mathbb{F}$ and obtain an expanded filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ such that τ is a $\mathbb{G}$-stopping time. For the remaining part of this section we suppose that τ is not an $\mathbb{F}$-stopping time.

We are interested in computing the compensator of the process $(I_{\{\tau\leq t\}})_{t\geq 0}$. Note that $(I_{\{\tau\leq t\}})_{t\geq 0}$ is a $\mathbb{G}$-submartingale such that this process admits a $\mathbb{G}$-compensator according to the Doob-Meyer decomposition theorem. We call this compensator *$\mathbb{G}$-compensator of τ*.

First, we distinguish between three different filtration expansions.

Definition 1.2.5. *The* minimal filtration expansion $\mathbb{G}' = (\mathcal{G}'_t)_{t\geq 0}$ *is the smallest expansion such that τ is a stopping time with respect to this filtration, i.e., it is given by*

$$\mathcal{G}'_t := \mathcal{F}_t \vee \sigma(\tau \wedge t) = \mathcal{F}_t \vee \sigma(\{\tau \leq s\} : s \leq t).$$

The progressive filtration expansion *is defined as the filtration $\mathbb{G}^\tau = (\mathcal{G}^\tau_t)_{t\geq 0}$ such that*

$$\mathcal{G}^\tau_t := \{A \in \mathcal{G}_\infty | \exists F_t \in \mathcal{F}_t, A \cap \{\tau > t\} = F_t \cap \{\tau > t\}\}$$

with $\mathcal{G}_\infty := \mathcal{F}_\infty \vee \sigma(\tau)$. Finally, we say that a filtration expansion $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is of the Guo-Zeng type *if τ is a $\mathbb{G}$-stopping time and*

$$\mathcal{G}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}.$$

For a detailed discussion of these filtration expansions we refer to Guo and Zeng (2008), who focus on the last filtration expansion, and Okhrati (2013). Chapter VI in Protter (2005) considers filtration expansions in general.

Note that the progressive filtration expansion is right-continuous; see also Chapter VI in Protter (2005). By the definition of the progressive filtration expansion, we have $\mathcal{F}_\infty \cap \{\tau \leq t\} \subset \mathcal{G}^\tau_t$. This means that on the event $\{\tau \leq t\}$, the σ-algebra $\mathcal{G}^\tau_t$ contains the entire information included in $\mathcal{F}_\infty$, i.e., it contains the whole information that is encoded in the smaller filtration $\mathbb{F}$ up to time ∞. Guo and Zeng (2008) point out that this is not realistic from a modeling point of view. In contrast to the progressive filtration expansion, the minimal filtration expansion does not satisfy this critical property. Moreover, both the minimal filtration expansion and the progressive filtration expansion are expansions of the Guo-Zeng type. We refer again to Guo and Zeng (2008) for more details.

Now, we come back to our problem of computing the compensator of the process $(I_{\{\tau\leq t\}})_{t\geq 0}$. If we define the $\mathbb{F}$-*conditional survival probability* by

$$Z_t := \mathbb{E}[I_{\{\tau>t\}} | \mathcal{F}_t],$$

then Z is an $\mathbb{F}$-supermartingale since for all $s \le t$, we have

$$\begin{aligned} Z_s &= \mathbb{E}[1 - I_{\{\tau \le s\}} | \mathcal{F}_s] \ge \mathbb{E}[1 - I_{\{\tau \le t\}} | \mathcal{F}_s] \\ &= \mathbb{E}[\mathbb{E}[1 - I_{\{\tau \le t\}} | \mathcal{F}_t] | \mathcal{F}_s] = \mathbb{E}[Z_t | \mathcal{F}_s] \quad \mathbb{P} - \text{a.s.} \end{aligned}$$

The following definition is closely related to this $\mathbb{F}$-supermartingale. This definition is based on Definition 5.1 in Giesecke (2006), and we will see below that the introduced process will play an important role in computing the $\mathbb{G}$-compensator of τ.

Definition 1.2.6. *Suppose that $Z_t > 0$ $\mathbb{P}$-a.s. for all $t \ge 0$ and let $Z_{t-} := \lim_{s \uparrow t} Z_s$ and $Z_{0-} := 1$. If $\mathscr{C}^Z$ denotes the $\mathbb{F}$-compensator of Z, then the process $\mathscr{A}$ defined by*

$$\mathscr{A}_t := \int_0^t \frac{1}{Z_{s-}} d\mathscr{C}_s^Z \tag{1.4}$$

is called default trend.

Jeulin and Yor (1978) proved in case of a progressive expansion $\mathbb{G} = \mathbb{G}^\tau$ that the problem of computing the $\mathbb{G}$-compensator of τ can be transformed to the problem of computing the $\mathbb{F}$-compensator of Z. This result was extended by Guo and Zeng (2008) to the more general expansions of the Guo-Zeng type.

Theorem 1.2.7 (Extended Jeulin-Yor theorem; see Theorem 1.1 in Guo and Zeng (2008))**.** *Let the filtration $\mathbb{F}$ satisfy the usual conditions with $\mathcal{F}_\infty \subset \mathcal{F}$ and let $\mathbb{G}$ be a filtration expansion of the Guo-Zeng type of $\mathbb{F}$. Then the $\mathbb{G}$-compensator of $(I_{\{\tau \le t\}})_{t \ge 0}$ is given by*

$$\int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\mathscr{C}_s^Z .$$

2. Modeling (portfolio) credit risk

In this chapter we first take a closer look at the two main types of models among the standard credit risk modeling approaches. We discuss the main ideas of structural and reduced form models in Section 2.1. For this purpose it suffices to consider the case of a single counterparty. In Section 2.2 we analyze the two main types for modeling portfolio credit risk, i.e., bottom up and top down models.

Throughout this chapter let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a filtered probability space and let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ satisfy the usual conditions. From now on, we assume that the filtration $\mathbb{G}$ models the flow of information over time which is available to an investor in the market. Hence, we call this filtration *information filtration* or *investor filtration.*

2.1. Structural versus reduced form models

2.1.1. Important structural models

The basis of all structural models is the definition of the default event. In structural models a firm defaults if its firm value is lower than a given trigger level. Since all structural models have such a default barrier in common, they are also called threshold models. In this subsection we shortly introduce two important structural models: the *Merton model*, which can be considered as the first structural model, and the *Black-Cox model*, the first first-passage model. The second model is of special interest since our model in Chapter 3 is strongly connected to the ideas of this approach.

In this subsection let the filtration $\mathbb{G}$ be generated by $\sigma(W_s : s \leq t)$ where W is a standard Brownian motion with respect to the risk neutral probability measure $\mathbb{P}$.

Merton (1974) used the setting of the standard Black-Scholes model with time horizon T'. To define the time of default of a given firm, the main structural variables considered by Merton are the firm's asset value and the value of the firm's debt. In Merton's model a firm's asset value V is described by a continuous-time geometric Brownian motion, i.e.,

$$dV_t = V_t r dt + V_t \sigma dW_t$$

where r is a constant risk-free rate of a money market account and $\sigma > 0$. The firm's capital structure is assumed to be given by equity and debt which is a zero coupon bond with maturity $T \leq T'$ and face value K. The equity value at time T is given by $S_T = \max\{V_T - K, 0\}$, and the payoff of the bond is equal to $B_T = \min\{K, V_T\}$. In the Merton model a firm defaults at time T if the firm's asset value is too low to pay back the face value of the debt to the bondholders, i.e., if $V_T < K$. Therefore, the probability of default as a function of T can be easily computed by

$$\mathbb{P}[\tau \leq T] = \mathbb{P}[V_T < K] = \Phi\left(\frac{-\log(V_0/K) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

Black and Cox (1976) extended the Merton model in the sense that a firm can also default at points in time with $t < T$, i.e., a default can occur before the maturity T of the bond. Default is defined as the first time the firm value falls below a deterministic barrier. Originally, the barrier was thought of as the point at which safety covenants of the bond are responsible for the default of the firm. Again, the firm value is given by a geometric Brownian motion as in Merton (1974), and the default barrier is modeled by

$$K_t = \bar{K}\exp(-\gamma(T - t)),$$

for $\gamma > 0$ and a constant $\bar{K}$ such that $\bar{K}\exp(-\gamma(T-t)) \leq K\exp(-r(T-t))$ for all $t \in [0, T]$. Hence, the payoff to debt at time T is equal to $B_T = \min\{K, V_T\}I_{\{\tau > T\}}$. Since the default time is given by

$$\tau = \inf\{t \in [0,T] | V_t \leq K_t\} \quad (\inf \emptyset := +\infty),$$

we obtain in case of a constant default barrier (i.e., $K_t = K$ for all $t \in [0, T]$) that

$$\mathbb{P}[\tau \leq T] = \mathbb{P}\left[\inf_{0 \leq s \leq T}(\log(V_0/K) + (r - \sigma^2/2)s + \sigma W_s) \leq 0\right].$$

This probability is well known (see, for instance, Chapter 3 in Jeanblanc et al. (2009) or Section 2.8 in Karatzas and Shreve (1988)) and given by

$$\begin{aligned}\mathbb{P}[\tau \leq T] = \Phi&\left(\frac{-\log(V_0/K) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ &+ e^{-2(r-\sigma^2/2)\log(V_0/K)/\sigma^2}\,\Phi\left(\frac{-\log(V_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right).\end{aligned}$$

Finally, note that this probability of default is equal to the Merton probability of default plus an additional term. This additional term is justified by the fact that in the Black-Cox model default can also occur prior to T. Obviously, this implies that the probability $\mathbb{P}[\tau \leq T]$ has to be higher than in Merton's model.

2.1.2. Reduced form models

In contrast to structural models, reduced form models, which were pioneered by Jarrow and Turnbull (1992), Artzner and Delbaen (1995) and Duffie and Singleton (1999), do not explicitly consider the relationship between default and the firm's financial situation. Here, default is indicated by a jump process which is given exogenously.

As discussed in Jeanblanc and Le Cam (2008), in the context of reduced form models, we can distinguish between two types: intensity based and hazard process models. In this subsection we shortly introduce both approaches and highlight important aspects. For more details we refer the reader, for instance, to Elliot et al. (2000), Jeanblanc and Rutkowski (1999, 2000), Bielecki and Rutkowski (2004) and Lando (2004).

The intensity based approach: An important feature of intensity based models, in particular in comparison to hazard process approaches, is that solely the investor filtration $\mathbb{G}$ is considered. In intensity based approaches the default time τ is a positive random variable. Its conditional distribution can be described by using the intensity process λ; see, for instance, Lando (2004). We can interpret the intensity λ as a conditional default rate in the sense that

$$\mathbb{P}[\tau \leq s + \Delta s | \mathcal{G}_s] \approx I_{\{\tau > s\}} \lambda_s \Delta s. \tag{2.1}$$

More precisely, we assume that the $\mathbb{G}$-stopping time τ denotes the time of a firm's default. The corresponding jump process is given by $(I_{\{\tau \leq t\}})_{t \geq 0}$. As a nondecreasing indicator process, $(I_{\{\tau \leq t\}})_{t \geq 0}$ is a $\mathbb{G}$-submartingale such that by the Doob-Meyer decomposition theorem (see Theorem 1.2.1), there exists a compensator $C^{(\tau)}$ such that $(I_{\{\tau \leq t\}} - C_t^{(\tau)})_{t \geq 0}$ is a $\mathbb{G}$-martingale. Since $I_{\{\tau \leq t \wedge \tau\}} = I_{\{\tau \leq t\}}$ for each $t \geq 0$ and $(I_{\{\tau \leq t \wedge \tau\}} - C_{t \wedge \tau}^{(\tau)})_{t \geq 0}$ is still a $\mathbb{G}$-martingale, it follows from the uniqueness of the compensator that $C_t^{(\tau)} = C_{t \wedge \tau}^{(\tau)}$ for all $t \geq 0$.

In intensity based models it is assumed that $C^{(\tau)}$ is absolutely continuous with respect to the Lebesgue measure ($\mathbb{P}$-a.s.), i.e., the compensator $C^{(\tau)}$ is given by

$$C_t^{(\tau)} = \int_0^t \lambda_s ds \quad \text{for all } t \geq 0$$

($\mathbb{P}$-a.s.) for a $\mathbb{G}$-progressive, nonnegative stochastic process λ, which is called the *($\mathbb{G}$-)intensity of* τ.

Since $(I_{\{\tau \leq t\}})_{t \geq 0}$ is a $\mathbb{G}$-adapted, nonexplosive point process $\mathscr{N}$ with $\zeta_1 = \tau$ and $\zeta_m = +\infty$ for $m > 1$, we know from Theorem 1.1.7 that λ is the $\mathbb{G}$-intensity of $(I_{\{\tau \leq t\}})_{t \geq 0}$ in the sense of Definition 1.1.5. If λ is additionally bounded and right-continuous, then it follows from Remark 1.2.3 that

$$\lambda_s = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}[I_{\{\tau \leq t\}} - I_{\{\tau \leq s\}} | \mathcal{G}_s]. \tag{2.2}$$

Since τ is a $\mathbb{G}$-stopping time, we have $\{\tau > s\} \in \mathcal{G}_s$. It follows

$$\begin{aligned} I_{\{\tau > s\}} \lambda_s &= \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}[(I_{\{\tau \leq t\}} - I_{\{\tau \leq s\}}) I_{\{\tau > s\}} | \mathcal{G}_s] = I_{\{\tau > s\}} \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E}[I_{\{\tau \leq t\}} | \mathcal{G}_s] \\ &= I_{\{\tau > s\}} \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{P}[\tau \leq t | \mathcal{G}_s] = I_{\{\tau > s\}} \lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}[\tau \leq s + h | \mathcal{G}_s]. \end{aligned}$$

Thus, Equation (2.2) is the exact version of Equation (2.1) and clarifies why the intensity is said to express the conditional probability of a default in the next instant.

An important advantage of reduced form models is the availability of more tractable pricing formulas for defaultable contingent claims. The following pricing formula was originally proved in Duffie et al. (1996).

Proposition 2.1.1 (See, e.g., Section 1.1 in Jeanblanc and Le Cam (2008))**.** *Consider a $\mathcal{G}_T$-measurable, integrable random variable X which describes a defaultable promised payoff at time T. Then, in absence of a risk-free interest rate, the price of X at time $t < T$ is given by*

$$PV(t,T) = \mathbb{E}[XI_{\{T<\tau\}}|\mathcal{G}_t] = I_{\{t<\tau\}}(Y_t - \mathbb{E}[\Delta Y_\tau I_{\{\tau \leq T\}}|\mathcal{G}_t])$$

where the process Y is defined by

$$Y_t := \exp(C_t^{(\tau)})\mathbb{E}[X \exp(-C_T^{(\tau)})|\mathcal{G}_t]$$

where ΔY_τ denotes the jump of Y at τ.

In an intensity based model with $C_t^{(\tau)} = \int_0^t \lambda_s ds$ where Y is continuous at τ, the price of the defaultable contingent claim X is given by

$$PV(t,T) = I_{\{t<\tau\}}\mathbb{E}\left[X \exp\left(-\int_t^T \lambda_s ds \right) \middle| \mathcal{G}_t \right]$$

in case of a zero interest rate and by

$$PV(t,T) = I_{\{t<\tau\}}\mathbb{E}\left[X \exp\left(-\int_t^T r_s + \lambda_s ds \right) \middle| \mathcal{G}_t \right]$$

in case of an interest rate r which is not equal to zero. Because of the last equation, a defaultable contingent claim can be priced as a risk-free security if the risk-free rate r is replaced by the adjusted rate $r + \lambda$.

The hazard process approach: In contrast to intensity based models, the hazard process approach, in general, relies on two filtrations. In addition to the investor filtration $\mathbb{G}$, a smaller filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is considered. From now on, we assume that the filtration $\mathbb{F}$, as well as the filtration $\mathbb{G}$, satisfies the usual conditions. Moreover, the default time τ is supposed to be a nonnegative random variable on $(\Omega, \mathcal{G}, \mathbb{P})$ such that $\mathbb{P}[\tau = 0] = 0$ and $\mathbb{P}[\tau > 0] > 0$ for all $t \geq 0$. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is given by

$$\mathcal{G}_t := \bigcap_{u>t} \mathcal{F}_u \vee \sigma(I_{\{\tau \leq s\}} : s \leq u).$$

Due to this definition, $\mathbb{G}$ is split into two filtrations. Obviously, the filtration generated by $\sigma(I_{\{\tau \leq s\}} : s \leq t)$ contains the default information, and a common assumption in applications is that the filtration $\mathbb{F}$ represents the default-free information which is available in the market. Note that τ is not necessarily an $\mathbb{F}$-stopping time, but τ is a $\mathbb{G}$-stopping time by construction.

In all hazard process approaches the process F defined by

$$F_t := \mathbb{P}[\tau \leq t | \mathcal{F}_t]$$

plays a prominent role. This process F is a bounded, nonnegative $\mathbb{F}$-submartingale with $\mathbb{E}[F_t] = \mathbb{P}[\tau \leq t]$. Thus, we can work with the càdlàg modification of F; see, for instance, Theorem 9, Chapter I in Protter (2005).

Definition 2.1.2. *Suppose that the process F satisfies $F_t < 1$ for all $t \geq 0$. Then the process Γ defined by $\Gamma_t := -\log(1 - F_t)$ is called the* $\mathbb{F}$-hazard process *of τ.*

For the remaining part of this subsection it is assumed that the condition $F_t < 1$ is satisfied for all $t \geq 0$ such that the $\mathbb{F}$-hazard process always exists. In particular, this means that τ is not an $\mathbb{F}$-stopping time. The following lemma clarifies the relationship between conditional expectations with respect to the filtrations $\mathbb{F}$ and $\mathbb{G}$.

Lemma 2.1.3 (See, e.g., Lemma 3.2 in Jeanblanc and Rutkowski (2000))**.** *Consider a $\mathcal{G}$-measurable, integrable random variable X. Then we have for all $t \geq 0$*

$$\mathbb{E}[I_{\{t<\tau\}}X|\mathcal{G}_t] = I_{\{t<\tau\}}\frac{\mathbb{E}[I_{\{t<\tau\}}X|\mathcal{F}_t]}{1 - F_t} = I_{\{t<\tau\}}e^{\Gamma_t}\mathbb{E}[I_{\{t<\tau\}}X|\mathcal{F}_t].$$

The pricing rule in the context of hazard process based models reads as follows.

Proposition 2.1.4 (See, e.g., Proposition 3.1 in Elliot et al. (2000))**.** *Consider an $\mathcal{F}_T$-measurable, integrable random variable X which describes a defaultable promised payoff at T. Then, in absence of a risk-free interest rate, the price of X at time $t < T$ is given by*

$$PV(t,T) = \mathbb{E}[XI_{\{T<\tau\}}|\mathcal{G}_t] = I_{\{t<\tau\}}\mathbb{E}[X\exp(\Gamma_t - \Gamma_T)|\mathcal{F}_t].$$

If we additionally assume that the $\mathbb{F}$-hazard process is absolutely continuous, i.e., $\Gamma_t = \int_0^t \gamma_s ds$ for all $t \geq 0$ and an $\mathbb{F}$-progressively measurable process γ, then this process γ is called $\mathbb{F}$-*intensity* of τ. Moreover, we obtain

$$PV(t,T) = I_{\{t<\tau\}}\mathbb{E}\left[X\exp\left(-\int_t^T \gamma_s ds\right)\middle|\mathcal{F}_t\right].$$

In order to study martingales with respect to the filtrations $\mathbb{F}$ and $\mathbb{G}$, an important requirement in hazard process approaches is the following hypothesis:

(H) Every square integrable $\mathbb{F}$-martingale is a square integrable $\mathbb{G}$-martingale.

It is well known (see, for instance, Lemma 6.4 in Jeanblanc and Rutkowski (1999)) that this hypothesis is equivalent to

$$\mathbb{P}[\tau \leq s|\mathcal{F}_\infty] = \mathbb{P}[\tau \leq s|\mathcal{F}_t] \quad \text{for all } s \leq t. \tag{2.3}$$

Furthermore, the (H)-hypothesis implies that every $\mathbb{F}$-Brownian motion is also a Brownian motion with respect to the larger filtration $\mathbb{G}$.

Let us now consider the following definition.

Definition 2.1.5. *An* $\mathbb{F}$-martingale hazard process *of a random time τ is defined as an $\mathbb{F}$-predictable, right-continuous, increasing process Λ with $\Lambda_0 = 0$ such that the process $(I_{\{\tau\leq t\}} - \Lambda_{t\wedge\tau})_{t\geq 0}$ is a $\mathbb{G}$-martingale.*

According to the Doob-Meyer decomposition theorem, the $\mathbb{G}$-compensator $C^{(\tau)}$ of $(I_{\{\tau\leq t\}})_{t\geq 0}$ is unique. Thus, the $\mathbb{F}$-martingale hazard process is unique up to the random time τ. More precisely, for two $\mathbb{F}$-martingale hazard processes Λ and Λ' we have $\Lambda_{\tau\wedge t} = \Lambda'_{\tau\wedge t}$ for all $t \geq 0$.

In order to compute this $\mathbb{F}$-martingale hazard process, we first consider the case in which the following hypothesis is satisfied:

(G) F admits a modification with increasing paths.

Proposition 2.1.6 (See, e.g., Proposition 6.1.1 in Bielecki and Rutkowski (2004))**.** *Suppose that F is an increasing and $\mathbb{F}$-predictable process. Then the $\mathbb{F}$-martingale hazard process Λ of τ is given by*

$$\Lambda_t = \int_{(0,t]} \frac{dF_u}{1-F_{u-}} = \int_{(0,t]} \frac{d\mathbb{P}[\tau \leq u|\mathcal{F}_u]}{1-\mathbb{P}[\tau < u|\mathcal{F}_u]}. \tag{2.4}$$

Note that the $\mathbb{F}$-hazard process Γ is continuous if and only if the process F is continuous. Therefore, we distinguish between two cases.

Proposition 2.1.7 (See, e.g., Proposition 6.2.1 in Bielecki and Rutkowski (2004))**.** *Suppose that (G) holds. Then the following statements are satisfied:*

1. *If the increasing process F is continuous, then we obtain the following equality: $\Gamma_t = \Lambda_t = -\log(1-F_t)$ for all $t \geq 0$. In particular, the $\mathbb{F}$-martingale hazard process Λ is also continuous.*
2. *If the increasing process F is $\mathbb{F}$-predictable and discontinuous, then*

$$\exp(-\Gamma_t) = \exp(-\Lambda^c_t) \prod_{0<u\leq t} (1-\Delta\Lambda_u)$$

where Λ^c denotes the continuous component of Λ, i.e., $\Lambda^c_t := \Lambda_t - \sum_{0\leq u\leq t} \Delta\Lambda_u$.

In general, (G) is not necessarily satisfied. Nevertheless, we can specify the $\mathbb{F}$-martingale hazard process by a formula which is similar to (2.4).

Let $\mathscr{C}^F$ be the $\mathbb{F}$-compensator of F, i.e., $\mathscr{C}^F$ is the unique, $\mathbb{F}$-predictable, increasing process with $\mathscr{C}^F_0 = 0$ such that $F - \mathscr{C}^F$ is an $\mathbb{F}$-martingale.

Proposition 2.1.8 (See, e.g., Proposition 6.1.2 in Bielecki and Rutkowski (2004))**.** *Suppose that one of the following two conditions holds:*

- *(G) is not satisfied.*
- *(G) is satisfied and the increasing process F is not $\mathbb{F}$-predictable.*

Then the following statements hold:

1. *The $\mathbb{F}$-martingale hazard process Λ of τ is given by*

$$\Lambda_t = \int_{(0,t]} \frac{d\mathscr{C}^F_u}{1-F_{u-}}. \tag{2.5}$$

2. *Suppose that $\mathscr{C}^F_t = \mathscr{C}^F_{t\wedge\tau}$ for all $t \geq 0$. Then the $\mathbb{F}$-martingale hazard process Λ of τ is given by $\Lambda = \mathscr{C}^F$.*

Note that if $\mathbb{F}$ satisfies the requirements from the extended Jeulin-Yor Theorem (see Theorem 1.2.7), then we already know from this theorem that $(I_{\{\tau\leq t\}} - \Lambda_{t\wedge\tau})_{t\geq 0}$ with Λ given in (2.5) is a $\mathbb{G}$-martingale.

Let us now consider the relationship between hypothesis (G) and (H). Since the (H)-hypothesis is equivalent to (2.3), it is obvious that we can find a modification of F which is increasing if hypothesis (H) is satisfied. Hence, the (H)-hypothesis is stronger than the (G)-hypothesis.

Proposition 2.1.9 (See, e.g., Proposition 6.7 in Jeanblanc and Rutkowski (1999))**.** *Let the process F be continuous. Then the following statements are equivalent:*

1. *The process F is increasing.*
2. *If the process $(Y_t)_{t\geq 0}$ is an $\mathbb{F}$-martingale, then the stopped process $(Y_{t\wedge\tau})_{t\geq 0}$ is a $\mathbb{G}$-martingale.*

In particular, the previous result states that models in which (G) is satisfied and F is continuous admit the property that stopped $\mathbb{F}$-martingales are still martingales with respect to the enlarged filtration $\mathbb{G}$.

Finally, we will discuss the relationship to intensity based models. More precisely, we are interested in the $\mathbb{G}$-intensity process λ, i.e., the $\mathbb{G}$-progressive, nonnegative process such that the $\mathbb{G}$-compensator $C^{(\tau)}$ of $(I_{\{\tau\leq t\}})_{t\geq 0}$ satisfies $C^{(\tau)}_t = \int_0^t \lambda_s ds$ for all $t \geq 0$. In the following, we will consider two specific cases.

First, let F be increasing and continuous. Then we know from Proposition 2.1.7 that

$$\Lambda_t = \Gamma_t = -\log(1 - F_t) \quad \text{for all } t \geq 0.$$

If additionally $\Gamma_t = \int_0^t \gamma_s ds$ for all $t \geq 0$, then the $\mathbb{G}$-intensity of τ is, for instance, given by $\lambda_t := I_{\{t<\tau\}}\gamma_t$.

In case of Proposition 2.1.8, we have

$$\Lambda_t = \int_{(0,t]} \frac{d\mathscr{C}^F_u}{1 - F_{u-}} \quad \text{for all } t \geq 0.$$

Now, let us suppose that the process $\mathscr{C}^F$ is absolutely continuous with respect to the Lebesgue measure, i.e., $\mathscr{C}^F_t = \int_0^t c_s ds$ for all $t \geq 0$ and an appropriate process c. Then the $\mathbb{G}$-intensity λ of τ can be specified by $\lambda_t := c_t/(1 - F_{t-})I_{\{t<\tau\}}$.

2.1.3. The role of information

An important advantage of structural models is that they link the default event with structural variables of the firm. Because of this dependence, the default time is economically founded. Therefore, these models are also called “cause and effect approach[es]” (Giesecke and Goldberg (2004a), p. 11). On the other hand, in traditional structural models it is assumed that the necessary structural variables, which are, for instance, the firm’s asset value and the value of the firm’s debt, are available for everyone in the market. As Jarrow and Protter (2004) point out, this means that the modeler's information set corresponds with the information set of a firm's manager. This so called complete information approach has two well known problems: First of all, it is unrealistic that an investor or a modeler has full information. For example, in practice an investor faces always some uncertainty about the true asset value of a firm. Moreover, Jarrow and Protter (2004) argue that the approach of publicly available information very often leads to a predictable default time, which means that there exists an announcing sequence of stopping times. An exception are approaches in which the asset value is modeled by a process that has jumps; see, for instance, Zhou (2001). In case of a continuous asset value process V, the sequence (τ_m) defined by $\tau_m := \inf\{t > 0 | V_t \leq K + 1/m\}$ is a possible announcing sequence for the first-passage default time τ. As a consequence, the default is never a real surprise because an investor is always aware of the "distance of the firm to default" (Giesecke (2006), p. 2285). Moreover, it is well known that under some technical requirements, predictable default times lead to short credit spreads which are equal to zero; see, for instance, Proposition 3.2 in Giesecke (2006). But zero short spreads mean that for short time horizons, investors in defaultable bonds do not ask for an additional compensation in form of a higher yield compared to the risk-free yield. This is intuitively very unrealistic and contradicts the outcome of empirical studies; see, for example, Sarig and Warga (1989).

In contrast to structural models, reduced form models are based on the assumption that much less information is available. Jarrow and Protter (2004) make clear that the modeler's information set coincides with the information set of the market. Technically, this means that in reduced form models an exogenously given jump process is used to model the default of a firm such that it is not necessary to model the firm's asset value. As a consequence, an investor does not know how close the firm is to a default, and hence the default event occurs surprisingly, which is much more realistic. Indeed, in most reduced form models the default time τ of a specific firm is totally inaccessible, which means that

$$\mathbb{P}[\tau = \theta < \infty] = 0 \quad \text{for every predictable stopping time } \theta.$$

Another advantage of reduced form models is the existence of tractable pricing formulas.

An information based distinction between structural and reduced form models can, for example, be found in Jarrow and Protter (2004) and Elizalde (2006). The authors of the former paper point out that "structural models can be transformed into reduced form models as the information set changes and becomes less refined, from that observable by the firm's management to that which is observed by the market" (Jarrow and Protter (2004), p. 2). Thus, a link between these two approaches is given by so called incomplete information models. Incomplete information approaches are structural models which assume that only partial information about the structural variables is available to the modeler. Since these models based on incomplete information can be arranged between structural and reduced form models, they have the potential to combine the advantages of both approaches. That is, the default event is economically explainable, and in addition, incomplete information models provide applicable pricing formulas; see also Giesecke and Goldberg (2004a,b).

The model in Duffie and Lando (2001) can be seen as the first structural incomplete information model where the true asset value is not observable. The authors assume that discrete asset information is available to bondholders only. In this case, the default time is totally inaccessible as in most reduced form models. Moreover, the authors are able to compute a default intensity for their model. Kusuoka (1999) and Nakagawa (2001) have introduced early filtering models assuming that investors cannot observe the asset value process directly. Nevertheless, they observe another process continuously in time which is related to the asset value. Other incomplete information approaches are Cetin et al. (2004), Giesecke (2006), Coculescu et al. (2008), Frey and Schmidt (2009), Cetin (2012) and Frey and Lu (2012) to name a few.

In our top down first-passage model we also work with such an incomplete information framework. More precisely, we consider the setting of Giesecke (2006), who introduces the "extreme" cases of incomplete information.

As stated by Jarrow and Protter (2004), incomplete information models have in common that they focus on two filtrations $\mathbb{F}$ and $\mathbb{G}$ satisfying $\mathbb{F} \subset \mathbb{G}$. Here, the larger filtration $\mathbb{G}$ represents again the investor filtration. In case of so called first-passage default models, the filtration $\mathbb{F}$ is considered in more detail below. For the following definition see also Definition 2.1 (and the subsequent remarks) in Giesecke (2006).

Definition 2.1.10. *Let $\mathbb{G}$ be the investor filtration and consider a firm with asset value process V and random default barrier K. Let the $\mathbb{G}$-stopping time τ be the time of the firm's default which is given by*

$$\tau = \inf\{t > 0 | V_t \leq K\}. \tag{2.6}$$

Moreover, let the subfiltration $\mathbb{F} \subset \mathbb{G}$ describe the information available relative to (2.6)*, i.e., $\mathbb{F}$ contains some kind of information about V and K. Then the pair $(\tau, \mathbb{F})$ is called* (first-passage) default model *with* model filtration $\mathbb{F}$.

Varying the information in $\mathbb{F}$ results in different default models. As already mentioned, Giesecke (2006) discusses the "extreme" cases of incomplete information. This means that the following scenarios are considered:

Complete information: The asset value V and the default barrier K are publicly available. This means that $\mathbb{F}$ is generated by $\sigma(V_s : s \leq t) \vee \sigma(K)$. In this case, τ is an $\mathbb{F}$-stopping time. Consequently, $\mathbb{G} = \mathbb{F}$ is a possible choice.

Incomplete information 1: The default barrier K is publicly available, but the asset value V is not known. Hence, $\mathbb{F}$ is generated by $\sigma(K)$.

Incomplete information 2: Neither the asset value nor the default barrier is publicly available. This means that $\mathbb{F}$ is generated by the trivial σ-algebra.

Incomplete information 3: The asset value V is publicly available, but the default barrier is unknown. In this scenario $\mathbb{F}$ is generated by $\sigma(V_s : s \leq t)$.

2.2. Bottom up versus top down

To classify our model correctly in Chapter 3, the understanding of the differences between top down and bottom up approaches in portfolio credit risk modeling is essential. Good references for this topic are Giesecke (2008) and Bielecki et al. (2010).

From now on, we focus on a portfolio consisting of n names. As already pointed out in the introduction, we can distinguish between two types of default times.

Notation 2.2.1. The *default time of name k* is a random time denoted by τ_k for $k \in \{1, \ldots, n\}$. Moreover, we denote the *time of the ith default* in the portfolio by T_i for $i \in \{1, \ldots, n\}$. The *ordered default times* $T_1, \ldots, T_n$ satisfy $T_1 \leq \ldots \leq T_n$.

With these random times we define the *default indicator processes* $N^{(k)}$ and N^i by

$$N_t^{(k)} := I_{\{\tau_k \leq t\}}, \quad N_t^i := I_{\{T_i \leq t\}} \quad \text{for } k, i \in \{1, \ldots, n\}$$

and the *default counting process* N by

$$N_t := \sum_{k=1}^{n} I_{\{\tau_k \leq t\}} = \sum_{i=1}^{n} I_{\{T_i \leq t\}}.$$

The first process $N^{(k)}$ indicates the default of firm k, the second process N^i indicates the ith default in the portfolio and N counts the absolute number of defaulted names.

The intuitive idea from Brigo et al. (2010) to distinguish between bottom up and top down approaches is the following: In bottom up models we consider every single name in the portfolio, and by choosing a specific dependence structure between the default times of each firm, the relationship between the different firms is defined. In contrast to this, in top down models the focus changes. We are not interested in single-name modeling any more, but we solely study the overall process that counts the defaults in the underlying portfolio and, if necessary, the losses occurring at a default event.

For a more technical distinction, we have to take a look at the investor filtration $\mathbb{G}$ and the different types of default times introduced above. The difference between bottom up and top down approaches is based on the content of the investor filtration $\mathbb{G}$ that models the information which is available to an investor. In both approaches the investor filtration contains enough information to identify a default event. But in bottom up models the investor is also informed about the identity of the defaulted name. In conclusion, the investor knows which firm has defaulted. Technically, this means that in bottom up approaches the investor filtration $\mathbb{G}$ is finer than in the top down framework. As a consequence, in bottom up models the default times τ_k, $k \in \{1, \ldots, n\}$, are stopping times with respect to $\mathbb{G}$. In contrast to this, in top down models the investor does not have information about the identity of a defaulted name. This implies that T_i, $i \in \{1, \dots, n\}$, are $\mathbb{G}$-stopping times and τ_k, $k \in \{1, \dots, n\}$, are only random times, i.e., random variables with values in $[0, \infty]$. We can summarize this aspect by stating that the difference between bottom up and top down models is information based.

There exist reduced form models in both approaches. Examples of top down reduced form approaches are Schönbucher (2005), Bennani (2005, 2006), Giesecke and Tomecek (2005), Lopatin and Misirpashaev (2008), Sidenius et al. (2008), Arnsdorf and Halperin (2009), Laurent et al. (2011), Giesecke et al. (2011) and Cont and Minca (2013) to name a few. Typically, these models admit a portfolio intensity which is sensitive to default events. This means that there is a possible change in the structure of the portfolio intensity after a default. This can be justified by two facts: On the one hand, there are less firms in the underlying portfolio due to the default. In other words, the portfolio contains less potential defaulters. On the other hand, a modified intensity could establish real feedback of default events; see Giesecke (2008). Consequently, the model incorporates possible default contagion.

Bottom up reduced form models have been studied in Duffie and Gârleanu (2001), Jarrow and Yu (2001), Frey and Backhaus (2008, 2010) and Eckner (2009) among many others. In case of these models, there exists the possibility to incorporate feedback of default events, too. We refer the reader to Giesecke (2008) and Bielecki et al. (2010) for a more detailed classification of reduced form models.

Finally, we can also find bottom up structural approaches in the literature; see, for instance, Giesecke and Goldberg (2004b), Overbeck and Schmidt (2005), Luciano and Schoutens (2006) or Hurd (2009).

The approach of Giesecke and Goldberg (2004b) works with the incomplete information model from Giesecke (2006), which also provides an important basis for our model in Chapter 3. Moreover, this bottom up structural model incorporates possible feedback of default events to prices of credit sensitive contingent claims, i.e., it covers financial contagion. The models proposed in Overbeck and Schmidt (2005), Luciano and Schoutens (2006) and Hurd (2009) share the idea to use time change techniques with our approach in the following chapter. Overbeck and Schmidt (2005) consider a threshold model based on time changed Brownian motions and obtain an analytic solution for the probability of joint default in case of two entities. In this approach the time change differs for each firm. Luciano and Schoutens (2006) choose another model based on the idea of Black and Cox (1976) where the same gamma time change is used for all names in the portfolio. Finally, Hurd (2009) models the so called log-leverage ratio process as a time changed Brownian motion where each time change is the weighted sum of a firm specific time change and a common time change which is shared by all names.

The aim of the following chapter is to find a top down first-passage model for portfolio credit risk which incorporates possible feedback of default events on future defaults. We will see that this is implemented by using a specific stochastic time change which was originally introduced by Giesecke and Tomecek (2005).

3. A top down first-passage default model

Chapter 3 is the main contribution of the first part of this thesis. Here, we construct and study the new top down first-passage default model. After motivating and discussing the idea of a time changed portfolio value process in Section 3.1, we introduce our model in Section 3.2. This model relies on some ideas in Giesecke and Tomecek (2005) and Giesecke (2006). We specify the different default events, the incomplete information model and the time change in Subsections 3.2.1-3.2.4. In Section 3.3 we derive conditional distributions of the corresponding arrival times. These are an important building block for the following section, where we consider default trends conditional on prior defaults. The specific form of these trends allows us to determine tractable solutions for prices of default sensitive securities. In detail, we derive prices of contingent claims that pay a specific amount at time T if the ith default in the underlying pool of names did not occur up to this point in time, i.e., if $T < T_i$; otherwise, the payout is equal to zero. At the end of Section 3.4 we apply our results to the default trends and determine the compensator process of the default counting process of the underlying portfolio. After discussing an additional incomplete information model in detail in Section 3.5, we study in Section 3.6 more specific examples of the time change. Finally, based on the results of Sections 3.4 and 3.5, we introduce an algorithm to simulate our default times which is similar to the algorithm in the bottom up first-passage structural model in Giesecke and Goldberg (2004b).

3.1. Motivation

Our aim is the construction of a top down first-passage default model. In this section we consider the naive way to derive such a model and finally discuss why this approach is not appropriate for our purposes.

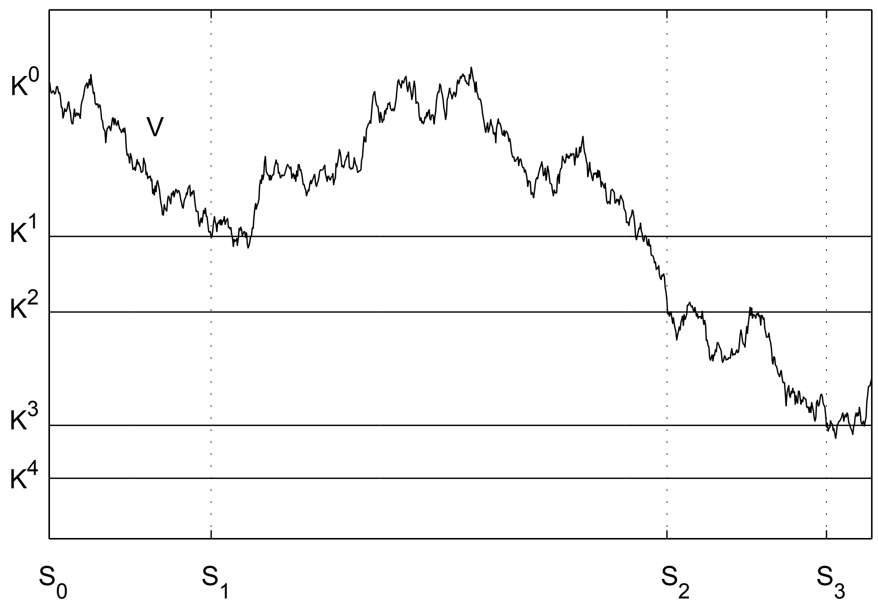
Throughout this section fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ that supports a standard Brownian motion W. We focus on a portfolio consisting of n defaultable firms. More precisely, we consider securities issued by these firms. Suppose that the *portfolio value process* V is given by the geometric Brownian motion

$$dV_t = V_t \mu_V dt + V_t \sigma_V dW_t$$

with constants $\mu_V \in \mathbb{R}$ and $\sigma_V > 0$. Note that $V_t = V_0 \exp\{(\mu_V - \sigma_V^2/2)t + \sigma_V W_t\}$. Without loss of generality, we assume that $V_0 = v_0 > 0$ is constant.

We define default times as first hitting times; see Black and Cox (1976). This means that the *time of the ith default* in our portfolio is the first time the portfolio

Figure 3.1.1.: Portfolio value process and default barriers (without time change)



value process V hits the deterministic barrier K^i. Let $V_0 = K^0 > K^1 > \ldots > K^n > 0$ and set $\kappa^i := \log(K^{i-1}/K^i)$ for $i \in \{1, \ldots, n\}$. Eventually, we define the ith default time by

$$S_i := \inf\{t \geq 0 | V_t \leq K^i\} \quad \text{for any } i \in \{1, \ldots, n\}$$

and set $S_0 := 0$. Figure 3.1.1 visualizes the first idea of the model. If we set $\mu := \mu_V - \sigma_V^2/2$ and $\sigma := \sigma_V$, then S_i, $i \in \{1, \ldots, n\}$, satisfy

$$\begin{aligned} S_i &= \min\{t \geq 0 | \log(V_0/K^i) + \mu t + \sigma W_t = 0\} \\ &= \min\{t \geq 0 | (\mu/\sigma)t + W_t = -\log(V_0/K^i)/\sigma\}. \end{aligned}$$

Note that for each $i \in \{1, \ldots, n\}$, the random time S_i is a stopping time with respect to the standard Brownian filtration $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \geq 0}$.

Remark 3.1.1. In general, S_i satisfies $\mathbb{P}[S_i = \infty] > 0$ for each $i \in \{1, \ldots, n\}$: It is well known (see, for instance, Section 3.2.3 in Jeanblanc et al. (2009)) that

$$\mathbb{P}[S_i < \infty] = \exp(-2\mu \log(V_0/K^i)/\sigma^2) \quad \text{if } -\mu \log(V_0/K^i)/\sigma^2 < 0.$$

Since $\log(V_0/K^i)/\sigma^2 > 0$, this means $\mathbb{P}[S_i = \infty] = 1 - \mathbb{P}[S_i < \infty] > 0$ if $\mu > 0$. On the other hand, we have $\mathbb{P}[S_i = \infty] = 1 - \mathbb{P}[S_i < \infty] = 0$ if $\mu \leq 0$.

We can now easily compute the distribution of the inter-arrival times $S_i - S_{i-1}$ for each $i \in \{1, \ldots, n\}$. But note that the difference $S_i - S_{i-1}$ is only well defined on the event $\{S_{i-1} < \infty\}$. Therefore, we introduce a new probability measure $\mathbb{P}^{i-1}$ on $(\Omega, \mathcal{A})$ by

$$\mathbb{P}^{i-1}[A] := \mathbb{P}[A|S_{i-1} < \infty] \quad \text{for } A \in \mathcal{A}.$$

Let us consider the corresponding probability space $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$ with $\Omega^{i-1} := \{S_{i-1} < \infty\}$ and $\mathcal{A}^{i-1} := \mathcal{A} \cap \Omega^{i-1} := \{A \cap \Omega^{i-1} | A \in \mathcal{A}\}$.

Remark 3.1.2. Fix $i \in \{1, \dots, n\}$.

1. For each $A \in \mathcal{A}$, we have

$$\mathbb{P}^{i-1}[A] = \mathbb{P}[A|\Omega^{i-1}] = \mathbb{P}[A \cap \Omega^{i-1}|\Omega^{i-1}] = \mathbb{P}^{i-1}[A \cap \Omega^{i-1}].$$

   Hence, if we want to compute the probability of $A \in \mathcal{A}$ under $\mathbb{P}^{i-1}$, it suffices to compute the probability of the corresponding element $A \cap \Omega^{i-1} \in \mathcal{A}^{i-1}$.
2. Let $\mathcal{E} \subset \mathcal{A}$ be a sub-σ-algebra with $\Omega^{i-1} \in \mathcal{E}$ and consider $X : \Omega^{i-1} \to \mathbb{R}$. Then X is $\mathcal{E} \cap \Omega^{i-1}$-measurable if and only if $X I_{\Omega^{i-1}} : \Omega \to \mathbb{R}$ is $\mathcal{E}$-measurable.

The following lemma will be very helpful in the remaining part of this chapter.

Lemma 3.1.3. *Fix $i \in \{1, \dots, n\}$.*

1. *We have $\mathbb{E}^{i-1}[I_A I_{\Omega^{i-1}}] = \mathbb{E}^{i-1}[I_A]$ for all $A \in \mathcal{A}$.*
2. *If $\mathcal{E} \subset \mathcal{A}$ is a sub-σ-algebra with $\Omega^{i-1} \in \mathcal{E}$, then for all $A \in \mathcal{A}$, it follows that*

$$\mathbb{E}^{i-1}[I_A|\mathcal{E}] = \mathbb{E}^{i-1}[I_A|\mathcal{E} \cap \Omega^{i-1}] I_{\Omega^{i-1}} \quad \mathbb{P}^{i-1} - a.s. \tag{3.1}$$

3. *Let $\mathcal{E} \subset \mathcal{A}$ be a sub-σ-algebra and $X, Y \geq 0$ random variables on the probability space $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. Moreover, assume that Y is independent of $\mathcal{E} \cap \Omega^{i-1}$ and let X be $\mathcal{E} \cap \Omega^{i-1}$-measurable. If $F(t) := \mathbb{P}^{i-1}[Y \leq t]$ for all $t \geq 0$, then*

$$\mathbb{P}^{i-1}[Y \leq X|\mathcal{E} \cap \Omega^{i-1}] = F(X) \quad \mathbb{P}^{i-1} - a.s.$$

Proof. The first assertion follows directly from Remark 3.1.2. Moreover, for each $E \in \mathcal{E}$ and $A \in \mathcal{A}$, we have

$$\begin{aligned} &\mathbb{E}^{i-1}[\mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E}]I_E] = \mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}I_E] = \mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}I_{E\cap\Omega^{i-1}}] \\ &= \mathbb{E}^{i-1}[\mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E} \cap \Omega^{i-1}]I_{E\cap\Omega^{i-1}}] = \mathbb{E}^{i-1}[\mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E} \cap \Omega^{i-1}]I_{\Omega^{i-1}}I_E]. \end{aligned}$$

Thus, Remark 3.1.2 yields $\mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E}] = \mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E} \cap \Omega^{i-1}]I_{\Omega^{i-1}}$ for each $A \in \mathcal{A}$. Equation (3.1) follows from

$$\begin{aligned} &\mathbb{E}^{i-1}[I_A|\mathcal{E}] = \mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E}] + \mathbb{E}^{i-1}[I_{A\cap\{S_{i-1}=\infty\}}|\mathcal{E}] \\ &= \mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E}] = \mathbb{E}^{i-1}[I_{A\cap\Omega^{i-1}}|\mathcal{E} \cap \Omega^{i-1}]I_{\Omega^{i-1}} = \mathbb{E}^{i-1}[I_A|\mathcal{E} \cap \Omega^{i-1}]I_{\Omega^{i-1}}. \end{aligned}$$

It remains to show the third assertion. To this end, we have to verify

$$\mathbb{E}^{i-1}[\mathbb{E}^{i-1}[I_{\{Y\leq X\}}|\mathcal{E} \cap \Omega^{i-1}]I_A] = \mathbb{E}^{i-1}[F(X)I_A]$$

for all $A \in \mathcal{E} \cap \Omega^{i-1}$. Let $\sigma_{\Omega^{i-1}}(X)$ be the smallest sub-σ-algebra of $\mathcal{A}^{i-1}$ such that X is measurable and define $\sigma_{\Omega^{i-1}}(A) := \{\Omega^{i-1}, \emptyset, A, \Omega^{i-1} \backslash A\}$. Then $\sigma_{\Omega^{i-1}}(X) \vee \sigma_{\Omega^{i-1}}(A) \subset \mathcal{E} \cap \Omega^{i-1}$ for all $A \in \mathcal{E} \cap \Omega^{i-1} \subset \mathcal{A} \cap \Omega^{i-1} = \mathcal{A}^{i-1}$. The $\mathbb{P}^{i-1}$-independence of Y and $\mathcal{E} \cap \Omega^{i-1}$ implies that Y is $\mathbb{P}^{i-1}$-independent of $\sigma_{\Omega^{i-1}}(X) \vee \sigma_{\Omega^{i-1}}(A)$. If we apply Lemma A.1.2 for the measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := I_{\{y \leq x\}}$, then

$$\int_{\omega_1 \in A} \int_{\omega_2 \in \Omega^{i-1}} I_{\{Y(\omega_2) \leq X(\omega_1)\}} d\mathbb{P}^{i-1}(\omega_1) d\mathbb{P}^{i-1}(\omega_2) = \int_A I_{\{Y \leq X\}} d\mathbb{P}^{i-1}. \tag{3.2}$$

The right hand side of the previous equation is equal to

$$\int_A I_{\{Y \leq X\}} d\mathbb{P}^{i-1} = \mathbb{E}^{i-1}[I_{\{Y \leq X\}} I_A] = \mathbb{E}^{i-1}[\mathbb{E}^{i-1}[I_{\{Y \leq X\}} | \mathcal{E} \cap \Omega^{i-1}] I_A].$$

Since $F : \mathbb{R}_+ \to [0,1]$ is monotone, F is measurable. This means $F(X)(\omega_1) = F(X(\omega_1))$ for $\mathbb{P}^{i-1}$-a.e. $\omega_1 \in \Omega^{i-1}$. Hence, the left hand side of Equation (3.2) is equal to

$$\int_{\omega_1 \in A} \left(\int_{\omega_2 \in \Omega^{i-1}} I_{\{Y(\omega_2) \leq X(\omega_1)\}} d\mathbb{P}^{i-1}(\omega_2) \right) d\mathbb{P}^{i-1}(\omega_1) = \int_{\omega_1 \in A} F(X(\omega_1)) d\mathbb{P}^{i-1}(\omega_1)$$
$$= \int_{\omega_1 \in \Omega^{i-1}} F(X)(\omega_1) I_A(\omega_1) d\mathbb{P}^{i-1}(\omega_1) = \mathbb{E}^{i-1}[F(X) I_A],$$

which yields the third assertion. □

Remark 3.1.4. Note that the third part of the previous lemma is also true for the probability space $(\Omega, \mathcal{A}, \mathbb{P})$: Let $\mathcal{E} \subset \mathcal{A}$ be a sub-σ-algebra and $X, Y \geq 0$ random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Moreover, assume that Y is independent of $\mathcal{E}$ and let X be $\mathcal{E}$-measurable. If $F(t) := \mathbb{P}[Y \leq t]$ for all $t \geq 0$, then

$$\mathbb{P}[Y \leq X | \mathcal{E}] = F(X) \quad \mathbb{P} - \text{a.s.}$$

Proposition 3.1.5. *For each $i \in \{1, \dots, n\}$, the $\mathbb{P}^{i-1}$-distribution of $S_i - S_{i-1}$ is given by*

$$\mathbb{P}^{i-1}\left[S_i - S_{i-1} \leq t\right] = \mathbb{P}\left[\min_{s \leq t}(\kappa^i + \sigma W_s + \mu s) \leq 0\right] \quad \textit{for } t \geq 0. \tag{3.3}$$

Proof. Fix $i \in \{1, \dots, n\}$. We can easily see that

$$(S_i - S_{i-1}) I_{\{S_{i-1} < \infty\}} = \inf\{s \geq 0 | V_{S_{i-1}+s} = K^i\} I_{\{S_{i-1} < \infty\}}.$$

Since V satisfies $V_t = V_0 \exp(\mu t + \sigma W_t)$ for all $t \geq 0$, we obtain the following equations on $\{S_{i-1} < \infty\}$:

$$\begin{aligned} V_{S_{i-1}+s} &= V_0 \exp(\sigma W_{S_{i-1}+s} + \mu(S_{i-1} + s)) \\ &= V_0 \exp(\sigma W_{S_{i-1}+s} - \sigma W_{S_{i-1}} + \sigma W_{S_{i-1}} + \mu(S_{i-1} + s)) \\ &= V_0 \exp(\mu S_{i-1} + \sigma W_{S_{i-1}}) \exp(\sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s). \end{aligned}$$

$K^{i-1} = V_0 \exp(\mu S_{i-1} + \sigma W_{S_{i-1}})$ on $\{S_{i-1} < \infty\}$ yields

$$\begin{aligned}
&(S_i - S_{i-1}) I_{\{S_{i-1}<\infty\}} \\
&= \inf\{s \geq 0 | K^{i-1} \exp(\sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s) = K^i\} I_{\{S_{i-1}<\infty\}} \\
&= \inf\{s \geq 0 | \log(K^{i-1}/K^i) + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} I_{\{S_{i-1}<\infty\}}.
\end{aligned}$$

Hence, we obtain for all $t \geq 0$

$$\begin{aligned}
&\mathbb{P}^{i-1}\left[S_i - S_{i-1} \leq t\right] \\
&= \frac{\mathbb{P}[\{\inf\{s \geq 0 | \kappa^i + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} \leq t\} \cap \{S_{i-1} < \infty\}]}{\mathbb{P}[S_{i-1} < \infty]} \\
&= \mathbb{P}^{i-1}[\{\inf\{s \geq 0 | \kappa^i + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} \leq t\}] \\
&= \mathbb{P}[\inf\{s \geq 0 | \kappa^i + \sigma W_s + \mu s = 0\} \leq t].
\end{aligned}$$

To the last equality: S_{i-1} is an $\mathbb{F}^W$-stopping time and $(W_t, \mathcal{F}_t^W)_{t\geq 0}$ is a Brownian motion. According to the strong Markov property (see Theorem A.1.1), the process $(W_{S_{i-1}+s} - W_{S_{i-1}})_{s\geq 0}$ is a Brownian motion on the probability space $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$ which is $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^W_{S_{i-1}} \cap \Omega^{i-1}$. Since $\Omega^{i-1} = \{S_{i-1} < \infty\} \in \mathcal{F}^W_{S_{i-1}} \cap \Omega^{i-1}$, the last equality holds.

Finally, Equation (3.3) follows from

$$\mathbb{P}[\inf\{s \geq 0 | \kappa^i + \sigma W_s + \mu s = 0\} \leq t] = \mathbb{P}\left[\min_{s\leq t}(\kappa^i + \sigma W_s + \mu s) \leq 0\right].$$

□

Notation 3.1.6. We denote

$$F^{\Delta S}(t,x) := \mathbb{P}\left[\min_{s\leq t}(x + \sigma W_s + \mu s) \leq 0\right] \quad \text{for } x > 0 \text{ and } t \geq 0.$$

The function $F^{\Delta S}(t,x)$ is well known for $x > 0$ and $t \geq 0$; see, for instance, Chapter 3 in Jeanblanc et al. (2009) or Section 2.8 in Karatzas and Shreve (1988) (see also Appendix A.1). It is given by

$$F^{\Delta S}(t,x) = \begin{cases} \Phi\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right) + e^{-2\mu x/\sigma^2}\Phi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right) & \text{for } x > 0 \text{ and } t > 0 \\ 0 & \text{for } x > 0 \text{ and } t = 0 \end{cases}.$$

Since κ^i is deterministic, we obtain the following corollary.

Corollary 3.1.7. *The distribution in* (3.3) *is given by*

$$\mathbb{P}^{i-1}[S_i - S_{i-1} \leq t] = F^{\Delta S}(t, \kappa^i) \quad \textit{for } t \geq 0.$$

Moreover, the $\mathcal{F}^W_{S_{i-1}}$*-conditional* $\mathbb{P}^{i-1}$*-distribution of* S_i *satisfies*

$$\mathbb{P}^{i-1}[S_i \leq t | \mathcal{F}^W_{S_{i-1}}] = F^{\Delta S}(t - S_{i-1}, \kappa^i) I_{\{S_{i-1}<t\}} \quad \mathbb{P}^{i-1} - a.s.$$

for each $t \geq 0$ *.*

Proof. Fix $i \in \{1, \ldots, n\}$ and $t \geq 0$. The first equation is obvious. For the second equation note that S_{i-1} is an $\mathbb{F}^W$-stopping time, which yields

$$\begin{aligned}\mathbb{P}^{i-1}[S_i \leq t|\mathcal{F}^W_{S_{i-1}}] &= \mathbb{P}^{i-1}[S_i \leq t|\mathcal{F}^W_{S_{i-1}}]I_{\{S_{i-1}<t\}} + \mathbb{P}^{i-1}[S_i \leq t|\mathcal{F}^W_{S_{i-1}}]I_{\{S_{i-1}\geq t\}} \\ &= \mathbb{P}^{i-1}[S_i \leq t|\mathcal{F}^W_{S_{i-1}}]I_{\{S_{i-1}<t\}} \\ &= \mathbb{P}^{i-1}[(S_i - S_{i-1})I_{\{S_{i-1}<\infty\}} \leq (t - S_{i-1})^+|\mathcal{F}^W_{S_{i-1}}]I_{\{S_{i-1}<t\}} \\ &= \mathbb{P}^{i-1}[S_i - S_{i-1} \leq (t - S_{i-1})^+|\mathcal{F}^W_{S_{i-1}} \cap \Omega^{i-1}]I_{\{S_{i-1}<t\}}. \qquad (3.4)\end{aligned}$$

The last equality in (3.4) follows from the second statement of Lemma 3.1.3. Moreover, we know from the proof of Proposition 3.1.5 that

$$\begin{aligned}&(S_i - S_{i-1})I_{\{S_{i-1}<\infty\}} \\ &= \inf\{s \geq 0| \log(K^{i-1}/K^i) + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\}I_{\{S_{i-1}<\infty\}}\end{aligned}$$

and that $(W_{S_{i-1}+s} - W_{S_{i-1}})_{s\geq 0}$ is a Brownian motion on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$ which is $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^W_{S_{i-1}} \cap \Omega^{i-1}$. This implies that $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^W_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. Since $(t - S_{i-1})^+$ is $\mathcal{F}^W_{S_{i-1}}$-measurable, it follows directly that $(t - S_{i-1})^+$ considered as a mapping on Ω^{i-1} is also $\mathcal{F}^W_{S_{i-1}} \cap \Omega^{i-1}$-measurable. The third statement of Lemma 3.1.3 implies

$$\mathbb{P}^{i-1}[S_i - S_{i-1} \leq (t - S_{i-1})^+|\mathcal{F}^W_{S_{i-1}} \cap \Omega^{i-1}] = F^{\Delta S}((t - S_{i-1})^+, \kappa^i).$$

Together with Equation (3.4), we obtain

$$\mathbb{P}^{i-1}[S_i \leq t|\mathcal{F}^W_{S_{i-1}}] = F^{\Delta S}((t - S_{i-1})^+, \kappa^i)I_{\{S_{i-1}<t\}} = F^{\Delta S}(t - S_{i-1}, \kappa^i)I_{\{S_{i-1}<t\}}.$$

□

The previous corollary shows that the $\mathbb{P}^{i-1}$-distribution of the inter-arrival times $S_i - S_{i-1}$ only depends on $\kappa^i = \log(K^{i-1}/K^i)$ and the parameters μ and σ of V. Similarly, the $\mathcal{F}^W_{S_{i-1}}$-conditional $\mathbb{P}^{i-1}$-distribution of S_i depends on κ^i, μ, σ and S_{i-1}. This means, apart from the dependence on the time which has passed since the last default, prior default events do not influence the conditional default probability of the remaining names in the portfolio. To be more specific, if, for instance, $n = 10$ and $\kappa^1 = \kappa^2$ and if we consider the conditional distribution functions of S_1 and S_2, then only the time which has passed since the last default is relevant for the conditional distribution (note that $S_{i-1} = S_0 = 0$ if $i = 1$; hence, there is no prior default), but it makes no difference whether there has been no default or one default in the portfolio so far. This is obviously not a desirable property. Our intention is to construct a top down first-passage default model that incorporates default contagion, which means that a default of one name influences future defaults in the underlying portfolio directly. In particular, the model should allow for a switch of regimes after each default. To generate such contagion effects, the model above is not suitable. It is obvious that we have to include more stochastic features to obtain the desired property.

A possible solution to this problem provides the approach of Giesecke and Tomecek (2005) which considers a specific time change model. In the following sections we adapt their reduced form approach to our first-passage model. Therefore, we suppose that the portfolio value process is a time changed geometric Brownian motion and analyze the corresponding first hitting times. As a result of this definition, our top down first-passage model is flexible enough to incorporate default contagion.

3.2. Model definition

As already pointed out at the end of the previous section, the most important object of our model is a time change which depends on prior defaults. The next definition clarifies this term in our setting.

Definition 3.2.1. *Fix a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*. A right-continuous, strictly increasing,* $[0, \infty]$*-valued process* G *with* $G_0 = 0$ *is called* time change. *The time change is called* continuous *if* G *has* $\mathbb{P}$*-a.s. continuous paths and* absolutely continuous *if* G *has* $\mathbb{P}$*-a.s. absolutely continuous paths, i.e., there exists an appropriate process* g *such that for all* $t \geq 0$*, we have*

$$G_t = \int_0^t g_s ds \quad \mathbb{P} - a.s.$$

Remark 3.2.2. Note that if a continuous time change G satisfies $G_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t \to \infty} G_t = \infty$ $\mathbb{P}$-a.s., then the *inverse process* G^{-1} defined by

$$G_t^{-1}(\omega) := \inf\{s \geq 0 | G_s(\omega) > t\} \quad \text{for } t \geq 0 \text{ and } \omega \in \Omega$$

is again a continuous time change that satisfies $G_t^{-1} < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t \to \infty} G_t^{-1} = \infty$ $\mathbb{P}$-a.s. Moreover, G^{-1} satisfies $G_t^{-1} = \min\{s \geq 0 | G_s = t\}$ and $G_{G_t^{-1}} = t$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Since we want to construct a top down first-passage model where the default of one firm can influence the default probability of the remaining firms in our portfolio, we have to choose a specific time change G for our model. In the following subsection we introduce our first-passage model with a general (absolutely continuous) time change G. A more detailed specification of G follows in Subsections 3.2.2 and 3.2.4.

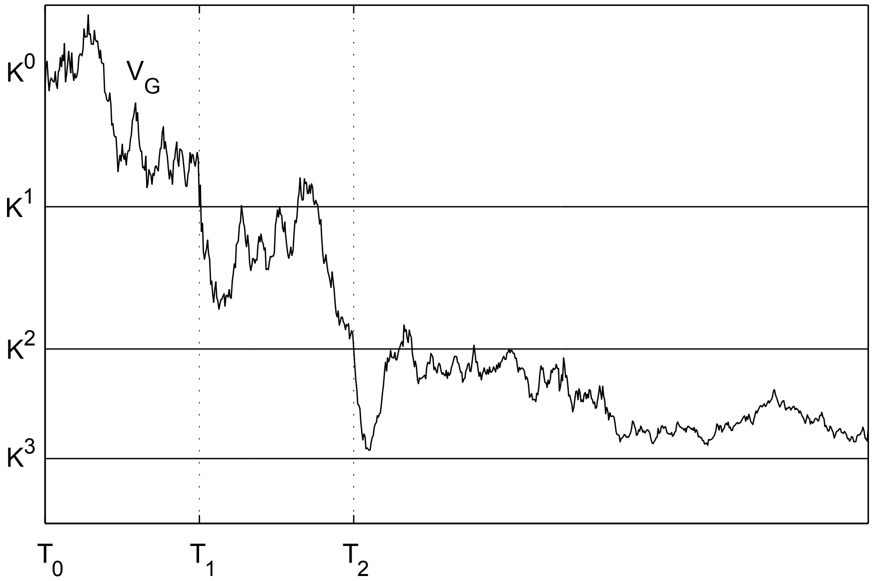
3.2.1. Default events

The construction of the following top down structural model is strongly connected to the top down reduced form model of Giesecke and Tomecek (2005), which is one of the main building blocks of our model.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space that supports a standard Brownian motion W and an absolutely continuous time change G that satisfies $G_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t \to \infty} G_t = \infty$ $\mathbb{P}$-a.s. Moreover, consider the underlying geometric Brownian motion V given by

$$dV_t = V_t \mu_V dt + V_t \sigma_V dW_t$$

Figure 3.2.1.: Portfolio value process and default barriers (with time change)



with constants $\mu_V \in \mathbb{R}$ and $\sigma_V > 0$. We assume that $V_0 = v_0 > 0$ is constant. Let $\kappa^1, \ldots, \kappa^n$ be independent random variables with values in $(0, \infty)$ that are independent of $\sigma(W_t : t \geq 0)$.

Again, our starting point is a portfolio consisting of n defaultable firms. We assume that the *portfolio value process* of this portfolio is given by the time changed process V_G. Moreover, the *ith default barrier,* $i \in \{1, \ldots, n\}$, is defined by $K^i := K^{i-1} \exp(-\kappa^i)$ with $K^0 := V_0$. Note that in this case, $\kappa^i = \log(K^{i-1}/K^i)$ and $K^0 > K^1 > \ldots > K^n > 0$. A default occurs if the portfolio value process hits one of the barriers K^i, $i \in \{1, \ldots, n\}$. More precisely, the ith default occurs if $V_{G_t} \leq K^i$ for the first time such that

$$T_i := \inf\{t \geq 0 | V_{G_t} \leq K^i\} \qquad (T_0 := 0)$$

describes the *time of the ith default* in the underlying portfolio, see Figure 3.2.1 for an illustration. In addition, we define the first hitting time of barrier K^i by V by

$$S_i := \inf\{t \geq 0 | V_t \leq K^i\} \qquad (S_0 := 0).$$

Remark 3.2.3. Fix $i \in \{1, \ldots, n\}$.

1. S_i and T_i are random times, i.e., $\mathcal{A}$-measurable random variables with values in $[0, \infty]$, which can be represented in the following way:

$$\begin{aligned} S_i &= \inf\{t \geq 0 | \log(V_t/K^i) \leq 0\} \\ &= \inf\{t \geq 0 | \log(V_0/K^i) + (\mu_V - \sigma_V^2/2)t + \sigma_V W_t \leq 0\} \end{aligned}$$

and

$$\begin{aligned}T_i &= \inf\{t \geq 0 | \log(V_{G_t}/K^i) \leq 0\}\\ &= \inf\{t \geq 0 | \log(V_0/K^i) + (\mu_V - \sigma_V^2/2)G_t + \sigma_V W_{G_t} \leq 0\}.\end{aligned}$$

If we define the process X^i by $X_t^i := \log(V_0/K^i)+\mu t+\sigma W_t$ with $\mu := \mu_V - \sigma_V^2/2$ and $\sigma := \sigma_V$, then $S_i = \inf\{s \geq 0 | X_s^i \leq 0\}$ and $T_i = \inf\{s \geq 0 | X_{G_s}^i \leq 0\}$.

2. Following the same steps described at the beginning of Section 3.1, we obtain that in general $\mathbb{P}[S_i = \infty] > 0$.
3. If we define $G_\infty := \infty$, then the properties of G imply $S_i = G_{T_i}$ and $T_i = G_{S_i}^{-1}$ $\mathbb{P}$-a.s.
4. The previous item implies $\mathbb{P}[S_i = \infty] = \mathbb{P}[T_i = \infty]$. As a consequence of this, we obtain that in general $\mathbb{P}[T_i = \infty] > 0$.

So far, our default model has obviously a structural character. Nevertheless, in the following subsections we focus on the default counting process

$$N_t := \sum_{i=1}^{n} I_{\{T_i \leq t\}}$$

and the adjunct process $N_t^0 := \sum_{i=1}^{n} I_{\{S_i \leq t\}}$. Note that by definition, the default counting process N is nonexplosive.

Remark 3.2.4. In Giesecke and Tomecek (2005) the authors define a reduced form model directly: Their starting point is a standard (marked) Poisson process with respect to a specific filtration $\mathbb{H}^{GT}$. This process is from now on denoted by $N^{GT,0}$, and the corresponding arrival times of this $\mathbb{H}^{GT}$-Poisson process are denoted by $(S_i^{GT})_{i \in \mathbb{N}_0}$. The default counting process N^{GT} in the approach of Giesecke and Tomecek (2005) is a time changed Poisson process. This means

$$N_t^{GT} = N_{G_t^{GT}}^{GT,0} \quad \text{for } t \geq 0$$

where G^{GT} denotes a specific time change. Moreover, the arrival times of N^{GT}, i.e., the times T_i^{GT} in which this process jumps, satisfy $S_i^{GT} = G_{T_i^{GT}}^{GT}$ $\mathbb{P}$-a.s. Note that in our model the hitting times S_i and T_i, $i \in \{1, \ldots, n\}$, depend on each other in a similar way (see 3. in Remark 3.2.3).

Nevertheless, in our first-passage model the arrival time T_i or, in other words, the point in time in which the process N jumps for the ith time, can be interpreted as the first time the portfolio value process hits the ith barrier. In the reduced form model of Giesecke and Tomecek (2005) there does not exist an explanation for the occurrence of a jump of the default counting process N^{GT}. Another important difference between the arrival times S_i^{GT}, $i \in \mathbb{N}$, and our first hitting times S_i, $i \in \{1, \ldots, n\}$, is that all S_i^{GT} are $\mathbb{P}$-a.s. finite as arrival times of a standard $\mathbb{H}^{GT}$-Poisson process (see Remark 1.1.4), while in our model we have in general $\mathbb{P}[S_i = \infty] > 0$ (see 2. in Remark 3.2.3).

Because of the specific construction of the time change G^{GT} in the model of Giesecke and Tomecek (2005), the resulting time changed process N^{GT} is self-affecting. Hence, the time change G in our model is constructed in a similar way to the time change G^{GT}.

3.2.2. Time change

In this subsection we discuss the time change G in more detail. The main idea is that the time change should have two important properties. On the one hand, G should depend on the first hitting times T_i, $i \in \{1, \dots, n\}$, which results in a self-affecting default counting process. On the other hand, G should be general enough to depend on other stochastic factors. As a consequence, the model is able to take into account additional dependencies on the random environment.

In order to implement this property of G, we assume that there exists a filtration $\mathbb{K} := (\mathcal{K}_t)_{t \geq 0} \subset \mathcal{A}$ satisfying the usual conditions. Additionally, $\mathcal{K}_\infty$ and $\sigma(W_t : t \geq 0)$ are assumed to be independent. A possible choice for $\mathbb{K}$ is the filtration generated by a Brownian motion B that is independent of W.

In the following, we introduce a general assumption that has to be satisfied by the underlying time change G. As a consequence, if we specify a time change G (see, for instance, Subsection 3.2.4), then we have to verify that G meets this requirement.

Assumption 3.2.5. *The time change G is adapted to the filtration generated by the σ-algebras*

$$\mathcal{K}_t \vee \sigma(I_{\{T_i \leq s\}} : s \leq t, i \leq n).$$

Note that the previous assumption allows for the time change G_t to depend on the arrivals that occurred before t and on the random environment described by $\mathbb{K}$. We will see later in more detail (see Section 3.6) that if the filtration $\mathbb{K}$ is trivial, then the time change G solely depends on the arrival times and t. For instance, a time change being deterministic between arrival times is an appropriate choice. On the other hand, if $\mathbb{K}$ is not trivial, then the time change might depend on additional stochastic variables.

3.2.3. Setting of the first incomplete information model (IIM1)

So far, we have developed a top down first-passage structural model where the portfolio value process is given by a time changed geometric Brownian motion. In this subsection we specify important features of the incomplete information model.

For each random time T_i, $i \in \{1, \dots, n\}$, we define a default model $(T_i, \mathbb{F}^{i-1})$ in the sense of Giesecke (2006) (see also Definition 2.1.10). In the following, we will specify the investor filtration $\mathbb{G}$ and the model filtrations $\mathbb{F}^{i-1}$, $i \in \{1, \dots, n\}$, such that $\mathbb{F}^{i-1} \subset \mathbb{G}$. For each $i \in \{1, \dots, n\}$, the model filtration $\mathbb{F}^{i-1}$ is the key to define our incomplete information model. If we apply Definition 2.1.10 to our context, then we have to specify the information which is available **with respect to**

$$T_i = \inf\{t \geq 0 | V_{G_t} \leq K^i\} \tag{3.5}$$

for each $i \in \{1, \dots, n\}$. Consequently, in order to define $\mathbb{F}^{i-1}$, we have to specify how much information is available with respect to V, G and K^i.

Note that the investor filtration $\mathbb{G}$ models the information which is available to an investor and not only the information which is available with respect to (3.5). Since

$\mathbb{F}^{i-1} \subset \mathbb{G}$ and since an investor indeed observes all defaults, the investor filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ has to satisfy

$$\mathcal{F}_t^{i-1} \vee \sigma(I_{\{T_k \leq s\}} : s \leq t, k \leq n) \subset \mathcal{G}_t \quad \text{for all } t \geq 0 \text{ and } i \in \{1, \ldots, n\}.$$

Specification of the first incomplete information model: In our model the following information should be available on $\{t \leq T_{i-1}\}$:

- $K^1, \ldots, K^n$
- $\mathbb{K}$ up to time t
- Number of defaults up to time t
- Time of all defaults that occurred up to time t
- G_s for $s \leq t$

On $\{t > T_{i-1}\}$, the following information should additionally be available:

- $\mathbb{K}$ up to time t

This means that an investor always knows the barriers which trigger a default in the underlying portfolio. As long as $t \leq T_{i-1}$, we also suppose complete information about the random environment represented by $\mathbb{K}$ and the defaults that occurred up to time t. As a consequence, the time change G is observable up to this point in time. In case of $t > T_{i-1}$, the flow of new information is much smaller: There is additional information available about the random environment described by $\mathbb{K}$ only.

Since an investor never receives any information about the underlying process V, the portfolio value process V_G is unobservable. Hence, the ith default is not observable with the information available with respect to (3.5). Note that by Assumption 3.2.5, the time change G_t might depend on all defaults that occurred up to time t. In this case, the fact that the ith default is not observable implies that the information which is available with respect to (3.5) contains only partial information about the time change after the $(i-1)$st default.

The following definitions guarantee the requirements from above: Let $\kappa^1, \ldots, \kappa^n > 0$ be deterministic, which leads to deterministic default barriers $K^1, \ldots, K^n$. Define the model filtrations $\mathbb{F}^{i-1} := (\mathcal{F}_t^{i-1})_{t \geq 0}$, $i \in \{1, \ldots, n\}$, by

$$\mathcal{F}_t^{i-1} := \bigcap_{u > t} \mathcal{K}_u \vee \sigma(I_{\{T_k \leq s\}} : s \leq u, k \leq i-1).$$

The investor filtration $\mathbb{G}$, which models the information available to an investor and not only the information available with respect to (3.5), is assumed to be given by

$$\mathcal{G}_t := \bigcap_{u > t} \mathcal{K}_u \vee \sigma(I_{\{T_k \leq s\}} : s \leq u, k \leq n).$$

Moreover, to specify the time change G in the next subsection, we need the filtrations $\mathbb{G}^{i-1} := (\mathcal{F}_{T_{i-1}+t}^{i-1})_{t \geq 0}$ for $i \in \{1, \ldots, n\}$.

Remark 3.2.6. 1. Note that the σ-algebra $\mathcal{K}_\infty$ is independent of $\sigma(W_s : s \geq 0)$ by definition. Because $K^0, K^1, \ldots, K^n$ and $\kappa^1, \ldots, \kappa^n$ are deterministic, the σ-algebra $\sigma(\kappa^1, \ldots, \kappa^n)$ is trivial. Furthermore, S_i is a stopping time with respect to the right-continuous filtration $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0} \subset \mathcal{A}$ generated by $\sigma(W_s : s \leq t) \vee \sigma(\kappa^1, \ldots, \kappa^n) \vee \mathcal{K}_\infty$, and W is still a Brownian motion with respect to this filtration.

2. The default counting process N is $\mathbb{G}$-adapted and T_{i-1} is a stopping time with respect to $\mathbb{F}^{i-1} \subset \mathbb{G}^{i-1}$ for $i \in \{1, \ldots, n\}$. Furthermore, T_i is not an $\mathbb{F}^{i-1}$-stopping time. Later, we will see that these default times are totally inaccessible in $\mathbb{G}$.

In order to analyze this incomplete information model, we have to consider the filtration $\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$ defined by

$$\mathcal{H}_t := \mathcal{K}_\infty \vee \sigma(I_{\{S_i \leq s\}} : s \leq t, i \leq n).$$

By definition, each S_i, $i \in \{1, \ldots, n\}$, is an $\mathbb{H}$-stopping time. Moreover, the filtration $(\sigma(I_{\{S_i \leq s\}} : s \leq t, i \leq n))_{t\geq 0}$ is right-continuous; hence, so is $\mathbb{H}$. Given $\mathcal{H}_t$, we know the number of jumps of N^0 up to time t and at which points in time these jumps have occurred, but we do not know the exact value of V.

3.2.4. Exact definition and properties of the time change

In the following definition we specify the time change G by using the filtrations $\mathbb{G}^{i-1}$, $i \in \{1, \ldots, n\}$, from the previous subsection. An important aim in this subsection is to prove that this time change satisfies Assumption 3.2.5. This specific form of the time change is not entirely new: Indeed, Giesecke and Tomecek (2005) introduced time changes of this type and pointed out that such time changes are especially tractable in view of simulating the arrival times $(T_i^{GT})_{i\in\mathbb{N}_0}$. Remark 3.2.8 discusses the approach in Giesecke and Tomecek (2005), and Section 3.6 presents the advantages of this choice for G.

Definition 3.2.7. *For each $i \in \{1, \ldots, n\}$, consider a $\mathbb{G}^{i-1}$-adapted, absolutely continuous time change G^i with $G^i_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t\to\infty} G^i_t = \infty$ $\mathbb{P}$-a.s. The* (overall) time change *G is defined by*

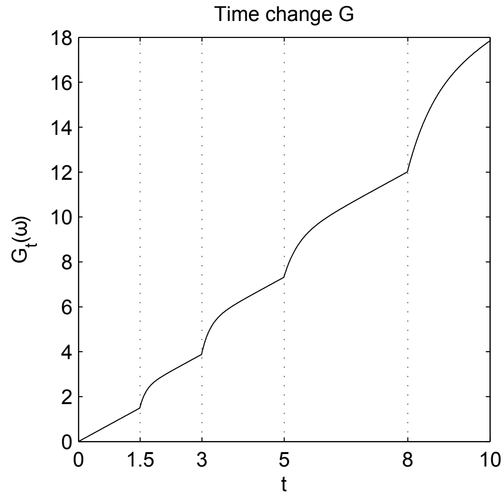
$$G_t := \begin{cases} \sum_{k=1}^{i-1} G^k_{T_k - T_{k-1}} + G^i_{t-T_{i-1}} & \text{on } \{T_{i-1} \leq t < T_i\} \text{ for } i \in \{1, \ldots, n\} \\ \sum_{k=1}^{n-1} G^k_{T_k - T_{k-1}} + G^n_{t-T_{n-1}} & \text{on } \{T_n \leq t\} \end{cases}.$$

Since each G^i is $\mathbb{G}^{i-1}$-adapted, G^i_t depends on the information in $\mathbb{K}$ up to time $T_{i-1} + t$ and on $T_0, \ldots, T_{i-1}$. By this definition, G is absolutely continuous, strictly increasing with $G_0 = 0$ and satisfies $G_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t\to\infty} G_t = \infty$ $\mathbb{P}$-a.s.

If $i > 1$, then for each $j \in \{1, \ldots, i-1\}$, T_j is given by $T_j = \inf\{t \geq 0 | V_{G_t} \leq K^j\} = \inf\{t \geq T_{j-1} | V_{G_t} = K^j\}$ on $\{T_{j-1} < \infty\}$. Definition 3.2.7 and $T_0 = 0$ imply

$$T_j = \inf\Big\{t \geq T_{j-1} \Big| V_{\sum_{k=1}^{j-1} G^k_{T_k - T_{k-1}} + G^j_{t-T_{j-1}}} = K^j\Big\} \quad \text{on } \{T_{j-1} < \infty\}.$$

Figure 3.2.2.: Example of time change G



Hence, for each $j \in \{1, \ldots, i-1\}$, T_j depends on V, K^k and G^k for $k \leq j$ only.

Moreover, G is constructed such that after each default, a new term is added to the time change. This property is illustrated in Figure 3.2.2, which shows a possible path of G. In this example it is assumed that $\mathbb{K}$ is trivial, and the arrival times are given by $T_0 = 0$, $T_1 = 1.5$, $T_2 = 3$, $T_3 = 5$ and $T_4 = 8$. We observe that after each default, the time change G changes its slope. In more detail, the time evolves faster after each default. The change in the slope of G after the $(i-1)$st arrival time is determined by the specific form of the ith time change G^i. We will discuss this and other examples of G in more detail in Section 3.6.

In the subsequent study we will see that this property of G, i.e., the possible change in the evolution of financial time after each default, leads to a self-affecting default counting process.

Remark 3.2.8. In Giesecke and Tomecek (2005) the authors also consider continuous time changes $(G^{GT,i})_{i\in\mathbb{N}}$ that satisfy $G_t^{GT,i} < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t\to\infty} G_t^{GT,i} = \infty$ $\mathbb{P}$-a.s. for all $i \in \mathbb{N}$. Here, the overall time change G^{GT} is defined by countably many $(S_i^{GT})_{i\in\mathbb{N}_0}$ which are arrival times of a standard (marked) $\mathbb{H}^{GT}$-Poisson process, and $(G^{GT,i})_{i\in\mathbb{N}}$ is such that

$$T_i^{GT} - T_{i-1}^{GT} = (G^{GT,i})^{-1}_{S_i^{GT} - S_{i-1}^{GT}} \quad \text{for } i \in \mathbb{N} \tag{3.6}$$

and $\lim_{i\to\infty} T_i^{GT} = \infty$ $\mathbb{P}$-a.s. More precisely, in Giesecke and Tomecek (2005) the time change G^{GT} is given by

$$G_t^{GT} = \sum_{k=1}^{i-1} G^{GT,k}_{T_k^{GT} - T_{k-1}^{GT}} + G^{GT,i}_{t - T_{i-1}^{GT}} \quad \text{on } \{T_{i-1}^{GT} \leq t < T_i^{GT}\} \text{ for } i \in \mathbb{N}.$$

Note that $(G^{GT,i})_t^{-1} < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $i \in \mathbb{N}$. Moreover, since S_i^{GT} are arrival times of a standard (marked) $\mathbb{H}^{GT}$-Poisson process, we know from Remark

1.1.4 that $\mathbb{P}[S_i^{GT} = \infty] = 0$ for all $i \in \mathbb{N}_0$. Therefore, Equation (3.6) implies $\mathbb{P}[T_i^{GT} = \infty] = 0$ for all $i \in \mathbb{N}$.

The next proposition shows that Equation (3.6) is also satisfied in our setting on the event $\{S_{i-1} < \infty\}$.

Proposition 3.2.9. *For each $i \in \{1, \dots, n\}$, the time change G^i satisfies $\mathbb{P}$-a.s.*

$$T_i - T_{i-1} = (G^i)^{-1}_{S_i - S_{i-1}} \quad \text{on } \{S_{i-1} < \infty\}$$

Proof. Fix $i \in \{1, \dots, n\}$. Since $S_i = G_{T_i}$, we have $\{S_i < \infty\} = \{T_i < \infty\}$. Moreover, together with $T_i = \inf\{t > T_{i-1} | V_{G_t} = K^i\}$ on $\{T_{i-1} < \infty\}$, we obtain

$$\begin{aligned} T_i - T_{i-1} &= \inf\Big\{t > T_{i-1} \Big| V_{\sum_{k=1}^{i-1} G^k_{T_k - T_{k-1}} + G^i_{t - T_{i-1}}} = K^i\Big\} - T_{i-1} \\ &= \inf\{s > 0 | V_{G_{T_{i-1}} + G^i_s} = K^i\} \end{aligned}$$

on $\{S_{i-1} < \infty\}$. Note that the equality $V_{G_{T_{i-1}}} = V_{S_{i-1}}$ holds on $\{S_{i-1} < \infty\}$. This implies that $T_i - T_{i-1} = (G^i)^{-1}_{\inf\{u>0 | V_{S_{i-1}+u} = K^i\}}$ on $\{S_{i-1} < \infty\}$. On the other hand, we can easily compute that $S_i - S_{i-1} = \inf\{u > 0 | V_{S_{i-1}+u} = K^i\}$ on $\{S_{i-1} < \infty\}$. Together, we see directly that $T_i - T_{i-1} = (G^i)^{-1}_{S_i - S_{i-1}}$ on $\{S_{i-1} < \infty\}$. □

Next, we will show that G defined in Definition 3.2.7 satisfies Assumption 3.2.5. We deduce this property from the following lemmas. Note that in the model setting of Giesecke and Tomecek (2005) the authors discuss analogous results in Remark A.4, Lemma A.5 and Lemma A.6. Nevertheless, as stated in Remark 3.2.8, T_i^{GT} satisfy $\mathbb{P}[T_i^{GT} = \infty] = 0$ for all $i \in \mathbb{N}$, and in our model we have in general $\mathbb{P}[T_i = \infty] > 0$ for $i \in \{1, \dots, n\}$ (see 4. in Remark 3.2.3). Therefore, we prove the relevant results in our model setting.

First, consider the random variables $R_t^1, \dots, R_t^n$ defined by

$$R_t^i := \begin{cases} \min\{T_i - T_{i-1}, (t - T_{i-1})^+\} I_{\{T_{i-1} < \infty\}} & \text{for } i \in \{1, \dots, n-1\} \\ (t - T_{n-1})^+ & \text{for } i = n \end{cases}$$

for $t \geq 0$.

Lemma 3.2.10. *For any $i \in \{1, \dots, n\}$ and $t \geq 0$, $(t - T_{i-1})^+$ is a finite $\mathbb{G}^{i-1}$-stopping time, and R_t^i is a finite $\mathbb{G}^i$-stopping time.*

Proof. Fix $i \in \{1, \dots, n\}$ and $t \geq 0$. The random times $(t - T_{i-1})^+ = (t - T_{i-1})^+ I_{\{T_{i-1} < \infty\}}$ and R_t^i are finite by definition. Moreover, we know that $(t - T_{i-1})^+ I_{\{T_{i-1} < \infty\}} \geq 0$ and $R_t^i \geq 0$. Then we have for $u \geq 0$

$$\begin{aligned} \{(t - T_{i-1})^+ \leq u\} &= \{T_{i-1} = \infty\} \cup (\{(t - T_{i-1})^+ \leq u\} \cap \{T_{i-1} < \infty\}) \\ &= \{T_{i-1} = \infty\} \cup (\{t \leq T_{i-1} + u\} \cap \{T_{i-1} < \infty\}). \end{aligned}$$

Since T_{i-1} is a stopping time with respect to $\mathbb{F}^{i-1}$, we have $\{T_{i-1} = \infty\} \in \mathcal{F}^{i-1}_{T_{i-1}} = \mathcal{G}^{i-1}_0 \subset \mathcal{G}^{i-1}_u$. Similarly, we get $\{T_{i-1} < \infty\}, \{t \leq T_{i-1} + u\} \in \mathcal{G}^{i-1}_u$ such that $(t - T_{i-1})^+$ is a $\mathbb{G}^{i-1}$-stopping time. Now, consider $i \in \{1, \dots, n-1\}$. Then

$$\begin{aligned}\{R^i_t \leq u\} &= \{\min\{T_i - T_{i-1}, (t - T_{i-1})^+\} I_{\{T_{i-1}<\infty\}} \leq u\} \\ &= \{T_{i-1} = \infty\} \cup (\{\min\{T_i - T_{i-1}, (t - T_{i-1})^+\} \leq u\} \cap \{T_{i-1} < \infty\}) \\ &= \{T_{i-1} = \infty\} \cup ([\{T_i - T_{i-1} \leq u\} \cup \{t - T_{i-1} \leq u\}] \cap \{T_{i-1} < \infty\}) \\ &= \{T_{i-1} = \infty\} \cup ([\{T_i \leq T_{i-1} + u\} \cup \{t \leq T_{i-1} + u\}] \cap \{T_{i-1} < \infty\}).\end{aligned} \tag{3.7}$$

Since T_i and T_{i-1} are stopping times with respect to $\mathbb{F}^i$, we have $\{T_i \leq T_{i-1} + u\} \in \mathcal{F}^i_{T_{i-1}+u} \subset \mathcal{F}^i_{T_i+u} = \mathcal{G}^i_u$. Similarly, we obtain $\{T_{i-1} = \infty\}, \{t \leq T_{i-1} + u\}, \{T_{i-1} < \infty\} \in \mathcal{G}^i_u$, which proves the second assertion in case of $i \in \{1, \dots, n-1\}$. If $i = n$, then $R^n_t = (t - T_{n-1})^+$. Because of the first part of this proof, $(t - T_{n-1})^+$ is a $\mathbb{G}^{n-1}$-stopping time, which implies the $\mathbb{G}^n$-stopping time property. □

Lemma 3.2.11. *For any $i \in \{1, \dots, n\}$ and $t \geq 0$, $G^i_{(t-T_{i-1})^+}$ is $\mathcal{F}^{i-1}_t$-measurable.*

Proof. Fix $i \in \{1, \dots, n\}$ and $t \geq 0$. According to the previous lemma, $(t - T_{i-1})^+$ is a finite $\mathbb{G}^{i-1}$-stopping time. Moreover, G^i is continuous and $\mathbb{G}^{i-1}$-adapted by definition. This implies that $G^i_{(t-T_{i-1})^+}$ is measurable with respect to $\mathcal{G}^{i-1}_{(t-T_{i-1})^+} = \mathcal{F}^{i-1}_{T_{i-1}+(t-T_{i-1})^+}$. Since $T_{i-1}+(t-T_{i-1})^+ \leq t \vee T_{i-1}$, it follows directly that $G^i_{(t-T_{i-1})^+}$ is measurable with respect to $\mathcal{F}^{i-1}_{t\vee T_{i-1}}$. This yields

$$\{G^i_{(t-T_{i-1})^+} \leq u\} \in \mathcal{F}^{i-1}_{t\vee T_{i-1}} \quad \text{for all } u \geq 0.$$

Due to $\mathcal{F}^{i-1}_{t\vee T_{i-1}} = \{A \in \mathcal{A} | A \cap \{t \vee T_{i-1} \leq u\} \in \mathcal{F}^{i-1}_u \text{ for all } u \geq 0\}$, we obtain

$$\{G^i_{(t-T_{i-1})^+} \leq u\} \cap \{t \vee T_{i-1} \leq t\} \in \mathcal{F}^{i-1}_t \quad \text{for all } u \geq 0.$$

Since $\{t \vee T_{i-1} \leq t\} = \{T_{i-1} \leq t\} \in \mathcal{F}^{i-1}_t$, it follows

$$\begin{aligned}&\{G^i_{(t-T_{i-1})^+} \leq u\} \\ &= (\{G^i_{(t-T_{i-1})^+} \leq u\} \cap \{T_{i-1} \leq t\}) \cup (\{G^i_{(t-T_{i-1})^+} \leq u\} \cap \{t < T_{i-1}\}) \\ &= (\{G^i_{(t-T_{i-1})^+} \leq u\} \cap \{T_{i-1} \leq t\}) \cup (\{G^i_0 \leq u\} \cap \{t < T_{i-1}\}) \\ &= (\{G^i_{(t-T_{i-1})^+} \leq u\} \cap \{T_{i-1} \leq t\}) \cup \{t < T_{i-1}\} \in \mathcal{F}^{i-1}_t.\end{aligned}$$

□

Lemma 3.2.12. *For all $i \in \{1, \dots, n\}$ and $t \geq 0$, $G^i_{R^i_t}$ is $\mathcal{F}^i_t$-measurable.*

Proof. Fix $i \in \{1, \dots, n\}$ and $t \geq 0$. Equation (3.7) yields that R^i_t is a finite $(\mathcal{F}^i_{T_{i-1}+u})_{u\geq 0}$-stopping time. Moreover, G^i is continuous and $\mathbb{G}^{i-1}(\subset (\mathcal{F}^i_{T_{i-1}+u})_{u\geq 0})$-adapted by definition. Hence, $G^i_{R^i_t}$ is $\mathcal{F}^i_{T_{i-1}+R^i_t}$-measurable. Since $T_{i-1} + R^i_t \leq T_{i-1} + (t - T_{i-1})^+ \leq t \vee T_{i-1}$, the remaining part of the proof is analogous to the proof of Lemma 3.2.11 if we replace $(t - T_{i-1})^+$ by R^i_t and $\mathcal{F}^{i-1}_t$ by $\mathcal{F}^i_t$. □

With the previous lemmas we can conclude as Giesecke and Tomecek (2005).

Proposition 3.2.13. *The overall time change G satisfies Assumption 3.2.5, and G_t^{-1} is a $\mathbb{G}$-stopping time for each $t \geq 0$.*

Proof. First of all, note that

$$G^i_{R^i_t} I_{\{T_{i-1}<\infty\}} = \begin{cases} G^i_{t-T_{i-1}} I_{\{T_{i-1}\leq t<T_i\}} + G^i_{T_i-T_{i-1}} I_{\{T_i\leq t\}} & \text{for } i \in \{1,\dots,n-1\} \\ G^n_{t-T_{n-1}} I_{\{T_{n-1}\leq t\}} & \text{for } i = n \end{cases}.$$

Moreover, the overall time change G satisfies $G_t = \sum_{k=1}^n G^k_{R^k_t}$ for each $t \geq 0$ since

$$\begin{aligned}
&\sum_{k=1}^{n-1} G^k_{\min\{T_k-T_{k-1},(t-T_{k-1})^+\}} I_{\{T_{k-1}<\infty\}} + G^n_{(t-T_{n-1})^+} \\
&= \sum_{i=1}^{n-1} \left(\sum_{k=1}^{n-1} G^k_{\min\{T_k-T_{k-1},(t-T_{k-1})^+\}} I_{\{T_{k-1}<\infty\}} + G^n_{(t-T_{n-1})^+} \right) I_{\{T_{i-1}\leq t<T_i\}} \\
&\quad + \left(\sum_{k=1}^{n-1} G^k_{\min\{T_k-T_{k-1},(t-T_{k-1})^+\}} I_{\{T_{k-1}<\infty\}} + G^n_{(t-T_{n-1})^+} \right) I_{\{T_{n-1}\leq t\}} \\
&= \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i-1} G^k_{T_k-T_{k-1}} + G^i_{t-T_{i-1}} \right) I_{\{T_{i-1}\leq t<T_i\}} \\
&\quad + \left(\sum_{k=1}^{n-1} G^k_{T_k-T_{k-1}} + G^n_{t-T_{n-1}} \right) I_{\{T_{n-1}\leq t\}}.
\end{aligned}$$

Lemma 3.2.12 yields that $G^k_{R^k_t}$ is $\mathcal{F}^k_t(\subset \mathcal{G}_t)$-measurable for each $k \in \{1,\dots,n\}$ and $t \geq 0$. It follows immediately that G is $\mathbb{G}$-adapted. Furthermore, it follows from $G_t^{-1} = \inf\{s \geq 0 | G_s > t\} = \min\{s \geq 0 | G_s = t\}$ that

$$\{G_t^{-1} \leq s\} = \{t \leq G_s\} \in \mathcal{G}_s \quad \text{for all } s \geq 0.$$

This means that G_t^{-1} is a $\mathbb{G}$-stopping time for each $t \geq 0$. □

To prove the last proposition in this subsection, we need the following lemma which is similar to Theorem T28, Chapter A2 in Brémaud (1981) (see Theorem 1.1.9).

Lemma 3.2.14. *Let $\mathscr{N}$ be a nonexplosive point process and define the filtration $\mathbb{E}^{\infty,\mathscr{N}} := (\mathcal{E}^{\infty,\mathscr{N}}_t)_{t\geq 0}$ by*

$$\mathcal{E}^{\infty,\mathscr{N}}_t := \mathcal{K}_\infty \vee \sigma(\mathscr{N}_s : s \leq t).$$

Then for every $\mathbb{E}^{\infty,\mathscr{N}}$-stopping time θ, the following equality holds:

$$\mathcal{E}^{\infty,\mathscr{N}}_\theta = \mathcal{K}_\infty \vee \sigma(\mathscr{N}_{\theta\wedge s} : s \geq 0).$$

Proof. Since $\mathcal{K}_\infty \subset \mathcal{E}_t^{\infty,\mathscr{N}}$ and $\mathscr{N}_{\theta\wedge t}$ is $\mathcal{E}_{\theta\wedge t}^{\infty,\mathscr{N}}(\subset \mathcal{E}_\theta^{\infty,\mathscr{N}})$-measurable for each $t \geq 0$, the inclusion "$\supset$" is trivial. The proof of the inclusion "$\subset$" is analogous to the proof of Theorem T28, Chapter A2 in Brémaud (1981) if one recalls that

$$\mathcal{E}_a^{\infty,\mathscr{N}} = \sigma(\mathscr{N}_s, I_C : s \leq a, C \in \mathcal{K}_\infty) \quad \text{for every } a \in \mathbb{R}_+$$

since $\mathcal{K}_\infty = \sigma(I_C : C \in \mathcal{K}_\infty)$. □

Remark 3.2.15. For $\mathscr{N} = N^0$, we obtain from Theorem 1.1.8 that $\mathcal{E}_t^{\infty,\mathscr{N}} = \mathcal{H}_t$. Thus, Lemma 3.2.14 yields $\mathcal{H}_\theta = \mathcal{K}_\infty \vee \sigma(N^0_{\theta\wedge s} : s \geq 0)$ for every $\mathbb{H}$-stopping time θ.

Proposition 3.2.16. *For each $i \in \{1,\dots,n\}$, $T_1,\dots,T_i$ are measurable with respect to $\mathcal{H}_{S_i}$. In particular, we have $\mathcal{G}_{T_i} \subset \mathcal{H}_{S_i}$.*

Proof. Fix $i \in \{1,\dots,n\}$. We know from Remark 3.2.15 that $\mathcal{H}_{S_i} = \mathcal{K}_\infty \vee \sigma(N^0_{S_i\wedge s} : s \geq 0)$. Moreover, because $\mathcal{F}_t^{N^0} = \sigma(N_s^0 : s \leq t)$, Theorem 1.1.9 yields $\mathcal{F}_{S_i}^{N^0} = \sigma(N^0_{S_i\wedge s} : s \geq 0)$, and from Theorem 1.1.10 we know that $\mathcal{F}_{S_i}^{N^0} = \sigma(S_j : j \leq i)$. Hence, we obtain

$$\mathcal{H}_{S_i} = \mathcal{K}_\infty \vee \sigma(S_j : j \leq i).$$

Measurability of $T_1,\dots,T_i$ follows directly from the definition of G by

$$G_t = \begin{cases} \sum_{k=1}^{i-1} G^k_{T_k - T_{k-1}} + G^i_{t-T_{i-1}} & \text{on } \{T_{i-1} \leq t < T_i\} \text{ for } i \in \{1,\dots,n\} \\ \sum_{k=1}^{n-1} G^k_{T_k - T_{k-1}} + G^n_{t-T_{n-1}} & \text{on } \{T_n \leq t\} \end{cases}$$

and the fact that

$$T_j = G^{-1}_{S_j} = \inf\{t \geq 0 | G_t = S_j\} \quad \text{for all } j \leq i:$$

First, consider T_1 which is given by $T_1 = \inf\{t \geq 0 | G_t = S_1\} = \inf\{t \geq 0 | G^1_t = S_1\}$. Then for each $s \geq 0$, we have

$$\{T_1 \leq s\} = \{\inf\{t \geq 0 | G^1_t = S_1\} \leq s\} = \{S_1 \leq G^1_s\}.$$

Since S_1 is $\sigma(S_j : j \leq i)$ measurable and G^1_s is $\mathcal{K}_\infty$-measurable by definition, this yields $\{T_1 \leq s\} \in \mathcal{K}_\infty \vee \sigma(S_j : j \leq i)$ for each $s \geq 0$. Thus, T_1 is measurable with respect to $\mathcal{K}_\infty \vee \sigma(S_j : j \leq i)$.

If $i > 1$, then measurability of T_2 follows from measurability of T_1: Due to the definition of G, we have $G_{T_1} = G^1_{T_1}$. Therefore, $G^1_{T_1} = S_1$. Moreover, we have

$$\begin{aligned} T_2 &= \inf\{t \geq 0 | G_t = S_2\} = \inf\{t \geq T_1 | G^1_{T_1} + G^2_{t-T_1} = S_2\} \\ &= \inf\{t \geq T_1 | G^1_{T_1} + G^2_{(t-T_1)^+} = S_2\} \quad \text{on } \{T_1 < \infty\}. \end{aligned}$$

Together, we obtain for all $s \geq 0$

$$\begin{aligned} \{T_2 \leq s\} &= \{T_1 \leq s\} \cap \{\inf\{t \geq T_1 | G^1_{T_1} + G^2_{(t-T_1)^+} = S_2\} \leq s\} \\ &= \{T_1 \leq s\} \cap \{\inf\{t \geq T_1 | S_1 + G^2_{(t-T_1)^+} = S_2\} \leq s\} \\ &= \{T_1 \leq s\} \cap \{S_2 \leq S_1 + G^2_{(s-T_1)^+}\}. \end{aligned}$$

We have already proved that T_1 is measurable with respect to $\mathcal{K}_\infty \vee \sigma(S_j : j \leq i)$. Furthermore, it is obvious that S_1 and S_2 are $\sigma(S_j : j \leq i)$-measurable. By Lemma 3.2.11, $G^2_{(s-T_1)^+}$ is $\mathcal{F}^1_s$-measurable. Since

$$\mathcal{F}^1_s = \bigcap_{u>s} \mathcal{K}_u \vee \sigma(I_{\{T_1 \leq v\}} : v \leq u) \subset \mathcal{K}_\infty \vee \sigma(T_1) \subset \mathcal{K}_\infty \vee \sigma(S_j : j \leq i)$$

for all $s \geq 0$, we finally obtain that $\{T_2 \leq s\} \in \mathcal{K}_\infty \vee \sigma(S_j : j \leq i)$ for all $s \geq 0$. This means that T_2 is measurable with respect to $\mathcal{K}_\infty \vee \sigma(S_j : j \leq i)$.

By this procedure, we successively obtain that all T_k for $k \leq i$ are $\mathcal{K}_\infty \vee \sigma(S_j : j \leq i)$-measurable. This yields the first assertion.

The inclusion $\mathcal{G}_{T_i} \subset \mathcal{H}_{S_i}$ follows from Lemma 3.2.14: Note that $\mathcal{G}_t \subset \mathcal{E}^{\infty,N}_t = \mathcal{K}_\infty \vee \sigma(N_s : s \leq t)$. According to Lemma 3.2.14, we have $\mathcal{E}^{\infty,N}_{T_i} = \mathcal{K}_\infty \vee \sigma(N_{T_i \wedge s} : s \geq 0)$. Since $\mathcal{K}_\infty \vee \sigma(N_{T_i \wedge s} : s \geq 0) = \mathcal{K}_\infty \vee \sigma(T_j : j \leq i)$, it follows

$$\mathcal{E}^{\infty,N}_{T_i} = \mathcal{K}_\infty \vee \sigma(T_j : j \leq i).$$

Finally, the second assertion holds due to $\mathcal{G}_{T_i} \subset \mathcal{E}^{\infty,N}_{T_i}$ and because $T_1, \ldots, T_i$ are measurable with respect to $\mathcal{H}_{S_i}$. □

3.3. Conditional distribution of the arrival times

In this section we determine conditional $\mathbb{P}^{i-1}$-distributions of the inter-arrival times $T_i - T_{i-1}$ and the arrival times T_i by using the results from Section 3.1.

Lemma 3.3.1. *For each $i \in \{1, \ldots, n\}$, the inter-arrival time $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$.*

Proof. Fix $i \in \{1, \ldots, n\}$ and consider $\mathbb{A} = (\mathcal{A}_t)_{t \geq 0}$ defined in Subsection 3.2.3 as the right-continuous filtration generated by $\sigma(W_s : s \leq t) \vee \sigma(\kappa^1, \ldots, \kappa^n) \vee \mathcal{K}_\infty$ (Note that because $\kappa^1, \ldots, \kappa^n$ are deterministic, the σ-algebra $\sigma(\kappa^1, \ldots, \kappa^n)$ is trivial.). Since S_j is an $\mathbb{A}$-stopping time, S_j is $\mathcal{A}_{S_j}$-measurable for each $j \in \{0, \ldots, i-1\}$. This implies that $S_0, \ldots, S_{i-1}$ are $\mathcal{A}_{S_{i-1}}$-measurable, and hence $\mathcal{H}_{S_{i-1}} = \mathcal{K}_\infty \vee \sigma(S_j : j \leq i-1) \subset \mathcal{A}_{S_{i-1}}$. Together with the $\mathbb{P}^{i-1}$-independence of $(W_{S_{i-1}+s} - W_{S_{i-1}})_{s \geq 0}$ and $\mathcal{A}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$, this yields that $(W_{S_{i-1}+s} - W_{S_{i-1}})_{s \geq 0}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. From the proof of Proposition 3.1.5 we know that

$$(S_i - S_{i-1}) I_{\{S_{i-1} < \infty\}} = \inf\{s \geq 0 | \kappa^i + \sigma \cdot (W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} I_{\{S_{i-1} < \infty\}}.$$

Since κ^i is deterministic for each $i \in \{1, \ldots, n\}$, we finally obtain the assertion. □

Before we arrive at the main result of this section, we have to consider the following lemma which is related to Lemma A.11 in Giesecke and Tomecek (2005).

Lemma 3.3.2. *For each $i \in \{1, \ldots, n\}$ and $t \geq 0$, $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}$ and $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^{i-1}_{t \vee T_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$.*

Proof. Fix $i \in \{1, \dots, n\}$ and $t \geq 0$. We know from Lemma 3.3.1 that $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. Now, consider the counting process $\mathscr{N}_t := \sum_{k=1}^{i-1} I_{\{T_k \leq t\}}$. Due to Lemma 3.2.14, we have $\mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}} = \mathcal{K}_\infty \vee \sigma(\mathscr{N}_{T_{i-1}\wedge s} : s \geq 0)$. Moreover, Theorem 1.1.9 yields $\mathcal{F}^{\mathscr{N}}_{T_{i-1}} = \sigma(\mathscr{N}_{T_{i-1}\wedge s} : s \geq 0)$, and from Theorem 1.1.10 we know that $\mathcal{F}^{\mathscr{N}}_{T_{i-1}} = \sigma(T_j : j \leq i-1)$. It follows

$$\mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}} = \mathcal{K}_\infty \vee \sigma(T_j : j \leq i-1).$$

According to Theorem 1.1.10, we have $\mathcal{E}^{\infty,\mathscr{N}}_{\infty} = \mathcal{K}_\infty \vee \sigma(T_j : j \leq i-1)$. Therefore, the inclusions $\mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}} \subset \mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}+t} \subset \mathcal{E}^{\infty,\mathscr{N}}_{\infty}$ and $\mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}} \subset \mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}\vee t} \subset \mathcal{E}^{\infty,\mathscr{N}}_{\infty}$ imply

$$\mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}+t} = \mathcal{E}^{\infty,\mathcal{N}}_{T_{i-1}\vee t} = \mathcal{K}_\infty \vee \sigma(T_j : j \leq i-1).$$

Furthermore, by definition of $\mathbb{F}^{i-1}$ and $\mathbb{E}^{\infty,\mathscr{N}}$, we have $\mathcal{F}^{i-1}_s \subset \mathcal{E}^{\infty,\mathscr{N}}_s$ for all $s \geq 0$. As a consequence, we obtain

$$\begin{aligned} \mathcal{F}^{i-1}_{T_{i-1}+t} &\subset \mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}+t} = \mathcal{K}_\infty \vee \sigma(T_j : j \leq i-1) \quad \text{and} \\ \mathcal{F}^{i-1}_{T_{i-1}\vee t} &\subset \mathcal{E}^{\infty,\mathscr{N}}_{T_{i-1}\vee t} = \mathcal{K}_\infty \vee \sigma(T_j : j \leq i-1). \end{aligned}$$

Since T_j, $j \leq i-1$, are $\mathcal{H}_{S_{i-1}}$-measurable by Proposition 3.2.16, this means

$$\mathcal{F}^{i-1}_{T_{i-1}+t} \subset \mathcal{H}_{S_{i-1}} \quad \text{and} \quad \mathcal{F}^{i-1}_{T_{i-1}\vee t} \subset \mathcal{H}_{S_{i-1}}. \tag{3.8}$$

Finally, the $\mathbb{P}^{i-1}$-independence of $S_i - S_{i-1}$ and $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$ and (3.8) yield the assertion. □

The next theorem is the main result of this section. In particular, we derive the $\mathcal{F}^{i-1}_{T_{i-1}}$-conditional distribution function of the inter-arrival times $T_i - T_{i-1}$ and the $\mathbb{F}^{i-1}$-conditional distribution function of T_i under the probability measure $\mathbb{P}^{i-1}$ for each $i \in \{1, \dots, n\}$. Later on, we will use this result to compute intensities. Note that Giesecke and Tomecek (2005) proved a similar statement for their inter-arrival times.

We have already determined the $\mathbb{P}^{i-1}$-distribution function of $S_i - S_{i-1}$ in Proposition 3.1.5 and Corollary 3.1.7, namely

$$\mathbb{P}^{i-1}[S_i - S_{i-1} \leq t] = F^{\Delta S}(t, \kappa^i) = \mathbb{P}\left[\min_{s \leq t}(\kappa^i + \sigma W_s + \mu s) \leq 0\right] \quad \text{for } t \geq 0$$

and

$$F^{\Delta S}(t, \kappa^i) = \begin{cases} \Phi\left(\frac{-\kappa^i - \mu t}{\sigma\sqrt{t}}\right) + e^{-2\mu\kappa^i/\sigma^2}\Phi\left(\frac{-\kappa^i + \mu t}{\sigma\sqrt{t}}\right) & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}.$$

Theorem 3.3.3. *For each $i \in \{1, \dots, n\}$ and $0 \leq s \leq t$, we have*

$$\begin{aligned} &\mathbb{P}^{i-1}[(T_i - T_{i-1})I_{\{T_{i-1}<\infty\}} \leq t | \mathcal{F}^{i-1}_{T_{i-1}+s}] \\ &= \mathbb{E}^{i-1}[F^{\Delta S}(G^i_t, \kappa^i) I_{\{T_{i-1}<\infty\}} | \mathcal{F}^{i-1}_{T_{i-1}+s}] \quad \mathbb{P}^{i-1} - a.s., \end{aligned}$$

and the $\mathbb{F}^{i-1}$-conditional $\mathbb{P}^{i-1}$-distribution function of T_i is given by

$$\mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}^{i-1}_t] = F^{\Delta S}(G^i_{t-T_{i-1}}, \kappa^i) I_{\{T_{i-1}<t\}} \quad \mathbb{P}^{i-1} - a.s.$$

Proof. Fix $i \in \{1, \dots, n\}$ and $t \geq 0$. First, note that Lemma 3.1.3 and Proposition 3.2.9 yield

$$\begin{aligned}
&\mathbb{P}^{i-1}[(T_i - T_{i-1}) I_{\{T_{i-1}<\infty\}} \leq t | \mathcal{F}^{i-1}_{T_{i-1}+t}] \\
&= \mathbb{P}^{i-1}[T_i - T_{i-1} \leq t | \mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}] I_{\{T_{i-1}<\infty\}} \\
&= \mathbb{P}^{i-1}[S_i - S_{i-1} \leq G^i_t | \mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}] I_{\{T_{i-1}<\infty\}}.
\end{aligned}$$

Since G^i is $\mathbb{G}^{i-1}$-adapted, G^i_t is $\mathcal{F}^{i-1}_{T_{i-1}+t}$-measurable. It follows that G^i_t considered as a random variable on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$ is measurable with respect to $\mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}$. Due to Lemma 3.3.2, $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. Hence, we obtain from Lemma 3.1.3 (in a similar way to the proof of Corollary 3.1.7)

$$\mathbb{P}^{i-1}[(T_i - T_{i-1}) I_{\{T_{i-1}<\infty\}} \leq t | \mathcal{F}^{i-1}_{T_{i-1}+t}] = F^{\Delta S}(G^i_t, \kappa^i) I_{\{T_{i-1}<\infty\}}.$$

Since $\mathcal{F}^{i-1}_{T_{i-1}+s} \subset \mathcal{F}^{i-1}_{T_{i-1}+t}$ for $s \leq t$, it follows that

$$\begin{aligned}
&\mathbb{P}^{i-1}[(T_i - T_{i-1}) I_{\{T_{i-1}<\infty\}} \leq t | \mathcal{F}^{i-1}_{T_{i-1}+s}] \\
&= \mathbb{E}^{i-1}[\mathbb{P}^{i-1}[(T_i - T_{i-1}) I_{\{T_{i-1}<\infty\}} \leq t | \mathcal{F}^{i-1}_{T_{i-1}+t}] | \mathcal{F}^{i-1}_{T_{i-1}+s}] \\
&= \mathbb{E}^{i-1}[F^{\Delta S}(G^i_t, \kappa^i) I_{\{T_{i-1}<\infty\}} | \mathcal{F}^{i-1}_{T_{i-1}+s}].
\end{aligned}$$

In order to prove the second assertion, note that since T_{i-1} is an $\mathbb{F}^{i-1}$-stopping time, $I_{\{T_{i-1}<t\}}$ is $\mathcal{F}^{i-1}_t$-measurable. Hence,

$$\begin{aligned}
\mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}^{i-1}_t] &= \mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}^{i-1}_t] I_{\{T_{i-1}<t\}} + \mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}^{i-1}_t] I_{\{T_{i-1}\geq t\}} \\
&= \mathbb{P}^{i-1}[(T_i - T_{i-1}) I_{\{T_{i-1}<\infty\}} \leq (t - T_{i-1})^+ | \mathcal{F}^{i-1}_t] I_{\{T_{i-1}<t\}}.
\end{aligned}$$

Because $T_i - T_{i-1} = (G^i)^{-1}_{S_i - S_{i-1}}$ on $\{S_{i-1} < \infty\}$ and $\{S_{i-1} < \infty\} = \{T_{i-1} < \infty\}$, this means that

$$\mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}^{i-1}_t] = \mathbb{P}^{i-1}[(S_i - S_{i-1}) I_{\{S_{i-1}<\infty\}} \leq G^i_{(t-T_{i-1})^+} | \mathcal{F}^{i-1}_t] I_{\{T_{i-1}<t\}}.$$

With Lemma A.1.3 and the second part of Lemma 3.1.3, we obtain

$$\begin{aligned}
\mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}^{i-1}_t] &= \mathbb{P}^{i-1}[(S_i - S_{i-1}) I_{\{S_{i-1}<\infty\}} \leq G^i_{(t-T_{i-1})^+} | \mathcal{F}^{i-1}_{t \vee T_{i-1}}] I_{\{T_{i-1}<t\}} \\
&= \mathbb{P}^{i-1}[S_i - S_{i-1} \leq G^i_{(t-T_{i-1})^+} | \mathcal{F}^{i-1}_{t \vee T_{i-1}} \cap \Omega^{i-1}] I_{\{T_{i-1}<t\}}.
\end{aligned}$$

Furthermore, we know from Lemma 3.2.11 that $G^i_{(t-T_{i-1})^+}$ is $\mathcal{F}^{i-1}_t$ $(\subset \mathcal{F}^{i-1}_{t \vee T_{i-1}})$-measurable, and Lemma 3.3.2 yields $\mathbb{P}^{i-1}$-independence of $S_i - S_{i-1}$ and $\mathcal{F}^{i-1}_{t \vee T_{i-1}} \cap \Omega^{i-1}$ on the probability space $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. As above, this implies by Lemma 3.1.3

$$\mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}^{i-1}_t] = F^{\Delta S}(G^i_{(t-T_{i-1})^+}, \kappa^i) I_{\{T_{i-1}<t\}} = F^{\Delta S}(G^i_{t-T_{i-1}}, \kappa^i) I_{\{T_{i-1}<t\}}.$$

□

Especially in contrast to Corollary 3.1.7 that considers the case of a portfolio value process which is given by a simple geometric Brownian motion, the previous result clarifies the advantage of the time change approach.

In Section 3.1 we have defined the time of the ith default in our underlying portfolio by the first time the geometric Brownian motion V hits the deterministic barrier K^i. This point in time is denoted by S_i. Moreover, we have stated after Corollary 3.1.7 that the $\mathcal{F}^W_{S_{i-1}}$-conditional $\mathbb{P}^{i-1}$-distribution of S_i depends on κ^i, μ, σ and S_{i-1}. Hence, apart from the dependence on the time since the last default, there is no influence of the previous defaults on the conditional default probability of the remaining firms in the portfolio.

In order to overcome this property, we have introduced the time change G and defined the time of the ith default as the first point in time T_i in which the time changed geometric Brownian motion V_G hits the deterministic barrier K^i. As a consequence, Theorem 3.3.3 states that the $\mathbb{F}^{i-1}$-conditional $\mathbb{P}^{i-1}$-distribution of the ith default time depends on κ^i, μ, σ (which are deterministic) and T_{i-1} and additionally on the ith time change G^i. This dependence on G^i is new and the key property of our model.

In Subsection 3.2.4 we have pointed out that the ith time change G^i is responsible for a possible change in the slope of the overall time change G. Now, Theorem 3.3.3 clarifies that by choosing an appropriate time change G^i, we can specify how a default of one firm influences the conditional distribution of the next default. Examples of such time changes G^i will be considered later on in Section 3.6.

3.4. Conditional survival probabilities, default trends and intensities

This section is related to the work of Giesecke (2006) and focuses on the determination of the ith default trend for each default time T_i, $i \in \{1, \dots, n\}$. These trends are used to derive tractable solutions for prices of credit sensitive securities which depend on the ith default in the underlying pool of names and to construct an algorithm to simulate default times.

We start with a short introduction to the terminology which is needed later on. Thereafter, we apply the results of Giesecke (2006) to our model setting. This means that we determine the ith default trend for each default time T_i, $i \in \{1, \dots, n\}$, and prove that default intensities with respect to the different model filtrations $\mathbb{F}^{i-1}$ exist. Moreover, we will see that our default times are totally inaccessible in $\mathbb{G}$, which is typically for reduced form models. Finally, in Subsection 3.4.3 we derive the $(\mathbb{P}, \mathbb{G})$-compensator C of the default counting process N by using the results from the previous sections. Since $N - C$ is a martingale, C encodes the upwards tendency of the default counting process. Thus, N can be specified in terms of its compensator.

3.4.1. General definitions

In this subsection we introduce important terms which are discussed in the following subsections in more detail. We also refer to Section 1.2, where the case of a single firm is considered.

Notation 3.4.1. For each $i \in \{1, \ldots, n\}$, the $\mathbb{F}^{i-1}$*-conditional survival probability of* T_i is denoted by Z^i, i.e., $Z_t^i := \mathbb{E}[1 - N_t^i | \mathcal{F}_t^{i-1}]$.

Remark 3.4.2. We know from Section 1.2 that Z^i is a $(\mathbb{P}, \mathbb{F}^{i-1})$-supermartingale for each $i \in \{1, \ldots, n\}$. Moreover, since each default indicator process N^i is a nondecreasing process, N^i is a $(\mathbb{P}, \mathbb{G})$-submartingale.

Because of the previous remark, each N^i admits a $(\mathbb{P}, \mathbb{G})$-compensator C^i such that $N^i - C^i$ are $(\mathbb{P}, \mathbb{G})$-martingales. If $\mathbb{G}$ is for each $\mathbb{F}^{i-1}$ a filtration expansion of the Guo-Zeng type (see Definition 1.2.5), then we obtain these compensators by using the extended Jeulin-Yor theorem (see Theorem 1.2.7).

Hereafter, we show that $\mathbb{G}$ is indeed a filtration expansion of $\mathbb{F}^{i-1}$ of the Guo-Zeng type for each $i \in \{1, \ldots, n\}$. To this end, recall that $0 = T_0 \leq T_1 \leq \ldots \leq T_n \leq \infty$ and $\mathbb{F}^0 \subset \mathbb{F}^1 \subset \ldots \subset \mathbb{F}^n := \mathbb{G}$.

Lemma 3.4.3. *For every* $i \in \{1, \ldots, n\}$ *and every* $t \geq 0$*, we have*

$$\mathcal{F}_t^{i-1} \cap \{T_i > t\} = \mathcal{F}_t^i \cap \{T_i > t\}.$$

Proof. Fix $i \in \{1, \ldots, n\}$ and $t \geq 0$ and let $\mathbb{G}^{T_i}$ be the progressive filtration expansion of $\mathbb{F}^{i-1}$, i.e.,

$$\mathcal{G}_t^{T_i} = \{A \in \mathcal{E}_\infty | \exists F_t \in \mathcal{F}_t^{i-1}, A \cap \{T_i > t\} = F_t \cap \{T_i > t\}\}$$

with $\mathcal{E}_\infty := \mathcal{F}_\infty^{i-1} \vee \sigma(T^i)$. We have

$$\begin{aligned} \mathcal{F}_t^i &= \bigcap_{u>t} \mathcal{K}_u \vee \sigma(I_{\{T_k \leq s\}} : s \leq u, k \leq i) \\ &= \bigcap_{u>t} \mathcal{F}_u^{i-1} \vee \sigma(I_{\{T_i \leq s\}} : s \leq u) = \bigcap_{u>t} (\mathcal{G}^{i-1})'_u \end{aligned} \tag{3.9}$$

where $(\mathbb{G}^{i-1})' = ((\mathcal{G}^{i-1})'_t)_{t \geq 0}$ denotes the minimal filtration expansion of $\mathbb{F}^{i-1}$. Since $(\mathbb{G}^{i-1})'$ is the smallest filtration expansion of $\mathbb{F}^{i-1}$ such that T_i is a $(\mathbb{G}^{i-1})'$-stopping time, we have $(\mathbb{G}^{i-1})' \subset \mathbb{G}^{T_i}$. Moreover, the progressive filtration expansion $\mathbb{G}^{T_i}$ is right-continuous (see Chapter VI in Protter (2005)). Therefore, it follows from (3.9) that $\mathcal{F}_t^i \subset \mathcal{G}_t^{T_i}$. This implies

$$\mathcal{F}_t^i \cap \{T^i > t\} \subset \mathcal{G}_t^{T_i} \cap \{T^i > t\} = \mathcal{F}_t^{i-1} \cap \{T^i > t\}.$$

The assertion follows by noting that the reverse implication $\mathcal{F}_t^i \cap \{T^i > t\} \supset \mathcal{F}_t^{i-1} \cap \{T^i > t\}$ is obvious. $\square$

The following lemma is a special case of Lemma 9, Chapter 1 in Kchia (2011).

Lemma 3.4.4. *For every j with $1 \leq i \leq j \leq n$ and every $t \geq 0$, we have*

$$\mathcal{F}_t^j \cap \{T_i > t\} = \mathcal{F}_t^{i-1} \cap \{T_i > t\}. \tag{3.10}$$

Especially, we have

$$\mathcal{G}_t \cap \{T_i > t\} = \mathcal{F}_t^{i-1} \cap \{T_i > t\} \quad \text{for all } i \in \{1, \ldots, n\}.$$

Proof. Note that the assertion holds for $j = i$ due to Lemma 3.4.3. Moreover, Lemma 3.4.3 yields $\mathcal{F}_t^j \cap \{T_{j+1} > t\} = \mathcal{F}_t^{j+1} \cap \{T_{j+1} > t\}$ for each $t \geq 0$ and $j \in \{i, \ldots, n-1\}$, which implies that $\mathcal{F}_t^j \cap \{T_{j+1} > t\} \cap \{T_i > t\} = \mathcal{F}_t^{j+1} \cap \{T_{j+1} > t\} \cap \{T_i > t\}$. Since the default times are increasing, we have $\{T_i > t\} \subset \{T_{j+1} > t\}$. Hence, the last equation is equivalent to $\mathcal{F}_t^j \cap \{T_i > t\} = \mathcal{F}_t^{j+1} \cap \{T_i > t\}$. But this implies that if (3.10) is true for j, then it is also satisfied for $j+1$. Since the statement holds for $j = i$, the proof of the first assertion is completed. The second statement is the special case $j = n$. □

According to the previous lemma, $\mathbb{G}$ is indeed a filtration expansion of $\mathbb{F}^{i-1}$ that is of the Guo-Zeng type for each $i \in \{1, \ldots, n\}$. Hence, we can apply the extended Jeulin-Yor theorem as described above.

To compute the $(\mathbb{P}, \mathbb{G})$-compensator processes C^i of the indicator processes N^i for $i \in \{1, \ldots, n\}$, we adopt definitions from Giesecke (2006) to our default models $(T_i, \mathbb{F}^{i-1})$. Giesecke (2006) considers the case of only one defaultable firm and uses progressive filtration expansions. Nevertheless, we will see that we obtain similar results in our multi-firm setting due to the extended Jeulin-Yor theorem.

Definition 3.4.5. *Fix $i \in \{1, \ldots, n\}$ and let $Z_{t-}^i := \lim_{s \uparrow t} Z_s^i$ and $Z_{0-}^i := 1$. The* ith default trend *$\mathscr{A}^i$ is defined by*

$$\mathscr{A}_t^i := \int_0^t \frac{1}{Z_{s-}^i} d\mathscr{C}_s^i \tag{3.11}$$

where the process $\mathscr{C}^i$ is the $(\mathbb{P}, \mathbb{F}^{i-1})$-compensator of Z^i. We say that $(T_i, \mathbb{F}^{i-1})$ is an intensity based default model *if there exists an $\mathbb{F}^{i-1}$-progressive and nonnegative process λ^i such that $\mathscr{A}_t^i = \int_0^t \lambda_s^i ds$ $\mathbb{P}$-a.s. for all $t \geq 0$. The process λ^i is called the* ith intensity process.

Remark 3.4.6. We will see later on that $Z_t^i > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ (see Lemma 3.4.15). Thus, we do not divide by 0 in (3.11).

Definition 3.4.7. *For $i \in \{1, \ldots, n\}$, we say that the default model $(T_i, \mathbb{F}^{i-1})$ is* strongly intensity based *if there exists an $\mathbb{F}^{i-1}$-progressive and nonnegative process λ^i such that for each $t \geq 0$, we have*

$$Z_t^i = \exp\left(-\int_0^t \lambda_s^i ds\right) \quad \mathbb{P} - a.s.$$

If $(T_i, \mathbb{F}^{i-1})$ is an intensity based default model, then it follows from the extended Jeulin-Yor theorem that $C^i = (\mathscr{A}^i)^{T_i}$ with $(\mathscr{A}^i_t)^{T_i} = \mathscr{A}^i_{t\wedge T_i} = \int_0^{t\wedge T_i} \lambda^i_s ds$ $\mathbb{P}$-a.s. is the $(\mathbb{P}, \mathbb{G})$-compensator of N^i. Because of that, we are interested in the detailed form of $\mathscr{A}^i$ and λ^i. But first, let us state a few remarks concerning the previous definitions.

Remark 3.4.8.
1. Although the previous definitions are based on the definitions in Giesecke (2006), we have modified the required properties of the involved intensity. If we would adopt the definition from Giesecke (2006) directly to our approach, then the intensities would be bounded, nonnegative and $\mathbb{F}^{i-1}$-predictable. Boundedness is needed in Proposition 5.10 in Giesecke (2006) which proves that intensities in the sense of Definition 3.4.7 are indeed short credit spreads. Moreover, Giesecke (2006) uses a progressive filtration expansion to obtain this result. For a more general result with respect to the filtration expansion which also requires relaxed predictability and boundedness assumptions, we refer to Okhrati (2013).
2. Note that the ith default trend $\mathscr{A}^i$ defined in Definition 3.4.5 corresponds with the $\mathbb{F}$-martingale hazard process Λ in Proposition 2.1.8. Indeed, the model and the results in Giesecke (2006) and the model in this chapter are closely related to the hazard process approach.
3. Let $\bar{\lambda}^i$ and $\tilde{\lambda}^i$ be two $\mathbb{F}^{i-1}$-predictable ith intensity processes. Then $\bar{\lambda}^i$ and $\tilde{\lambda}^i$ are also $\mathbb{G}$-predictable since $\mathbb{F}^{i-1} \subset \mathbb{G}$. Moreover, note that for every ith intensity process λ^i and every $t \geq 0$, we have

$$C^i_t = \mathscr{A}^i_{t\wedge T_i} = \int_0^{t\wedge T_i} \lambda^i_s ds = \int_0^t \lambda^i_s I_{\{s\leq T_i\}} ds \quad \mathbb{P}-\text{a.s.}$$

   Therefore, it follows from Theorem T12, Chapter II in Brémaud (1981) that the $\mathbb{G}$-predictable processes $(\bar{\lambda}^i_t I_{\{t\leq T_i\}})_{t\geq 0}$ and $(\tilde{\lambda}^i_t I_{\{t\leq T_i\}})_{t\geq 0}$ satisfy

$$(\bar{\lambda}^i_t I_{\{t\leq T_i\}})(\omega) = (\tilde{\lambda}^i_t I_{\{t\leq T_i\}})(\omega) \quad \mathbb{P}(d\omega) \times dN^i_t(\omega)\text{-a.e.}$$

The next lemma, which was proved in Giesecke (2006), states that the "strongly intensity based" property of $(T_i, \mathbb{F}^{i-1})$ is really stronger than the "intensity based" property.

Lemma 3.4.9 (See Proposition 5.8 in Giesecke (2006))**.** *For every $i \in \{1, \dots, n\}$, the following statements are satisfied:*

1. *If a default model $(T_i, \mathbb{F}^{i-1})$ is strongly intensity based, then it is intensity based in the sense of Definition 3.4.5.*
2. *If a default model $(T_i, \mathbb{F}^{i-1})$ is intensity based in the sense of Definition 3.4.5 and additionally satisfies $\mathscr{C}^i = 1 - Z^i$, then it is also strongly intensity based.*

3.4.2. Default trends

In this subsection we determine the default trends (and hence the compensators of the processes N^i) in our default model with deterministic barriers and unknown portfolio value process. Moreover, at the end of this subsection we use these trends to determine prices of contingent claims depending on the ith default in the underlying pool of names.

In Theorem 3.3.3 we have already computed the $\mathbb{F}^{i-1}$-conditional $\mathbb{P}^{i-1}$-distribution function of T_i for each $i \in \{1, \ldots, n\}$. The next corollary follows from Theorem 3.3.3 and addresses the $\mathbb{F}^{i-1}$-conditional survival probability of T_i.

Corollary 3.4.10. *For each $i \in \{1, \ldots, n\}$ and each $t \geq 0$, we have*

$$\mathbb{P}[T_i \leq t | \mathcal{F}_t^{i-1}] = \mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}_t^{i-1}] I_{\{T_{i-1}<t\}} = F^{\Delta S}(G^i_{t-T_{i-1}}, \kappa^i) I_{\{T_{i-1}<t\}} \quad \mathbb{P} - a.s.$$

In particular, for each $t \geq 0$, Z^i satisfies

$$Z_t = 1 - F^{\Delta S}(G^i_{t-T_{i-1}}, \kappa^i) I_{\{T_{i-1}<t\}} \quad \mathbb{P} - a.s.$$

Proof. Fix $i \in \{1, \ldots, n\}$ and $t \geq 0$ and note that

$$\mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}_t^{i-1}] = \mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}_t^{i-1}] I_{\{T_{i-1}<t\}} \quad \mathbb{P}^{i-1} - \text{a.s.}$$

It follows for $B := \{T_i \leq t\}$ and every $F \in \mathcal{F}_t^{i-1}$

$$\begin{aligned}
\mathbb{E}^{i-1}[\mathbb{E}[I_B I_{\{T_{i-1}<t\}} | \mathcal{F}_t^{i-1}] I_F] &= \frac{\mathbb{E}[\mathbb{E}[I_B I_{\{T_{i-1}<t\}} | \mathcal{F}_t^{i-1}] I_F I_{\{T_{i-1}<\infty\}}]}{\mathbb{P}[T_{i-1} < \infty]} \\
&= \frac{\mathbb{E}[I_B I_{\{T_{i-1}<t\}} I_F I_{\{T_{i-1}<\infty\}}]}{\mathbb{P}[T_{i-1} < \infty]} \\
&= \mathbb{E}^{i-1}[I_B I_{\{T_{i-1}<t\}} I_F] \\
&= \mathbb{E}^{i-1}[\mathbb{E}^{i-1}[I_B I_{\{T_{i-1}<t\}} | \mathcal{F}_t^{i-1}] I_F].
\end{aligned}$$

By the definition of the conditional expectation, this means

$$\mathbb{E}[I_B I_{\{T_{i-1}<t\}} | \mathcal{F}_t^{i-1}] = \mathbb{E}^{i-1}[I_B I_{\{T_{i-1}<t\}} | \mathcal{F}_t^{i-1}] \quad \mathbb{P}^{i-1} - \text{a.s.}$$

As a consequence, we have

$$\begin{aligned}
\mathbb{P}[T_i \leq t | \mathcal{F}_t^{i-1}] I_{\{T_{i-1}<t\}} &= \mathbb{P}^{i-1}[T_i \leq t | \mathcal{F}_t^{i-1}] I_{\{T_{i-1}<t\}} \\
&= F^{\Delta S}(G^i_{t-T_{i-1}}, \kappa^i) I_{\{T_{i-1}<t\}} \quad \mathbb{P}^{i-1} - \text{a.s.},
\end{aligned}$$

where the last equality follows from Theorem 3.3.3. Finally, Lemma 3.4.11 yields the desired result. □

Lemma 3.4.11. *Let X and Y be random variables on $(\Omega, \mathcal{A}, \mathbb{P}^{i-1})$ for some $i \in \{1, \ldots, n\}$. Then $X I_{\{T_{i-1}<\infty\}} = Y I_{\{T_{i-1}<\infty\}}$ $\mathbb{P}^{i-1}$-a.s. implies that $X I_{\{T_{i-1}<\infty\}} = Y I_{\{T_{i-1}<\infty\}}$ $\mathbb{P}$-a.s.*

Proof. Fix $i \in \{1, \ldots, n\}$ and note that $\mathbb{P}^{i-1}[A] = \mathbb{P}[A \cap \{T_{i-1} < \infty\}] / \mathbb{P}[T_{i-1} < \infty]$ for each $A \in \mathcal{A}$. Because

$$\begin{aligned}
\{X I_{\{T_{i-1}<\infty\}} = Y I_{\{T_{i-1}<\infty\}}\} &= (\{X I_{\{T_{i-1}<\infty\}} = Y I_{\{T_{i-1}<\infty\}}\} \cap \{T_{i-1} < \infty\}) \\
&\quad \cup (\{X I_{\{T_{i-1}<\infty\}} = Y I_{\{T_{i-1}<\infty\}}\} \cap \{T_{i-1} = \infty\}) \\
&= (\{X I_{\{T_{i-1}<\infty\}} = Y I_{\{T_{i-1}<\infty\}}\} \cap \{T_{i-1} < \infty\}) \\
&\quad \cup \{T_{i-1} = \infty\},
\end{aligned}$$

we obtain

$$\begin{aligned}\mathbb{P}[XI_{\{T_{i-1}<\infty\}} = YI_{\{T_{i-1}<\infty\}}] &= \mathbb{P}^{i-1}[XI_{\{T_{i-1}<\infty\}} = YI_{\{T_{i-1}<\infty\}}]\mathbb{P}[T_{i-1}<\infty] \\ &\quad + \mathbb{P}[T_{i-1}=\infty] \\ &= \mathbb{P}[T_{i-1}<\infty] + \mathbb{P}[T_{i-1}=\infty] \\ &= 1.\end{aligned}$$

□

Remark 3.4.12. Because of the definition of T_i, we have

$$\mathbb{P}[T_i \leq t|\mathcal{F}_t^{i-1}] = \mathbb{P}\left[\min_{0\leq s\leq t} V_{G_s} \leq K^i \middle| \mathcal{F}_t^{i-1}\right] \quad \mathbb{P}-\text{a.s.}$$

Hence, if we fix $t \geq 0$ and consider $\mathbb{P}[T_i \leq t|\mathcal{F}_t^{i-1}]$ as a function in K^i, then we can interpret this function as the $\mathbb{F}^{i-1}$-*conditional distribution function of the portfolio minimum at time* t given by $M_t := \min_{0\leq s\leq t} V_{G_s}$. Giesecke (2006) studies this perspective in detail.

To derive the ith default trend $\mathscr{A}^i$, the process $F^i(x)$ defined for each $x > 0$ by

$$F_t^i(x) := F^{\Delta S}(G^i_{t-T_{i-1}}, x)I_{\{T_{i-1}<t\}} \tag{3.12}$$

plays an important role. In the following, we discuss this process in detail and verify important properties which are essential to obtain the desired ith default trend. Since the process $F^i(x)$ is strongly connected to the function $F^{\Delta S}(\cdot, x)$, we start with important properties of this function.

Lemma 3.4.13. *For each $x > 0$, $F^{\Delta S}(\cdot, x)$ is continuous and increasing and satisfies $F^{\Delta S}(t, x) < 1$ for each $t \geq 0$. Moreover, $F^{\Delta S}(\cdot, x)$ is absolutely continuous with derivative $f^{\Delta S}(\cdot, x)$ which is given by*

$$f^{\Delta S}(t,x) = e^{-2\mu x/\sigma^2}\frac{x}{\sigma}t^{-3/2}\varphi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right) \quad \textit{for } t > 0. \tag{3.13}$$

In particular, $f^{\Delta S}(\cdot, x)$ satisfies $\lim_{t\downarrow 0} f^{\Delta S}(t, x) = 0$.

Proof. Fix $x > 0$. Continuity and monotonicity are obvious. Moreover, $F^{\Delta S}(0, x) = 0$ and $F^{\Delta S}(t, x) \in [0, 1]$ for each $t > 0$. Therefore, $F^{\Delta S}(t, x) < 1$ for $t > 0$ if and only if

$$F^{\Delta S}(t,x) = 1 - \Phi\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right) + e^{-2\mu x/\sigma^2}\Phi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right) \neq 1.$$

This means that we have to prove that $\Upsilon(a, b, c) := e^{2ab/c^2}\Phi((a + b)/c) - \Phi((-a + b)/c) \neq 0$ for $a < 0$, $b \in \mathbb{R}$ and $c > 0$. On the one hand, we have

$$e^{\frac{2ab}{c^2}}\Phi\left(\frac{a+b}{c}\right) = \frac{1}{c\sqrt{2\pi}}\int_{-\infty}^{a} e^{\frac{2ab}{c^2}}e^{-\frac{1}{2}\left(\frac{y+b}{c}\right)^2}dy = \frac{1}{c\sqrt{2\pi}}\int_{-\infty}^{a} e^{\frac{2ab}{c^2}-\frac{y^2}{2c^2}-\frac{by}{c^2}-\frac{b^2}{2c^2}}dy.$$

On the other hand, we obtain

$$\begin{aligned}\Phi\left(\frac{-a+b}{c}\right) &= \frac{1}{c\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{1}{2}\left(\frac{y-2a+b}{c}\right)^2} dy \\ &= \frac{1}{c\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{y^2}{2c^2}+\frac{2ay}{c^2}-\frac{by}{c^2}-\frac{2a^2}{c^2}+\frac{2ab}{c^2}-\frac{b^2}{2c^2}} dy.\end{aligned}$$

If we define $l\,(y) := \exp\left((2ab-by)/c^2 - (y^2+b^2)/(2c^2)\right)$, then we get

$$\Upsilon(a,b,c) = \frac{1}{c\sqrt{2\pi}}\int_{-\infty}^{a} l\,(y)\left[1-e^{\frac{2ay-2a^2}{c^2}}\right] dy.$$

Since $a<0$, we have $2ay-2a^2>0$ for all $y\in(-\infty,a)$. Together with $l(y)>0$ this yields $\Upsilon(a,b,c)\neq 0$.

Moreover, from $\frac{\partial}{\partial t}[(-x\pm\mu t)/(\sigma\sqrt{t})] = (x/2\sigma)t^{-3/2} \pm (\mu/2\sigma)t^{-1/2}$ it follows that

$$\begin{aligned}f^{\Delta S}\,(t,x) &= \frac{\partial}{\partial t}\left[\Phi\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right) + e^{-2\mu x/\sigma^2}\Phi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right)\right] \\ &= \left(\frac{x}{2\sigma}t^{-3/2} - \frac{\mu}{2\sigma}t^{-1/2}\right)\varphi\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right) \\ &\quad + e^{-2\mu x/\sigma^2}\left(\frac{x}{2\sigma}t^{-3/2} + \frac{\mu}{2\sigma}t^{-1/2}\right)\varphi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right).\end{aligned}$$

Since

$$\begin{aligned}\varphi\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right)^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{2\mu x}{\sigma^2}-\frac{x^2}{2\sigma^2 t}+\frac{x\mu}{\sigma^2}-\frac{\mu^2 t}{2\sigma^2}} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{2\mu x}{\sigma^2}}e^{-\frac{1}{2}\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right)^2} = e^{-\frac{2\mu x}{\sigma^2}}\varphi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right),\end{aligned}$$

this yields Equation (3.13).

Finally, we consider the limit $t\downarrow 0$. First, note that for each $t>0$, we have

$$t^{-3/2}\varphi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{3}{2}\log t-\frac{x^2}{2\sigma^2 t}+\frac{x\mu}{\sigma^2}-\frac{\mu^2 t}{2\sigma^2}}.$$

Since $\mu^2 t/(2\sigma^2)\to 0$ and

$$3\log t + \frac{x^2}{\sigma^2 t} = \log t^3 + \log e^{\frac{x^2}{\sigma^2 t}} = \log\left(t^3 e^{\frac{x^2}{\sigma^2 t}}\right) \to \infty,$$

we obtain $\lim_{t\downarrow 0} f^{\Delta S}(t,x) = 0$. □

For the remaining part of this chapter we only consider time changes G^i, $i\in\{1,\ldots,n\}$, that satisfy the following assumption.

Assumption 3.4.14. *For each $i\in\{1,\ldots,n\}$ and all $t\geq 0$, the time change G^i satisfies*

$$G^i_t = \int_0^t g^i_s ds \quad \mathbb{P}-a.s. \tag{3.14}$$

for a $\mathbb{G}^{i-1}$-adapted, right-continuous and positive density process g^i.

Lemma 3.4.15. *For each $i \in \{1,\dots,n\}$ and each $x > 0$, the process $F^i(x)$ satisfies the following properties:*

1. *$F^i_t(x) < 1$ $\mathbb{P}$-a.s. for all $t \geq 0$.*
2. *$F^i(x)$ is continuous and increasing.*
3. *$F^i(x)$ is absolutely continuous with a nonnegative and right-continuous density process $f^i(x)$ which is for $t \geq 0$ given by*

$$f^i_t(x) = g^i_{t-T_{i-1}} e^{-2\mu x/\sigma^2} \frac{x}{\sigma} (G^i_{t-T_{i-1}})^{-3/2} \varphi\left(\frac{-x+\mu G^i_{t-T_{i-1}}}{\sigma\sqrt{G^i_{t-T_{i-1}}}}\right) I_{\{T_{i-1}<t\}} \tag{3.15}$$

$\mathbb{P}$-a.s.

Moreover, $f^i(x)$ is $\mathbb{F}^{i-1}$-predictable if g^i is $\mathbb{G}^{i-1}$-predictable.

Proof. Fix $i \in \{1,\dots,n\}$ and $x > 0$. Note that by Lemma 3.4.13, $F^{\Delta S}(\cdot,x)$ is increasing, continuous and satisfies $F^{\Delta S}(t,x) < 1$ for all $t \geq 0$ and $\lim_{t\downarrow 0} F^{\Delta S}(t,x) = 0$. Since G^i has monotone and continuous paths by definition, the first two properties are obvious.

For $\omega \in \{T_{i-1} \geq t\}$, we obviously have $f^i_t(\omega) = 0$. According to Lemma 3.4.13, $F^{\Delta S}(\cdot,x)$ is absolutely continuous with derivative

$$f^{\Delta S}(s,x) = e^{-2\mu x/\sigma^2} \frac{x}{\sigma} s^{-3/2} \varphi\left(\frac{-x+\mu s}{\sigma\sqrt{s}}\right) \quad \text{for } s > 0.$$

For $\omega \in \{T_{i-1} < t\}$, we have $[F^i_t(x)](\omega) = F^{\Delta S}(G^i_{t-T_{i-1}(\omega)}(\omega), x)$. Because of Equation (3.14), we obtain by the chain rule that

$$[f^i_t(x)](\omega) = g^i_{t-T_{i-1}(\omega)}(\omega) e^{-2\mu x/\sigma^2} \frac{x}{\sigma} (G^i_{t-T_{i-1}(\omega)}(\omega))^{-3/2} \varphi\left(\frac{-x+\mu G^i_{t-T_{i-1}(\omega)}(\omega)}{\sigma\sqrt{G^i_{t-T_{i-1}(\omega)}(\omega)}}\right).$$

Since g^i is right-continuous by Assumption 3.4.14, it follows from $\lim_{t\downarrow 0} f^{\Delta S}(t,x) = 0$ (see Lemma 3.4.13) that $f^i(x)$ is also right-continuous. Thus, the third property is satisfied. The last assertion follows directly from Lemma 3.4.16. □

Lemma 3.4.16. *Fix $i \in \{1,\dots,n\}$ and let g^i be $\mathbb{G}^{i-1}$-predictable. Then the process*

$$g^i_{(\cdot - T_{i-1})^+} I_{\{T_{i-1}<\cdot\}} : (0,\infty) \times \Omega \to \mathbb{R},$$

$g^i_{(\cdot - T_{i-1})^+} I_{\{T_{i-1}<\cdot\}}(t,\omega) = g^i_{(t-T_{i-1}(\omega))^+}(\omega) I_{\{T_{i-1}(\omega)<t\}}$ is $\mathbb{F}^{i-1}$-predictable.

Proof. Fix $i \in \{1,\dots,n\}$. The predictable σ-algebra is generated by processes f which are $\mathbb{G}^{i-1}$-adapted and càglàd (left-continuous with right limits) on $(0,\infty)$. Hence, it suffices to prove the assertion for such processes f.

Since $(t-T_{i-1})^+$ is a finite $\mathbb{G}^{i-1}$-stopping time for every $t > 0$, $f_{(t-T_{i-1})^+}$ is measurable with respect to $\mathcal{G}^{i-1}_{(t-T_{i-1})^+}$. From $T_{i-1} + (t-T_{i-1})^+ \leq t \vee T_{i-1}$ it follows directly that $f_{(t-T_{i-1})^+}$ is measurable with respect to $\mathcal{F}^{i-1}_{t\vee T_{i-1}}$. Moreover, because $I_{\{T_{i-1}<t\}}$ is

$\mathcal{F}_t^{i-1}(\subset \mathcal{F}_{t\vee T_{i-1}}^{i-1})$-measurable, we obtain that $f_{(t-T_{i-1})^+} I_{\{T_{i-1}<t\}}$ is measurable with respect to $\mathcal{F}_{t\vee T_{i-1}}^{i-1}$.

Now, it follows analogously to the proof of Lemma 3.2.11 that $f_{(t-T_{i-1})^+} I_{\{T_{i-1}<t\}}$ is $\mathcal{F}_t^{i-1}$-measurable. As $(f_{(t-T_{i-1})^+} I_{\{T_{i-1}<t\}})_{t>0}$ is also càglàd on $(0,\infty)$, this implies the assertion.

□

In the following, we determine the ith default trend and ith default intensity. For the corresponding result in the single-firm setting of Giesecke (2006) we refer to Proposition 6.4 in this paper.

Proposition 3.4.17. *For each $i \in \{1,\dots,n\}$, the ith default time T_i is totally inaccessible in $\mathbb{G}$, and the ith default trend $\mathscr{A}^i$ is continuous. For each $t \geq 0$, we have*

$$\mathscr{A}_t^i = -\log(1 - F_t^i(\kappa^i)) \quad \mathbb{P}-a.s. \tag{3.16}$$

Moreover, the default model $(T_i, \mathbb{F}^{i-1})$ is strongly intensity based, and $\mathscr{A}^i$ admits a right-continuous ith intensity process λ^i given by

$$\lambda_t^i = \frac{f_t^i(\kappa^i)}{1 - F_t^i(\kappa^i)} \quad \mathbb{P}-a.s. \tag{3.17}$$

for $t \geq 0$.

Proof. Fix $i \in \{1,\dots,n\}$. From Lemma 3.4.15 we know that $F_t^i(\kappa^i) < 1$ for all $t \geq 0$. Consequently, we have $Z_t^i = 1 - F_t^i(\kappa^i) \in (0,1]$ for all $t \geq 0$. Moreover, $F^i(\kappa^i)$ is continuous and increasing by Lemma 3.4.15. Thus, Z^i is continuous and decreasing. Because Z^i is $\mathbb{F}^{i-1}$-adapted by definition, it follows from the continuity property of Z^i that Z^i is $\mathbb{F}^{i-1}$-predictable.

Since by the Doob-Meyer decomposition theorem (see Theorem 1.2.1) the compensator $\mathscr{C}^i$ is unique, $\mathbb{F}^{i-1}$-predictability and monotonicity of Z^i are sufficient for $\mathscr{C}^i = 1 - Z^i$. In this case, we obtain

$$\mathscr{A}_t^i = \int_0^t \frac{1}{Z_s^i} d(1 - Z_s^i) = -\int_0^t \frac{1}{Z_s^i} dZ_s^i = -\log Z_t^i,$$

where the last equality follows by change of variables (see, for instance, Theorem 54, Chapter I in Protter (2005)). The previous equation yields (3.16). Furthermore, continuity of Z^i implies continuity of $(\mathscr{A}^i)^{T_i}$, which is the $(\mathbb{P},\mathbb{G})$-compensator of N^i according to the extended Jeulin-Yor theorem. Therefore, T_i is a totally inaccessible $\mathbb{G}$-stopping time (see, for instance, Chapter V, Theorem T40 in Dellacherie (1972)).

Since $F^i(\kappa^i)$ satisfies the third property in Lemma 3.4.15, $\mathscr{A}^i$ inherits absolute continuity from $F^i(\kappa^i)$. Moreover, λ^i is the derivative of $\mathscr{A}^i$ with respect to t $\mathbb{P}$-a.s. and is given by (3.17).

At last, note that we have an intensity based model with $\mathscr{C}^i = 1 - Z^i$. Thus, it follows from Lemma 3.4.9 that the model is also strongly intensity based. □

Remark 3.4.18. Note that

$$\mathscr{A}_t^i = -\log(1 - F_t^i(\kappa^i))I_{\{T_{i-1}<t\}} = -\log(Z_t^i)I_{\{T_{i-1}<t\}} \quad \mathbb{P}-\text{a.s.}$$

since $Z_t^i I_{\{T_{i-1}\geq t\}} = \mathbb{P}[T_i > t|\mathcal{F}_t^{i-1}]I_{\{T_{i-1}\geq t\}} = I_{\{T_{i-1}\geq t\}}$ $\mathbb{P}$-a.s. Hence,

$$\begin{aligned}
-\log(Z_t^i) &= -\log(Z_t^i I_{\{T_{i-1}<t\}} + I_{\{T_{i-1}\geq t\}}) \\
&= -\log(Z_t^i)I_{\{T_{i-1}<t\}} + 0 \cdot I_{\{T_{i-1}\geq t\}} = -\log(Z_t^i)I_{\{T_{i-1}<t\}} \quad \mathbb{P}-\text{a.s.}
\end{aligned}$$

Moreover, from (3.15) it follows directly that $\lambda_t^i = f_t^i(\kappa^i)/(1-F_t^i(\kappa^i))I_{\{T_{i-1}<t\}}$ $\mathbb{P}$-a.s.

Explicit form of $\mathscr{A}^i$ and λ^i: We know exactly how $\mathscr{A}^i$ and λ^i in Proposition 3.4.17 look like. Because of (3.12) and (3.15), we obtain $\mathbb{P}$-a.s.

$$\mathscr{A}_t^i = -\log\left(\Phi\left(\frac{\kappa^i + \mu G_{t-T_{i-1}}^i}{\sigma\sqrt{G_{t-T_{i-1}}^i}}\right) - e^{-2\mu\kappa^i/\sigma^2}\Phi\left(\frac{-\kappa^i + \mu G_{t-T_{i-1}}^i}{\sigma\sqrt{G_{t-T_{i-1}}^i}}\right)\right) I_{\{T_{i-1}<t\}} \tag{3.18}$$

and

$$\lambda_t^i = \frac{g_{t-T_{i-1}}^i e^{-2\mu\kappa^i/\sigma^2}\frac{\kappa^i}{\sigma}(G_{t-T_{i-1}}^i)^{-3/2}\varphi\left(\frac{-\kappa^i+\mu G_{t-T_{i-1}}^i}{\sigma\sqrt{G_{t-T_{i-1}}^i}}\right)}{\Phi\left(\frac{\kappa^i+\mu G_{t-T_{i-1}}^i}{\sigma\sqrt{G_{t-T_{i-1}}^i}}\right) - e^{-2\mu\kappa^i/\sigma^2}\Phi\left(\frac{-\kappa^i+\mu G_{t-T_{i-1}}^i}{\sigma\sqrt{G_{t-T_{i-1}}^i}}\right)} I_{\{T_{i-1}<t\}}.$$

According to Proposition 3.4.17, the ith default trend $\mathscr{A}^i$ satisfies $\mathscr{A}_t^i = -\log(1 - F_t^i(\kappa^i))$ $\mathbb{P}$-a.s. for $i \in \{1, \dots, n\}$. The following proposition uses this result to determine prices of default sensitive contingent claims. More precisely, we consider securities paying out the bounded, $\mathcal{G}_T$-measurable amount X at time T if the ith default in the underlying pool of names did not occur up to this point in time, i.e., if $T < T_i$. In case of $T \geq T_i$, the payout is equal to zero.

In general, the price of this security at time $t \leq T$ is given by

$$PV(t,T) = \mathbb{E}\left[X\exp\left(-\int_t^T r_s ds\right)I_{\{T<T_i\}}\middle|\mathcal{G}_t\right]$$

if r denotes the deterministic risk-free interest rate and $\mathbb{P}$ is assumed to be the risk neutral probability measure. The subsequent proposition and the corresponding proof are closely related to Proposition 5 in Giesecke and Goldberg (2004b) and Proposition 5.4 and Corollary 5.5 in Giesecke (2006).

Proposition 3.4.19. *Consider a defaultable promised payoff X at time T and let X be bounded and $\mathcal{G}_T$-measurable. Moreover, suppose that the risk free interest rate $r = (r_s)_{s\geq 0}$ is deterministic and fix $i \in \{1, \dots, n\}$. If the process Y defined by*

$Y_t := \mathbb{E}[X \exp(\mathscr{A}^i_t - \mathscr{A}^i_T)|\mathcal{F}^i_t]$ is $\mathbb{P}$-a.s. continuous at T_i, then the price of X at time $t \le T$ is given by

$$PV(t,T) = \mathbb{E}\left[X \exp\left(-\int_t^T r_s ds + \mathscr{A}^i_t - \mathscr{A}^i_T\right)\middle|\mathcal{F}^i_t\right] I_{\{t<T_i\}} \quad \mathbb{P}-a.s. \tag{3.19}$$

On $\{t < T_i\}$, we have for every $T' \ge t$

$$\mathbb{P}[T_i \le T'|\mathcal{G}_t] = 1 - \mathbb{E}[\exp(\mathscr{A}^i_t - \mathscr{A}^i_{T'})|\mathcal{F}^i_t] \quad \mathbb{P}-a.s. \tag{3.20}$$

Proof. It suffices to consider the case of a zero interest rate, i.e., $r = 0$. Define the processes L by $L_t := \mathbb{E}[X \exp(-\mathscr{A}^i_T)|\mathcal{F}^i_t]$. Since $\mathscr{A}^i_t$ is $\mathcal{F}^i_t$-measurable, this means

$$Y_t = \mathbb{E}[X \exp(\mathscr{A}^i_t - \mathscr{A}^i_T)|\mathcal{F}^i_t] = \exp(\mathscr{A}^i_t)\mathbb{E}[X \exp(-\mathscr{A}^i_T)|\mathcal{F}^i_t] = \exp(\mathscr{A}^i_t)L_t$$

for all $t \ge 0$. Moreover, $\mathscr{A}^i$ is continuous and of finite variation such that the quadratic covariation of L and $\exp(\mathscr{A}^i)$ satisfies $d[L, \exp(\mathscr{A}^i)]_t = 0$. Integration by parts leads to

$$dY_t = \exp(\mathscr{A}^i_t)dL_t + L_{t-}d(\exp(\mathscr{A}^i_t)) + d[L, \exp(\mathscr{A}^i)]_t = \exp(\mathscr{A}^i_t)dL_t + L_{t-}d(\exp(\mathscr{A}^i_t)).$$

Moreover, by change of variables, we obtain

$$dY_t = \exp(\mathscr{A}^i_t)dL_t + Y_{t-}d\mathscr{A}^i_t.$$

From Theorem 28, Chapter II in Protter (2005) we know that

$$[Y, 1-N^i]_t = Y_0(1-N^i_0) + \sum_{0<s\le t} \Delta Y_s \Delta(1-N^i_s)$$

where $\Delta Y_t := Y_t - Y_{t-}$. By assumption, the process Y does not jump at T_i. Therefore, we obtain $d[Y, 1-N^i]_t = 0$ for all $t \le T$. If we define the process U by $U_t := Y_t(1-N^i_t)$, then integration by parts yields

$$\begin{aligned}
dU_t &= -Y_{t-}dN^i_t + (1-N^i_{t-})dY_t + d[Y, 1-N^i]_t \\
&= -Y_{t-}dN^i_t + (1-N^i_{t-})(\exp(\mathscr{A}^i_t)dL_t + Y_{t-}d\mathscr{A}^i_t) \\
&= -Y_{t-}dN^i_t + (1-N^i_{t-})\exp(\mathscr{A}^i_t)dL_t + Y_{t-}d\mathscr{A}^i_{t\wedge T_i} \\
&= (1-N^i_{t-})\exp(\mathscr{A}^i_t)dL_t - Y_{t-}(dN^i_t - d\mathscr{A}^i_{t\wedge T_i}).
\end{aligned}$$

In other words, we have

$$U_T - U_t = \int_t^T (1-N^i_{s-})\exp(\mathscr{A}^i_s)dL_s - \int_t^T Y_{s-}(dN^i_s - d\mathscr{A}^i_{s\wedge T_i}) \tag{3.21}$$

for all $t \le T$. Note that both integrators in Equation (3.21) are square-integrable $\mathbb{F}^i$-martingales: It is easily seen that L is a bounded $\mathbb{F}^i$-martingale. Furthermore, we know from Lemma 3.4.4 that $\mathbb{F}^i$ is a filtration expansion of the Guo-Zeng type of $\mathbb{F}^{i-1}$. Thus, according to the extended Jeulin-Yor theorem, $(N^i_t - \mathscr{A}^i_{t\wedge T_i})_{t\ge 0}$ is an $\mathbb{F}^i$-martingale, and Lemma 1.2.4 states that $(N^i_t - \mathscr{A}^i_{t\wedge T_i})_{t\ge 0}$ is square-integrable with quadratic variation $[N^i - (\mathscr{A}^i)^{T_i}, N^i - (\mathscr{A}^i)^{T_i}] = N^i$. Moreover, the integrands in Equation (3.21) are bounded and $\mathbb{F}^i$-predictable. Together, we obtain that U is an $\mathbb{F}^i$-martingale. Hence, it follows for all $t \le T$ that

$$\begin{aligned} Y_t(1 - N_t^i) = U_t = \mathbb{E}[U_T|\mathcal{F}_t^i] = \mathbb{E}[Y_T(1 - N_T^i)|\mathcal{F}_t^i] \\ = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_T^i](1 - N_T^i)|\mathcal{F}_t^i] = \mathbb{E}[X(1 - N_T^i)|\mathcal{F}_t^i], \end{aligned}$$

i.e.,

$$Y_t I_{\{t<T_i\}} = \mathbb{E}[X \exp(\mathscr{A}_t^i - \mathscr{A}_T^i)|\mathcal{F}_t^i] I_{\{t<T_i\}} = \mathbb{E}[X I_{\{T<T_i\}}|\mathcal{F}_t^i].$$

By Lemma 3.4.4, we have $\mathcal{G}_t \cap \{T_i > t\} = \mathcal{F}_t^i \cap \{T_i > t\}$ for all $t \ge 0$, which implies $\mathbb{E}[X I_{\{T<T_i\}}|\mathcal{F}_t^i] = \mathbb{E}[X I_{\{T<T_i\}}|\mathcal{G}_t]$ on $\{t < T_i\}$. This yields the first assertion since

$$PV(t,T) = \mathbb{E}[X I_{\{T<T_i\}}|\mathcal{G}_t] = \mathbb{E}[X I_{\{T<T_i\}}|\mathcal{F}_t^i] = \mathbb{E}[X \exp(\mathscr{A}_t^i - \mathscr{A}_T^i)|\mathcal{F}_t^i] I_{\{t<T_i\}}.$$

If we use $r = 0$, $X = 1$ and $T = T'$ in Equation (3.19), we arrive at

$$PV(t,T') = \mathbb{E}[I_{\{T'<T_i\}}|\mathcal{F}_t^i] I_{\{t<T_i\}} = \mathbb{E}[\exp(\mathscr{A}_t^i - \mathscr{A}_{T'}^i)|\mathcal{F}_t^i] I_{\{t<T_i\}},$$

which implies Equation (3.20). □

3.4.3. The compensator of the default counting process

So far, we have determined the default trends $\mathscr{A}^i$ and the $(\mathbb{P},\mathbb{G})$-compensators C^i of the indicator processes N^i for $i \in \{1,\dots,n\}$. It remains to consider the default counting process N given by

$$N_t = \sum_{i=1}^n I_{\{T_i \le t\}} \quad \text{for } t \ge 0$$

and to compute the $(\mathbb{P},\mathbb{G})$-compensator C of this process. For a similar result, we refer to Theorem 12, Chapter 1 in Kchia (2011). Moreover, note that our default trends $\mathscr{A}^i$ are absolutely continuous and admit the density λ^i for all $i \in \{1,\dots,n\}$. The following theorem shows that the $(\mathbb{P},\mathbb{G})$-compensator of N is also an absolutely continuous process.

Theorem 3.4.20. *The $(\mathbb{P},\mathbb{G})$-compensator C of the default counting process N is for each $t \ge 0$ given by*

$$C_t = \sum_{i=1}^n \mathscr{A}_{T_i \wedge t}^i \quad \mathbb{P} - a.s.$$

Moreover, N admits the right-continuous $(\mathbb{P},\mathbb{G})$-intensity λ^N given by

$$\lambda_t^N = \sum_{i=1}^n \lambda_t^i I_{\{T_{i-1} \le t < T_i\}} \quad \mathbb{P} - a.s.$$

for $t \ge 0$.

Proof. We know from Proposition 3.4.17 that the $(\mathbb{P},\mathbb{G})$-compensator of N^i is given by $C_t^i = \mathscr{A}^i_{t\wedge T_i} = \int_0^{T_i\wedge t}\lambda_s^i ds$ and λ^i satisfies

$$\lambda_t^i = \lambda_t^i I_{\{T_{i-1}<t\}} = \frac{f_t^i(\kappa^i)}{1-F_t^i(\kappa^i)} \tag{3.22}$$

with $f^i(x)$ given in (3.15) and $F^i(x)$ given in (3.12) for $x>0$. As $F^i(x)$ is continuous and $f^i(x)$ is right-continuous (see Lemma 3.4.15), it follows directly that λ^i has $\mathbb{P}$-a.s. right-continuous paths. Therefore, we obtain

$$\sum_{i=1}^n \mathscr{A}^i_{T_i\wedge t} = \sum_{i=1}^n \int_0^{T_i\wedge t}\lambda_s^i ds = \int_0^t \sum_{i=1}^n \lambda_t^i I_{\{T_{i-1}\le t<T_i\}}ds = \int_0^{t\wedge T_n}\sum_{i=1}^n \lambda_t^i I_{\{T_{i-1}\le t<T_i\}}ds.$$

The process $(\sum_{i=1}^n \lambda_t^i I_{\{T_{i-1}\le t<T_i\}})_{t\ge 0}$ is obviously $\mathbb{G}$-adapted and nonnegative. Moreover, it inherits right-continuity from λ^i and $(I_{\{T_{i-1}\le t<T_i\}})_{t\ge 0}$, $i\in\{1,\dots,n\}$. In particular, $(\sum_{i=1}^n \lambda_t^i I_{\{T_{i-1}\le t<T_i\}})_{t\ge 0}$ is $\mathbb{G}$-progressive.

The process $(\sum_{i=1}^n \mathscr{A}^i_{T_i\wedge t})_{t\ge 0}$ is increasing, continuous and $\mathbb{G}$-adapted, which implies $\mathbb{G}$-predictability. Furthermore, we have

$$N_t - \sum_{i=1}^n \mathscr{A}^i_{T_i\wedge t} = \sum_{i=1}^n N_t^i - \sum_{i=1}^n \mathscr{A}^i_{T_i\wedge t} = \sum_{i=1}^n (N_t^i - C_t^i).$$

Since $N^i - C^i$ is a $(\mathbb{P},\mathbb{G})$-martingale for every $i\in\{1,\dots,n\}$, so is the process $(N_t - \sum_{i=1}^n \mathscr{A}^i_{T_i\wedge t})_{t\ge 0}$. Because the compensator is unique, this completes the proof. □

Remark 3.4.21. As the compensator C is $\mathbb{G}$-predictable by definition, we are able to find a $\mathbb{G}$-predictable intensity $\bar\lambda^N$; see, for instance, Proposition 3.13, Chapter I in Jacod and Shiryaev (2003). If all g^i, $i\in\{1,\dots,n\}$, are additionally $\mathbb{G}^{i-1}$-predictable, then this $\mathbb{G}$-predictable intensity is given by

$$\bar\lambda_t^N = \sum_{i=1}^n \lambda_t^i I_{\{T_{i-1}<t\le T_i\}} \quad \mathbb{P}-\text{a.s.}$$

for $t\ge 0$. This can be verified as follows: We know from Lemma 3.4.15 that $f^i(\kappa^i)$ is $\mathbb{F}^{i-1}$-predictable if g^i is $\mathbb{G}^{i-1}$-predictable. Since $\lambda_t^i = f_t^i(\kappa^i)/(1-F_t^i(\kappa^i))$ $\mathbb{P}$-a.s. for all $t\ge 0$ and $F^i(\kappa^i)$ is continuous, this means that λ^i is $\mathbb{F}^{i-1}$-predictable. The inclusion $\mathbb{F}^{i-1}\subset\mathbb{G}$ implies that λ^i is $\mathbb{G}$-predictable, and hence the process $(\sum_{i=1}^n \lambda_t^i I_{\{T_{i-1}<t\le T_i\}})_{t\ge 0}$ satisfies the same property.

Remark 3.4.22. Giesecke and Tomecek (2005) prove for their time changed Poisson process N^{GT} (see Remark 3.2.4) that for each $i\in\mathbb{N}$, $(N^{GT}_{t\wedge T_i^{GT}} - G^{GT}_{t\wedge T_i^{GT}})_{t\ge 0}$ is a martingale with respect to their investor filtration $\mathbb{G}^{GT}$. Moreover, if $T_i^{GT}\to\infty$ $\mathbb{P}$-a.s., then $N^{GT}-G^{GT}$ is a $\mathbb{G}^{GT}$-local martingale. In this case, G^{GT} is the $(\mathbb{P},\mathbb{G}^{GT})$-compensator of N^{GT}. We refer to Theorem 3.3 in Giesecke and Tomecek (2005) for more details.

3.5. Another incomplete information model

Note that in Subsection 3.2.3 the incomplete information model in the definition of our top down first-passage model (IIM1) was constructed such that the default barriers $K^1, \dots, K^n$ (and hence $\kappa^1, \dots, \kappa^n$) are deterministic and the portfolio value process V_G is not observable by the investor. But if we take into account Subsection 2.1.3, there exist two more extreme cases of incomplete information:

- Information model in which neither the portfolio value process nor the default barriers are publicly available.
- Information model in which the portfolio value process is publicly available, but the default barriers are unobservable.

Replacing the incomplete information model (IIM1 from Subsection 3.2.3) by another incomplete information model results in a new top down first-passage approach.

In general, we could study both additional cases of incomplete information. Nevertheless, assuming that the portfolio value process is observable, the corresponding model in Giesecke (2006) relies on independence between the single default barrier and the firm's asset value process. But this is not an appropriate requirement in our multi-firm setting because the portfolio value process V_G, modeled as a time changed Brownian motion, depends on the default barriers. This is a consequence of the time change construction in Subsection 3.2.4. More precisely, G satisfies

$$G_t = \begin{cases} \sum_{k=1}^{i-1} G^k_{T_k - T_{k-1}} + G^i_{t-T_{i-1}} & \text{on } \{T_{i-1} \leq t < T_i\} \text{ for } i \in \{1, \dots, n\} \\ \sum_{k=1}^{n-1} G^k_{T_k - T_{k-1}} + G^n_{t-T_{n-1}} & \text{on } \{T_n \leq t\} \end{cases}$$

by definition, and hence the overall time change is closely connected to the arrival times $T_1, \dots, T_{n-1}$. Since our default times are defined as first hitting times, they also depend on the default barriers $K^1, \dots, K^{n-1}$. Thus, the model setting in the third incomplete information model in Giesecke (2006) is not suitable for our top down first-passage approach.

In conclusion, we just focus on the incomplete information model in which neither the portfolio value process nor the default barriers are available. After defining this additional incomplete information model in detail, we discuss the results of Sections 3.3 and 3.4 in the context of this new model.

3.5.1. Setting of the second incomplete information model (IIM2)

In order to study the model with stochastic and unobservable default barriers and unobservable portfolio value process V_G, we have to specify the model filtration $\mathbb{F}^{i-1}$ of the default model $(T_i, \mathbb{F}^{i-1})$ and the investor filtration $\mathbb{G}$ which satisfies $\mathbb{F}^{i-1} \subset \mathbb{G}$ for each $i \in \{1, \dots, n\}$. Recall that in case of the model filtration $\mathbb{F}^{i-1}$, this means that we have to specify how much information is available **with respect to**

$$T_i = \inf\{t \geq 0 | V_{G_t} \leq K^i\}, \tag{3.23}$$

i.e., how much information is available with respect to V, G and K^i.

Specification of the second incomplete information model: In IIM2 the following information should be available on $\{t \leq T_{i-1}\}$:

- $\mathbb{K}$ up to time t
- Number of defaults up to time t
- Time of all defaults that occurred up to time t
- G_s for $s \leq t$

On $\{t > T_{i-1}\}$, the following information should additionally be available:

- $\mathbb{K}$ up to time t

By these assumptions, an investor cannot observe the barriers any more which trigger the default in the underlying portfolio. Since an investor never has any information about the underlying process V, the portfolio value process V_G is unobservable as in IIM1. The available information with respect to G and the defaults also coincides with IIM1. Note that with the information which is available with respect to (3.23) the ith default is not observable.

The requirements from above are satisfied if we define the model filtrations $\mathbb{F}^{i-1}$ and $\mathbb{G}^{i-1}$ for $i \in \{1, \ldots, n\}$ exactly as in IIM1 and suppose that κ^i, $i \in \{1, \ldots, n\}$, are independent random variables with values in $(0, \infty)$. In addition to the independence of $\sigma(W_t : t \geq 0)$, which has already been claimed at the beginning of Section 3.2, we assume that $\kappa^1, \ldots, \kappa^n, \sigma(W_t : t \geq 0)$ and $\mathcal{K}_\infty$ are independent. The distribution function of each κ^i is based on the approach in Yi et al. (2011). We assume that it is given by

$$\psi^i\left(x; a_i, v_i, \sigma_i\right) := \begin{cases} \frac{\varphi(x; a_i+v_i, \sigma_i) - e^{-2a_i v_i/\sigma_i^2}\varphi(x; v_i-a_i, \sigma_i)}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_i v_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

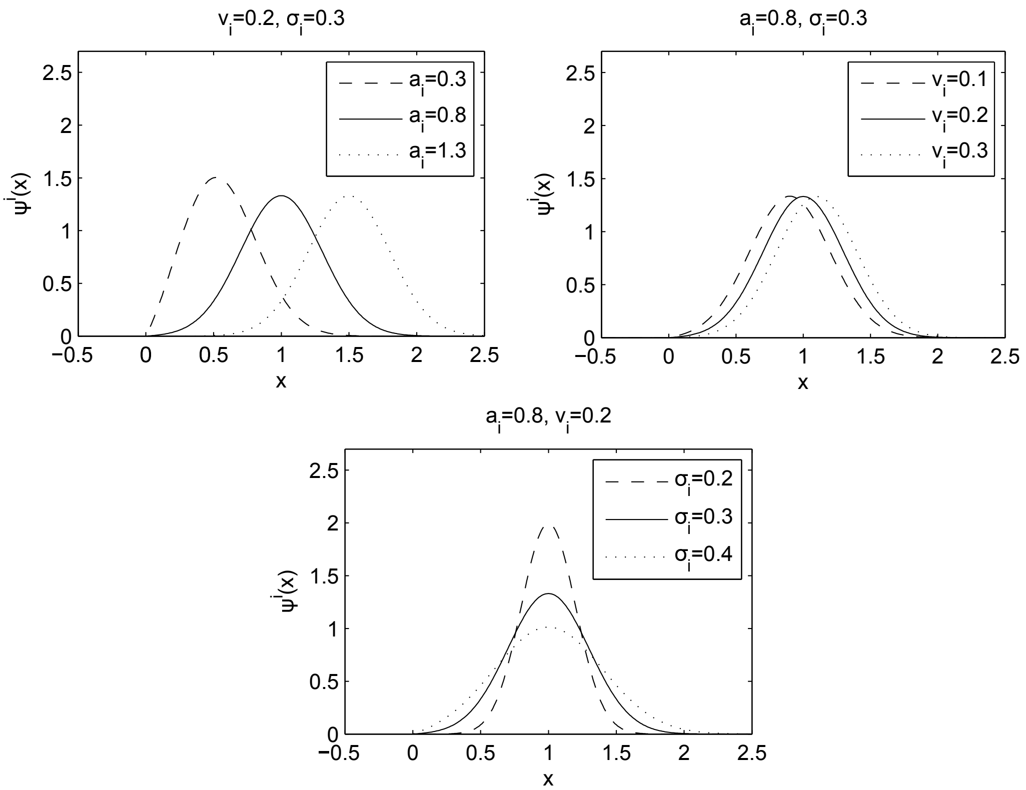
with $\sigma_i > 0$ and $a_i > |v_i|$. Here, $\varphi(x; \bar{\mu}, \bar{\sigma})$ denotes the probability density function of the normal distribution $\mathcal{N}(\bar{\mu}, \bar{\sigma}^2)$. Figure 3.5.1 shows some examples of the density ψ^i for different parameters. If we set $\Psi^i\left(x\right) := \mathbb{P}[\kappa^i \leq x]$, then $\Psi^i(x) = 0$ if $x < 0$ and

$$\Psi^i\left(x\right) = \frac{\Phi\left(\frac{x-a_i-v_i}{\sigma_i}\right) - \Phi\left(\frac{-a_i-v_i}{\sigma_i}\right) - e^{-2a_i v_i/\sigma_i^2}\left(\Phi\left(\frac{x+a_i-v_i}{\sigma_i}\right) - \Phi\left(\frac{a_i-v_i}{\sigma_i}\right)\right)}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_i v_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}$$

if $x \geq 0$. Finally, recall that $K^i = K^{i-1}\exp(-\kappa^i)$. Consequently, the default barriers $K^1, \ldots, K^n$ are also random.

Remark 3.5.1. The random variable κ^i is $\mathbb{P}$-independent of $\mathcal{H}_{S_{i-1}}$ for all $i \in \{1, \ldots, n\}$. If $i = 1$, this follows directly from the assumptions on κ^1 and $\mathcal{H}_{S_0} = \mathcal{H}_0 = \mathcal{K}_\infty$. Therefore, let us consider $i \in \{2, \ldots, n\}$. We know from the proof of Proposition 3.2.16 that $\mathcal{H}_{S_{i-1}} = \mathcal{K}_\infty \vee \sigma(S_j : j \leq i-1)$. Because S_j is measurable with respect to $\sigma(W_t : t \geq 0) \vee \sigma(\kappa^1, \ldots, \kappa^j)$ for $j \in \{1, \ldots, i-1\}$, we obtain $\mathcal{H}_{S_{i-1}} \subset \sigma(W_t : t \geq 0) \vee \sigma(\kappa^1, \ldots, \kappa^{i-1}) \vee \mathcal{K}_\infty$. Again, the assumptions on κ^i imply that κ^i and $\mathcal{H}_{S_{i-1}}$ are $\mathbb{P}$-independent.

Figure 3.5.1.: Density functions $\psi^i\,(x; a_i, v_i, \sigma_i)$ of κ^i



3.5.2. Conditional distribution of the arrival times

IIM2 differs from IIM1 by stochastic κ^i for $i \in \{1, \ldots, n\}$. Suppose that $i \in \{1, \ldots, n\}$ is fixed. We know from the proof of Proposition 3.1.5 that

$$(S_i - S_{i-1})\, I_{\{S_{i-1}<\infty\}} = \inf\{s \geq 0 | \kappa^i + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} I_{\{S_{i-1}<\infty\}}.$$

Moreover, Proposition 3.1.5 and Corollary 3.1.7 yield

$$\begin{aligned}
&\mathbb{P}^{i-1}[S_i - S_{i-1} \leq t | \kappa^i = x] \\
&= \mathbb{P}^{i-1}[\inf\{s \geq 0 | \kappa^i + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} \leq t | \kappa^i = x] \\
&= F^{\Delta S}(t, x) \quad \text{for } x > 0.
\end{aligned}$$

Since κ^i is $\mathbb{P}$-independent of $\mathcal{H}_{S_{i-1}} = \mathcal{K}_\infty \vee \sigma(S_j : j \leq i-1)$ (see Remark 3.5.1), κ^i and S_{i-1} are $\mathbb{P}$-independent. Hence,

$$\mathbb{P}^{i-1}[\kappa^i \leq x] = \mathbb{P}[\kappa^i \leq x | S_{i-1} < \infty] = \mathbb{P}[\kappa^i \leq x] = \Psi^i(x) \quad \text{for } x \in \mathbb{R}.$$

Altogether, we obtain the $\mathbb{P}^{i-1}$-distribution of the inter-arrival times $S_i - S_{i-1}$ by integrating over possible values of κ^i, i.e., for each $t \geq 0$, we have

$$\begin{aligned}
&\mathbb{P}^{i-1}[S_i - S_{i-1} \leq t] \\
&= \mathbb{P}^{i-1}[\inf\{s \geq 0 | \kappa^i + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} \leq t] \\
&= \mathbb{E}^{i-1}[\mathbb{P}^{i-1}[\inf\{s \geq 0 | \kappa^i + \sigma(W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} \leq t | \kappa^i]] \\
&= \int_0^\infty F^{\Delta S}(t,x) d\Psi^i(x). \qquad (3.24)
\end{aligned}$$

This integral was computed by Yi et al. (2011) (see Theorem A.1.5 and Proposition A.1.7) such that we arrive at the following corollary.

Corollary 3.5.2. *In IIM2 the distribution in* (3.24) *is given by*

$$\mathbb{P}^{i-1}\left[S_i - S_{i-1} \leq t\right] = \int_0^\infty F^{\Delta S}(t,x) d\Psi^i(x) = \frac{\Upsilon^i(t)}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_i v_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}$$

for $t > 0$ *where* $\Upsilon^i(t) := A^i(t) + B^i(t) - C^i(t) - D^i(t)$ *with*

$$A^i(t) = \Phi_2\left(-\frac{a_i + v_i + \mu t}{\sqrt{\sigma_i^2 + \sigma^2 t}}, -\frac{-a_i - v_i}{\sigma_i}, \rho^i(t)\right),$$

$$B^i(t) = \Phi_2\left(-\frac{a_i + v_i - 2\mu\sigma_i^2/\sigma^2 - \mu t}{\sqrt{\sigma_i^2 + \sigma^2 t}}, -\frac{-a_i - v_i + 2\mu\sigma_i^2/\sigma^2}{\sigma_i}, \rho^i(t)\right) \cdot e^{-2\mu(a_i+v_i)/\sigma^2 + 2\mu^2\sigma_i^2/\sigma^4},$$

$$C^i(t) = \Phi_2\left(-\frac{v_i - a_i + \mu t}{\sqrt{\sigma_i^2 + \sigma^2 t}}, -\frac{-v_i + a_i}{\sigma_i}, \rho^i(t)\right) e^{-2a_i v_i/\sigma_i^2},$$

$$D^i(t) = \Phi_2\left(-\frac{v_i - a_i - 2\mu\sigma_i^2/\sigma^2 - \mu t}{\sqrt{\sigma_i^2 + \sigma^2 t}}, -\frac{-v_i + a_i + 2\mu\sigma_i^2/\sigma^2}{\sigma_i}, \rho^i(t)\right) \cdot e^{-2a_i v_i/\sigma_i^2 - 2\mu(v_i-a_i)/\sigma^2 + 2\mu^2\sigma_i^2/\sigma^4}$$

and $\rho^i(t) = \frac{-\sigma_i}{\sqrt{\sigma_i^2+\sigma^2 t}}$. *Here,* $\Phi_2(x_1, x_2, \rho)$ *denotes the* 2*-dimensional normal distribution function with standard normal marginal distributions and correlation coefficient* ρ. *In the special case of* $\mu/\sigma^2 = v_i/\sigma_i^2$, *we have*

$$\begin{aligned}
&\mathbb{P}^{i-1}\left[S_i - S_{i-1} \leq t\right] \\
&= \frac{\Phi\left(-\frac{a_i+v_i+\mu t}{\sqrt{\sigma^2 t+\sigma_i^2}}\right) + e^{-2a_i\mu/\sigma^2}\Phi\left(-\frac{a_i-v_i-\mu t}{\sqrt{\sigma^2 T+\sigma_i^2}}\right) - \Phi\left(-\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_i\mu/\sigma^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_i v_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}.
\end{aligned}$$

We are interested in conditional $\mathbb{P}^{i-1}$-distributions of $T_i - T_{i-1}$ and T_i for all $i \in \{1, \ldots, n\}$. Again, we compute these distributions by using the unconditional $\mathbb{P}^{i-1}$-distribution of the inter-arrival times $S_i - S_{i-1}$. But first, we have to prove a corresponding result to Lemma 3.3.2 from IIM1.

Lemma 3.5.3. *For each $i \in \{1,\dots,n\}$, the inter-arrival time $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$.*

Proof. Fix $i \in \{1,\dots,n\}$. We have

$$(S_i - S_{i-1})\, I_{\{S_{i-1}<\infty\}} = \inf\{s \geq 0 | \kappa^i + \sigma \cdot (W_{S_{i-1}+s} - W_{S_{i-1}}) + \mu s = 0\} I_{\{S_{i-1}<\infty\}}. \tag{3.25}$$

The assumptions on $\kappa^1,\dots,\kappa^n, \sigma(W_t : t \geq 0)$ and $\mathcal{K}_\infty$ imply that W is a Brownian motion with respect to $\mathbb{A}$ where the right-continuous filtration $\mathbb{A} = (\mathcal{A}_t)_{t\geq 0}$ is generated by $\sigma(W_s : s \leq t) \vee \sigma(\kappa^1,\dots,\kappa^n) \vee \mathcal{K}_\infty$. By the strong Markov property (see Theorem A.1.1), we know that $(W_{S_{i-1}+s} - W_{S_{i-1}})_{s\geq 0}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{A}_{S_{i-1}} \cap \Omega^{i-1}$ on the probability space $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. Since $\sigma(\kappa^i) \vee \mathcal{H}_{S_{i-1}} = \sigma(\kappa^i) \vee \mathcal{K}_\infty \vee \sigma(S_j : j \leq i-1) \subset \mathcal{A}_{S_{i-1}}$, it follows that $(W_{S_{i-1}+s} - W_{S_{i-1}})_{s\geq 0}$ is $\mathbb{P}^{i-1}$-independent of $(\sigma(\kappa^i) \vee \mathcal{H}_{S_{i-1}}) \cap \Omega^{i-1}$.

Furthermore, the σ-algebras $\sigma(\kappa^i) \cap \Omega^{i-1}$ and $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ are $\mathbb{P}^{i-1}$-independent on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$: From Remark 3.5.1 we know that κ^i is $\mathbb{P}$-independent of $\mathcal{H}_{S_{i-1}}$, i.e., $\mathbb{P}[C \cap H] = \mathbb{P}[C] \cdot \mathbb{P}[H]$ for each $C \in \sigma(\kappa^i)$ and each $H \in \mathcal{H}_{S_{i-1}}$. Consider $D \in \sigma(\kappa^i) \cap \Omega^{i-1}$ and $E \in \mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$. This means $D = C \cap \Omega^{i-1}$ and $E = H \cap \Omega^{i-1}$ for some $C \in \sigma(\kappa^i)$ and $H \in \mathcal{H}_{S_{i-1}}$. Because $\Omega^{i-1} \in \mathcal{H}_{S_{i-1}}$, we obtain $E \in \mathcal{H}_{S_{i-1}}$. It follows

$$\mathbb{P}^{i-1}[D \cap E] = \frac{\mathbb{P}[C \cap E \cap \Omega^{i-1}]}{\mathbb{P}[\Omega^{i-1}]} = \frac{\mathbb{P}[C] \cdot \mathbb{P}[E \cap \Omega^{i-1}]}{\mathbb{P}[\Omega^{i-1}]} = \mathbb{P}[C] \cdot \mathbb{P}^{i-1}[E]. \tag{3.26}$$

Moreover, note that

$$\mathbb{P}^{i-1}[D] = \mathbb{P}^{i-1}[C \cap \Omega^{i-1}] = \frac{\mathbb{P}[C \cap \Omega^{i-1}]}{\mathbb{P}[\Omega^{i-1}]}. \tag{3.27}$$

Since κ^i is $\mathbb{P}$-independent of $\mathcal{H}_{S_{i-1}}$ and $\Omega^{i-1} \in \mathcal{H}_{S_{i-1}}$, Equation (3.27) implies $\mathbb{P}^{i-1}[D] = \mathbb{P}[C]$. Together with (3.26), this yields $\mathbb{P}^{i-1}[D \cap E] = \mathbb{P}^{i-1}[D] \cdot \mathbb{P}^{i-1}[E]$ for all $D \in \sigma(\kappa^i) \cap \Omega^{i-1}$ and all $E \in \mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$.

To sum up, we obtain that the σ-algebra generated by κ^i and $(W_{S_{i-1}+s} - W_{S_{i-1}})_{s\geq 0}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. Together with (3.25), this implies the assertion. $\square$

Now, we obtain the desired result.

Lemma 3.5.4. *For each $i \in \{1,\dots,n\}$ and $t \geq 0$, in IIM2 the inter-arrival time $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}$ and $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^{i-1}_{t \vee T_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$.*

Proof. This follows analogously to the proof of Lemma 3.3.2 from the $\mathbb{P}^{i-1}$-independence of $S_i - S_{i-1}$ and $\mathcal{H}_{S_{i-1}} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$. $\square$

Finally, we are able to prove the following result with regard to the conditional $\mathbb{P}^{i-1}$-distributions.

Theorem 3.5.5. *For each $i \in \{1,\dots,n\}$ and $t \geq 0$, denote $F^{(2,i)}(t) := \mathbb{P}^{i-1}[S_i - S_{i-1} \leq t]$ in IIM2 and consider $0 \leq s \leq t$. Then we have*

$$\begin{aligned}&\mathbb{P}^{i-1}[(T_i - T_{i-1})I_{\{T_{i-1}<\infty\}} \leq t|\mathcal{F}^{i-1}_{T_{i-1}+s}]\\&= \mathbb{E}^{i-1}[F^{(2,i)}(G^i_t)I_{\{T_{i-1}<\infty\}}|\mathcal{F}^{i-1}_{T_{i-1}+s}] \quad \mathbb{P}^{i-1} - a.s.,\end{aligned} \tag{3.28}$$

and the $\mathbb{F}^{i-1}$-conditional $\mathbb{P}^{i-1}$-distribution function of T_i is given by

$$\mathbb{P}^{i-1}[T_i \leq t|\mathcal{F}^{i-1}_t] = F^{(2,i)}(G^i_{t-T_{i-1}})I_{\{T_{i-1}<t\}} \quad \mathbb{P}^{i-1} - a.s.$$

Proof. Fix $i \in \{1,\dots,n\}$ and $t \geq 0$. As in the proof of Theorem 3.3.3, we know that

$$\begin{aligned}&\mathbb{P}^{i-1}[(T_i - T_{i-1})I_{\{T_{i-1}<\infty\}} \leq t|\mathcal{F}^{i-1}_{T_{i-1}+t}]\\&= \mathbb{P}^{i-1}[S_i - S_{i-1} \leq G^i_t|\mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}]I_{\{T_{i-1}<\infty\}}.\end{aligned}$$

Since G^i_t is $\mathcal{F}^{i-1}_{T_{i-1}+t}$-measurable and $S_i - S_{i-1}$ is $\mathbb{P}^{i-1}$-independent of $\mathcal{F}^{i-1}_{T_{i-1}+t} \cap \Omega^{i-1}$ on $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$ (see Lemma 3.5.4), Lemma 3.1.3 yields

$$\mathbb{P}^{i-1}[(T_i - T_{i-1})I_{\{T_{i-1}<\infty\}} \leq t|\mathcal{F}^{i-1}_{T_{i-1}+t}] = F^{(2,i)}(G^i_t)I_{\{T_{i-1}<\infty\}}.$$

Equation (3.28) follows as in the proof of Theorem 3.3.3.

Moreover, we obtain analogously to proof of Theorem 3.3.3 that

$$\mathbb{P}^{i-1}[T_i \leq t|\mathcal{F}^{i-1}_t] = \mathbb{P}^{i-1}[S_i - S_{i-1} \leq G^i_{(t-T_{i-1})^+}|\mathcal{F}^{i-1}_{t\vee T_{i-1}} \cap \Omega^{i-1}]I_{\{T_{i-1}<t\}}.$$

Since $G^i_{(t-T_{i-1})^+}$ is $\mathcal{F}^{i-1}_t$ $(\subset \mathcal{F}^{i-1}_{t\vee T_{i-1}})$-measurable (see Lemma 3.2.11) and $S_i - S_{i-1}$ and $\mathcal{F}^{i-1}_{t\vee T_{i-1}} \cap \Omega^{i-1}$ are $\mathbb{P}^{i-1}$-independent on the probability space $(\Omega^{i-1}, \mathcal{A}^{i-1}, \mathbb{P}^{i-1})$ (see Lemma 3.5.4), Lemma 3.1.3 implies

$$\mathbb{P}^{i-1}[T_i \leq t|\mathcal{F}^{i-1}_t] = F^{(2,i)}(G^i_{(t-T_{i-1})^+})I_{\{T_{i-1}<t\}} = F^{(2,i)}(G^i_{t-T_{i-1}})I_{\{T_{i-1}<t\}}.$$

□

3.5.3. Default trends

An important variable for computing the ith default trend and the ith default intensity is Z^i for $i \in \{1,\dots,n\}$, namely the $\mathbb{F}^{i-1}$-conditional survival probability of T_i. In IIM1 Z^i is given in Corollary 3.4.10. Similarly, we obtain in IIM2 from Theorem 3.5.5 and Equation (3.24) that

$$\begin{aligned}Z^i_t &= 1 - \mathbb{P}^{i-1}[T_i \leq t|\mathcal{F}^{i-1}_t]\\&= 1 - F^{(2,i)}(G^i_{t-T_{i-1}})I_{\{T_{i-1}<t\}}\\&= 1 - \left[\int_0^\infty F^{\Delta S}(G^i_{t-T_{i-1}}, x)d\Psi^i(x)\right] I_{\{T_{i-1}<t\}}\\&= 1 - \int_0^\infty F^i_t(x)d\Psi^i(x) \quad \mathbb{P} - \text{a.s.}\end{aligned} \tag{3.29}$$

where $F^i(x)$ is defined in (3.12). Moreover, the previous equation and Corollary 3.5.2 yield

$$Z_t^i = 1 - \left(\frac{\Upsilon^i(G^i_{t-T_{i-1}})}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_iv_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)} \right) I_{\{T_{i-1}<t\}} \quad \mathbb{P}-\text{a.s.} \tag{3.30}$$

where $\Upsilon^i(t)$ is given in Corollary 3.5.2.

The following proposition yields the ith default trend and the ith default intensity in this model. See Proposition 6.5 in Giesecke (2006) for the corresponding result in the single-firm setting.

Proposition 3.5.6. *For each $i \in \{1, \dots, n\}$, in IIM2 the ith default time T_i is totally inaccessible in $\mathbb{G}$, and the ith default trend $\mathscr{A}^i$ is continuous. For each $t \geq 0$, we have*

$$\mathscr{A}_t^i = -\log\left(1 - \int_0^\infty F_t^i(x)d\Psi^i(x)\right) \quad \mathbb{P}-a.s. \tag{3.31}$$

Moreover, the default model $(T_i, \mathbb{F}^{i-1})$ is strongly intensity based, and $\mathscr{A}^i$ admits a right-continuous ith intensity process λ^i given by

$$\lambda_t^i = \frac{\int_0^\infty f_t^i(x)d\Psi^i(x)}{1 - \int_0^\infty F_t^i(x)d\Psi^i(x)} \quad \mathbb{P}-a.s. \tag{3.32}$$

for $t \geq 0$.

Proof. Fix $i \in \{1, \dots, n\}$. From Lemma 3.4.15 we know that $F_t^i(x) < 1$ for each $x > 0$ and $t \geq 0$. Hence, Z^i given in (3.29) satisfies $Z_t^i \in (0, 1]$ for all $t \geq 0$. Since $F^i(x)$ is a continuous and increasing process for each $x > 0$, Z^i is continuous and decreasing. Thus, the compensator $\mathscr{C}^i$ is given by $\mathscr{C}^i = 1 - Z^i$. As in the proof of Proposition 3.4.17, we obtain that $\mathscr{A}^i$ satisfies Equation (3.31) and that T_i is a totally inaccessible $\mathbb{G}$-stopping time. Again, λ^i is the derivative of the ith default trend $\mathscr{A}^i$ with respect to t $\mathbb{P}$-a.s. It follows that

$$\begin{aligned} \lambda_t^i &= -\frac{1}{Z_t^i}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left[1 - \int_0^\infty F_{t+\varepsilon}^i(x)d\Psi^i(x) - 1 + \int_0^\infty F_t^i(x)d\Psi^i(x)\right] \\ &= \frac{1}{Z_t^i}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^\infty (F_{t+\varepsilon}^i(x) - F_t^i(x))d\Psi^i(x) \quad \mathbb{P}-\text{a.s.} \end{aligned}$$

Since $f_t^i(x) \geq 0$ for all $x > 0$ and because it is easily seen that $f_t^i(\cdot)$ is pointwise bounded, i.e., for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $b(t, \omega) \in \mathbb{R}$ such that $|f_t^i(\cdot)(\omega)| \leq b(t, \omega)$, Equation (3.32) holds due to Lebesgue's dominated convergence theorem.

Finally, it follows analogously to the proof of Proposition 3.4.17 that the default model $(T_i, \mathbb{F}^{i-1})$ is strongly intensity based. □

Explicit form of $\mathscr{A}^i$: The previous proposition and Equations (3.29) and (3.30) yield $\mathbb{P}$-a.s.

$$\begin{aligned}\mathscr{A}_t^i &= -\log\left(1-\left(\frac{\Upsilon^i(G^i_{t-T_{i-1}})}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right)-e^{-2a_iv_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}\right)I_{\{T_{i-1}<t\}}\right)\\ &= -\log\left(1-\frac{\Upsilon^i(G^i_{t-T_{i-1}})}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right)-e^{-2a_iv_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}\right)I_{\{T_{i-1}<t\}} \qquad (3.33)\end{aligned}$$

where $\Upsilon^i(t)$ is given in Corollary 3.5.2. Note that the second equality follows analogously to Remark 3.4.18.

3.5.4. The compensator of the default counting process

It remains to consider the $(\mathbb{P},\mathbb{G})$-compensator of the default counting process N in IIM2. Note that in IIM1, as well as in IIM2, the $(\mathbb{P},\mathbb{G})$-compensator of N^i is given by C^i with $C_t^i = \mathscr{A}^i_{t\wedge T_i}$ $\mathbb{P}^{i-1}$-a.s. Moreover, in both models the ith default trend $\mathscr{A}^i$ is of the form $\mathscr{A}_t^i = -\log(Y_t)I_{\{T_{i-1}<t\}}$ $\mathbb{P}$-a.s. for a continuous process Y. Thus, $\mathscr{A}^i$ and λ^i in IIM2 have similar properties compared to IIM1. For instance, $\mathscr{A}^i$ is absolutely continuous and

$$\lambda_t^i = \frac{\int_0^\infty f_t^i(x)d\Psi^i(x)}{1-\int_0^\infty F_t^i(x)d\Psi^i(x)} = \frac{\int_0^\infty f_t^i(x)d\Psi^i(x)}{1-\int_0^\infty F_t^i(x)d\Psi^i(x)}I_{\{T_{i-1}<t\}} \quad \mathbb{P}-\text{a.s.}$$

$F^i(x)$ defined in Equation (3.12) is continuous and $f^i(x)$ is given in Equation (3.15) by

$$f_t^i(x) = g^i_{t-T_{i-1}}e^{-2\mu x/\sigma^2}\frac{x}{\sigma}(G^i_{t-T_{i-1}})^{-3/2}\varphi\left(\frac{-x+\mu G^i_{t-T_{i-1}}}{\sigma\sqrt{G^i_{t-T_{i-1}}}}\right)I_{\{T_{i-1}<t\}} \quad \mathbb{P}-\text{a.s.}$$

for $x>0$. Since G^i has a right-continuous density process g^i by Assumption 3.4.14, we know from Lemma 3.4.15 that $f^i(x)$ is also right-continuous. Lebesgue's dominated convergence theorem yields that λ^i is right-continuous, too.

If g^i is additionally $\mathbb{G}^{i-1}$-predictable, then $f^i(x)$ is $\mathbb{F}^{i-1}$-predictable for each $x>0$ due to Lemma 3.4.15. Note that $[f_t^i(\cdot)](\omega)$ is for $\mathbb{P}$-a.e. $\omega\in\Omega$ continuous and define $f_t^i(0) := \lim_{x\downarrow 0} f_t^i(x) = 0$. Then for every $y>0$, the Lebesgue-Stieltjes integral satisfies

$$\sum_{y_k^m, y_{k+1}^m\in\pi_m}[f_t^i(z_k^m)](\omega)(\Psi^i(y_{k+1}^m)-\Psi^i(y_k^m)) \to \int_0^y [f_t^i(x)](\omega)d\Psi^i(x) \quad (m\to\infty)$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$ and all $t>0$. Here, $\pi_m=\{y_0^m,y_1^m,\dots,y_m^m\}$ is a sequence of finite partitions of $[0,y]$, i.e., $0=y_0^m<y_1^m<\ldots<y_m^m=y$, with $\lim_{m\to\infty}(y_{k+1}^m - y_k^m)=0$ for all $k\in\{0,\dots,m-1\}$ and $z_k^m\in[y_k^m,y_{k+1}^m]$. This implies that the map $\int_0^y f^i(x)d\Psi^i(x):(0,\infty)\times\Omega\to\mathbb{R}$ is $\mathbb{F}^{i-1}$-predictable. Finally, it follows directly that $\int_0^\infty f^i(x)d\Psi^i(x):(0,\infty)\times\Omega\to\mathbb{R}$ is also $\mathbb{F}^{i-1}$-predictable.

Hence, the results from Subsection 3.4.3 are also satisfied in IIM2.

3.6. Examples of time changes

In this section we study the overall time change G in more detail. More precisely, we discuss explicit examples which are related to those in Giesecke and Tomecek (2005). All time changes are supposed to be absolutely continuous such that we have

$$G_t = \int_0^t g_s ds \quad \mathbb{P}-\text{a.s.} \quad \text{and} \quad G_t^i = \int_0^t g_s^i ds \quad \mathbb{P}-\text{a.s.} \tag{3.34}$$

for suitable processes g and g^i, $i \in \{1, \dots, n\}$. By Assumption 3.4.14, the density processes g^i are $\mathbb{G}^{i-1}$-adapted, right-continuous and positive.

In the following, we distinguish between two cases. In the first case, the filtration $\mathbb{K}$ is trivial and the time change G is deterministic between arrival times. In the second case, $\mathbb{K}$ is not trivial any more, which implies that G might depend on additional stochastic variables between the arrival times T_i, $i \in \{0, \dots, n\}$.

3.6.1. Preliminary remarks

By the definition of the overall time change in Subsection 3.2.4, we have

$$\begin{aligned} G_t = & \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i-1} G^k_{T_k - T_{k-1}} + G^i_{t-T_{i-1}} \right) I_{\{T_{i-1} \le t < T_i\}} \\ & + \left(\sum_{k=1}^{n-1} G^k_{T_k - T_{k-1}} + G^n_{t-T_{n-1}} \right) I_{\{T_{n-1} \le t\}}. \end{aligned} \tag{3.35}$$

Since G and G^i, $i \in \{1, \dots, n\}$, satisfy (3.34), this implies

$$g_t = \sum_{i=1}^{n-1} g^i_{t-T_{i-1}} I_{\{T_{i-1} \le t < T_i\}} + g^n_{t-T_{n-1}} I_{\{T_{n-1} \le t\}}. \tag{3.36}$$

Because all g^i are right-continuous and positive, so is g. Moreover, in a similar way to the proof of Lemma 3.2.11 we can show that $g^i_{t-T_{i-1}} I_{\{T_{i-1} \le t\}}$ is $\mathcal{F}_t^{i-1} (\subset \mathcal{G}_t)$-measurable for each $i \in \{1, \dots, n\}$ and $t \ge 0$, which implies that g is $\mathbb{G}$-adapted.

In this section we specify the time changes G^i and their corresponding density processes g^i for $i \in \{1, \dots, n\}$. Since the time change G^i is only relevant on subsets of $\{T_{i-1} \le t\}$ in Equation (3.35), we specify G^i on the set $\{T_{i-1} < \infty\}$ and define

$$G_t^i := G_t^i I_{\{T_{i-1} < \infty\}} + t I_{\{T_{i-1} = \infty\}}. \tag{3.37}$$

Note that T_{i-1} is measurable with respect to $\mathcal{G}_t^{i-1} = \mathcal{F}^{i-1}_{T_{i-1}+t}$ for each $t \ge 0$ such that the sets $\{T_{i-1} < \infty\}$ and $\{T_{i-1} = \infty\}$ are elements in $\mathcal{G}_t^{i-1}$ for all $t \ge 0$. A consistent density process to G^i defined in (3.37) is given by

$$g_t^i = g_t^i I_{\{T_{i-1} < \infty\}} + I_{\{T_{i-1} = \infty\}}. \tag{3.38}$$

Now, let us consider the following properties of G^i, $i \in \{1, \dots, n\}$:

(i) $G^i I_{\{T_{i-1} < \infty\}}$ is $\mathbb{G}^{i-1}$-adapted.

(ii) $G^i I_{\{T_{i-1}<\infty\}}$ is absolutely continuous.
(iii) $G^i_t I_{\{T_{i-1}<\infty\}} < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.
(iv) $\lim_{t\to\infty} G^i_t = \infty$ $\mathbb{P}$-a.s. on $\{T_{i-1} < \infty\}$.

If G^i defined in (3.37) satisfies (i)-(iv), then G^i meets all requirements from Definition 3.2.7.

3.6.2. Time change that is deterministic between arrival times

In this subsection we focus on time changes that are deterministic between arrival times. Thus, we assume that $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$ is trivial. This means that $\mathcal{K}_t = \mathcal{N}$ for each $t \geq 0$ where $\mathcal{N}$ denotes the set of all $\mathbb{P}$-null sets of $\mathcal{A}$.

We adapt the idea of Giesecke and Tomecek (2005) and study time changes G^i, $i \in \{1, \dots, n\}$, of the form

$$G^i_t = \int_0^t \nu^0_{s+T_{i-1}} ds + \sum_{k=1}^{i-1} \int_0^t \nu^k_{s+T_{i-1}-T_k} ds \quad \text{on } \{T_{i-1} < \infty\} \tag{3.39}$$

for specific processes ν^i for $i \in \{0, \dots, n-1\}$. The next definition specifies these so called impact processes.

Definition 3.6.1. *For each $i \in \{0, \dots, n-1\}$, the* impact process *ν^i is given by $\nu^0_t := g^1_t$ and*

$$\nu^i_t := \begin{cases} g^{i+1}_t - g^i_{t+T_i-T_{i-1}} & \textit{on } \{T_i < \infty\} \\ 0 & \textit{on } \{T_i = \infty\} \end{cases}$$

for $i \in \{1, \dots, n-1\}$.

Note that the density process of the overall time change G satisfies for $i \in \{1, \dots, n-1\}$

$$\begin{aligned} g_t &= g^i_{t-T_{i-1}} \quad \text{on } \{T_{i-1} \leq t < T_i\}, \\ g_{T_i+t} &= g^{i+1}_t \qquad \text{on } \{T_i \leq T_i + t < T_{i+1}\} \cap \{T_i < \infty\}. \end{aligned}$$

Hence, it follows for each $i \in \{1, \dots, n-1\}$

$$\Delta g_{T_i} = g^{i+1}_0 - g^i_{T_i-T_{i-1}} = \nu^i_0 \quad \text{on } \{T_i < \infty\}.$$

This explains why the processes ν^i, $i \in \{1, \dots, n-1\}$, are called impact processes: The density process g jumps at time T_i by ν^i_0 on $\{T_i < \infty\}$, and ν^i_t encodes the *impact* of the ith default on g at time $T_i + t$. In case of $i = 0$, ν^0 is equal to the density process of the time change up to the first default. See also Giesecke and Tomecek (2005) for this interpretation of the impact processes.

Because after the nth default there is no possible future default in our underlying portfolio, it is not necessary to consider a jump of g at time T_n. Indeed, by construction, g does not jump at T_n: In case of $i = n$, we have $g_t = g^n_{t-T_{n-1}}$ on $\{T_{n-1} \leq t < T_n\}$ and $g_{T_n+t} = g^n_{T_n+t-T_{n-1}}$ on $\{T_n < \infty\}$. This means $\Delta g_{T_n} = g^n_{T_n-T_{n-1}} - g^n_{T_n-T_{n-1}} = 0$ on $\{T_n < \infty\}$.

Because of Definition 3.6.1, we obtain the following result.

Proposition 3.6.2. *For each $i \in \{0, \dots, n-1\}$, the impact process ν^i satisfies the following properties:*

1. *$\nu^i I_{\{T_i<\infty\}}$ is right-continuous and $\mathbb{G}^i$-adapted.*
2. *$\nu^i_t > -\sum_{k=0}^{i-1} \nu^k_{t+T_i-T_k}$ $\mathbb{P}$-a.s. on $\{T_i < \infty\}$ for all $t \geq 0$.*
3. *$\int_0^t \nu^i_s ds < \infty$ $\mathbb{P}$-a.s. on $\{T_i < \infty\}$ for all $t \geq 0$.*

Proof. Fix $t \geq 0$. The first and the last assertion are obvious for $i = 0$. Thus, let us consider $i \in \{1, \dots, n-1\}$. Right-continuity follows directly from the definition of ν^i, and the third assertion holds because G^i_t and G^{i+1}_t are finite for all $t \geq 0$. Note that $(t + T_i - T_{i-1})I_{\{T_{i-1}<\infty\}}$ is an $(\mathcal{F}^i_{T_{i-1}+s})_{s\geq 0}$-stopping time because

$$\begin{aligned}&\{(t + T_i - T_{i-1})I_{\{T_{i-1}<\infty\}} \leq s\}\\ &= \{T_{i-1} = \infty\} \cup (\{T_{i-1} < \infty\} \cap \{t + T_i \leq T_{i-1} + s\}) \in \mathcal{F}^i_{T_{i-1}+s} \quad \text{for all } s \geq 0.\end{aligned}$$

Since g^i is right-continuous and adapted with respect to the filtration $\mathbb{G}^{i-1} \subset (\mathcal{F}^i_{T_{i-1}+t})_{t\geq 0}$, the random variable $g^i_{(t+T_i-T_{i-1})I_{\{T_{i-1}<\infty\}}} I_{\{(t+T_i-T_{i-1})I_{\{T_{i-1}<\infty\}}<\infty\}}$ is $\mathcal{F}^i_{T_{i-1}+(t+T_i-T_{i-1})I_{\{T_{i-1}<\infty\}}}$-measurable. Furthermore, it follows from

$$T_{i-1} + (t + T_i - T_{i-1})I_{\{T_{i-1}<\infty\}} = T_{i-1}I_{\{T_{i-1}=\infty\}} + (T_i + t)I_{\{T_{i-1}<\infty\}} \leq T_i + t$$

that $g^i_{(t+T_i-T_{i-1})I_{\{T_{i-1}<\infty\}}} I_{\{(t+T_i-T_{i-1})I_{\{T_{i-1}<\infty\}}<\infty\}}$ is $\mathcal{F}^i_{T_i+t}(= \mathcal{G}^i_t)$-measurable. The inclusion

$$\{T_i < \infty\} \subset \{(t + T_i - T_{i-1})I_{\{T_{i-1}<\infty\}} < \infty\}$$

implies that

$$g^i_{(t+T_i-T_{i-1})I_{\{T_{i-1}<\infty\}}} I_{\{(t+T_i-T_{i-1})I_{\{T_{i-1}<\infty\}}<\infty\}} I_{\{T_i<\infty\}} = g^i_{t+T_i-T_{i-1}} I_{\{T_i<\infty\}}$$

is $\mathcal{G}^i_t$-measurable. Finally, we obtain that $\nu^i_t I_{\{T_i<\infty\}} = (g^{i+1}_t - g^i_{t+T_i-T_{i-1}})I_{\{T_i<\infty\}}$ is $\mathcal{G}^i_t$-measurable.

From now on, consider again $i \in \{0, \dots, n-1\}$. In order to prove the inequality in the second assertion, note that summing up over all ν^k for $k \in \{0, \dots, i\}$ yields on $\{T_i < \infty\}$

$$\nu^0_{t+T_i} + \sum_{k=1}^{i} \nu^k_{t+T_i-T_k} = g^1_{t+T_i} + \sum_{k=1}^{i}(g^{k+1}_{t+T_i-T_k} - g^k_{t+T_i-T_k+T_k-T_{k-1}}) = g^{i+1}_t. \tag{3.40}$$

The second assertion holds because g^{i+1} is positive. □

Remark 3.6.3. Equation (3.40) ensures that all $G^i_t = \int_0^t g^i_s ds$, $i \in \{1, \dots, n\}$, satisfy (3.39).

Finally, we are interested in how the density process g of the overall time change G depends on the impact processes. If we sum up over all ν^k for $k \in \{0, \dots, i-1\}$, we obtain on $\{T_{i-1} \leq t\}$

$$\nu^0_t + \sum_{k=1}^{i-1} \nu^k_{t-T_k} = g^1_t + \sum_{k=1}^{i-1}(g^{k+1}_{t-T_k} - g^k_{t-T_{k-1}}) = g^i_{t-T_{i-1}}.$$

Together with Equation (3.36), this yields

$$
\begin{aligned}
g_t &= \sum_{i=1}^{n-1}\left(\nu_t^0+\sum_{k=1}^{i-1}\nu_{t-T_k}^k\right)I_{\{T_{i-1}\le t<T_i\}}+\left(\nu_t^0+\sum_{k=1}^{n-1}\nu_{t-T_k}^k\right)I_{\{T_{n-1}\le t\}}\\
&= \nu_t^0+\sum_{k=1}^{n-1}\nu_{t-T_k}^k I_{\{T_k\le t\}}.
\end{aligned}
\tag{3.41}
$$

In the remaining part of this subsection we determine the time change G and the time changes G^i for $i \in \{1,\dots,n\}$ by specifying impact processes ν^i for $i \in \{0,\dots,n-1\}$ that satisfy the properties from Proposition 3.6.2 and additionally guarantee that $\lim_{t\to\infty} G_t^{i+1} = \infty$ $\mathbb{P}$-a.s. on $\{T_i < \infty\}$. Note that if ν^i, $i \in \{0,\dots,n-1\}$, meet all these requirements, then G^i, $i \in \{1,\dots,n\}$, given in Equation (3.39) satisfy (i)-(iv) defined in Subsection 3.6.1.

Giesecke and Tomecek (2005) state examples of positive and negative impact processes; see Examples 3.11 and 3.12 in Giesecke and Tomecek (2005) and Examples 3.6.4 and 3.6.6 below. In the following, we take a closer look at their proposed impact processes. Since we would like to model the situation in which one default in our portfolio increases the likelihood of further defaults, the case of positive impact processes is more relevant to us. Hence, we add another example of positive impact processes and then discuss differences and similarities.

Example 3.6.4 (Positive impact process I)**.** Define $\nu_t^i := \alpha_i e^{-\beta_i t}$ on $\{T_i < \infty\}$ with $\alpha_i \ge 0$ and $\beta_i > 0$ for $i \in \{1,\dots,n-1\}$ and set $\nu_t^0 := \alpha_0 e^{-\beta_0 t} = \alpha_0$ with $\beta_0 := 0$ and $\alpha_0 > 0$ constant. Moreover, α_i and β_i may depend on $T_0,\dots,T_i$.

Note that it is obvious that these ν^i, $i \in \{0,\dots,n-1\}$, satisfy the conditions from Proposition 3.6.2 and additionally guarantee that $\lim_{t\to\infty} G_t^{i+1} = \infty$ $\mathbb{P}$-a.s. on $\{T_i < \infty\}$. Moreover, G^i, $i \in \{1,\dots,n\}$, is on $\{T_{i-1} < \infty\}$ given by

$$
\begin{aligned}
G_t^i &= \int_0^t \sum_{k=0}^{i-1}\nu_{s+T_{i-1}-T_k}^k ds = \alpha_0 t+\sum_{k=1}^{i-1}\int_0^t \alpha_k e^{-\beta_k(s+T_{i-1}-T_k)}ds\\
&= \alpha_0 t+\sum_{k=1}^{i-1}\left(\frac{\alpha_k}{\beta_k}e^{-\beta_k(T_{i-1}-T_k)}-\frac{\alpha_k}{\beta_k}e^{-\beta_k(t+T_{i-1}-T_k)}\right),
\end{aligned}
\tag{3.42}
$$

and g^i satisfies on $\{T_{i-1} < \infty\}$

$$
g_t^i = \sum_{k=0}^{i-1}\nu_{t+T_{i-1}-T_k}^k = \sum_{k=0}^{i-1}\alpha_k e^{-\beta_k(t+T_{i-1}-T_k)}. \tag{3.43}
$$

Finally, Equation (3.41) yields

$$
g_t = \nu_t^0+\sum_{k=1}^{n-1}\nu_{t-T_k}^k I_{\{T_k\le t\}} = \alpha_0+\sum_{k=1}^{n-1}\alpha_k e^{-\beta_k(t-T_k)}I_{\{T_k\le t\}} = \sum_{k=0}^{n-1}\alpha_k e^{-\beta_k(t-T_k)}I_{\{T_k\le t\}}. \tag{3.44}
$$

Example 3.6.5 (Positive impact process II)**.** Define $\nu_t^i := \gamma_i(t+1)^{-\delta_i}$ on $\{T_i < \infty\}$ with $\gamma_i \geq 0$ and $\delta_i > 1$ for $i \in \{1, \ldots, n-1\}$ and set $\nu_t^0 := \gamma_0(t+1)^{-\delta_0} = \gamma_0$ with $\delta_0 := 0$ and $\gamma_0 > 0$ constant. Again, γ_i and δ_i may depend on $T_0, \ldots, T_i$.

Again, the conditions from Proposition 3.6.2 are satisfied and $\lim_{t\to\infty} G_t^{i+1} = \infty$ $\mathbb{P}$-a.s. on $\{T_i < \infty\}$. More precisely, G^i, $i \in \{1, \ldots, n\}$, is on $\{T_{i-1} < \infty\}$ given by

$$\begin{aligned}
G_t^i &= \gamma_0 t + \sum_{k=1}^{i-1} \int_0^t \gamma_k \left(s + T_{i-1} - T_k + 1\right)^{-\delta_k} ds \\
&= \gamma_0 t + \sum_{k=1}^{i-1} \left(\frac{\gamma_k}{-\delta_k + 1} (t + T_{i-1} - T_k + 1)^{-\delta_k + 1} - \frac{\gamma_k}{-\delta_k + 1} (T_{i-1} - T_k + 1)^{-\delta_k + 1} \right).
\end{aligned}$$

Furthermore, g^i satisfies on $\{T_{i-1} < \infty\}$

$$g_t^i = \sum_{k=0}^{i-1} \gamma_k (t + T_{i-1} - T_k + 1)^{-\delta_k}, \tag{3.45}$$

and g is given by

$$g_t = \sum_{k=0}^{n-1} \gamma_k (t - T_k + 1)^{-\delta_k} I_{\{T_k \leq t\}}.$$

Example 3.6.6 (Negative impact process)**.** Consider a constant impact process ν^0 that satisfies $\nu^0 > 0$ and $p_1, \ldots, p_{n-1} \in (0,1)$. We obtain a negative impact process by setting $\nu_t^i := -\nu^0(1-p_i)\prod_{j=1}^{i-1} p_j$ on $\{T_i < \infty\}$ for $i \in \{1, \ldots, n-1\}$. Moreover, p_i may depend on $T_0, \ldots, T_i$.

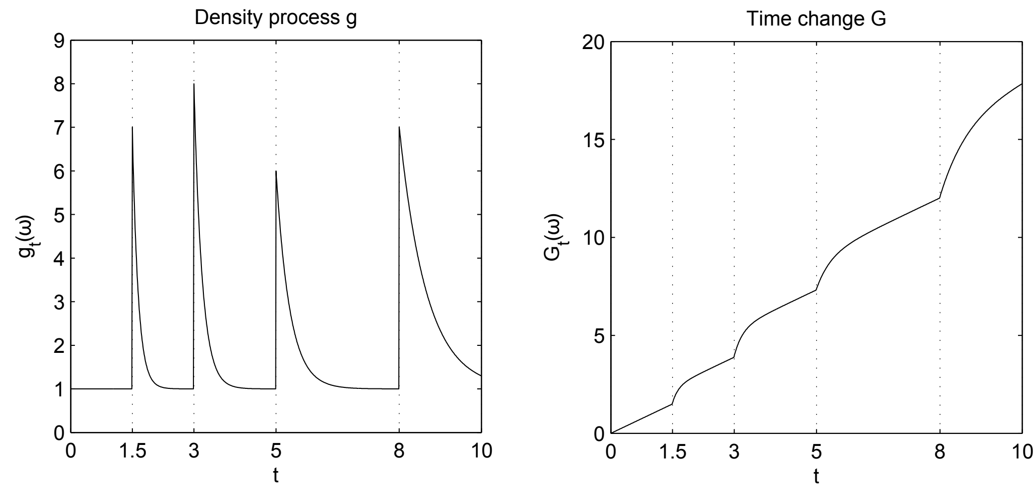
The impact processes ν^i, $i \in \{1, \ldots, n-1\}$, defined above obviously satisfy the first and third condition from Proposition 3.6.2 and guarantee that $\lim_{t\to\infty} G_t^{i+1} = \infty$ $\mathbb{P}$-a.s. on $\{T_i < \infty\}$. The second condition from Proposition 3.6.2 is satisfied since on $\{T_i < \infty\}$, we have

$$-\sum_{k=0}^{i-1} \nu_{t+T_i-T_k}^k = -\nu^0 + \sum_{k=1}^{i-1} \nu^0(1-p_k) \prod_{j=1}^{k-1} p_j = -\nu^0 \prod_{j=1}^{i-1} p_j < \nu_t^i.$$

Remark 3.6.7. Suppose that $g^i I_{\{T_{i-1}<\infty\}}$ is given by (3.43) or by (3.45). Then g^i specified in Equation (3.38) is continuous and $\mathbb{G}^{i-1}$-adapted, and hence $\mathbb{G}^{i-1}$-predictable. Moreover, g^i is bounded.

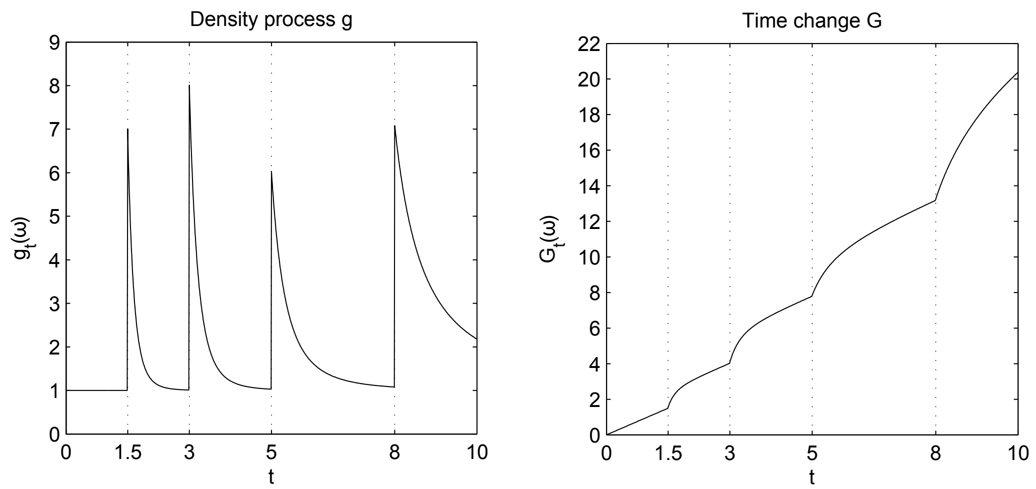
Figure 3.6.1 shows a possible path of g and G for the exponential impact processes from Example 3.6.4 and Figure 3.6.2 shows a realization of Example 3.6.5. In both figures we have identical default times $T_0 = 0$, $T_1 = 1.5$, $T_2 = 3$, $T_3 = 5$ and $T_4 = 8$, and the parameters of the time change satisfy $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for $i \in \{0, \ldots, 4\}$. We see that at the default time T_i, the path of the density process jumps by α_i (γ_i). As a consequence, the graph of the corresponding time change gets steeper after this default, i.e., time evolves faster. After the jump by α_i (γ_i), the density decreases again. This is controlled by the choice of the parameter β_i in case of Figure 3.6.1 and by the parameter δ_i in case of Figure 3.6.2: The greater β_i (δ_i), the faster decreases

Figure 3.6.1.: Example I: density process g and time change G for a portfolio of size $n > 4$



The default times are $T_0 = 0$, $T_1 = 1.5$, $T_2 = 3$, $T_3 = 5$ and $T_4 = 8$. The parameters of the time change are given by $\alpha_0 = 1$, $\alpha_1 = 6$, $\alpha_2 = 7$, $\alpha_3 = 5$, $\alpha_4 = 6$, $\beta_0 = 0$, $\beta_1 = 7$, $\beta_2 = 5$, $\beta_3 = 3$ and $\beta_4 = 1.5$.

Figure 3.6.2.: Example II: density process g and time change G for a portfolio of size $n > 4$



The default times are $T_0 = 0$, $T_1 = 1.5$, $T_2 = 3$, $T_3 = 5$ and $T_4 = 8$. The parameters of the time change are given by $\gamma_0 = 1$, $\gamma_1 = 6$, $\gamma_2 = 7$, $\gamma_3 = 5$, $\gamma_4 = 6$, $\delta_0 = 0$, $\delta_1 = 7$, $\delta_2 = 5$, $\delta_3 = 3$ and $\delta_4 = 1.5$.

the density process g after T_i. This shape of g is plausible for our model: Directly after a default, which occurs by surprise, there is uncertainty in the market. This uncertainty results in a financial time evolving faster than usual. After some time to adapt to the new situation, the market becomes more stable, and hence financial time evolves slower again.

If we compare the density processes of Figure 3.6.1 and Figure 3.6.2, then they look very similar at the first glance. But if we look more closely, then we see that in Figure 3.6.1 g decreases faster after the defaults. This is a natural consequence of the exponential form of the impact processes. Moreover, this also implies that in case of $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ and identical default times, the corresponding path of the time change in Example 3.6.4 is always greater than or equal to the corresponding path of the time change in Example 3.6.5. Indeed, this holds since $e^{-\beta x} \leq (x+1)^{-\beta}$ for $\beta \geq 0$ and each $x \geq 0$, which implies

$$\sum_{k=0}^{n-1} \alpha_k e^{-\beta_k (t - T_k(\omega))} I_{\{T_k(\omega) \leq t\}} \leq \sum_{k=0}^{n-1} \alpha_k (t - T_k(\omega) + 1)^{-\beta_k} I_{\{T_k(\omega) \leq t\}}.$$

In case of IIM1, Figure 3.6.3 shows the dependence of the conditional survival probability $\mathbb{P}[T_2 > t|\mathcal{F}_t^1]$ for $T_1 = 1$ on the parameters of the time change defined in Example 3.6.4. Since we set $T_1 = 1$, this conditional survival probability is equal to 1 up to time $T_1 = 1$. We see that the conditional survival probability decreases faster over the course of time if α_1 is higher. This comes up to our expectations since the higher the jump of g at time T_1, the faster the time change G evolves after the first default event and the likelier is the next default. The dependence on β_1 shows a reverse situation. The higher β_1, the steeper is the decrease of g after the first default in Figure 3.6.1. Therefore, the conditional survival probability increases with β_1.

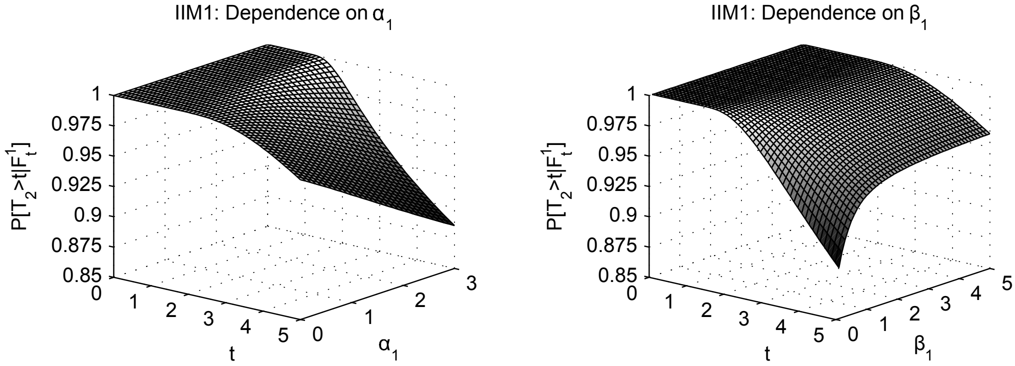
In IIM2 the (conditional) survival probability additionally depends on the parameters of the barrier distribution. Figure 3.6.4 shows this dependence of $\mathbb{P}[T_1 > t|\mathcal{F}_t^0] = \mathbb{P}[T_1 > t]$ in case of a piecewise deterministic time change with parameters $\alpha_0 = 1$ and $\beta_0 = 0$.

Simulating default times

We have constructed a top down first-passage model of default with features of reduced form models. Hence, we can simulate our totally inaccessible default times $T_1, \dots, T_n$ by an algorithm that is typical for reduced form models; see, for instance, Section 5.3 and 7.7 in Schönbucher (2003) for the case of doubly stochastic Poisson processes. A very similar algorithm was used in Giesecke and Goldberg (2004b) to generate the totally inaccessible default times in their bottom up first-passage model.

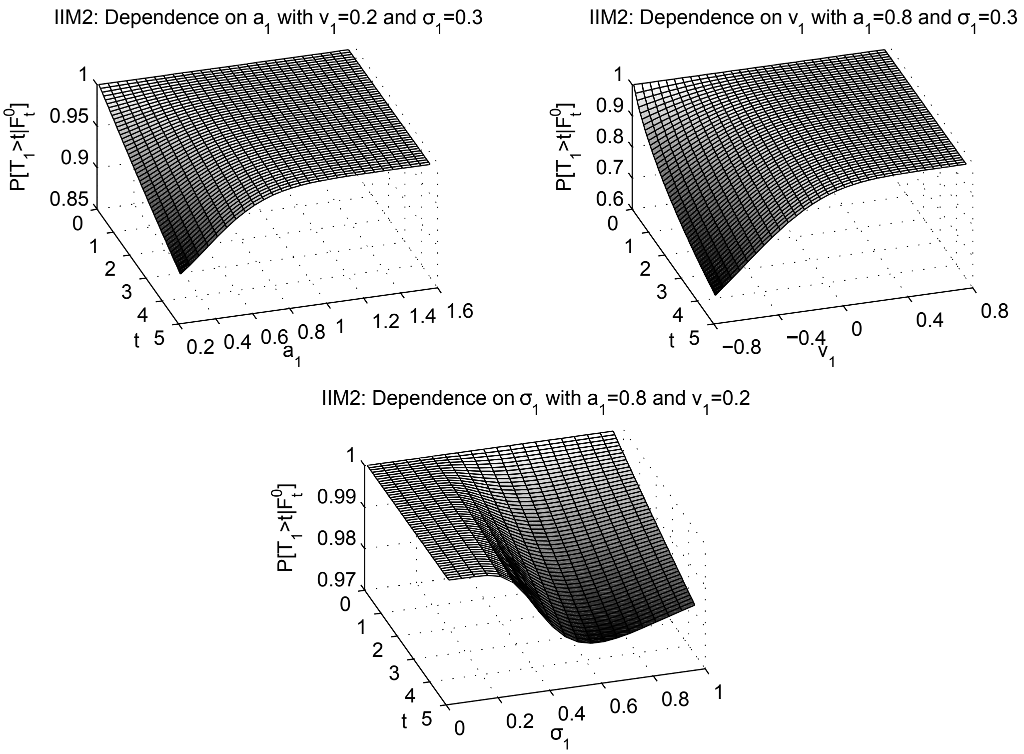
IIM1 and IIM2 have in common that there is no available information about the portfolio value process. As a consequence, the ith default trends $\mathscr{A}^i$, $i \in \{1, \dots, n\}$, do not depend on this process, which leads to a tractable simulation process. The

Figure 3.6.3.: Conditional survival probability $\mathbb{P}[T_2 > t|\mathcal{F}_t^1]$ for $T_1 = 1$ in IIM1 with time change that is deterministic between arrival times



In both figures we have $V_0 = 100$, $\mu_V = 0.01$, $\sigma_V = 0.1$, $K^1 = 60$, $K^2 = 40$, $\alpha_0 = 1$ and $\beta_0 = 0$. On the left hand side, we set $\beta_1 = 0.8$; on the right hand side, we set $\alpha_1 = 1$.

Figure 3.6.4.: Survival probability $\mathbb{P}[T_1 > t|\mathcal{F}_t^0] = \mathbb{P}[T_1 > t]$ in IIM2 with time change that is deterministic between arrival times



In all subfigures we have $V_0 = 100$, $\mu_V = 0.01$, $\sigma_V = 0.1$, $\alpha_0 = 1$ and $\beta_0 = 0$.

explicit form of $\mathscr{A}^i$ is given by

$$\mathscr{A}_t^i = -\log\left(\Phi\left(\frac{\kappa^i + \mu G^i_{t-T_{i-1}}}{\sigma\sqrt{G^i_{t-T_{i-1}}}}\right) - e^{-2\mu\kappa^i/\sigma^2}\Phi\left(\frac{-\kappa^i + \mu G^i_{t-T_{i-1}}}{\sigma\sqrt{G^i_{t-T_{i-1}}}}\right)\right) I_{\{T_{i-1}<t\}} \tag{3.46}$$

$\mathbb{P}$-a.s. in IIM1 (see Equation (3.18)) and by

$$\mathscr{A}_t^i = -\log\left(1 - \frac{\Upsilon^i(G^i_{t-T_{i-1}})}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_i v_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}\right) I_{\{T_{i-1}<t\}} \tag{3.47}$$

$\mathbb{P}$-a.s. in IIM2 (see Equation (3.33)) where $\Upsilon^i(t)$ is specified in Corollary 3.5.2. Moreover, the time changes G^i are given in Equation (3.39) and satisfy in case of Example 3.6.4 on $\{T_{i-1} < \infty\}$

$$G_t^i = \int_0^t \sum_{k=0}^{i-1} \nu^k_{s+T_{i-1}-T_k} ds = \alpha_0 t + \sum_{k=1}^{i-1}\left(\frac{\alpha_k}{\beta_k} e^{-\beta_k(T_{i-1}-T_k)} - \frac{\alpha_k}{\beta_k} e^{-\beta_k(t+T_{i-1}-T_k)}\right).$$

We generate each default time conditioned on the available information. In case of IIM1 and IIM2, this means that we generate the ith default, $i \in \{1,\dots,n\}$, conditioned on the information that is available up to the $(i-1)$st default because $\mathbb{K}$ is assumed to be trivial. In detail, the algorithm for simulating default times $\underset{\sim}{T}_1,\dots,\underset{\sim}{T}_n$ reads as follows:

1. Initialize $i = 1$ and $\underset{\sim}{T}_0 = 0$.
2. Simulate an independent standard uniform random variable U^i.
3. Set $\underset{\sim}{T}_i = \inf\{t > \underset{\sim}{T}_{i-1} | \underset{\sim}{\mathscr{A}}_t^i \geq -\log U^i\}$.
4. If $i = n$, then stop, else set $i = i + 1$.

$\underset{\sim}{\mathscr{A}}_t^i$ in step three is given as the ith default trend $\mathscr{A}_t^i$ (see (3.46) or (3.47)) where T_j is replaced by $\underset{\sim}{T}_j$ for $j \in \{0,\dots,i-1\}$.

3.6.3. Time change that is stochastic between arrival times

To construct a time change G that is stochastic between arrival times, we can apply the two fold time change used by Giesecke and Tomecek (2005) to construct time changed Hawkes processes. An important advantage of time changes of this specific type is an easier simulation of the default times T_i, $i \in \{1,\dots,n\}$, compared to models with a general time change.

As already mentioned at the beginning of this section, the time change might depend on additional stochastic factors apart from $T_1,\dots,T_{n-1}$ if $\mathbb{K}$ is not trivial any more. We assume that $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$ is generated by a stochastic process Y. A possible example of Y is a Brownian motion B. Note that $\mathcal{K}_\infty$ has to be $\mathbb{P}$-independent of $\sigma(W_s : s \geq 0)$ by assumption, which implies for $Y = B$ that W and B are independent Brownian motions.

Giesecke and Tomecek (2005) propose a time change G that satisfies

$$G_t = M_{H_t}$$

where M and H are both appropriately measurable, absolutely continuous time changes. In the subsequent section we show that if M is constructed similarly to G in Subsection 3.6.2 and if, given $\mathcal{K}_0$, H is an independent, absolutely continuous, $\mathbb{K}$-adapted time change, then $(M_{H_t})_{t\geq 0}$ satisfies Definition 3.2.7, and hence $(M_{H_t})_{t\geq 0}$ is adapted with respect to $\mathbb{G}$.

From now on, let H be an absolutely continuous time change admitting the form

$$H_t = \int_0^t h_s ds$$

for a $\mathbb{K}$-adapted process h such that H satisfies $H_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t\to\infty} H_t = \infty$ $\mathbb{P}$-a.s. In analogy to G in the previous subsection, let M be defined by

$$M_t := \begin{cases} \sum_{k=1}^{i-1} M^k_{\bar{T}_k - \bar{T}_{k-1}} + M^i_{t-\bar{T}_{i-1}} & \text{on } \{\bar{T}_{i-1} \leq t < \bar{T}_i\} \text{ for } i \in \{1,\dots,n\} \\ \sum_{k=1}^{n-1} M^k_{\bar{T}_k - \bar{T}_{k-1}} + M^n_{t-\bar{T}_{n-1}} & \text{on } \{\bar{T}_n \leq t\} \end{cases}$$

where $\bar{T}_i$ is given by $\bar{T}_i := M^{-1}_{S_i}$ for $i \in \{0,\dots,n\}$ (M^{-1} is the inverse process of M) and $M^1,\dots,M^n$ are absolutely continuous time changes that satisfy $M^i_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t\to\infty} M^i_t = \infty$ $\mathbb{P}$-a.s. for all $i \in \{1,\dots,n\}$. Additionally, we assume that each M^i, $i \in \{1,\dots,n\}$, is adapted to the filtration $\bar{\mathbb{G}}^{i-1} := (\bar{\mathcal{G}}^{i-1}_t)_{t\geq 0}$ where $\bar{\mathcal{G}}^{i-1}_t := \bar{\mathcal{F}}^{i-1}_{\bar{T}_{i-1}+t}$ and $\bar{\mathcal{F}}^{i-1}_t := \mathcal{K}_0 \vee \sigma(I_{\{\bar{T}_k \leq s\}} : s \leq t, k \leq i-1)$.

Now, let us consider the process G defined by $G_t := M_{H_t}$. Obviously, G is an absolutely continuous time change with $G_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t\to\infty} G_t = \infty$ $\mathbb{P}$-a.s. Using this time change to define the portfolio value process V_G leads to

$$T_i = G^{-1}_{S_i} = H^{-1}_{M^{-1}_{S_i}} \quad \text{for } i \in \{0,\dots,n\}$$

(H^{-1} is the inverse process of H). It follows

$$\bar{T}_i = M^{-1}_{S_i} = H_{T_i} \quad \text{for } i \in \{0,\dots,n\}.$$

In the following, we verify that G admits the representation from Definition 3.2.7.

Lemma 3.6.8. *For each $i \in \{1,\dots,n\}$, the process G^i defined by*

$$G^i_t := (M^i_{H_{t+T_{i-1}} - H_{T_{i-1}}}) I_{\{T_{i-1}<\infty\}} + t I_{\{T_{i-1}=\infty\}} \tag{3.48}$$

is a $\mathbb{G}^{i-1}$-adapted, absolutely continuous time change with $G^i_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $\lim_{t\to\infty} G^i_t = \infty$ $\mathbb{P}$-a.s.

Proof. Fix $i \in \{1,\dots,n\}$. Since M^i and H are time changes, G^i is strictly increasing and satisfies $G^i_0 = 0$. Absolute continuity, finiteness and $\lim_{t\to\infty} G^i_t = \infty$ $\mathbb{P}$-a.s. follow directly from the corresponding properties of M^i and H. To prove that G^i is $\mathbb{G}^{i-1}$-adapted, we have to show that $\{G^i_t \leq s\} \in \mathcal{F}^{i-1}_{T_{i-1}+t}$ for all $s \geq 0$ and $t \geq 0$. Hence, fix $s \geq 0$ and $t \geq 0$ and note that the properties of M^i imply

$$\{G^i_t \leq s\} = (\{H_{t+T_{i-1}} - H_{T_{i-1}} \leq (M^i)^{-1}_s\} \cap \{T_{i-1} < \infty\}) \cup (\{t \leq s\} \cap \{T_{i-1} = \infty\}).$$

Thus, we have to prove that

$$\{H_{t+T_{i-1}}I_{\{T_{i-1}<\infty\}} - H_{T_{i-1}}I_{\{T_{i-1}<\infty\}} \le (M^i)_s^{-1}\} \cap \{T_{i-1} < \infty\} \in \mathcal{F}^{i-1}_{T_{i-1}+t}. \quad (3.49)$$

Since M^i_u is $\bar{\mathcal{F}}^{i-1}_\infty (= \mathcal{K}_0 \vee \sigma(\bar{T}_j : j \le i-1))$-measurable for each $u \ge 0$, we have

$$\{(M^i)_s^{-1} \le u\} = \{s \le M^i_u\} \in \mathcal{K}_0 \vee \sigma(\bar{T}_j : j \le i-1) \quad \text{for each } u \ge 0.$$

It follows that $(M^i)_s^{-1}$ is $\mathcal{K}_0 \vee \sigma(\bar{T}_j : j \le i-1)$-measurable. Moreover, we know that H is $\mathbb{K}(\subset \mathbb{F}^{i-1})$-adapted and that $T_0, \dots, T_{i-1}$ and $T_{i-1}+t$ are $\mathbb{F}^{i-1}$-stopping times. This implies that $H_{T_k}I_{\{T_k<\infty\}}$ is $\mathcal{F}^{i-1}_{T_k}$-measurable for each $k \in \{0,\dots,i-1\}$ and that $H_{T_{i-1}+t}I_{\{T_{i-1}<\infty\}}$ is measurable with respect to $\mathcal{F}^{i-1}_{T_{i-1}+t}$. Since $\bar{T}_k = H_{T_k}I_{\{T_k<\infty\}} + \infty I_{\{T_k=\infty\}}$ for each $k \in \{0,\dots,i-1\}$, this means that $\bar{T}_0,\dots,\bar{T}_{i-1}$ are measurable with respect to $\mathcal{F}^{i-1}_{T_{i-1}+t}$. To sum up, we obtain:

1. $(M^i)_s^{-1}$ is measurable with respect to $\mathcal{K}_0 \vee \sigma(\bar{T}_j : j \le i-1) \subset \mathcal{F}^{i-1}_{T_{i-1}} \subset \mathcal{F}^{i-1}_{T_{i-1}+t}$.
2. $H_{t+T_{i-1}}I_{\{T_{i-1}<\infty\}} - H_{T_{i-1}}I_{\{T_{i-1}<\infty\}}$ is measurable with respect to $\mathcal{F}^{i-1}_{T_{i-1}+t}$.

But this implies (3.49). Therefore, G^i is $\mathbb{G}^{i-1}$-adapted. □

The following lemma specifies the overall time change which is obtained by using the time changes G^i, $i \in \{1,\dots,n\}$, given by Equation (3.48).

Lemma 3.6.9. *Suppose that G^i, $i \in \{1,\dots,n\}$, are given by Equation (3.48). Then $G_t = M_{H_t}$ satisfies*

$$G_t = \begin{cases} \sum_{k=1}^{i-1} G^k_{T_k-T_{k-1}} + G^i_{t-T_{i-1}} & \text{on } \{T_{i-1} \le t < T_i\} \text{ for } i \in \{1,\dots,n\} \\ \sum_{k=1}^{n-1} G^k_{T_k-T_{k-1}} + G^n_{t-T_{n-1}} & \text{on } \{T_n \le t\} \end{cases}$$
$$= \begin{cases} \sum_{k=1}^{i-1} M^k_{\bar{T}_k-\bar{T}_{k-1}} + M^i_{H_t-\bar{T}_{i-1}} & \text{on } \{\bar{T}_{i-1} \le H_t < \bar{T}_i\} \text{ for } i \in \{1,\dots,n\} \\ \sum_{k=1}^{n-1} M^k_{\bar{T}_k-\bar{T}_{k-1}} + M^n_{H_t-\bar{T}_{n-1}} & \text{on } \{\bar{T}_n \le H_t\} \end{cases}.$$

In particular, $(G_t)_{t\ge0} = (M_{H_t})_{t\ge0}$ is adapted with respect to $\mathbb{G}$.

Proof. For $j \in \{1,\dots,n\}$ consider $\sum_{k=1}^{j-1} G^k_{T_k-T_{k-1}} + G^j_{t-T_{j-1}}$ on $\{T_{j-1} \le t\}$. If we plug in (3.48), then we obtain

$$\sum_{k=1}^{j-1} G^k_{T_k-T_{k-1}} + G^j_{t-T_{j-1}} = \sum_{k=1}^{j-1} M^k_{H_{T_k-T_{k-1}+T_{k-1}}-H_{T_{k-1}}} + M^j_{H_{t-T_{j-1}+T_{j-1}}-H_{T_{j-1}}}$$
$$= \sum_{k=1}^{j-1} M^k_{\bar{T}_k-\bar{T}_{k-1}} + M^j_{H_t-\bar{T}_{j-1}}.$$

The assertion holds since $\bar{T}_i = H_{T_i}$ and since continuity and strict monotonicity of H imply $\{\bar{T}_{i-1} \le H_t < \bar{T}_i\} = \{T_{i-1} \le t < T_i\}$ for $i \in \{1,\dots,n\}$ and $\{\bar{T}_n \le H_t\} = \{T_n \le t\}$. □

Example 3.6.10. A possible example of h is a squared Brownian motion, i.e., $h = \hat{\sigma} B^2$ and

$$H_t = \hat{\sigma} \int_0^t B_s^2 ds \quad \text{for } \hat{\sigma} > 0.$$

Moreover, applying the results from Subsection 3.6.2 yields possible examples of M^i, $i \in \{1, \dots, n\}$. For instance, suppose that M^i on $\{\bar{T}_{i-1} < \infty\}$ is given by

$$\begin{aligned} M_t^i &= \int_0^t \sum_{k=0}^{i-1} \nu^k_{s+\bar{T}_{i-1}-\bar{T}_k} ds = \alpha_0 t + \sum_{k=1}^{i-1} \int_0^t \alpha_k e^{-\beta_k (s+\bar{T}_{i-1}-\bar{T}_k)} ds \\ &= \alpha_0 t + \sum_{k=1}^{i-1} \left(\frac{\alpha_k}{\beta_k} e^{-\beta_k(\bar{T}_{i-1}-\bar{T}_k)} - \frac{\alpha_k}{\beta_k} e^{-\beta_k(t+\bar{T}_{i-1}-\bar{T}_k)} \right) \end{aligned}$$

with $\alpha_0 > 0$, $\beta_0 = 0$, $\alpha_k \geq 0$ and $\beta_k > 0$ for $k \in \{1, \dots, n-1\}$. Note that the impact processes ν^k, $k \in \{0, \dots, n-1\}$, are defined as in Example 3.6.4. Furthermore, the properties of H and M imply $\{S_i < \infty\} = \{\bar{T}_i < \infty\} = \{T_i < \infty\}$ for all $i \in \{1, \dots, n\}$.

The density process of M is given by Equation (3.44) in Subsection 3.6.2. More precisely, we have $M_t = \int_0^t \eta_s ds$ with

$$\eta_t = \sum_{k=0}^{n-1} \alpha_k e^{-\beta_k(t-\bar{T}_k)} I_{\{\bar{T}_k \leq t\}}.$$

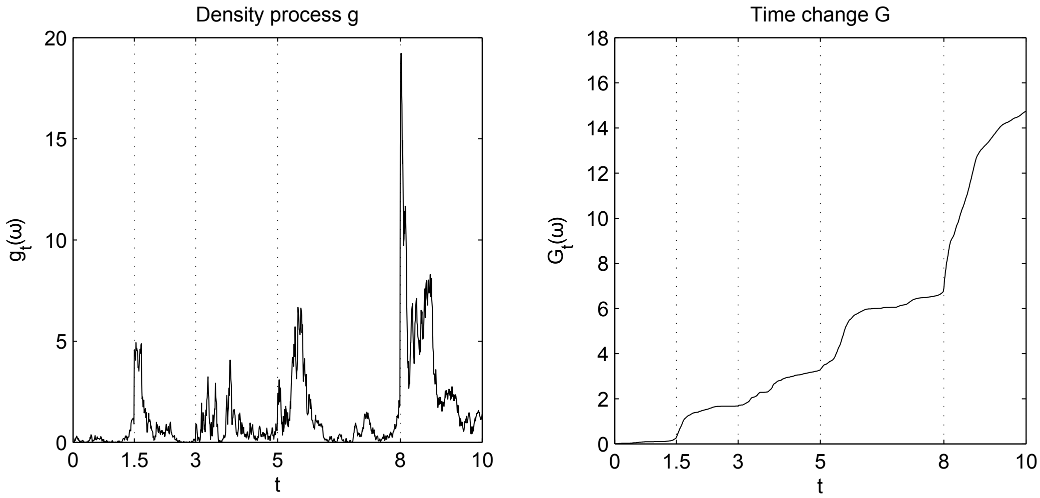
As a consequence, the overall time change G satisfies $G_t = M_{H_t} = \int_0^{H_t} \eta_s ds$, which implies that $G_t = \int_0^t g_s ds$ with

$$g_t = h_t \eta_{H_t} = h_t \sum_{k=0}^{n-1} \alpha_k e^{-\beta_k(H_t-\bar{T}_k)} I_{\{\bar{T}_k \leq H_t\}}.$$

Figure 3.6.5 shows a possible path of the density process g and the time change G in case of Example 3.6.10. Note that the parameters of the time change M correspond with the parameters in Figure 3.6.1. Because of the additional randomness in the density process g, the path of G evolves less smooth than in Figure 3.6.1. We can still observe changes in the slope of the time change after T_1 and T_4, but obviously, these are not only induced by the choice of the parameters α_i. In case of the arrival times T_2 and T_3, we see almost no immediate change in the slope of G any more. Hence, the influence of the impact processes ν_i is apparently smaller than in case of piecewise deterministic time changes.

Figure 3.6.6 shows in a similar way to Figure 3.6.3 the dependence of the conditional survival probability $\mathbb{P}[T_2 > t | \mathcal{F}_t^1]$ for $T_1 = 1$ on the parameters of G (in particular, the parameters of M) in case of IIM1. But now, we consider the time change introduced in Example 3.6.10 which is stochastic between arrival times. Figure 3.6.6 presents only one path of $\mathbb{P}[T_2 > t | \mathcal{F}_t^1]$ for $T_1 = 1$. We see immediately that the general shape is similar to the piecewise deterministic case, but the surface is less smooth, which is obviously a consequence of the additional randomness in the time change.

Figure 3.6.5.: Example of density process g and time change G that is stochastic between arrival times for a portfolio of size $n > 4$



The default times are $T_0 = 0$, $T_1 = 1.5$, $T_2 = 3$, $T_3 = 5$ and $T_4 = 8$. The parameters of the time change are given by $\alpha_0 = 1$, $\alpha_1 = 6$, $\alpha_2 = 7$, $\alpha_3 = 5$, $\alpha_4 = 6$, $\beta_0 = 0$, $\beta_1 = 7$, $\beta_2 = 5$, $\beta_3 = 3$, $\beta_4 = 1.5$ and $H_t = \hat{\sigma} \int_0^t B_s^2 ds$ with $\hat{\sigma} = 1$.

Simulating default times

Recall again that $\mathscr{A}^i$ satisfies

$$\mathscr{A}_t^i = -\log\left(\Phi\left(\frac{\kappa^i + \mu G^i_{t-T_{i-1}}}{\sigma\sqrt{G^i_{t-T_{i-1}}}}\right) - e^{-2\mu\kappa^i/\sigma^2}\Phi\left(\frac{-\kappa^i + \mu G^i_{t-T_{i-1}}}{\sigma\sqrt{G^i_{t-T_{i-1}}}}\right)\right) I_{\{T_{i-1}<t\}}$$

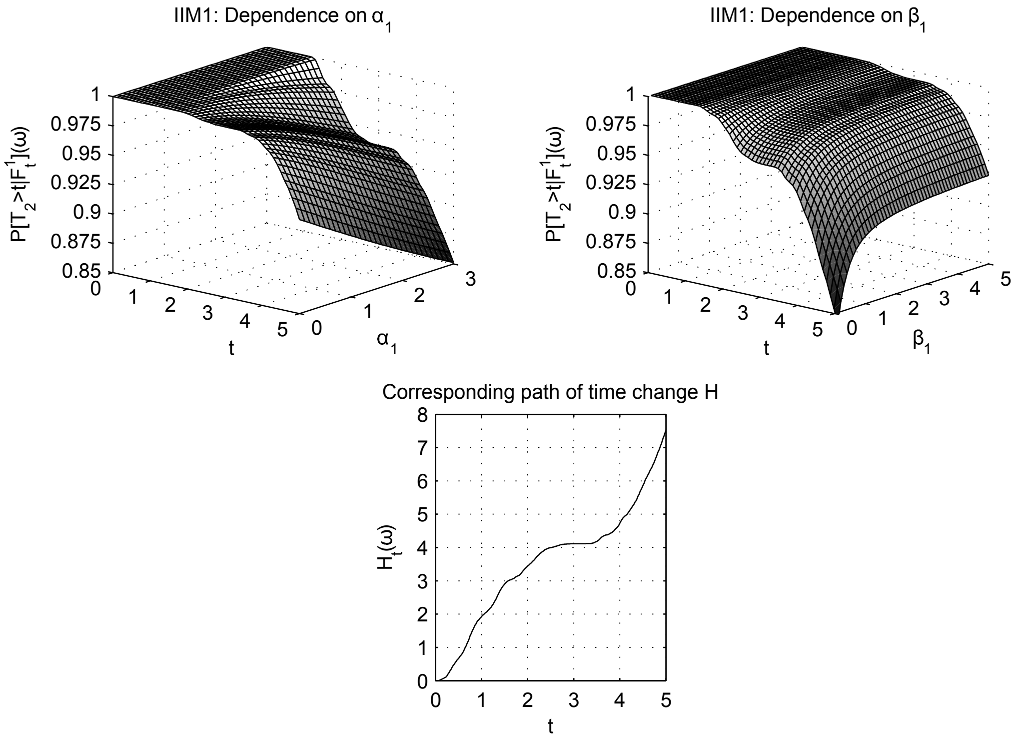
$\mathbb{P}$-a.s. in IIM1 (see Equation (3.18)) and

$$\mathscr{A}_t^i = -\log\left(1 - \frac{\Upsilon^i(G^i_{t-T_{i-1}})}{\Phi\left(\frac{a_i+v_i}{\sigma_i}\right) - e^{-2a_i v_i/\sigma_i^2}\Phi\left(\frac{v_i-a_i}{\sigma_i}\right)}\right) I_{\{T_{i-1}<t\}}$$

$\mathbb{P}$-a.s. in IIM2 (see Equation (3.33)). In Example 3.6.10 where $h = \hat{\sigma}B^2$ for an independent Brownian motion B, the time change G^i given in Equation (3.48) satisfies on $\{T_{i-1} < \infty\}$

$$\begin{aligned}G_t^i &= \alpha_0[H_{t+T_{i-1}} - H_{T_{i-1}}] + \sum_{k=1}^{i-1}\left(\frac{\alpha_k}{\beta_k}e^{-\beta_k(\bar{T}_{i-1}-\bar{T}_k)}\right.\\ &\quad\left. - \frac{\alpha_k}{\beta_k}e^{-\beta_k([H_{t+T_{i-1}}-H_{T_{i-1}}]+\bar{T}_{i-1}-\bar{T}_k)}\right)\\ &= \alpha_0\left[\hat{\sigma}\int_{T_{i-1}}^{t+T_{i-1}} B_s^2 ds\right] + \sum_{k=1}^{i-1}\left(\frac{\alpha_k}{\beta_k}e^{-\beta_k\hat{\sigma}\int_{T_k}^{T_{i-1}} B_s^2 ds} - \frac{\alpha_k}{\beta_k}e^{-\beta_k\hat{\sigma}\int_{T_k}^{t+T_{i-1}} B_s^2 ds}\right).\end{aligned}$$

Figure 3.6.6.: Path of conditional survival probability $\mathbb{P}[T_2 > t|\mathcal{F}_t^1]$ for $T_1 = 1$ in IIM1 with time change that is deterministic between arrival times



In both surface plots we have $V_0 = 100$, $\mu_V = 0.01$, $\sigma_V = 0.1$, $K^1 = 60$, $K^2 = 40$, $\alpha_0 = 1$ and $\beta_0 = 0$. On the left hand side, we set $\beta_1 = 0.8$; on the right hand side, we set $\alpha_1 = 1$. The third subfigure shows the corresponding path of the time change $H_t = \hat{\sigma} \int_0^t B_s^2 ds$ with $\hat{\sigma} = 1$.

If the overall time change G is defined by these G^i, $i \in \{1, \ldots, n\}$, then the density process of the time change H is an additional stochastic variable that also has to be simulated. In order to simulate the totally inaccessible default times in IIM1 and IIM2, we can now apply the algorithm from Subsection 3.6.2 with an additional step at the beginning:

1. Simulate a path of B.
2. Initialize $i = 1$ and $\underset{\sim}{T}_0 = 0$.
3. Simulate an independent standard uniform random variable U^i.
4. Set $\underset{\sim}{T}_i = \inf\{t > \underset{\sim}{T}_{i-1} | \underset{\sim}{\mathscr{A}}_t^i \geq -\log U^i\}$.
5. If $i = n$, then stop, else set $i = i + 1$.

The above two fold time change is more tractable than a general time change that is stochastic between arrival times. Giesecke and Tomecek (2005) point out that this time change is very applicable if we want to make a comparison between different parametrizations of the impact processes ν^i, $i \in \{0, \ldots, n-1\}$. This is illustrated in an example in which each parametrization corresponds to a specific assumption of how the default counting process N depends on prior defaults:

Let us consider k different parametrizations of the impact processes. With the algorithm described above we are able to simulate default times $(\underset{\sim}{T}_i^1)_{i\in\{1,\dots,n\}}$, ..., $(\underset{\sim}{T}_i^k)_{i\in\{1,\dots,n\}}$ for only one simulation of the process H by repeating 2.-5. for each parametrization. As a result, we can compare the different parametrizations by using the same simulation of H for all parametrizations. In addition, different results for $(\underset{\sim}{T}_i^1)_{i\in\{1,\dots,n\}}$, ... $(\underset{\sim}{T}_i^k)_{i\in\{1,\dots,n\}}$ can be attributed to different assumptions on ν^i, $i \in \{0,\dots,n-1\}$. In particular, different realizations of H do not complicate the comparison, which leads to a much easier situation for the modeler.

Part II.

Systemic risk measures

An important feature of our top down first-passage model is the self-affecting property of the default counting process, which means that a default can increase the likelihood of the next default. Thus, our model in Part I incorporates possible feedback of events to prices of credit sensitive securities. The reasons for this phenomenon are different forms of *contagion*. Direct contagion between counterparties is the most obvious form since the default of a firm leads to direct financial losses for its creditors and a loss of funding for its borrowers. But there are also several types of indirect contagion. Consider, for example, two firms A and B and suppose that firm A is a creditor of firm B. If now firm B defaults, then market participants could fear possible negative effects of this default to firm A. As a consequence, a market participant might not be willing to lend money to firm A any more. Hence, the situation has deteriorated for firm A although direct contagion has only minor or no impact. Related effects could occur to firms that are similar to the defaulted firm because market participants could expect difficulties for such firms in the near future. Another important form of indirect contagion shows the recent financial crisis: If market participants lose trust in each other and start to question their own models, then contagion can occur as a result of panic. For a more detailed discussion concerning these and other forms of contagion we refer to Staum (2013).

Especially the last mentioned form of contagion clarifies the contribution of contagion to *systemic risk* in a financial system. The recent financial crisis has revealed multiple problems concerning identification, measuring and controlling this specific form of risk. Thus, this research topic became more and more important. The complex interactions between the different entities of a given system and the various possible perspectives on this topic lead to various aspects that can be analyzed in the context of systemic risk. Consequently, many research approaches exist that study these different aspects:

One possibility is to consider the whole financial system as a network consisting of nodes and edges. The nodes represent the different firms which are interconnected by edges representing exposures between these firms. Authors who adopt this network modeling point of view are especially interested in different types of contagion that – in the worst case scenario – lead to the destabilization of the whole financial system; see, for instance, Nier et al. (2007), Gai and Kapadia (2010), Amini et al. (2013), Hurd and Gleeson (2011) and Cont et al. (2013). Another point of view, which is closely connected to the network approaches, is taken in so called clearing models. In these approaches some sort of clearing mechanism is modeled, and the most important objects are so called clearing vectors. Examples of these kind of models are Eisenberg and Noe (2001) and generalizations of this approach in Cifuentes et al. (2005) and Rogers and Veraart (2013). An excellent overview of this far reaching field of research is provided in Staum (2013).

In the second part of this thesis we study systemic risk from the perspective of financial regulators or central banks. In contrast to Part I, here we do not study portfolios from the perspective of a modeler or an investor. Instead, we study whole financial systems from the viewpoint of financial regulators. Regulators are interested in measuring and managing the risk in order to maintain the stability of the financial system. Closely connected to this subject is the attribution of systemic risk to the different entities contained in the underlying system.

In case of single-firm risk modeling, risk measurement from the perspective of a financial regulator is an important and far reaching field of research. Artzner et al. (1999) introduced an axiomatic approach to this topic and defined so called *coherent risk measures* as maps ρ which assign risk to random payments and satisfy four economically desirable properties. These are monotonicity, a translation property, subadditivity and positive homogeneity. Because of the monotonicity property, higher payments lead to less risk. The translation property ensures that adding a sure amount a to a random payment reduces the risk by a. Subadditivity and positive homogeneity guarantee that the risk measure rewards diversification, which means that diversification of capital to different investments results in a position that is less risky. In case of a finite probability space, Artzner et al. (1999) derived a so called dual representation by which a coherent risk measure can be characterized as the largest expected loss with respect to a given family of probability measures. Delbaen (2000, 2002) extended the approach and the results of Artzner et al. (1999) to general probability spaces. A further generalization was obtained by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) by replacing the subadditivity and positive homogeneity axioms by the weaker condition of convexity. Risk measures of this kind are called *convex risk measures*. Again, one of the key results is a dual representation of convex risk measures. Based on these fundamental papers, the research area of axiomatic single-firm risk measurement has developed very fast, and there exists a large number of research approaches.

In connection to systemic risk, Chen et al. (2013) introduced an axiomatic approach and studied so called *systemic risk measures* which satisfy desirable properties. This axiomatic approach considers the topic of systemic risk from an entirely new perspective. Therefore, systemic risk measures provide a new tool for measuring and managing the risk contained in a financial system, in particular in view of regulatory issues. Moreover, this approach is an extension of the traditional portfolio approach in which the whole economy is considered as a portfolio managed by the financial regulator. Several drawbacks of this traditional portfolio framework are discussed in Chen et al. (2013): The most problematic issue is the possibility to compensate the losses of one firm with the profits of another. Although this procedure may be reasonable for a real portfolio manager, a financial regulator might disagree. Since, in general, different firms with different owners are pursuing quite different interests, in most cases a regulator "is not able to directly cross-subsidize different firms" (Chen et al. (2013), p. 1373). Furthermore, a regulator might prefer a specific loss distribution between the different entities in a financial system. In conclusion, offsetting gains and losses is not an appropriate concept. This problem of cross-subsidizing is avoided by introducing systemic risk measures that are strongly connected to so called aggregation functions. These provide more flexible possibilities to pool the losses of the individual firms.

Similar to single-firm risk measurement, the axiomatic approach to systemic risk has the advantage of not depending on the choice of a specific risk measure. On the contrary, every functional that satisfies the defining properties is covered by this approach. Therefore, the systemic risk measures introduced in Tarashev et al. (2010), Acharya et al. (2012) and Gauthier et al. (2012) can be regarded as special cases of systemic risk measures in the sense of Chen et al. (2013). However, the approach in Chen et al. (2013) admits several important drawbacks: Only the case of a finite probability space is studied and all systemic risk measures have to be positively homogeneous. Furthermore, they only consider static systemic risk measures which means that no time-dynamic aspects are taken into account.

The range of possible loss distributions of the different entities in the financial system is tremendously reduced by considering a finite probability space. For example, the normal distribution is not covered by this modeling approach. This emphasizes the importance of extending the approach in Chen et al. (2013) to a *general probability space*. Therefore, Chapter 5 is dedicated to systemic risk measures defined on multi-dimensional L^p-spaces. This chapter is based on the paper "Systemic risk measures on general probability spaces" which is joint work with E. Kromer and L. Overbeck. By studying systemic risk measures in conjunction with a general probability space, we have to work in a much more technical framework compared to Chen et al. (2013). This is rewarded by the possibility of applying the concept of systemic risk measures to more general loss distributions.

A second point of criticism concerns the fact that Chen et al. (2013) solely consider systemic risk measures which are positively homogeneous. Although this property may be desirable in some cases, positively homogeneous systemic risk measures are not suitable for every risk measurement framework. To address this problem, we introduce in Chapter 5 so called *convex systemic risk measures* which not necessarily have to be positively homogeneous. Nevertheless, by dropping the axiom of positive homogeneity, we have to introduce a new property. This is strongly connected to the constancy property of standard single-firm risk measures, which was originally studied in Frittelli and Rosazza Gianin (2002). Constancy on a set $A \subset \mathbb{R}$ means that the risk measure assigns the risk $-a$ to the fixed payment $a \in A$. In the context of convex systemic risk measures, we introduce and discuss a generalized version of this property. Nonetheless, the introduction of an appropriate constancy property for systemic risk measures does not seem to restrict the scope of possible choices for these risk measures.

Similar to the approach in Chen et al. (2013), our convex systemic risk measures can be decomposed into a so called convex single-firm risk measure and a convex aggregation function. Here, the single-firm risk measure is essentially identical with standard single-firm risk measures and, as indicated above, the aggregation function specifies how the losses of the different firms in the system are pooled. Based on this fundamental characterization of convex systemic risk measures, we generalize the representation results of Chen et al. (2013). In particular, we provide a primal and a dual representation result for convex systemic risk measures.

The last aspect of systemic risk studied in Chapter 5 is the problem of risk attribution. We consider the question of what fraction each firm contributes to the systemic risk of the whole financial system; see Staum and Liu (2012) and Drehmann and Tarashev (2013). In conjunction with systemic risk measures, a possible solution to this problem provides our approach in Chapter 5. We will see that if the supremum in the dual representation of systemic risk measures is attained, then the different summands of this optimal value can be used to define a risk attribution method that satisfies, for instance, the full allocation property. This property states that by summing up the risk that is attributed to each firm, one obtains the systemic risk contained in the entire financial system. Obviously, this is a desired property. Moreover, in the context of the traditional portfolio framework, the full allocation property is well known and, for instance, was studied in Denault (2001), Tasche (2004), Kalkbrener (2005), Cheridito and Kromer (2011) and Kromer and Overbeck (2014).

So far, we have always focused on so called static risk measures. This means we consider a one period model and valuate, in case of standard single-firm risk measures, the future value of a financial position or, in case of systemic risk measures, a random vector where each component represents the loss of a specific firm in the financial system. There exist many research studies extending the fundamental results of Artzner et al. (1999) to a *dynamic framework*. In this context, the term dynamic can be understood in several ways since there exist various dynamic aspects of risk measurement.

A first extension of the static approach is the theory of conditional risk measures. Here, the focus lies on informational aspects. Over time, additional information is available, and conditional risk measures take into account this information. As a consequence, a conditional risk measure is a map ρ_t where for each random payment X, $\rho_t(X)$ depends on the information that is available at time t. In this framework one can specify a dynamic risk measure as a family (ρ_t) of conditional risk measures such that each $\rho_t(X)$ is measurable with respect to the σ-algebra $\mathcal{F}_t$ representing the available information at time t. Examples of such studies are Bion-Nadal (2004), Detlefsen and Scandolo (2005) and Föllmer and Penner (2006) among many others.

Another possibility to introduce dynamics is to study risk measures that are defined on discrete-time or continuous-time stochastic processes. These processes represent, for instance, the market or accounting value of a firm's equity or the market value of selected financial securities; see Artzner et al. (2007) and Cheridito et al. (2006). Examples of research that studies risk measures on discrete-time processes are Riedel (2004), Artzner et al. (2007), Cheridito et al. (2006), Cheridito and Kupper (2011), Jobert and Rogers (2008) and Acciaio et al. (2012). Continuous-time processes have been discussed in Cheridito et al. (2004, 2005). As, for instance, Acciaio and Penner (2011) point out, an important advantage of these risk measures on processes is the possibility to consider the "time value of money". Moreover, since bounded discrete-time processes can be identified with random variables on a specific product space, there exists also a connection between risk measures on such processes and risk measures on random variables. For more details we refer to Artzner et al. (2007) and Acciaio et al. (2012). A nice overview of dynamic risk measures in discrete time is given in Acciaio and Penner (2011).

Finally, one can combine risk measurement on processes with the theory of conditional risk measures. In this way, one obtains dynamic risk measures on processes; see, for instance, Cheridito et al. (2006), Cheridito and Kupper (2011), Jobert and Rogers (2008), Acciaio and Penner (2011) and Acciaio et al. (2012).

Note that in the dynamic setting one always considers risk measures at different points in time. As a consequence, a natural question in the context of dynamic risk measurement is how these risk measures at different points in time are connected. Thus, one has to introduce an appropriate time-consistency concept. Looking at the relevant literature, we find different suggestions for this issue of time-consistency. The most widely used approach is the so called strong time-consistency, which is connected to the dynamic programming principle; see, for instance, Artzner et al. (2007). This time-consistency concept has been used in most approaches mentioned so far and can be characterized in several ways, for example, by an additivity property of the corresponding acceptance sets or by a supermartingale property of the risk measures; see, for instance, Delbaen (2006), Cheridito et al. (2006), Penner (2007) and Acciaio et al. (2012) and the references therein. Since this form of time-consistency is a rather strong requirement, in particular in view of the existence of consistent updates, other forms of weaker time-consistency properties were studied, among others, in Weber (2006), Artzner et al. (2007), Roorda and Schumacher (2007), Penner (2007) and Roorda and Schumacher (2013).

Static risk measures introduced in Chapter 4 and studied in Chapter 5 in the context of systemic risk do not allow for any dynamic features. Furthermore, note that all the approaches to systemic risk in the previously mentioned papers are essentially static. Thus, we develop in Chapter 7 the first dynamic approach to systemic risk. The results of this chapter are summarized in the paper "Dynamic systemic risk measures for bounded discrete-time processes" which is joint work with E. Kromer and L. Overbeck. We extend the approach from Chapter 5 and consider conditional convex and positively homogeneous systemic risk measures on multi-dimensional discrete-time stochastic processes. Note that our systemic risk measures from Chapter 5 evaluate losses for each node of the underlying network. Consequently, in the conditional setting in Chapter 7 we assume that each firm in the network admits a discrete-time stochastic process representing its losses over time. Here, we use some of the techniques from Cheridito et al. (2006) who have studied dynamic single-firm risk measures in a non systemic context. After generalizing the main results from Chapter 5, in particular the decomposition and the dual representation result, we follow the dynamic approaches for standard single-firm risk measures and introduce dynamic systemic risk measures as families of conditional systemic risk measures. Moreover, adjusted to our setting, we consider the concept of strong time-consistency. Since our dynamic convex systemic risk measures can be decomposed into a dynamic convex single-firm risk measure and a dynamic convex aggregation function, we introduce a time-consistency property for both components and study how these properties depend on each other.

Another possible way to extend the results from Chen et al. (2013) and Kromer et al. (2014a) to a conditional setting, which was considered independently of the approach in Chapter 7 and Kromer et al. (2014b), can be found in Hoffmann et al. (2014). Here, the authors solely focus on the decomposition result in conjunction with generalized conditional aggregation functions.

The second part of this thesis is organized as follows. First, we introduce static single-firm risk measures in Chapter 4. We discuss important representation results and the most important examples. Chapter 5 studies systemic risk measures on general probability spaces and extends the results from Chen et al. (2013) to convex, not necessarily positively homogeneous, systemic risk measures. In Chapter 6 we discuss some of the concepts in Cheridito et al. (2006) where risk measures on discrete-time processes are covered. These ideas are used in Chapter 7 in which we generalize our axiomatic approach to systemic risk regarding different dynamic aspects. That is, we focus in Chapter 7 on conditional and dynamic convex systemic risk measures.

4. Introduction to static risk measures

The aim of this chapter is to introduce static risk measures that take into account the risk of a single firm or one financial position. Most definitions and theorems of Sections 4.1-4.3 are based on Chapter 4 in Föllmer and Schied (2011), which provides an excellent overview of the research topic on convex risk measures.

We start in Section 4.1 with the basic axioms and define convex and coherent risk measures. Moreover, we discuss the connection between risk measures and the corresponding set of acceptable positions. In Section 4.2 we repeat well known results concerning the dual representations of static risk measures. Sections 4.3 and 4.4 consider risk measures on L^∞- and L^p-spaces in more detail. Finally, in Section 4.5 we illustrate the previous results by presenting the most important examples of convex and coherent risk measures.

From now on, a financial position X is considered as a map $X : \Omega \to \mathbb{R}$, and $X(\omega)$ represents the discounted net worth of the financial position X at scenario $\omega \in \Omega$. Furthermore, we suppose that X is an element in the space $\mathcal{X}^{\mathrm{fp}}$ of all financial positions which will be described in detail in the following sections. First, unless explicitly stated otherwise, we assume that $\mathcal{X}^{\mathrm{fp}}$ is the linear space of bounded functions on Ω.

4.1. Definitions and important properties

Consider the following properties of a function $\rho : \mathcal{X}^{\mathrm{fp}} \to \mathbb{R}$:

(M) Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$ for all $X, Y \in \mathcal{X}^{\mathrm{fp}}$.

(T) Translation property: $\rho(X + a) = \rho(X) - a$ for all $X \in \mathcal{X}^{\mathrm{fp}}$ and $a \in \mathbb{R}$.

(C) Convexity: $\rho(aX + (1-a)Y) \leq a\rho(X) + (1-a)\rho(Y)$ for all $X, Y \in \mathcal{X}^{\mathrm{fp}}$ and $a \in [0, 1]$.

(PH) Positive homogeneity: $\rho(aX) = a\rho(X)$ for all $X \in \mathcal{X}^{\mathrm{fp}}$ and $a \in \mathbb{R}_+$.

Definition 4.1.1. *A* risk measure *is a function $\rho : \mathcal{X}^{\mathrm{fp}} \to \mathbb{R}$ that satisfies the properties (M) and (T). A* convex risk measure *is a risk measure that additionally satisfies the property (C), and a* coherent risk measure *is a convex risk measure that additionally satisfies the property (PH).*

Because of the monotonicity property, a position with a higher net worth is associated with less risk. The translation property ensures that adding a fixed amount of money a to a financial position X leads to a risk reduction by this amount a. We can motivate this property by the idea that the risk of a position X represents the amount which has to be added to X such that the new position $X + \rho(X)$ is acceptable. For example, we could specify acceptable positions by including all positions X the risk of which does not hit a specific barrier. Note that the translation property yields $\rho(X+\rho(X)) = \rho(X) - \rho(X) = 0$. A risk measure satisfying the convexity property does not penalize diversification: The risk of the diversified position $aX + (1-a)Y$ is less than or equal to the weighted risk of the positions X and Y. The last property, i.e., positive homogeneity, states that the risk of a financial position increases linear with its size. Nevertheless, in many cases this property is not satisfied. This was the reason for developing the theory of convex risk measures. Finally, note that if ρ is positively homogeneous, then the following property holds:

(N) Normalization: $\rho(0) = 0$.

Remark 4.1.2. Artzner et al. (1999) were the first who introduced coherent risk measures. They defined coherent risk measures by using the properties of monotonicity, the translation property, positive homogeneity and the following property:

(SA) Subadditivity: $\rho(X+Y) = \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}^{\text{fp}}$.

The definition of Artzner et al. (1999) is equivalent to our definition above since under positive homogeneity, convexity and subadditivity are equivalent.

Let $\|\cdot\|$ be the supremum norm defined by $\|X\| := \sup_{\omega\in\Omega} |X(\omega)|$ for $X \in \mathcal{X}^{\text{fp}}$ and consider a risk measure $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$. Since $X \leq Y + \|X - Y\|$ implies that $\rho(X) \geq \rho(Y) - \|X - Y\|$ for all $X, Y \in \mathcal{X}^{\text{fp}}$, we obtain $|\rho(X) - \rho(Y)| \leq \|X - Y\|$ for all $X, Y \in \mathcal{X}^{\text{fp}}$. This means that every risk measure $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ is (1-)Lipschitz continuous with respect to the supremum norm $\|\cdot\|$

Now, we take a closer look at financial positions that are acceptable. At first, we define a so called acceptance set which characterizes acceptable positions as positions X satisfying $\rho(X) \leq 0$. This means that the position X is acceptable if no capital has to be added. Thereafter, we discuss the relationship between acceptance set and the corresponding risk measure.

Definition 4.1.3. *The* acceptance set *of a risk measure ρ is defined by*

$$\mathcal{A}_\rho := \{X \in \mathcal{X}^{\text{fp}} | \rho(X) \leq 0\}.$$

Proposition 4.1.4 (See Proposition 4.6 in Föllmer and Schied (2011))**.** *Let $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ be a risk measure and $\mathcal{A}_\rho$ the corresponding acceptance set. Then the following properties are satisfied:*

1. *$\mathcal{A}_\rho \neq \emptyset$ and*
   a) $\inf\{r \in \mathbb{R} | r \in \mathcal{A}_\rho\} > -\infty$,
   b) if $X \in \mathcal{A}_\rho$ and $Y \in \mathcal{X}^{\text{fp}}$ with $Y \geq X$, then $Y \in \mathcal{A}_\rho$,
   c) $\mathcal{A}_\rho$ is $\|\cdot\|$-closed.
2. *ρ admits the representation*

$$\rho(X) = \inf\{r \in \mathbb{R} | r + X \in \mathcal{A}_\rho\} \quad \textit{for all } X \in \mathcal{X}^{\text{fp}}.$$

3. *ρ is a convex risk measure if and only if the set $\mathcal{A}_\rho$ is convex, i.e., $aX + (1-a)X' \in \mathcal{A}_\rho$ for all $X, X' \in \mathcal{A}_\rho$ and $a \in [0,1]$.*
4. *ρ is positively homogeneous if and only if the set $\mathcal{A}_\rho$ is a cone, i.e., $aX \in \mathcal{A}_\rho$ for all $X \in \mathcal{A}_\rho$ and $a \in \mathbb{R}_+$. In particular, ρ is a coherent risk measure if and only if $\mathcal{A}_\rho$ is a convex cone.*

On the other hand, one can define for every set $\mathcal{A} \subset \mathcal{X}^{\text{fp}}$ of acceptable positions a risk measure ρ by using the idea that $\rho(X)$ represents the smallest amount which has to be added to X such that the new position $X + \rho(X)$ is acceptable.

Proposition 4.1.5 (See Proposition 4.7 in Föllmer and Schied (2011))**.** *Consider $\emptyset \neq \mathcal{A} \subset \mathcal{X}^{\text{fp}}$ that satisfies the properties 1.a) and 1.b) from Proposition 4.1.4. Then the so called* capital requirement *$\rho_{\mathcal{A}}$ defined by*

$$\rho_{\mathcal{A}}(X) := \inf\{r \in \mathbb{R} | r + X \in \mathcal{A}\} \quad \textit{for } X \in \mathcal{X}^{\text{fp}}$$

satisfies the following properties:

1. *$\rho_{\mathcal{A}} : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ is a risk measure.*
2. *If the set $\mathcal{A}$ is convex, then $\rho_{\mathcal{A}}$ is a convex risk measure.*
3. *If the set $\mathcal{A}$ is a cone, then $\rho_{\mathcal{A}}$ is positively homogeneous. Especially, if $\mathcal{A}$ is a convex cone, then $\rho_{\mathcal{A}}$ is a coherent risk measure.*
4. *$\mathcal{A}$ is a subset of the acceptance set $\mathcal{A}_{\rho_{\mathcal{A}}}$, and we have $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$ if and only if $\mathcal{A}$ satisfies property 1.c) from Proposition 4.1.4.*

4.2. Representations of risk measures

In this section we first consider representations of risk measures for $\mathcal{X}^{\text{fp}} = \{X | X : \Omega \to \mathbb{R}\}$ where the state space Ω is supposed to be finite. We will see that this case of a finite state space is also the starting point in Chapter 5, which studies systemic risk measures on general probability spaces. A first general representation of coherent risk measures can be found in Artzner et al. (1999). Recall that $X(\omega)$ is the discounted value of the position X. Therefore, we set the total return r in Artzner et al. (1999) equal to 1.

Theorem 4.2.1 (See Proposition 4.1 in Artzner et al. (1999))**.** *Suppose that Ω is finite. A risk measure $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ is coherent if and only if there exists a family $\mathcal{Q}$ of probability measures on $(\Omega, \mathfrak{P}(\Omega))$ ($\mathfrak{P}(\Omega)$ denotes the set of all subsets of Ω) such that*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X] \quad \textit{for all } X \in \mathcal{X}^{\text{fp}}.$$

Föllmer and Schied (2002) proved the corresponding theorem for convex risk measures.

Theorem 4.2.2 (See Theorem 5 in Föllmer and Schied (2002))**.** *Suppose that Ω is finite and let $\mathcal{M}_1(\Omega, \mathfrak{P}(\Omega))$ be the set of all probability measures on $(\Omega, \mathfrak{P}(\Omega))$.*

A risk measure $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ *is convex if and only if there exists a function* $\alpha : \mathcal{M}_1(\Omega, \mathfrak{P}(\Omega)) \to (-\infty, +\infty]$ *such that*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{M}_1(\Omega, \mathfrak{P}(\Omega))} \{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha(\mathbb{Q})\} \quad \textit{for all } X \in \mathcal{X}^{\text{fp}}.$$

Remark 4.2.3. Note that the representation of coherent risk measures in Theorem 4.2.1 is a special case of the representation of convex risk measures in Theorem 4.2.2. If we consider for a coherent risk measure ρ the function $\alpha(\mathbb{Q}) := \sup_{X \in \mathcal{X}^{\text{fp}}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \rho(X)\} = \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbb{Q}}[-X]$ for $\mathbb{Q} \in \mathcal{M}_1(\Omega, \mathfrak{P}(\Omega))$ (see, for instance, Föllmer and Schied (2002)), then we can easily prove that $\alpha(\mathbb{Q}) \in \{0, +\infty\}$ for all $\mathbb{Q} \in \mathcal{M}_1(\Omega, \mathfrak{P}(\Omega))$. Hence, the representation in Theorem 4.2.1 with $\mathcal{Q} = \{\mathbb{Q} \in \mathcal{M}_1(\Omega, \mathfrak{P}(\Omega)) | \alpha(\mathbb{Q}) = 0\}$ follows from Theorem 4.2.2.

From now on, we consider a general measurable space $(\Omega, \mathcal{F})$ and suppose that $\mathcal{X}^{\text{fp}}$ is the space of all bounded measurable functions on $(\Omega, \mathcal{F})$. Moreover, let $\mathcal{M}_{1,f} := \mathcal{M}_{1,f}(\Omega, \mathcal{F})$ be the space of all finitely additive set functions $Q : \mathcal{F} \to [0, 1]$ with $Q[\Omega] = 1$ and let $\mathcal{M}_1 := \mathcal{M}_1(\Omega, \mathcal{F})$ be the space of all probability measures on $(\Omega, \mathcal{F})$. In order to facilitate the notation, we adapt the notation from Föllmer and Schied (2011) and denote the integral of $X \in \mathcal{X}^{\text{fp}}$ with respect to $Q \in \mathcal{M}_{1,f}$ by $\mathbb{E}_Q[X]$.

The representation in the following theorem is called *robust representation.*

Theorem 4.2.4 (See Theorem 4.16 in Föllmer and Schied (2011))**.** *Let* $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ *be a convex risk measure. Then* ρ *admits the representation*

$$\rho(X) = \max_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}_Q[-X] - \alpha_{min}(Q)\} \quad \textit{for all } X \in \mathcal{X}^{\text{fp}} \tag{4.1}$$

where the function α_{min} *is given by*

$$\alpha_{min}(Q) := \sup_{X \in \mathcal{A}_\rho} E_Q[-X] = \sup_{X \in \mathcal{X}^{fp}} \{\mathbb{E}_Q[-X] - \rho(X)\} \quad \textit{for } Q \in \mathcal{M}_{1,f}.$$

Moreover, this function is minimal in the sense that $\alpha_{min}(Q) \leq \alpha(Q)$ *for all* $Q \in \mathcal{M}_{1,f}$ *and all functions* $\alpha : \mathcal{M}_{1,f} \to \mathbb{R} \cup \{+\infty\}$ *that satisfy representation* (4.1) *with* α *instead of* α_{min} *and* $\inf_{Q \in \mathcal{M}_{1,f}} \alpha(Q) \in \mathbb{R}$.

Remark 4.2.5. The functions $\alpha : \mathcal{M}_{1,f} \to \mathbb{R} \cup \{+\infty\}$ from the previous theorem that satisfy representation (4.1) with α instead of α_{min} and $\inf_{Q \in \mathcal{M}_{1,f}} \alpha(Q) \in \mathbb{R}$ are called *penalty functions* of the risk measure ρ.

The coherent case is a special case of the previous theorem. To prove this special case, one has to show that the penalty function α_{min} takes only the values 0 and $+\infty$ (see Remark 4.2.3).

Corollary 4.2.6 (See Corollary 4.19 in Föllmer and Schied (2011))**.** *Let* $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ *be a coherent risk measure. Then* ρ *admits the representation*

$$\rho(X) = \max_{Q \in \mathcal{Q}_{max}} \mathbb{E}_Q[-X] \quad \textit{for all } X \in \mathcal{X}^{\text{fp}} \tag{4.2}$$

where the convex set $\mathcal{Q}_{max}$ is defined by

$$\mathcal{Q}_{max} := \{Q \in \mathcal{M}_{1,f} | \alpha_{min}(Q) = 0\}.$$

Moreover, the largest set $\mathcal{Q} \subset \mathcal{M}_{1,f}$ which satisfies representation (4.2) *with $\mathcal{Q}$ instead of $\mathcal{Q}_{max}$ is equal to $\mathcal{Q}_{max}$.*

Finally, the set $\mathcal{M}_{1,f}$ in representation (4.1) can be reduced to the set $\mathcal{M}_1$ if the risk measure satisfies an additional condition.

Theorem 4.2.7 (See Theorem 4.22 in Föllmer and Schied (2011))**.** *Let $\rho : \mathcal{X}^{\text{fp}} \to \mathbb{R}$ be a convex risk measure. Then the following statements are equivalent:*

1. *ρ is* continuous from below*, i.e., $X_m \uparrow X$ pointwise implies $\rho(X_m) \downarrow \rho(X)$.*
2. *The minimal penalty function α_{min} of ρ is concentrated on the set $\mathcal{M}_1$, i.e., $\alpha_{min}(Q) < +\infty$ for $Q \in \mathcal{M}_{1,f}$ implies that Q is σ-additive.*

In particular, if one of these properties is satisfied, then

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{M}_1} \{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha_{min}(\mathbb{Q})\} \quad \textit{for all } X \in \mathcal{X}^{\text{fp}}.$$

4.3. Risk measures on L^∞

From now on, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a general probability space and set $\mathcal{X}^{\text{fp}} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. This setting was considered first of all in Delbaen (2000, 2002), Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002). The results in this section are again based on Föllmer and Schied (2011).

In case of $\mathcal{X}^{\text{fp}} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, we focus on risk measures $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ that satisfy

$$\rho(X) = \rho(Y) \quad \text{if } X = Y \quad \mathbb{P} - \text{a.s.} \tag{4.3}$$

Let us write for short $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $1 \leq p < \infty$. These spaces are endowed with the usual norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$, respectively. These are defined by $\|X\|_\infty := \inf\{r \in \mathbb{R} | |X| \leq r \ \mathbb{P}\text{-a.s.}\}$ for $X \in L^\infty$ and $\|X\|_p := (\int |X|^p d\mathbb{P})^{1/p}$ for $X \in L^p$.

Define $\mathcal{M}_{1,f}(\mathbb{P}) := \mathcal{M}_{1,f}(\Omega, \mathcal{F}, \mathbb{P})$ as the set of finitely additive set functions $Q : \mathcal{F} \to [0,1]$ with $Q[\Omega] = 1$ which are absolutely continuous with respect to $\mathbb{P}$. Similarly, let $\mathcal{M}_1(\mathbb{P}) := \mathcal{M}_1(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$.

First, we state an important property which follows from (4.3).

Lemma 4.3.1 (See Lemma 4.32 in Föllmer and Schied (2011))**.** *Consider a convex risk measure ρ that satisfies* (4.3) *and admits the representation*

$$\rho(X) = \sup_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}_Q[-X] - \alpha(Q)\} \quad \textit{for all } X \in L^\infty$$

for a penalty function α. Then we have $\alpha(Q) = +\infty$ for all $Q \in \mathcal{M}_{1,f} \backslash \mathcal{M}_{1,f}(\mathbb{P})$.

In what follows, we need several continuity properties. Since we discuss functions $\upsilon : L^p \to \mathbb{R} \cup \{+\infty\}$ for $1 \leq p < \infty$ in the next section, the following definition includes such functions.

Definition 4.3.2. *Consider a function $\upsilon : L^p \to \mathbb{R} \cup \{+\infty\}$ for $1 \leq p \leq \infty$.*

1. *υ is called* lower semicontinuous (l.s.c.) at X_0 (with respect to the norm topology) *if for all sequences $(X_m) \subset L^p$ with $\|X_m - X_0\|_p \to 0$, it follows that*
$$\upsilon(X_0) \leq \liminf_{m\to\infty} \upsilon(X_m).$$
υ is called lower semicontinuous (with respect to the norm topology) *if υ is lower semicontinuous at all $X \in L^p$ (with respect to the norm topology).*
2. *υ is called* continuous from above *if for every sequence $(X_m) \subset L^p$ with $X_m \downarrow X$ $\mathbb{P}$-a.s. for some $X \in L^p$, it follows that $\upsilon(X_m) \to \upsilon(X)$.*
3. *υ is called* continuous from below *if for every sequence $(X_m) \subset L^p$ with $X_m \uparrow X$ $\mathbb{P}$-a.s. for some $X \in L^p$, it follows that $\upsilon(X_m) \to \upsilon(X)$.*
4. *υ satisfies the* Fatou-property *if for every sequence $(X_m) \subset L^p$ with $|X_m| \leq Y$ $\mathbb{P}$-a.s. for some $Y \in L^p$ and $X_m \to X$ $\mathbb{P}$-a.s. for some $X \in L^p$, it follows that*
$$\upsilon(X) \leq \liminf_{m\to\infty} \upsilon(X_m).$$

If we focus on the space L^∞, then we obtain the following representation result.

Theorem 4.3.3 (See Theorem 4.33 in Föllmer and Schied (2011))**.** *Let $\rho : L^\infty \to \mathbb{R}$ be a convex risk measure. Then the following statements are equivalent:*

1. *There exists a penalty function $\alpha : \mathcal{M}_1(\mathbb{P}) \to \mathbb{R} \cup \{+\infty\}$ such that*
$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \{\mathbb{E}_\mathbb{Q}[-X] - \alpha(\mathbb{Q})\} \quad \textit{for all } X \in L^\infty. \tag{4.4}$$
2. *ρ is continuous from above.*
3. *ρ satisfies the Fatou-property.*
4. *The acceptance set $\mathcal{A}_\rho$ is weak* closed, i.e., it is closed with respect to the topology $\sigma(L^\infty, L^1)$.*

In particular, if one of these properties is satisfied, then ρ can be represented with the minimal penalty function restricted on $\mathcal{M}_1(\mathbb{P})$, i.e.,
$$\alpha_{min}(\mathbb{Q}) = \sup_{X\in\mathcal{A}_\rho} \mathbb{E}_\mathbb{Q}[-X] = \sup_{X\in L^\infty} \{\mathbb{E}_\mathbb{Q}[-X] - \rho(X)\} \quad \textit{for } \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}).$$

Föllmer and Penner (2006) provide in their introduction a well known interpretation of representation (4.4): "[...] the risk of a position is evaluated as the worst expected loss, suitably modified, under a whole class of probabilistic models. These alternative models are described by probability measures Q [$\mathbb{Q}$ with our notation] on the underlying set of scenarios. But they are taken seriously at a different degree, and this is made precise by the non-negative penalty function $\alpha(Q)$ [$\alpha(\mathbb{Q})$]." (Föllmer and Penner (2006), p. 61)

It remains to consider the coherent case.

Theorem 4.3.4 (See Corollary 4.37 in Föllmer and Schied (2011))**.** *Let $\rho : L^\infty \to \mathbb{R}$ be a coherent risk measure. Then there exists a set $\mathcal{Q} \subset \mathcal{M}_1(\mathbb{P})$ such that ρ admits the representation*

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X] \quad \text{for all } X \in L^\infty \tag{4.5}$$

if and only if the equivalent properties from Theorem 4.3.3 are satisfied. In this case, the set

$$\mathcal{Q}_{max} := \{Q \in \mathcal{M}_1(\mathbb{P}) | \alpha_{min}(Q) = 0\}$$

is the maximal subset of $\mathcal{M}_1(\mathbb{P})$ which satisfies representation (4.5) *with $\mathcal{Q}_{max}$ instead of $\mathcal{Q}$.*

Note that it follows from Theorem 4.2.7 and Lemma 4.3.1 that the supremum in (4.4) (and hence also in (4.5)) is attained if ρ is continuous from below.

4.4. Risk measures on L^p

Filipovic and Svindland (2007) point out that considering risk measures on L^∞ reduces the scope of possible risk models dramatically. For example, normal distributed random variables are excluded by such approaches. Hence, the theory of risk measures was extended to the theory of risk measures on L^p for $1 \leq p < \infty$; see, for instance, Filipovic and Svindland (2007) and Kaina and Rüschendorf (2009) and the references therein. In this section we introduce risk measures on L^p-spaces and repeat important results from the previously mentioned papers.

In case of L^p-spaces, we modify the definition of risk measures in the sense that risk measures now map from L^p to $\mathbb{R}\cup\{+\infty\}$. Delbaen (2002) motivated this change of definition by the existence of positions which are so risky that they will never be acceptable regardless of how much capital is put aside. As a consequence, such positions X satisfy $\rho(X) = +\infty$.

For the remaining part of this section we understand equalities and inequalities between random variables $\mathbb{P}$-a.s. Now, let us consider the corresponding properties to (M), (T), (C) and (PH) for functions $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ and $1 \leq p \leq \infty$:

(M') Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$ for all $X, Y \in L^p$.

(T') Translation property: $\rho(X + a) = \rho(X) - a$ for all $X \in L^p$ and $a \in \mathbb{R}$.

(C') Convexity: $\rho(aX + (1-a)Y) \leq a\rho(X) + (1-a)\rho(Y)$ for all $X, Y \in L^p$ and $a \in [0, 1]$.

(PH') Positive homogeneity: $\rho(aX) = a\rho(X)$ for all $X \in L^p$ and $a \in \mathbb{R}_+$.

Definition 4.4.1. *Let $1 \leq p < \infty$. A* risk measure *(on L^p) is a function $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ that satisfies the properties (M') and (T'). A* convex risk measure *(on L^p) is a risk measure (on L^p) that additionally satisfies the property (C'), and a* coherent risk measure *(on L^p) is a convex risk measure (on L^p) that additionally satisfies the property (PH').*

The following lemma is well known and states that the previous definition can also be used to define convex risk measures on L^∞.

Lemma 4.4.2. *For every proper function $\rho : L^\infty \to \mathbb{R} \cup \{+\infty\}$ that satisfies (M'), (T') and (C'), we have $\rho(X) < \infty$ for all $X \in L^\infty$, i.e., $\rho : L^\infty \to \mathbb{R}$.*

In particular, this means that ρ is a convex risk measure in the sense of Definition 4.1.1.

Proof. For each $X \in L^\infty$, we have $X \leq \|X\|_\infty < \infty$. Moreover, because ρ is proper, there exists some $Y \in L^\infty$ with $\rho(Y) < \infty$ and $\|Y\|_\infty < \infty$. Since $0 \geq Y - \|Y\|_\infty$, the properties (M') and (T') imply $\rho(0) \leq \rho(Y - \|Y\|_\infty) = \rho(Y) + \|Y\|_\infty < \infty$. If we apply again (M') and (T'), then we obtain $\rho(X) \leq \rho(-\|X\|_\infty) = \rho(0) + \|X\|_\infty < \infty$. □

Again, we can study the relationship between risk measures $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ and their acceptance sets given by

$$\mathcal{A}_\rho := \{X \in L^p | \rho(X) \leq 0\}.$$

The subsequent propositions follow from the remarks in Chapter 1 in Kaina and Rüschendorf (2009).

Proposition 4.4.3. *Let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ be a proper risk measure and $\mathcal{A}_\rho$ the corresponding acceptance set. Then the following properties are satisfied:*

1. *$\mathcal{A}_\rho \neq \emptyset$ and*
   a) *$\inf\{r \in \mathbb{R} | r + Y \in \mathcal{A}_\rho\} > -\infty$ for all $Y \in L^p$ where $\inf \emptyset := +\infty$,*
   b) *if $X \in \mathcal{A}_\rho$ and $Y \in L^p$ with $Y \geq X$, then $Y \in \mathcal{A}_\rho$.*
2. *ρ admits the representation*
$$\rho(X) = \inf\{r \in \mathbb{R} | r + X \in \mathcal{A}_\rho\} \quad \text{for all } X \in L^p$$
   where $\inf \emptyset := +\infty$.
3. *ρ is a convex risk measure if and only if $\mathcal{A}_\rho$ is convex.*
4. *ρ is positively homogeneous if and only if $\mathcal{A}_\rho$ is a cone. In particular, ρ is a coherent risk measure if and only if $\mathcal{A}_\rho$ is a convex cone.*

Proposition 4.4.4. *Consider $\emptyset \neq \mathcal{A} \subset L^p$ that satisfies the properties 1.a) and 1.b) from Proposition 4.4.3. Then*

$$\rho_{\mathcal{A}}(X) := \inf\{r \in \mathbb{R} | r + X \in \mathcal{A}\} \quad \text{for } X \in L^p$$

satisfies the following properties:

1. *$\rho_{\mathcal{A}}$ is a risk measure.*
2. *If $\mathcal{A}$ is convex, then $\rho_{\mathcal{A}}$ is a convex risk measure.*
3. *If $\mathcal{A}$ is a cone, then $\rho_{\mathcal{A}}$ is positively homogeneous. Especially, if $\mathcal{A}$ is a convex cone, then $\rho_{\mathcal{A}}$ is a coherent risk measure.*

4. *$\mathcal{A}$ is a subset of the acceptance set $\mathcal{A}_{\rho_{\mathcal{A}}}$.*

Note that in case of $\mathcal{X}^{\mathrm{fp}} = L^\infty$, it suffices to claim that $\inf\{m \in \mathbb{R} | m+Y \in \mathcal{A}_\rho\} > -\infty$ for $Y = 0$ only (see Proposition 4.1.5).

At the end of this section we repeat important representation results for risk measures on L^p. For $1 \leq p < \infty$, the dual space of L^p is given by L^q with $1/p+1/q = 1$ (up to isomorphism). We define

$$\mathcal{M}_1^q(\mathbb{P}) := \left\{ \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \middle| \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q \right\} \quad \text{for } 1 \leq p < \infty.$$

By using the duality theorem for conjugate functions (see Theorem A.2.9), we obtain a first representation result.

Theorem 4.4.5 (See Theorem 2.4 in Kaina and Rüschendorf (2009))**.** *Fix $1 \leq p < \infty$ and let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ be a proper convex risk measure on L^p. Then the following statements are equivalent:*

1. *ρ is l.s.c.*
2. *There exists a subset $\mathcal{Q} \subset \mathcal{M}_1^q(\mathbb{P})$ and a function $\alpha : \mathcal{Q} \to \mathbb{R} \cup \{+\infty\}$ such that $\inf_{\mathbb{Q} \in \mathcal{Q}} \alpha(\mathbb{Q}) \in \mathbb{R}$ and*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \alpha(\mathbb{Q})\} \quad \textit{for all } X \in L^p.$$

In particular, if one of these properties is satisfied, then ρ admits the representation

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{M}_1^q(\mathbb{P})} \{\mathbb{E}_{\mathbb{Q}}[-X] - \rho^*(\mathbb{Q})\} \quad \textit{for all } X \in L^p \tag{4.6}$$

with $\rho^*(\mathbb{Q}) := \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbb{Q}}[-X]$ for $\mathbb{Q} \in \mathcal{M}_1^q(\mathbb{P})$.

Remark 4.4.6. If we replace the proper convex risk measure on L^p by a convex risk measure on L^∞ in the previous theorem, then the theorem still holds with $\mathcal{M}_1^q(\mathbb{P})$ replaced by the space $\mathcal{M}_{1,f}(\mathbb{P})$. Additionally, the supremum in (4.6) is attained. This follows from Theorem 4.2.4 and Lemma 4.3.1. But since risk measures on L^∞ are automatically Lipschitz continuous, this representation does always exist. Finally, note that the last representation result for convex risk measures on L^p, $p < \infty$, is based on probability measures, which is not the case for the corresponding representation result for convex risk measures on L^∞.

In case of coherent risk measures, Kaina and Rüschendorf (2009) prove a more specific representation result.

Theorem 4.4.7 (See Theorem 2.9 in Kaina and Rüschendorf (2009))**.** *Fix $1 \leq p < \infty$ and let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ be a proper coherent risk measure on L^p. Then the following properties are equivalent:*

1. *ρ is l.s.c.*

2. *There exists a subset* $\mathcal{Q} \subset \mathcal{M}_1^q(\mathbb{P})$ *such that*

$$\rho(X) = \max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X] \quad \textit{for all } X \in L^p.$$

3. ρ *is finite and continuous with respect to the norm topology.*

The previous result shows that the supremum in the dual representation is always attained if ρ is a proper and l.s.c. coherent risk measure on L^p, $p < \infty$.

4.5. Examples of convex and coherent risk measures

In this section we discuss important examples of convex and coherent risk measures. Again, we refer to Föllmer and Schied (2011) for more details and additional examples.

A risk measure that is frequently used in practice is the Value at Risk.

Example 4.5.1 (Value at Risk)**.** Let $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. The *Value at Risk* of a financial position $X \in \mathcal{X}^{\text{fp}} = L^0$ is defined by

$$\mathrm{VaR}_\lambda(X) := \inf\{r \in \mathbb{R} | \mathbb{P}[X + r < 0] \le \lambda\} \quad \text{for } \lambda \in (0,1).$$

This means that the Value at Risk is the minimal amount one has to add to the position X such that the probability of a loss of this new position is bounded by λ. One can easily prove that the Value at Risk is monotone, satisfies the translation property and is positively homogeneous. But it does not satisfy the convexity property. For a counterexample see, for instance, Example 4.46 in Föllmer and Schied (2011). As a consequence, the Value at Risk does not reward diversification, which is not a desired property. Another disadvantage is that it does not provide any information on the size of a loss.

Another risk measure which is based on the Value at Risk is the so called Average Value at Risk.

Example 4.5.2 (Average Value at Risk)**.** The *Average Value at Risk* of a position $X \in L^1$ is defined by

$$\mathrm{AVaR}_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\gamma(X) d\gamma \quad \text{for } \lambda \in (0,1].$$

This risk measure, which is indeed a coherent risk measure, is also called *Conditional Value at Risk* or *Expected Shortfall.* The corresponding dual representation is given by

$$\mathrm{AVaR}_\lambda(X) = \max_{\mathbb{Q}\in Q_\lambda} \mathbb{E}_{\mathbb{Q}}[-X] \quad \text{for } X \in \mathcal{X}^{\text{fp}}$$

where $\mathcal{Q}_\lambda$ is the set of all probability measures $\mathbb{Q}$ which are absolutely continuous with respect to $\mathbb{P}$ and satisfy $d\mathbb{Q}/d\mathbb{P} \le 1/\lambda$ $\mathbb{P}$-a.s; see, for instance, Theorem 4.52 in Föllmer and Schied (2011).

An important example of a convex risk measure is the following.

Example 4.5.3 (Entropic risk measure)**.** Assume that the preferences of an investor are given by the function $U_\gamma(X) := \mathbb{E}_\mathbb{P}[1 - e^{-\gamma X}]$ where $X \in L^\infty$ and $\gamma > 0$. If we consider the set

$$\mathcal{A}_\gamma := \{X \in L^\infty | U_\gamma(X) \geq U_\gamma(0)\} = \{X \in L^\infty | \mathbb{E}_\mathbb{P}[e^{-\gamma X}] \leq 1\},$$

then it is easily seen that $\mathcal{A}_\gamma$ is convex and satisfies the conditions from Proposition 4.1.5. Hence, the capital requirement $\rho_{\mathcal{A}_\gamma}$ defined by this acceptance set is a convex risk measure according to Proposition 4.1.5. This risk measures is called *entropic risk measure.* Moreover,

$$\rho_{\mathcal{A}_\gamma}(X) = \inf\{r \in \mathbb{R} | r + X \in \mathcal{A}_\gamma\} = \inf\{r \in \mathbb{R} | \mathbb{E}_\mathbb{P}[e^{-\gamma X}] \leq e^{\gamma r}\} = \frac{1}{\gamma} \log \mathbb{E}_\mathbb{P}[e^{-\gamma X}].$$

The dual representation of this convex risk measure is given by

$$\rho_{\mathcal{A}_\gamma}(X) = \sup_{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})} \{\mathbb{E}_\mathbb{Q}[-X] - \alpha^{min}(\mathbb{Q})\} \quad \text{for } X \in L^\infty$$

where the minimal penalty function α_{min} satisfies

$$\alpha_{min}(\mathbb{Q}) = \frac{1}{\gamma} H(\mathbb{Q}|\mathbb{P}) \quad \text{for } \mathbb{Q} \in \mathcal{M}_1(\mathbb{P}).$$

Here, $H(\mathbb{Q}|\mathbb{P})$ denotes the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ defined by

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{else} \end{cases}.$$

For more details we refer again to Föllmer and Schied (2011).

In our last example we consider a further generalization of the entropic risk measure.

Example 4.5.4 (Distortion entropic risk measure)**.** Let $g : [0,1] \to [0,1]$ be a non-decreasing function with $g(0) = 0$ and $g(1) = 1$. Then g induces the so called distorted probability $\mathbb{P}_g$ defined by $\mathbb{P}_g[A] := g \circ \mathbb{P}[A]$ for $A \in \mathcal{F}$. By using this distorted probability, we can define the so called distorted expectation $\mathbb{E}_g$ as a Choquet integral (see, for instance, Example 4.14 in Föllmer and Schied (2011)):

$$\mathbb{E}_g[X] := \int_0^\infty \mathbb{P}_g(X > t)dt + \int_{-\infty}^0 (\mathbb{P}_g(X > t) - 1)dt \quad \text{for } X \in L^\infty.$$

Furthermore, this distorted expectation enables us to define the *distortion entropic risk measure*, which is a generalization of the entropic risk measure introduced in Example 4.5.3 and given by

$$\rho_{g,\gamma}(X) := \frac{1}{\gamma} \log \mathbb{E}_g[e^{-\gamma X}] \quad \text{for } X \in L^\infty \text{ and } \gamma > 0.$$

5. Static systemic risk measures on general probability spaces

This chapter is based on the paper “Systemic risk measures on general probability spaces” which is joint work with E. Kromer and L. Overbeck. We study the extension of the approach in Chen et al. (2013) to general probability spaces and convex, not necessarily positively homogeneous, systemic risk measures. We will see that the static risk measures introduced in the previous chapter are an important building block for the construction of systemic risk measures.

The first section in this chapter is dedicated to the introduction of our main objects of interest: convex and positively homogeneous systemic risk measures. After introducing our notation, we generalize several different axioms from the theory of standard single-firm risk measurement. Section 5.2 provides a basic decomposition result which is essential for the remaining part of this chapter. We will see that each systemic risk measure can be decomposed into a single-firm risk measure and a so called aggregation function. Thereafter, we illustrate in Section 5.3 the construction of systemic risk measures by considering some examples. Since we generalize the results of Chen et al. (2013), their examples can still be used in our setting. Nevertheless, we study the larger class of convex systemic risk measures, and therefore we can also add new examples. Section 5.4 is dedicated to different representation results. We first define acceptance sets corresponding to a given convex systemic risk measure and then provide a primal representation, which illustrates the connection between convex systemic risk measures and the corresponding acceptance sets. Based on this first representation result, we finally prove the dual representation of convex systemic risk measures. An important application of this dual representation is considered in Section 5.5. We will see that if the supremum in the dual representation is attained, then the different summands of the corresponding optimal value can be used to define a risk attribution method in a sensible way. Thus, our approach in this chapter enables us to allocate the risk of the financial system to the different firms contained in this system.

5.1. Model and notation

Throughout this chapter fix the underlying general probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We work in a one-period model and consider a finite set of n firms. From a network modeling point of view this means that we consider a financial network which consists of n nodes.

Let $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $1 \leq p \leq \infty$. The random vector $\bar{X} = (\bar{X}_1, \dots, \bar{X}_n) \in (L^p)^n$ represents the *losses* of the different firms, i.e., $\bar{X}_i$ is assumed to be the random loss of firm $i \in \{1, \dots, n\}$. It is important to note that in this chapter we valuate losses $\bar{X}_1, \dots, \bar{X}_n$. In contrast to this, in Chapter 4 the random variable X represents the net *worth* of a specific financial position. We will see later on that this change of perspective affects the properties of the risk measures studied in this chapter.

From now on, we interpret equalities and inequalities between random vectors $\bar{X}, \bar{Y} \in (L^p)^m$, $m \in \mathbb{N}$, componentwise $\mathbb{P}$-a.s. This means $\bar{X}, \bar{Y} \in (L^p)^m$ satisfy $\bar{X} \leq \bar{Y}$ if and only if $\bar{X}_i \leq \bar{Y}_i$ $\mathbb{P}$-a.s. for all $i \in \{1, \dots, m\}$.

In case of $1 \leq p < \infty$, we endow the space L^p with the usual L^p-norm given by $\|X\|_p := (\int |X|^p d\mathbb{P})^{1/p} = \mathbb{E}[|X|^p]^{1/p}$ for $X \in L^p$. Similarly, if $p = \infty$, then the space L^∞ is endowed with the norm $\|X\|_\infty := \inf\{r \in \mathbb{R} | |X| \leq r \ \mathbb{P}\text{-a.s.}\}$ for $X \in L^\infty$. Let us now consider the dual space of L^p for $1 \leq p \leq \infty$, i.e., the space of all continuous and linear functionals on L^p. It is well known that in case of $1 \leq p < \infty$, the dual space of L^p satisfies $(L^p)^* = L^q$ (up to isomorphism) where $q \in (1, \infty]$ is such that $1/p + 1/q = 1$, and the pairing $\langle \cdot, \cdot \rangle : L^p \times L^q \to \mathbb{R}$ between L^p and L^q is given by

$$\langle X, \xi \rangle := \mathbb{E}[X\xi];$$

see, for instance, Theorem IV.8.1 and Theorem IV.8.5 in Dunford and Schwartz (1957). If we set $p = \infty$, then the dual space of L^∞ satisfies $(L^\infty)^* = ba$ (up to isomorphism) where the Banach space $ba := ba(\Omega, \mathcal{F}, \mathbb{P})$ is the space of all bounded, finitely additive functions μ on $(\Omega, \mathcal{F})$ with the property that $\mathbb{P}[A] = 0$ implies $\mu[A] = 0$. In this case, the pairing $\langle \cdot, \cdot \rangle : L^\infty \times ba \to \mathbb{R}$ between L^∞ and ba is given by

$$\langle X, \mu \rangle := \int_\Omega X(\omega) d\mu(\omega);$$

see Theorem IV.8.16 in Dunford and Schwartz (1957). In order to simplify the notation, we use

$$\langle X, \xi \rangle = \mathbb{E}[X\xi] \quad \text{for } X \in L^p, \xi \in (L^p)^* \text{ and all } 1 \leq p \leq \infty$$

for the pairing between L^p and $(L^p)^*$.

Now, let us consider the multi-dimensional case. For $m \in \mathbb{N}$, in particular in case of $m > 1$, we equip the space $(L^p)^m$ for $1 \leq p \leq \infty$ with the norm

$$\|\bar{X}\|_{p,m} := \sum_{i=1}^m \|\bar{X}_i\|_p \quad \text{for } \bar{X} = (\bar{X}_1, \dots, \bar{X}_m) \in (L^p)^m.$$

The dual spaces satisfy $((L^p)^m)^* = (L^q)^m$ (up to isomorphism) for $1 \leq p < \infty$ and $((L^\infty)^m)^* = (ba)^m$ (up to isomorphism) for $p = \infty$, and the pairing $\langle \cdot, \cdot \rangle_m : (L^p)^m \times ((L^p)^m)^* \to \mathbb{R}$ between $(L^p)^m$ and $((L^p)^m)^*$ is given by

$$\langle \bar{X}, \bar{\xi} \rangle_m := \sum_{i=1}^m \langle \bar{X}_i, \bar{\xi}_i \rangle = \sum_{i=1}^m \mathbb{E}[\bar{X}_i \bar{\xi}_i] \quad \text{for } \bar{X} = (\bar{X}_1, \dots, \bar{X}_m) \in (L^p)^m$$
$$\text{and } \bar{\xi} = (\bar{\xi}_1, \dots, \bar{\xi}_m) \in ((L^p)^m)^*.$$

For all $m \in \mathbb{N}$, we use the corresponding norm topology on $((L^p)^m)^*$ in case of $1 < p < \infty$. If $p = 1$ or $p = \infty$, we consider the weak*-topologies $\sigma((L^\infty)^m, (L^1)^m)$

and $\sigma((ba)^m, (L^\infty)^m)$, respectively. Both topologies are compatible with the corresponding pairing above; see Appendix A.2 for more details. Note that with these notations and definitions we have determined the paired spaces $((L^p)^m, ((L^p)^m)^*)$ in the sense of Definition A.2.1.

Finally, in the multi-dimensional case, we will frequently use the notations $1_m := (1, \dots, 1)$ and $0_m := (0, \dots, 0)$ (m-times) for $m \in \mathbb{N}$.

Now, let us come back to the main objects in this chapter. Recall that we focus on a network consisting of n firms. Our aim is to define systemic risk measures as mappings

$$\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$$

which quantify the risk associated with an economy $\bar{X} \in (L^p)^n$. Moreover, this economy is specified by the losses $\bar{X}_1, \dots, \bar{X}_n$ of the individual firms in the underlying system. In what follows, we will see that systemic risk measures are connected to convex and coherent risk measures which were discussed in Chapter 4. More precisely, every systemic risk measure is a decomposition of a single-firm risk measure, which is essentially identical to a risk measure from Chapter 4, and a so called aggregation function.

First of all, let us consider important properties of a function $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$:

(R1) Monotonicity: If $X \geq Y$, then $\rho_0(X) \geq \rho_0(Y)$ for all $X, Y \in L^p$.

(R2) Convexity: $\rho_0(aX + (1-a)Y) \leq a\rho_0(X) + (1-a)\rho_0(Y)$ for all $X, Y \in L^p$ and $a \in [0, 1]$.

(R3) Translation property: $\rho_0(X + a) = \rho_0(X) + a$ for all $X \in L^p$ and $a \in \mathbb{R}$.

(R4) Positive homogeneity: $\rho_0(aX) = a\rho_0(X)$ for all $X \in L^p$ and $a \in \mathbb{R}_+$.

(R5) Constancy on $\mathcal{R} \subset \mathbb{R}$: $\rho_0(a) = a$ for all $a \in \mathcal{R}$.

(R6) Normalization: $\rho_0(1) = 1$.

As we want to quantify the risk of a given economy that is represented by individual losses, the properties above slightly differ from the corresponding properties in Chapter 4, where we have considered the worth of a financial position. Because of these different viewpoints, the inequality in property (R1) is reversed compared to the monotonicity property from Chapter 4. Similarly, in the translation property (R3) we add a to $\rho_0(X)$ instead of subtracting a from $\rho_0(X)$. Nevertheless, the motivation behind the properties (R1)-(R4) above is analogous to the motivation of the corresponding properties in Chapter 4.

The constancy property (R5) was originally introduced and studied in Frittelli and Rosazza Gianin (2002). Note that the translation property and $\rho_0(0) = 0$ imply constancy on $\mathbb{R}$. Moreover, the normalization property (R3) is equivalent to constancy on $\{1\}$.

Definition 5.1.1. *A* convex single-firm risk measure *is a function $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$ that satisfies the properties (R1) and (R2). A* positively homogeneous single-firm risk measure *is a convex single-firm risk measure that additionally satisfies the properties (R4) and (R6). A* coherent single-firm risk measure *is a positively homogeneous single-firm risk measure that additionally satisfies the property (R3).*

This definition of convex single-firm risk measures (that satisfy monotonicity and convexity) is neither equivalent to the definition of standard risk measures in Chapter 4 nor to the definition of standard convex risk measures in Chapter 4. Solely the term coherent single-firm risk measure coincides with the term coherent risk measure defined in Chapter 4 (up to the sign change and normalization). We will explain this difference in detail below. Nevertheless, note that every standard convex risk measure $\hat{\rho}$ defined in Chapter 4 induces a convex single-firm risk measure ρ_0 in the sense of Definition 5.1.1 by $\rho_0(X) := \hat{\rho}(-X)$ for $X \in L^p$.

Let us consider the following properties of a function $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$:

(S1) Monotonicity: If $\bar{X} \geq \bar{Y}$, then $\rho(\bar{X}) \geq \rho(\bar{Y})$ for all $\bar{X}, \bar{Y} \in (L^p)^n$.

(S2) Preference consistency: If $\rho(\bar{X}(\omega)) \geq \rho(\bar{Y}(\omega))$ for $\bar{X}, \bar{Y} \in (L^p)^n$ and a.e. $\omega \in \Omega$, then $\rho(\bar{X}) \geq \rho(\bar{Y})$.

(S3) f_ρ-constancy: Either $\operatorname{Im} \rho|_{\mathbb{R}^n} = \mathbb{R}$ and there exists a surjective function $f_\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(a1_n) = f_\rho(a)$ for all $a \in \mathbb{R}$ or $\operatorname{Im} \rho|_{\mathbb{R}^n} = \mathbb{R}_+$ and there exists a function $f_\rho : \mathbb{R} \to \mathbb{R}_+$ and $b \in \mathbb{R}_+$ such that f_ρ is surjective and strictly increasing on $[b, \infty)$, $f_\rho(a) = 0$ for $a \leq b$ and $\rho(a1_n) = f_\rho(a)$ for all $a \in \mathbb{R}$.

(S4) Convexity:

- **(S4a)** Outcome convexity: $\rho(a\bar{X} + (1-a)\bar{Y}) \leq a\rho(\bar{X}) + (1-a)\rho(\bar{Y})$ for all $\bar{X}, \bar{Y} \in (L^p)^n$ and $a \in [0, 1]$.
- **(S4b)** Risk convexity: Suppose $\rho(\bar{Z}(\omega)) = a\rho(\bar{X}(\omega)) + (1-a)\rho(\bar{Y}(\omega))$ for $\bar{X}, \bar{Y}, \bar{Z} \in (L^p)^n$, a given scalar $a \in [0, 1]$ and for a.e. $\omega \in \Omega$. Then $\rho(\bar{Z}) \leq a\rho(\bar{X}) + (1-a)\,\rho(\bar{Y})$.

(S5) Positive homogeneity: $\rho(a\bar{X}) = a\rho(\bar{X})$ for all $\bar{X} \in (L^p)^n$ and $a \in \mathbb{R}_+$.

(S6) Normalization: $\rho(1_n) = n$.

We understand (S2) and (S4b) in the following way: If the property (S2) is satisfied, then $\mathbb{P}[\{\omega \in \Omega | \rho(\bar{x}) \geq \rho(\bar{y}), (\bar{x}, \bar{y}) = (\bar{X}(\omega), \bar{Y}(\omega))\}] = 1$ implies that $\rho(\bar{X}) \geq \rho(\bar{Y})$. Similarly, if the property (S4b) is satisfied, then $\mathbb{P}[\{\omega \in \Omega | \rho(\bar{z}) = a\rho(\bar{x}) + (1-a)\rho(\bar{y}), (\bar{z}, \bar{x}, \bar{y}) = (\bar{Z}(\omega), \bar{X}(\omega), \bar{Y}(\omega))\}] = 1$ implies that $\rho(\bar{Z}) \leq a\rho(\bar{X}) + (1-a)\,\rho(\bar{Y})$.

Note that $\rho|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. As a consequence, $\rho|_{\mathbb{R}^n} \circ \bar{X} : \Omega \to \mathbb{R}$ is also measurable.

For the remaining part of this chapter it is essential to know that the properties (S1), (S3) and (S4a) guarantee the existence of the inverse function f_ρ^{-1} of f_ρ. Moreover, there exist two different cases: If $\operatorname{Im} \rho|_{\mathbb{R}^n} = \mathbb{R}$, then f_ρ and the inverse function f_ρ^{-1} are maps from $\mathbb{R}$ to $\mathbb{R}$. On the other hand, if $\operatorname{Im} \rho|_{\mathbb{R}^n} = \mathbb{R}_+$, then the function f_ρ is surjective and strictly increasing on $[b, \infty)$ and the inverse function f_ρ^{-1} maps from $\mathbb{R}_+$ into $[b, \infty)$. Occasionally, we do not distinguish between these two cases for simplicity.

Definition 5.1.2. *A* positively homogeneous systemic risk measure *is a function $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$ that satisfies the properties (S1), (S2), (S4), (S5) and (S6). If ρ satisfies the properties (S1)-(S4) and the function f_ρ from property (S3) satisfies $\|f_\rho(Z)\|_p < \infty$ and $\|f_\rho^{-1}(Z)\|_p < \infty$ for all $Z \in L^p$ ($\|f_\rho^{-1}(Z)\|_p < \infty$ for* *all* $Z \in L^p_+ := \{X \in L^p | X \geq 0\}$ *if* $\operatorname{Im} f_\rho = \mathbb{R}_+$*), we call* ρ *a* convex systemic risk measure.

Monotonicity (S1), outcome convexity (S4a) and positive homogeneity (S5) have already been considered in case of single-firm risk measurement, and we can interpret these properties in the same way. The normalization property (S6) is the systemic counterpart to property (R6). Preference consistency (S2) and risk convexity (S4b) were originally introduced, motivated and studied in Chen et al. (2013). To interpret the preference consistency, consider two economies $\bar{X}$ and $\bar{Y}$. If in a.e. scenario $\omega \in \Omega$ the systemic risk of an economy $\bar{Y}(\omega) \in \mathbb{R}^n$ is less than or equal to the systemic risk of another economy $\bar{X}(\omega) \in \mathbb{R}^n$, then this relation should still be satisfied by the systemic risk of the random economies $\bar{X} \in (L^p)^n$ and $\bar{Y} \in (L^p)^n$. Similarly, the risk convexity property is based on assumptions on realizations of economies $\bar{X}$, $\bar{Y}$ and $\bar{Z}$. If for a.e. $\omega \in \Omega$ the systemic risk of $\bar{Z}(\omega)$ is equal to the convex combination of the systemic risk of $\bar{X}(\omega)$ and $\bar{Y}(\omega)$, then the systemic risk of the random economy $\bar{Z}$ is bounded from above by the convex combination of the systemic risk of $\bar{X}$ and $\bar{Y}$. Note that the transition from constant economies $\bar{X}(\omega)$, $\bar{Y}(\omega)$ and $\bar{Z}(\omega)$, $\omega \in \Omega$, to random economies $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ can be interpreted as introduction of "randomness". Due to property (S4b), this process of transition does not lead to an increase of the systemic risk of $\bar{Z}$ beyond the convex combination of the systemic risk of $\bar{X}$ and $\bar{Y}$.

Chen et al. (2013) point out that in case of an economy that consists of a single firm, i.e., $n = 1$, outcome convexity and constancy on $\mathbb{R}$ imply risk convexity directly. Indeed, in this case, $\rho(Z(\omega)) = a\rho(X(\omega)) + (1-a)\,\rho(Y(\omega))$ for $X, Y, Z \in L^p$ is equivalent to $Z(\omega) = aX(\omega) + (1-a)Y(\omega)$ for a.e. $\omega \in \Omega$, which means that $Z = aX + (1-a)Y$. Now, the inequality $\rho(Z) \leq a\rho(X) + (1-a)\rho(Y)$ follows directly from outcome convexity.

It remains to discuss the f_ρ-constancy property (S3). This property is new, and we will see that it is essential in the decomposition result below. Note that the f_ρ-constancy property tells us something about the behavior of ρ on constants: Let us assume that each firm in the financial network has the same constant loss $a \in \mathbb{R}$. Then the systemic risk of this economy is equal to the value of a function $f_\rho : \mathbb{R} \to \mathbb{R}$ in a. Since for every function from $\mathbb{R}$ to $\mathbb{R}$ surjectivity, monotonicity and convexity imply strict monotonicity, we know that f_ρ is strictly increasing and unbounded. Let us consider a second economy in which each firm has the same constant loss $c \in \mathbb{R}$ with $c > a$. Then strict monotonicity of f_ρ implies that the systemic risk of the second economy is strictly greater than the systemic risk of the first economy, i.e., $\rho(c1_n) > \rho(a1_n)$. A direct consequence of the unboundedness of f_ρ is that the systemic risk of $a1_n$ increases to infinity if the constant loss a increases to infinity. Therefore, we solely study systemic risk measures without an upper bound.

Finally, let us consider the following properties of a function $\Lambda : \mathbb{R}^n \to \mathbb{R}$:

(A1) Monotonicity: If $\bar{x} \geq \bar{y}$, then $\Lambda(\bar{x}) \geq \Lambda(\bar{y})$ for all $\bar{x}, \bar{y} \in \mathbb{R}^n$.

(A2) Convexity: $\Lambda(a\bar{x} + (1-a)\bar{y}) \leq a\Lambda(\bar{x}) + (1-a)\Lambda(\bar{y})$ for all $\bar{x}, \bar{y} \in \mathbb{R}^n$ and $a \in [0,1]$.

(A3) f_Λ-constancy: Either $\operatorname{Im}\Lambda = \mathbb{R}$ and there exists a surjective function $f_\Lambda : \mathbb{R} \to \mathbb{R}$ such that $\Lambda(a1_n) = f_\Lambda(a)$ for all $a \in \mathbb{R}$ or $\operatorname{Im}\Lambda = \mathbb{R}_+$ and there exists a function $f_\Lambda : \mathbb{R} \to \mathbb{R}_+$ and $b \in \mathbb{R}_+$ such that f_Λ is surjective and strictly increasing on $[b, \infty)$, $f_\Lambda(a) = 0$ for $a \leq b$ and $\Lambda(a1_n) = f_\Lambda(a)$ for all $a \in \mathbb{R}$.

(A4) Positive homogeneity: $\Lambda(a\bar{x}) = a\Lambda(\bar{x})$ for all $\bar{x} \in \mathbb{R}^n$ and $a \in \mathbb{R}_+$.

(A5) Normalization: $\Lambda(1_n) = n$.

Definition 5.1.3. *A* positively homogeneous aggregation function *is a function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ that satisfies the properties (A1), (A2), (A4) and (A5). If Λ satisfies the properties (A1)-(A3) and the function f_Λ from (A3) satisfies $\|f_\Lambda(Z)\|_p < \infty$ and $\|f_\Lambda^{-1}(Z)\|_p < \infty$ for all $Z \in L^p$ ($\|f_\Lambda^{-1}(Z)\|_p < \infty$ for all $Z \in L^p_+$ if $\operatorname{Im} f_\Lambda = \mathbb{R}_+$), we call Λ a* convex aggregation function.

Remark 5.1.4. If $p = \infty$, the properties (A1)-(A3) imply that $\|f_\Lambda(Z)\|_p < \infty$ and $\|f_\Lambda^{-1}(Z)\|_p < \infty$ for all $Z \in L^p$ ($\|f_\Lambda^{-1}(Z)\|_p < \infty$ for all $Z \in L^p_+$ if $\operatorname{Im} f_\Lambda = \mathbb{R}_+$). Therefore, we do not have to claim these additional properties in the previous definition in case of $p = \infty$.

It is well known that every convex and finite valued function on $\mathbb{R}^n$ is continuous. Thus, the convex aggregation function Λ is continuous and measurable. Note that the f_Λ-constancy property (A3) is similar to the f_ρ-constancy property (S3), and the motivation for both properties is the same: The value of the aggregation function of an economy in which every firm has the same constant loss $a \in \mathbb{R}$ is equal to the value of $f_\rho : \mathbb{R} \to \mathbb{R}$ in a. We will see in the decomposition theorem below (see Theorem 5.2.1) that the constancy properties (R5), (S3) and (A3) are highly dependent on each other. Moreover, the constancy properties are key properties that enable us to drop the positive homogeneity property and to consider convex systemic risk measures that are not necessarily positively homogeneous. At this point, it is important to note that standard convex risk measures $\hat{\rho}$ defined in Chapter 4 satisfy the translation property. In addition to this, many examples of standard convex risk measures also satisfy $\hat{\rho}(0) = 0$. Since these two properties imply $\hat{\rho}(a) = -a$ for $a \in \mathbb{R}$, these standard convex risk measures satisfy the corresponding constancy property on $\mathbb{R}$. Nonetheless, if we consider single-firm risk measures in conjunction with systemic risk measures, then the translation property, which is satisfied by all standard convex risk measures, is not required any more (see again Theorem 5.2.1). This is precisely the reason for changing the definition of convex single-firm risk measures in this chapter compared to the standard approach in Chapter 4.

The following lemma clarifies why we claim the f_Λ-constancy property for convex aggregation functions.

Lemma 5.1.5. *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a convex aggregation function with $\operatorname{Im}\Lambda = \mathbb{R}$ [$\operatorname{Im}\Lambda = \mathbb{R}_+$]. Then $\Lambda((L^p)^n) = L^p$ [$\Lambda((L^p)^n) = L^p_+$].*

Proof. Let us suppose that $\operatorname{Im}\Lambda = \mathbb{R}$. The proof of the other case is analogous. Fix $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_n) \in (L^p)^n$ and define the random variable $Z_{\bar{X}}$ by

$$Z_{\bar{X}}(\omega) := \max\{\bar{X}_1(\omega), \ldots, \bar{X}_n(\omega)\} I_A(\omega) + \min\{\bar{X}_1(\omega), \ldots, \bar{X}_n(\omega)\} I_{A^c}(\omega)$$

where $A := \{\omega \in \Omega | \Lambda(\bar{X}(\omega)) \geq 0\}$. Then $Z_{\bar{X}} \in L^p$ because the maximum and the minimum of L^p-integrable (bounded) random variables are L^p-integrable (bounded). The monotonicity property of Λ implies $0 \leq \Lambda(\bar{X}(\omega)) \leq \Lambda(Z_{\bar{X}}(\omega)1_n)$ for all $\omega \in A$ and $0 > \Lambda(\bar{X}(\omega)) \geq \Lambda(Z_{\bar{X}}(\omega)1_n)$ for all $\omega \in A^c$. It follows

$$|\Lambda(\bar{X}(\omega))| \leq |\Lambda(Z_{\bar{X}}(\omega)1_n)| = |f_\Lambda(Z_{\bar{X}}(\omega))| \quad \text{for all } \omega \in \Omega,$$

which yields $\mathbb{E}[|\Lambda(\bar{X})|^p] \leq \mathbb{E}[|f_\Lambda(Z_{\bar{X}})|^p]$ in case of $p < \infty$ and $\inf\{r \in \mathbb{R} | |\Lambda(\bar{X})| \leq r\} \leq \inf\{r \in \mathbb{R} | |f_\Lambda(Z_{\bar{X}})| \leq r\}$ in case of $p = \infty$. Since $\|f_\Lambda(Z_{\bar{X}})\|_p < \infty$ for all $Z \in L^p$, we obtain

$$\|\Lambda(\bar{X})\|_p \leq \|f_\Lambda(Z_{\bar{X}})\|_p < \infty.$$

But this means that $\Lambda(\bar{X}) \in L^p$. Thus, $\Lambda((L^p)^n) \subset L^p$.

To prove the other inclusion, consider an arbitrary random variable $X \in L^p$. The properties (A1)-(A3) imply the existence of the inverse function f_Λ^{-1}. Therefore, we can define the random variable Y_X by

$$Y_X(\omega) := f_\Lambda^{-1}(X(\omega)) \quad \text{for } \omega \in \Omega.$$

Since $\|f_\Lambda^{-1}(Z)\|_p < \infty$ for all $Z \in L^p$, we obtain that $Y_X \in L^p$. The definition of Y_X implies $f_\Lambda(Y_X) = X$ and the f_Λ-constancy property of Λ yields $\Lambda(Y_X 1_n) = f_\Lambda(Y_X) = X$. This means that $L^p \subset \Lambda((L^p)^n)$. Together with the first part of this proof, we have $\Lambda((L^p)^n) = L^p$. □

The previous lemma guarantees that for every $\bar{X} \in L^p$, the image $\Lambda(\bar{X})$ is again an element in the space L^p. On the other hand, the surjectivity of the real valued function Λ is transferred to Λ considered as a mapping from $(L^p)^n$ to L^p.

Lemma 5.1.6. *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous aggregation function. Then Λ also satisfies the f_Λ-constancy property (A3). The corresponding function f_Λ is given by*

$$f_\Lambda(a) = \begin{cases} an & \text{if } a \geq 0 \\ a(-\Lambda(-1_n)) & \text{if } a < 0 \end{cases} \tag{5.1}$$

in case of $\operatorname{Im}\Lambda = \mathbb{R}$ *and* $f_\Lambda(a) = na^+$ *in case of* $\operatorname{Im}\Lambda = \mathbb{R}_+$.

Proof. For every positively homogeneous aggregation function Λ, we have $\Lambda(a1_n) = a\Lambda(1_n) = an$ for all $a \in \mathbb{R}_+$. Therefore, $\mathbb{R}_+ \subset \operatorname{Im}\Lambda$. If $\Lambda(\bar{x}) \geq 0$ for all $\bar{x} \in \mathbb{R}^n$, then monotonicity and positive homogeneity lead to $0 \leq \Lambda(a1_n) \leq \Lambda(0_n) = 0$ for $a < 0$. This means that $f_\Lambda(a) = na^+$.

If there exists $\bar{x} \in \mathbb{R}^n$ such that $\Lambda(\bar{x}) < 0$, then there exists $i \in \{1, \dots, n\}$ with $\bar{x}_i < 0$. If we define $y := \min_{i \in \{1,\dots,n\}} \bar{x}_i$, then by monotonicity, we have $\Lambda(y1_n) \leq \Lambda(\bar{x}) < 0$, and positive homogeneity implies $\Lambda(y1_n) = (-y)\Lambda(-1_n)$. Thus, $\Lambda(-1_n) < 0$ and $\operatorname{Im}\Lambda = \mathbb{R}$. It remains to prove that f_Λ given by (5.1) is convex. In other words, we have to show that $-\Lambda(-1_n) \leq n$. But this inequality follows from convexity and positive homogeneity of Λ, which imply that $0 = \Lambda(0_n) = \Lambda(1_n + (-1_n)) \leq \Lambda(1_n) + \Lambda(-1_n)$. □

Remark 5.1.7. Note that $\|f_\Lambda(Z)\|_p < \infty$ and $\|f_\Lambda^{-1}(Z)\|_p < \infty$ for all $Z \in L^p$ ($\|f_\Lambda^{-1}(Z)\|_p < \infty$ for all $Z \in L_+^p$ if $\operatorname{Im} f_\Lambda = \mathbb{R}_+$) are automatically satisfied if Λ is a positively homogeneous aggregation function. As a consequence, every positively homogeneous aggregation function is also a convex aggregation function. By repeating the same argument for systemic risk measures, we obtain that every positively homogeneous systemic risk measure is also a convex systemic risk measure.

5.2. Structural decomposition

The aim of this section is to generalize the decomposition result in Chen et al. (2013) for convex systemic risk measures defined on a general probability space. We will see that each convex systemic risk measure is a composition of a convex single-firm risk measure and a convex aggregation function. By the final remark in the previous section, every positively homogeneous systemic risk measure is also a convex systemic risk measure. Therefore, the positively homogeneous case considered in Chen et al. (2013) can be regarded as a special case of our decomposition theorem for convex systemic risk measures.

The proof of the decomposition theorem below is in several arguments similar to the proof of Theorem 1 in Chen et al. (2013). Nevertheless, the extension from positively homogeneous systemic risk measures to convex systemic risk measures which are not necessarily positively homogeneous requires an application of Lemma 5.1.5, which is new in the context of convex systemic risk measurement. Since this lemma strongly depends on the f_Λ-constancy property of Λ, the constancy properties (f_ρ-constancy, f_Λ-constancy and constancy of ρ_0) play a key role in our decomposition result for convex systemic risk measures.

Theorem 5.2.1 (Convex structural decomposition)**.**

a) *A function $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\mathbb{R}^n) = \mathbb{R}$ is a convex systemic risk measure if and only if there exists a convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}$ and a convex single-firm risk measure $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$ that satisfies the constancy property on $\mathbb{R}$ such that ρ is the composition of ρ_0 and Λ, i.e.,*
$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in (L^p)^n.$$

b) *A function $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\mathbb{R}^n) = \mathbb{R}_+$ is a convex systemic risk measure if and only if there exists a convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}_+$ and a convex single-firm risk measure $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$ that satisfies the constancy property on $\mathbb{R}_+$ such that ρ is the composition of ρ_0 and Λ, i.e.,*
$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in (L^p)^n.$$

Proof. In case of part a), set $\mathcal{R} = \mathcal{S} = \mathbb{R}$, and in case of part b), set $\mathcal{R} = \mathbb{R}_+$ and $\mathcal{S} = [b, \infty)$ for $b \in \mathbb{R}_+$. Let ρ be a convex systemic risk measure with $f_\rho : \mathbb{R} \to \mathcal{R}$ such that f_ρ is surjective and strictly increasing on $\mathcal{S}$. We define the function Λ by
$$\Lambda(\bar{x}) := \rho(\bar{x}) \quad \text{for } \bar{x} \in \mathbb{R}^n. \tag{5.2}$$

Then the convexity property (A2) follows from the corresponding property of ρ since

$$\Lambda(a\bar{x}+(1-a)\bar{y})=\rho(a\bar{x}+(1-a)\bar{y})\leq a\rho(\bar{x})+(1-a)\rho(\bar{y})=a\Lambda(\bar{x})+(1-a)\Lambda(\bar{y})$$

for all $\bar{x},\bar{y}\in\mathbb{R}^n$ and $a\in[0,1]$. Similarly, monotonicity of ρ yields the monotonicity property (A1) of Λ.

Because ρ satisfies f_ρ-constancy, the following equality is satisfied for all $a\in\mathbb{R}$:

$$\Lambda(a1_n)=\rho(a1_n)=f_\rho\left(a\right).$$

By setting $f_\Lambda:=f_\rho$, f_Λ-constancy (A3) of Λ follows directly. Therefore, $\Lambda(\mathbb{R}^n)=\mathcal{R}$. Furthermore, Lemma 5.1.5 gives $\Lambda((L^p)^n)=L^p$ in case of a) and $\Lambda((L^p)^n)=L^p_+$ in case of b).

Now, let us define $\tilde{\rho}_0:\Lambda((L^p)^n)\to\mathbb{R}\cup\{+\infty\}$ by

$$\tilde{\rho}_0(X):=\rho(\bar{X})\quad\text{where }\bar{X}\in(L^p)^n\text{ satisfies }\Lambda(\bar{X})=X. \tag{5.3}$$

Moreover, define $\rho_0:L^p\to\mathbb{R}\cup\{+\infty\}$ by

$$\rho_0(X):=\begin{cases}\tilde{\rho}_0(X) & \text{if }\Lambda((L^p)^n)=L^p\\ \tilde{\rho}_0(X^+) & \text{if }\Lambda((L^p)^n)=L^p_+\end{cases}. \tag{5.4}$$

Then $\tilde{\rho}_0$ is well-defined: For two economies $\bar{X},\bar{Y}\in(L^p)^n$ with $\Lambda(\bar{X})=\Lambda(\bar{Y})$, we obtain

$$\begin{aligned}&\rho(\bar{X}(\omega))=\Lambda(\bar{X})(\omega)\geq\Lambda(\bar{Y})(\omega)=\rho(\bar{Y}(\omega))\quad\text{and}\\&\rho(\bar{X}(\omega))=\Lambda(\bar{X})(\omega)\leq\Lambda(\bar{Y})(\omega)=\rho(\bar{Y}(\omega))\end{aligned}$$

for a.e. $\omega\in\Omega$. Furthermore, preference consistency of ρ implies that $\rho(\bar{X})=\rho(\bar{Y})$.

In the following, we will show that $\tilde{\rho}_0$ defined by (5.3) is monotone, convex and satisfies constancy on $\mathcal{R}$. Then it follows immediately that ρ_0 defined by (5.4) satisfies the monotonicity property (R1), convexity (R2) and constancy on $\mathcal{R}$ (R5).

In order to prove the monotonicity property, consider $X,Y\in\Lambda((L^p)^n)$ with $\Lambda(\bar{X})=X$, $\Lambda(\bar{Y})=Y$ and $X\leq Y$. Then

$$\rho(\bar{X}(\omega))=\Lambda(\bar{X})(\omega)\leq\Lambda(\bar{Y})(\omega)=\rho(\bar{Y}(\omega))$$

for a.e. $\omega\in\Omega$, and preference consistency of ρ yields $\tilde{\rho}_0(X)\leq\tilde{\rho}_0(Y)$.

It remains to prove the convexity property and constancy on $\mathcal{R}$. First, let $X,Y\in\Lambda((L^p)^n)$ and $a\in[0,1]$ and define $Z:=aX+(1-a)Y$. Moreover, let $\bar{X},\bar{Y},\bar{Z}\in(L^p)^n$ be such that

$$\tilde{\rho}_0(X)=\rho(\bar{X}),\quad\tilde{\rho}_0(Y)=\rho(\bar{Y}),\quad\tilde{\rho}_0(Z)=\rho(\bar{Z})$$

where $\Lambda(\bar{X})=X$, $\Lambda(\bar{Y})=Y$ and $\Lambda(\bar{Z})=Z$. Then we obtain

$$\begin{aligned}\rho(\bar{Z}(\omega))&=\Lambda(\bar{Z})(\omega)=Z(\omega)=aX(\omega)+(1-a)Y(\omega)\\&=a\Lambda(\bar{X})(\omega)+(1-a)\Lambda(\bar{Y})(\omega)=a\rho(\bar{X}(\omega))+(1-a)\rho(\bar{Y}(\omega))\end{aligned}$$

for a.e. $\omega \in \Omega$, and risk convexity of ρ implies

$$\tilde{\rho}_0(Z) = \rho(\bar{Z}) \leq a\rho(\bar{X}) + (1-a)\rho(\bar{Y}) = a\tilde{\rho}_0(X) + (1-a)\tilde{\rho}_0(Y).$$

This means that $\tilde{\rho}_0$ satisfies the convexity property. Finally, note that for all $a \in \mathcal{R}$, there exists $\bar{x} \in \mathbb{R}^n$ with $\Lambda(\bar{x}) = a$ and $\tilde{\rho}_0(a) = \rho(\bar{x})$. Since $a = \Lambda(\bar{x}) = \rho(\bar{x})$, we obtain $\tilde{\rho}_0(a) = a$ for all $a \in \mathcal{R}$. Therefore, $\tilde{\rho}_0$ satisfies the constancy property on $\mathcal{R}$. The equality $\rho = \rho_0 \circ \Lambda$ follows immediately from the definition of ρ_0 and Λ.

For the second part of the proof consider a convex aggregation function Λ with $f_\Lambda : \mathbb{R} \to \mathcal{R}$ that is surjective and strictly increasing on $\mathcal{S}$ and suppose that ρ_0 is a convex single-firm risk measure with $\rho_0(a) = a$ for all $a \in \mathcal{R}$. Monotonicity (S1) and outcome convexity (S4a) of ρ are satisfied due to the corresponding properties of ρ_0 and Λ. To prove preference consistency (S2), consider $\bar{X}, \bar{Y} \in (L^p)^n$ with

$$(\rho_0 \circ \Lambda)(\bar{X}(\omega)) = \rho(\bar{X}(\omega)) \geq \rho(\bar{Y}(\omega)) = (\rho_0 \circ \Lambda)(\bar{Y}(\omega))$$

for a.e. $\omega \in \Omega$. Because ρ_0 satisfies constancy on $\mathcal{R}$ and $\Lambda(\mathbb{R}^n) = \mathcal{R}$, we obtain $\Lambda(\bar{X}(\omega)) \geq \Lambda(\bar{Y}(\omega))$ for a.e. $\omega \in \Omega$. The monotonicity property of ρ_0 leads to

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \geq (\rho_0 \circ \Lambda)(\bar{Y}) = \rho(\bar{Y}),$$

which means that property (S2) is satisfied.

Now, consider $\bar{X}, \bar{Y}, \bar{Z} \in (L^p)^n$ and $a \in [0,1]$ and suppose that $\rho(\bar{Z}(\omega)) = a\rho(\bar{X}(\omega)) + (1-a)\rho(\bar{Y}(\omega))$ for a.e. $\omega \in \Omega$. Since $\rho = \rho_0 \circ \Lambda$, this means

$$\rho_0(\Lambda(\bar{Z}(\omega))) = a\rho_0(\Lambda(\bar{X}(\omega))) + (1-a)\rho_0(\Lambda(\bar{Y}(\omega))) \tag{5.5}$$

for a.e. $\omega \in \Omega$. Note again that $\Lambda(\mathbb{R}^n) = \mathcal{R}$ and $\rho_0\,(c) = c$ for all $c \in \mathcal{R}$. Therefore, Equation (5.5) yields

$$\Lambda(\bar{Z}(\omega)) = a\Lambda(\bar{X}(\omega)) + (1-a)\Lambda(\bar{Y}(\omega))$$

for a.e. $\omega \in \Omega$, i.e., $\Lambda(\bar{Z}) = a\Lambda(\bar{X}) + (1-a)\Lambda(\bar{Y})$. The convexity property of ρ_0 implies

$$\rho(\bar{Z}) = \rho_0(\Lambda(\bar{Z})) \leq a\rho_0(\Lambda(\bar{X})) + (1-a)\rho_0(\Lambda(\bar{Y})) = a\rho(\bar{X}) + (1-a)\rho(\bar{Y}).$$

Hence, ρ satisfies the risk convexity property (S4b). It remains to show the f_ρ-constancy property (S3). Since Λ satisfies the f_Λ-constancy property and $\rho_0(c) = c$ for all $c \in \mathcal{R}$, we have

$$\rho(a1_n) = \rho_0(\Lambda(a1_n)) = f_\Lambda(a) \quad \text{for all } a \in \mathbb{R}.$$

But this means that f_ρ-constancy (S3) is satisfied with $f_\rho := f_\Lambda$. □

Remark 5.2.2. In the previous proof Lemma 5.1.5 ensures that $\tilde{\rho}_0$ specified in Equation (5.3) is a mapping defined on the entire space L^p or on the entire space L^p_+, respectively. In the positively homogeneous case studied in Chen et al. (2013), this property is an immediate consequence of positive homogeneity and normalization.

The following corollary addresses the positively homogeneous special case of Theorem 5.2.1.

Corollary 5.2.3 (Positively homogeneous structural decomposition)**.**

a) A function $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\mathbb{R}^n) = \mathbb{R}$ is a positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}$ and a coherent single-firm risk measure $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in (L^p)^n.$$

b) A function $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\mathbb{R}^n) = \mathbb{R}_+$ is a positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}_+$ and a positively homogeneous single-firm risk measure $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in (L^p)^n.$$

Proof. Note that every positively homogeneous systemic risk measure ρ is also a convex systemic risk measure, and every positively homogeneous aggregation function Λ is a convex aggregation function. Similarly, every coherent [positively homogeneous] single-firm risk measure ρ_0 is also a convex systemic risk measure which satisfies constancy on $\mathbb{R}$ [$\mathbb{R}_+$]. Therefore, we can apply the convex decomposition theorem (see Theorem 5.2.1).

First, let us suppose that ρ is a positively homogeneous systemic risk measure. Then it remains to prove that Λ defined by (5.2) and $\tilde{\rho}_0$ defined by (5.3) are positively homogeneous and normalized. Note that ρ_0 defined by (5.4) inherits both properties from $\tilde{\rho}_0$. In case of part a), we additionally have to verify the translation property (R3) for $\rho_0 = \tilde{\rho}_0$.

Positive homogeneity (A4) and normalization (A5) of Λ follow directly from the corresponding property of ρ. Moreover, $\Lambda(1_n/n) = (1/n)\Lambda(1_n) = 1$ and $\rho(1_n/n) = (1/n)\rho(1_n) = 1$ such that $\tilde{\rho}_0(1) = \rho(1_n/n) = 1$, which means that $\tilde{\rho}_0$ is normalized. Now, consider $X \in \Lambda((L^p)^n)$ and $\bar{X} \in (L^p)^n$ such that $\Lambda(\bar{X}) = X$. Since Λ is positively homogeneous, we have $\Lambda(a\bar{X}) = a\Lambda(\bar{X}) = aX$ for all $a \in \mathbb{R}_+$. Hence, the positive homogeneity property of ρ implies $\tilde{\rho}_0(aX) = \rho(a\bar{X}) = a\rho(\bar{X}) = a\tilde{\rho}_0(X)$ for all $a \in \mathbb{R}_+$.

To prove the translation property (R3) in case of part a), note that we know from Theorem 5.2.1 that $\rho_0\,(a) = a$ for all $a \in \mathbb{R}$. Since ρ_0 is convex and positively homogenous, ρ_0 is also subadditive (see Remark 4.1.2). It follows for $X \in L^p$ and all $a \in \mathbb{R}$ that

$$\begin{aligned} &\rho_0(X+a) \le \rho_0(X) + \rho_0(a) = \rho_0(X) + a \quad \text{and} \\ &\rho_0(X+a) = \rho_0(X - (-a)) \ge \rho_0(X) - \rho_0(-a) = \rho_0(X) + a. \end{aligned}$$

This means that ρ_0 satisfies the translation property (R3).

For the second part of the proof it remains to show the normalization property (S6) and positive homogeneity (S5). Obviously, ρ is positively homogeneous since Λ and ρ_0 satisfy the corresponding property. Finally, ρ_0 has the positive homogeneity and the translation property in case of a), and ρ_0 is positively homogeneous and satisfies $\rho_0(1) = 1$ in case of b). Therefore, we have $\rho_0(a) = a$ for all $a \in \mathcal{R}$ in both cases. This property together with the normalization property of Λ yields that ρ is normalized. □

Remark 5.2.4. In the first part of the proof of the previous corollary we have shown that every positively homogeneous single-firm risk measure ρ_0 which satisfies constancy on $\mathbb{R}$ also admits the translation property. Consequently, an alternative formulation for part a) of Corollary 5.2.3 is the following:

a') A function $\rho : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\mathbb{R}^n) = \mathbb{R}$ is a positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}$ and a positively homogeneous single-firm risk measure $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$ that satisfies the constancy property on $\mathbb{R}$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \text{for all } \bar{X} \in (L^p)^n.$$

5.3. Examples of systemic risk measures

In this section we provide examples of convex and positively homogeneous systemic risk measures. Since the positively homogeneous systemic risk measures considered in Chen et al. (2013) are special cases of our convex systemic risk measures, we can carry over their examples to our setting. Nevertheless, our systemic risk measures do not necessarily have to be positively homogeneous such that we can add completely new examples. We will see that all examples in the subsequent section are based on the decomposition results above. This means that we construct convex [positively homogeneous] systemic risk measures by defining the corresponding convex [positively homogeneous] single-firm risk measure and the corresponding convex [positively homogeneous] aggregation function.

Example 5.3.1. The first possibility to obtain a simple positively homogeneous aggregation function is to sum up the losses/ profits of each firm $i \in \{1, \dots, n\}$. The resulting aggregation function $\Lambda_{\text{sum}} : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\Lambda_{\text{sum}}(\bar{x}) := \sum_{i=1}^{n} \bar{x}_i. \tag{5.6}$$

Note that Λ_{sum} is a linear function. We can generalize this positively homogeneous aggregation function by considering $\Lambda_{\text{aff}} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Lambda_{\text{aff}}(\bar{x}) := \bar{b}^t \bar{x} + c \quad \text{for } \bar{b} \in \mathbb{R}^n, \bar{b} > 0_n \text{ and } c \in \mathbb{R}.$$

We know from Example 4.5.2 that the Average Value at Risk is given by

$$\text{AVaR}_\lambda(X) = \max_{\mathbb{Q} \in \mathcal{Q}_\lambda} \mathbb{E}_{\mathbb{Q}}[X] \quad \text{for all } X \in L^p \text{ and } \lambda \in (0, 1] \tag{5.7}$$

where $\mathcal{Q}_\lambda$ is the set of all probability measures $\mathbb{Q}$ which are absolutely continuous with respect to $\mathbb{P}$ and satisfy $d\mathbb{Q}/d\mathbb{P} \leq 1/\lambda$ $\mathbb{P}$-a.s. Note that we interpret $\bar{X}_1, \dots, \bar{X}_n$ as losses while in Example 4.5.2 the perspective is reversed. This explains why we have $\mathbb{E}_{\mathbb{Q}}[X]$ instead of $\mathbb{E}_{\mathbb{Q}}[-X]$ in representation (5.7).

If we now consider the corresponding composition, then we obtain the positively homogeneous systemic risk measure

$$\rho_{\mathrm{SEM}}(\bar{X}) := \mathrm{AVaR}_\lambda\left(\sum_{i=1}^{n} \bar{X}_i\right) \quad \text{for } \bar{X} \in (L^p)^n \text{ and } \lambda \in (0,1].$$

This risk measure, also referred to as *Systemic Expected Shortfall*, has already been discussed in Acharya et al. (2010). Nevertheless, Chen et al. (2013) point out that a financial regulator, in general, does not want to compensate the losses of one firm with the profits of another. In conclusion, the positively homogeneous systemic risk measure ρ_{SEM} is not an appropriate choice to meet these specific demands.

In the next example we provide a positively homogeneous aggregation function that covers the criticism above. Moreover, by considering this positively homogeneous aggregation function in conjunction with a convex systemic risk measure, we obtain a first example of a convex systemic risk measure which is not positively homogeneous.

Example 5.3.2. Let the positively homogeneous aggregation function $\Lambda_{\mathrm{loss}} : \mathbb{R}^n \to \mathbb{R}_+$ be defined by

$$\Lambda_{\mathrm{loss}}(\bar{x}) := \sum_{i=1}^{n} \bar{x}_i^+. \tag{5.8}$$

Since this aggregation function sets all profits equal to 0, it is not possible to cross-subsidize the losses of one firm with the gains of another. The previous aggregation function defined in (5.8) can be modified by introducing a lower bound which is not necessarily equal to 0. In this case, we obtain the convex aggregation function $\Lambda_{b,\mathrm{loss}} : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$\Lambda_{b,\mathrm{loss}}(\bar{x}) := \sum_{i=1}^{n} (\bar{x}_i - b)^+ \quad \text{for } b \in \mathbb{R}_+.$$

If this convex aggregation function is applied, then the losses above b are aggregated only. Since $\Lambda_{b,\mathrm{loss}}$ is convex but not positively homogeneous and $\operatorname{Im}\Lambda = \mathbb{R}_+$, these aggregation functions are covered by the second part of Theorem 5.2.1.

In order to define a convex systemic risk measure, we can use the aggregation function Λ_{loss} in conjunction with the distortion entropic risk measure defined in Example 4.5.4. This convex single-firm risk measure is given by

$$\rho_0^{g,\gamma}(X) = \frac{1}{\gamma} \log \mathbb{E}_g[e^{\gamma X}] \quad \text{for } X \in L^\infty \text{ and } \gamma > 0$$

where $\mathbb{E}_g$ denotes the distorted expectation for a nondecreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$. Together with the positively homogeneous aggregation function defined in (5.8), we obtain the *distortion entropic systemic risk* *measure*

$$\rho_{g,\gamma}(\bar{X}) := \frac{1}{\gamma} \log \mathbb{E}_g \left[e^{\gamma \sum_{i=1}^{n} \bar{X}_i^+} \right] \quad \text{for } \bar{X} \in (L^\infty)^n \text{ and } \gamma > 0. \tag{5.9}$$

This systemic risk measure is convex but not positively homogeneous. This follows from the fact that the underlying distortion entropic risk measure $\rho_0^{g,\gamma}$ is not positively homogeneous. Finally, note that the $f_{\rho_{g,\gamma}}$-constancy property is satisfied with $f_{\rho_{g,\gamma}}(a) = f_{\Lambda_{\text{loss}}}(a) = na^+$ for $a \in \mathbb{R}$.

Another critical property, which has already been pointed out in Chen et al. (2013), is that both positively homogeneous aggregation functions Λ_{sum} and Λ_{loss} do not distinguish between large losses and small losses. As a consequence, one large loss is as bad as several smaller losses summing up to the same amount. The following two examples consider convex aggregation functions without this property.

Example 5.3.3. It seems to be reasonable that a financial regulator strongly prefers several smaller losses against one large loss. Then the convex aggregation function $\Lambda_{\text{exp}} : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$\Lambda_{\text{exp}}(\bar{x}) := \sum_{i=1}^{n} (e^{\gamma \bar{x}_i^+} - 1) \quad \text{for } \gamma > 0$$

is a possible choice. Here, the regulator penalizes losses exponentially, and hence the convex aggregation function Λ_{exp} is not positively homogeneous. The corresponding function $f_{\Lambda_{\text{exp}}}$ is given by $f_{\Lambda_{\text{exp}}}(a) = n(e^{\gamma a^+} - 1)$ for $a \in \mathbb{R}$. Since $\Lambda_{\text{exp}}(\bar{X}) \in L^\infty$ for all $\bar{X} \in (L^\infty)^n$, this convex aggregation function is suitable for convex systemic risk measures defined on $(L^\infty)^n$. As in the previous examples, this aggregation function does not allow the compensation of losses of one firm with gains of another.

Example 5.3.4. Let us consider the piecewise linear convex aggregation function $\Lambda_{\text{plin}} : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$\Lambda_{\text{plin}}(\bar{x}) := \sum_{i=1}^{n} \lambda(\bar{x}_i)$$

where $\lambda : \mathbb{R} \to \mathbb{R}_+$ is given by

$$\lambda(x) := \begin{cases} 0 & \text{for } x < 0 \\ ax & \text{for } 0 \leq x < c \\ b(x-c) + ac & \text{for } x \geq c \end{cases}$$

with $0 < a < b$ and $c > 0$. Again, this convex aggregation function is not positively homogeneous. Furthermore, by applying the aggregation function Λ_{plin}, we distinguish between losses being below or above the barrier c: If a loss increases above c, then we pay greater attention to those losses. This is implemented by increasing the slope of the convex aggregation function Λ_{plin}.

The following example has already been discussed in Chen et al. (2013) and is motivated by the clearing model of Eisenberg and Noe (2001).

Example 5.3.5. Let us consider a financial network consisting of n nodes. These nodes represent the different firms that are interconnected by the matrix $\Pi = (\Pi_{ij})_{i,j\in\{1,\dots,n\}}$ denoting the interfirm liabilities. For every firm $i \in \{1,\dots,n\}$, Π_{ij} is the proportion of its total liabilities that firm i has to pay to firm j. We suppose that an external regulator is able to intervene in the system by injecting capital. As usual, we interpret $\bar{x} \in \mathbb{R}^n$ as the realized loss, i.e., $\bar{x}_i$ represents the loss of firm $i \in \{1,\dots,n\}$. To cover these losses, each firm has two options: The first possibility is to receive money (the amount $\bar{b}_i$) from the regulator; the other possibility is to reduce payments to other firms (by $\bar{y}_i$). However, reducing the payments to other firms leads to new losses of these firms. More precisely, firm j additionally loses the amount $\Pi_{ij}\bar{y}_i$.

In this context, the following function measures the "net systemic cost of the contagion" (Chen et al. (2013), p. 1380):

$$\tilde{\Lambda}_{\mathrm{CM}}(\bar{x}) := \min_{(\bar{y},\bar{b})\in A^{\bar{x}}} \left\{ \sum_{i=1}^{n} \bar{y}_i + \gamma \bar{b}_i \right\} \quad \text{for } \bar{x} \in \mathbb{R}^n \text{ and } \gamma > 1 \tag{5.10}$$

where $A^{\bar{x}}$ is defined by

$$A^{\bar{x}} := \{(\bar{y},\bar{b}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ | \bar{b}_i + \bar{y}_i \geq \bar{x}_i + \sum_{j=1}^{n} \Pi_{ji}\bar{y}_j \text{ for all } i \in \{1,\dots,n\}\}.$$

Moreover, we can normalize $\tilde{\Lambda}_{\mathrm{CM}}$ to

$$\Lambda_{\mathrm{CM}}(\bar{x}) = \frac{\tilde{\Lambda}_{\mathrm{CM}}(\bar{x}) \cdot n}{\tilde{\Lambda}_{\mathrm{CM}}(1_n)} \quad \text{for } \bar{x} \in \mathbb{R}^n.$$

By varying the parameter γ in Equation (5.10), a regulator can balance between losses arising from payment reductions between counterparties ($\sum_{i=1}^n \bar{y}_i$) and the costs of supporting the system by injecting new capital ($\sum_{i=1}^n \bar{b}_i$).

The function Λ_{CM} is a positively homogeneous aggregation function in the sense of Definition 5.1.3. This can be verified as follows:

The normalization property (A5) is obvious. To show positive homogeneity (A4), fix $\bar{x} \in \mathbb{R}^n$ and $a > 0$. Then

$$a\tilde{\Lambda}_{\mathrm{CM}}(\bar{x}) = \min_{(\bar{z}/a,\bar{c}/a)\in A^{\bar{x}}} \left\{ \sum_{i=1}^{n} \bar{z}_i + \gamma \bar{c}_i \right\} = \tilde{\Lambda}_{\mathrm{CM}}(a\bar{x})$$

and $\tilde{\Lambda}_{\mathrm{CM}}(0 \cdot \bar{x}) = \tilde{\Lambda}_{\mathrm{CM}}(0_n) = 0 = 0 \cdot \tilde{\Lambda}_{\mathrm{CM}}(\bar{x})$. Hence, $\tilde{\Lambda}_{\mathrm{CM}}$ is positively homogeneous, which implies positive homogeneity of Λ_{CM}.

Now, note that if $\bar{x}, \bar{v} \in \mathbb{R}^n$ with $\bar{x} \geq \bar{v}$, then $A^{\bar{x}} \subset A^{\bar{v}}$ since every $(\bar{y},\bar{b}) \in A^{\bar{x}}$ satisfies $\bar{b}_i + \bar{y}_i - \sum_{j=1}^n \Pi_{ji}\bar{y}_j \geq \bar{x}_i \geq \bar{v}_i$ for all $i \in \{1,\dots,n\}$. It follows directly that $\tilde{\Lambda}_{\mathrm{CM}}(\bar{x}) \geq \tilde{\Lambda}_{\mathrm{CM}}(\bar{v})$, which means that Λ_{CM} satisfies the monotonicity property (A1).

Similarly, if $(\bar{z},\bar{c}) \in A^{a\bar{x}}$ and $(\bar{w},\bar{d}) \in A^{(1-a)\bar{v}}$ for $\bar{x},\bar{v} \in \mathbb{R}^n$ and $a \in [0,1]$, then the following inequality holds for all $i \in \{1,\dots,n\}$:

$$(\bar{c}_i + \bar{d}_i) + (\bar{z}_i + \bar{w}_i) \geq a\bar{x}_i + (1-a)\bar{v}_i + \sum_{j=1}^{n} \Pi_{ji}(\bar{z}_j + \bar{w}_j).$$

It follows that $(\bar{z}+\bar{w},\bar{c}+\bar{d}) \in A^{a\bar{x}+(1-a)\bar{v}}$, and thus we obtain for all $\bar{x},\bar{y} \in \mathbb{R}^n$ and $a \in [0,1]$ that

$$\begin{aligned}
\tilde{\Lambda}_{\mathrm{CM}}(a\bar{x}+(1-a)\bar{v}) &= \min_{(\bar{y},\bar{b})\in A^{a\bar{x}+(1-a)\bar{v}}} \left\{ \sum_{i=1}^{n} \bar{y}_i + \gamma\bar{b}_i \right\} \\
&\leq \min_{(\bar{z},\bar{c})\in A^{a\bar{x}},(\bar{w},\bar{d})\in A^{(1-a)\bar{v}}} \left\{ \sum_{i=1}^{n} (\bar{z}_i + \bar{w}_i) + \gamma(\bar{c}_i + \bar{d}_i) \right\} \\
&= \min_{(\bar{z},\bar{c})\in A^{a\bar{x}}} \left\{ \sum_{i=1}^{n} \bar{z}_i + \gamma\bar{c}_i \right\} + \min_{(\bar{w},\bar{d})\in A^{(1-a)\bar{v}}} \left\{ \sum_{i=1}^{n} \bar{w}_i + \gamma\bar{d}_i \right\} \\
&= a\tilde{\Lambda}_{\mathrm{CM}}(\bar{x}) + (1-a)\tilde{\Lambda}_{\mathrm{CM}}(\bar{v}).
\end{aligned}$$

Therefore, Λ_{CM} satisfies the convexity property (A2).

5.4. Representations of systemic risk measures

This section is dedicated to different representation results for convex and positively homogeneous systemic risk measures. After defining the acceptance sets of a convex systemic risk measure, we first provide a primal representation which clarifies the connection between these acceptance sets and the convex systemic risk measure. Then, based on this primal representation, we deduce a dual representation result.

From now on, we consider convex [positively homogeneous] systemic risk measures $\rho = \rho_0 \circ \Lambda$ with convex [positively homogeneous] single-firm risk measure ρ_0 and convex [positively homogeneous] aggregation function Λ. This means $\operatorname{Im}\Lambda = \mathbb{R}$ or $\operatorname{Im}\Lambda = \mathbb{R}_+$ and $\rho_0 : L^p \to \mathbb{R} \cup \{+\infty\}$. Because of the decomposition results (see Theorem 5.2.1 and Corollary 5.2.3), ρ_0 satisfies constancy on $\mathbb{R}$ in case of $\operatorname{Im}\Lambda = \mathbb{R}$ and constancy on $\mathbb{R}_+$ in case of $\operatorname{Im}\Lambda = \mathbb{R}_+$.

Definition 5.4.1. *The* acceptance sets *of the convex systemic risk measure $\rho = \rho_0 \circ \Lambda$ with convex single-firm risk measure ρ_0 and convex aggregation function Λ are given by*

$$\mathcal{A}_{\rho_0} := \{(r,X) \in \mathbb{R}\times L^p | r \geq \rho_0(X)\} \quad \textit{and} \quad \mathcal{A}_{\Lambda} := \{(Y,\bar{Z}) \in L^p\times(L^p)^n | Y \geq \Lambda(\bar{Z})\}.$$

Before we study the connection between these acceptance sets and the corresponding convex systemic risk measure in detail, let us define the following properties for a subset of a linear vector space $\mathcal{X} \times \mathcal{Y}$.

Definition 5.4.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two linear spaces. A set $\mathcal{S} \subset \mathcal{X} \times \mathcal{Y}$ satisfies the* monotonicity property *if $(x,y_1) \in \mathcal{S}$, $y_2 \in \mathcal{Y}$ and $y_1 \geq y_2$ imply $(x,y_2) \in \mathcal{S}$. A set $\mathcal{S} \subset \mathcal{X} \times \mathcal{Y}$ satisfies the* epigraph property *if $(x_1,y) \in \mathcal{S}$, $x_2 \in \mathcal{X}$ and $x_2 \geq x_1$ imply $(x_2,y) \in \mathcal{S}$.*

Proposition 5.4.3. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a convex systemic risk measure with convex single-firm risk measure ρ_0 and convex aggregation function Λ. Let $\mathcal{A}_{\rho_0}$ and $\mathcal{A}_\Lambda$ be the corresponding acceptance sets.*

1. *$\mathcal{A}_{\rho_0}$ and $\mathcal{A}_\Lambda$ satisfy the following properties:*
   - *a) $\mathcal{A}_{\rho_0}$ and $\mathcal{A}_\Lambda$ satisfy the monotonicity property.*
   - *b) $\mathcal{A}_{\rho_0}$ and $\mathcal{A}_\Lambda$ satisfy the epigraph property.*
   - *c) $\mathcal{A}_{\rho_0}$ and $\mathcal{A}_\Lambda$ are convex sets.*
   - *d) $(a,a) \in \mathcal{A}_{\rho_0}$ with $\inf\{r \in \mathbb{R} | (r,a) \in \mathcal{A}_{\rho_0}\} = a$ for all $a \in \operatorname{Im}\Lambda$ and $(f_\Lambda(a), a1_n) \in \mathcal{A}_\Lambda$ with $\operatorname{ess\,inf}\{Y \in L^p | (Y, a1_n) \in \mathcal{A}_\Lambda\} = f_\Lambda(a)$ for all $a \in \mathbb{R}$.*

   If $\rho = \rho_0 \circ \Lambda$ is a positively homogeneous systemic risk measure, then the following properties are additionally satisfied:
   - *e) $\mathcal{A}_{\rho_0}$ and $\mathcal{A}_\Lambda$ are cones.*
   - *f) $(n, 1_n) \in \mathcal{A}_\Lambda$ with $\operatorname{ess\,inf}\{Y \in L^p | (Y, 1_n) \in \mathcal{A}_\Lambda\} = n$.*
2. *ρ admits the so called* primal representation

$$\rho(\bar{X}) = \inf\{r \in \mathbb{R} | (r, Y) \in \mathcal{A}_{\rho_0}, (Y, \bar{X}) \in \mathcal{A}_\Lambda\} \quad \text{for all } \bar{X} \in (L^p)^n \tag{5.11}$$

   where we set $\inf \emptyset := +\infty$.

Proof. The monotonicity properties, the epigraph properties, convexity and the properties in 1.d) follow directly from the properties of ρ_0 and Λ. Similarly, positive homogeneity and normalization of ρ_0 and Λ imply the additional properties in case of positively homogeneous systemic risk measures. It remains to prove the primal representation.

We know that ρ_0 is for all $X \in L^p$ representable by

$$\rho_0(X) = \inf\{r \in \mathbb{R} | r \geq \rho_0(X)\} = \inf\{r \in \mathbb{R} | (r, X) \in \mathcal{A}_{\rho_0}\}.$$

Analogously, we obtain for all $\bar{Z} \in (L^p)^n$

$$\Lambda(\bar{Z}) = \operatorname{ess\,inf}\{Y \in L^p | Y \geq \Lambda(\bar{Z})\} = \operatorname{ess\,inf}\{Y \in L^p | (Y, \bar{Z}) \in \mathcal{A}_\Lambda\}.$$

Together with the equality $\rho = \rho_0 \circ \Lambda$, the previous representations imply that for all $\bar{X} \in (L^p)^n$, we have

$$\begin{aligned}\rho(\bar{X}) &= \inf\{r \in \mathbb{R} | r \geq (\rho_0 \circ \Lambda)(\bar{X})\} = \inf\{r \in \mathbb{R} | (r, \Lambda(\bar{X})) \in \mathcal{A}_{\rho_0}\} \\ &= \inf\{r \in \mathbb{R} | (r, \operatorname{ess\,inf}\{Y \in L^p | (Y, \bar{X}) \in \mathcal{A}_\Lambda\}) \in \mathcal{A}_{\rho_0}\}.\end{aligned}$$

Finally, representation (5.11) follows from the monotonicity property of $\mathcal{A}_{\rho_0}$. □

In the following proposition we start with subsets of $\mathbb{R} \times L^p$ and $\mathbb{R} \times \mathbb{R}^n$ and study in which cases these sets induce convex or positively homogeneous systemic risk measures.

Proposition 5.4.4. *Assume that* $\emptyset \neq \mathcal{B} \subset \mathbb{R} \times L^p$ *and* $\emptyset \neq \mathcal{C} \subset \mathbb{R} \times \mathbb{R}^n$ *and define*

$$\rho_0^{\mathcal{B}}(X) := \inf\{r \in \mathbb{R} |\, (r, X) \in \mathcal{B}\} \quad \text{for } X \in L^p,$$
$$\Lambda^{\mathcal{C}}(\bar{x}) := \inf\{r \in \mathbb{R} | (r, \bar{x}) \in \mathcal{C}\} \quad \text{for } \bar{x} \in \mathbb{R}^n.$$

Suppose that $\inf\{r \in \mathbb{R} |\, (r, X) \in \mathcal{B}\} > -\infty$ *for all* $X \in L^p$ *and* $\inf\{r \in \mathbb{R} | (r, \bar{x}) \in \mathcal{C}\} > -\infty$ *for all* $\bar{x} \in \mathbb{R}^n$. *Moreover, let* $\mathcal{B}$ *be such that there exists* $r \in \mathbb{R}$ *with* $(r, 0) \in \mathcal{B}$ *and let* $\mathcal{C}$ *be such that for all* $\bar{x} \in \mathbb{R}^n$, *there exists* $r \in \mathbb{R}$ *with* $(r, \bar{x}) \in \mathcal{C}$. *Then the following statements are satisfied:*

1. *If* $\mathcal{B}$ *and* $\mathcal{C}$ *satisfy the monotonicity property, then* $\rho_0^{\mathcal{B}}$ *and* $\Lambda^{\mathcal{C}}$ *are monotone.*
2. *If* $\mathcal{B}$ *and* $\mathcal{C}$ *are convex, then* $\rho_0^{\mathcal{B}}$ *and* $\Lambda^{\mathcal{C}}$ *are convex.*
3. *If* $\mathcal{B}$ *and* $\mathcal{C}$ *are cones, then* $\rho_0^{\mathcal{B}}$ *and* $\Lambda^{\mathcal{C}}$ *are positively homogeneous.*
4. *If* $(1, 1) \in \mathcal{B}$ *with* $\inf\{r \in \mathbb{R} | (r, 1) \in \mathcal{B}\} = 1$ *and* $(n, 1_n) \in \mathcal{C}$ *with* $\inf\{r \in \mathbb{R} | (r, 1_n) \in \mathcal{C}\} = n$, *then* $\rho_0^{\mathcal{B}}$ *and* $\Lambda^{\mathcal{C}}$ *are normalized.*
5. *If* $(a, a) \in \mathcal{B}$ *and* $\inf\{r \in \mathbb{R} | (r, a) \in \mathcal{B}\} = a$ *for all* $a \in \mathbb{R}$, *then* $\rho_0^{\mathcal{B}}(a) = a$.
6. *Define* $f_{\mathcal{C}}(a) := \inf\{r \in \mathbb{R} | (r, a1_n) \in \mathcal{C}\}$ *for all* $a \in \mathbb{R}$ *and suppose that the function* $f_{\mathcal{C}} : \mathbb{R} \to \mathbb{R}$ *is surjective. Then* $\Lambda^{\mathcal{C}}$ *satisfies* f_Λ*-constancy with* $f_{\Lambda^{\mathcal{C}}} = f_{\mathcal{C}}$.

In particular, if $\mathcal{B}$ *and* $\mathcal{C}$ *satisfy all of the properties from 1.-5., then* $\rho^{\mathcal{B},\mathcal{C}} : (L^p)^n \to \mathbb{R} \cup \{+\infty\}$ *defined by*

$$\rho^{\mathcal{B},\mathcal{C}}(\bar{X}) := \inf\{r \in \mathbb{R} | (r, Y) \in \mathcal{B}, (Y, \bar{X}) \in \mathcal{A}_{\Lambda^{\mathcal{C}}}\} \quad \text{for } \bar{X} \in (L^p)^n \tag{5.12}$$

(where $\mathcal{A}_{\Lambda^{\mathcal{C}}}$ *is the acceptance set of* $\Lambda^{\mathcal{C}}$*) is a positively homogeneous systemic risk measure with* $\rho^{\mathcal{B},\mathcal{C}} = \rho_0^{\mathcal{B}} \circ \Lambda^{\mathcal{C}}$. *If* $\mathcal{B}$ *and* $\mathcal{C}$ *satisfy the properties from 1., 2., 5. and 6. and* $\|f_{\mathcal{C}}(Z)\|_p < \infty$ *and* $\|f_{\mathcal{C}}^{-1}(Z)\|_p < \infty$ *for all* $Z \in L^p$, *then* $\rho^{\mathcal{B},\mathcal{C}}$ *defined in* (5.12) *is a convex systemic risk measure with* $\rho^{\mathcal{B},\mathcal{C}} = \rho_0^{\mathcal{B}} \circ \Lambda^{\mathcal{C}}$.

Furthermore, $\mathcal{B}$ *is a subset of* $\mathcal{A}_{\rho_0^{\mathcal{B}}}$, *and* $\mathcal{C}$ *is a subset of* $\mathcal{A}_{\Lambda^{\mathcal{C}}}$.

Proof. First, let $\mathcal{B}$ satisfy the monotonicity property and consider $X, Y \in L^p$ with $X \geq Y$. Then

$$\{r \in \mathbb{R} \,|(r, X) \in \mathcal{B}\} \subset \{r \in \mathbb{R} \,|(r, Y) \in \mathcal{B}\},$$

which implies that $\rho_0^{\mathcal{B}}(X) \geq \rho_0^{\mathcal{B}}(Y)$. If the set $\mathcal{B}$ is convex, then it follows for $X, Y \in L^p$ and $a \in [0, 1]$ that

$$\begin{aligned}\rho_0^{\mathcal{B}}(aX + (1-a)Y) &= \inf\{ax + (1-a)y \in \mathbb{R} \,| a\,(x, X) + (1-a)\,(y, Y) \in \mathcal{B}\}\\ &\leq \inf\{ax + (1-a)y \in \mathbb{R} \,|(x, X), (y, Y) \in \mathcal{B}\} = a\rho_0^{\mathcal{B}}(X) + (1-a)\rho_0^{\mathcal{B}}(Y).\end{aligned}$$

This means that $\rho_0^{\mathcal{B}}$ is convex. Now, suppose that $\mathcal{B}$ is a cone and let $a > 0$ and $X \in L^p$. Then we have

$$\rho_0^{\mathcal{B}}(aX) = \inf\{ar \in \mathbb{R} \,|(ar, aX) \in \mathcal{B}\} \leq \inf\{ar \in \mathbb{R} \,|(r, X) \in \mathcal{B}\} = a\rho_0^{\mathcal{B}}(X).$$

For the other inequality consider $x < \rho_0^{\mathcal{B}}(X) = \inf\{r \in \mathbb{R} \,|(r, X) \in \mathcal{B}\}$. Then $(r, X) \notin \mathcal{B}$, and thus $(ax, aX) \notin \mathcal{B}$. It follows that $ax < \rho_0^{\mathcal{B}}(aX)$ and finally

$\rho_0^{\mathcal{B}}(aX) = a\rho_0^{\mathcal{B}}(X)$. In conclusion, $\rho_0^{\mathcal{B}}$ satisfies $\rho_0^{\mathcal{B}}(aX) = a\rho_0^{\mathcal{B}}(X)$ for all $X \in L^p$ and $a > 0$. This implies $\rho_0^{\mathcal{B}}(0) = \rho_0^{\mathcal{B}}(a \cdot 0) = a\rho_0^{\mathcal{B}}(0)$ for all $a > 0$. Since $\rho_0^{\mathcal{B}}(0) < \infty$ due to the assumptions on $\mathcal{B}$, this means that $\rho_0^{\mathcal{B}}(0) = 0$. Altogether, $\rho_0^{\mathcal{B}}$ is positively homogeneous.

By using the same arguments, we can show that $\Lambda^{\mathcal{C}}$ inherits monotonicity, convexity and positive homogeneity from the corresponding properties of $\mathcal{C}$.

Note that part 4, part 5 and part 6 are trivial and that the assumptions on $\mathcal{C}$ imply that $\Lambda^{\mathcal{C}}(\bar{x}) = \inf\{r \in \mathbb{R} | (r, \bar{x}) \in \mathcal{C}\} \in \mathbb{R}$ for every $\bar{x} \in \mathbb{R}^n$. Now, suppose that all of the properties from 1.-5. are satisfied. Then we can consider $\Lambda^{\mathcal{C}}$ as a mapping on $(L^p)^n$ that maps into L^p, and the composition of $\rho_0^{\mathcal{B}}$ and $\Lambda^{\mathcal{C}}$ leads to

$$\begin{aligned}(\rho_0^{\mathcal{B}} \circ \Lambda^{\mathcal{C}})(\bar{X}) &= \inf\{r \in \mathbb{R} | (r, \Lambda^{\mathcal{C}}(\bar{X})) \in \mathcal{B}\} \\ &= \inf\{r \in \mathbb{R} | (r, \operatorname{ess\,inf}\{Y \in L^p | (Y, \bar{X}) \in \mathcal{A}_{\Lambda^{\mathcal{C}}}\}) \in \mathcal{B}\} \\ &= \inf\{r \in \mathbb{R} | (r, Y) \in \mathcal{B}, (Y, \bar{X}) \in \mathcal{A}_{\Lambda^{\mathcal{C}}}\} \end{aligned} \tag{5.13}$$

for $\bar{X} \in (L^p)^n$. Remark 5.2.4 implies that $\rho^{\mathcal{B},\mathcal{C}}$ is a positively homogeneous systemic risk measure. Similarly, if the properties from 1., 2., 5. and 6. are satisfied and $\|f_{\mathcal{C}}(Z)\|_p < \infty$ and $\|f_{\mathcal{C}}^{-1}(Z)\|_p < \infty$ for all $Z \in L^p$, then it follows from (5.13) and Theorem 5.2.1 that $\rho^{\mathcal{B},\mathcal{C}}$ is a convex systemic risk measure.

It remains to show that $\mathcal{B} \subset \mathcal{A}_{\rho_0^{\mathcal{B}}}$ and $\mathcal{C} \subset \mathcal{A}_{\Lambda^{\mathcal{C}}}$. To this end, consider $(x, X) \in \mathcal{B}$. By definition of $\rho_0^{\mathcal{B}}$, this implies $\rho_0^{\mathcal{B}}(X) \leq x$. Since $\mathcal{A}_{\rho_0^{\mathcal{B}}} = \{(r, X) \in \mathbb{R} \times L^p | r \geq \rho_0^{\mathcal{B}}(X)\}$, we can conclude that $(x, X) \in \mathcal{A}_{\rho_0^{\mathcal{B}}}$. The inclusion $\mathcal{C} \subset \mathcal{A}_{\Lambda^{\mathcal{C}}}$ is verified analogously. □

The dual representation of convex systemic risk measures requires additional continuity properties of the convex single-firm risk measure ρ_0 and the convex aggregation function Λ. More precisely, ρ_0 is supposed to be lower semicontinuous and Λ, considered as a mapping on $(L^p)^n$, is supposed to be continuous. Note that in Chen et al. (2013) single-firm risk measures are convex functions from $\mathbb{R}^{|\Omega|}$ to $\mathbb{R}$ and aggregation functions map from $\mathbb{R}^{n\times|\Omega|}$ into $\mathbb{R}$ where Ω is a finite probability space. Since all finite, convex functions on $\mathbb{R}^m$, $m \in \mathbb{N}$, are continuous, Chen et al. (2013) do not need to claim any additional properties in their dual representation result.

In the subsequent theorem we consider the dual representation result for convex systemic risk measures. Thereafter, the positively homogeneous case can be deduced as a special case. In what follows, we need the indicator function $\iota_{\mathcal{E}} : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\iota_{\mathcal{E}}(x) := \begin{cases} 0 & \text{for } x \in \mathcal{E} \\ \infty & \text{else} \end{cases}$$

where $\mathcal{E} \subset \mathcal{X}$ is a subset of a linear vector space $\mathcal{X}$.

Theorem 5.4.5. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a convex systemic risk measure characterized by a l.s.c. convex single-firm risk measure ρ_0 and a convex aggregation function Λ that is continuous on $(L^p)^n$. Then ρ admits the representation*

$$\rho(\bar{X}) = \sup_{(\xi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*} \left\{ \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i] - \alpha_n(\xi, \bar{\xi}) \right\} \quad \text{for all } \bar{X} \in (L^p)^n \tag{5.14}$$

where $\alpha_n : (L^p)^* \times ((L^p)^n)^* \to \mathbb{R} \cup \{+\infty\}$ *is given by*

$$\alpha_n(\xi,\bar{\xi}) := \sup_{(r,Y)\in\mathcal{A}_{\rho_0},(V,\bar{Z})\in\mathcal{A}_\Lambda} \left\{ -r + \mathbb{E}[(Y-V)\xi] + \sum_{i=1}^{n} \mathbb{E}[\bar{Z}_i\bar{\xi}_i] \right\}. \tag{5.15}$$

In addition, a feasible solution $(\xi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*$ *to optimization problem* (5.14) *satisfies*

$$\xi \geq 0, \quad \mathbb{E}[\xi] \leq 1 \quad \textit{and} \quad \bar{\xi} \geq 0.$$

If additionally $\rho(\mathbb{R}^n) = \mathbb{R}$*, then we obtain* $\mathbb{E}[\xi] = 1$.

Remark. Note that in case of $p = \infty$, $\mathbb{E}[\xi] \leq 1$ means $\int 1 d\xi = \xi[\Omega] \leq 1$. If additionally $\rho(\mathbb{R}^n) = \mathbb{R}$, then we obtain $\xi[\Omega] = 1$, which means that $\xi : \mathcal{F} \to [0,1]$ is a finitely additive set function which is absolutely continuous with respect to $\mathbb{P}$ and normalized to 1. In other words, $\xi \in \mathcal{M}_{1,f}(\mathbb{P})$. In case of $\rho(\mathbb{R}^n) = \mathbb{R}$ and $1 \leq p < \infty$, the first component of a feasible solution $(\xi,\bar{\xi})$ to (5.14) represents a density function.

Proof. Fix $\bar{X} \in (L^p)^n$. *Part 1:* First of all, we will verify representation (5.14). Since every convex systemic risk measure admits the primal representation from Proposition 5.4.3, we have

$$\begin{aligned} \rho(\bar{X}) &= \inf\{r \in \mathbb{R} \,|\, (r,Y) \in \mathcal{A}_{\rho_0}, (Y,\bar{X}) \in \mathcal{A}_\Lambda\} \\ &= \inf_{(r,Y)\in\mathbb{R}\times L^p} \{r + \iota_{\mathcal{A}_{\rho_0}}(r,Y) + \iota_{\mathcal{A}_\Lambda}(Y,\bar{X})\}. \end{aligned}$$

By definition, the convex conjugate $\iota^*_{\mathcal{A}_{\rho_0}} : \mathbb{R} \times (L^p)^* \to \overline{\mathbb{R}}$ of $\iota_{\mathcal{A}_{\rho_0}}$ satisfies

$$\iota^*_{\mathcal{A}_{\rho_0}}(-x,\xi) = \sup_{(r,Y)\in\mathbb{R}\times L^p} \{-rx + \mathbb{E}[Y\xi] - \iota_{\mathcal{A}_{\rho_0}}(r,Y)\} = \sup_{(r,Y)\in\mathcal{A}_{\rho_0}} \{-rx + \mathbb{E}[Y\xi]\}$$

for $(x,\xi) \in \mathbb{R} \times (L^p)^*$. Note that $\iota_{\mathcal{A}_{\rho_0}}$ is convex because $\mathcal{A}_{\rho_0}$ is convex. Moreover, since ρ_0 is assumed to be l.s.c., we know that $\iota_{\mathcal{A}_{\rho_0}}$ is closed. It follows from the duality theorem for conjugate functions (see Theorem A.2.9) that

$$\begin{aligned} \iota_{\mathcal{A}_{\rho_0}}(r,Y) &= \sup_{(x,\xi)\in\mathbb{R}\times(L^p)^*} \{rx + \mathbb{E}[Y\xi] - \iota^*_{\mathcal{A}_{\rho_0}}(x,\xi)\} \\ &= \sup_{(x,\xi)\in\mathbb{R}\times(L^p)^*} \{-rx + \mathbb{E}[Y\xi] - \iota^*_{\mathcal{A}_{\rho_0}}(-x,\xi)\} \end{aligned}$$

for all $(r,Y) \in \mathbb{R} \times L^p$. Similarly, the convex conjugate $\iota^*_{\mathcal{A}_\Lambda} : (L^p)^* \times ((L^p)^n)^* \to \overline{\mathbb{R}}$ of $\iota_{\mathcal{A}_\Lambda}$ satisfies

$$\begin{aligned} \iota^*_{\mathcal{A}_\Lambda}(-\psi,\bar{\xi}) &= \sup_{(V,\bar{Z})\in L^p\times(L^p)^n} \left\{ -\mathbb{E}[V\psi] + \sum_{i=1}^{n} \mathbb{E}[\bar{Z}_i\bar{\xi}_i] - \iota_{\mathcal{A}_\Lambda}(V,\bar{Z}) \right\} \\ &= \sup_{(V,\bar{Z})\in\mathcal{A}_\Lambda} \left\{ -\mathbb{E}[V\psi] + \sum_{i=1}^{n} \mathbb{E}[\bar{Z}_i\bar{\xi}_i] \right\} \end{aligned}$$

for $(-\psi, \bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*$. Furthermore, the continuity property of Λ implies closedness of $\mathcal{A}_\Lambda$, which means that the convex function $\iota_{\mathcal{A}_\Lambda}$ is closed. Therefore,

$$\iota_{\mathcal{A}_\Lambda}(Y, \bar{X}) = \sup_{(\psi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*} \left\{ -\mathbb{E}[Y\psi] + \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i] - \iota^*_{\mathcal{A}_\Lambda}(-\psi, \bar{\xi}) \right\}$$

for $(Y, \bar{X}) \in L^p \times (L^p)^n$. Together, we obtain that the systemic risk measure ρ satisfies

$$\begin{aligned}
\rho(\bar{X}) &= \inf_{(r,Y) \in \mathbb{R} \times L^p} \{ r + \iota_{\mathcal{A}_{\rho_0}}(r, Y) + \iota_{\mathcal{A}_\Lambda}(Y, \bar{X}) \} \\
&= \inf_{(r,Y) \in \mathbb{R} \times L^p} \sup_{\substack{(x,\xi) \in \mathbb{R} \times (L^p)^*, \\ (\psi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*}} \Bigg\{ r - rx + \mathbb{E}[Y\xi] - \mathbb{E}[Y\psi] + \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i] \\
&\qquad - \iota^*_{\mathcal{A}_{\rho_0}}(-x, \xi) - \iota^*_{\mathcal{A}_\Lambda}(-\psi, \bar{\xi}) \Bigg\} \\
&= \sup_{\substack{(x,\xi) \in \mathbb{R} \times (L^p)^*, \\ (\psi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*}} \inf_{(r,Y) \in \mathbb{R} \times L^p} \Bigg\{ r - rx + \mathbb{E}[Y\xi] - \mathbb{E}[Y\psi] + \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i] \\
&\qquad - \iota^*_{\mathcal{A}_{\rho_0}}(-x, \xi) - \iota^*_{\mathcal{A}_\Lambda}(-\psi, \bar{\xi}) \Bigg\} \qquad (5.16) \\
&= \sup_{\substack{(1,\xi) \in \mathbb{R} \times (L^p)^*, \\ (\xi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*}} \left\{ \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i] - \iota^*_{\mathcal{A}_{\rho_0}}(-1, \xi) - \iota^*_{\mathcal{A}_\Lambda}(-\xi, \bar{\xi}) \right\}.
\end{aligned}$$

We are allowed to interchange infimum and supremum in (5.16) due to Lemma 5.4.6. Representation (5.14) follows directly from

$$\begin{aligned}
\iota^*_{\mathcal{A}_{\rho_0}}(-1, \xi) + \iota^*_{\mathcal{A}_\Lambda}(-\xi, \bar{\xi}) &= \sup_{(r,Y) \in \mathcal{A}_{\rho_0}, (V,\bar{Z}) \in \mathcal{A}_\Lambda} \left\{ -r + \mathbb{E}[(Y - V)\xi] + \sum_{i=1}^n \mathbb{E}[\bar{Z}_i \bar{\xi}_i] \right\} \\
&= \alpha_n(\xi, \bar{\xi}).
\end{aligned}$$

Part 2: In this part of the proof we will verify the claimed properties of feasible solutions to optimization problem (5.14). In case of $p < \infty$, assume that there exists $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$ and $\xi < 0$ on A. We will show that this implies $\alpha_n(\xi, \bar{\xi}) = \infty$: Consider an arbitrary element $(r, Y) \in \mathcal{A}_{\rho_0}$ and define $Z_{(Y,m)} \in L^p$ for $m \in \mathbb{N}$ by $Z_{(Y,m)} := (-|Y| - m)I_A + YI_{A^c}$. Since $Y \geq Z_{(Y,m)}$, the monotonicity property of $\mathcal{A}_{\rho_0}$ yields $(r, Z_{(Y,m)}) \in \mathcal{A}_{\rho_0}$ for every $m \in \mathbb{N}$. Moreover, we have

$$\mathbb{E}[Z_{(Y,m)}\xi] = \mathbb{E}[(-|Y| - m)\xi I_A] + \mathbb{E}[Y\xi I_{A^c}] = \mathbb{E}[(-|Y| - m)\xi | A]\mathbb{P}[A] + \mathbb{E}[Y\xi I_{A^c}].$$

Because $\mathbb{E}[(-|Y| - m)\xi|A]\mathbb{P}[A] > 0$ and $\mathbb{E}[Y\xi I_{A^c}] \in \mathbb{R}$, letting m tend to ∞ leads to $\lim_{m\to\infty} \mathbb{E}[Z_{(Y,m)}\xi] = \infty$, and therefore $\alpha_n(\xi, \bar{\xi}) = \infty$. In conclusion, it suffices to consider $\xi \geq 0$ in optimization problem (5.14).

Similarly, considering an element $\bar{\xi} \in (L^p)^n$ that satisfies $\bar{\xi}_j < 0$ on a set $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$ for $j \in \{1, \ldots, n\}$ leads to $\alpha_n(\xi, \bar{\xi}) = \infty$. This means that $\bar{\xi} \geq 0$.

If $p = \infty$, assume that there exists $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$ and $\xi[A] < 0$. Consider again an arbitrary element $(r, Y) \in \mathcal{A}_{\rho_0}$ and define $Z_{(Y,m)} \in L^\infty$ by $Z_{(Y,m)} := (-\|Y\|_\infty - m)I_A - \|Y\|_\infty I_{A^c}(\leq Y)$. The monotonicity property of $\mathcal{A}_{\rho_0}$ implies that $(r, Z_{(Y,m)}) \in \mathcal{A}_{\rho_0}$ for every $m \in \mathbb{N}$. Furthermore, we have $\int Z_{(Y,m)} d\xi = (-\|Y\|_\infty - m)\xi[A] - \|Y\|_\infty \xi[A^c]$, and boundedness of ξ yields $|\xi[A^c]| \leq M_{(\xi)}$ for some $M_{(\xi)} \in \mathbb{R}$. Therefore, we obtain $\lim_{m\to\infty} \int Z_{(Y,m)} d\xi = \infty$, which implies $\alpha_n(\xi, \bar{\xi}) = \infty$. In conclusion, we only have to consider $\xi \geq 0$. $\bar{\xi} \geq 0$ follows analogously.

From now on, let $1 \leq p \leq \infty$. Assume that $\xi \in (L^p)^*$ is such that $-1 + \mathbb{E}[\xi] > 0$. From the constancy property of ρ_0 we know that $(\lambda, \lambda) \in \mathcal{A}_{\rho_0}$ for all $\lambda > 0$. Moreover, we have $\lim_{\lambda\to\infty}(-\lambda + \mathbb{E}[\lambda\xi]) = \lim_{\lambda\to\infty}(\lambda(-1 + \mathbb{E}[\xi])) = \infty$, which implies that $\alpha_n(\xi, \bar{\xi}) = \infty$. Therefore, it suffices to consider $\xi \in (L^p)^*$ with $\mathbb{E}[\xi] \leq 1$.

Finally, suppose that $\rho(\mathbb{R}^n) = \mathbb{R}$ and let $\xi \in (L^p)^*$ be such that $1 - \mathbb{E}[\xi] > 0$. The constancy property of ρ_0 implies that $(-\lambda, -\lambda) \in \mathcal{A}_{\rho_0}$ for every $\lambda > 0$. Furthermore, we have $\lim_{\lambda\to\infty}(\lambda - \mathbb{E}[\lambda\xi]) = \lim_{\lambda\to\infty}(\lambda(1 - \mathbb{E}[\xi])) = \infty$. Together, it follows that $\alpha_n(\xi, \bar{\xi}) = \infty$, and hence $\mathbb{E}[\xi] \geq 1$. $\square$

Lemma 5.4.6. *Suppose that the requirements from Theorem 5.4.5 are satisfied. Define $\mathcal{X} := \mathbb{R} \times L^p$ and $\mathcal{U} := \mathbb{R} \times L^p \times L^p \times (L^p)^n$ and consider the paired spaces $(\mathcal{X}, \mathcal{X}^*)$ and $(\mathcal{U}, \mathcal{U}^*)$. If $1 < p < \infty$, the spaces $\mathcal{X}, \mathcal{X}^*, \mathcal{U}$ and $\mathcal{U}^*$ are endowed with the corresponding norm topology. If $p \in \{1, \infty\}$, then we endow $\mathcal{X}$ and $\mathcal{U}$ with the corresponding norm topology and $\mathcal{X}^*$ and $\mathcal{U}^*$ are endowed with the weak* topologies $\sigma(\mathcal{X}^*, \mathcal{X})$ and $\sigma(\mathcal{U}^*, \mathcal{U})$, respectively. Let $\bar{X} \in (L^p)^n$ and $K : \mathcal{X} \times \mathcal{U}^* \to \overline{\mathbb{R}}$ be defined by*

$$K((r,Y)\,(x,\xi,\psi,\bar{\xi})) := r - rx + \mathbb{E}[Y\xi] - \mathbb{E}[Y\psi] + \sum_{i=1}^n \mathbb{E}[\bar{X}_i\bar{\xi}_i] - \iota^*_{\mathcal{A}_{\rho_0}}(-x,\xi) - \iota^*_{\mathcal{A}_\Lambda}(-\psi,\bar{\xi}).$$

Then we have

$$\inf_{(r,Y)\in\mathcal{X}} \sup_{(x,\xi,\psi,\bar{\xi})\in\mathcal{U}^*} K((r,Y),(x,\xi,\psi,\bar{\xi})) = \sup_{(x,\xi,\psi,\bar{\xi})\in\mathcal{U}^*} \inf_{(r,Y)\in\mathcal{X}} K((r,Y),(x,\xi,\psi,\bar{\xi})).$$

Proof. Fix $\bar{X} \in (L^p)^n$. First, note that for every $(r, Y) \in \mathcal{X}$, $K((r,Y), \cdot)$ is upper semicontinuous (u.s.c.): This follows from continuity of the linear functional $(x, \xi, \psi, \bar{\xi}) \mapsto r - rx + \mathbb{E}[Y\xi] - \mathbb{E}[Y\psi] + \sum_{i=1}^n \mathbb{E}[\bar{X}_i\bar{\xi}_i]$ and closedness (lower semicontinuity) of $\iota^*_{\mathcal{A}_{\rho_0}}$ and $\iota^*_{\mathcal{A}_\Lambda}$; see, for instance, Theorem 5 in Rockafellar (1974) (see Theorem A.2.9). Moreover, K is concave in the second argument since $(x, \xi, \psi, \bar{\xi}) \mapsto r - rx + \mathbb{E}[Y\xi] - \mathbb{E}[Y\psi] + \sum_{i=1}^n \mathbb{E}[\bar{X}_i\bar{\xi}_i]$ is linear and $\iota^*_{\mathcal{A}_{\rho_0}}$ and $\iota^*_{\mathcal{A}_\Lambda}$ are convex; see again Theorem 5 in Rockafellar (1974). By Theorem 6 in Rockafellar (1974) (see Theorem A.2.12), K is the Lagrangian of the minimization problem "minimize f over $\mathcal{X}$" where f is given by $f(r,Y) = F((r,Y), 0_{n+3})$ for $F : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ defined by

$$\begin{aligned} &F((r,Y),(s,V,X,\bar{Z})) \\ &:= \sup_{(x,\xi,\psi,\bar{\xi})\in\mathcal{U}^*} \left\{ K((r,Y),(x,\xi,\psi,\bar{\xi})) - sx - \mathbb{E}[V\xi] - \mathbb{E}[X\psi] - \sum_{i=1}^n \mathbb{E}[\bar{Z}_i\bar{\xi}_i] \right\}. \end{aligned}$$

Moreover, Theorem 6 in Rockafellar (1974) states that $F((r,Y),\cdot)$ is closed and convex. In addition, F satisfies for $(r,Y)\in\mathcal{X}$ and $(s,V,X,\bar{Z})\in\mathcal{U}$

$$\begin{aligned}
&F((r,Y),(s,V,X,\bar{Z}))\\
&=\sup_{(x,\xi,\psi,\bar{\xi})\in\mathcal{U}^*}\Bigg\{r-rx+\mathbb{E}[Y\xi]-\mathbb{E}[Y\psi]+\sum_{i=1}^n\mathbb{E}[\bar{X}_i\bar{\xi}_i]-\iota^*_{\mathcal{A}_{\rho_0}}(-x,\xi)-\iota^*_{\mathcal{A}_\Lambda}(-\psi,\bar{\xi})\\
&\qquad-sx-\mathbb{E}[V\xi]-\mathbb{E}[X\psi]-\sum_{i=1}^n\mathbb{E}[\bar{Z}_i\bar{\xi}_i]\Bigg\}\\
&=r+\sup_{(x,\xi,\psi,\bar{\xi})\in\mathcal{U}^*}\Bigg\{-(r+s)x+\mathbb{E}[(Y-V)\xi]-\mathbb{E}[(Y+X)\psi]\\
&\qquad+\sum_{i=1}^n\mathbb{E}[(\bar{X}_i-\bar{Z}_i)\bar{\xi}_i]-\iota^*_{\mathcal{A}_{\rho_0}}(-x,\xi)-\iota^*_{\mathcal{A}_\Lambda}(-\psi,\bar{\xi})\Bigg\}\\
&=r+\sup_{(x,\xi)\in\mathbb{R}\times(L^p)^*}\{-(r+s)x+\mathbb{E}[(Y-V)\xi]-\iota^*_{\mathcal{A}_{\rho_0}}(-x,\xi)\}\\
&\qquad+\sup_{(\psi,\bar{\xi})\in(L^p)^*\times((L^p)^n)^*}\Bigg\{-\mathbb{E}[(Y+X)\psi]+\sum_{i=1}^n\mathbb{E}[(\bar{X}_i-\bar{Z}_i)\bar{\xi}_i]-\iota^*_{\mathcal{A}_\Lambda}(-\psi,\bar{\xi})\Bigg\}\\
&=r+\iota_{\mathcal{A}_{\rho_0}}(r+s,Y-V)+\iota_{\mathcal{A}_\Lambda}(Y+X,\bar{X}-\bar{Z}).
\end{aligned}$$

Hence, F is convex in both arguments. Define the convex function φ by

$$\begin{aligned}
\varphi(s,V,X,\bar{Z})&:=\inf_{(r,Y)\in\mathcal{X}}F((r,Y),(s,V,X,\bar{Z}))\\
&=\inf_{(r,Y)\in\mathcal{X}}\{r+\iota_{\mathcal{A}_{\rho_0}}(r+s,Y-V)+\iota_{\mathcal{A}_\Lambda}(Y+X,\bar{X}-\bar{Z})\}
\end{aligned}$$

for $(s,V,X,\bar{Z})\in\mathcal{U}$. Since $\mathcal{A}_{\rho_0}=\{(r,X)\in\mathbb{R}\times L^p|r\geq\rho_0(X)\}$ and $\mathcal{A}_\Lambda=\{(Y,\bar{Z})\in L^p\times(L^p)^n|Y\geq\Lambda(\bar{Z})\}$, we obtain for φ that

$$\begin{aligned}
\varphi(s,V,X,\bar{Z})&=\inf_{(r,Y)\in\mathcal{X}}\{r+\iota_{\mathcal{A}_{\rho_0}}(r+s,Y-V)+\iota_{\mathcal{A}_\Lambda}(Y+X,\bar{X}-\bar{Z})\}\\
&=\inf_{r\in\mathbb{R}}\{r+\iota_{\mathcal{A}_{\rho_0}}(r+s,\Lambda(\bar{X}-\bar{Z})-X-V)\}\\
&=\rho_0(\Lambda(\bar{X}-\bar{Z})-X-V)-s.
\end{aligned}$$

Note that $\rho_0(0)=0$ because ρ_0 satisfies the constancy property. This implies that $\varphi(0,0,\Lambda(\bar{X}-\bar{Z}),\bar{Z})=0$, and therefore φ is proper. It follows that φ is closed if φ is l.s.c. Hence, if we can show that φ is l.s.c., then it follows from Theorem 7 in Rockafellar (1974) (see Theorem A.2.13) that we are allowed to interchange supremum and infimum. We will show that the function $\varrho:(V,X,\bar{Z})\mapsto\rho_0(\Lambda(\bar{X}-\bar{Z})-X-V)$ is l.s.c., which implies the desired lower semicontinuity property of φ immediately. Consider a sequence $(V_{(m)},X_{(m)},\bar{Z}_{(m)})\subset L^p\times L^p\times(L^p)^n$ with $(V_{(m)},X_{(m)},\bar{Z}_{(m)})\to(V,X,\bar{Z})$ in $L^p\times L^p\times(L^p)^n$. Because $\bar{X}-\bar{Z}_{(m)}\to\bar{X}-\bar{Z}$ in $(L^p)^n$, the continuity property of Λ implies $\Lambda(\bar{X}-\bar{Z}_{(m)})\to\Lambda(\bar{X}-\bar{Z})$ in L^p. As a consequence, we have

$$\Lambda(\bar{X}-\bar{Z}_{(m)})-X_{(m)}-V_{(m)}\to\Lambda(\bar{X}-\bar{Z})-X-V\quad\text{in }L^p.$$

Finally, lower semicontinuity of ρ_0 yields

$$\begin{aligned}&\varrho(V,X,\bar{Z}) = \rho_0(\Lambda(\bar{X}-\bar{Z})-X-V)\\&\le \liminf_{m\to\infty} \rho_0(\Lambda(\bar{X}-\bar{Z}_{(m)})-X_{(m)}-V_{(m)}) = \liminf_{m\to\infty} \varrho(V_{(m)},X_{(m)},\bar{Z}_{(m)}),\end{aligned}$$

which means that ϱ is l.s.c. □

Remark 5.4.7. Theorem 5.4.5 holds for convex aggregation functions Λ satisfying L^p-continuity. Alternatively, we could claim 1-Lipschitz continuity for $\Lambda : \mathbb{R}^n \to \mathbb{R}$ because this property implies the required L^p-continuity.

This implication can be verified as follows: Suppose that the convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is 1-Lipschitz continuous, i.e., $|\Lambda(\bar{x}) - \Lambda(\bar{y})| \le \|\bar{x}-\bar{y}\| = \sum_{i=1}^n |\bar{x}_i - \bar{y}_i|$ for all $\bar{x}, \bar{y} \in \mathbb{R}^n$, and consider a sequence $(\bar{X}_{(m)}) \subset (L^p)^n$ with $\bar{X}_{(m)} \to \bar{X}$ in L^p. First, let us suppose that $p < \infty$. Since Λ is measurable, 1-Lipschitz continuity and Hölder's inequality yield

$$|\Lambda(\bar{X}_{(m)}) - \Lambda(\bar{X})| \le \sum_{i=1}^n |(\bar{X}_{(m)})_i - \bar{X}_i| \le n^{(p-1)/p} \left(\sum_{i=1}^n |(\bar{X}_{(m)})_i - \bar{X}_i|^p \right)^{1/p}.$$

As a consequence, we obtain

$$\begin{aligned}&\left\|\Lambda(\bar{X}_{(m)}) - \Lambda(\bar{X})\right\|_p^p = \mathbb{E}[|\Lambda(\bar{X}_{(m)}) - \Lambda(\bar{X})|^p]\\&\le n^{p-1}\mathbb{E}\left[\sum_{i=1}^n |(\bar{X}_{(m)})_i - \bar{X}_i|^p\right] = n^{p-1}\sum_{i=1}^n \left\|(\bar{X}_{(m)})_i - \bar{X}_i\right\|_p^p.\end{aligned}$$

Now, let $p=\infty$. Then $|\Lambda(\bar{X}_{(m)}) - \Lambda(\bar{X})| \le \sum_{i=1}^n |(\bar{X}_{(m)})_i - \bar{X}_i|$ and $\|Y\|_\infty = \inf\{r \in \mathbb{R} | |Y| \le r\}$ for $Y \in L^\infty$ imply

$$\left\|\Lambda(\bar{X}_{(m)}) - \Lambda(\bar{X})\right\|_\infty \le \sum_{i=1}^n \left\|(\bar{X}_{(m)})_i - \bar{X}_i\right\|_\infty.$$

Thus, $\left\|\bar{X}_{(m)} - \bar{X}\right\|_p \to 0$ leads to $\left\|\Lambda(\bar{X}_{(m)}) - \Lambda(\bar{X})\right\|_p \to 0$ for all $p \in [1,\infty]$, which means that Λ is L^p-continuous.

The following remark connects the previous theorem with the positively homogeneous special case.

Remark 5.4.8. Suppose that the function $f_\Lambda : \mathbb{R} \to \mathbb{R}$ (see property (A3)) is positively homogeneous. Note that this does not automatically imply positive homogeneity of the corresponding convex aggregation function Λ. Then every feasible solution $(\xi, \bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*$ to optimization problem (5.14) additionally satisfies

$$\sum_{i=1}^n \mathbb{E}[\bar{\xi}_i] \le f_\Lambda(1)\mathbb{E}[\xi]. \tag{5.17}$$

The proof of this inequality is similar to the proof of $\mathbb{E}[\xi] \le 1$: Suppose that $(\xi, \bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*$ is such that $-f_\Lambda(1)\mathbb{E}[\xi] + \sum_{i=1}^n \mathbb{E}[\bar{\xi}_i] > 0$. Since $\Lambda(\lambda 1_n) = f_\Lambda(\lambda) =$

$\lambda f_\Lambda(1)$, we have $(\lambda f_\Lambda(1), \lambda 1_n) \in \mathcal{A}_\Lambda$ for every $\lambda > 0$. Hence, $\lim_{\lambda\to\infty}(-\mathbb{E}[\lambda f_\Lambda(1)\xi] + \sum_{i=1}^n \mathbb{E}[\lambda\bar{\xi}_i]) = \lim_{\lambda\to\infty}(\lambda(-f_\Lambda(1)\mathbb{E}[\xi] + \sum_{i=1}^n \mathbb{E}[\bar{\xi}_i])) = \infty$ implies that $\alpha_n(\xi,\bar{\xi}) = \infty$. As a consequence, we only have to consider $(\xi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*$ with (5.17).

If we additionally assume that Λ is positively homogeneous, then it follows from the normalization property that $f_\Lambda(1) = \Lambda(1_n) = n$. Therefore, in case of a positively homogeneous aggregation function Λ, we obtain

$$\sum_{i=1}^n \mathbb{E}[\bar{\xi}_i] \leq n\mathbb{E}[\xi]$$

for every feasible solution to optimization problem (5.14).

In the following dual representation result for positively homogeneous systemic risk measures the following sets play a key role:

$$\mathcal{A}^*_{\rho_0} := \{(x,\psi) \in \mathbb{R} \times (L^p)^* | rx - \mathbb{E}[Y\psi] \geq 0 \text{ for all } (r,Y) \in \mathcal{A}_{\rho_0}\},$$

$$\mathcal{A}^*_\Lambda := \left\{(\xi,\bar{\xi}) \in (L^p)^* \times ((L^p)^n)^* \,\middle|\, \mathbb{E}[V\xi] - \sum_{i=1}^n \mathbb{E}[\bar{Z}_i\bar{\xi}_i] \geq 0 \text{ for all } (V,\bar{Z}) \in \mathcal{A}_\Lambda\right\}.$$

Theorem 5.4.9. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a positively homogeneous systemic risk measure characterized by a l.s.c. positively homogeneous single-firm risk measure ρ_0 and a positively homogeneous aggregation function Λ that is continuous on $(L^p)^n$. Then ρ admits the representation*

$$\rho(\bar{X}) = \sup_{(1,\xi)\in\mathcal{A}^*_{\rho_0}, (\xi,\bar{\xi})\in\mathcal{A}^*_\Lambda} \sum_{i=1}^n \mathbb{E}[\bar{X}_i\bar{\xi}_i] \quad \text{for all } \bar{X} \in (L^p)^n. \tag{5.18}$$

In addition, a feasible solution $(\xi,\bar{\xi})$ to this optimization problem satisfies

$$\xi \geq 0, \quad \mathbb{E}[\xi] \leq 1, \quad \bar{\xi} \geq 0 \quad \text{and} \quad \sum_{i=1}^n \mathbb{E}[\bar{\xi}_i] \leq n\mathbb{E}[\xi].$$

If additionally $\rho(\mathbb{R}^n) = \mathbb{R}$, then we obtain $\mathbb{E}[\xi] = 1$.

Proof. Since every positively homogeneous systemic risk measure ρ which is a composition of a positively homogeneous single-firm risk measure ρ_0 and a positively homogeneous aggregation function Λ is also a convex systemic risk measure which is a composition of the convex single-firm risk measure ρ_0 and the convex aggregation function Λ, we can apply the convex dual representation result (see Theorem 5.4.5). As a consequence, ρ admits the representation

$$\rho(\bar{X}) = \sup_{(\xi,\bar{\xi})\in(L^p)^*\times((L^p)^n)^*} \left\{\sum_{i=1}^n \mathbb{E}[\bar{X}_i\bar{\xi}_i] - \alpha_n(\xi,\bar{\xi})\right\} \quad \text{for all } \bar{X} \in (L^p)^n$$

where $\alpha_n : (L^p)^* \times ((L^p)^n)^* \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\alpha_n(\xi,\bar{\xi}) = \sup_{(r,Y)\in\mathcal{A}_{\rho_0}, (V,\bar{Z})\in\mathcal{A}_\Lambda} \left\{-r + \mathbb{E}[(Y-V)\xi] + \sum_{i=1}^n \mathbb{E}[\bar{Z}_i\bar{\xi}_i]\right\}.$$

According to Proposition 5.4.3, positive homogeneity of ρ_0 and Λ implies that $\mathcal{A}_{\rho_0}$ and $\mathcal{A}_\Lambda$ are cones. Then we can easily verify that

$$\alpha_n(\xi, \bar{\xi}) = \begin{cases} 0 & \text{if } (1,\xi) \in \mathcal{A}^*_{\rho_0} \text{ and } (\xi,\bar{\xi}) \in \mathcal{A}^*_\Lambda \\ \infty & \text{otherwise} \end{cases}. \tag{5.19}$$

In case of $(1,\xi) \in \mathcal{A}^*_{\rho_0}$ and $(\xi,\bar{\xi}) \in \mathcal{A}^*_\Lambda$, we have $-r + \mathbb{E}[(Y-V)\xi] + \sum_{i=1}^n \mathbb{E}[\bar{Z}_i\bar{\xi}_i] \leq 0$ for all $(r,Y) \in \mathcal{A}_{\rho_0}$ and $(V,\bar{Z}) \in \mathcal{A}_\Lambda$. Since $\rho_0(0) = 0$ and $\Lambda(0_n) = 0$, we obtain $\alpha_n(\xi,\bar{\xi}) = 0$. On the other hand, suppose that $(1,\xi) \notin \mathcal{A}^*_{\rho_0}$. Then there exists $(r,Y) \in \mathcal{A}_{\rho_0}$ such that $-r + \mathbb{E}[Y\xi] > 0$. Moreover, we have $\lambda\,(r,Y) \in \mathcal{A}_{\rho_0}$ for $\lambda \geq 0$ because $\mathcal{A}_{\rho_0}$ is a cone. But this implies that $\alpha_n(\xi,\bar{\xi}) = \infty$. The same argumentation applies if we suppose that $(\xi,\bar{\xi}) \notin \mathcal{A}^*_\Lambda$.

Since (5.19) leads to representation (5.18), it remains to prove the assertions for feasible solutions to optimization problem (5.18). But these follow from Theorem 5.4.5 and Remark 5.4.8, which completes the proof. □

Remark 5.4.10. According to Corollary 2.3 in Kaina and Rüschendorf (2009), convex single-firm risk measures ρ_0 on L^∞ that satisfy monotonicity (R1), convexity (R2), the translation property (R3) and $\rho_0(0) = 0$ are finite and continuous on L^∞. Consequently, if we consider convex systemic risk measures $\rho = \Lambda \circ \rho_0$ on $(L^\infty)^n$ in conjunction with single-firm risk measures ρ_0 satisfying the translation property, then we do not require any additional continuity property of ρ_0 in Theorem 5.4.5 and Theorem 5.4.9.

We are interested in the relationship between the dual representation of ρ, in particular the $\bar{\xi}$-component of a feasible solution $(\xi,\bar{\xi})$ to (5.18), and the subdifferential of ρ specified in the following definition.

Definition 5.4.11. *Let $m \in \mathbb{N}$ and suppose that $\upsilon : (L^p)^m \to \overline{\mathbb{R}}$ is proper and convex. The* subdifferential of υ at $\bar{X} \in \operatorname{dom}\upsilon$ *is defined as the set*

$$\begin{aligned}\partial\upsilon(\bar{X}) &:= \left\{ \bar{\xi} \in ((L^p)^m)^* \,\middle|\, \sum_{i=1}^m \mathbb{E}[(\bar{Y}_i - \bar{X}_i)\bar{\xi}_i] \leq \upsilon(\bar{Y}) - \upsilon(\bar{X}) \text{ for all } \bar{Y} \in (L^p)^m \right\} \\ &= \left\{ \bar{\xi} \in ((L^p)^m)^* \,\middle|\, \sum_{i=1}^n \mathbb{E}[\bar{Z}_i\bar{\xi}_i] + \upsilon(\bar{X}) \leq \upsilon(\bar{X} + \bar{Z}) \text{ for all } \bar{Z} \in (L^p)^m \right\}.\end{aligned}$$

If $\bar{\xi} \in \partial\upsilon(\bar{X})$, then $\bar{\xi}$ is called subgradient of υ at $\bar{X} \in \operatorname{dom}\upsilon$. *The function υ is called* subdifferentiable at $\bar{X} \in \operatorname{dom}\upsilon$ *if $\partial\upsilon(\bar{X}) \neq \emptyset$.*

Corollary 5.4.12. *Let ρ be a positively homogeneous systemic risk measure characterized by a l.s.c. positively homogeneous single-firm risk measure ρ_0 and a positively homogeneous aggregation function Λ that is continuous on $(L^p)^n$. Fix an arbitrary economy $\bar{X} \in \operatorname{dom}\rho$. Then for every optimal solution $(\xi^o, \bar{\xi}^o)$ to (5.18), $\bar{\xi}^o$ is a subgradient of ρ at $\bar{X}$, i.e., $\bar{\xi}^o \in \partial\rho(\bar{X})$.*

Proof. Fix $\bar{X} \in \operatorname{dom} \rho$ and let $(\xi^o, \bar{\xi}^o)$ be an optimal solution to (5.18). The assertion holds since we have for every $\bar{Y} \in (L^p)^n$

$$\sum_{i=1}^n \mathbb{E}[(\bar{Y}_i - \bar{X}_i)\bar{\xi}_i^o] = \sum_{i=1}^n \mathbb{E}[\bar{Y}_i \bar{\xi}_i^o] - \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i^o] = \sum_{i=1}^n \mathbb{E}[\bar{Y}_i \bar{\xi}_i^o] - \rho(\bar{X}) \leq \rho(\bar{Y}) - \rho(\bar{X}).$$

□

5.5. Risk attribution

In this section we return to the problem of risk attribution, i.e., we are interested in the question of what fraction each firm contributes to the systemic risk of the whole financial system. We will see that the dual representation results from the previous section provide a possible solution to this problem.

Before we discuss this specific risk attribution method in detail, let us formally define the term systemic risk attribution in our setting. Similar definitions can be found in the traditional portfolio framework; see, for instance, Denault (2001), Tasche (2004), Kalkbrener (2005), Cheridito and Kromer (2011) and Kromer and Overbeck (2014).

Definition 5.5.1. *Let us consider a network of n firms and fix a convex systemic risk measure ρ. A* systemic risk attribution of $\bar{X} \in \operatorname{dom} \rho$ *is a vector $k(\bar{X}) = (k_1(\bar{X}), \ldots, k_n(\bar{X})) \in \mathbb{R}^n$ where $k_i(\bar{X})$ represents the systemic risk attributed to firm $i \in \{1, \ldots, n\}$. The systemic risk attribution $k(\bar{X})$ satisfies the* full allocation property (with respect to ρ) *if*

$$\rho(\bar{X}) = \sum_{i=1}^n k_i(\bar{X}).$$

The full allocation property states that the risk which is attributed to the different firms adds up to the systemic risk contained in the entire financial system. Because we try to answer the question of what fraction is caused by which firm in the financial system, this full allocation property is clearly a desirable property of systemic risk attributions.

In the following, we propose a possible risk attribution method for both the positively homogeneous and the convex case.

Positively homogeneous case: Let $\rho = \rho_0 \circ \Lambda$ be a positively homogeneous systemic risk measure with corresponding positively homogeneous single-firm risk measure ρ_0 and positively homogeneous aggregation function Λ and fix an economy $\bar{X} \in \operatorname{dom} \rho$. Moreover, let us consider the case in which there exists an optimal solution $(\xi^o, \bar{\xi}^o)$ to dual problem (5.18), i.e.,

$$\rho(\bar{X}) = \sup_{(1,\xi) \in \mathcal{A}^*_{\rho_0}, (\xi, \bar{\xi}) \in \mathcal{A}^*_{\Lambda}} \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i] = \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i^o].$$

Then, based on the idea in Chen et al. (2013), we can define a systemic risk attribution by

$$k(\bar{X}) := k(\bar{X}, \bar{\xi}^o) := (k_1(\bar{X}, \bar{\xi}^o), \ldots, k_n(\bar{X}, \bar{\xi}^o))$$

where the risk attributed to firm $i \in \{1, \ldots, n\}$ is given by

$$k_i(\bar{X}, \bar{\xi}^o) := \mathbb{E}[\bar{X}_i \bar{\xi}_i^o]. \tag{5.20}$$

Obviously, the systemic risk attribution defined above is unique if the corresponding optimal solution to (5.18) is unique, and the definition of $k(\bar{X}, \bar{\xi}^o)$ leads to

$$\rho(\bar{X}) = \sum_{i=1}^n k_i(\bar{X}, \bar{\xi}^o).$$

Thus, the full allocation property with respect to ρ is satisfied.

Convex case: Let $\rho = \rho_0 \circ \Lambda$ be a convex systemic risk measure with corresponding convex single-firm risk measure ρ_0 and convex aggregation function Λ and fix again an economy $\bar{X} \in \operatorname{dom} \rho$. Now, consider the case in which there exists an optimal solution $(\xi^o, \bar{\xi}^o)$ to dual problem (5.14), i.e.,

$$\rho(\bar{X}) = \sup_{(\xi, \bar{\xi}) \in (L^p)^* \times ((L^p)^n)^*} \left\{ \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i] - \alpha_n(\xi, \bar{\xi}) \right\} = \sum_{i=1}^n \mathbb{E}[\bar{X}_i \bar{\xi}_i^o] - \alpha_n(\xi^o, \bar{\xi}^o).$$

A similar risk attribution method to (5.20) that satisfies the full allocation property can be defined by

$$k(\bar{X}) := k(\bar{X}, \xi^o, \bar{\xi}^o) := (k_1(\bar{X}, \xi^o, \bar{\xi}^o), \ldots, k_n(\bar{X}, \xi^o, \bar{\xi}^o))$$

where the risk attributed to firm $i \in \{1, \ldots, n\}$ is given by

$$k_i(\bar{X}, \xi^o, \bar{\xi}^o) := \mathbb{E}[\bar{X}_i \bar{\xi}_i^o] - \gamma_i \alpha_n(\xi^o, \bar{\xi}^o). \tag{5.21}$$

Here, $\gamma_i \in \mathbb{R}$ are chosen such that $\sum_{i=1}^n \gamma_i = 1$.

In the following, we generalize Theorem 4 in Chen et al. (2013) for positively homogenous systemic risk measures on general probability spaces. This result is closely related to the so called no-undercut property of a risk attribution method which was, for instance, studied in Delbaen (2000) and Denault (2001).

Theorem 5.5.2. *Let $\rho = \rho_0 \circ \Lambda$ be a positively homogeneous systemic risk measure that admits representation* (5.18) *and fix $\bar{X} \in \operatorname{dom} \rho$. For $\bar{a} \in \mathbb{R}^n_+$, let us define $\bar{a} \star \bar{X} := (\bar{a}_1 \bar{X}_1, \ldots, \bar{a}_n \bar{X}_n) \in (L^p)^n$. If $(\xi^o, \bar{\xi}^o)$ is an optimal solution to* (5.18)*, then*

$$\sum_{i=1}^n \bar{a}_i k_i(\bar{X}, \bar{\xi}^o) \leq \rho(\bar{a} \star \bar{X}).$$

Proof. Fix $\bar{X} \in \operatorname{dom}\rho$. If $\rho(\bar{a} \star \bar{X}) = +\infty$, the assertion is trivial. Hence, suppose that $\bar{a} \star \bar{X} \in \operatorname{dom}\rho$. The dual representation result for positively homogeneous systemic risk measures yields

$$\rho(\bar{a} \star \bar{X}) = \sup_{(1,\xi)\in\mathcal{A}^*_{\rho_0},(\xi,\bar{\xi})\in\mathcal{A}^*_{\Lambda}} \sum_{i=1}^{n} \bar{a}_i \mathbb{E}[\bar{X}_i\bar{\xi}_i]. \tag{5.22}$$

Let $(\xi^o, \bar{\xi}^o)$ be an optimal solution to (5.18), i.e.,

$$\rho(\bar{X}) = \sup_{(1,\xi)\in\mathcal{A}^*_{\rho_0},(\xi,\bar{\xi})\in\mathcal{A}^*_{\Lambda}} \sum_{i=1}^{n} \mathbb{E}[\bar{X}_i\bar{\xi}_i] = \sum_{i=1}^{n} \mathbb{E}[\bar{X}_i\bar{\xi}^o_i],$$

and let $k(\bar{X}) = k(\bar{X}, \bar{\xi}^o)$ be the corresponding systemic risk attribution. Then $(1, \xi^o) \in \mathcal{A}^*_{\rho_0}$ and $(\xi^o, \bar{\xi}^o) \in \mathcal{A}^*_{\Lambda}$, which implies that $(\xi^o, \bar{\xi}^o)$ is a feasible solution to optimization problem (5.22) for all $\bar{a} \in \mathbb{R}^n_+$. Thus, we obtain

$$\rho(\bar{a} \star \bar{X}) \geq \sum_{i=1}^{n} \bar{a}_i \mathbb{E}[\bar{X}_i\bar{\xi}^o_i] = \sum_{i=1}^{n} \bar{a}_i k_i(\bar{X}, \bar{\xi}^o).$$

□

The previous result can be interpreted as follows: Fix an arbitrary economy $\bar{X} \in \operatorname{dom}\rho$ and define a new economy $\bar{a} \star \bar{X}$ by scaling the original economy $\bar{X}$ componentwise by the vector $\bar{a} \in \mathbb{R}^n_+$. Then the systemic risk of this new economy $\bar{a} \star \bar{X}$, i.e., $\rho(\bar{a} \star \bar{X})$, is always bounded from below by the weighted sum of the attributed risk components of the original economy $\bar{X}$.

Finally, we deal with the question of differentiability of positively homogeneous systemic risk measures and study the relation to systemic risk attribution. Again, the dual representation proved in Theorem 5.4.9 plays a key role to answer this question. A definition of the necessary notions of differentiability can be found in Appendix A.2.3.

Let us now consider the set

$$\mathcal{Z}^{\#} := \{(\xi, \bar{\xi}) \in (L^p)^* \times ((L^p)^n)^* | (1, \xi) \in \mathcal{A}^*_{\rho_0}, (\xi, \bar{\xi}) \in \mathcal{A}^*_{\Lambda}\}$$

of all feasible solutions to optimization problem (5.18) and the set

$$\mathcal{Z}^{\#}(\bar{X}) := \left\{ (\xi, \bar{\xi}) \in \mathcal{Z}^{\#} \,\middle|\, \rho(\bar{X}) = \sum_{i=1}^{n} \mathbb{E}[\bar{X}_i\bar{\xi}_i] \right\}$$

of all optimal solutions to (5.18) for a fixed economy $\bar{X} \in (L^p)^n$.

Theorem 5.5.3. *Let $\rho = \rho_0 \circ \Lambda$ be a finite valued positively homogeneous systemic risk measure that admits representation* (5.18)*. Then ρ has a directional derivative at $\bar{X} \in (L^p)^n$ in the direction $\bar{Y} \in (L^p)^n$ that is given by*

$$d^+\rho(\bar{X})(\bar{Y}) = \max_{(\xi,\bar{\xi})\in\mathcal{Z}^{\#}(\bar{X})} \sum_{i=1}^{n} \mathbb{E}[\bar{Y}_i\bar{\xi}_i].$$

Proof. Consider an arbitrary element $(U, \bar{Z}) \in L^p \times (L^p)^n$. Since ρ is finite valued, this is also true for the positively homogeneous single-firm risk measure ρ_0. As a consequence, there exists $r_U \in \mathbb{R}$ such that $\rho_0(U) \leq r_U$, and thus $(r_U, U) \in \mathcal{A}_{\rho_0}$. From $(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}$ it follows that $(1, \xi) \in \mathcal{A}^*_{\rho_0}$ and $(\xi, \bar{\xi}) \in \mathcal{A}^*_{\Lambda}$. Furthermore, $(1, \xi) \in \mathcal{A}^*_{\rho_0}$ implies that $\mathbb{E}[U\xi] \leq r_U$. Moreover, since $\Lambda(\bar{Z}) \in L^p$, there exists $Y_{\bar{Z}} \in L^p$ such that $\Lambda(\bar{Z}) \leq Y_{\bar{Z}}$, and $(\xi, \bar{\xi}) \in \mathcal{A}^*_{\Lambda}$ yields $\sum_{i=1}^n \mathbb{E}[\bar{Z}_i \bar{\xi}_i] \leq \mathbb{E}[Y_{\bar{Z}}\xi]$. Since ρ_0 is finite, we can find an element $r_{\bar{Z}} \in \mathbb{R}$ such that $\rho_0(Y_{\bar{Z}}) \leq r_{\bar{Z}}$. As above, $(1, \xi) \in \mathcal{A}^*_{\rho_0}$ again implies $\mathbb{E}[Y_{\bar{Z}}\xi] \leq r_{\bar{Z}}$. Thus, there exists $M_{(U,\bar{Z})} > 0$ with

$$\mathbb{E}[U\xi] + \sum_{i=1}^n \mathbb{E}[\bar{Z}_i \bar{\xi}_i] \leq M_{(U,\bar{Z})} \quad \text{for all } (\xi, \bar{\xi}) \in \mathcal{Z}^{\#}.$$

It follows that for each $(U, \bar{Z}) \in L^p \times (L^p)^n$, there exists $M > 0$ such that

$$\left| \mathbb{E}[U\xi] + \sum_{i=1}^n \mathbb{E}[\bar{Z}_i \bar{\xi}_i] \right| \leq M \quad \text{for all } (\xi, \bar{\xi}) \in \mathcal{Z}^{\#}.$$

This means that $\mathcal{Z}^{\#}$ is pointwise bounded. Moreover, this property is equivalent to norm boundedness of $\mathcal{Z}^{\#}$ due to the uniform boundedness principle (see Theorem A.2.3). Since $\mathcal{Z}^{\#}$ is convex, weak*-closed and norm bounded, Alaoglu's theorem (see Theorem A.2.4) implies weak*-compactness of $\mathcal{Z}^{\#}$. Now, consider the function $J : (L^p)^n \times \mathcal{Z}^{\#} \to \mathbb{R}$ defined by

$$J(\bar{Z}, (\xi, \bar{\xi})) := \sum_{i=1}^n \mathbb{E}[\bar{Z}_i \bar{\xi}_i] = \langle \bar{Z}, \bar{\xi} \rangle_n.$$

If we fix $\bar{X}, \bar{Y} \in (L^p)^n$ and endow $(L^p)^* \times ((L^p)^n)^*$ with the weak* topology $\sigma((L^p)^* \times ((L^p)^n)^*, L^p \times (L^p)^n)$, then the function J satisfies assumption D1 from Bernhard and Rapaport (1995) (see Condition A.2.18): The first two properties are trivial since $\mathcal{Z}^{\#}$ is weak*-compact and for all $(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}$, the map $(t, (\xi, \bar{\xi})) \mapsto J(\bar{X} + t\bar{Y}, (\xi, \bar{\xi}))$ is continuous at $(0, (\xi, \bar{\xi}))$. The third condition is satisfied with

$$\begin{aligned} d_1^+ J(\bar{X} + t\bar{Y}, (\xi, \bar{\xi}))(\bar{Y}) &= \lim_{u \downarrow 0} \frac{J(\bar{X} + (t+u)\bar{Y}, (\xi, \bar{\xi})) - J(\bar{X} + t\bar{Y}, (\xi, \bar{\xi}))}{u} \\ &= \lim_{u \downarrow 0} \frac{\sum_{i=1}^n \mathbb{E}[(\bar{X}_i + (t+u)\bar{Y}_i)\bar{\xi}_i] - \sum_{i=1}^n \mathbb{E}[(\bar{X}_i + t\bar{Y}_i)\bar{\xi}_i]}{u} \\ &= \lim_{u \downarrow 0} \frac{\sum_{i=1}^n \mathbb{E}[u\bar{Y}_i \bar{\xi}_i]}{u} \\ &= J(\bar{Y}, (\xi, \bar{\xi})) \end{aligned} \tag{5.23}$$

for $t \geq 0$ and $(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}$. (5.23) implies that $(t, (\xi, \bar{\xi})) \mapsto d_1^+ J(\bar{X} + t\bar{Y}, (\xi, \bar{\xi}))(\bar{Y})$ is continuous at $(0, (\xi, \bar{\xi}))$ for each $(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}$, which yields the forth condition. Finally, we obtain with Theorem D1 in Bernhard and Rapaport (1995) (see Theorem A.2.19) that ρ given by $\rho(\bar{Z}) = \sup_{(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}} J(\bar{Z}, (\xi, \bar{\xi}))$ for $\bar{Z} \in (L^p)^n$ satisfies

$$d^+ \rho(\bar{X})(\bar{Y}) = \max_{(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}(\bar{X})} d_1^+ J(\bar{X}, (\xi, \bar{\xi}))(\bar{Y}).$$

The assertion follows from

$$d_1^+ J(\bar{X}, (\xi, \bar{\xi}))(\bar{Y}) = \lim_{u \downarrow 0} \frac{J(\bar{X} + u\bar{Y}, (\xi, \bar{\xi})) - J(\bar{X}, (\xi, \bar{\xi}))}{u} = \sum_{i=1}^{n} \mathbb{E}[\bar{Y}_i \bar{\xi}_i].$$

□

Finally, we obtain the following corollary.

Corollary 5.5.4. *Let $\rho = \rho_0 \circ \Lambda$ be a finite valued positively homogeneous systemic risk measure that admits representation* (5.18)*. If* (5.18) *has a unique solution $(\xi^o, \bar{\xi}^o) \in \mathcal{Z}^\#$ at $\bar{X} \in (L^p)^n$, then ρ is Gâteaux differentiable at $\bar{X}$. Moreover, the Gâteaux derivative of ρ at $\bar{X}$ is given by*

$$D_G\rho(\bar{X})(\bar{Y}) = \sum_{i=1}^{n} \mathbb{E}[\bar{Y}_i \bar{\xi}_i^o] \quad \textit{for all } \bar{Y} \in (L^p)^n.$$

Proof. If (5.18) has a unique solution $(\xi^o, \bar{\xi}^o) \in \mathcal{Z}^\#$ at $\bar{X} \in (L^p)^n$, then we know from Theorem 5.5.3 that

$$d^+\rho(\bar{X})(\bar{Y}) = \sum_{i=1}^{n} \mathbb{E}[\bar{Y}_i \bar{\xi}_i^o] \quad \text{for all } \bar{Y} \in (L^p)^n.$$

Hence, the mapping $d^+\rho(\bar{X}) : (L^p)^n \to \mathbb{R}$ is linear and continuous. As a consequence, the assertion holds with $D_G\rho(\bar{X}) = d^+\rho(\bar{X})$. □

Remark 5.5.5. If we consider convex systemic risk measures on $(L^\infty)^n$ where $\rho = \rho_0 \circ \Lambda$ and ρ_0 satisfies (R1)-(R3), then finiteness of ρ follows from properness of ρ (see Lemma 4.4.2). Thus, in this specific case it is sufficient to require proper positively homogeneous systemic risk measures in Theorem 5.5.3 and Corollary 5.5.4.

Note that Corollary 5.5.4 provides a link between our systemic risk attribution proposed in this section and the differentiability of the corresponding systemic risk measure. By fixing $\bar{X} \in (L^p)^n$ and assuming that a finite, positively homogeneous systemic risk measure $\rho = \rho_0 \circ \Lambda$ admits representation (5.18) with the unique optimal solution $(\xi^o, \bar{\xi}^o) \in \mathcal{Z}^\#$, we obtain for each $i \in \{1, \dots, n\}$ that

$$k_i(\bar{X}, \bar{\xi}^o) = \mathbb{E}[\bar{X}_i \bar{\xi}_i^o] = D_G\rho(\bar{X})(1_{(n,i)} \star \bar{X})$$

where $1_{(n,i)} \in \mathbb{R}_+^n$ is defined by

$$(1_{(n,i)})_j := \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \quad \text{for } j \in \{1, \dots, n\}.$$

That is, the ith component of the systemic risk attribution $k(\bar{X}, \xi^o)$ defined in Equation (5.20) is equal to the Gâteaux derivative of ρ at $\bar{X}$ in the direction $1_{(n,i)} \star \bar{X}$. Let us consider the function $r : \mathbb{R}_+^n \to \mathbb{R}$ defined by $r(\bar{a}) := \rho(\bar{a} \star \bar{X})$. Then the Gâteaux derivative of ρ at $\bar{X}$ in the direction $1_{(n,i)} \star \bar{X}$ corresponds with the ith partial derivative of r at 1_n, i.e., $D_G\rho(\bar{X})(1_{(n,i)} \star \bar{X}) = \partial_i r(1_n)$. Therefore, as already pointed out in Chen et al. (2013), for positively homogeneous systemic risk measures which satisfy the requirements from Corollary 5.5.4, the risk attribution method defined in (5.20) corresponds to the well known Aumann-Shapley prices given by

$$k_i^{\mathrm{AS}} := \int_0^1 \partial_i r(t1_n)dt = \partial_i r(1_n) \quad \text{for } i \in \{1, \ldots, n\}.$$

This risk attribution method is also called Euler allocation rule. For additional information concerning this specific method we refer the reader, for instance, to Denault (2001) and Tasche (2008).

6. From static to dynamic risk measures

Static risk measures introduced in Chapter 4 do not take into account any dynamic features like the availability of additional information over a specific period of time. Another possibility is to consider not only random variables as input for risk measures but discrete-time or continuous-time stochastic processes. These processes represent, for instance, the market or accounting value of a firm's equity or the market value of selected financial securities; see Artzner et al. (2007) and Cheridito et al. (2006).

We have already pointed out in the introduction to this part of the thesis that there exist multiple approaches studying these different dynamic aspects of risk measurement. For an overview we refer to Acciaio and Penner (2011).

In this chapter we discuss the approach introduced in Cheridito et al. (2006) and Cheridito and Kupper (2011). This approach provides an excellent starting point for a dynamic generalization of our systemic risk measures studied in Chapter 5. From now on, the main objects of interest are conditional risk measures on discrete-time stochastic processes. Thus, we combine risk measurement on processes with the theory of conditional risk measures where the focus lies on informational aspects. As a consequence of this, the corresponding risk measures depend at time t on the available information at this point in time.

Focusing on risk measures which depend on different (discrete) points in time leads to the new problem of time-consistency of these risk measures. Consequently, we have to study how the different risk measures depend on each other. As a possible solution, Cheridito et al. (2006) use the concept of strong time-consistency. In this chapter we repeat this notion of time-consistency and highlight important results from Cheridito et al. (2006).

The outline of this chapter is the following: In Section 6.1 we introduce the notation, important definitions and properties of conditional risk measures for bounded discrete-time processes. Thereafter, Section 6.2 repeats dual representation results for the convex and the coherent case. After introducing dynamic risk measures in this context in Section 6.3, we consider the strong time-consistency property. Finally, in Section 6.4 we present some examples from Cheridito et al. (2006) and Cheridito and Kupper (2011).

6.1. Notation, definitions and important properties

Throughout this chapter we fix the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{N}_0}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We denote the space of all extended random variables by $L^0(\overline{\mathbb{R}})$, i.e., $L^0(\overline{\mathbb{R}})$ contains all measurable functions $\gamma : (\Omega, \mathcal{F}, \mathbb{P}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, and $\mathcal{B}(\overline{\mathbb{R}})$ denotes the corresponding Borel-σ-

algebra. We equip the space $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ with the usual L^1-norm, i.e., $\|\varsigma\|_1 := \int |\varsigma| d\mathbb{P}$ for $\varsigma \in L^1$, and the space $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with the norm $\|\gamma\|_\infty := \inf\{r \in \mathbb{R} | |\gamma| \leq r \ \mathbb{P}\text{-a.s.}\}$ for $\gamma \in L^\infty$.

In this chapter we understand equalities and inequalities between random variables and stochastic processes $\mathbb{P}$-a.s. For processes X and Y this means that $X \leq Y$ if and only if $X_t \leq Y_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$. Let $\mathcal{R}^0$ be the space of $\mathbb{F}$-adapted processes and define

$$\mathcal{R}^\infty := \{X \in \mathcal{R}^0 | \|X\|_{\mathcal{R}^\infty} < \infty\} \quad \text{and} \quad \mathcal{A}^1 := \{\xi \in \mathcal{R}^0 | \|\xi\|_{\mathcal{A}^1} < \infty\}$$

with

$$\|X\|_{\mathcal{R}^\infty} := \inf\left\{r \in \mathbb{R} \,\middle|\, \sup_{t \in \mathbb{N}_0} |X_t| \leq r\right\} \quad \text{and}$$
$$\|\xi\|_{\mathcal{A}^1} := \mathbb{E}\left[\sum_{t \in \mathbb{N}_0} |\Delta \xi_t|\right] \quad \text{where } \xi_{-1} := 0,\ \Delta\xi_t := \xi_t - \xi_{t-1} \quad \text{for } t \in \mathbb{N}_0.$$

If we consider the bilinear form $\langle \cdot, \cdot \rangle : \mathcal{R}^\infty \times \mathcal{A}^1 \to \mathbb{R}$ defined by

$$\langle X, \xi \rangle := \mathbb{E}\left[\sum_{t \in \mathbb{N}_0} X_t \Delta \xi_t\right],$$

then the topology $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$ denotes the weakest topology on $\mathcal{R}^\infty$ such that for all $\xi \in \mathcal{A}^1$, the functional $X \mapsto \langle X, \xi \rangle$ on $\mathcal{R}^\infty$ is continuous and linear. Similarly, the topology $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$ denotes the weakest topology on $\mathcal{A}^1$ such that for all $X \in \mathcal{R}^\infty$, the functional $\xi \mapsto \langle X, \xi \rangle$ on $\mathcal{A}^1$ is continuous and linear. For additional information concerning these definitions we refer to Section 7.1 and Appendix A.2.

Let τ be a finite ($\mathbb{F}$-)stopping time and θ be an ($\mathbb{F}$-)stopping time such that $0 \leq \tau \leq \theta \leq \infty$. Then the projection $p^{\tau,\theta} : \mathcal{R}^0 \to \mathcal{R}^0$ is given by

$$p^{\tau,\theta}(X)_t := I_{\{\tau \leq t\}} X_{t \wedge \theta} \quad \text{for } t \in \mathbb{N}_0.$$

Furthermore, we define

$$\|X\|_{\tau,\theta} := \operatorname{ess\,inf}\left\{\gamma \in L^\infty_\tau \,\middle|\, \sup_{t \in \mathbb{N}_0} |p^{\tau,\theta}(X_t)| \leq \gamma\right\} \quad \text{for } X \in \mathcal{R}^\infty$$

with $L^\infty_\tau := L^\infty(\Omega, \mathcal{F}_\tau, \mathbb{P})$. At last, define $\mathcal{R}^\infty_{\tau,\theta} \subset \mathcal{R}^\infty$ and $\mathcal{A}^1_{\tau,\theta} \subset \mathcal{A}^1$ by

$$\mathcal{R}^\infty_{\tau,\theta} := p^{\tau,\theta}\mathcal{R}^\infty \quad \text{and} \quad \mathcal{A}^1_{\tau,\theta} := p^{\tau,\theta}\mathcal{A}^1.$$

Cheridito et al. (2006) define risk measures on the space $\mathcal{R}^\infty_{\tau,\theta}$. Thus, the main objects of interest are processes on the interval $[\tau, \theta] \cap \mathbb{N}_0 := \{(t, \omega) \in \mathbb{N}_0 \times \Omega | \tau(\omega) \leq t \leq \theta(\omega)\}$. It is important to note that in accordance with the static setting in Chapter 4, these processes are assumed to represent the evolution of different financial values. This means that we focus on value processes in this chapter.

For $\gamma \in L^\infty_\tau$ and $X \in \mathcal{R}^\infty$, the processes $Y := \gamma I_{[\tau,\theta)}$ and $Z := XI_{[\tau,\theta)}$ are understood in the following way:

$$Y_t(\omega) = \gamma(\omega)I_{[\tau,\theta)}(t,\omega) \quad \text{and} \quad Z_t(\omega) = X_t(\omega)I_{[\tau,\theta)}(t,\omega) \quad \text{for } t \in \mathbb{N}_0, \omega \in \Omega.$$

Now, consider the following properties of a mapping $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$:

(n) Normalization: $\rho(0) = 0$.

(m) Monotonicity: If $X \geq Y$, then $\rho(X) \leq \rho(Y)$ for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$.

(t) $\mathcal{F}_\tau$-translation property: $\rho(X + \gamma I_{[\tau,\infty)}) = \rho(X) - \gamma$ for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $\gamma \in L^\infty_\tau$.

(c) $\mathcal{F}_\tau$-convexity: $\rho(\gamma X + (1-\gamma)Y) \leq \gamma\rho(X) + (1-\gamma)\rho(Y)$ for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ and $\gamma \in L^\infty_\tau$ with $0 \leq \gamma \leq 1$.

(ph) $\mathcal{F}_\tau$-positive homogeneity: $\rho(\gamma X) = \gamma\rho(X)$ for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $\gamma \in (L^\infty_\tau)_+ := \{\gamma \in L^\infty_\tau | \gamma \geq 0\}$.

Definition 6.1.1. *A* conditional risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) *is a mapping $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ that satisfies the properties (n), (m) and (t). A* conditional convex risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) *is a conditional risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) that additionally satisfies the property (c), and a* conditional coherent risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) *is a conditional convex risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) that additionally satisfies the property (ph). For a conditional risk measure ρ on $\mathcal{R}^\infty_{\tau,\theta}$ and $X \in \mathcal{R}^\infty$, we set $\rho(X) := \rho(p^{\tau,\theta}(X))$.*

We can interpret the properties above as in the static case. Nevertheless, in contrast to the static approach, conditional risk measures take into account the information available at the stopping time τ. For technical reasons, Definition 6.1.1, which is based on Definition 3.1 in Cheridito et al. (2006), requires the normalization property (n) for conditional risk measures on $\mathcal{R}^\infty_{\tau,\theta}$. Furthermore, note that the main objects in Cheridito et al. (2006) are so called *conditional monetary utility functions (on $\mathcal{R}^\infty_{\tau,\theta}$)* defined by $\phi := -\rho$ for conditional risk measures ρ on $\mathcal{R}^\infty_{\tau,\theta}$. Since we are studying risk measures throughout this thesis, we adapt the results from Cheridito et al. (2006) to our setting.

In the conditional setting in this chapter the translation property holds for random variables $\gamma \in L^\infty_\tau$. Thus, we obtain a stronger notion of Lipschitz continuity.

Proposition 6.1.2 (See Proposition 3.3 in Cheridito et al. (2006))**.** *Every mapping $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ with (m) and (t) satisfies the following properties:*

(lc) *$\mathcal{F}_\tau$-Lipschitz continuity: $|\rho(X) - \rho(Y)| \leq \|X - Y\|_{\tau,\theta}$ for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$.*

(lp) *Local property: $\rho(I_A X + I_{A^c} Y) = I_A\rho(X) + I_{A^c}\rho(Y)$ for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ and $A \in \mathcal{F}_\tau$.*

First of all, let us discuss the dependence between the set of acceptable positions and the corresponding conditional risk measure ρ on $\mathcal{R}^\infty_{\tau,\theta}$.

Definition 6.1.3. *The* acceptance set *of a conditional risk measure ρ on $\mathcal{R}^\infty_{\tau,\theta}$ is defined by*

$$\mathcal{B}_\rho := \{X \in \mathcal{R}^\infty_{\tau,\theta} | \rho(X) \leq 0\}.$$

The following two propositions are based on Propositions 3.6, 3.9 and 3.10 in Cheridito et al. (2006). For the corresponding results regarding static risk measures see Proposition 4.1.4 and Proposition 4.1.5.

Proposition 6.1.4. *Let $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ be a conditional risk measure and $\mathcal{B}_\rho$ the corresponding acceptance set. Then the following properties are satisfied:*

1. *$\mathcal{B}_\rho \neq \emptyset$ and*
   a) *$0 \in \mathcal{B}_\rho$ and $\operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | \gamma I_{[\tau,\infty)} \in \mathcal{B}_\rho\} = 0$,*
   b) *if $X \in \mathcal{B}_\rho$ and $Y \in \mathcal{R}^\infty_{\tau,\theta}$ with $Y \geq X$, then $Y \in \mathcal{B}_\rho$,*
   c) *if $(X^{(m)}) \subset \mathcal{B}_\rho$ and $X \in \mathcal{R}^\infty_{\tau,\theta}$ with $\|X^{(m)} - X\|_{\tau,\theta} \to 0$ $\mathbb{P}$-a.s., then $X \in \mathcal{B}_\rho$,*
   d) *if $A \in \mathcal{F}_\tau$ and $X, Y \in \mathcal{B}_\rho$, then $I_A X + I_{A^c} Y \in \mathcal{B}_\rho$.*
2. *ρ admits the representation*

$$\rho(X) = \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | X + \gamma I_{[\tau,\infty)} \in \mathcal{B}_\rho\} \quad \textit{for all } X \in \mathcal{R}^\infty_{\tau,\theta}.$$

3. *ρ is a conditional convex risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ if and only if $\mathcal{B}_\rho$ is $\mathcal{F}_\tau$-convex, i.e., $\gamma X + (1-\gamma)Y$ for all $X, Y \in \mathcal{B}_\rho$ and $\gamma \in L^\infty_\tau$ with $0 \leq \gamma \leq 1$.*
4. *ρ is positively homogeneous if and only if $\mathcal{B}_\rho$ is an $\mathcal{F}_\tau$-cone, i.e., $\gamma X \in \mathcal{B}_\rho$ for all $X \in \mathcal{B}_\rho$ and $\gamma \in (L^\infty_\tau)_+$. In particular, ρ is a conditional coherent risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ if and only if $\mathcal{B}_\rho$ is an $\mathcal{F}_\tau$-convex $\mathcal{F}_\tau$-cone.*

Now, suppose that $\mathcal{B} \subset \mathcal{R}^\infty_{\tau,\theta}$ is a set of acceptable positions. If we define a conditional risk measure ρ on $\mathcal{R}^\infty_{\tau,\theta}$ by using the idea that $\rho(X)$ is the smallest amount that has to be added to the position X such that the new position is acceptable, then we obtain the following proposition.

Proposition 6.1.5. *Consider $\emptyset \neq \mathcal{B} \subset \mathcal{R}^\infty_{\tau,\theta}$ that satisfies the properties 1.a) and 1.b) from Proposition 6.1.4. Then*

$$\rho_{\mathcal{B}}(X) := \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | X + \gamma I_{[\tau,\infty)} \in \mathcal{B}\} \quad \textit{for } X \in \mathcal{R}^\infty_{\tau,\theta}$$

satisfies the following properties:

1. *$\rho_{\mathcal{B}}$ is a conditional risk measure on $\mathcal{R}^\infty_{\tau,\theta}$.*
2. *If $\mathcal{B}$ is $\mathcal{F}_\tau$-convex, then $\rho_{\mathcal{B}}$ is a conditional convex risk measure on $\mathcal{R}^\infty_{\tau,\theta}$.*
3. *If $\mathcal{B}$ is an $\mathcal{F}_\tau$-cone, then $\rho_{\mathcal{B}}$ is positively homogeneous. Especially, if $\mathcal{B}$ is an $\mathcal{F}_\tau$-convex $\mathcal{F}_\tau$-cone, then $\rho_{\mathcal{B}}$ is a conditional coherent risk measure on $\mathcal{R}^\infty_{\tau,\theta}$.*
4. *$\mathcal{B}$ is a subset of the acceptance set $\mathcal{B}_{\rho_{\mathcal{B}}}$, and the smallest subset of $\mathcal{R}^\infty_{\tau,\theta}$ that contains $\mathcal{B}$ and satisfies the properties 1.c) and 1.d) from Proposition 6.1.4 is equal to $\mathcal{B}_{\rho_{\mathcal{B}}}$.*

6.2. Representation of conditional risk measures

In the conditional setting studied in this chapter there exist several dual representation results for conditional risk measures on $\mathcal{R}^\infty_{\tau,\theta}$. In this section we present selected representation results from Cheridito et al. (2006). But first, let us consider the following definition.

Definition 6.2.1. *For $X \in \mathcal{R}^\infty$ and $\xi \in \mathcal{A}^1$, we define*

$$\langle X, \xi\rangle^{\tau,\theta} := \mathbb{E}\left[\sum_{t\in[\tau,\theta]\cap\mathbb{N}_0} X_t\Delta\xi_t \middle| \mathcal{F}_\tau\right].$$

The sets $\mathcal{A}^1_+$, $(\mathcal{A}^1_{\tau,\theta})_+$ and $\mathcal{D}_{\tau,\theta}$ are defined by

$$\begin{aligned}\mathcal{A}^1_+ :=& \{\xi \in \mathcal{A}^1 | \Delta\xi_t \geq 0 \text{ for all } t \in \mathbb{N}_0\},\\ (\mathcal{A}^1_{\tau,\theta})_+ :=& p^{\tau,\theta}\mathcal{A}^1_+,\\ \mathcal{D}_{\tau,\theta} :=& \{\xi \in (\mathcal{A}^1_{\tau,\theta})_+ | \langle 1, \xi\rangle^{\tau,\theta} = 1\}.\end{aligned}$$

Remark 6.2.2.

1. For $X \in \mathcal{R}^\infty$ and $\xi \in \mathcal{A}^1_{\tau,\theta}$, we have

$$\langle X, \xi\rangle = \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} X_t\Delta\xi_t\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{t\in\mathbb{N}_0} X_t\Delta\xi_t \middle| \mathcal{F}_\tau\right]\right] = \mathbb{E}[\langle X, \xi\rangle^{\tau,\theta}].$$

2. Because of the Radon-Nikodym theorem, we can identify probability measures $\mathbb{Q}$ which are absolutely continuous with respect to $\mathbb{P}$ by its densities $\frac{d\mathbb{Q}}{d\mathbb{P}}$ where $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \{\varsigma \in L^1 | \varsigma \geq 0, \mathbb{E}[\varsigma] = 1\}$. Hence, Cheridito et al. (2006) point out that we can consider processes in $\mathcal{D}_{\tau,\theta}$ as conditional probability densities on the product space $\mathbb{N}_0 \times \Omega$. We will see below that regarding the dual representations, the set $\mathcal{D}_{\tau,\theta}$ replaces the set of absolutely continuous probability measures from the static results.

To obtain a dual representation, Cheridito et al. (2006) introduce a continuity property for mappings $\upsilon : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$.

Definition 6.2.3. *A mapping $\upsilon : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ is called* continuous for bounded decreasing sequences *if for $X \in \mathcal{R}^\infty_{\tau,\theta}$ and every decreasing sequence $(X^{(m)}) \subset \mathcal{R}^\infty_{\tau,\theta}$ with $X^{(m)}_t \downarrow X_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$, it follows that $\lim_{m\to\infty} \upsilon(X^{(m)}) = \upsilon(X)$ $\mathbb{P}$-a.s.*

We also need generalized penalty functions which are appropriate for the setting in this chapter.

Definition 6.2.4. *Define the spaces $L^0_\tau(\overline{\mathbb{R}})$ and $L^0_\tau(\overline{\mathbb{R}}_+)$ by*

$$L^0_\tau(\overline{\mathbb{R}}) := \{\gamma \in L^0(\overline{\mathbb{R}}) | \gamma \text{ is } \mathcal{F}_\tau\text{-measurable}\} \quad \text{and} \quad L^0_\tau(\overline{\mathbb{R}}_+) := \{\gamma \in L^0_\tau(\overline{\mathbb{R}}) | \gamma \geq 0\}.$$

Then a penalty function (on $\mathcal{D}_{\tau,\theta}$) *is a mapping $\alpha^{\tau,\theta} : \mathcal{D}_{\tau,\theta} \to L^0_\tau(\overline{\mathbb{R}}_+)$ that satisfies*

$$\operatorname*{ess\,inf}_{\xi\in\mathcal{D}_{\tau,\theta}} \alpha^{\tau,\theta}(\xi) = 0.$$

The following theorem is the adaption of the corresponding result in Cheridito et al. (2006) to our setting.

Theorem 6.2.5 (See Theorem 3.16 in Cheridito et al. (2006)). *Let $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ be a conditional convex risk measure. Then the following properties are equivalent:*

1. *ρ admits the representation*
$$\rho(X) = \operatorname*{ess\,sup}_{\xi \in \mathcal{D}_{\tau,\theta}} \{\langle -X, \xi\rangle^{\tau,\theta} - \alpha^{\tau,\theta}(\xi)\} \quad \text{for } X \in \mathcal{R}^\infty_{\tau,\theta} \tag{6.1}$$
where $\alpha^{\tau,\theta}$ is a penalty function on $\mathcal{D}_{\tau,\theta}$.
2. *The acceptance set $\mathcal{B}_\rho$ is a $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed subset of $\mathcal{R}^\infty$.*
3. *ρ is continuous for bounded decreasing sequences.*

In particular, if one of these properties is satisfied, then ρ can be represented with the minimal penalty function
$$\alpha^{\tau,\theta}_{min}(\xi) := \operatorname*{ess\,sup}_{X \in \mathcal{B}_\rho} \langle -X, \xi\rangle^{\tau,\theta} = \operatorname*{ess\,sup}_{X \in \mathcal{R}^\infty_{\tau,\theta}} \{\langle -X, \xi\rangle^{\tau,\theta} - \rho(X)\} \quad \text{for } \xi \in \mathcal{D}_{\tau,\theta}.$$
Moreover, this penalty function on $\mathcal{D}_{\tau,\theta}$ is minimal in the sense that $\alpha^{\tau,\theta}_{min}(\xi) \leq \alpha^{\tau,\theta}(\xi)$ for all $\xi \in \mathcal{D}_{\tau,\theta}$ and all penalty functions $\alpha^{\tau,\theta}$ on $\mathcal{D}_{\tau,\theta}$ that satisfy representation (6.1)*.*

The coherent special case reads as follows.

Corollary 6.2.6 (See Theorem 3.18 in Cheridito et al. (2006)). *Let $\rho : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ be a conditional coherent risk measure. Then there exists a set $\emptyset \neq \mathcal{Q}_{\tau,\theta} \subset \mathcal{D}_{\tau,\theta}$ such that ρ admits the representation*
$$\rho(X) = \operatorname*{ess\,sup}_{\xi \in \mathcal{Q}_{\tau,\theta}} \langle -X, \xi\rangle^{\tau,\theta} \quad \text{for all } X \in \mathcal{R}^\infty_{\tau,\theta} \tag{6.2}$$
if and only if the equivalent properties from Theorem 6.2.5 are satisfied. In this case, the set
$$\mathcal{Q}^0_{\tau,\theta} := \{\xi \in \mathcal{D}_{\tau,\theta} | \alpha^{\tau,\theta}_{min}(\xi) = 0\}$$
is the smallest $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closed, $\mathcal{F}_\tau$-convex subset of $\mathcal{D}_{\tau,\theta}$ that contains $\mathcal{Q}_{\tau,\theta}$ and satisfies representation (6.2) *with $\mathcal{Q}^0_{\tau,\theta}$ instead of $\mathcal{Q}_{\tau,\theta}$.*

Let us compare the previous dual representation results for conditional convex and coherent risk measures on $\mathcal{R}^\infty_{\tau,\theta}$ with the dual representation results for convex and coherent risk measures on L^∞ from Section 4.3. The additional information which is available at time τ is represented by the fact that we use conditional expectations in (6.1) and (6.2) (recall that $\langle -X, \xi\rangle^{\tau,\theta}$ is defined as a conditional expectation) instead of the usual expectations in (4.4) and (4.5), respectively. Moreover, the penalty function on $\mathcal{D}_{\tau,\theta}$ is also random. If we interpret the elements in $\mathcal{D}_{\tau,\theta}$ as conditional probability measures on the product space $\mathbb{N}_0 \times \Omega$ (see Remark 6.2.2), then we obtain the conditional convex risk measure ρ on $\mathcal{R}^\infty_{\tau,\theta}$ by considering the essential supremum over different probabilistic models.

6.3. Time-consistent dynamic risk measures

This section addresses another dynamic aspect in the axiomatic risk measurement approach. We adopt the definition from Cheridito et al. (2006) and define dynamic risk measures as families of conditional risk measures at different points in time. In order to clarify the relationship between these different conditional risk measures, we subsequently introduce and discuss an appropriate time-consistency concept. For a detailed analysis we refer the reader to Cheridito et al. (2006) and Cheridito and Kupper (2011).

Throughout this section let $S \in \mathbb{N}_0$ and $T \in \mathbb{N}_0 \cup \{+\infty\}$ with $S \leq T$.

Definition 6.3.1. *For every $t \in [S,T] \cap \mathbb{N}_0$, let $\rho_{t,T} : \mathcal{R}^\infty_{t,T} \to L^\infty_t$ be a conditional risk measure on $\mathcal{R}^\infty_{t,T}$ with acceptance set $\mathcal{B}_{t,T}$. A* dynamic risk measure *is defined as a family $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$. The* corresponding family of acceptance sets *is given by $(\mathcal{B}_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$. Let τ and θ be stopping times with $\tau < \infty$ and $S \leq \tau \leq \theta \leq T$ and define $\rho_{\tau,\theta} : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ by*

$$\rho_{\tau,\theta}(X) := \sum_{t\in[S,T]\cap\mathbb{N}_0} \rho_{t,T}(I_{\{\tau=t\}}X). \tag{6.3}$$

The corresponding set $\mathcal{B}_{\tau,\theta} \subset \mathcal{R}^\infty_{\tau,\theta}$ is given by

$$\mathcal{B}_{\tau,\theta} := \{X \in \mathcal{R}^\infty_{\tau,\theta} | I_{\{\tau=t\}}X \in \mathcal{B}_{t,T} \text{ for all } t \in [S,T] \cap \mathbb{N}_0\}.$$

A dynamic risk measure $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$ is called convex [coherent] *if all $\rho_{t,T}$, $t \in [S,T] \cap \mathbb{N}_0$, are conditional convex [coherent] risk measures on $\mathcal{R}^\infty_{t,T}$.*

Cheridito et al. (2006) point out that $\rho_{\tau,\theta}$ defined in (6.3) is a conditional risk measure on $\mathcal{R}^\infty_{\tau,\theta}$ with acceptance set $\mathcal{B}_{\tau,\theta}$ that inherits convexity and positive homogeneity from $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$. Similarly, $\rho_{\tau,\theta}$ is continuous for bounded decreasing sequences if all $\rho_{t,T}$, $t \in [S,T] \cap \mathbb{N}_0$, satisfy this property.

Definition 6.3.2. *A dynamic risk measure $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$ is called* time-consistent *if the following property is satisfied for all finite stopping times τ, θ with $S \leq \tau \leq \theta \leq T$:*

(TC) *If $X, Y \in \mathcal{R}^\infty_{\tau,T}$ with*

$$XI_{[\tau,\theta)} = YI_{[\tau,\theta)} \quad \text{and} \quad \rho_{\theta,T}(X) \leq \rho_{\theta,T}(Y),$$

then $\rho_{\tau,T}(X) \leq \rho_{\tau,T}(Y)$.

Time-consistency is equivalent to the following property (TC'). For a proof see Proposition 4.4 in Cheridito et al. (2006).

(TC') For each $t \in [S,T] \cap \mathbb{N}_0$ and every finite stopping time θ that satisfies $t \leq \theta \leq T$, equation

$$\rho_{t,T}(X) = \rho_{t,T}(XI_{[t,\theta)} - \rho_{\theta,T}(X)I_{[\theta,\infty)})$$

holds for all $X \in \mathcal{R}^\infty_{t,T}$.

The time-consistency property (TC) can be described as follows: Let us consider two processes X and Y that are equal up to the stopping time θ, i.e., $XI_{[\tau,\theta)} = YI_{[\tau,\theta)}$, and let the time-$\theta$-risk of one process be greater than or equal to the time-θ-risk of the other process, i.e., $\rho_{\theta,T}(X) \leq \rho_{\theta,T}(Y)$. Then this relation is still satisfied for the risk of these processes at the smaller stopping time τ.

In case of $T \in \mathbb{N}$ or in case of a dynamic risk measure $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$ where all conditional risk measures are continuous for bounded decreasing sequences, time-consistency and the weaker property of so called one-step time-consistency are equivalent. More precisely, the following result is satisfied.

Proposition 6.3.3 (See Proposition 4.5 in Cheridito et al. (2006))**.** *Consider a dynamic risk measure* $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$ *that satisfies* one-step time-consistency*, i.e.,*

$$\rho_{t,T}(X) = \rho_{t,T}(XI_{\{t\}} - \rho_{t+1,T}(X)I_{[t+1,\infty)}) \quad \textit{for all } t \in [S, T-1] \cap \mathbb{N}_0 \textit{ and } X \in \mathcal{R}^{\infty}_{t,T}.$$

Additionally, suppose that at least one of the following conditions holds:

1. $T \in \mathbb{N}$.
2. *All conditional risk measures* $\rho_{t,T}$*,* $t \in [S, T] \cap \mathbb{N}_0$*, are continuous for bounded decreasing sequences.*

Then the dynamic risk measure $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$ *is time-consistent.*

The connection between time-consistent dynamic risk measures and their acceptance sets is discussed in Theorem 4.6 in Cheridito et al. (2006).

Finally, we introduce the term relevant risk measures, which was originally studied in Artzner et al. (1999). Following the idea that a position X with $X \leq 0$ and $X \neq 0$ has positive risk, i.e., $\rho(X) > 0$, Cheridito et al. (2006) adapt the definition from Artzner et al. (1999) to their setting of conditional risk measurement.

We will see that this property guarantees the existence of time-consistent dynamic extensions of conditional risk measures on $\mathcal{R}^{\infty}_{S,T}$.

Definition 6.3.4. *Consider a conditional risk measure* $\rho : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}_{\tau}$*. Then* ρ *is called* θ-relevant *if*

$$A \subset \{\rho(-\varepsilon I_A I_{[t\wedge\theta,\infty)}) > 0\}$$

for all $\varepsilon > 0$*,* $t \in \mathbb{N}_0$ *and* $A \in \mathcal{F}_{t\wedge\theta}$*. The set* $\mathcal{D}^{rel}_{\tau,\theta}$ *is defined by*

$$\mathcal{D}^{rel}_{\tau,\theta} := \left\{ \xi \in \mathcal{D}_{\tau,\theta} \,\middle|\, \mathbb{P}\left[\sum_{j\geq t\wedge\theta} \Delta\xi_j > 0\right] = 1 \textit{ for all } t \in \mathbb{N}_0 \right\}.$$

Remark 6.3.5. 1. Cheridito et al. (2006) point out that if θ is finite, then θ-relevance of a conditional risk measure ρ on $\mathcal{R}^{\infty}_{\tau,\theta}$ is equivalent to $A \subset \{\rho(-\varepsilon I_A I_{[\theta,\infty)}) > 0\}$ for all $\varepsilon > 0$ and $A \in \mathcal{F}_\theta$. Furthermore, supposing that the previous property is satisfied, we have

$$\mathcal{D}^{rel}_{\tau,\theta} = \{\xi \in \mathcal{D}_{\tau,\theta} | \mathbb{P}[\Delta\xi_\theta > 0] = 1\}.$$

2. There exist specific dual representations of θ-relevant conditional convex and coherent risk measures on $\mathcal{R}^{\infty}_{\tau,\theta}$. In these representations the set $\mathcal{D}^{rel}_{\tau,\theta}$ replaces the set $\mathcal{D}_{\tau,\theta}$ from representations of general conditional risk measures on $\mathcal{R}^{\infty}_{\tau,\theta}$. For more details we refer to Section 3.3 in Cheridito et al. (2006).

We close this section by stating an important result for T-relevant risk measures.

Proposition 6.3.6 (See Corollary 4.8 in Cheridito et al. (2006))**.** *Consider a T-relevant conditional risk measure ρ on $\mathcal{R}^{\infty}_{S,T}$. Then we can find at most one time-consistent dynamic risk measure $(\rho_{t,T})_{t\in[S,T]\cap\mathbb{N}_0}$ that satisfies $\rho_{S,T} = \rho$.*

6.4. Examples of dynamic risk measures

In this section we present examples of dynamic convex and coherent risk measures. To this end, we have to introduce two additional spaces and the pasting of probability measures or the pasting of the corresponding densities, respectively.

Definition 6.4.1. *For $T \in \mathbb{N}_0 \cup \{+\infty\}$, we define the sets*

$$\tilde{\mathcal{D}}_T := \{\varsigma \in L^1_T | \varsigma \geq 0, \mathbb{E}[\varsigma] = 1\} \quad \text{and} \quad \tilde{\mathcal{D}}^{rel}_T := \{\varsigma \in L^1_T | \varsigma > 0, \mathbb{E}[\varsigma] = 1\}$$

where $L^1_T := L^1(\Omega, \mathcal{F}_T, \mathbb{P})$. If $T = \infty$, we use $\mathcal{F}_\infty := \sigma(\bigcup_{t\in\mathbb{N}_0} \mathcal{F}_t)$. Moreover, for $\varsigma, \vartheta \in \tilde{\mathcal{D}}_T$, $s \in [0,T] \cap \mathbb{N}_0$ and $A \in \mathcal{F}_s$, let us define the pasting *$\varsigma \otimes^s_A \vartheta$ by*

$$\varsigma \otimes^s_A \vartheta := \begin{cases} \varsigma & \text{on } A^c \cup \{\mathbb{E}[\vartheta|\mathcal{F}_s] = 0\} \\ \mathbb{E}[\varsigma|\mathcal{F}_s] \frac{\vartheta}{\mathbb{E}[\vartheta|\mathcal{F}_s]} & \text{on } A \cap \{\mathbb{E}[\vartheta|\mathcal{F}_s] > 0\} \end{cases}.$$

A subset $\mathcal{P}$ of $\tilde{\mathcal{D}}_T$ is called m1-stable *if $\varsigma \otimes^s_A \vartheta \in \mathcal{P}$ for all $\varsigma, \vartheta \in \mathcal{P}$, $s \in [0,T] \cap \mathbb{N}_0$ and $A \in \mathcal{F}_s$.*

Remark 6.4.2. Suppose that $T \in \mathbb{N}$. Then we can identify $\varsigma \in \tilde{\mathcal{D}}_T$ with $\xi^\varsigma \in \mathcal{A}^1_{0,T}$ defined by

$$\xi^\varsigma_t := \begin{cases} 0 & \text{for } t \in [0, T-1] \cap \mathbb{N}_0 \\ \varsigma & \text{for } t \in [T, \infty) \cap \mathbb{N}_0 \end{cases}.$$

Thus, we have $\tilde{\mathcal{D}}_T \subset \mathcal{D}_{0,T} = \{\xi \in (\mathcal{A}^1_{0,T})_+ | \langle 1, \xi \rangle^{0,T} = 1\}$ and $\tilde{\mathcal{D}}^{rel}_T \subset \mathcal{D}^{rel}_{0,T} = \{\xi \in \mathcal{D}_{0,T} | \mathbb{P}[\xi_T - \xi_{T-1} > 0] = 1\}$ in case of $T \in \mathbb{N}$.

The following three examples are considered in detail in Section 5 in Cheridito et al. (2006). In order to verify the assertions from the following examples, one needs additional results from Sections 3 and 4 in Cheridito et al. (2006). Repeating these results would go beyond the scope of this introductory chapter. Nevertheless, the following examples give a good impression of the structure of time-consistent dynamic convex and coherent risk measures.

Example 6.4.3 (Dynamic coherent risk measures defined by worst stopping)**.** Consider $T \in \mathbb{N}_0 \cup \{+\infty\}$ and a set $\mathcal{P}^{rel}$ that satisfies $\emptyset \neq \mathcal{P}^{rel} \subset \tilde{\mathcal{D}}_T^{rel}$ and define for all $t \in [0,T] \cap \mathbb{N}_0$

$$\Xi_t(\gamma) := \operatorname*{ess\,sup}_{\varsigma \in \mathcal{P}^{rel}} \frac{\mathbb{E}[-\varsigma\gamma|\mathcal{F}_t]}{\mathbb{E}[\varsigma|\mathcal{F}_t]} \quad \text{for } \gamma \in L_T^\infty. \tag{6.4}$$

If $\mathcal{P}^{rel}$ is m1-stable and if we set

$$\rho_{t,T}^{\mathrm{WS}}(X) := \operatorname{ess\,sup}\{\Xi_t(X_\theta) | \theta \text{ is a finite stopping time with } t \leq \theta \leq T\}$$

for all $t \in [0,T] \cap \mathbb{N}_0$ and $X \in \mathcal{R}_{t,T}^\infty$, then $(\rho_{t,T}^{\mathrm{WS}})_{t \in [0,T] \cap \mathbb{N}_0}$ is a time-consistent dynamic coherent risk measure and every $\rho_{t,T}^{\mathrm{WS}}$ is T-relevant.

Example 6.4.4 (Dynamic coherent risk measures that depend on an average over time)**.** Define for $T \in \mathbb{N}_0 \cup \{+\infty\}$, $\emptyset \neq \mathcal{P}^{rel} \subset \tilde{\mathcal{D}}_T^{rel}$ and $t \in [0,T] \cap \mathbb{N}_0$

$$\rho_{t,T}^{\mathrm{Av}}(X) := \Xi_t \left(\frac{\sum_{s \in [t,T] \cap \mathbb{N}_0} r_s X_s}{\sum_{s \in [t,T] \cap \mathbb{N}_0} r_s} \right) \quad \text{for } X \in \mathcal{R}_{t,T}^\infty$$

where Ξ_t is defined by (6.4) and $(r_s)_{s \in [0,T] \cap \mathbb{N}_0}$ is a sequence of nonnegative real numbers that satisfy

$$\sum_{s \in [0,T] \cap \mathbb{N}_0} r_s = 1 \quad \text{and} \quad \sum_{s \in [t,T] \cap \mathbb{N}_0} r_s > 0 \quad \text{for all } t \in [0,T] \cap \mathbb{N}_0.$$

Then $(\rho_{t,T}^{\mathrm{Av}})_{t \in [0,T] \cap \mathbb{N}_0}$ defines a dynamic coherent risk measure and every $\rho_{t,T}^{\mathrm{Av}}$ is T-relevant. If additionally $\mathcal{P}^{rel}$ is m1-stable, then $(\rho_{t,T}^{\mathrm{Av}})_{t \in [0,T] \cap \mathbb{N}_0}$ is time-consistent.

From now on, we focus on dynamic convex risk measures.

Example 6.4.5 (Dynamic robust entropic risk measure)**.** Consider $T \in \mathbb{N}_0$, a set $\mathcal{P}^{rel}$ that satisfies $\emptyset \neq \mathcal{P}^{rel} \subset \tilde{\mathcal{D}}_T^{rel}$ and $a > 0$. For each $t \in [0,T] \cap \mathbb{N}_0$, we define $\rho_{t,T}^{\mathrm{entr}}$ by

$$\rho_{t,T}^{\mathrm{entr}}(X) := \operatorname*{ess\,sup}_{\varsigma \in \mathcal{P}^{rel}} \left\{ \frac{1}{a} \log \frac{\mathbb{E}[\varsigma \exp(-aX_T)|\mathcal{F}_t]}{\mathbb{E}[\varsigma|\mathcal{F}_t]} \right\} \quad \text{for } X \in \mathcal{R}_{t,T}^\infty.$$

Cheridito et al. (2006) prove that for every $t \in [0,T] \cap \mathbb{N}_0$, $\rho_{t,T}^{\mathrm{entr}}$ is a T-relevant conditional convex risk measure on $\mathcal{R}_{t,T}^\infty$. Moreover, each $\rho_{t,T}^{\mathrm{entr}}$ is continuous for bounded decreasing sequences, and if $\mathcal{P}^{rel}$ is m1-stable, then $(\rho_{t,T}^{\mathrm{entr}})_{t \in [0,T] \cap \mathbb{N}_0}$ is time-consistent.

In addition, for every $t \in [0,T] \cap \mathbb{N}_0$, the conditional convex risk measure $\rho_{t,T}^{\mathrm{entr}}$ satisfies $\rho_{t,T}^{\mathrm{entr}}(X) = \tilde{\rho}_{t,T}^{\mathrm{entr}}(X_T)$ for $X \in \mathcal{R}_{t,T}^\infty$ where $\tilde{\rho}_{t,T}^{\mathrm{entr}} : L_T^\infty \to L_t^\infty$ admits the representation

$$\tilde{\rho}_{t,T}^{\mathrm{entr}}(\gamma) = \operatorname*{ess\,sup}_{\varsigma \in \mathcal{P}^{rel}, \vartheta \in \tilde{\mathcal{D}}_T^{rel}} \left\{ \mathbb{E}[-\vartheta\gamma|\mathcal{F}_t] - \frac{1}{a} H_t(\vartheta|\varsigma) \right\} \quad \text{for } \gamma \in L_T^\infty.$$

Here, $H_t(\vartheta|\varsigma)$ denotes the conditional relative entropy defined by

$$H_t(\vartheta|\varsigma) := \mathbb{E}\left[\vartheta \log \frac{\vartheta}{\varsigma} \middle| \mathcal{F}_t \right] \quad \text{for } \varsigma, \vartheta \in \tilde{\mathcal{D}}_T^{rel}.$$

The last example in this section shows how we can generate time-consistent dynamic convex risk measures by using so called generators. For proofs and additional information we refer to Cheridito and Kupper (2011).

Example 6.4.6 (Generating time-consistent dynamic convex risk measures)**.** Fix $T \in \mathbb{N}$ and consider for $t \in [0, T-1] \cap \mathbb{N}_0$ mappings $H_t : L^\infty_{t+1} \to L^\infty_t$ that satisfy the following properties:

(H1) $H_t(0) = 0$.

(H2) If $X_{t+1} \geq Y_{t+1}$, then $H_t(X_{t+1}) \leq H_t(Y_{t+1})$ for all $X_{t+1}, Y_{t+1} \in L^\infty_{t+1}$.

(H3) $H_t(X_{t+1} + a) \geq H_t(X_{t+1}) - a$ for all $X_{t+1} \in L^\infty_{t+1}$ and $a \in (L^\infty_t)_+$.

Then the family $(H_t)_{t\in[0,T-1]\cap\mathbb{N}_0}$ generates a time-consistent dynamic risk measure $(\rho_{t,T})_{t\in[0,T]\cap\mathbb{N}_0}$ by

$$\rho_{t,T}(X) := \begin{cases} -X_T & \text{if } t = T \\ -X_t + H_t(-\rho_{t+1,T}(X) - X_t) & \text{if } t < T \end{cases} \quad \text{for } X \in \mathcal{R}^\infty_{t,T}.$$

Moreover, $(\rho_{t,T})_{t\in[0,T]\cap\mathbb{N}_0}$ is convex if each H_t satisfies

(H4) $H_t(aX_{t+1} + (1-a)\,Y_{t+1}) \leq aH_t(X_{t+1}) + (1-a)\,H_t(Y_{t+1})$ for all $X_{t+1}, Y_{t+1} \in L^\infty_{t+1}$ and $a \in L^\infty_t$ with $0 \leq a \leq 1$.

We call mappings $H_t : L^\infty_{t+1} \to L^\infty_t$ that satisfy (H1)-(H3) *generators* of the time-consistent dynamic risk measure $(\rho_{t,T})_{t\in[0,T]\cap\mathbb{N}_0}$. A specific kind of these generators are *composed generators* which are defined in the following way:

For $t \in [0, T-1] \cap \mathbb{N}_0$, consider a map $F_t : L^\infty_{t+1} \to L^\infty_t$ that satisfies the following properties:

(F1) $F_t(0) = 0$.

(F2) If $X_{t+1} \geq Y_{t+1}$, then $F_t(X_{t+1}) \leq F_t(Y_{t+1})$ for all $X_{t+1}, Y_{t+1} \in L^\infty_{t+1}$.

(F3) $F_t(X_{t+1} + a) = F_t(X_{t+1}) - a$ for all $X_{t+1} \in L^\infty_{t+1}$ and $a \in L^\infty_t$.

Let $h_t : \mathbb{R} \to \mathbb{R}$, $t \in [0, T-1] \cap \mathbb{N}_0$, be such that

(h1) $h_t(0) = 0$,

(h2) h_t is monotonically decreasing,

(h3) $|h_t(r_1) - h_t(r_2)| \leq |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$.

Then $H_t := h_t \circ (-F_t) : L^\infty_{t+1} \to L^\infty_t$ satisfies the properties (H1)-(H3). In particular, $(h_t, F_t)_{t\in[0,T-1]\cap\mathbb{N}_0}$ induces a time-consistent dynamic risk measure. Moreover, if additionally all F_t satisfy

(F4) $F_t(aX_{t+1} + (1-a)\,Y_{t+1}) \leq aF_t(X_{t+1}) + (1-a)\,F_t(Y_{t+1})$ for all $X_{t+1}, Y_{t+1} \in L^\infty_{t+1}$ and $a \in L^\infty_t$ with $0 \leq a \leq 1$

and all h_t are convex, then H_t satisfies (H4), which means that we obtain a time-consistent dynamic convex risk measure.

7. Conditional and dynamic systemic risk measures

In this chapter we generalize the model from Chapter 5 by using the setting of Cheridito et al. (2006) presented in Chapter 6 and study conditional and dynamic systemic risk measures on multi-dimensional bounded discrete-time processes. We refer the reader to the paper "Dynamic systemic risk measures for bounded discrete-time processes" which is joint work with E. Kromer and L. Overbeck and provides a summary of the results in this chapter.

After introducing general notations, we define in Section 7.1 conditional convex and positively homogeneous systemic risk measures. In Section 7.2 we prove a decomposition result for conditional convex systemic risk measures which is similar to the corresponding result in the static case. In conclusion, we can represent every conditional convex systemic risk measure as a composition of a conditional convex single-firm risk measure and a convex aggregation function. This decomposition is the basis for the subsequent study. In Section 7.3 we discuss different representation results for conditional convex systemic risk measures. Section 7.3.1 provides the primal representation result, which is proved analogously to the corresponding result in Chapter 5. In the following subsection we apply techniques from Cheridito et al. (2006) and introduce and study continuity properties of the underlying conditional convex single-firm risk measure and the underlying convex aggregation function. Based on these results, we finally prove a dual representation result for conditional convex systemic risk measures in Subsection 7.3.3.

In Section 7.4 we go one step further and introduce dynamic systemic risk measures as families of conditional systemic risk measures. Furthermore, we use the concept of strong time-consistency and study time-consistent systemic risk measures. In this context, we introduce in Subsection 7.4.1 a time-consistency property for our convex aggregation function and examine how this property depends on the time-consistency property of the corresponding systemic risk measure. In Subsection 7.4.2 we consider examples of time-consistent dynamic aggregation functions and finally we discuss our results concerning time-consistent dynamic systemic risk measures.

7.1. Notation and definitions

In this chapter we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{N}_0}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. As in Chapter 5, we consider a finite set of n firms or, in other words, n nodes in a financial network. The n-dimensional discrete-time process $\bar{X} = (\bar{X}^1, \ldots, \bar{X}^n)$ represents the losses of these n firms over time. Here, $\bar{X}^i$ is supposed to be an element in the space $\mathcal{R}^\infty$ defined in Chapter 6 and describes the loss process of firm $i \in \{1, \dots, n\}$.

From now on, let $m \in \mathbb{N}$ be arbitrary. In this chapter we need the following generalizations of the spaces introduced in Chapter 6:

Let $\mathcal{R}^{0,m}$ be the space of $\mathbb{F}$-adapted m-dimensional processes $\bar{X} = (\bar{X}_t)_{t\in\mathbb{N}_0}$ such that for each $i \in \{1, \dots, m\}$, $\bar{X}^i = (\bar{X}^i_t)_{t\in\mathbb{N}_0}$ is an element in $\mathcal{R}^0$. Again, we understand equalities and inequalities between random variables and (multi-dimensional) stochastic processes $\mathbb{P}$-a.s. This means m-dimensional processes $\bar{X} = (\bar{X}_t)_{t\in\mathbb{N}_0} \in \mathcal{R}^{0,m}$ and $\bar{Y} = (\bar{Y}_t)_{t\in\mathbb{N}_0} \in \mathcal{R}^{0,m}$ satisfy $\bar{X} \leq \bar{Y}$ if and only if $\bar{X}^i_t \leq \bar{Y}^i_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$ and all $i \in \{1, \dots, m\}$. Furthermore, define

$$\mathcal{R}^{\infty,m} := \{\bar{X} \in \mathcal{R}^{0,m} | \|\bar{X}\|_{\mathcal{R}^{\infty,m}} < \infty\} \quad \text{and} \quad \mathcal{A}^{1,m} := \{\bar{\xi} \in \mathcal{R}^{0,m} | \|\bar{\xi}\|_{\mathcal{A}^{1,m}} < \infty\}$$

with

$$\|\bar{X}\|_{\mathcal{R}^{\infty,m}} := \max_{i\in\{1,\dots,m\}} \|\bar{X}^i\|_{\mathcal{R}^\infty} \quad \text{and} \quad \|\bar{\xi}\|_{\mathcal{A}^{1,m}} := \sum_{i=1}^m \|\bar{\xi}^i\|_{\mathcal{A}^1}$$

and note that $\mathcal{R}^{\infty,1}$ and $\mathcal{A}^{1,1}$ are equal to $\mathcal{R}^\infty$ and $\mathcal{A}^1$, respectively.

By definition of the space $\mathcal{R}^{\infty,m}$, each $\bar{X} \in \mathcal{R}^{\infty,m}$ satisfies $\bar{X}^i \in \mathcal{R}^\infty$ for all $i \in \{1, \dots, m\}$. Thus, $\bar{X}_t = (\bar{X}^1_t, \dots, \bar{X}^m_t)$ is an element in $(L^\infty)^m := (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^m$ for each $t \in \mathbb{N}_0$.

We define the bilinear form $\langle \cdot, \cdot \rangle_m : \mathcal{R}^{\infty,m} \times \mathcal{A}^{1,m} \to \mathbb{R}$ by

$$\langle \bar{X}, \bar{\xi} \rangle_m := \sum_{i=1}^m \langle \bar{X}^i, \bar{\xi}^i \rangle = \sum_{i=1}^m \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} \bar{X}^i_t \Delta \bar{\xi}^i_t\right].$$

The following remark is based on statements in the proof of Lemma 3.17 in Cheridito et al. (2006).

Remark 7.1.1 (Connection between processes in $\mathcal{R}^\infty$ and random variables)**.**

1. Note that any stochastic process $X = (X_t)_{t\in\mathbb{N}_0} \in \mathcal{R}^\infty$ is a map $X : \mathbb{N}_0 \times \Omega \to \mathbb{R}$ with $X(t,\omega) = X_t(\omega)$. Let $\mathcal{H}$ be the σ-algebra on $\mathbb{N}_0 \times \Omega$ generated by the sets $\{t\} \times B$, $t \in \mathbb{N}_0$, $B \in \mathcal{F}_t$ and define the measure η on $(\mathbb{N}_0 \times \Omega, \mathcal{H})$ by

$$\eta(\{t\} \times B) := 2^{-(t+1)} \mathbb{P}[B] \quad \text{for } t \in \mathbb{N}_0 \text{ and } B \in \mathcal{F}_t.$$

Then $\mathcal{R}^\infty = L^\infty_{\mathcal{H}} := L^\infty(\mathbb{N}_0 \times \Omega, \mathcal{H}, \eta)$ since for every $X \in \mathcal{R}^\infty$, we have

$$\begin{aligned}
\|X\|_{L^\infty_{\mathcal{H}}} &= \inf\{r \in \mathbb{R} | |X_t(\omega)| \leq r \text{ for } \eta\text{-a.e. } (t,\omega) \in \mathbb{N}_0 \times \Omega\} \\
&= \inf\{r \in \mathbb{R} | |X_t(\omega)| \leq r \text{ for all } t \in \mathbb{N}_0 \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega\} \\
&= \inf\left\{r \in \mathbb{R} \middle| \sup_{t\in\mathbb{N}_0} |X_t(\omega)| \leq r \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega\right\} \\
&= \|X\|_{\mathcal{R}^\infty}.
\end{aligned}$$

Moreover, $\mathcal{A}^1$ can be identified with $L^1_{\mathcal{H}} := L^1(\mathbb{N}_0 \times \Omega, \mathcal{H}, \eta)$ by identifying each $\Xi \in L^1_{\mathcal{H}}$ with $\xi^\Xi \in \mathcal{A}^1$ where $\xi^\Xi_{-1} := 0$ and $\xi^\Xi_t(\omega) := 2^{-(t+1)} \Xi(t,\omega) + \xi^\Xi_{t-1}(\omega)$

for $t \in \mathbb{N}_0$ and $\omega \in \Omega$. In this case, we obtain

$$\begin{aligned}\|\Xi\|_{L^1_{\mathcal{H}}} &= \int_{(t,\omega)\in\mathbb{N}_0\times\Omega} |\Xi(t,\omega)| d\eta(t,\omega) = \sum_{t\in\mathbb{N}_0} \int_{\omega\in\Omega} |\Xi(t,\omega)| 2^{-(t+1)} d\mathbb{P}(\omega) \\ &= \sum_{t\in\mathbb{N}_0} \int_{\omega\in\Omega} |\Delta\xi^{\Xi}_t(\omega) 2^{(t+1)}| 2^{-(t+1)} d\mathbb{P}(\omega) = \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} |\Delta\xi^{\Xi}_t|\right] = \|\xi^{\Xi}\|_{\mathcal{A}^1}.\end{aligned}$$

Similarly, the pairing $\langle\cdot,\cdot\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}} : L^\infty_{\mathcal{H}} \times L^1_{\mathcal{H}} \to \mathbb{R}$ given by $\langle X, \Xi\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}} := \int X\Xi d\eta$ satisfies

$$\langle X, \Xi\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}} = \int X\Xi d\eta = \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} X_t \Delta\xi^{\Xi}_t\right] = \langle X, \xi^{\Xi}\rangle.$$

2. By repeating the same arguments, $\mathcal{R}^{\infty,m} = (L^\infty_{\mathcal{H}})^m$ and $\mathcal{A}^{1,m}$ can be identified with $(L^1_{\mathcal{H}})^m$ where $(L^\infty_{\mathcal{H}})^m$ and $(L^1_{\mathcal{H}})^m$ are equipped with the norms

$$\|\bar{X}\|_{(L^\infty_{\mathcal{H}})^m} := \max_{i\in\{1,\dots,m\}} \|\bar{X}^i\|_{L^\infty_{\mathcal{H}}} \quad \text{and} \quad \|\bar{\Xi}\|_{(L^1_{\mathcal{H}})^m} := \sum_{i=1}^m \|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}}$$

for $\bar{X} \in (L^\infty_{\mathcal{H}})^m$ and $\bar{\Xi} \in (L^1_{\mathcal{H}})^m$. Furthermore, the pairing $\langle\cdot,\cdot\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m} : (L^\infty_{\mathcal{H}})^m \times (L^1_{\mathcal{H}})^m \to \mathbb{R}$ defined by $\langle \bar{X}, \bar{\Xi}\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m} := \sum_{i=1}^m \int \bar{X}^i \bar{\Xi}^i d\eta$ satisfies $\langle \bar{X}, \bar{\Xi}\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m} = \langle \bar{X}, \xi^{\bar{\Xi}}\rangle_m$ for $\bar{X} \in (L^\infty_{\mathcal{H}})^m$, $\bar{\Xi} \in (L^1_{\mathcal{H}})^m$ and $\xi^{\bar{\Xi}} \in \mathcal{A}^{1,m}$ defined by $(\xi^{\bar{\Xi}})^i_{-1} := 0$ and $(\xi^{\bar{\Xi}})^i_t(\omega) := 2^{-(t+1)}\bar{\Xi}^i(t,\omega) + (\xi^{\bar{\Xi}})^i_{t-1}(\omega)$ for $t \in \mathbb{N}_0$, $\omega \in \Omega$ and $i \in \{1,\dots,n\}$.
3. By these definitions, the norm on $(L^\infty_{\mathcal{H}})^m$ satisfies

$$\|\bar{X}\|_{(L^\infty_{\mathcal{H}})^m} = \sup_{\|\bar{\Xi}\|_{(L^1_{\mathcal{H}})^m}\le 1} \left|\langle \bar{X}, \bar{\Xi}\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m}\right| \quad \text{for all } \bar{X} \in (L^\infty_{\mathcal{H}})^m.$$

This can be verified as follows: It is easily seen that

$$\begin{aligned}&\sup_{\|\bar{\Xi}\|_{(L^1_{\mathcal{H}})^m}\le 1} \left|\langle \bar{X}, \bar{\Xi}\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m}\right| = \sup_{\sum_{i=1}^m \|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}}\le 1} \left|\sum_{i=1}^m \langle \bar{X}^i, \bar{\Xi}^i\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}}\right| \\ &\ge \max_{i\in\{1,\dots,m\}} \sup_{\|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}}\le 1} \left|\langle \bar{X}^i, \bar{\Xi}^i\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}}\right| = \max_{i\in\{1,\dots,m\}} \|\bar{X}^i\|_{L^\infty_{\mathcal{H}}}.\end{aligned}$$

On the other hand, for every $i \in \{1,\dots,m\}$ and $\bar{\Xi} \in (L^1_{\mathcal{H}})^m$, we have

$$\left|\langle \bar{X}^i, \bar{\Xi}^i\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}}\right| / \|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}} = \left|\langle \bar{X}^i, \bar{\Xi}^i/\|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}}\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}}\right| \le \|\bar{X}^i\|_{L^\infty_{\mathcal{H}}}$$

due to $\|\bar{X}^i\|_{L^\infty_{\mathcal{H}}} = \sup_{\|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}}\le 1} |\langle \bar{X}^i, \bar{\Xi}^i\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}}|$. But this leads to

$$\begin{aligned}\sup_{\sum_{i=1}^m \|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}}\le 1} \left|\sum_{i=1}^m \langle \bar{X}^i, \bar{\Xi}^i\rangle_{L^\infty_{\mathcal{H}},L^1_{\mathcal{H}}}\right| &\le \sup_{\sum_{i=1}^m \|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}}\le 1} \sum_{i=1}^m \|\bar{X}^i\|_{L^\infty_{\mathcal{H}}} \|\bar{\Xi}^i\|_{L^1_{\mathcal{H}}} \\ &\le \max_{i\in\{1,\dots,m\}} \|\bar{X}^i\|_{L^\infty_{\mathcal{H}}}.\end{aligned}$$

4. Note that $(L^1_{\mathcal{H}})^m$ separates points of $(L^\infty_{\mathcal{H}})^m$ and $(L^\infty_{\mathcal{H}})^m$ separates points of $(L^1_{\mathcal{H}})^m$ under $\langle\cdot,\cdot\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m}$, which means that

$$\bar{X} \in (L^\infty_{\mathcal{H}})^m \text{ and } \langle \bar{X}, \bar{\Xi}\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m} = 0 \text{ for all } \bar{\Xi} \in (L^1_{\mathcal{H}})^m \quad \Rightarrow \quad \bar{X} = 0,$$
$$\bar{\Xi} \in (L^1_{\mathcal{H}})^m \text{ and } \langle \bar{X}, \bar{\Xi}\rangle_{(L^\infty_{\mathcal{H}})^m,(L^1_{\mathcal{H}})^m} = 0 \text{ for all } \bar{X} \in (L^\infty_{\mathcal{H}})^m \quad \Rightarrow \quad \bar{\Xi} = 0.$$

Because of the previous remark, $\mathcal{A}^{1,m}$ separates points of $\mathcal{R}^{\infty,m}$ and $\mathcal{R}^{\infty,m}$ separates points of $\mathcal{A}^{1,m}$ under $\langle\cdot,\cdot\rangle_m$. Hence, we can define $\sigma(\mathcal{R}^{\infty,m},\mathcal{A}^{1,m})$ as the weakest topology on $\mathcal{R}^{\infty,m}$ such that for all $\bar{\xi} \in \mathcal{A}^{1,m}$, the functional $\bar{X} \mapsto \langle \bar{X}, \bar{\xi}\rangle_m$ on $\mathcal{R}^{\infty,m}$ is continuous and linear. Analogously, the topology $\sigma(\mathcal{A}^{1,m},\mathcal{R}^{\infty,m})$ denotes the weakest topology on $\mathcal{A}^{1,m}$ such that for all $\bar{X} \in \mathcal{R}^{\infty,m}$, the functional $\bar{\xi} \mapsto \langle \bar{X}, \bar{\xi}\rangle_m$ on $\mathcal{A}^{1,m}$ is continuous and linear.

From now on, let τ be a finite ($\mathbb{F}$-)stopping time and θ be an ($\mathbb{F}$-)stopping time such that $0 \leq \tau \leq \theta \leq \infty$. Then we define the projection $p^{\tau,\theta}_m : \mathcal{R}^{0,m} \to \mathcal{R}^{0,m}$ by

$$p^{\tau,\theta}_m(\bar{X})_t := I_{\{\tau \leq t\}}\bar{X}_{t\wedge\theta} \quad \text{for } t \in \mathbb{N}_0.$$

The spaces $\mathcal{R}^{\infty,m}_{\tau,\theta} \subset \mathcal{R}^{\infty,m}$ and $\mathcal{A}^{1,m}_{\tau,\theta} \subset \mathcal{A}^{1,m}$ are defined by

$$\mathcal{R}^{\infty,m}_{\tau,\theta} := p^{\tau,\theta}_m\mathcal{R}^{\infty,m} \quad \text{and} \quad \mathcal{A}^{1,m}_{\tau,\theta} := p^{\tau,\theta}_m\mathcal{A}^{1,m},$$

and the set $(\mathcal{R}^\infty_{\tau,\theta})_+ \subset \mathcal{R}^\infty_{\tau,\theta}$ is given by

$$(\mathcal{R}^\infty_{\tau,\theta})_+ := \{X \in \mathcal{R}^\infty_{\tau,\theta} | X \geq 0\}.$$

In this chapter we want to define conditional convex and positively homogeneous systemic risk measures on $\mathcal{R}^{\infty,n}_{\tau,\theta}$, i.e., we focus on stochastic processes $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$, which describe the evolution of losses in the interval $[\tau,\theta] \cap \mathbb{N}_0$. In analogy to the static case discussed in Chapter 5, we change the perspective from value processes, considered in the standard approach in Chapter 6, to loss processes.

Before we introduce the defining properties of conditional convex and positively homogeneous systemic and single-firm risk measures, note that for $\bar{\gamma} \in (L^\infty_\tau)^m$ and $\bar{X} \in \mathcal{R}^{\infty,m}$, we understand $\bar{Y} := \bar{\gamma}I_{[\tau,\theta)}$ and $\bar{Z} := \bar{X}I_{[\tau,\theta)}$ in the following way:

$$\bar{Y}^i_t(\omega) = \bar{\gamma}^i(\omega)I_{[\tau,\theta)}(t,\omega) \quad \text{and}$$
$$\bar{Z}^i_t(\omega) = \bar{X}^i_t(\omega)I_{[\tau,\theta)}(t,\omega) \quad \text{for } t \in \mathbb{N}_0, \omega \in \Omega, i \in \{1,\ldots,n\}.$$

Let us consider the following properties of a mapping $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$:

(r1) Monotonicity: If $X \geq Y$, then $\rho_0(X) \geq \rho_0(Y)$ for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$.

(r2) $\mathcal{F}_\tau$-convexity: $\rho_0(\gamma X + (1-\gamma)Y) \leq \gamma\rho_0(X) + (1-\gamma)\rho_0(Y)$ for all $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ and $\gamma \in L^\infty_\tau$ with $0 \leq \gamma \leq 1$.

(r3) $\mathcal{F}_\tau$-translation property: $\rho_0(X + \gamma I_{[\tau,\infty)}) = \rho_0(X) + \gamma$ for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $\gamma \in L^\infty_\tau$.

(r4) $\mathcal{F}_\tau$-positive homogeneity: $\rho_0(\gamma X) = \gamma\rho_0(X)$ for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ and $\gamma \in (L^\infty_\tau)_+$.

(r5) Constancy on $\mathfrak{R} \subset L_\tau^\infty$: $\rho_0(\gamma I_{[\tau,\infty)}) = \gamma$ for all $\gamma \in \mathfrak{R}$.

(r6) Normalization: $\rho_0(I_{[\tau,\infty)}) = 1$.

Most of these properties, i.e., (r1)-(r4), have already been introduced in Chapter 6. The constancy property (r5) is known from Chapter 5, and a special case of constancy on $\mathfrak{R} \subset L_\tau^\infty$ is constancy on $\{1\}$, which corresponds with property (r6).

Definition 7.1.2. *A* conditional convex single-firm risk measure (on $\mathcal{R}_{\tau,\theta}^\infty$) *is a mapping $\rho_0 : \mathcal{R}_{\tau,\theta}^\infty \to L_\tau^\infty$ that satisfies the properties (r1) and (r2). A* conditional positively homogeneous single-firm risk measure (on $\mathcal{R}_{\tau,\theta}^\infty$) *is a conditional convex single-firm risk measure (on $\mathcal{R}_{\tau,\theta}^\infty$) that additionally satisfies the properties (r4) and (r6). A* conditional coherent single-firm risk measure (on $\mathcal{R}_{\tau,\theta}^\infty$) *is a conditional positively homogeneous single-firm risk measure (on $\mathcal{R}_{\tau,\theta}^\infty$) that additionally satisfies the property (r3). For $X \in \mathcal{R}^\infty$, we set $\rho_0(X) := \rho_0(p^{\tau,\theta}(X))$.*

In general, we do not postulate $\rho_0(0) = 0$ (this is the normalization property (n) from Chapter 6) for conditional convex single-firm risk measures. Nevertheless, in Section 7.2 we will see that conditional convex single-firm risk measures ρ_0 that are a part of the decomposition of a conditional convex systemic risk measure (see below) indeed satisfy this additional property.

Moreover, if $\rho_0(0) = 0$ holds, then constancy on L_τ^∞ is equivalent to the translation property for $X = 0$ since

$$\rho_0(\gamma I_{[\tau,\infty)}) = \rho_0(0 + \gamma I_{[\tau,\infty)}) = \rho_0(0) + \gamma = \gamma \quad \text{for } \gamma \in L_\tau^\infty.$$

Now, consider the following properties of a mapping $\rho : \mathcal{R}_{\tau,\theta}^{\infty,n} \to L_\tau^\infty$:

(s1) Monotonicity: If $\bar{X} \geq \bar{Y}$, then $\rho(\bar{X}) \geq \rho(\bar{Y})$ for all $\bar{X}, \bar{Y} \in \mathcal{R}_{\tau,\theta}^{\infty,n}$.

(s2) Preference consistency: If $\rho(\bar{X}_t(\omega) I_{[\tau,\infty)}) \geq \rho(\bar{Y}_t(\omega) I_{[\tau,\infty)})$ for $\bar{X}, \bar{Y} \in \mathcal{R}_{\tau,\theta}^{\infty,n}$, all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$, then $\rho(\bar{X}) \geq \rho(\bar{Y})$.

(s3) f_ρ-constancy: Either $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} = \mathbb{R}$ and there exists a surjective function $f_\rho : \mathbb{R} \to \mathbb{R}$ with $f_\rho(0) = 0$ such that $\rho(a 1_n I_{[\tau,\infty)}) = f_\rho(a)$ for all $a \in \mathbb{R}$ or $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} = \mathbb{R}_+$ and there exists a function $f_\rho : \mathbb{R} \to \mathbb{R}_+$ and $b \in \mathbb{R}_+$ such that f_ρ is surjective and strictly increasing on $[b, \infty)$, $f_\rho(a) = 0$ for $a \leq b$ and $\rho(a 1_n I_{[\tau,\infty)}) = f_\rho(a)$ for all $a \in \mathbb{R}$.

(s4) $\mathcal{F}_\tau$-convexity:

- **(s4a)** $\mathcal{F}_\tau$-outcome convexity: $\rho(\gamma\bar{X} + (1-\gamma)\bar{Y}) \leq \gamma\rho(\bar{X}) + (1-\gamma)\rho(\bar{Y})$ for all $\bar{X}, \bar{Y} \in \mathcal{R}_{\tau,\theta}^{\infty,n}$ and $\gamma \in L_\tau^\infty$ with $0 \leq \gamma \leq 1$.
- **(s4b)** $\mathcal{F}_\tau$-risk convexity: Suppose $\rho(\bar{Z}_t(\omega) I_{[\tau,\infty)}) = \gamma(\omega)\rho(\bar{X}_t(\omega) I_{[\tau,\infty)}) + (1-\gamma(\omega))\rho(\bar{Y}_t(\omega) I_{[\tau,\infty)})$ for $\bar{X}, \bar{Y}, \bar{Z} \in \mathcal{R}_{\tau,\theta}^{\infty,n}$, $\gamma \in L_\tau^\infty$ with $0 \leq \gamma \leq 1$ and for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. Then $\rho(\bar{Z}) \leq \gamma\rho(\bar{X}) + (1-\gamma)\rho(\bar{Y})$.

(s5) $\mathcal{F}_\tau$-positive homogeneity: $\rho(\gamma\bar{X}) = \gamma\rho(\bar{X})$ for all $\bar{X} \in \mathcal{R}_{\tau,\theta}^{\infty,n}$ and all $\gamma \in (L_\tau^\infty)_+$.

(s6) Normalization: $\rho(1_n I_{[\tau,\infty)}) = n$.

As in the static case, property (s2) means that if

$$\mathbb{P}[\{\omega \in \Omega | \rho(\bar{x}_t I_{[\tau,\infty)}) \geq \rho(\bar{y}_t I_{[\tau,\infty)}), (\bar{x}_t, \bar{y}_t) = (\bar{X}_t(\omega), \bar{Y}_t(\omega)) \text{ for all } t \in \mathbb{N}_0\}] = 1,$$

then $\rho(\bar{X}) \geq \rho(\bar{Y})$, and (s4b) means that

$$\mathbb{P}\left[\begin{array}{c}\{\omega \in \Omega | \rho(\bar{z}_t I_{[\tau,\infty)}) = \gamma\rho(\bar{x}_t I_{[\tau,\infty)}) + (1-\gamma)\rho(\bar{y}_t I_{[\tau,\infty)}),\\ (\bar{z}_t, \bar{x}_t, \bar{y}_t) = (\bar{Z}_t(\omega), \bar{X}_t(\omega), \bar{Y}_t(\omega)) \text{ for all } t \in \mathbb{N}_0\}\end{array}\right] = 1$$

implies $\rho(\bar{Z}) \leq \gamma\rho(\bar{X}) + (1-\gamma)\,\rho(\bar{Y})$. Furthermore, it is important to note that the properties (s1), (s3) and (s4a) guarantee the existence of an inverse function f_ρ^{-1} of f_ρ.

Definition 7.1.3. *A* conditional positively homogeneous systemic risk measure (on $\mathcal{R}^{\infty,n}_{\tau,\theta}$) *is a mapping* $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_\tau$ *that satisfies the properties (s1), (s2), (s4), (s5), (s6) and* $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} \subset \mathbb{R}$. *If* ρ *satisfies the properties (s1)-(s4), we call* ρ *a* conditional convex systemic risk measure (on $\mathcal{R}^{\infty,n}_{\tau,\theta}$). *For* $\bar{X} \in \mathcal{R}^{\infty,n}$, *we set* $\rho(\bar{X}) := \rho(p_n^{\tau,\theta}(\bar{X}))$.

Remark 7.1.4. Every conditional positively homogeneous systemic risk measure ρ satisfies the f_ρ-constancy property (s3). The corresponding function f_ρ is given by

$$f_\rho(a) = \begin{cases} an & \text{if } a \geq 0 \\ a(-\rho(-1_n I_{[\tau,\infty)})) & \text{if } a < 0 \end{cases} \tag{7.1}$$

in case of $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} = \mathbb{R}$ and $f_\rho(a) = na^+$ in case of $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} = \mathbb{R}_+$. This can be verified analogously to Lemma 5.1.6. As a consequence, every conditional positively homogeneous systemic risk measure is also a convex systemic risk measure.

It remains to consider aggregation functions. In Chapter 5 we have introduced in Definition 5.1.3 convex and positively homogeneous aggregation functions as functions from $\mathbb{R}^n$ to $\mathbb{R}$. In the conditional case, we can use these aggregation functions as well. To be more precisely, because of Remark 7.1.1 and $\mathcal{R}^{\infty,n}_{\tau,\theta} \subset \mathcal{R}^{\infty,n}$, we use the aggregation functions from Chapter 5 for $p = \infty$.

Finally, note that in contrast to the definition of property (S3) in Chapter 5, we additionally assume $f_\rho(0) = 0$ in property (s3). The reason for this additional requirement is the dependence of the convex systemic risk measure ρ and the corresponding convex aggregation function Λ. This convex aggregation function has to guarantee that $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^{\infty}_{\tau,\theta}$. In particular, Λ has to map every $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ to a process $\Lambda(\bar{X})$ that is equal to 0 before time τ, i.e., $\Lambda(\bar{X}) = \Lambda(\bar{X}) I_{[\tau,\infty)}$. We will see below that Λ satisfies this property if we assume that $f_\Lambda(0) = 0$.

Remark 7.1.5. Since $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is convex, Λ is continuous and measurable. If we consider $\bar{X} \in \mathcal{R}^{\infty,m}$ as a function $\bar{X} : (\mathbb{N}_0 \times \Omega, \mathcal{H}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and Λ as a function $\Lambda : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then $\Lambda(\bar{X}) := \Lambda \circ \bar{X}$ is a measurable function

$$\Lambda(\bar{X}) : (\mathbb{N}_0 \times \Omega, \mathcal{H}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

such that $\Lambda(\bar{X}_t(\omega)) = \Lambda(\bar{X})_t(\omega)$ for all $t \in \mathbb{N}_0$ and all $\omega \in \Omega$.

In the remaining part of this section we will show that every convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ induces a mapping $\Lambda : \mathcal{R}^{\infty,n}_{\tau,\theta} \to \mathcal{R}^{\infty}_{\tau,\theta}$.

Lemma 7.1.6. *Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. Then we have $g(\mathcal{R}^\infty) \subset \mathcal{R}^\infty$. If additionally $g(0) = 0$, then $g(\mathcal{R}^\infty_{\tau,\theta}) \subset \mathcal{R}^\infty_{\tau,\theta}$.*

Proof. Note that g is measurable since each increasing function $g : \mathbb{R} \to \mathbb{R}$ is measurable. For each $X \in \mathcal{R}^\infty$, there exists $r_X \in \mathbb{R}_+$ such that $\sup_{t\in\mathbb{N}_0}|X_t| \leq r_X$. Monotonicity of g implies $g(-r_X) \leq g(X_t) \leq g(r_X)$ for all $t \in \mathbb{N}_0$. This means $\|g(X)\|_{\mathcal{R}^\infty} < \infty$ such that $g(X) \in \mathcal{R}^\infty$. $g(X) \in \mathcal{R}^\infty_{\tau,\theta}$ follows directly from $g(0) = 0$. □

The following lemma can easily be deduced from Lemma 5.1.5. Moreover, note that we need the additional assumption $f_\Lambda(0) = 0$ to guarantee that $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^\infty_{\tau,\theta}$.

Lemma 7.1.7. *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a convex aggregation function with $f_\Lambda(0) = 0$ and $\operatorname{Im}\Lambda = \mathbb{R}$ [$\operatorname{Im}\Lambda = \mathbb{R}_+$]. Then $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^\infty_{\tau,\theta}$ [$\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = (\mathcal{R}^\infty_{\tau,\theta})_+$].*

Proof. First, let $\operatorname{Im}\Lambda = \mathbb{R}$. Because of Remark 7.1.1, we can consider every $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ as an element in the space $(L^\infty_{\mathcal{H}})^n$. Lemma 5.1.5 yields $\Lambda((L^\infty_{\mathcal{H}})^n) \subset L^\infty_{\mathcal{H}}$ or, in other words, $\Lambda(\mathcal{R}^{\infty,n}) \subset \mathcal{R}^\infty$. Moreover, the additional assumption $f_\Lambda(0) = 0$ implies that $\Lambda(0_n) = f_\Lambda(0) = 0$. As a consequence, we obtain $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) \subset \mathcal{R}^\infty_{\tau,\theta}$.

On the other hand, fix an arbitrary process $X \in \mathcal{R}^\infty_{\tau,\theta}$. Since f_Λ is a bijective function, the inverse function f_Λ^{-1} exists and we can define a process Y^X by

$$Y^X_t(\omega) := f_\Lambda^{-1}(X_t(\omega)) \quad \text{for } t \in \mathbb{N}_0 \text{ and } \omega \in \Omega. \tag{7.2}$$

Because of Lemma 7.1.6, we know that $f_\Lambda^{-1}(Z) \in \mathcal{R}^\infty_{\tau,\theta}$ for all $Z \in \mathcal{R}^\infty_{\tau,\theta}$. It follows $Y^X \in \mathcal{R}^\infty_{\tau,\theta}$ and $f_\Lambda(Y^X) = X$. By using the f_Λ-constancy property of Λ, we obtain for all $X \in \mathcal{R}^\infty_{\tau,\theta}$ a process $Y^X 1_n \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ such that

$$\Lambda(Y^X 1_n) = f_\Lambda(Y^X) = X.$$

This yields $\mathcal{R}^\infty_{\tau,\theta} \subset \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta})$. Summing up, we arrive at $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^\infty_{\tau,\theta}$.

If $\operatorname{Im}\Lambda = \mathbb{R}_+$, we have $f_\Lambda^{-1} : \mathbb{R}_+ \to [b,\infty)$ and $f_\Lambda^{-1}(0) = b$. Hence, the process defined in (7.2) satisfies $Y^X \in \mathcal{R}^\infty_{0,\theta}$ for $X \in (\mathcal{R}^\infty_{\tau,\theta})_+$. Nevertheless, the process $(Y^X)'$ defined by $(Y^X)' := Y^X I_{[\tau,\infty)}$ satisfies $(Y^X)' \in \mathcal{R}^\infty_{\tau,\theta}$ and $f_\Lambda((Y^X)') = X$. □

7.2. Structural decomposition

In this section we extend the structural decomposition results from Chapter 5 to our conditional setting. Although the proof of the following theorem is similar to the proof of the static decomposition, we provide the proof in order to obtain a deeper understanding of the relationship and dependence between conditional systemic risk measures, conditional single-firm risk measures and aggregation functions.

Theorem 7.2.1 (Convex structural decomposition)**.**

a) A mapping $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_\tau$ with $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$ is a conditional convex systemic risk measure if and only if there exists a convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $f_\Lambda(0) = 0$ and $\Lambda(\mathbb{R}^n) = \mathbb{R}$ and a conditional convex single-firm risk measure $\rho_0 : \mathcal{R}^{\infty}_{\tau,\theta} \to L^\infty_\tau$ that satisfies the constancy property on $\mathbb{R}$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}.$$

b) A mapping $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_\tau$ with $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}_+$ is a conditional convex systemic risk measure if and only if there exists a convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}_+$ and a conditional convex single-firm risk measure $\rho_0 : \mathcal{R}^{\infty}_{\tau,\theta} \to L^\infty_\tau$ that satisfies the constancy property on $\mathbb{R}_+$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}.$$

Proof. In case of part a), set $\mathfrak{R} = \mathfrak{S} = \mathbb{R}$, and in case of part b), set $\mathfrak{R} = \mathbb{R}_+$ and $\mathfrak{S} = [b, \infty)$ for $b \in \mathbb{R}_+$. Let ρ be a conditional convex systemic risk measure with $f_\rho : \mathbb{R} \to \mathfrak{R}$ such that f_ρ is surjective and strictly increasing on $\mathfrak{S}$. We define the function Λ by

$$\Lambda(\bar{x}) := \rho(\bar{x} I_{[\tau,\infty)}) \quad \text{for } \bar{x} \in \mathbb{R}^n.$$

The monotonicity property (A1) and the convexity property (A2) follow directly from the corresponding property of ρ. Because ρ satisfies the f_ρ-constancy property, we have

$$\Lambda(a 1_n) = \rho(a 1_n I_{[\tau,\infty)}) = f_\rho\,(a) \quad \text{for all } a \in \mathbb{R}.$$

If we set $f_\Lambda := f_\rho$, then we obtain the f_Λ-constancy property (A3). Because $f_\rho(0) = 0$, we also obtain the required property of f_Λ. Moreover, Lemma 7.1.7 gives $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^{\infty}_{\tau,\theta}$ in case of a) and $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = (\mathcal{R}^{\infty}_{\tau,\theta})_+$ in case of b).

Now, define $\tilde{\rho}_0 : \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) \to L^\infty_\tau$ by

$$\tilde{\rho}_0\,(X) := \rho(\bar{X}) \quad \text{where } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta} \text{ satisfies } \Lambda(\bar{X}) = X. \tag{7.3}$$

Furthermore, define $\rho_0 : \mathcal{R}^{\infty}_{\tau,\theta} \to L^\infty_\tau$ by

$$\rho_0(X) := \begin{cases} \tilde{\rho}_0(X) & \text{if } \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^{\infty}_{\tau,\theta} \\ \tilde{\rho}_0(X^+) & \text{if } \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = (\mathcal{R}^{\infty}_{\tau,\theta})_+ \end{cases} \tag{7.4}$$

where the stochastic process $X^+ \in (\mathcal{R}^{\infty}_{\tau,\theta})_+$ is defined by $X^+_t(\omega) := \max\{X_t(\omega), 0\}$ for $t \in \mathbb{N}_0$ and $\omega \in \Omega$.

First, note that $\tilde{\rho}_0$ is well-defined: Let $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ be such that $\Lambda(\bar{X}) = \Lambda(\bar{Y})$. Then

$$\rho(\bar{X}_t(\omega) I_{[\tau,\infty)}) = \Lambda(\bar{X}_t(\omega)) = \Lambda(\bar{X})_t(\omega) = \Lambda(\bar{Y})_t(\omega) = \Lambda(\bar{Y}_t(\omega)) = \rho(\bar{Y}_t(\omega) I_{[\tau,\infty)})$$

for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$, and preference consistency of ρ yields $\rho(\bar{X}) = \rho(\bar{Y})$.

In what follows, we will verify monotonicity, $\mathcal{F}_\tau$-convexity and constancy on $\mathfrak{R}$ for the mapping $\tilde{\rho}_0$ defined by (7.3). Then it follows directly that ρ_0 defined by (7.4) satisfies the monotonicity property (r1), $\mathcal{F}_\tau$-convexity (r2) and constancy on $\mathfrak{R}$ (r5).

To prove the monotonicity property for $\tilde{\rho}_0$, let $X, Y \in \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta})$ with $\Lambda(\bar{X}) = X$, $\Lambda(\bar{Y}) = Y$ and $X \leq Y$. Then we have

$$\rho(\bar{X}_t(\omega)I_{[\tau,\infty)}) = \Lambda(\bar{X}_t(\omega)) = \Lambda(\bar{X})_t(\omega) \leq \Lambda(\bar{Y})_t(\omega) = \Lambda(\bar{Y}_t(\omega)) = \rho(\bar{Y}_t(\omega)I_{[\tau,\infty)})$$

for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. Furthermore, preference consistency of ρ implies $\tilde{\rho}_0(X) \leq \tilde{\rho}_0(Y)$.

It remains to show $\mathcal{F}_\tau$-convexity and constancy on $\mathfrak{R}$. First, consider $X, Y \in \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta})$ and $\gamma \in L^\infty_\tau$ with $0 \leq \gamma \leq 1$ and set $Z := \gamma X + (1-\gamma)\, Y$. Let $\bar{X}, \bar{Y}, \bar{Z} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ be such that

$$\tilde{\rho}_0\,(X) = \rho(\bar{X}), \quad \tilde{\rho}_0\,(Y) = \rho(\bar{Y}), \quad \tilde{\rho}_0\,(Z) = \rho(\bar{Z})$$

where $\Lambda(\bar{X}) = X$, $\Lambda(\bar{Y}) = Y$ and $\Lambda(\bar{Z}) = Z$. Then we obtain

$$\begin{aligned}
\rho(\bar{Z}_t(\omega)I_{[\tau,\infty)}) &= \Lambda(\bar{Z})_t(\omega) \\
&= Z_t\,(\omega) \\
&= \gamma(\omega)X_t\,(\omega) + (1-\gamma(\omega))\,Y_t\,(\omega) \\
&= \gamma(\omega)\Lambda(\bar{X})_t(\omega) + (1-\gamma(\omega))\,\Lambda(\bar{Y})_t(\omega) \\
&= \gamma(\omega)\rho(\bar{X}_t(\omega)I_{[\tau,\infty)}) + (1-\gamma(\omega))\,\rho(\bar{Y}_t(\omega)I_{[\tau,\infty)})
\end{aligned}$$

for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. Now, $\mathcal{F}_\tau$-convexity follows from the $\mathcal{F}_\tau$-risk convexity property of ρ since

$$\tilde{\rho}_0\,(Z) = \rho(\bar{Z}) \leq \gamma\rho(\bar{X}) + (1-\gamma)\,\rho(\bar{Y}) = \gamma\tilde{\rho}_0\,(X) + (1-\gamma)\,\tilde{\rho}_0\,(Y)\,.$$

Finally, note that for all $a \in \mathfrak{R}$, there exists $\bar{x} \in \mathbb{R}^n$ with $\Lambda(\bar{x}) = a$ and $\tilde{\rho}_0(aI_{[\tau,\infty)}) = \rho(\bar{x}I_{[\tau,\infty)})$. Since $a = \Lambda(\bar{x}) = \rho(\bar{x}I_{[\tau,\infty)})$, we obtain $\tilde{\rho}_0(aI_{[\tau,\infty)}) = a$ for all $a \in \mathfrak{R}$. This means that $\tilde{\rho}_0$ satisfies the constancy property on $\mathfrak{R}$. The equality $\rho = \rho_0 \circ \Lambda$ follows immediately from the definition of ρ_0 and Λ.

For the second part of the proof let Λ be a convex aggregation function with $f_\Lambda : \mathbb{R} \to \mathfrak{R}$ that is surjective and strictly increasing on $\mathfrak{S}$ and satisfies $f_\Lambda(0) = 0$. Moreover, assume that ρ_0 is a conditional convex single-firm risk measure with $\rho_0(aI_{[\tau,\infty)}) = a$ for all $a \in \mathfrak{R}$. Monotonicity (s1) and $\mathcal{F}_\tau$-outcome convexity (s4a) of ρ follow from the corresponding properties of Λ and ρ_0. In order to prove preference consistency (s2), fix $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ with

$$(\rho_0 \circ \Lambda)(\bar{X}_t\,(\omega)\,I_{[\tau,\infty)}) = \rho(\bar{X}_t\,(\omega)\,I_{[\tau,\infty)}) \geq \rho(\bar{Y}_t\,(\omega)\,I_{[\tau,\infty)}) = (\rho_0 \circ \Lambda)(\bar{Y}_t\,(\omega)\,I_{[\tau,\infty)})$$

for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. Note that $\Lambda(\mathbb{R}^n) = \mathfrak{R}$ (and thus $\Lambda(\mathbb{R}^n I_{[\tau,\infty)}) = \mathfrak{R}I_{[\tau,\infty)}$) and $\rho_0(aI_{[\tau,\infty)}) = a$ for all $a \in \mathfrak{R}$. It follows that $\Lambda(\bar{X}_t\,(\omega)) \geq \Lambda(\bar{Y}_t\,(\omega))$ for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. The monotonicity property of ρ_0 yields

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \geq (\rho_0 \circ \Lambda)(\bar{Y}) = \rho(\bar{Y}),$$

which means that property (s2) is satisfied.

Now, consider $\bar{X}, \bar{Y}, \bar{Z} \in \mathcal{R}^{\infty;n}_{\tau,\theta}$ and $\gamma \in L^\infty_\tau$ with $0 \leq \gamma \leq 1$ and assume that $\rho(\bar{Z}_t(\omega)I_{[\tau,\infty)}) = \gamma(\omega)\rho(\bar{X}_t(\omega)I_{[\tau,\infty)}) + (1-\gamma(\omega))\,\rho(\bar{Y}_t(\omega)I_{[\tau,\infty)})$ for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. Since $\rho = \rho_0 \circ \Lambda$, this means

$$\rho_0(\Lambda(\bar{Z}_t(\omega)I_{[\tau,\infty)})) = \gamma(\omega)\rho_0(\Lambda(\bar{X}_t(\omega)I_{[\tau,\infty)})) + (1-\gamma(\omega))\rho_0(\Lambda(\bar{Y}_t(\omega)I_{[\tau,\infty)})) \quad (7.5)$$

for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. Together with $\Lambda(\mathbb{R}^n I_{[\tau,\infty)}) = \mathfrak{R} I_{[\tau,\infty)}$ and $\rho_0(aI_{[\tau,\infty)}) = a$ for all $a \in \mathfrak{R}$, Equation (7.5) yields

$$\Lambda(\bar{Z}_t(\omega)) = \gamma(\omega)\Lambda(\bar{X}_t(\omega)) + (1-\gamma(\omega))\,\Lambda(\bar{Y}_t(\omega))$$

for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. The $\mathcal{F}_\tau$-convexity property of ρ_0 implies

$$\rho(\bar{Z}) = \rho_0(\Lambda(\bar{Z})) \leq \gamma\rho_0(\Lambda(\bar{X})) + (1-\gamma)\,\rho_0(\Lambda(\bar{Y})) = \gamma\rho(\bar{X}) + (1-\gamma)\,\rho(\bar{Y}).$$

Hence, ρ satisfies $\mathcal{F}_\tau$-risk convexity (s4b). It remains to show the f_ρ-constancy property (s3). The constancy properties of Λ and ρ_0 and $f_\Lambda(0) = 0$ imply

$$\rho(a1_n I_{[\tau,\infty)}) = \rho_0(\Lambda(a1_n I_{[\tau,\infty)})) = \rho_0(f_\Lambda(a)I_{[\tau,\infty)}) = f_\Lambda(a) \quad \text{for all } a \in \mathbb{R}.$$

In conclusion, f_ρ-constancy is satisfied with $f_\rho := f_\Lambda$. □

The positively homogeneous case reads as follows. The proof is analogous to the proof of Corollary 5.2.3.

Corollary 7.2.2 (Positively homogeneous structural decomposition)**.**

a) A mapping $\rho : \mathcal{R}^{\infty;n}_{\tau,\theta} \to L^\infty_\tau$ with $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$ is a conditional positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}$ and a conditional coherent single-firm risk measure $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \text{for all } \bar{X} \in \mathcal{R}^{\infty;n}_{\tau,\theta}.$$

b) A mapping $\rho : \mathcal{R}^{\infty;n}_{\tau,\theta} \to L^\infty_\tau$ with $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}_+$ is a conditional positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}_+$ and a conditional positively homogeneous single-firm risk measure $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \text{for all } \bar{X} \in \mathcal{R}^{\infty;n}_{\tau,\theta}.$$

Remark 7.2.3. In the same way to Remark 5.2.4, there exists an alternative formulation for part a) of Corollary 7.2.2:

a') A mapping $\rho : \mathcal{R}^{\infty;n}_{\tau,\theta} \to L^\infty_\tau$ with $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$ is a conditional positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ with $\Lambda(\mathbb{R}^n) = \mathbb{R}$ and a conditional positively homogeneous single-firm risk measure $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ that satisfies the constancy property on $\mathbb{R}$ such that ρ is the composition of ρ_0 and Λ, i.e.,

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \text{for all } \bar{X} \in \mathcal{R}^{\infty;n}_{\tau,\theta}.$$

7.3. Representations of conditional systemic risk measures

In this section we focus on different representation results for conditional convex and positively homogeneous systemic risk measures ρ. From now on, we solely consider compositions $\rho = \rho_0 \circ \Lambda$ in the sense of Theorem 7.2.1 and Corollary 7.2.2. This means that the conditional convex [positively homogeneous] systemic risk measure ρ is characterized by a convex [positively homogeneous] single-firm risk measure ρ_0 and a convex [positively homogeneous] aggregation function Λ. Moreover, $\operatorname{Im}\Lambda = \mathbb{R}$ and $f_\Lambda(0) = 0$ or $\operatorname{Im}\Lambda = \mathbb{R}_+$, and $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ satisfies constancy on $\mathbb{R}$ in case of $\operatorname{Im}\Lambda = \mathbb{R}$ and constancy on $\mathbb{R}_+$ in case of $\operatorname{Im}\Lambda = \mathbb{R}_+$.

7.3.1. Primal representation

The aim of this subsection is to obtain a first representation of conditional convex systemic risk measures in terms of their acceptance sets.

Definition 7.3.1. *The* acceptance sets *of the conditional convex systemic risk measure $\rho = \rho_0 \circ \Lambda$ with conditional convex single-firm risk measure ρ_0 and convex aggregation function Λ are given by*

$$\tilde{\mathcal{B}}_{\rho_0} := \{(\gamma, X) \in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta} | \gamma \geq \rho_0(X)\} \quad \textit{and}$$
$$\mathcal{B}_\Lambda := \{(Y, \bar{X}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty;n}_{\tau,\theta} | Y \geq \Lambda(\bar{X})\}.$$

Note that $\mathcal{B}_\Lambda = \{(Y, \bar{X}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty;n}_{\tau,\theta} | Y_t \geq \Lambda(\bar{X}_t) \text{ for all } t \in \mathbb{N}_0\}$. In the following proposition we use the properties from Definition 5.4.2 for subsets of $L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}$ and $\mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$.

Proposition 7.3.2. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a conditional convex systemic risk measure with conditional convex single-firm risk measure $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ and convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$. Let $\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ be the corresponding acceptance sets.*

1. *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ satisfy the following properties:*
   a) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ satisfy the monotonicity property.*
   b) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ satisfy the epigraph property.*
   c) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ are $\mathcal{F}_\tau$-convex sets.*
   d) *$(a, aI_{[\tau,\infty)}) \in \tilde{\mathcal{B}}_{\rho_0}$ with $\operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma, aI_{[\tau,\infty)}) \in \tilde{\mathcal{B}}_{\rho_0}\} = a$ for all $a \in \operatorname{Im}\Lambda$ and*
   $(f_\Lambda(a)I_{[\tau,\infty)}, a1_n I_{[\tau,\infty)}) \in \mathcal{B}_\Lambda$ with $\operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma I_{[\tau,\infty)}, a1_n I_{[\tau,\infty)}) \in \mathcal{B}_\Lambda\} = f_\Lambda(a)$ for all $a \in \mathbb{R}$.

   If $\rho = \rho_0 \circ \Lambda$ is a conditional positively homogeneous systemic risk measure, then the following properties are additionally satisfied:
   e) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ are $\mathcal{F}_\tau$-cones.*
   f) *$(nI_{[\tau,\infty)}, 1_n I_{[\tau,\infty)}) \in \mathcal{B}_\Lambda$ with $\operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma I_{[\tau,\infty)}, 1_n I_{[\tau,\infty)}) \in \mathcal{B}_\Lambda\} = n$.*

2. *ρ admits the so called* primal representation

$$\rho(\bar{X}) = \operatorname{ess\,inf}\{\gamma \in L^{\infty}_{\tau} | (\gamma, Y) \in \tilde{\mathcal{B}}_{\rho_0}, (Y, \bar{X}) \in \mathcal{B}_{\Lambda}\} \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty;n}_{\tau,\theta} \tag{7.6}$$

where we set $\operatorname{ess\,inf} \emptyset := +\infty$.

Proof. All properties in 1.a)-1.f) for $\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_{\Lambda}$ are satisfied due to the properties of ρ_0 and Λ. Furthermore, we know that ρ_0 is for all $X \in \mathcal{R}^{\infty}_{\tau,\theta}$ representable by

$$\rho_0(X) = \operatorname{ess\,inf}\{\gamma \in L^{\infty}_{\tau} | \gamma \geq \rho_0(X)\} = \operatorname{ess\,inf}\{\gamma \in L^{\infty}_{\tau} | (\gamma, X) \in \tilde{\mathcal{B}}_{\rho_0}\}.$$

Since $\mathcal{R}^{\infty} = L^{\infty}_{\mathcal{H}}$, we can consider $V \in \mathcal{R}^{\infty}_{\tau,\theta}$ as an element in the larger space $L^{\infty}_{\mathcal{H}}$. It follows that

$$\begin{aligned}\Lambda(\bar{Z}) &= \operatorname{ess\,inf}\{V \in L^{\infty}_{\mathcal{H}} | V \geq \Lambda(\bar{Z})\} = \operatorname{ess\,inf}\{V \in L^{\infty}_{\mathcal{H}} | V \geq \Lambda(\bar{Z}), V \in \mathcal{R}^{\infty}_{\tau,\theta}\} \\ &= \operatorname{ess\,inf}\{V \in L^{\infty}_{\mathcal{H}} | (V, \bar{Z}) \in \mathcal{B}_{\Lambda}\}.\end{aligned}$$

for all $\bar{Z} \in \mathcal{R}^{\infty;n}_{\tau,\theta}$. Because equality $\rho = \rho_0 \circ \Lambda$ is satisfied, the representations above yield

$$\begin{aligned}\rho(\bar{X}) &= \operatorname{ess\,inf}\{\gamma \in L^{\infty}_{\tau} | \gamma \geq (\rho_0 \circ \Lambda)(\bar{X})\} = \operatorname{ess\,inf}\{\gamma \in L^{\infty}_{\tau} | (\gamma, \Lambda(\bar{X})) \in \tilde{\mathcal{B}}_{\rho_0}\} \\ &= \operatorname{ess\,inf}\{\gamma \in L^{\infty}_{\tau} | (\gamma, \operatorname{ess\,inf}\{V \in L^{\infty}_{\mathcal{H}} | (V, \bar{X}) \in \mathcal{B}_{\Lambda}\}) \in \tilde{\mathcal{B}}_{\rho_0}\}\end{aligned}$$

for all $\bar{X} \in \mathcal{R}^{\infty;n}_{\tau,\theta}$. Representation (7.6) follows from the monotonicity property of $\tilde{\mathcal{B}}_{\rho_0}$. □

7.3.2. Continuity and closedness

In this subsection we provide closedness results for the acceptance sets of the conditional convex systemic risk measure $\rho = \rho_0 \circ \Lambda$. These results will be applied in the next subsection to obtain a dual representation of ρ.

But first, note that for any $\gamma \in L^{\infty}_{\tau}$, we can identify the element $\gamma I_{[\tau,\infty)} \in \mathcal{R}^{\infty}_{\tau,\theta}$. If we define

$$\mathcal{B}_{\rho_0} := \{(X, Z) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} | X = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^{\infty}_{\tau}, \gamma \geq \rho_0(Z)\},$$

then we have a one-to-one relation between the sets $\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_{\rho_0}$. From now on, we focus on the set $\mathcal{B}_{\rho_0}$.

The following lemma is a generalization of Lemma A.65 in Föllmer and Schied (2011) for multi-dimensional L^{∞}-spaces.

Remark 7.3.3 (See, e.g., Section 5.14 in Aliprantis and Border (2006))**.** Let $(\mathcal{X}, \mathcal{V})$ be paired spaces and suppose that $\langle \cdot, \cdot \rangle$ is the corresponding bilinear form. Consider a net $(x_\iota) \subset \mathcal{X}$ and $x \in \mathcal{X}$. Then $x_\iota \to x$ in $\sigma(\mathcal{X}, \mathcal{V})$ if and only if

$$\langle x_\iota, v \rangle \to \langle x, v \rangle \text{ in } \mathbb{R} \quad \text{for all } v \in \mathcal{V}.$$

Similarly, if we suppose that $(v_\iota) \subset \mathcal{V}$ is a net and $v \in \mathcal{V}$, then $v_\iota \to v$ in $\sigma(\mathcal{V}, \mathcal{X})$ if and only if

$$\langle x, v_\iota \rangle \to \langle x, v \rangle \text{ in } \mathbb{R} \quad \text{for all } x \in \mathcal{X}.$$

Lemma 7.3.4. *Define the set* $\mathcal{B}^r_{\infty,m} := \{\bar{X} \in (L^\infty_\mathcal{H})^m | \|\bar{X}\|_{(L^\infty_\mathcal{H})^m} \leq r\}$ *for* $r > 0$.

1. *For every* $r > 0$*, the set* $\mathcal{B}^r_{\infty,m}$ *is closed in* $(L^1_\mathcal{H})^m$*, i.e., for every sequence* $(\bar{Y}^{(k)}) \subset \mathcal{B}^r_{\infty,m}$ *that converges to* $\bar{Y}$ *in* $(L^1_\mathcal{H})^m$*, we have* $\bar{Y} \in \mathcal{B}^r_{\infty,m}$.
2. *A convex subset* $\mathcal{C}$ *of* $(L^\infty_\mathcal{H})^m$ *is* $\sigma((L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m)$*-closed if the set* $\mathcal{C}_r := \mathcal{C} \cap \mathcal{B}^r_{\infty,m}$ *is closed in* $(L^1_\mathcal{H})^m$ *for every* $r > 0$.

Proof. Part 1: Fix $r > 0$ and let $(\bar{Y}^{(k)}) \subset \mathcal{B}^r_{\infty,m}$ with $\bar{Y}^{(k)} \to \bar{Y}$ in $(L^1_\mathcal{H})^m$. This means that $\|\bar{Y}^{(k)} - \bar{Y}\|_{(L^1_\mathcal{H})^m} = \sum_{i=1}^m \|(\bar{Y}^{(k)})^i - \bar{Y}^i\|_{L^1_\mathcal{H}} \to 0$. Hence, $(\bar{Y}^{(k)})^i \to \bar{Y}^i$ in $L^1_\mathcal{H}$ for each $i \in \{1, \dots, m\}$. Moreover, there exists a subsequence $\bar{Y}^{(k_l)}$ such that $(\bar{Y}^{(k_l)})^i \to \bar{Y}^i$ η-a.s. and in $L^1_\mathcal{H}$ for each $i \in \{1, \dots, m\}$. In addition, the inequality

$$|\bar{Y}^i| \leq |(\bar{Y}^{(k_l)})^i| + |(\bar{Y}^{(k_l)})^i - \bar{Y}^i| \leq \|(\bar{Y}^{(k_l)})^i\|_{L^\infty_\mathcal{H}} + |(\bar{Y}^{(k_l)})^i - \bar{Y}^i| \quad \eta - \text{a.s.}$$

implies

$$\begin{aligned}
\max_{i \in \{1,\dots,m\}} |\bar{Y}^i| &\leq \max_{i \in \{1,\dots,m\}} \left(\|(\bar{Y}^{(k_l)})^i\|_{L^\infty_\mathcal{H}} + |(\bar{Y}^{(k_l)})^i - \bar{Y}^i| \right) \\
&\leq \max_{i \in \{1,\dots,m\}} \|(\bar{Y}^{(k_l)})^i\|_{L^\infty_\mathcal{H}} + \max_{i \in \{1,\dots,m\}} |(\bar{Y}^{(k_l)})^i - \bar{Y}^i| \\
&\leq r + \max_{i \in \{1,\dots,m\}} |(\bar{Y}^{(k_l)})^i - \bar{Y}^i| \quad \eta - \text{a.s.}
\end{aligned}$$

Since the convergence $(\bar{Y}^{(k_l)})^i \to \bar{Y}^i$ holds η-a.s., it follows $\max_{i \in \{1,\dots,m\}} |\bar{Y}^i| \leq r$ η-a.s. Therefore, we obtain $|\bar{Y}^i| \leq r$ η-a.s. for all $i \in \{1, \dots, m\}$, which implies $\|\bar{Y}^i\|_{L^\infty_\mathcal{H}} \leq r$ for all $i \in \{1, \dots, m\}$ because $\|\bar{Y}^i\|_{L^\infty_\mathcal{H}} = \inf\{c \in \mathbb{R} | |\bar{Y}^i| \leq c \ \eta\text{-a.s.}\}$. But this leads to $\max_{i \in \{1,\dots,m\}} \|\bar{Y}^i\|_{L^\infty_\mathcal{H}} \leq r$, which shows that $\bar{Y} \in \mathcal{B}^r_{\infty,m}$.

Part 2: First, note that the natural injection

$$\Upsilon : ((L^\infty_\mathcal{H})^m, \sigma((L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m)) \to ((L^1_\mathcal{H})^m, \sigma((L^1_\mathcal{H})^m, (L^\infty_\mathcal{H})^m))$$

is continuous: To this end, consider a net $(\bar{X}^{(\iota)}) \subset (L^\infty_\mathcal{H})^m$ with $\bar{X}^{(\iota)} \to \bar{X} \in (L^\infty_\mathcal{H})^m$ in the topological space $((L^\infty_\mathcal{H})^m, \sigma((L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m))$. Then

$$\langle \bar{X}^{(\iota)}, \bar{\Xi} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} \to \langle \bar{X}, \bar{\Xi} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} \quad \text{for all } \bar{\Xi} \in (L^1_\mathcal{H})^m.$$

Since $\langle \bar{Z}, \bar{Y} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} = \langle \bar{Y}, \bar{Z} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m}$ for all $\bar{Y}, \bar{Z} \in (L^\infty_\mathcal{H})^m$, this implies

$$\begin{aligned}
\langle \bar{Z}, \bar{X}^{(\iota)} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} &= \langle \bar{X}^{(\iota)}, \bar{Z} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} \\
\to \langle \bar{X}, \bar{Z} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} &= \langle \bar{Z}, \bar{X} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m}
\end{aligned}$$

for all $\bar{Z} \in (L^\infty_\mathcal{H})^m$. But this means that $\bar{X}^{(\iota)} \to \bar{X}$ in $((L^1_\mathcal{H})^m, \sigma((L^1_\mathcal{H})^m, (L^\infty_\mathcal{H})^m))$. Hence, Υ is continuous.

Now, suppose that $\mathcal{C}_r$ is closed in $(L^1_{\mathcal{H}})^m$ for every $r > 0$ and let $\mathcal{C}$ be convex. Since for $\bar{X}, \bar{Y} \in \mathcal{C}_r$ and $a \in [0,1]$, we have

$$\max_{i\in\{1,\dots,n\}} \|a\bar{X}^i + (1-a)\bar{Y}^i\|_{L^\infty_{\mathcal{H}}} \leq \max_{i\in\{1,\dots,n\}} \Big(a\|\bar{X}^i\|_{L^\infty_{\mathcal{H}}} + (1-a)\|\bar{Y}^i\|_{L^\infty_{\mathcal{H}}}\Big),$$
$$\leq a \max_{i\in\{1,\dots,n\}} \|\bar{X}^i\|_{L^\infty_{\mathcal{H}}} + (1-a) \max_{i\in\{1,\dots,n\}} \|\bar{Y}^i\|_{L^\infty_{\mathcal{H}}} \leq r,$$

the set $\mathcal{C}_r$ is convex for every $r > 0$. It follows that each $\mathcal{C}_r$ is also $\sigma((L^1_{\mathcal{H}})^m, (L^\infty_{\mathcal{H}})^m)$-closed; see, for instance, Theorem A.60 in Föllmer and Schied (2011). Continuity of Υ implies that $\mathcal{C}_r$ is $\sigma((L^\infty_{\mathcal{H}})^m, (L^1_{\mathcal{H}})^m)$-closed in $(L^\infty_{\mathcal{H}})^m$. Finally, the Krein-Šmulian theorem (see Theorem A.2.5) yields that $\mathcal{C}$ is $\sigma((L^\infty_{\mathcal{H}})^m, (L^1_{\mathcal{H}})^m)$-closed. □

In the following definition, we introduce continuity properties for mappings defined on $\mathcal{R}^{\infty,m}_{\tau,\theta}$ and $\mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,m}_{\tau,\theta}$. This definition is closely related to Definition 3.15 in Cheridito et al. (2006) (see Definition 6.2.3).

Definition 7.3.5. *We say that a sequence $(\bar{X}^{(k)}) \subset \mathcal{R}^{\infty,m}_{\tau,\theta}$ is* increasing (decreasing) *if each $(\bar{X}^{(k)})^i \subset \mathcal{R}^{\infty}_{\tau,\theta}$, $i \in \{1,\dots,m\}$, is increasing (decreasing) in k. Moreover, for every $s \in \mathbb{N}_0$, $\bar{X}^{(k)}_s \uparrow \bar{X}_s$ $\mathbb{P}$-a.s. for some $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ means that $(\bar{X}^{(k)}_s)^i \uparrow (\bar{X}_s)^i$ $\mathbb{P}$-a.s. for all $i \in \{1,\dots,m\}$.*

We call a mapping $\upsilon : \mathcal{R}^{\infty,m}_{\tau,\theta} \to L^\infty$ continuous for bounded increasing sequences *if for $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ and every increasing sequence $(\bar{X}^{(k)}) \subset \mathcal{R}^{\infty,m}_{\tau,\theta}$ with $\bar{X}^{(k)}_t \uparrow \bar{X}_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$, we obtain $\lim_{k\to\infty} \upsilon(\bar{X}^{(k)}) = \upsilon(\bar{X})$ $\mathbb{P}$-a.s.*

Similarly, a mapping $\Upsilon : \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,m}_{\tau,\theta} \to L^\infty$ is called continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument *if for $(X, \bar{Y}) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,m}_{\tau,\theta}$ and every sequence $(X^{(k)}, \bar{Y}^{(k)}) \subset \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,m}_{\tau,\theta}$ that satisfies $X^{(k)}_t \downarrow X_t$ and $\bar{Y}^{(k)}_t \uparrow \bar{Y}_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$, we obtain $\lim_{k\to\infty} \Upsilon(X^{(k)}, \bar{Y}^{(k)}) = \Upsilon(X, \bar{Y})$ $\mathbb{P}$-a.s.*

The subsequent lemma is one of the main results in this subsection and yields a closedness result for the acceptance set $\mathcal{B}_{\rho_0}$. For the proof we borrow ideas from the proof of Lemma 3.17 in Cheridito et al. (2006) and Section 4 in Cheridito et al. (2004).

Lemma 7.3.6. *Let ρ_0 be a conditional convex single-firm risk measure. If ρ_0 is continuous for bounded increasing sequences, then the acceptance set $\mathcal{B}_{\rho_0}$ is $\sigma(\mathcal{R}^\infty \times \mathcal{R}^\infty, \mathcal{A}^1 \times \mathcal{A}^1)$-closed.*

Proof. Consider the map $\varrho : \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} \to L^\infty_\tau$ defined by $\varrho(X, Y) := \rho_0(Y) - X_\tau$. If ρ_0 is continuous for bounded increasing sequences, then ϱ is continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument. Moreover, since for each $\gamma \in L^\infty_\tau$ and $Z \in \mathcal{R}^{\infty}_{\tau,\theta}$, $\gamma \geq \rho_0(Z)$ if and only if $0 \geq \varrho(\gamma I_{[\tau,\infty)}, Z)$, we obtain

$$\mathcal{B}_{\rho_0} = \{(X, Z) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} | X = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^\infty_\tau \text{ and } 0 \geq \varrho(\gamma I_{[\tau,\infty)}, Z)\}.$$

Consider a net $(X^\iota, Y^\iota) \subset \mathcal{B}_{\rho_0}$ and $(X^\#, Y^\#) \in \mathcal{R}^\infty$ with $(X^\iota, Y^\iota) \to (X^\#, Y^\#)$ in the topological space $(\mathcal{R}^\infty \times \mathcal{R}^\infty, \sigma(\mathcal{R}^\infty \times \mathcal{R}^\infty, \mathcal{A}^1 \times \mathcal{A}^1))$. Then $(X^\#, Y^\#) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta}$ and $X^\# = \gamma^\# I_{[\tau,\infty)}$ for some $\gamma^\# \in L^\infty_\tau$. Assume that $0 < \varrho(X^\#, Y^\#)$ on a set $A \in \mathcal{F}_\tau$ with $\mathbb{P}[A] > 0$. The function $\tilde{\varrho} : \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} \to \mathbb{R}$ defined by

$$\tilde{\varrho}(X, Z) := \mathbb{E}[\varrho(X_\tau I_{[\tau,\infty)}, Z) I_A]$$

is obviously decreasing in the first and increasing in the second argument and convex. Moreover, $\tilde{\rho}$ is continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument: Consider $(X, Y) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta}$ and a sequence $(X^{(k)}, Y^{(k)}) \subset \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta}$ that satisfies $X^{(k)}_t \downarrow X_t$ and $Y^{(k)}_t \uparrow Y_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$. Then $\varrho(X^{(k)}_\tau I_{[\tau,\infty)}, Y^{(k)}) \uparrow \varrho(X_\tau I_{[\tau,\infty)}, Y)$ $\mathbb{P}$-a.s. Hence, by the monotone convergence theorem, we obtain $\tilde{\varrho}(X^{(k)}, Y^{(k)}) \uparrow \tilde{\varrho}(X, Y)$.

Now, note that $\mathcal{R}^\infty = L^\infty_{\mathcal{H}}$ and identify $\mathcal{A}^1$ with $L^1_{\mathcal{H}}$. Define the convex set $\mathcal{D}_{\tilde{\varrho}} := \{(X, Z) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} | X = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^\infty_\tau \text{ and } 0 \geq \tilde{\varrho}(\gamma I_{[\tau,\infty)}, Z)\}$. We will show that the set

$$\mathcal{C}_r := \mathcal{D}_{\tilde{\varrho}} \cap \{(X, Z) \in L^\infty_{\mathcal{H}} \times L^\infty_{\mathcal{H}} | \, \|(X, Z)\|_{L^\infty_{\mathcal{H}} \times L^\infty_{\mathcal{H}}} \leq r\}$$

is closed in $L^1_{\mathcal{H}} \times L^1_{\mathcal{H}}$ for each $r > 0$. Then it follows from the second part of Lemma 7.3.4 that $\mathcal{D}_{\tilde{\varrho}}$ is $\sigma(L^\infty_{\mathcal{H}} \times L^\infty_{\mathcal{H}}, L^1_{\mathcal{H}} \times L^1_{\mathcal{H}})$-closed.

To this end, fix $r > 0$, consider $\mathcal{C}_r$ as a subset of $L^1_{\mathcal{H}} \times L^1_{\mathcal{H}}$ and let $(\gamma^{(k)} I_{[\tau,\infty)}, Z^{(k)}) \subset \mathcal{C}_r$ be a sequence which converges to $(\gamma I_{[\tau,\infty)}, Z)$ in $L^1_{\mathcal{H}} \times L^1_{\mathcal{H}}$.

By the first part of Lemma 7.3.4, we have $\|(\gamma I_{[\tau,\infty)}, Z)\|_{L^\infty_{\mathcal{H}} \times L^\infty_{\mathcal{H}}} \leq r$. It remains to show that $0 \geq \tilde{\varrho}(\gamma I_{[\tau,\infty)}, Z)$. Since $(\gamma^{(k)} I_{[\tau,\infty)}, Z^{(k)}) \subset \mathcal{C}_r$ converges to $(\gamma I_{[\tau,\infty)}, Z)$ in $L^1_{\mathcal{H}} \times L^1_{\mathcal{H}}$, there exists a subsequence $(\gamma^{(k_l)} I_{[\tau,\infty)}, Z^{(k_l)}) \subset L^1_{\mathcal{H}} \times L^1_{\mathcal{H}}$ such that $(\gamma^{(k_l)} I_{\{\tau \leq t\}}, Z^{(k_l)}_t) \to (\gamma I_{\{\tau \leq t\}}, Z_t)$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$. If we define $Y^{(m)}_t := \inf_{l \geq m} (Z^{(k_l)}_t \wedge Z_t)$ for all $t \in \mathbb{N}_0$, then $(Y^{(m)})$ is increasing with $Y^{(m)}_t \uparrow Z_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$ and $Y^{(m)}_t \leq Z^{(k_m)}_t$ for all $m \in \mathbb{N}$ and $t \in \mathbb{N}_0$. Similarly, define $\phi^{(m)} := \sup_{l \geq m} (\gamma^{(k_l)} \vee \gamma)$ such that $(\phi^{(m)})$ is decreasing with $\phi^{(m)} \downarrow \gamma$ $\mathbb{P}$-a.s. and $\phi^{(m)} \geq \gamma^{(k_m)}$ for all $m \in \mathbb{N}$.

The monotonicity property of $\tilde{\varrho}$ yields $\tilde{\varrho}(\gamma^{(k_m)} I_{[\tau,\infty)}, Z^{(k_m)}) \geq \tilde{\varrho}(\phi^{(m)} I_{[\tau,\infty)}, Y^{(m)})$, and together with the continuity property of $\tilde{\varrho}$, we obtain

$$0 \geq \liminf_{m \to \infty} \tilde{\varrho}(\gamma^{(k_m)} I_{[\tau,\infty)}, Z^{(k_m)}) \geq \lim_{m \to \infty} \tilde{\varrho}(\phi^{(m)} I_{[\tau,\infty)}, Y^{(m)}) = \tilde{\varrho}(\gamma I_{[\tau,\infty)}, Z).$$

Thus, $\mathcal{C}_r$ is closed in $L^1_{\mathcal{H}} \times L^1_{\mathcal{H}}$.

As $(X^\iota, Y^\iota) \in \mathcal{B}_{\rho_0}$, we know that $X^\iota = \gamma^\iota I_{[\tau,\infty)}$ and $0 \geq \varrho(\gamma^\iota I_{[\tau,\infty)}, Y^\iota)$. This implies $\tilde{\varrho}(\gamma^\iota I_{[\tau,\infty)}, Y^\iota) = \mathbb{E}[\varrho(\gamma^\iota I_{[\tau,\infty)}, Y^\iota) I_A] \leq 0$, i.e., $(X^\iota, Y^\iota) \in \mathcal{D}_{\tilde{\varrho}}$. Because $\mathcal{D}_{\tilde{\varrho}}$ is $\sigma(L^\infty_{\mathcal{H}} \times L^\infty_{\mathcal{H}}, L^1_{\mathcal{H}} \times L^1_{\mathcal{H}})$-closed, this means that $(X^\#, Y^\#) \in \mathcal{D}_{\tilde{\varrho}}$ and

$$0 \geq \tilde{\varrho}(X^\#, Y^\#) = \mathbb{E}[\varrho(\gamma^\# I_{[\tau,\infty)}, Y^\#) I_A] = \mathbb{E}[\varrho(\gamma^\# I_{[\tau,\infty)}, Y^\#) | A] \mathbb{P}[A].$$

Since $\mathbb{P}[A] > 0$ and $0 < \varrho(X^\#, Y^\#)$ on A, this is a contradiction. It follows $\varrho(X^\#, Y^\#) \leq 0$, i.e., $\mathcal{B}_{\rho_0}$ is $\sigma(\mathcal{R}^\infty \times \mathcal{R}^\infty, \mathcal{A}^1 \times \mathcal{A}^1)$-closed. $\square$

Before we prove the analogous result to Lemma 7.3.6 for the acceptance set $\mathcal{B}_\Lambda$, note that every convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is automatically continuous. This implies that Λ satisfies the following property: For $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ and every increasing sequence $(\bar{X}^{(k)}) \subset \mathcal{R}^{\infty,m}_{\tau,\theta}$ with $\bar{X}^{(k)}_t \uparrow \bar{X}_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$, it follows that $\Lambda(\bar{X}^{(k)}_t) \uparrow \Lambda(\bar{X}_t)$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$.

Lemma 7.3.7. *Let Λ be a convex aggregation function. Then the acceptance set $\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}, \mathcal{A}^{1,n+1})$-closed.*

Proof. Define the mapping $\Upsilon_t : \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_t$ by $\Upsilon_t(X, \bar{Z}) := \Lambda(\bar{Z}_t) - X_t$ for all $t \in \mathbb{N}_0$. Since Λ is continuous, each Υ_t is continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument. Moreover, since for each $(X, \bar{Z}) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$, $X_t \geq \Lambda(\bar{Z}_t)$ for all $t \in \mathbb{N}_0$ if and only if $0 \geq \Upsilon_t(X, \bar{Z})$ for all $t \in \mathbb{N}_0$, we obtain

$$\mathcal{B}_\Lambda = \{(X, \bar{Z}) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} | 0 \geq \Upsilon_t(X, \bar{Z}) \text{ for all } t \in \mathbb{N}_0\}.$$

Consider a net $(Y^\iota, \bar{X}^\iota) \subset \mathcal{B}_\Lambda$ and $(Y^\#, \bar{X}^\#) \in \mathcal{R}^{\infty,n+1}$ with $(Y^\iota, \bar{X}^\iota) \to (Y^\#, \bar{X}^\#)$ in the topological space $(\mathcal{R}^{\infty,n+1}, \sigma(\mathcal{R}^{\infty,n+1}, \mathcal{A}^{1,n+1}))$. Then $(Y^\#, \bar{X}^\#) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$. Assume that there exists $s \in \mathbb{N}_0$ such that $0 < \Upsilon_s(Y^\#, \bar{X}^\#)$ on a set $A_s \in \mathcal{F}_s$ with $\mathbb{P}[A_s] > 0$. For all other $(s \neq) t \in \mathbb{N}_0$, we set $A_t := \emptyset$. For each $t \in \mathbb{N}_0$, the function $\tilde{\Upsilon}_t : \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} \to \mathbb{R}$ defined by

$$\tilde{\Upsilon}_t(Y, \bar{X}) := \mathbb{E}[\Upsilon_t(Y, \bar{X}) I_{A_t}]$$

is decreasing in the first and increasing in the second argument, convex and continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument.

From now on, we use that $\mathcal{R}^\infty = L^\infty_{\mathcal{H}}$ and identify $\mathcal{A}^1$ with $L^1_{\mathcal{H}}$. Define $\mathcal{D}_{\tilde{\Upsilon}} := \{(Y, \bar{X}) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} | 0 \geq \tilde{\Upsilon}_t(Y, \bar{X}) \text{ for all } t \in \mathbb{N}_0\}$ and consider the set

$$\mathcal{C}_r := \mathcal{D}_{\tilde{\Upsilon}} \cap \{(Y, \bar{X}) \in L^\infty_{\mathcal{H}} \times (L^\infty_{\mathcal{H}})^n | \|(Y, \bar{X})\|_{(L^\infty_{\mathcal{H}})^{n+1}} \leq r\}.$$

We will show that $\mathcal{C}_r$ is closed in $(L^1_{\mathcal{H}})^{n+1}$ for each $r > 0$. According to the second part of Lemma 7.3.4, this implies that $\mathcal{D}_{\tilde{\Upsilon}}$ is $\sigma((L^\infty_{\mathcal{H}})^{n+1}, (L^1_{\mathcal{H}})^{n+1})$-closed.

Fix $r > 0$ and let $(Z^{(k)}, \bar{U}^{(k)}) \subset \mathcal{C}_r$ be a sequence which converges to $(Z, \bar{U})$ in $(L^1_{\mathcal{H}})^{n+1}$. By the first part of Lemma 7.3.4, we know that $\|(Z, \bar{U})\|_{(L^\infty_{\mathcal{H}})^{n+1}} \leq r$. Hence, it remains to verify that $0 \geq \tilde{\Upsilon}_t(Z, \bar{U})$ for all $t \in \mathbb{N}_0$. For $t \neq s$ this is trivial. In conclusion, we have to consider the case $t = s$.

Since $(Z^{(k)}, \bar{U}^{(k)}) \subset \mathcal{C}_r$ converges to $(Z, \bar{U})$ in the space $(L^1_{\mathcal{H}})^{n+1}$, we can find a subsequence $(Z^{(k_l)}, \bar{U}^{(k_l)}) \subset (L^1_{\mathcal{H}})^{n+1}$ such that $(Z^{(k_l)}_u, \bar{U}^{(k_l)}_u) \to (Z_u, \bar{U}_u)$ $\mathbb{P}$-a.s. for all $u \in \mathbb{N}_0$. Define $(\bar{Y}^{(m)})^i_u := \inf_{l \geq m}((\bar{U}^{(k_l)})^i_u \wedge \bar{U}^i_u)$ for each $i \in \{1, \ldots, n\}$ and $u \in \mathbb{N}_0$. Then $\bar{Y}^{(m)}_u \uparrow \bar{U}_u$ $\mathbb{P}$-a.s. for all $u \in \mathbb{N}_0$ and $\bar{Y}^{(m)}_u \leq \bar{U}^{(k_m)}_u$ for all $m \in \mathbb{N}$ and $u \in \mathbb{N}_0$. Similarly, for $V^{(m)}_u := \sup_{l \geq m}(Z^{(k_l)}_u \vee Z_u)$, $u \in \mathbb{N}_0$, we obtain $V^{(m)}_u \downarrow Z_u$ $\mathbb{P}$-a.s. for all $u \in \mathbb{N}_0$ and $V^{(m)}_u \geq Z^{(k_m)}_u$ for all $m \in \mathbb{N}$ and $u \in \mathbb{N}_0$. The monotonicity property of

$\tilde{\Upsilon}_s$ implies $\tilde{\Upsilon}_s(Z^{(k_m)}, \bar{U}^{(k_m)}) \geq \tilde{\Upsilon}_s(V^{(m)}, \bar{Y}^{(m)})$, and the continuity property of $\tilde{\Upsilon}_s$ yields

$$0 \geq \liminf_{m\to\infty} \tilde{\Upsilon}_s(Z^{(k_m)}, \bar{U}^{(k_m)}) \geq \lim_{m\to\infty} \tilde{\Upsilon}_s(V^{(m)}, \bar{Y}^{(m)}) = \tilde{\Upsilon}_s(Z, \bar{U}).$$

Altogether, we get the closedness of $\mathcal{C}_r$ in $(L^1_{\mathcal{H}})^{n+1}$.

Since $(Y^\iota, \bar{X}^\iota) \in \mathcal{B}_\Lambda$, we know that $0 \geq \Upsilon_t(Y^\iota, \bar{X}^\iota)$ for all $t \in \mathbb{N}_0$. This implies $\tilde{\Upsilon}_t(Y^\iota, \bar{X}^\iota) = \mathbb{E}[\Upsilon_t(Y^\iota, \bar{X}^\iota)I_{A_t}] \leq 0$ for all $t \in \mathbb{N}_0$, i.e., $(Y^\iota, \bar{X}^\iota) \in \mathcal{D}_{\tilde{\Upsilon}}$. Because $\mathcal{D}_{\tilde{\Upsilon}}$ is $\sigma((L^\infty_{\mathcal{H}})^{n+1}, (L^1_{\mathcal{H}})^{n+1})$-closed, this means that $(Y^\#, \bar{X}^\#) \in \mathcal{D}_{\tilde{\Upsilon}}$ and

$$0 \geq \tilde{\Upsilon}_t(Y^\#, \bar{X}^\#) = \mathbb{E}[\Upsilon_t(Y^\#, \bar{X}^\#)I_{A_t}] = \mathbb{E}[\Upsilon_t(Y^\#, \bar{X}^\#)|A_t]\mathbb{P}[A_t] \quad \text{for all } t \in \mathbb{N}_0.$$

But this is a contradiction to $\mathbb{P}[A_s] > 0$ and $0 < \Upsilon_s(Y^\#, \bar{X}^\#)$ on A_s. It follows $\Upsilon_t(Y^\#, \bar{X}^\#) \leq 0$ for all $t \in \mathbb{N}_0$. In other words, $\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}, \mathcal{A}^{1,n+1})$-closed. □

The next lemma is another building block for the proof of the dual representation result in the following subsection.

Lemma 7.3.8. *Let $\rho = \rho_0 \circ \Lambda$ be a conditional convex systemic risk measure characterized by a conditional convex single-firm risk measure ρ_0 and a convex aggregation function Λ. If ρ_0 is continuous for bounded increasing sequences, then the convex set*

$$\begin{aligned}\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda} := \{&(V, X, Z, \bar{Z}) \in \mathcal{R}^{\infty,n+3}_{\tau,\theta} | V = \phi I_{[\tau,\infty)} \textit{ for some } \phi \in L^\infty_\tau \textit{ and} \\ &\mathbb{E}[\rho_0(\Lambda(\bar{X} - \bar{Z}) - X - Z)] - \mathbb{E}[\phi] \leq d\}\end{aligned}$$

is $\sigma(\mathcal{R}^{\infty,n+3}, \mathcal{A}^{1,n+3})$-closed for each $d \in \mathbb{R}$ and each $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$.

Proof. Fix $d \in \mathbb{R}$ and $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ and consider the function $\varrho : \mathcal{R}^{\infty,n+3}_{\tau,\theta} \to \mathbb{R}$ defined by $\varrho(V, X, Z, \bar{Z}) := \mathbb{E}[\rho_0(\Lambda(\bar{X} - \bar{Z}) - X - Z)] - \mathbb{E}[V_\tau] - d$. Then ϱ is decreasing due to the monotonicity properties of ρ_0 and Λ. Let $(\tilde{U}^{(k)}) := (V^{(k)}, X^{(k)}, Z^{(k)}, \bar{Z}^{(k)}) \subset \mathcal{R}^{\infty,n+3}_{\tau,\theta}$ be a decreasing sequence with $\tilde{U}^{(k)}_t \downarrow \tilde{U}_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$ and $\tilde{U} := (V, X, Z, \bar{Z}) \in \mathcal{R}^{\infty,n+3}_{\tau,\theta}$. Then continuity and monotonicity of Λ imply $\Lambda(\bar{X} - \bar{Z}^{(k)})_t \uparrow \Lambda(\bar{X} - \bar{Z})_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$. Furthermore, the continuity for bounded increasing sequences and the monotonicity property of ρ_0 yield $\rho_0(\Lambda(\bar{X} - \bar{Z}^{(k)}) - X^{(k)} - Z^{(k)}) \uparrow \rho_0(\Lambda(\bar{X} - \bar{Z}) - X - Z)$ $\mathbb{P}$-a.s. By the monotone convergence theorem, we obtain $\varrho(V^{(k)}, X^{(k)}, Z^{(k)}, \bar{Z}^{(k)}) \uparrow \varrho(V, X, Z, \bar{Z})$.

Since for each $(V, X, Z, \bar{Z}) \in \mathcal{R}^{\infty,n+3}_{\tau,\theta}$, $d \geq \mathbb{E}[\rho_0(\Lambda(\bar{X} - \bar{Z}) - X - Z)] - \mathbb{E}[V_\tau]$ if and only if $0 \geq \varrho(V, X, Z, \bar{Z})$, we have

$$\begin{aligned}\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda} = \{&(V, X, Z, \bar{Z}) \in \mathcal{R}^{\infty,n+3}_{\tau,\theta} | V = \phi I_{[\tau,\infty)} \text{ for some } \phi \in L^\infty_\tau \text{ and} \\ &0 \geq \varrho(\phi I_{[\tau,\infty)}, X, Z, \bar{Z})\}.\end{aligned}$$

We will show that the set

$$\mathcal{C}_r := \mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda} \cap \{(V, X, Z, \bar{Z}) \in (L^\infty_{\mathcal{H}})^{n+3} | \|(V, X, Z, \bar{Z})\|_{(L^\infty_{\mathcal{H}})^{n+3}} \leq r\}$$

is closed in $(L^1_{\mathcal{H}})^{n+3}$ for each $r > 0$. Then $\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda}$ is $\sigma((L^\infty_{\mathcal{H}})^{n+3}, (L^1_{\mathcal{H}})^{n+3})$-closed due to the second part of Lemma 7.3.4.

Now, fix $r > 0$ and let $(\tilde{U}^{(k)}) := (\phi^{(k)} I_{[\tau,\infty)}, X^{(k)}, Z^{(k)}, \bar{Z}^{(k)}) \subset \mathcal{C}_r$ be a sequence that satisfies $(\tilde{U}^{(k)}) \to \tilde{U}$ in $(L^1_{\mathcal{H}})^{n+3}$ for $\tilde{U} = (\phi I_{[\tau,\infty)}, X, Z, \bar{Z}) \in (L^1_{\mathcal{H}})^{n+3}$. It remains to prove that $0 \geq \varrho(\tilde{U})$.

Since $(\tilde{U}^{(k)})^i \to \tilde{U}^i$ in $L^1_{\mathcal{H}}$ for each $i \in \{1, \ldots, n+3\}$, there exists a subsequence $(\tilde{U}^{(k_l)})$ such that $(\tilde{U}^{(k_l)})^i \to \tilde{U}^i$ η-a.s. and in $L^1_{\mathcal{H}}$ for each $i \in \{1, \ldots, n+3\}$. This means that $(\tilde{U}^{(k_l)})^i_t \to \tilde{U}^i_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$. Define $(\tilde{Y}^{(m)})^i_t := \sup_{l \geq m}((\tilde{U}^{(k_l)})^i_t \vee \tilde{U}^i_t)$ for all $t \in \mathbb{N}_0$ and $i \in \{1, \ldots, n+3\}$. Then $(\tilde{Y}^{(m)})^i \subset \mathcal{R}^\infty$ is decreasing and $(\tilde{Y}^{(m)})^i_t \downarrow \tilde{U}^i_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$ and $i \in \{1, \ldots, n+3\}$. By the monotonicity property of ϱ, we have $\varrho(\tilde{U}^{(k_m)}_t) \geq \varrho(\tilde{Y}^{(m)})$, and the continuity property of ϱ implies

$$0 \geq \liminf_{m\to\infty} \varrho(\tilde{U}^{(k_m)}) \geq \lim_{m\to\infty} \varrho(\tilde{Y}^{(m)}) = \varrho(\tilde{U}).$$

□

Note that the pairing $\langle \cdot, \cdot \rangle_m$ defined on $\mathcal{R}^{\infty,m} \times \mathcal{A}^{1,m}$ induces the pairing

$$\langle \cdot, \cdot \rangle_m |_{\mathcal{R}^{\infty,m}_{\tau,\theta} \times \mathcal{A}^{1,m}_{\tau,\theta}} : \mathcal{R}^{\infty,m}_{\tau,\theta} \times \mathcal{A}^{1,m}_{\tau,\theta} \to \mathbb{R}$$

defined on $\mathcal{R}^{\infty,m}_{\tau,\theta} \times \mathcal{A}^{1,m}_{\tau,\theta}$. In the following, we do not distinguish between these two cases. The next lemma provides a basic property of the spaces $\mathcal{R}^{\infty,m}_{\tau,\theta}$ and $\mathcal{A}^{1,m}_{\tau,\theta}$.

Lemma 7.3.9. *$\mathcal{R}^{\infty,m}_{\tau,\theta}$ and $\mathcal{A}^{1,m}_{\tau,\theta}$ satisfy the following properties:*

1. *$\mathcal{R}^{\infty,m}_{\tau,\theta}$ separates points of $\mathcal{A}^{1,m}_{\tau,\theta}$ under $\langle \cdot, \cdot \rangle_m$: If $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ and $\langle \bar{X}, \bar{\xi} \rangle_m = 0$ for all $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$, then $\bar{\xi} = 0$.*
2. *$\mathcal{A}^{1,m}_{\tau,\theta}$ separates points of $\mathcal{R}^{\infty,m}_{\tau,\theta}$ under $\langle \cdot, \cdot \rangle_m$: If $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ and $\langle \bar{X}, \bar{\xi} \rangle_m = 0$ for all $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$, then $\bar{X} = 0$.*

Proof. We know from Remark 7.1.1 that $\mathcal{R}^{\infty,m}$ separates points of $\mathcal{A}^{1,m}$ and $\mathcal{A}^{1,m}$ separates points of $\mathcal{R}^{\infty,m}$ under $\langle \cdot, \cdot \rangle_m$. This means

$$\bar{\xi} \in \mathcal{A}^{1,m} \text{ and } \langle \bar{X}, \bar{\xi} \rangle_m = 0 \text{ for all } \bar{X} \in \mathcal{R}^{\infty,m} \quad \Rightarrow \quad \bar{\xi} = 0, \tag{7.7}$$

$$\bar{X} \in \mathcal{R}^{\infty,m} \text{ and } \langle \bar{X}, \bar{\xi} \rangle_m = 0 \text{ for all } \bar{\xi} \in \mathcal{A}^{1,m} \quad \Rightarrow \quad \bar{X} = 0. \tag{7.8}$$

Let us first consider $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ and assume that $\langle \bar{X}, \bar{\xi} \rangle_m = 0$ for all $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$. Then every $\bar{Y} \in \mathcal{R}^{\infty,m}$ satisfies

$$\langle \bar{Y}, \bar{\xi} \rangle_m = \sum_{i=1}^m \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} \bar{Y}^i_t \Delta \bar{\xi}^i_t\right] = \sum_{i=1}^m \mathbb{E}\left[\sum_{t\in[\tau,\theta]\cap\mathbb{N}_0} \bar{Y}^i_t \Delta \bar{\xi}^i_t\right] = \langle \bar{Z}, \bar{\xi} \rangle_m$$

for $\bar{Z} := \bar{Y} I_{[\tau,\theta]} + \bar{Y}_\theta I_{(\theta,\infty)} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$. Hence, it follows $\langle \bar{Y}, \bar{\xi} \rangle_m = 0$ for all $\bar{Y} \in \mathcal{R}^{\infty,m}$ and (7.7) yields $\bar{\xi} = 0$. Therefore, $\mathcal{R}^{\infty,m}_{\tau,\theta}$ separates points of $\mathcal{A}^{1,m}_{\tau,\theta}$ under $\langle \cdot, \cdot \rangle_m$.

On the other hand, let $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ and assume that $\langle \bar{X}, \bar{\xi} \rangle_m = 0$ for all $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$. Now, consider an arbitrary $\bar{\psi} \in \mathcal{A}^{1,m}$. Since $\langle \bar{X}, \bar{\psi} \rangle_m = \sum_{i=1}^m \mathbb{E}[\sum_{t\in\mathbb{N}_0} \bar{X}^i_t \Delta\bar{\psi}^i_t]$ and $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$, we can find an element $\bar{\phi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ such that $\langle \bar{X}, \bar{\psi} \rangle_m = \langle \bar{X}, \bar{\phi} \rangle_m$: Define $\bar{\phi} \in \mathcal{A}^{1,m}$ by $\bar{\phi}^i_t := 0$ on $\{t < \tau\}$, $\Delta\bar{\phi}^i_t := \Delta\bar{\psi}^i_t$ on $\{\tau \le t < \theta\}$, $\Delta\bar{\phi}^i_\theta := \mathbb{E}[\sum_{t\in[\theta,\infty)\cap\mathbb{N}_0} \Delta\bar{\psi}^i_t | \mathcal{F}_\theta]$ and $\bar{\phi}^i_t := \bar{\phi}^i_\theta$ on $\{t > \theta\}$ for $i \in \{1, \dots, m\}$. Then $\bar{\phi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ because $\bar{\phi} = p^{\tau,\theta}_m(\bar{\phi})$ and

$$
\begin{aligned}
\|\bar{\phi}\|_{\mathcal{A}^{1,m}} &= \sum_{i=1}^m \mathbb{E}\left[\sum_{t\in[\tau,\theta)\cap\mathbb{N}_0} |\Delta\bar{\psi}^i_t| + \left|\mathbb{E}\left[\sum_{t\in[\theta,\infty)\cap\mathbb{N}_0} \Delta\bar{\psi}^i_t \middle| \mathcal{F}_\theta\right]\right|\right] \\
&\le \sum_{i=1}^m \mathbb{E}\left[\sum_{t\in[\tau,\theta)\cap\mathbb{N}_0} |\Delta\bar{\psi}^i_t| + \sum_{t\in[\theta,\infty)\cap\mathbb{N}_0} |\Delta\bar{\psi}^i_t|\right] \le \|\bar{\psi}\|_{\mathcal{A}^{1,m}} < \infty.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\langle \bar{X}, \bar{\psi} \rangle_m &= \sum_{i=1}^m \left(\mathbb{E}\left[\sum_{t\in[\tau,\theta)\cap\mathbb{N}_0} \bar{X}^i_t \Delta\bar{\psi}^i_t\right] + \mathbb{E}\left[\bar{X}^i_\theta \sum_{t\in[\theta,\infty)\cap\mathbb{N}_0} \Delta\bar{\psi}^i_t\right]\right) \\
&= \sum_{i=1}^m \left(\mathbb{E}\left[\sum_{t\in[\tau,\theta)\cap\mathbb{N}_0} \bar{X}^i_t \Delta\bar{\psi}^i_t\right] + \mathbb{E}\left[\bar{X}^i_\theta \mathbb{E}\left[\sum_{t\in[\theta,\infty)\cap\mathbb{N}_0} \Delta\bar{\psi}^i_t \middle| \mathcal{F}_\theta\right]\right]\right) \\
&= \sum_{i=1}^m \left(\mathbb{E}\left[\sum_{t\in[\tau,\theta)\cap\mathbb{N}_0} \bar{X}^i_t \Delta\bar{\phi}^i_t\right] + \mathbb{E}[\bar{X}^i_\theta \Delta\bar{\phi}^i_\theta]\right) \\
&= \langle \bar{X}, \bar{\phi} \rangle_m.
\end{aligned}
$$

Hence, for any $\bar{\psi} \in \mathcal{A}^{1,m}$, there exists an element $\bar{\phi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ such that $\langle \bar{X}, \bar{\psi} \rangle_m = \langle \bar{X}, \bar{\phi} \rangle_m$. This implies $\langle \bar{X}, \bar{\psi} \rangle_m = 0$ for any $\bar{\psi} \in \mathcal{A}^{1,m}$. Thus, (7.8) yields $\bar{X} = 0$. In other words, $\mathcal{A}^{1,m}_{\tau,\theta}$ separates points of $\mathcal{R}^{\infty,m}_{\tau,\theta}$ under $\langle \cdot, \cdot \rangle_m$. □

Because of the previous Lemma, we can define the topology $\sigma(\mathcal{R}^{\infty,m}_{\tau,\theta}, \mathcal{A}^{1,m}_{\tau,\theta})$ on $\mathcal{R}^{\infty,m}_{\tau,\theta}$ and the topology $\sigma(\mathcal{A}^{1,m}_{\tau,\theta}, \mathcal{R}^{\infty,m}_{\tau,\theta})$ on $\mathcal{A}^{1,m}_{\tau,\theta}$ (see Definition A.2.1). Both topologies are compatible with the pairing $\langle \cdot, \cdot \rangle_m$. Let us consider the so called *subspace topology* $\mathfrak{T}^m_{\tau,\theta}$ on $\mathcal{R}^{\infty,m}_{\tau,\theta}$ defined by

$$
\mathfrak{T}^m_{\tau,\theta} := \{\mathcal{U} \cap \mathcal{R}^{\infty,m}_{\tau,\theta} | \mathcal{U} \in \sigma(\mathcal{R}^{\infty,m}, \mathcal{A}^{1,m})\}.
$$

The next remark clarifies the relationship between $\sigma(\mathcal{R}^{\infty,m}_{\tau,\theta}, \mathcal{A}^{1,m}_{\tau,\theta})$ and $\mathfrak{T}^m_{\tau,\theta}$.

Remark 7.3.10. It is well known that $\sigma(\mathcal{R}^{\infty,m}, \mathcal{A}^{1,m}) = \sigma(\mathcal{R}^{\infty,m}, \mathscr{F})$ where $\mathscr{F} = \{f_{\bar{\xi}} : \mathcal{R}^{\infty,m} \to \mathbb{R} | f_{\bar{\xi}}(\cdot) = \langle \cdot, \bar{\xi} \rangle_m$ for $\bar{\xi} \in \mathcal{A}^{1,m}\}$. Here, the topology $\sigma(\mathcal{R}^{\infty,m}, \mathscr{F})$ denotes the topology on $\mathcal{R}^{\infty,m}$ defined by the base which is given by all sets of the form

$$
\{\bar{Y} \in \mathcal{R}^{\infty,m} || f_{\bar{\xi}^{(i)}}(\bar{Y}) - f_{\bar{\xi}^{(i)}}(\bar{X})| < \epsilon, i = 1, \dots, n\}
$$

for $n \in \mathbb{N}$, $\bar{X} \in \mathcal{R}^{\infty,m}$ and $f_{\bar{\xi}^{(i)}} \in \mathscr{F}$; see, for instance, Section V.3 in Dunford and Schwartz (1957). Lemma 2.53 in Aliprantis and Border (2006) states that $\mathfrak{T}^m_{\tau,\theta} = \sigma(\mathcal{R}^{\infty,m}_{\tau,\theta}, \mathscr{F}|_{\mathcal{R}^{\infty,m}_{\tau,\theta}})$ where $\mathscr{F}|_{\mathcal{R}^{\infty,m}_{\tau,\theta}} := \{f_{\bar{\xi}}|_{\mathcal{R}^{\infty,m}_{\tau,\theta}} : \mathcal{R}^{\infty,m}_{\tau,\theta} \to \mathbb{R} | f_{\bar{\xi}} \in \mathscr{F}\}$. Moreover, in the proof of the previous lemma we have verified that for each $\bar{\xi} \in \mathcal{A}^{1,m}$, we can find an element $\bar{\phi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ such that $\langle \bar{X}, \bar{\xi} \rangle_m = \langle \bar{X}, \bar{\phi} \rangle_m$ for all $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$. This implies $\mathscr{F}|_{\mathcal{R}^{\infty,m}_{\tau,\theta}} = \{g_{\bar{\xi}} : \mathcal{R}^{\infty,m}_{\tau,\theta} \to \mathbb{R} | g_{\bar{\xi}}(\cdot) = \langle \cdot, \bar{\xi} \rangle_m$ for $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}\}$. As a consequence, we have

$$\mathfrak{T}^m_{\tau,\theta} = \sigma(\mathcal{R}^{\infty,m}_{\tau,\theta}, \mathscr{F}|_{\mathcal{R}^{\infty,m}_{\tau,\theta}}) = \sigma(\mathcal{R}^{\infty,m}_{\tau,\theta}, \mathcal{A}^{1,m}_{\tau,\theta}).$$

Finally, the previous results are combined to the following lemma.

Lemma 7.3.11. *Let ρ_0 be a conditional convex single-firm risk measure and Λ be a convex aggregation function.*

1. *If ρ_0 is continuous for bounded increasing sequences, then the acceptance set $\mathcal{B}_{\rho_0}$ is $\sigma(\mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta}, \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta})$-closed.*
2. *The acceptance set $\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}_{\tau,\theta}, \mathcal{A}^{1,n+1}_{\tau,\theta})$-closed.*
3. *If ρ_0 is continuous for bounded increasing sequences, then the set $\mathcal{C}^{d,\bar{X}}_{\rho_0 \circ \Lambda}$ is $\sigma(\mathcal{R}^{\infty,n+3}_{\tau,\theta}, \mathcal{A}^{1,n+3}_{\tau,\theta})$-closed for all $d \in \mathbb{R}$ and $\bar{X} \in \mathcal{R}^{\infty}_{\tau,\theta}$.*

Proof. Note that the $\mathfrak{T}^m_{\tau,\theta}$-closed subsets $\mathcal{C}$ of $\mathcal{R}^{\infty,m}_{\tau,\theta}$ are the sets $\mathcal{C} = \mathcal{D} \cap \mathcal{R}^{\infty,m}_{\tau,\theta}$ where $\mathcal{D}$ is some $\sigma(\mathcal{R}^{\infty,m}, \mathcal{A}^{1,m})$-closed subset of $\mathcal{R}^{\infty,m}$; see, for instance, Section 2.1 in Aliprantis and Border (2006). Since $\mathcal{B}_{\rho_0} = \mathcal{B}_{\rho_0} \cap (\mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta})$, it follows from Lemma 7.3.6 that $\mathcal{B}_{\rho_0}$ is $\mathfrak{T}^2_{\tau,\theta}$-closed. Similarly, we obtain that $\mathcal{B}_\Lambda$ is $\mathfrak{T}^{n+1}_{\tau,\theta}$-closed and $\mathcal{C}^{d,\bar{X}}_{\rho_0 \circ \Lambda}$ is $\mathfrak{T}^{n+3}_{\tau,\theta}$-closed for all $d \in \mathbb{R}$ and $\bar{X} \in \mathcal{R}^{\infty}_{\tau,\theta}$. Now, all assertions follow from the previous remark. □

7.3.3. Dual representation

In order to formulate a dual representation result for conditional convex systemic risk measures, we have to generalize the definitions from Section 6.2 for m dimensions. Hence, define for $\bar{X} \in \mathcal{R}^{\infty,m}$ and $\bar{\xi} \in \mathcal{A}^{1,m}$

$$\langle \bar{X}, \bar{\xi} \rangle^{\tau,\theta}_m := \sum_{i=1}^m \langle \bar{X}^i, \bar{\xi}^i \rangle^{\tau,\theta} = \sum_{i=1}^m \mathbb{E}\left[\sum_{t \in [\tau,\theta] \cap \mathbb{N}_0} \bar{X}^i_t \Delta \bar{\xi}^i_t \middle| \mathcal{F}_\tau \right].$$

As in one-dimensional case, we obtain for $\bar{X} \in \mathcal{R}^{\infty,m}$ and $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$

$$\langle \bar{X}, \bar{\xi} \rangle_m = \sum_{i=1}^m \langle \bar{X}^i, \bar{\xi}^i \rangle = \sum_{i=1}^m \mathbb{E}[\langle \bar{X}^i, \bar{\xi}^i \rangle^{\tau,\theta}] = \mathbb{E}[\langle \bar{X}, \bar{\xi} \rangle^{\tau,\theta}_m].$$

Finally, consider the following natural extensions of the spaces $\mathcal{A}^1_+$ and $(\mathcal{A}^1_{\tau,\theta})_+$ for m dimensions

$$\mathcal{A}^{1,m}_+ := \{\bar{\xi} \in \mathcal{A}^{1,m} | \Delta \bar{\xi}^i_t \geq 0 \text{ for all } t \in \mathbb{N}_0, i \in \{1, \dots, m\}\}, \tag{7.9}$$

$$(\mathcal{A}^{1,m}_{\tau,\theta})_+ := p^{\tau,\theta}_m \mathcal{A}^{1,m}_+ \tag{7.10}$$

and define

$$\mathcal{E}_{\tau,\theta} := \{\xi \in (\mathcal{A}^1_{\tau,\theta})_+ | \langle 1, \xi \rangle^{\tau,\theta} \leq 1\}.$$

To prove the dual representation result, we need the following technical lemma. For a proof see, for instance, Detlefsen and Scandolo (2005).

Lemma 7.3.12. *Consider a set $\mathcal{Y} \subset L^0(\overline{\mathbb{R}})$ and suppose that $\mathcal{Y}$ is directed upwards, i.e., for all $\gamma, \gamma' \in \mathcal{Y}$, there exists $\gamma'' \in \mathcal{Y}$ with $\gamma'' \geq \gamma \vee \gamma'$. Then*

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{\gamma \in \mathcal{Y}} \gamma\right] = \sup_{\gamma \in \mathcal{Y}} \mathbb{E}\left[\gamma\right]$$

if the expectations exist (finite or infinite).

Theorem 7.3.13. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a conditional convex systemic risk measure characterized by a conditional convex single-firm risk measure ρ_0 and a convex aggregation function Λ. If ρ_0 is continuous for bounded increasing sequences, then ρ admits the representation*

$$\rho(\bar{X}) = \operatorname*{ess\,sup}_{(\xi,\bar{\xi}) \in \mathcal{E}_{\tau,\theta} \times (\mathcal{A}^{1,n}_{\tau,\theta})_+} \{\langle \bar{X}, \bar{\xi} \rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi, \bar{\xi})\} \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta} \tag{7.11}$$

where $\alpha^{\tau,\theta}_n : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \to L^0_\tau(\overline{\mathbb{R}}_+)$ is given by

$$\alpha^{\tau,\theta}_n(\xi, \bar{\xi}) = \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)}, Y) \in \mathcal{B}_{\rho_0}, (V, \bar{Z}) \in \mathcal{B}_\Lambda} \{-\gamma + \langle Y - V, \xi \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} \rangle^{\tau,\theta}_n\}. \tag{7.12}$$

If $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$, then a feasible solution to optimization problem (7.11) *additionally satisfies $\xi \in \mathcal{D}_{\tau,\theta}$.*

Remark. Note that unlike in the static case, we do not need an additional continuity requirement for Λ. In Chapter 5 we additionally claim L^p-continuity of $\Lambda : (L^p)^n \to L^p$. The reason for this difference is that in case of conditional convex systemic risk measures, we solely consider the space of bounded processes. In contrast to this, convex systemic risk measures are defined on $(L^p)^n$-spaces for all $1 \leq p \leq \infty$ in the static framework. Moreover, in case of convex systemic risk measures on $(L^\infty)^n$, we study the paired space $((L^\infty)^n, (ba)^n)$ where $(L^\infty)^n$ is endowed with the corresponding norm topology, and in this chapter we study the paired space $(\mathcal{R}^{\infty,n}, \mathcal{A}^{1,n})$ where $\mathcal{R}^{\infty,n}$ is endowed with the topology $\sigma(\mathcal{R}^{\infty,n}, \mathcal{A}^{1,n})$.

Proof. Fix $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$. *Part 1:* At first, we will verify the following equation:

$$\begin{aligned}\mathbb{E}[\rho(\bar{X})] &= \sup_{(\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}} \{\mathbb{E}[\langle \bar{X}, \bar{\xi} \rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi, \bar{\xi})]\} \\ &= \mathbb{E}\left[\operatorname*{ess\,sup}_{(\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}} \{\langle \bar{X}, \bar{\xi} \rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi, \bar{\xi})\}\right]. \end{aligned} \tag{7.13}$$

By the primal representation of ρ in Proposition 7.3.2, we have

$$\begin{aligned}\rho(\bar{X}) &= \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma I_{[\tau,\infty)}, Y) \in \mathcal{B}_{\rho_0}, (Y, \bar{X}) \in \mathcal{B}_\Lambda\} \\ &= \operatorname*{ess\,inf}_{(\gamma,Y)\in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}} \{\gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y, \bar{X})\}.\end{aligned}$$

Define the mapping $s_{\mathcal{B}_{\rho_0}} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta} \to L^0_\tau(\overline{\mathbb{R}})$ for the convex set $\mathcal{B}_{\rho_0}$ by

$$s_{\mathcal{B}_{\rho_0}}(\psi, \xi) := \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)}, Y)\in\mathcal{B}_{\rho_0}} \{\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y, \xi\rangle^{\tau,\theta}\}.$$

Consider the set $\mathcal{C} := \mathcal{C}^{\psi,\xi} := \{-\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y, \xi\rangle^{\tau,\theta} | (\gamma I_{[\tau,\infty)}, Y) \in \mathcal{B}_{\rho_0}\}$ and let $C := -\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y, \xi\rangle^{\tau,\theta}$ and $C' := -\langle \gamma' I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y', \xi\rangle^{\tau,\theta}$ be such that $C, C' \in \mathcal{C}$. Moreover, define $A := \{C \geq C'\} \in \mathcal{F}_\tau$. Then we have

$$CI_A + C'I_{A^c} = -\langle (\gamma I_A + \gamma' I_{A^c}) I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y I_A + Y' I_{A^c}, \xi\rangle^{\tau,\theta}$$

and $((\gamma I_A + \gamma' I_{A^c}) I_{[\tau,\infty)}, Y I_A + Y' I_{A^c}) = I_A(\gamma I_{[\tau,\infty)}, Y) + I_{A^c}(\gamma' I_{[\tau,\infty)}, Y') \in \mathcal{B}_{\rho_0}$ since $\mathcal{B}_{\rho_0}$ is convex. This means that $\mathcal{C}$ is directed upwards and that we can apply Lemma 7.3.12. Then we obtain a function $\tilde{s}_{\mathcal{B}_{\rho_0}} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta} \to \mathbb{R} \cup \{+\infty\}$ by setting $\tilde{s}_{\mathcal{B}_{\rho_0}}(\psi, \xi) := \mathbb{E}[s_{\mathcal{B}_{\rho_0}}(\psi, \xi)]$, and $\tilde{s}_{\mathcal{B}_{\rho_0}}$ satisfies

$$\begin{aligned}\tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi, \xi) &= \sup_{(\gamma I_{[\tau,\infty)}, Y)\in\mathcal{B}_{\rho_0}} \mathbb{E}[-\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y, \xi\rangle^{\tau,\theta}] \\ &= \sup_{(\gamma I_{[\tau,\infty)}, Y)\in\mathcal{B}_{\rho_0}} \{-\langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y, \xi\rangle\}\end{aligned}$$

for all $(\psi, \xi) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta}$. Furthermore, the convex conjugate of $\iota_{\mathcal{B}_{\rho_0}}$ is given by

$$\begin{aligned}\iota^*_{\mathcal{B}_{\rho_0}}(-\psi, \xi) &= \sup_{(X,Y)\in\mathcal{R}^\infty_{\tau,\theta}\times\mathcal{R}^\infty_{\tau,\theta}} \{-\langle X, \psi\rangle + \langle Y, \xi\rangle - \iota_{\mathcal{B}_{\rho_0}}(X, Y)\} \\ &= \sup_{(\gamma I_{[\tau,\infty)}, Y)\in\mathcal{B}_{\rho_0}} \{-\langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y, \xi\rangle\} = \tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi, \xi)\end{aligned}$$

for $(\psi, \xi) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta}$. Since ρ_0 is continuous for bounded increasing sequences, it follows from Lemma 7.3.11 that the acceptance set $\mathcal{B}_{\rho_0}$ is $\sigma(\mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta}, \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta})$-closed. This implies $\sigma(\mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta}, \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta})$-closedness of the function $\iota_{\mathcal{B}_{\rho_0}}$. Furthermore, the duality theorem for conjugate functions (see Theorem A.2.9) leads to

$$\begin{aligned}\iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) &= \iota^{**}_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) \\ &= \sup_{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}} \{\langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y, \xi\rangle - \iota^*_{\mathcal{B}_{\rho_0}}(\psi, \xi)\} \\ &= \sup_{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}} \{-\langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y, \xi\rangle - \tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi, \xi)\} \qquad (7.14)\end{aligned}$$

for $(\gamma, Y) \in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}$.

Similarly, we can define the mapping $s_{\mathcal{B}_\Lambda} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \to L^0_\tau(\overline{\mathbb{R}})$ for the convex set $\mathcal{B}_\Lambda$ by

$$s_{\mathcal{B}_\Lambda}(\phi, \bar{\xi}) := \operatorname*{ess\,sup}_{(Y,\bar{Z})\in\mathcal{B}_\Lambda} \left\{ \langle Y, \phi \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} \rangle^{\tau,\theta}_n \right\}$$

and show that the set $\mathcal{M}^{\phi,\bar{\xi}} := \{ -\langle Y, \phi \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} \rangle^{\tau,\theta}_n | (Y, \bar{Z}) \in \mathcal{B}_\Lambda \}$ is directed upwards. Then the function $\tilde{s}_{\mathcal{B}_\Lambda} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \to \mathbb{R} \cup \{+\infty\}$ defined by $\tilde{s}_{\mathcal{B}_\Lambda}(\phi, \bar{\xi}) := \mathbb{E}[s_{\mathcal{B}_\Lambda}(\phi, \bar{\xi})]$ satisfies

$$\tilde{s}_{\mathcal{B}_\Lambda}(-\phi, \bar{\xi}) = \sup_{(Y,\bar{Z})\in\mathcal{B}_\Lambda} \mathbb{E}[-\langle Y, \phi \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} \rangle^{\tau,\theta}_n] = \sup_{(Y,\bar{Z})\in\mathcal{B}_\Lambda} \{ -\langle Y, \phi \rangle + \langle \bar{Z}, \bar{\xi} \rangle_n \}$$

and

$$\iota^*_{\mathcal{B}_\Lambda}(-\phi, \bar{\xi}) = \tilde{s}_{\mathcal{B}_\Lambda}(-\phi, \bar{\xi})$$

for $(\phi, \bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}$. According to Lemma 7.3.11, the acceptance set $\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}_{\tau,\theta}, \mathcal{A}^{1,n+1}_{\tau,\theta})$-closed. It follows directly that the function $\iota_{\mathcal{B}_\Lambda}$ is closed with respect to the topology $\sigma(\mathcal{R}^{\infty,n+1}_{\tau,\theta}, \mathcal{A}^{1,n+1}_{\tau,\theta})$, and by the duality theorem for conjugate functions, we obtain

$$\begin{aligned}
\iota_{\mathcal{B}_\Lambda}(Y, \bar{X}) &= \iota^{**}_{\mathcal{B}_\Lambda}(Y, \bar{X}) \\
&= \sup_{(\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} \{ \langle Y, \phi \rangle + \langle \bar{X}, \bar{\xi} \rangle_n - \iota^*_{\mathcal{B}_\Lambda}(\phi, \bar{\xi}) \} \\
&= \sup_{(\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} \{ -\langle Y, \phi \rangle + \langle \bar{X}, \bar{\xi} \rangle_n - \tilde{s}_{\mathcal{B}_\Lambda}(-\phi, \bar{\xi}) \} \qquad (7.15)
\end{aligned}$$

for $(Y, \bar{X}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$. With Lemma 7.3.15, Lemma 7.3.12 and Equations (7.14) and (7.15) it follows that

$$\begin{aligned}
\mathbb{E}[\rho(\bar{X})] &= \mathbb{E}\left[\operatorname*{ess\,inf}_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}} \{ \gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y, \bar{X}) \right] \\
&= \inf_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}} \mathbb{E}\left[\gamma + \sup_{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}} \{ -\langle \gamma I_{[\tau,\infty)}, \psi \rangle + \langle Y, \xi \rangle - \tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi, \xi) \} \right. \\
&\quad \left. + \sup_{(\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} \{ -\langle Y, \phi \rangle + \langle \bar{X}, \bar{\xi} \rangle_n - \tilde{s}_{\mathcal{B}_\Lambda}(-\phi, \bar{\xi}) \} \right] \\
&= \inf_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}} \sup_{\substack{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}, \\ (\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}} \Big\{ \mathbb{E}[\gamma] - \langle \gamma I_{[\tau,\infty)}, \psi \rangle + \langle Y, \xi \rangle - \langle Y, \phi \rangle \\
&\quad + \langle \bar{X}, \bar{\xi} \rangle_n - \tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi, \xi) - \tilde{s}_{\mathcal{B}_\Lambda}(-\phi, \bar{\xi})] \Big\}. \qquad (7.16)
\end{aligned}$$

Now, Lemma 7.3.14 yields

$$\begin{aligned}
\mathbb{E}[\rho(\bar{X})] &= \sup_{\substack{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta},\\ (\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}} \inf_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}} \Big\{\mathbb{E}\left[\gamma\right] - \langle \gamma I_{[\tau,\infty)},\psi\rangle + \langle Y,\xi\rangle - \langle Y,\phi\rangle \\
&\qquad + \langle \bar{X},\bar{\xi}\rangle_n - \tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - \tilde{s}_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})]\Big\} \\
&= \sup_{\substack{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta},\\ (\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}} \inf_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}} \Big\{\mathbb{E}[\gamma(1-\langle 1,\psi\rangle^{\tau,\theta})] + \langle Y,\xi-\phi\rangle \\
&\qquad + \mathbb{E}\big[\langle \bar{X},\bar{\xi}\rangle^{\tau,\theta}_n - s_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - s_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})]\Big\} \\
&= \sup_{\substack{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta},\langle 1,\psi\rangle^{\tau,\theta}=1\\ (\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}} \Big\{\mathbb{E}\big[\langle \bar{X},\bar{\xi}\rangle^{\tau,\theta}_n - s_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - s_{\mathcal{B}_\Lambda}(-\xi,\bar{\xi})]\Big\}.
\end{aligned} \tag{7.17}$$

The last equation can be verified as follows: First, assume that $A := \{\langle 1,\psi\rangle^{\tau,\theta} > 1\} \in \mathcal{F}_\tau$ satisfies $\mathbb{P}[A] > 0$. Then $\gamma^{(m)}$ defined by $\gamma^{(m)} := mI_A$ satisfies $\gamma^{(m)} \in L^\infty_\tau$ for all $m \in \mathbb{N}$ and $\mathbb{E}[\gamma^{(m)}(1-\langle 1,\psi\rangle^{\tau,\theta})] < 0$. As a consequence, we have $\lim_{m\to\infty}\mathbb{E}[\gamma^{(m)}(1-\langle 1,\psi\rangle^{\tau,\theta})] = -\infty$. The same argumentation applies if $A := \{\langle 1,\psi\rangle^{\tau,\theta} < 1\}$. Hence, it suffices to consider $\psi \in \mathcal{A}^1_{\tau,\theta}$ with $\langle 1,\psi\rangle^{\tau,\theta} = 1$ in the supremum.
Now, let us assume that there exists $s \in \mathbb{N}_0$ such that $A_s := \{\Delta\xi_s < \Delta\phi_s\} \in \mathcal{F}_s$ satisfies $\mathbb{P}[A] > 0$ and define $Y^{(m)} \in \mathcal{R}^\infty_{\tau,\theta}$ by $Y^{(m)}_t := mI_{A_s}I_{\{t=s\}}$ for $t \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then

$$\langle Y^{(m)},\xi-\phi\rangle = \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} Y^{(m)}_t\Delta(\xi_t-\phi_t)\right] = \mathbb{E}[mI_A\Delta(\xi_s-\phi_s)] < 0,$$

which implies that $\lim_{m\to\infty}\langle Y^{(m)},\xi-\phi\rangle = -\infty$. With the same argumentation for $A := \{\Delta\xi_s > \Delta\phi_s\}$ it follows that we only have to consider $(\psi,\xi) \in \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}$ such that $\Delta\xi_t = \Delta\phi_t$ for all $t \in \mathbb{N}_0$. Hence, $\xi_t = \phi_t$ for all $t \in \mathbb{N}_0$.

For $\psi \in \mathcal{A}^1_{\tau,\theta}$ with $\langle 1,\psi\rangle^{\tau,\theta} = 1$ and $(\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}$, we obtain

$$\begin{aligned}
& s_{\mathcal{B}_{\rho_0}}(-\psi,\xi) + s_{\mathcal{B}_\Lambda}(-\xi,\bar{\xi}) \\
&= \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0}} \{-\gamma\langle 1,\psi\rangle^{\tau,\theta} + \langle Y,\xi\rangle^{\tau,\theta}\} + \operatorname*{ess\,sup}_{(V,\bar{Z})\in\mathcal{B}_\Lambda} \{-\langle V,\xi\rangle^{\tau,\theta} + \langle \bar{Z},\bar{\xi}\rangle^{\tau,\theta}_n\} \\
&= \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda} \{-\gamma + \langle Y-V,\xi\rangle^{\tau,\theta} + \langle \bar{Z},\bar{\xi}\rangle^{\tau,\theta}_n\} \\
&= \alpha^{\tau,\theta}_n(\xi,\bar{\xi}).
\end{aligned}$$

Thus, (7.13) follows from Equation (7.17) and an application of Lemma 7.3.16 and Lemma 7.3.12.

Part 2: Now, we will show that

$$\begin{aligned}\mathbb{E}[\rho(\bar{X})] &= \sup_{(\xi,\bar{\xi})\in\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+}\{\mathbb{E}[\langle\bar{X},\bar{\xi}\rangle_n^{\tau,\theta}-\alpha_n^{\tau,\theta}(\xi,\bar{\xi})]\}\\ &= \mathbb{E}\left[\operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+}\{\langle\bar{X},\bar{\xi}\rangle_n^{\tau,\theta}-\alpha_n^{\tau,\theta}(\xi,\bar{\xi})\}\right].\end{aligned}$$

First, note that for every $(\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}$, we have

$$\begin{aligned}\mathbb{E}[\alpha_n^{\tau,\theta}(\xi,\bar{\xi})] &= \mathbb{E}\left[\operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda}\{-\gamma+\langle Y-V,\xi\rangle^{\tau,\theta}+\langle\bar{Z},\bar{\xi}\rangle_n^{\tau,\theta}\}\right]\\ &= \sup_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda}\mathbb{E}[-\gamma+\langle Y-V,\xi\rangle^{\tau,\theta}+\langle\bar{Z},\bar{\xi}\rangle_n^{\tau,\theta}]\\ &= \sup_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda}\{-\mathbb{E}[\gamma]+\langle Y-V,\xi\rangle+\langle\bar{Z},\bar{\xi}\rangle_n\}\end{aligned}$$

because the set $\{-\gamma+\langle Y-V,\xi\rangle^{\tau,\theta}+\langle\bar{Z},\bar{\xi}\rangle_n^{\tau,\theta}|(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda\}$ is directed upwards.

Assume that there exists $s\in\mathbb{N}_0$ such that $\Delta\xi_s<0$ on $A_s\in\mathcal{F}_s$ with $\mathbb{P}[A_s]>0$. We will show that this implies $\mathbb{E}[\alpha_n^{\tau,\theta}(\xi,\bar{\xi})]=\infty$: The constancy property of ρ_0 yields $\rho_0(0)=0$, and hence $(0,0)\in\mathcal{B}_{\rho_0}$. Define $Y^{(m)}\in\mathcal{R}^\infty_{\tau,\theta}$ by $Y_t^{(m)}:=-mI_{A_s}I_{\{t=s\}}$ for $t\in\mathbb{N}_0$ and $m\in\mathbb{N}$. Then $0\geq Y^{(m)}$, and $(0,Y^{(m)})\in\mathcal{B}_{\rho_0}$ for every $m\in\mathbb{N}$ due to the monotonicity property of $\mathcal{B}_{\rho_0}$. Moreover, we have

$$\langle Y^{(m)},\xi\rangle=\mathbb{E}\left[\sum_{t\in\mathbb{N}_0}Y_t^{(m)}\Delta\xi_t\right]=\mathbb{E}\left[-mI_{A_s}\Delta\xi_s\right]>0,$$

which implies $\lim_{m\to\infty}\langle Y^{(m)},\xi\rangle=\infty$. Together, we obtain $\mathbb{E}[\alpha_n^{\tau,\theta}(\xi,\bar{\xi})]=\infty$. In conclusion, we only have to consider $\xi\in(\mathcal{A}^1_{\tau,\theta})_+$.

The same argumentation applies if we suppose that there exists $s\in\mathbb{N}_0$ and $j\in\{1,\ldots,n\}$ such that $\Delta\bar{\xi}^j_s<0$ on $A_s\in\mathcal{F}_s$ with $\mathbb{P}[A_s]>0$. Then monotonicity of the acceptance set $\mathcal{B}_\Lambda$ implies $\mathbb{E}[\alpha_n^{\tau,\theta}(\xi,\bar{\xi})]=\infty$, such that we only have to consider $\bar{\xi}\in(\mathcal{A}^{1,n}_{\tau,\theta})_+$.

Now, let $\xi\in(\mathcal{A}^1_{\tau,\theta})_+$ be such that $-1+\langle 1,\xi\rangle^{\tau,\theta}>0$ on $A\in\mathcal{F}_\tau$ with $\mathbb{P}[A]>0$. Since $(\lambda I_{[\tau,\infty)},\lambda I_{[\tau,\infty)})\in\mathcal{B}_{\rho_0}$ for all $\lambda\geq 0$ and since $\mathcal{B}_{\rho_0}$ is $\mathcal{F}_\tau$-convex, we obtain $I_A(\lambda I_{[\tau,\infty)},\lambda I_{[\tau,\infty)})+I_{A^c}(0,0)=I_A(\lambda I_{[\tau,\infty)},\lambda I_{[\tau,\infty)})\in\mathcal{B}_{\rho_0}$. Moreover, we have for all $\lambda>0$

$$\begin{aligned}&-\mathbb{E}[\lambda I_A]+\langle\lambda I_A I_{[\tau,\infty)},\xi\rangle=-\mathbb{E}[\lambda I_A]+\mathbb{E}[\langle\lambda I_A I_{[\tau,\infty)},\xi\rangle^{\tau,\theta}]\\ &=\lambda\mathbb{E}[(-1+\langle 1,\xi\rangle^{\tau,\theta})I_A]>0,\end{aligned}$$

and $\lim_{\lambda\to\infty}(-\mathbb{E}[\lambda I_A] + \langle \lambda I_A I_{[\tau,\infty)}, \xi\rangle) = \infty$ implies again that $\mathbb{E}[\alpha_n^{\tau,\theta}(\xi,\bar{\xi})] = \infty$. Hence, $\xi \in \mathcal{E}_{\tau,\theta}$.

Finally, assume that $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$. It remains to show that $\langle 1, \xi\rangle^{\tau,\theta} \geq 1$. Consider $\xi \in \mathcal{E}_{\tau,\theta}$ with $1 - \langle 1,\xi\rangle^{\tau,\theta} > 0$ on $A \in \mathcal{F}_\tau$ with $\mathbb{P}[A] > 0$. Because $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$, we know that $\rho_0(-\lambda I_{[\tau,\infty)}) = -\lambda$ for all $\lambda \geq 0$. Thus, $(-\lambda I_{[\tau,\infty)}, -\lambda I_{[\tau,\infty)}) \in \mathcal{B}_{\rho_0}$ for all $\lambda \geq 0$. $\mathcal{F}_\tau$-convexity of $\mathcal{B}_{\rho_0}$ yields $I_A(-\lambda I_{[\tau,\infty)}, -\lambda I_{[\tau,\infty)}) \in \mathcal{B}_{\rho_0}$ for all $\lambda > 0$. Furthermore,

$$-\mathbb{E}[(-\lambda)I_A] + \langle (-\lambda) I_A I_{[\tau,\infty)}, \xi\rangle = \lambda \mathbb{E}[(1 - \langle 1,\xi\rangle^{\tau,\theta}) I_A] > 0$$

for all $\lambda > 0$ and $\lim_{\lambda\to\infty}(-\mathbb{E}[(-\lambda)I_A] + \langle (-\lambda)I_A I_{[\tau,\infty)}, \xi\rangle) = \infty$. Therefore, we obtain $\mathbb{E}[\alpha_n^{\tau,\theta}(\xi,\bar{\xi})] = \infty$. Altogether, we arrive at $\langle 1,\xi\rangle^{\tau,\theta} = 1$.

Part 3: It remains to prove that

$$\rho(\bar{X}) = \operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+} \{\langle \bar{X},\bar{\xi}\rangle_n^{\tau,\theta} - \alpha_n^{\tau,\theta}(\xi,\bar{\xi})\}.$$

Let us begin with " $\geq$ ": For $(\xi,\bar{\xi}) \in \mathcal{E}_{\tau,\theta} \times (\mathcal{A}^{1,n}_{\tau,\theta})_+$, we have

$$\begin{aligned}
&\alpha_n^{\tau,\theta}(\xi,\bar{\xi})\\
&= \operatorname*{ess\,sup}_{\substack{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta},\\ (V,\bar{Z})\in\mathcal{R}^\infty_{\tau,\theta}\times\mathcal{R}^{\infty,n}_{\tau,\theta}}} \{-\gamma + \langle Y - V,\xi\rangle^{\tau,\theta} + \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) - \iota_{\mathcal{B}_\Lambda}(V,\bar{Z})\}\\
&\geq \operatorname*{ess\,sup}_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta},\, \bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \{-\gamma + \langle Y - Y,\xi\rangle^{\tau,\theta} + \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y)\\
&\quad - \iota_{\mathcal{B}_\Lambda}(Y,\bar{Z})\}\\
&= \operatorname*{ess\,sup}_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta},\, \bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \{-\gamma + \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) - \iota_{\mathcal{B}_\Lambda}(Y,\bar{Z})\}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \{\langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \rho(\bar{Z})\}\\
&= \operatorname*{ess\,sup}_{\bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \left\{\langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} + \operatorname*{ess\,sup}_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}} \{-\gamma - \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) - \iota_{\mathcal{B}_\Lambda}(Y,\bar{Z})\}\right\}\\
&= \operatorname*{ess\,sup}_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta},\, \bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \{-\gamma + \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) - \iota_{\mathcal{B}_\Lambda}(Y,\bar{Z})\}.
\end{aligned}$$

Together, we obtain $\alpha_n^{\tau,\theta}(\xi,\bar{\xi}) \geq \operatorname{ess\,sup}_{\bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \{\langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \rho(\bar{Z})\}$. This inequality implies $\alpha_n^{\tau,\theta}(\xi,\bar{\xi}) \geq \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \rho(\bar{Z})$, i.e., $\rho(\bar{Z}) \geq \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta} - \alpha_n^{\tau,\theta}(\xi,\bar{\xi})$ for all $\bar{Z} \in \mathcal{R}^\infty_{\tau,\theta}$ and $(\xi,\bar{\xi}) \in \mathcal{E}_{\tau,\theta} \times (\mathcal{A}^{1,n}_{\tau,\theta})_+$. Therefore, we have

$$\rho(\bar{X}) \geq \operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+} \{\langle \bar{X},\bar{\xi}\rangle_n^{\tau,\theta} - \alpha_n^{\tau,\theta}(\xi,\bar{\xi})\}. \tag{7.18}$$

Since the second part of this proof states that the expectations of both sides of (7.18) are equal, we get equality in (7.18). $\square$

Lemma 7.3.14. *Suppose that the requirements from Theorem 7.3.13 are satisfied. Define* $\mathcal{X} := \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta}$, $\mathcal{X}' := \mathcal{A}^{1}_{\tau,\theta} \times \mathcal{A}^{1}_{\tau,\theta}$, $\mathcal{U} := \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$ *and* $\mathcal{U}' := \mathcal{A}^{1}_{\tau,\theta} \times \mathcal{A}^{1}_{\tau,\theta} \times \mathcal{A}^{1}_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}$ *and consider the paired spaces* $(\mathcal{X},\mathcal{X}')$, $(\mathcal{U},\mathcal{U}')$. *Let* $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ *and* $K : \mathcal{X} \times \mathcal{U}' \to \overline{\mathbb{R}}$ *be defined by*

$$K((X,Y),(\psi,\xi,\phi,\bar{\xi}))$$
$$:= \begin{cases} \Big[\mathbb{E}[\gamma] - \langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y,\xi\rangle - \langle Y,\phi\rangle + \langle \bar{X},\bar{\xi}\rangle_n \\ -\tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - \tilde{s}_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})\Big] & \text{if } X = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^\infty_\tau \\ \infty & \text{else} \end{cases}.$$

Then we have

$$\inf_{(\gamma,Y)\in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}} \sup_{(\psi,\xi,\phi,\bar{\xi})\in\mathcal{U}'} K((\gamma I_{[\tau,\infty)},Y),(\psi,\xi,\phi,\bar{\xi}))$$
$$= \sup_{(\psi,\xi,\phi,\bar{\xi})\in\mathcal{U}'} \inf_{(\gamma,Y)\in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}} K((\gamma I_{[\tau,\infty)},Y),(\psi,\xi,\phi,\bar{\xi})).$$

Proof. The proof of this lemma is similar to the proof of Lemma 5.4.6. Fix $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ and note that K is concave in the second argument since $\tilde{s}_{\mathcal{B}_{\rho_0}} = \iota^*_{\mathcal{B}_{\rho_0}}$ and $\tilde{s}_{\mathcal{B}_\Lambda} = I^*_{\mathcal{B}_\Lambda}$ are convex; see, for instance, Theorem 5 in Rockafellar (1974) (see Theorem A.2.9). Moreover, $-K((X,Y),\cdot)$ is $\sigma(\mathcal{U}',\mathcal{U})$-closed for each $(X,Y) \in \mathcal{X}$: If $-K((X,Y),\cdot) = -\infty$, this holds by definition. Otherwise, we have $-K((X,Y),\cdot) = -K((\gamma I_{[\tau,\infty)},Y),\cdot) > -\infty$ for some $\gamma \in L^\infty_\tau$. In this case, the function

$$(\psi,\xi,\phi,\bar{\xi}) \mapsto \mathbb{E}[\gamma] - \langle \gamma I_{[\tau,\infty)},\psi\rangle + \langle Y,\xi\rangle - \langle Y,\phi\rangle + \langle \bar{X},\bar{\xi}\rangle_n$$

is $\sigma(\mathcal{U}',\mathcal{U})$-continuous. Moreover, according to Theorem 5 in Rockafellar (1974), the function $\tilde{s}_{\mathcal{B}_{\rho_0}} = \iota^*_{\mathcal{B}_{\rho_0}}$ is $\sigma(\mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta}, \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta})$-closed. The properness of $\tilde{s}_{\mathcal{B}_{\rho_0}}$ (we have $\tilde{s}_{\mathcal{B}_{\rho_0}}(0,0) = 0$) implies that $\tilde{s}_{\mathcal{B}_{\rho_0}}$ is $\sigma(\mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta}, \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta})$-l.s.c. Similarly, it follows that $\tilde{s}_{\mathcal{B}_\Lambda} = \iota^*_{\mathcal{B}_\Lambda}$ is $\sigma(\mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}, \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta})$-l.s.c. Altogether, we obtain that $-K((\gamma I_{[\tau,\infty)},Y),\cdot)$ is $\sigma(\mathcal{U}',\mathcal{U})$-l.s.c. Therefore, $-K((\gamma I_{[\tau,\infty)},Y),\cdot)$ is $\sigma(\mathcal{U}',\mathcal{U})$-closed.

By Theorem 6 in Rockafellar (1974) (see Theorem A.2.12), we know that K is the Lagrangian of the minimization problem "minimize f over $\mathcal{X}$" where f is given by $f(W,Y) = F((W,Y),0_{n+3})$ for $F : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ defined by

$$F((W,Y),(V,X,Z,\bar{Z}))$$
$$:= \sup_{(\psi,\xi,\phi,\bar{\xi})\in\mathcal{U}'} \{K((W,Y),(\psi,\xi,\phi,\bar{\xi})) - \langle V,\psi\rangle - \langle X,\xi\rangle - \langle Z,\phi\rangle - \langle \bar{Z},\bar{\xi}\rangle_n\}$$
$$= \begin{cases} \Big[\sup_{(\psi,\xi,\phi,\bar{\xi})\in\mathcal{U}'}\{\mathbb{E}[\gamma] - \langle \gamma I_{[\tau,\infty)},\psi\rangle + \langle Y,\xi\rangle \\ -\langle Y,\phi\rangle + \langle \bar{X},\bar{\xi}\rangle_n - \tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - \tilde{s}_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi}) \\ -\langle V,\psi\rangle - \langle X,\xi\rangle - \langle Z,\phi\rangle - \langle \bar{Z},\bar{\xi}\rangle_n\}\Big] & \text{if } W = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^\infty_\tau \\ \infty & \text{else} \end{cases}.$$

Moreover, Theorem 6 in Rockafellar (1974) states that $F((W,Y),\cdot)$ is $\sigma(\mathcal{U},\mathcal{U}')$-closed and convex. For $(W,Y)\in\mathcal{X}$ with $W=\gamma I_{[\tau,\infty)}$ for some $\gamma\in L^\infty_\tau$ and $(V,X,Z,\bar{Z})\in\mathcal{U}$, the function F satisfies

$$\begin{aligned}
&F((W,Y),(V,X,Z,\bar{Z}))\\
&=\mathbb{E}[\gamma]+\sup_{(\psi,\xi,\phi,\bar{\xi})\in\mathcal{U}'}\{-\langle\gamma I_{[\tau,\infty)}+V,\psi\rangle+\langle Y-X,\xi\rangle-\langle Y+Z,\phi\rangle\\
&\quad+\langle\bar{X}-\bar{Z},\bar{\xi}\rangle_n-\tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi,\xi)-\tilde{s}_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})\}\\
&=\mathbb{E}[\gamma]+\sup_{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}}\{-\langle\gamma I_{[\tau,\infty)}+V,\psi\rangle+\langle Y-X,\xi\rangle-\tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi,\xi)\}\\
&\quad+\sup_{(\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}\{-\langle Y+Z,\phi\rangle+\langle\bar{X}-\bar{Z},\bar{\xi}\rangle^{\tau,\theta}_n-\tilde{s}_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})\}\\
&=\mathbb{E}[\gamma]+\iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}+V,Y-X)+\iota_{\mathcal{B}_\Lambda}(Y+Z,\bar{X}-\bar{Z}).
\end{aligned}$$

Hence, F is convex in both arguments. Now, define the convex function φ by

$$\begin{aligned}
&\varphi(V,X,Z,\bar{Z})\\
&:=\inf_{(W,Y)\in\mathcal{X}}F((W,Y),(V,X,Z,\bar{Z}))\\
&=\inf_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}}\{\mathbb{E}[\gamma]+\iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}+V,Y-X)+\iota_{\mathcal{B}_\Lambda}(Y+Z,\bar{X}-\bar{Z})\}
\end{aligned}$$

for $(V,X,Z,\bar{Z})\in\mathcal{U}$. The definition of $\mathcal{B}_{\rho_0}$ implies $\varphi(V,X,Z,\bar{Z})=\infty$ if $V\neq\phi I_{[\tau,\infty)}$ for all $\phi\in L^\infty_\tau$. On the other hand, if $V=\phi I_{[\tau,\infty)}$ for $\phi\in L^\infty_\tau$, then

$$\begin{aligned}
&\varphi(\phi I_{[\tau,\infty)},X,Z,\bar{Z})\\
&=\inf_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}}\{\mathbb{E}[\gamma]+\iota_{\mathcal{B}_{\rho_0}}((\gamma+\phi)I_{[\tau,\infty)},Y-X)+\iota_{\mathcal{B}_\Lambda}(Y+Z,\bar{X}-\bar{Z})\}\\
&=\inf_{\gamma\in L^\infty_\tau}\{\mathbb{E}[\gamma]+\iota_{\mathcal{B}_{\rho_0}}((\gamma+\phi)I_{[\tau,\infty)},\Lambda(\bar{X}-\bar{Z})-X-Z)\}\\
&=\mathbb{E}[\rho_0(\Lambda(\bar{X}-\bar{Z})-X-Z)]-\mathbb{E}[\phi].
\end{aligned}$$

Therefore, φ satisfies

$$\varphi(V,X,Z,\bar{Z})=\begin{cases}\mathbb{E}[\rho_0(\Lambda(\bar{X}-\bar{Z})-X-Z)]-\mathbb{E}[\phi] & \text{if } V=\phi I_{[\tau,\infty)} \text{ for } \phi\in L^\infty_\tau\\ \infty & \text{else}\end{cases}. \tag{7.19}$$

Since ρ_0 maps into L^∞_τ and $\phi\in L^\infty_\tau$, this means that $\varphi(\phi I_{[\tau,\infty)},X,Z,\bar{Z})<\infty$. Hence, φ is proper. If we can show that φ is $\sigma(\mathcal{U},\mathcal{U}')$-l.s.c., then the assertion follows from Theorem 7 in Rockafellar (1974) (see Theorem A.2.13). To prove that φ is $\sigma(\mathcal{U},\mathcal{U}')$-l.s.c., we have to show that the set $\{(V,X,Z,\bar{Z})\in\mathcal{U}|\varphi(V,X,Z,\bar{Z})\leq d\}$ is $\sigma(\mathcal{U},\mathcal{U}')$-closed for all $d\in\mathbb{R}$. Equation (7.19) yields for every $d\in\mathbb{R}$

$$\begin{aligned}
&\{(V,X,Z,\bar{Z})\in\mathcal{U}|\varphi(V,X,Z,\bar{Z})\leq d\}\\
&=\{(V,X,Z,\bar{Z})\in\mathcal{U}|V=\phi I_{[\tau,\infty)} \text{ for some } \phi\in L^\infty_\tau \text{ and}\\
&\quad\mathbb{E}[\rho_0(\Lambda(\bar{X}-\bar{Z})-X-Z)]-\mathbb{E}[\phi]\leq d\}\\
&=\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda}
\end{aligned}$$

where the set $\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda}$ is defined in Lemma 7.3.8. $\sigma(\mathcal{U},\mathcal{U}')$-closedness of $\mathcal{C}^{d,\bar{X}}_{\rho_0\circ\Lambda}$ follows immediately from Lemma 7.3.11. □

Lemma 7.3.15. *For each $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$, the set*

$$\mathcal{N} := \{\gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y, \bar{X}) | \, (\gamma, Y) \in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}\}$$

is directed downwards.

Proof. Fix $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ and let $N, N' \in \mathcal{N}$ with $N := \gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y, \bar{X})$ and $N' = \gamma' + \iota_{\mathcal{B}_{\rho_0}}(\gamma' I_{[\tau,\infty)}, Y') + \iota_{\mathcal{B}_\Lambda}(Y', \bar{X})$. In order to show that $\mathcal{N}$ is directed downwards, we have to find an $N'' = \gamma'' + \iota_{\mathcal{B}_{\rho_0}}(\gamma'' I_{[\tau,\infty)}, Y'') + \iota_{\mathcal{B}_\Lambda}(Y'', \bar{X}) \in \mathcal{N}$ such that $N'' \leq N \wedge N'$. We distinguish between four different cases: If $N = N' = \infty$, set $Y'' = \Lambda(\bar{X})$ and $\gamma'' = \rho_0(Y'')$. If $N = \infty$ and $N' < \infty$ [$N' = \infty$ and $N < \infty$], then define $N'' = N'$ [$N'' = N$]. At last, consider the case in which $N, N' < \infty$: Since $N, N' < \infty$, we have $(\gamma I_{[\tau,\infty)}, Y), (\gamma' I_{[\tau,\infty)}, Y') \in \mathcal{B}_{\rho_0}$ and $(Y, \bar{X}), (Y', \bar{X}) \in \mathcal{B}_\Lambda$. Define the set $A := \{N \geq N'\} \in \mathcal{F}_\tau$. Then

$$I_A \iota_{\mathcal{B}_{\rho_0}}(\gamma' I_{[\tau,\infty)}, Y') + I_{A^c} \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) = 0$$

and $\iota_{\mathcal{B}_{\rho_0}}(I_A(\gamma' I_{[\tau,\infty)}, Y') + I_{A^c}(\gamma I_{[\tau,\infty)}, Y)) = 0$ since $\mathcal{B}_{\rho_0}$ is $\mathcal{F}_\tau$-convex. Similarly, we obtain

$$I_A \iota_{\mathcal{B}_\Lambda}(Y', \bar{X}) + I_{A^c} \iota_{\mathcal{B}_\Lambda}(Y, \bar{X}) = 0$$

and $\iota_{\mathcal{B}_\Lambda}(I_A(Y', \bar{X}) + I_{A^c}(Y, \bar{X})) = 0$. This yields

$$\begin{aligned}
&N' I_A + N I_{A^c} \\
&= (\gamma' + \iota_{\mathcal{B}_{\rho_0}}(\gamma' I_{[\tau,\infty)}, Y') + \iota_{\mathcal{B}_\Lambda}(Y', \bar{X})) I_A + (\gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y, \bar{X})) I_{A^c} \\
&= \gamma' I_A + \gamma I_{A^c} \\
&= (\gamma' I_A + \gamma I_{A^c}) + \iota_{\mathcal{B}_{\rho_0}}((\gamma' I_A + \gamma I_{A^c}) I_{[\tau,\infty)}, Y' I_A + Y I_{A^c}) + \iota_{\mathcal{B}_\Lambda}(Y' I_A + Y I_{A^c}, \bar{X}).
\end{aligned}$$

Because $(\gamma' I_A + \gamma I_{A^c}, Y' I_A + Y I_{A^c}) \in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}$, it follows that $\mathcal{N}$ is directed downwards. □

Lemma 7.3.16. *For each $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$, the set*

$$\mathcal{M} := \{\langle \bar{X}, \bar{\xi} \rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi, \bar{\xi}) | (\xi, \bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}\}$$

is directed upwards.

Proof. Fix $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ and let $M := \langle \bar{X}, \bar{\xi} \rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi, \bar{\xi})$ and $M' := \langle \bar{X}, \bar{\phi} \rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\phi, \bar{\phi})$ be such that $M, M' \in \mathcal{M}$. Furthermore, define $A := \{M \geq M'\} \in \mathcal{F}_\tau$. Then $M'' := M I_A + M' I_{A^c} \geq M \vee M'$ and

$$\begin{aligned}
M'' = &\langle \bar{X}, \bar{\xi} I_A + \bar{\phi} I_{A^c} \rangle^{\tau,\theta}_n \\
&- \operatorname*{ess\,sup}_{(\gamma,Y)\in\mathcal{B}_{\rho_0},(\hat{\gamma},\hat{Y})\in\mathcal{B}_{\rho_0}} \{-\gamma I_A - \hat{\gamma} I_{A^c} + \langle Y, \xi I_A \rangle^{\tau,\theta} + \langle \hat{Y}, \phi I_{A^c} \rangle^{\tau,\theta}\} \\
&- \operatorname*{ess\,sup}_{(V,\bar{Z})\in\mathcal{B}_\Lambda,(\hat{V},\hat{\bar{Z}})\in\mathcal{B}_\Lambda} \{-\langle V, \xi I_A \rangle^{\tau,\theta} - \langle \hat{V}, \phi I_{A^c} \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} I_A \rangle^{\tau,\theta}_n + \langle \hat{\bar{Z}}, \bar{\phi} I_{A^c} \rangle^{\tau,\theta}_n\}.
\end{aligned}$$

Moreover, it is easily seen that

$$\langle YI_A + \hat{Y}I_{A^c}, \xi I_A + \phi I_{A^c}\rangle^{\tau,\theta} = \langle YI_A, \xi I_A\rangle^{\tau,\theta} + \langle \hat{Y}I_{A^c}, \phi I_{A^c}\rangle^{\tau,\theta} \quad \text{and}$$
$$\langle \bar{Z}I_A + \hat{\bar{Z}}I_{A^c}, \bar{\xi}I_A + \bar{\phi}I_{A^c}\rangle_n^{\tau,\theta} = \langle \bar{Z}I_A, \bar{\xi}I_A\rangle_n^{\tau,\theta} + \langle \hat{\bar{Z}}I_{A^c}, \bar{\phi}I_{A^c}\rangle_n^{\tau,\theta}.$$

It follows

$$\begin{aligned} M'' &= \langle \bar{X}, \bar{\xi}I_A + \bar{\phi}I_{A^c}\rangle_n^{\tau,\theta} \\ &\quad - \operatorname*{ess\,sup}_{(\gamma,Y)\in\mathcal{B}_{\rho_0},(\hat{\gamma},\hat{Y})\in\mathcal{B}_{\rho_0}} \{-\gamma I_A - \hat{\gamma}I_{A^c} + \langle YI_A + \hat{Y}I_{A^c}, \xi I_A + \phi I_{A^c}\rangle^{\tau,\theta}\} \\ &\quad - \operatorname*{ess\,sup}_{(V,\bar{Z})\in\mathcal{B}_\Lambda,(\hat{V},\hat{\bar{Z}})\in\mathcal{B}_\Lambda} \{-\langle VI_A + \hat{V}I_{A^c}, \xi I_A + \phi I_{A^c}\rangle^{\tau,\theta} \\ &\quad + \langle \bar{Z}I_A + \hat{\bar{Z}}I_{A^c}, \bar{\xi}I_A + \bar{\phi}I_{A^c}\rangle_n^{\tau,\theta}\} \\ &\leq \langle \bar{X}, \bar{\xi}I_A + \bar{\phi}I_{A^c}\rangle_n^{\tau,\theta} - \operatorname*{ess\,sup}_{(\gamma,Y)\in\mathcal{B}_{\rho_0}} \{-\gamma + \langle Y, \xi I_A + \phi I_{A^c}\rangle^{\tau,\theta}\} \\ &\quad - \operatorname*{ess\,sup}_{(V,\bar{Z})\in\mathcal{B}_\Lambda} \{-\langle V, \xi I_A + \phi I_{A^c}\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}I_A + \bar{\phi}I_{A^c}\rangle_n^{\tau,\theta}\}. \end{aligned}$$

Since $(\xi I_A + \phi I_{A^c}, \bar{\xi}I_A + \bar{\phi}I_{A^c}) = I_A(\xi,\bar{\xi}) + I_{A^c}(\phi,\bar{\phi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}$, the set $\mathcal{M}$ is directed upwards. □

In the remaining part of this subsection we consider the positively homogeneous special case of Theorem 7.3.13. In order to prove the corresponding result, we borrow ideas from the proof of Corollary 11.6 in Föllmer and Schied (2011). Let us define

$$\mathcal{Z} := \{(\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} | (\gamma I_{[\tau,\infty)}, Y) \in \mathcal{B}_{\rho_0}, (V,\bar{Z}) \in \mathcal{B}_\Lambda\}$$

and

$$\begin{aligned} \mathcal{Z}^\# := \{(\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} | -\gamma + \langle Y - V, \xi\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}\rangle_n^{\tau,\theta} \leq 0 \\ \text{for all } (\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{Z}\}. \end{aligned}$$

Theorem 7.3.17. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a conditional positively homogeneous systemic risk measure characterized by a conditional positively homogeneous single-firm risk measure ρ_0 and a positively homogeneous aggregation function Λ. If ρ_0 is continuous for bounded increasing sequences, then ρ admits the representation*

$$\rho(\bar{X}) = \operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{Z}^\#} \langle \bar{X}, \bar{\xi}\rangle_n^{\tau,\theta} \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}. \tag{7.20}$$

In addition, a feasible solution to optimization problem (7.20) *satisfies*

$$\xi \in \mathcal{E}_{\tau,\theta}, \quad \bar{\xi} \in (\mathcal{A}^{1,n}_{\tau,\theta})_+ \quad \textit{and} \quad \langle 1_n, \bar{\xi}\rangle_n^{\tau,\theta} \leq n\langle 1, \xi\rangle^{\tau,\theta}.$$

If additionally $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$, then $\xi \in \mathcal{D}_{\tau,\theta}$.

Proof. Since every conditional positively homogeneous systemic risk measure characterized by a conditional positively homogeneous single-firm risk measure ρ_0 and a positively homogeneous aggregation function Λ is also a conditional convex systemic risk measure characterized by the conditional convex single-firm risk measure ρ_0 and the convex aggregation function Λ, we can apply Theorem 7.3.13. Hence, ρ satisfies

$$\rho(\bar{X}) = \operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{E}^1_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+} \{\langle \bar{X},\bar{\xi}\rangle_n^{\tau,\theta} - \alpha_n^{\tau,\theta}(\xi,\bar{\xi})\} \quad \text{for all } \bar{X}\in\mathcal{R}^{\infty,n}_{\tau,\theta} \tag{7.21}$$

where $\alpha_n^{\tau,\theta}: \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}\to L^0_\tau(\overline{\mathbb{R}}_+)$ is given by

$$\alpha_n^{\tau,\theta}(\xi,\bar{\xi}) = \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y,V,\bar{Z})\in\mathcal{Z}} \{-\gamma + \langle Y-V,\xi\rangle^{\tau,\theta} + \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta}\}.$$

Moreover, Theorem 7.3.13 states that every feasible solution to (7.21) satisfies $\xi\in\mathcal{D}_{\tau,\theta}$ if $\rho(\mathbb{R}^n I_{[\tau,\infty)})=\mathbb{R}$. Since ρ_0 and Λ are ($\mathcal{F}_\tau$-)positively homogeneous, we know that $\mathcal{B}_{\rho_0}$ and $\mathcal{B}_\Lambda$ are $\mathcal{F}_\tau$-cones. This implies that

$$\{\alpha_n^{\tau,\theta}(\xi,\bar{\xi})=0\}\cup\{\alpha_n^{\tau,\theta}(\xi,\bar{\xi})=\infty\}=\Omega \quad \text{for all } (\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}.$$

To prove the previous equality, fix $(\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}$ and consider $A:=\{\alpha_n^{\tau,\theta}(\xi,\bar{\xi})>0\}\in\mathcal{F}_\tau$. For every $(\gamma I_{[\tau,\infty)},Y,V,\bar{Z})\in\mathcal{Z}$, we have $\lambda I_A(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0}$ and $\lambda I_A(V,\bar{Z})\in\mathcal{B}_\Lambda$ for all $\lambda>0$. Hence, we obtain

$$\begin{aligned}
\alpha_n^{\tau,\theta}(\xi,\bar{\xi}) &= \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y,V,\bar{Z})\in\mathcal{Z}} \{-\gamma + \langle Y-V,\xi\rangle^{\tau,\theta} + \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta}\}\\
&\geq \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y,V,\bar{Z})\in\mathcal{Z}} \{-\lambda\gamma I_A + \langle \lambda Y I_A-\lambda V I_A,\xi\rangle^{\tau,\theta} + \langle \lambda\bar{Z}I_A,\bar{\xi}\rangle_n^{\tau,\theta}\}\\
&= \lambda I_A \alpha_n^{\tau,\theta}(\xi,\bar{\xi}).
\end{aligned} \tag{7.22}$$

It follows that $\alpha_n^{\tau,\theta}(\xi,\bar{\xi})=\infty$ on A since $\lambda I_A\alpha_n^{\tau,\theta}(\xi,\bar{\xi})\to\infty I_A$ $\mathbb{P}$-a.s.

Now, let us consider $\alpha_n^{\tau,\theta}(\xi,\bar{\xi})$ for $(\xi,\bar{\xi})\in\mathcal{Z}^\#$. By definition of $\mathcal{Z}^\#$, we have $-\gamma+\langle Y-V,\xi\rangle^{\tau,\theta}+\langle\bar{Z},\bar{\xi}\rangle_n^{\tau,\theta}\leq 0$ for all $(\gamma I_{[\tau,\infty)},Y,V,\bar{Z})\in\mathcal{Z}$. Since $(0,0,0,0_n)\in\mathcal{Z}$, this implies $\alpha_n^{\tau,\theta}(\xi,\bar{\xi})=0$ for all $(\xi,\bar{\xi})\in\mathcal{Z}^\#$.

On the other hand, consider $(\xi,\bar{\xi})\notin\mathcal{Z}^\#$. Then there exists $(\gamma I_{[\tau,\infty)},Y,V,\bar{Z})\in\mathcal{Z}$ and $B\in\mathcal{F}_\tau$ with $\mathbb{P}[B]>0$ such that

$$(-\gamma+\langle Y-V,\xi\rangle^{\tau,\theta}+\langle\bar{Z},\bar{\xi}\rangle_n^{\tau,\theta})I_B = -\gamma I_B+\langle YI_B-VI_B,\xi\rangle^{\tau,\theta}+\langle\bar{Z}I_B,\bar{\xi}\rangle_n^{\tau,\theta}>0 \quad \text{on } B.$$

Note that Equation (7.22) is also true with B instead of A. As a consequence, $\alpha_n^{\tau,\theta}(\xi,\bar{\xi})=\infty$ on B. Hence, we only have to consider $(\xi,\bar{\xi})\in\mathcal{Z}^\#$ in Equation (7.21).

It remains to verify the inequality $\langle 1_n,\bar{\xi}\rangle_n^{\tau,\theta}\leq n\langle 1,\xi\rangle^{\tau,\theta}$ for feasible solutions $(\xi,\bar{\xi})$ to (7.20). Since $\mathcal{B}_{\rho_0}$ and $\mathcal{B}_\Lambda$ satisfy $(0,0)\in\mathcal{B}_{\rho_0}$ and $(f_\Lambda(1)I_{[\tau,\infty)},1_nI_{[\tau,\infty)})\in\mathcal{B}_\Lambda$, we know that $(0,0,f_\Lambda(1)I_{[\tau,\infty)},1_nI_{[\tau,\infty)})\in\mathcal{Z}$. Thus, it follows

$$\langle -f_\Lambda(1)I_{[\tau,\infty)},\xi\rangle^{\tau,\theta}+\langle 1_nI_{[\tau,\infty)},\bar{\xi}\rangle_n^{\tau,\theta} = \langle -f_\Lambda(1),\xi\rangle^{\tau,\theta}+\langle 1_n,\bar{\xi}\rangle_n^{\tau,\theta}\leq 0$$

for all $(\xi,\bar{\xi})\in\mathcal{Z}^\#$. Normalization of Λ leads to the desired inequality. □

7.4. Dynamic systemic risk measures

In this section we define dynamic systemic risk measures as families of conditional systemic risk measures at different points in time. This dynamization of conditional systemic risk measures requires an appropriate time-consistency concept, which is introduced and studied in Subsection 7.4.1.

For the remaining part of this section fix $S \in \mathbb{N}_0$ and $T \in \mathbb{N}_0$ such that $S \leq T$ and set $\mathcal{S} := [S, T] \cap \mathbb{N}_0$. Note that we analyze dynamic systemic risk measures on a finite interval only. This is similar to the model setting in Cheridito and Kupper (2011).

Definition 7.4.1. *For each $t \in \mathcal{S}$, let $\rho_{t,T} : \mathcal{R}^{\infty,n}_{t,T} \to L^\infty_t$ be a conditional convex [positively homogeneous] systemic risk measure with $\rho_{t,T} = \rho^0_{t,T} \circ \Lambda_{t,T}$ for a conditional convex [positively homogeneous] single-firm risk measure $\rho^0_{t,T} : \mathcal{R}^\infty_{t,T} \to L^\infty_t$ that satisfies the $\mathcal{F}_t$-translation property and a convex [positively homogeneous] aggregation function $\Lambda_{t,T} : \mathbb{R}^n \to \mathbb{R}$. Then we call the family $(\rho_{t,T})_{t\in\mathcal{S}}$* dynamic convex [positively homogeneous] systemic risk measure*. Moreover, we call the corresponding family $(\rho^0_{t,T})_{t\in\mathcal{S}}$* dynamic convex [coherent] single-firm risk measure*, and the family $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is called* dynamic convex [positively homogeneous] aggregation function.

Because of the previous definition, dynamic convex risk measures $(\rho_{t,T})_{t\in\mathcal{S}}$ can be decomposed into a dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and a dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. For the remaining part of this thesis, we solely consider dynamic convex single-firm risk measures $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation functions $(\Lambda_{t,T})_{t\in\mathcal{S}}$ being part of a dynamic convex systemic risk measure $(\rho_{t,T})_{t\in\mathcal{S}}$ in the sense of Theorem 7.2.1 and Corollary 7.2.2, i.e., we have $\rho_{t,T} = \rho^0_{t,T} \circ \Lambda_{t,T}$ for each $t \in \mathcal{S}$.

Thus, for every dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$, we have either $\operatorname{Im}\Lambda_{t,T} = \mathbb{R}$ and $f_{\Lambda_{t,T}}(0) = 0$ or $\operatorname{Im}\Lambda_{t,T} = \mathbb{R}_+$ for $t \in \mathcal{S}$. Furthermore, for every dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$, each $\rho^0_{t,T}$, $t \in \mathcal{S}$, additionally satisfies the $\mathcal{F}_t$-translation property. Because of Theorem 7.2.1, we know that every $\rho^0_{t,T}$ satisfies constancy on $\{0\}$. As a consequence, it follows directly that $\rho^0_{t,T}(aI_{[t,\infty)}) = \rho^0_{t,T}(0) + a = a$ for all $a \in \mathbb{R}$. In conclusion, for every dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$, each $\rho^0_{t,T}$ satisfies constancy on $\mathbb{R}$.

If $\operatorname{Im}\Lambda_{t,T} = \mathbb{R}_+$ for $t \in \mathcal{S}$, then the corresponding function $f_{\Lambda_{t,T}}$ in the $f_{\Lambda_{t,T}}$-constancy property is a map from $\mathbb{R}$ to $\mathbb{R}_+$. According to the properties of $\Lambda_{t,T}$, there exists $b_{t,T} \in \mathbb{R}_+$ such that $f_{\Lambda_{t,T}}|_{[b_{t,T},\infty)}$ is a bijective, strictly increasing function from $[b_{t,T}, \infty)$ to $\mathbb{R}_+$ with $f_{\Lambda_{t,T}}(a) = 0$ for all $a \leq b_{t,T}$. Moreover, the inverse function $f^{-1}_{\Lambda_{t,T}}$ maps from $\mathbb{R}_+$ to $[b_{t,T}, \infty)$ and is also strictly increasing.

We have already pointed out that in contrast to Sections 7.1-7.3, in this section we solely study conditional convex single-firm risk measures $\rho^0_{t,T}$ satisfying the $\mathcal{F}_t$-translation property. Note that standard dynamic convex risk measures from Chapter 6 (see Definition 6.1.1 and Definition 6.3.1) admit the corresponding $\mathcal{F}_t$-translation property by definition. Thus, the $\mathcal{F}_t$-translation property is a feasible assumption. Nonetheless, note that $\rho_{t,T} = \rho^0_{t,T} \circ \Lambda_{t,T}$ does not inherit this property because the convex aggregation function $\Lambda_{t,T}$ does not satisfy any sort of translation property.

Since $\Lambda_{t,T}(a1_n) = f_{\Lambda_{t,T}}(a)$ for all $a \in \mathbb{R}$, we obtain for $\gamma \in L_t^\infty$, $u \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$ that

$$\Lambda_{t,T}(\gamma 1_n I_{[t,\infty)})(u,\omega) = \Lambda_{t,T}((\gamma 1_n I_{[t,\infty)})(u,\omega)) = \begin{Bmatrix} \Lambda_{t,T}(\gamma(\omega)1_n) & \text{if } t \le u \\ \Lambda_{t,T}(0_n) & \text{if } t > u \end{Bmatrix}$$
$$= f_{\Lambda_{t,T}}((\gamma I_{[t,\infty)})(u,\omega)) = f_{\Lambda_{t,T}}(\gamma I_{[t,\infty)})(u,\omega).$$

From $\Lambda_{t,T}(0_n) = 0 = f_{\Lambda_{t,T}}(0)$ it follows that $\Lambda_{t,T}(\gamma 1_n I_{[t,\infty)}) = f_{\Lambda_{t,T}}(\gamma I_{[t,\infty)}) = f_{\Lambda_{t,T}}(\gamma) I_{[t,\infty)}$ for all $\gamma \in L_t^\infty$. Because each $\rho_{t,T}^0$ satisfies the $\mathcal{F}_t$-translation property and $\rho_{t,T}^0(0) = 0$, this implies

$$\rho_{t,T}(\gamma 1_n I_{[t,\infty)}) = \rho_{t,T}^0(\Lambda_{t,T}(\gamma 1_n I_{[t,\infty)})) = \rho_{t,T}^0(f_{\Lambda_{t,T}}(\gamma) I_{[t,\infty)})$$
$$= \rho_{t,T}^0(0) + f_{\Lambda_{t,T}}(\gamma) = f_{\Lambda_{t,T}}(\gamma)$$

for all $\gamma \in L_t^\infty$. Note that in case of a dynamic convex systemic risk measure $(\rho_{t,T})_{t\in\mathcal{S}}$ with $\rho_{t,T} = \rho_{t,T}^0 \circ \Lambda_{t,T}$ for all $t \in \mathcal{S}$, we have $f_{\Lambda_{t,T}} = f_{\rho_{t,T}}$. Therefore, the following property is satisfied:

(s7) $\rho_{t,T}(\gamma 1_n I_{[t,\infty)}) = f_{\rho_{t,T}}(\gamma)$ for all $t \in \mathcal{S}$ and $\gamma \in L_t^\infty$.

7.4.1. Time-consistency

Our next aim is to introduce an appropriate time-consistency concept for dynamic convex systemic risk measures which establishes a connection between the different conditional convex systemic risk measures in time. At the beginning of this part of the thesis we have already pointed out that the concept of strong time-consistency is frequently used in the literature about dynamic single-firm risk measurement. In line with these approaches, we introduce a version of this time-consistency property for our dynamic convex systemic risk measures. Since our dynamic convex single-firm risk measures additionally satisfy the $\mathcal{F}_t$-translation property and $\rho_{t,T}^0(0) = 0$, they correspond with the dynamic convex risk measures considered in Cheridito et al. (2006), Cheridito and Kupper (2011) and Section 6.3 in this thesis. More precisely, because of the different perspectives concerning the processes $\bar{X} \in \mathcal{R}_{t,T}^{\infty;n}$, for every dynamic convex risk measure $(\tilde{\rho}_{t,T})_{t\in\mathcal{S}}$ in the sense of Definition 6.3.1, we can define a dynamic convex single-firm risk measure $(\rho_{t,T}^0)_{t\in\mathcal{S}}$ by $\rho_{t,T}^0(X) := \tilde{\rho}_{t,T}(-X)$ for $t \in \mathcal{S}$ and $X \in \mathcal{R}_{t,T}^\infty$. Consequently, we can carry over the definition from Section 6.3 to dynamic convex single-firm risk measures and adapt this concept for dynamic convex systemic risk measures and dynamic convex aggregation functions.

Because of Proposition 6.3.3 and $T < +\infty$, we can use the following definition for time-consistent dynamic convex single-firm risk measures.

Definition 7.4.2. *A dynamic convex single-firm risk measure* $(\rho_{t,T}^0)_{t\in\mathcal{S}}$ *is called* time-consistent *if the following property is satisfied:*

(r-TC) *For every pair* $s, t \in \mathcal{S}$ *with* $s \leq t$ *and* $X, Y \in \mathcal{R}^{\infty}_{s,T}$,

$$XI_{[s,t)} = YI_{[s,t)} \quad \textit{and} \quad \rho^0_{t,T}(X) \leq \rho^0_{t,T}(Y)$$

imply $\rho^0_{s,T}(X) \leq \rho^0_{s,T}(Y)$.

Similarly, we define time-consistent dynamic convex aggregation functions.

Definition 7.4.3. *A dynamic convex aggregation function* $(\Lambda_{t,T})_{t\in\mathcal{S}}$ *is called* time-consistent *if the following two properties are satisfied:*

(a-TC1) *Either all* $\Lambda_{t,T}$, $t \in \mathcal{S}$, *map into* $\mathbb{R}$ *or all* $\Lambda_{t,T}$, $t \in \mathcal{S}$, *map into* $\mathbb{R}_+$.

(a-TC2) *For every pair* $s, t \in \mathcal{S}$ *with* $s \leq t$ *and* $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$,

$$\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)} \quad \textit{and} \quad \Lambda_{t,T}(\bar{X}I_{[t,\infty)}) \leq \Lambda_{t,T}(\bar{Y}I_{[t,\infty)})$$

imply $\Lambda_{s,T}(\bar{X}) \leq \Lambda_{s,T}(\bar{Y})$.

In property (a-TC1) we distinguish between $\mathbb{R}$-valued and $\mathbb{R}_+$-valued dynamic convex aggregation functions $(\Lambda_{t,T})_{t\in\mathcal{S}}$. We claim this property because a mixture of both approaches does not seem to be consistent. A possible interpretation of property (a-TC2) is the following: Consider two economies $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ which are equal up to time t. If additionally the time-t-aggregation of economy $\bar{X}I_{[t,\infty)}$ is less than or equal to the time-t-aggregation of the other economy $\bar{Y}I_{[t,\infty)}$, then this relation should still hold if we use the time-s-aggregation function and consider the processes $\bar{X}$ and $\bar{Y}$. At this point, note that $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$ and $\Lambda_{s,T}(0_n) = 0$ directly imply $\Lambda_{s,T}(\bar{X})I_{[s,t)} = \Lambda_{s,T}(\bar{X}I_{[s,t)}) = \Lambda_{s,T}(\bar{Y}I_{[s,t)}) = \Lambda_{s,T}(\bar{Y})I_{[s,t)}$. In addition, for every time-consistent dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$ with $\Lambda_{t,T}(\bar{X}) = 0$ for $\bar{X} \in \mathcal{R}^{\infty,n}_{t,T}$ and some $t \in \mathcal{S}$, we also have $\Lambda_{s,T}(\bar{X}) = 0$ for all $s \in \mathcal{S}$ with $s \leq t$.

Further, because each $\Lambda_{t,T}$, $t \in \mathcal{S}$, is measurable and satisfies $\Lambda_{t,T}(0_n) = 0$, property (a-TC2) is equivalent to the following property:

(a-TC2') For every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $\bar{x}, \bar{y} \in \mathbb{R}^n$,

$$\Lambda_{t,T}(\bar{x}) \leq \Lambda_{t,T}(\bar{y})$$

implies $\Lambda_{s,T}(\bar{x}) \leq \Lambda_{s,T}(\bar{y})$.

It seems reasonable to define time-consistent dynamic convex risk measures by using Definition 7.4.2 and Definition 7.4.3.

Definition 7.4.4. *A dynamic convex systemic risk measure* $(\rho_{t,T})_{t\in\mathcal{S}}$ *is called* time-consistent *if the corresponding dynamic convex single-firm risk measure* $(\rho^0_{t,T})_{t\in\mathcal{S}}$ *is time-consistent and if the corresponding dynamic convex aggregation function* $(\Lambda_{t,T})_{t\in\mathcal{S}}$ *is time-consistent.*

In the subsequent study we analyze the relationship between the properties (r-TC), (a-TC1), (a-TC2) and the following properties of a dynamic convex systemic risk measure $(\rho_{t,T})_{t\in\mathcal{S}}$:

(s-TC1) Either all $\rho_{t,T}$, $t \in \mathcal{S}$, satisfy $\rho_{t,T}(\mathbb{R}^n I_{[t,\infty)}) = \mathbb{R}$ or all $\rho_{t,T}$, $t \in \mathcal{S}$, satisfy $\rho_{t,T}(\mathbb{R}^n I_{[t,\infty)}) = \mathbb{R}_+$.

(s-TC2) For every pair $s,t \in \mathcal{S}$ with $s \leq t$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$,

$$\bar{X} I_{[s,t)} = \bar{Y} I_{[s,t)} \quad \text{and} \quad \rho_{t,T}(\bar{X}_u(\omega) I_{[t,\infty)}) \leq \rho_{t,T}(\bar{Y}_u(\omega) I_{[t,\infty)}) \qquad (7.23)$$
$$\text{for all } u \geq t \text{ and a.e. } \omega \in \Omega$$

imply $\rho_{s,T}(\bar{X}_u(\omega) I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Y}_u(\omega) I_{[s,\infty)})$ for all $u \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$.

(s-TC3) For every pair $s,t \in \mathcal{S}$ with $s \leq t$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$,

$$\bar{X} I_{[s,t)} = \bar{Y} I_{[s,t)} \quad \text{and} \quad \rho_{t,T}(\bar{X}) \leq \rho_{t,T}(\bar{Y})$$

imply $\rho_{s,T}(\bar{X}) \leq \rho_{s,T}(\bar{Y})$.

If the dynamic convex systemic risk measure $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC2), then (s-TC2) and preference consistency of $\rho_{t,T}$ and $\rho_{s,T}$ imply the following sequence of implications for $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$:

$$\begin{aligned}(7.23) &\Rightarrow \rho_{s,T}(\bar{X}_u(\omega) I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Y}_u(\omega) I_{[s,\infty)}) \quad \text{for all } u \in \mathbb{N}_0 \text{ and a.e. } \omega \in \Omega \\ &\Rightarrow \rho_{s,T}(\bar{X}) \leq \rho_{s,T}(\bar{Y}).\end{aligned}$$

We can interpret (s-TC3) similarly to (a-TC2): Consider two economies $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ that are equal up to time t, i.e., $\bar{X} I_{[s,t)} = \bar{Y} I_{[s,t)}$, and let the time-$t$-systemic risk of one economy be lesser than or equal to the time-t-systemic risk of the other economy, i.e., $\rho_{t,T}(\bar{X}) \leq \rho_{t,T}(\bar{Y})$. Then the time-$s$-systemic risk should satisfy the same relation.

We will see that time-consistent dynamic convex systemic risk measures $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfy property (s-TC2) (see Proposition 7.4.8). This property can be interpreted as follows: Fix economies $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ that are equal up to time t. For fixed $(u,\omega) \in ([t,T] \cap \mathbb{N}_0) \times \Omega$, consider the n-dimensional processes $\bar{X}_u(\omega) I_{[t,\infty)}$ and $\bar{Y}_u(\omega) I_{[t,\infty)}$. These are constant after time t and represent a possible realization of $\bar{X}$ (or $\bar{Y}$) at time $u \geq t$. Supposing that the time-t-systemic risk of one of these processes is lesser than or equal to the risk of the other process for each $u \geq t$ and a.e. $\omega \in \Omega$ leads to the same relation regarding the time-s-systemic risk of the processes $\bar{X}_u(\omega) I_{[s,\infty)}$ and $\bar{Y}_u(\omega) I_{[s,\infty)}$ for all $u \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. Note that $\rho_{s,T}(\bar{X}_u(\omega) I_{[s,\infty)}) = \rho_{s,T}(\bar{Y}_u(\omega) I_{[s,\infty)})$ is obviously satisfied for $u < t$ since $\bar{X} I_{[s,t)} = \bar{Y} I_{[s,t)}$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$.

Lemma 7.4.5. *For every dynamic convex systemic risk measure, property (s-TC3) implies property (s-TC2).*

Proof. Let $s,t \in \mathcal{S}$ with $s \leq t$ and consider processes $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ that satisfy (7.23). Obviously, we have

$$\rho_{s,T}(\bar{X}_u(\omega) I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Y}_u(\omega) I_{[s,\infty)}) \quad \text{for all } u < t \text{ and a.e. } \omega \in \Omega.$$

Moreover, for $u \geq t$ and a.e. $\omega \in \Omega$, define the processes $\bar{X}^{(\omega,u)}, \bar{Y}^{(\omega,u)} \in \mathcal{R}^{\infty,n}_{s,T}$ by $\bar{X}^{(\omega,u)} := \bar{X}_u(\omega)I_{[t,\infty)}$ and $\bar{Y}^{(\omega,u)} := \bar{Y}_u(\omega)I_{[t,\infty)}$. According to these definitions, we have $\bar{X}^{(\omega,u)}I_{[s,t)} = 0_n = \bar{Y}^{(\omega,u)}I_{[s,t)}$ and (7.23) means that

$$\rho_{t,T}(\bar{X}^{(\omega,u)}) = \rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}) \leq \rho_{t,T}(\bar{Y}_u(\omega)I_{[t,\infty)}) = \rho_{t,T}(\bar{Y}^{(\omega,u)}).$$

Since

$$\rho_{s,T}(\bar{Z}^{(\omega,u)}) = \rho_{s,T}(\bar{Z}_u(\omega)I_{[t,\infty)}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{Z}_u(\omega)I_{[t,\infty)})) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{Z}_u(\omega))I_{[t,\infty)})$$

is satisfied for $\bar{Z} = \bar{X}$ and $\bar{Z} = \bar{Y}$, property (s-TC3) yields $\rho^0_{s,T}(\Lambda_{s,T}(\bar{X}_u(\omega))I_{[t,\infty)}) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}_u(\omega))I_{[t,\infty)})$. Furthermore, $\rho^0_{s,T}$ satisfies constancy on $\mathbb{R}$, which implies $\Lambda_{s,T}(\bar{X}_u(\omega)) \leq \Lambda_{s,T}(\bar{Y}_u(\omega))$. The monotonicity property of $\rho^0_{s,T}$ yields

$$\begin{aligned}\rho_{s,T}(\bar{X}_u(\omega)I_{[s,\infty)}) &= \rho^0_{s,T}(\Lambda_{s,T}(\bar{X}_u(\omega))I_{[s,\infty)})\\ &\leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}_u(\omega))I_{[s,\infty)}) = \rho_{s,T}(\bar{Y}_u(\omega)I_{[s,\infty)}).\end{aligned}$$

Summing up, we get $\rho_{s,T}(\bar{X}_u(\omega)I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Y}_u(\omega)I_{[s,\infty)})$ for all $u \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. This means that property (s-TC2) is satisfied. □

The following lemma analyzes property (s-TC3) and provides a condition which implies equivalence between time-consistency of the dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and property (s-TC3) of the corresponding dynamic convex systemic risk measure $(\rho_{t,T})_{t\in\mathcal{S}}$.

Lemma 7.4.6. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with corresponding dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. Moreover, assume that the following property is satisfied for all $s,t \in \mathcal{S}$ with $s \leq t$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$:*

$$\rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{t,T}(\bar{Y})) \quad \Leftrightarrow \quad \rho^0_{s,T}(\Lambda_{s,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y})) \tag{7.24}$$

Then $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC3).

Proof. "⇒" Let $(\rho^0_{t,T})_{t\in\mathcal{S}}$ be time-consistent and consider $s,t \in \mathcal{S}$ with $s \leq t$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ with $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$ and $\rho_{t,T}(\bar{X}) \leq \rho_{t,T}(\bar{Y})$. Then $\rho^0_{t,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{t,T}(\Lambda_{t,T}(\bar{Y}))$. Since $\Lambda_{t,T}(\bar{X})I_{[s,t)} = \Lambda_{t,T}(\bar{Y})I_{[s,t)}$, property (r-TC) implies that $\rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{t,T}(\bar{Y}))$. Finally, (7.24) yields

$$\rho_{s,T}(\bar{X}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y})) = \rho_{s,T}(\bar{Y}).$$

"⇐" Let (s-TC3) be satisfied and consider $s,t \in \mathcal{S}$ with $s \leq t$ and $X, Y \in \mathcal{R}^{\infty}_{s,T}$ with $XI_{[s,t)} = YI_{[s,t)}$ and $\rho^0_{t,T}(X) \leq \rho^0_{t,T}(Y)$. Then there exist $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ with $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$, $\Lambda_{t,T}(\bar{X}) = X$ and $\Lambda_{t,T}(\bar{Y}) = Y$. Due to property (s-TC3), it follows $\rho^0_{s,T}(\Lambda_{s,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}))$, and (7.24) yields

$$\rho^0_{s,T}(X) = \rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{t,T}(\bar{Y})) = \rho^0_{s,T}(Y).$$

□

Example 7.4.7. The equivalence in (7.24) is satisfied if $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is a dynamic positively homogeneous single-firm risk measure and $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is given by $\Lambda_{t,T} = q_t\Lambda$ for $t \in \mathcal{S}$, a convex aggregation function Λ and $q_t \in \mathbb{R}_+\backslash\{0\}$. In this case, we have

$$\rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) = q_t\rho^0_{s,T}(\Lambda(\bar{X})) \quad \text{for each } s,t \in \mathcal{S} \text{ and } \bar{X} \in \mathcal{R}^{\infty,n}_{s,T}.$$

Therefore, for all $s,t \in \mathcal{S}$ and all $\bar{X},\bar{Y} \in \mathcal{R}^{\infty}_{s,T}$, the inequality $\rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{t,T}(\bar{Y}))$ is satisfied if and only if $\rho^0_{s,T}(\Lambda(\bar{X})) \leq \rho^0_{s,T}(\Lambda(\bar{Y}))$, which implies the equivalence in (7.24).

The next result provides a characterization of time-consistent dynamic convex aggregation functions.

Proposition 7.4.8. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with corresponding dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. Moreover, let $(\rho^0_{t,T})_{t\in\mathcal{S}}$ be time-consistent. Then $(\rho_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC1) and (s-TC2).*

Proof. First, we will show that (s-TC1) is equivalent to (a-TC1): Since all $\rho^0_{t,T}$ satisfy constancy on $\mathbb{R}$, we have $\rho_{t,T}(\bar{x}I_{[t,\infty)}) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{x}I_{[t,\infty)})) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{x})I_{[t,\infty)}) = \Lambda_{t,T}(\bar{x})$ for all $\bar{x} \in \mathbb{R}^n$ and $t \in \mathcal{S}$. This equality implies the desired equivalence.

Let $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfy (s-TC2). Instead of proving property (a-TC2), we will show the equivalent property (a-TC2'). For $s,t \in \mathcal{S}$ with $s \leq t$, consider $\bar{x},\bar{y} \in \mathbb{R}^n$ with $\Lambda_{t,T}(\bar{x}) \leq \Lambda_{t,T}(\bar{y})$ and define the process $\bar{Z} \in \mathcal{R}^{\infty,n}_{s,T}$ by $\bar{Z} := \bar{x}I_{[s,t)} + \bar{y}I_{[t,\infty)}$. Then $\bar{Z}I_{[s,t)} = \bar{x}I_{[s,t)}$ and $\Lambda_{t,T}(\bar{x}) \leq \Lambda_{t,T}(\bar{y}) = \Lambda_{t,T}(\bar{Z}_u(\omega))$ for all $u \geq t$ and a.e. $\omega \in \Omega$. Monotonicity of $\rho^0_{t,T}$ implies

$$\begin{aligned}
&\rho_{t,T}(\bar{x}I_{[t,\infty)}) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{x}I_{[t,\infty)})) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{x})I_{[t,\infty)}) \\
&\leq \rho^0_{t,T}(\Lambda_{t,T}(\bar{Z}_u(\omega))I_{[t,\infty)}) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{Z}_u(\omega)I_{[t,\infty)})) = \rho_{t,T}(\bar{Z}_u(\omega)I_{[t,\infty)})
\end{aligned}$$

for all $u \geq t$ and a.e. $\omega \in \Omega$. Due to property (s-TC2), it follows

$$\rho_{s,T}(\bar{x}I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Z}_u(\omega)I_{[s,\infty)}) \quad \text{for all } u \geq t \text{ and a.e. } \omega \in \Omega.$$

This means that

$$\begin{aligned}
&\rho^0_{s,T}(\Lambda_{s,T}(\bar{x})I_{[s,\infty)}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{x}I_{[s,\infty)})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Z}_u(\omega)I_{[s,\infty)})) \\
&= \rho^0_{s,T}(\Lambda_{s,T}(\bar{y}I_{[s,\infty)})) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{y})I_{[s,\infty)})
\end{aligned}$$

for all $u \geq t$ and a.e. $\omega \in \Omega$. Since $\rho^0_{s,T}$ satisfies constancy on $\mathbb{R}$, this implies $\Lambda_{s,T}(\bar{x}) \leq \Lambda_{s,T}(\bar{y})$. Therefore, $(\rho_{t,T})_{t\in\mathcal{S}}$ is time-consistent.

On the other hand, assume that $(\rho_{t,T})_{t\in\mathcal{S}}$ is time-consistent. To show (s-TC2), consider $s,t \in \mathcal{S}$ with $s \leq t$ and $\bar{X},\bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ with $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$ and

$$\begin{aligned}
&\rho^0_{t,T}(\Lambda_{t,T}(\bar{X}_u(\omega))I_{[t,\infty)}) = \rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}) \\
&\leq \rho_{t,T}(\bar{Y}_u(\omega)I_{[t,\infty)}) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{Y}_u(\omega))I_{[t,\infty)})
\end{aligned}$$

for all $u \geq t$ and a.e. $\omega \in \Omega$. Then the constancy property of $\rho^0_{t,T}$ on $\mathbb{R}$ yields $\Lambda_{t,T}(\bar{X}_u(\omega)) \leq \Lambda_{t,T}(\bar{Y}_u(\omega))$ for all $u \geq t$ and a.e. $\omega \in \Omega$. Hence, $\Lambda_{t,T}(\bar{X} I_{[t,\infty)}) \leq \Lambda_{t,T}(\bar{Y} I_{[t,\infty)})$. Time-consistency of $(\Lambda_{t,T})_{t\in\mathcal{S}}$ implies $\Lambda_{s,T}(\bar{X}) \leq \Lambda_{s,T}(\bar{Y})$, which means that

$$\Lambda_{s,T}(\bar{X}_u(\omega)) \leq \Lambda_{s,T}(\bar{Y}_u(\omega)) \quad \text{for all } u \in \mathbb{N}_0 \text{ and a.e. } \omega \in \Omega.$$

Finally, because $\rho^0_{s,T}$ is monotone, we arrive at

$$\begin{aligned}\rho_{s,T}(\bar{X}_u(\omega) I_{[s,\infty)}) &= \rho^0_{s,T}(\Lambda_{s,T}(\bar{X}_u(\omega)) I_{[s,\infty)}) \\ &\leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}_u(\omega)) I_{[s,\infty)}) = \rho_{s,T}(\bar{Y}_u(\omega) I_{[s,\infty)})\end{aligned}$$

for all $u \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. □

Summing up the previous results, we obtain the following corollary.

Corollary 7.4.9. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with corresponding dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. Moreover, assume that (a-TC1) is satisfied. Then $(\rho_{t,T})_{t\in\mathcal{S}}$ is a time-consistent dynamic convex systemic risk measure if $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is time-consistent and* (7.24) *is satisfied.*

Proof. If $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is time-consistent and (7.24) is satisfied, then we know from Lemma 7.4.6 that $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC3). Furthermore, Lemma 7.4.5 yields that (s-TC2) holds. Finally, we obtain with Proposition 7.4.8 that $(\rho_{t,T})_{t\in\mathcal{S}}$ is time-consistent. □

Next, we consider equivalent properties to (s-TC3), (s-TC2) and (a-TC2). The corresponding result in terms of standard dynamic risk measures was proved in Proposition 4.4 in Cheridito et al. (2006). For completeness, we reformulate their result and the corresponding proof for our dynamic convex single-firm risk measures.

Proposition 7.4.10. *Let $(\rho^0_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex single-firm risk measure. Then the following statements are equivalent:*

1. *$(\rho^0_{t,T})_{t\in\mathcal{S}}$ is time-consistent.*
2. *For every pair $s,t \in \mathcal{S}$ with $s \leq t$ and $X \in \mathcal{R}^\infty_{s,T}$, we have*

$$\rho^0_{s,T}(X) = \rho^0_{s,T}(X I_{[s,t)} + \rho^0_{t,T}(X) I_{[t,\infty)}).$$

Proof. 1. ⇒ 2. : Consider $X \in \mathcal{R}^{\infty,n}_{s,T}$ and define $Z := X I_{[s,t)} + \rho^0_{t,T}(X) I_{[t,\infty)}$. Then the $\mathcal{F}_t$-translation property of $\rho^0_{t,T}$ and $\rho^0_{t,T}(0) = 0$ imply

$$\rho^0_{t,T}(Z) = \rho^0_{t,T}(p^{t,T}(Z)) = \rho^0_{t,T}(\rho^0_{t,T}(X) I_{[t,\infty)}) = \rho^0_{t,T}(X).$$

Since (r-TC) is satisfied, it follows $\rho^0_{s,T}(X) = \rho_{s,T}(Z)$.

2. $\Rightarrow$ 1. : Let $X, Y \in \mathcal{R}^{\infty}_{s,T}$ with $XI_{[s,t)} = YI_{[s,t)}$ and $\rho^0_{t,T}(X) \le \rho^0_{t,T}(Y)$. By monotonicity of $\rho^0_{s,T}$, we obtain

$$\begin{aligned}&\rho^0_{s,T}(X) = \rho^0_{s,T}(XI_{[s,t)} + \rho^0_{t,T}(X)I_{[t,\infty)}) \le \rho^0_{s,T}(XI_{[s,t)} + \rho^0_{t,T}(Y)I_{[t,\infty)})\\ &\quad = \rho^0_{s,T}(YI_{[s,t)} + \rho^0_{t,T}(Y)I_{[t,\infty)}) = \rho^0_{s,T}(Y).\end{aligned}$$

□

The corresponding result for dynamic convex systemic risk measures reads as follows.

Proposition 7.4.11. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure. Then the following statements are equivalent:*

1. *$(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC3).*
2. *For every pair $s, t \in \mathcal{S}$ with $s \le t$ and $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$, we have*

$$\rho_{s,T}(\bar{X}) = \rho_{s,T}(\bar{X}I_{[s,t)} + f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}))1_n I_{[t,\infty)}).$$

Proof. 1. $\Rightarrow$ 2. : Let $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$ and define $\bar{Z} := \bar{X}I_{[s,t)} + f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}))1_n I_{[t,\infty)}$. Property (s7) yields

$$\rho_{t,T}(\bar{Z}) = \rho_{t,T}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}))1_n I_{[t,\infty)}) = f_{\rho_{t,T}}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}))) = \rho_{t,T}(\bar{X}).$$

Since (s-TC3) is satisfied, we have $\rho_{s,T}(\bar{X}) = \rho_{s,T}(\bar{Z})$.

2. $\Rightarrow$ 1. : Consider $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ with $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$ and $\rho_{t,T}(\bar{X}) \le \rho_{t,T}(\bar{Y})$. Monotonicity of $f^{-1}_{\rho_{t,T}}$ and $\rho_{s,T}$ yields

$$\begin{aligned}\rho_{s,T}(\bar{X}) &= \rho_{s,T}(\bar{X}I_{[s,t)} + f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}))1_n I_{[t,\infty)})\\ &= \rho_{s,T}(\bar{Y}I_{[s,t)} + f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}))1_n I_{[t,\infty)})\\ &\le \rho_{s,T}(\bar{Y}I_{[s,t)} + f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{Y}))1_n I_{[t,\infty)})\\ &= \rho_{s,T}(\bar{Y}).\end{aligned}$$

□

Analogously, we obtain the following proposition regarding property (s-TC2).

Proposition 7.4.12. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure. Then the following statements are equivalent:*

1. *$(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC2).*
2. *For every pair $s, t \in \mathcal{S}$ with $s \le t$ and $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$, we have*

$$\rho_{s,T}(\bar{X}_u(\omega)I_{[s,\infty)}) = \rho_{s,T}(\bar{Z}_u(\omega)I_{[s,\infty)}) \quad \text{for all } u \in \mathbb{N}_0 \text{ and a.e. } \omega \in \Omega$$

where $\bar{Z}_u(\omega) := \bar{X}_u(\omega)$ for $u < t$ and $\bar{Z}_u(\omega) := f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}))1_n$ for $u \ge t$.

Proof. 1. $\Rightarrow$ 2. : Let $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$ and define $\bar{Z}$ as above. For $u \geq t$ and a.e. $\omega \in \Omega$, property (s7) yields

$$\begin{aligned}\rho_{t,T}(\bar{Z}_u(\omega)I_{[t,\infty)}) &= \rho_{t,T}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}))1_n I_{[t,\infty)}) \\ &= f_{\rho_{t,T}}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}))) = \rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}).\end{aligned}$$

Since (s-TC2) is satisfied, we have $\rho_{s,T}(\bar{X}_u(\omega)I_{[s,\infty)}) = \rho_{s,T}(\bar{Z}_u(\omega)I_{[s,\infty)})$ for all $u \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$.

2. $\Rightarrow$ 1. : Consider $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ such that $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$ and $\rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}) \leq \rho_{t,T}(\bar{Y}_u(\omega)I_{[t,\infty)})$ for all $u \geq t$ and a.e. $\omega \in \Omega$. Obviously, $\rho_{s,T}(\bar{X}_u(\omega)I_{[s,\infty)}) = \rho_{s,T}(\bar{Y}_u(\omega)I_{[s,\infty)})$ for $u < t$ and a.e. $\omega \in \Omega$. Moreover, for $u \geq t$ and a.e. $\omega \in \Omega$, we get

$$\begin{aligned}\rho_{s,T}(\bar{X}_u(\omega)I_{[s,\infty)}) &= \rho_{s,T}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}))1_n I_{[s,\infty)}) \\ &\leq \rho_{s,T}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{Y}_u(\omega)I_{[t,\infty)}))1_n I_{[s,\infty)}) = \rho_{s,T}(\bar{Y}_u(\omega)I_{[s,\infty)})\end{aligned}$$

since $\rho_{s,T}$ and $f^{-1}_{\rho_{t,T}}$ are monotone. □

The following remark addresses another important property of dynamic convex aggregation functions $(\Lambda_{t,T})_{t\in\mathcal{S}}$ which is needed to derive an analogous result for $(\Lambda_{t,T})_{t\in\mathcal{S}}$.

Remark 7.4.13. Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex aggregation function and fix $t \in \mathcal{S}$. If $\operatorname{Im}\Lambda_{t,T} = \mathbb{R}$, we have

$$\Lambda_{t,T}(f^{-1}_{\Lambda_{t,T}}(X)\,1_n) = f_{\Lambda_{t,T}}(f^{-1}_{\Lambda_{t,T}}(X)) = X \quad \text{for all } X \in \mathcal{R}^\infty.$$

If $\operatorname{Im}\Lambda_{t,T} = \mathbb{R}_+$, then the corresponding inverse function $f^{-1}_{\Lambda_{t,T}}$ maps from $\mathbb{R}_+$ into $[b_{t,T}, \infty)$ for $b_{t,T} \in \mathbb{R}_+$ and

$$\Lambda_{t,T}(f^{-1}_{\Lambda_{t,T}}(X^+)1_n) = f_{\Lambda_{t,T}}(f^{-1}_{\Lambda_{t,T}}(X^+)) = X^+ \quad \text{for all } X \in \mathcal{R}^\infty$$

where X^+ is defined by $X^+_t(\omega) := \max\{X_t(\omega), 0\}$ for $t \in \mathbb{N}_0$ and $\omega \in \Omega$.

Proposition 7.4.14. *Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex aggregation function. Then the following statements are equivalent:*

1. *$(\Lambda_{t,T})_{t\in\mathcal{S}}$ satisfies (a-TC2).*
2. *For every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$, we have*

$$\Lambda_{s,T}(\bar{X}) = \Lambda_{s,T}(\bar{X}I_{[s,t)} + f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X}))1_n I_{[t,\infty)}). \tag{7.25}$$

3. *For every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $\bar{x} \in \mathbb{R}^n$, we have*

$$\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n). \tag{7.26}$$

Proof. 1. $\Rightarrow$ 2. : Let $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$ and define $\bar{Z} := \bar{X} I_{[s,t)} + f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X}))1_n I_{[t,\infty)}$. Remark 7.4.13 yields

$$\Lambda_{t,T}(\bar{Z} I_{[t,\infty)}) = \Lambda_{t,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X} I_{[t,\infty)}))1_n) = \Lambda_{t,T}(\bar{X} I_{[t,\infty)}).$$

Since (a-TC2) is satisfied, it follows $\Lambda_{s,T}(\bar{X}) = \Lambda_{s,T}(\bar{Z})$.

2. $\Rightarrow$ 3. : Fix $\bar{x} \in \mathbb{R}^n$ and consider the corresponding process $\bar{x} I_{[t,\infty)} \in \mathcal{R}^{\infty,n}_{s,T}$. For all $u \geq t$ and a.e. $\omega \in \Omega$, Equation (7.25) and the measurability property of $\Lambda_{s,T}$, $\Lambda_{t,T}$ and $f^{-1}_{\Lambda_{t,T}}$ imply

$$\begin{aligned}\Lambda_{s,T}(\bar{x}) &= \Lambda_{s,T}(\bar{x} I_{[t,\infty)})(u,\omega) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x} I_{[t,\infty)}))1_n I_{[t,\infty)})(u,\omega)\\ &= \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x} I_{[t,\infty)}(u,\omega)))1_n I_{[t,\infty)}(u,\omega)) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n).\end{aligned}$$

3. $\Rightarrow$ 1. : We will show that $(\Lambda_{t,T})_{t\in\mathcal{S}}$ satisfies (a-TC2') which is equivalent to (a-TC2). Consider $\bar{x}, \bar{y} \in \mathbb{R}^n$ with $\Lambda_{t,T}(\bar{x}) \leq \Lambda_{t,T}(\bar{y})$. Then monotonicity of $\Lambda_{s,T}$ and $f^{-1}_{\Lambda_{t,T}}$ and Equation (7.26) yield

$$\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n) \leq \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{y}))1_n) = \Lambda_{s,T}(\bar{y}).$$

□

The following corollary sums up the previous results.

Corollary 7.4.15. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with corresponding dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$ that satisfies (a-TC1). Then the following statements are equivalent:*

1. *$(\rho_{t,T})_{t\in\mathcal{S}}$ is time-consistent.*
2. *For every pair $s,t \in \mathcal{S}$ with $s \leq t$, the following equalities hold for every $X \in \mathcal{R}^{\infty}_{s,T}$ and every $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$:*

$$\begin{aligned}\rho^0_{s,T}(X) &= \rho^0_{s,T}(X I_{[s,t)} + \rho^0_{t,T}(X) I_{[t,\infty)}),\\ \Lambda_{s,T}(\bar{X}) &= \Lambda_{s,T}(\bar{X} I_{[s,t)} + f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X}))1_n I_{[t,\infty)}).\end{aligned}$$

3. *For every pair $s,t \in \mathcal{S}$ with $s \leq t$, the following equalities hold for every $X \in \mathcal{R}^{\infty}_{s,T}$ and every $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$:*

$$\begin{aligned}\rho^0_{s,T}(X) &= \rho^0_{s,T}(X I_{[s,t)} + \rho^0_{t,T}(X) I_{[t,\infty)}),\\ \rho_{s,T}(\bar{X}_u(\omega) I_{[s,\infty)}) &= \rho_{s,T}(\bar{Z}_u(\omega) I_{[s,\infty)}) \quad \text{for all } u \in \mathbb{N}_0 \text{ and a.e. } \omega \in \Omega\end{aligned}$$

where $\bar{Z}_u(\omega) := \bar{X}_u(\omega)$ for $u < t$ and $\bar{Z}_u(\omega) := f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}_u(\omega) I_{[t,\infty)}))1_n$ for $u \geq t$.

Proof. The equivalence 1. $\Leftrightarrow$ 2. follows from the definition of time-consistent dynamic systemic risk measures, Proposition 7.4.10 and Proposition 7.4.14. The equivalence 1. $\Leftrightarrow$ 3. is a consequence of Proposition 7.4.10, Proposition 7.4.12 and Proposition 7.4.8. □

Summary 7.4.16. Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with time-consistent dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. Moreover, let (7.24) be satisfied. Then Figure 7.4.1 sums up Lemma 7.4.5, Lemma 7.4.6 and Propositions 7.4.8, 7.4.11, 7.4.12 and 7.4.14.

Figure 7.4.1.: Summary: time-consistency properties

$$\boxed{\text{(r-TC)}} \Leftrightarrow \boxed{\text{(s-TC3)}} \Leftrightarrow \boxed{\rho_{s,T}(\bar{X}) = \rho_{s,T}(\bar{X}I_{[s,t)} + f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}))1_n I_{[t,\infty)})}$$

$$\Downarrow$$

$$\boxed{\text{(s-TC2)}} \Leftrightarrow \boxed{\begin{array}{c}\rho_{s,T}(\bar{X}_u(\omega)I_{[s,\infty)}) = \rho_{s,T}(\bar{Z}_u(\omega)I_{[s,\infty)}) \text{ for} \\ \bar{Z}_u(\omega) = \begin{cases}\bar{X}_u(\omega) & \text{if } u < t \\ f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar{X}_u(\omega)I_{[t,\infty)}))1_n & \text{if } u \geq t\end{cases}\end{array}}$$

$$\Updownarrow$$

$$\boxed{\text{(a-TC2)}} \Leftrightarrow \boxed{\Lambda_{s,T}(\bar{X}) = \Lambda_{s,T}(\bar{X}I_{[s,t)} + f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X}))1_n I_{[t,\infty)})}$$

According to Proposition 4.5 in Cheridito et al. (2006) (see Proposition 6.3.3), standard dynamic risk measures that satisfy so called one-step time-consistency are time-consistent if $T \in \mathbb{N}$. The next proposition states the corresponding result for dynamic convex aggregation functions $(\Lambda_{t,T})_{t\in\mathcal{S}}$. For the proof we borrow the idea from the proof of Proposition 4.5 in Cheridito et al. (2006).

Proposition 7.4.17. *Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex aggregation function that satisfies (a-TC1). Then the following properties are equivalent:*

(a-tc) $\Lambda_{s,T}(\bar{X}) = \Lambda_{s,T}(\bar{X}I_{\{s\}} + f^{-1}_{\Lambda_{s+1,T}}(\Lambda_{s+1,T}(\bar{X}))1_n I_{[s+1,\infty)})$ *for all* $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$ *and* $s \in [S, T-1] \cap \mathbb{N}_0$.

(a-tc') $\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{s+1,T}}(\Lambda_{s+1,T}(\bar{x}))1_n)$ *for all* $\bar{x} \in \mathbb{R}^n$ *and* $s \in [S, T-1] \cap \mathbb{N}_0$.

In addition, if one of these equivalent properties is satisfied, then the dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent.

Proof. The implication (a-tc) $\Rightarrow$ (a-tc') follows as in the proof of Proposition 7.4.14. In order to show the reverse implication, fix $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$ and $s \in [S, T-1] \cap \mathbb{N}_0$. Obviously, we have

$$\Lambda_{s,T}(\bar{X})(s,\omega) = \Lambda_{s,T}(\bar{X}_s(\omega)) = \Lambda_{s,T}(\bar{X}I_{\{s\}} + f^{-1}_{\Lambda_{s+1,T}}(\Lambda_{s+1,T}(\bar{X}))1_n I_{[s+1,\infty)})(s,\omega)$$

for a.e. $\omega \in \Omega$. Moreover, for $u \in [s+1,\infty) \cap \mathbb{N}_0$ and a.e. $\omega \in \Omega$, it is easily seen that (a-tc') and the measurability property of $\Lambda_{s,T}$, $f^{-1}_{\Lambda_{s+1,T}}$ and $\Lambda_{s+1,T}$ imply

$$\Lambda_{s,T}(\bar{X})(u,\omega) = \Lambda_{s,T}(\bar{X}_u(\omega)) = \Lambda_{s,T}(f^{-1}_{\Lambda_{s+1,T}}(\Lambda_{s+1,T}(\bar{X}_u(\omega)))1_n)$$
$$= \Lambda_{s,T}(\bar{X}I_{\{s\}} + f^{-1}_{\Lambda_{s+1,T}}(\Lambda_{s+1,T}(\bar{X}))1_n I_{[s+1,\infty)})(u,\omega).$$

This means that property (a-tc) holds.

Now, suppose that (a-tc') is satisfied and consider $s, t \in \mathcal{S}$ with $s \le t$, $\bar{x} \in \mathbb{R}^n$ and $\bar{y} := f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n$. Because of Proposition 7.4.14, it suffices to show that

$$\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(\bar{y})(= \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n)). \tag{7.27}$$

We will prove this equality by backwards induction: If $s = T$, then this equality is trivial because $s = t = T$. Therefore, consider $s \le T-1$ and for all $t' \in [s+1, T] \cap \mathbb{N}_0$, assume that

$$\Lambda_{s+1,T}(\bar{z}) = \Lambda_{s+1,T}(f^{-1}_{\Lambda_{t',T}}(\Lambda_{t',T}(\bar{z}))1_n) \quad \text{for all } \bar{z} \in \mathbb{R}^n. \tag{7.28}$$

Since $s \le t$, we know that either $t = s$ or $t \ge s+1$. Obviously, (7.27) is satisfied if $t = s$. Thus, assume that $t \ge s+1$. Then Equation (7.28) implies

$$\Lambda_{s+1,T}(\bar{x}) = \Lambda_{s+1,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n) = \Lambda_{s+1,T}(\bar{y}),$$

which yields together with (a-tc')

$$\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{s+1,T}}(\Lambda_{s+1,T}(\bar{x}))1_n) = \Lambda_{s,T}(f^{-1}_{\Lambda_{s+1,T}}(\Lambda_{s+1,T}(\bar{y}))1_n) = \Lambda_{s,T}(\bar{y}).$$

This means that $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent. □

7.4.2. Examples of time-consistent dynamic aggregation functions

According to Definition 7.4.4, we can construct time-consistent dynamic convex systemic risk measures as a composition of time-consistent dynamic convex single-firm risk measures and time-consistent dynamic convex aggregation functions.

Since every time-consistent standard convex risk measure $(\tilde{\rho}_t)_{t\in\mathcal{S}}$ defined in Section 6.3 induces a time-consistent dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ by $\rho^0_{t,T}(X) := \tilde{\rho}_{t,T}(-X)$ for $t \in \mathcal{S}$ and $X \in \mathcal{R}^\infty_{t,T}$, we have already discussed examples of time-consistent dynamic convex single-firm risk measures in Section 6.4. It remains the question of how time-consistent dynamic convex aggregation functions look like. A first step to answer this question is the following proposition, which yields equivalent properties to time-consistency.

Proposition 7.4.18. *Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex aggregation function that satisfies (a-TC1).*

1. *If all $\Lambda_{t,T}$, $t \in \mathcal{S}$, are $\mathbb{R}$-valued, then $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$ does not depend on $t \in \mathcal{S}$ for all $\bar{x} \in \mathbb{R}^n$.*
2. *If all $\Lambda_{t,T}$, $t \in \mathcal{S}$, are $\mathbb{R}_+$-valued, then $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if the following properties are satisfied:*

a) $b_{t,T} \leq b_{s,T}$ *for all* $s, t \in \mathcal{S}$ *with* $s \leq t$.

b) The following conditions hold for all $s, t \in \mathcal{S}$ *with* $s \leq t$*:*

i. If $\Lambda_{t,T}(\bar{x}) > 0$ *and* $\Lambda_{s,T}(\bar{x}) = 0$ *for* $\bar{x} \in \mathbb{R}^n$*, then* $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) \leq f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))(= b_{s,T})$.

ii. If $\Lambda_{t,T}(\bar{x}) > 0$ *and* $\Lambda_{s,T}(\bar{x}) > 0$ *for* $\bar{x} \in \mathbb{R}^n$*, then* $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$.

In particular, $(\Lambda_{t,T})_{t\in\mathcal{S}}$ *is time-consistent if* $\Lambda_{t,T} = r_t\Lambda$ *for a convex aggregation function* Λ *and* $r_t \in \mathbb{R}_+\backslash\{0\}$.

Proof. Part 1: Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be time-consistent and $\mathbb{R}$-valued and fix $\bar{x} \in \mathbb{R}^n$. Then it follows from Proposition 7.4.14 that

$$\begin{aligned}\Lambda_{s,T}(\bar{x}) &= \Lambda_{s,T}(f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))1_n)\\ &= \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n) \quad \text{for all } s, t \in \mathcal{S} \text{ with } s \leq t. \end{aligned} \tag{7.29}$$

Moreover, the $f_{\Lambda_{s,T}}$-constancy property implies

$$\Lambda_{s,T}(\bar{x}) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))) \quad \text{for all } s, t \in \mathcal{S} \text{ with } s \leq t. \tag{7.30}$$

Bijectivity of $f_{\Lambda_{s,T}} : \mathbb{R} \to \mathbb{R}$ leads to $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$. This yields the "only if" part of the assertion.

The "if" part follows from Proposition 7.4.14 since $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$ for all $s, t \in \mathcal{S}$ implies immediately that (7.30) holds for all $\bar{x} \in \mathbb{R}^n$.

Part 2: Now, assume that all $\Lambda_{t,T}$ map into $\mathbb{R}_+$. Note that in this case, each $f_{\Lambda_{t,T}} : \mathbb{R} \to \mathbb{R}_+$ is such that $f_{\Lambda_{t,T}}|_{[b_{t,T},\infty)}$ is a bijective, strictly increasing function from $[b_{t,T}, \infty)$ to $\mathbb{R}_+$ and $f_{\Lambda_{t,T}}(a) = 0$ for all $a \leq b_{t,T}$. Moreover, $f^{-1}_{\Lambda_{t,T}} : \mathbb{R}_+ \to [b_{t,T}, \infty)$ is strictly increasing, too.

At first, let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be time-consistent, fix $\bar{x} \in \mathbb{R}^n$ and consider $s, t \in \mathcal{S}$ with $s \leq t$. Then either $\Lambda_{t,T}(\bar{x}) = 0$ or $\Lambda_{t,T}(\bar{x}) > 0$.

If $\Lambda_{t,T}(\bar{x}) = 0$, then $\Lambda_{s,T}(\bar{x}) = 0$ due to (a-TC2'). Moreover, Equation (7.30) yields

$$0 = \Lambda_{s,T}(\bar{x}) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(0)).$$

$f^{-1}_{\Lambda_{t,T}}(0) = b_{t,T}$ now implies that $0 = f_{\Lambda_{s,T}}(b_{t,T})$. Therefore, we obtain $b_{t,T} \leq b_{s,T}$.

If $\Lambda_{t,T}(\bar{x}) > 0$, then we distinguish between two cases: Either $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) = 0$ or $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) > 0$. In both cases, we have $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) > b_{t,T}$ because $f^{-1}_{\Lambda_{t,T}}$ is a strictly increasing function.

If $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) = 0$, the equality

$$0 = \Lambda_{s,T}(\bar{x}) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})))$$

implies $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) \leq b_{s,T} = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$.

Now, suppose that $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) > 0$. Because $f_{\Lambda_{s,T}}$ is strictly increasing on $[b_{s,T}, \infty)$ and $f_{\Lambda_{s,T}}(a) = 0$ for all $a \leq b_{s,T}$, Equation (7.30) and $\Lambda_{s,T}(\bar{x}) > 0$

imply $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) > b_{s,T}$ and $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) > b_{s,T}$. Further, bijectivity and (7.30) lead to $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$. This yields the "only if" part of the assertion.

In order to prove the "if" part, it suffices again to show (7.30) for $\bar{x} \in \mathbb{R}^n$. Then the assertion follows from Proposition 7.4.14. To this end, consider $s, t \in \mathcal{S}$ with $s \leq t$. In case of $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) > 0$, Equation (7.30) is obvious. If $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) = 0$, we have $0 = \Lambda_{s,T}(\bar{x}) = f_{\Lambda_{s,T}}(f^{-1}_{s,T}(\Lambda_{s,T}(\bar{x}))) \geq f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})))$. Hence, it follows that $0 = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})))$. Finally, if $\Lambda_{t,T}(\bar{x}) = 0$ and $\Lambda_{s,T}(\bar{x}) = 0$, then $b_{t,T} \leq b_{s,T}$ implies that $\Lambda_{s,T}(\bar{x}) = 0 = f_{\Lambda_{s,T}}(b_{t,T}) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})))$.

To the last assertion: Consider $\Lambda_{t,T} = r_t \Lambda$ for $r_t \in \mathbb{R}_+ \backslash \{0\}$. At first, assume that all $\Lambda_{t,T}$, $t \in \mathcal{S}$, are $\mathbb{R}$-valued. Then $f_{\Lambda_{t,T}}(a) = r_t f_\Lambda(a)$ and $f^{-1}_{\Lambda_{t,T}}(a) = f^{-1}_\Lambda(a/r_t)$ for all $t \in \mathcal{S}$ and $a \in \mathbb{R}$. This means

$$f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_\Lambda(\Lambda_{t,T}(\bar{x})/r_t) = f^{-1}_\Lambda(r_t\Lambda(\bar{x})/r_t) = f^{-1}_\Lambda(\Lambda(\bar{x})) \quad \text{for } \bar{x} \in \mathbb{R}^n.$$

Thus, $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$ does not depend on t for each $\bar{x} \in \mathbb{R}^n$. The first part of this proposition implies that our dynamic convex aggregation function is time-consistent.

On the other hand, let all $\Lambda_{t,T}$, $t \in \mathcal{S}$, be $\mathbb{R}_+$-valued. Then for each $t \in \mathcal{S}$, we have $f_{\Lambda_{t,T}}(a) = r_t f_\Lambda(a)$ for all $a \in \mathbb{R}$ and $f^{-1}_{\Lambda_{t,T}}(a) = f^{-1}_\Lambda(a/r_t)$ for all $a \in \mathbb{R}_+$. Moreover, $b = b_{t,T}$ and $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_\Lambda(\Lambda(\bar{x}))$ for $\bar{x} \in \mathbb{R}^n$ and all $t \in \mathcal{S}$. Hence, the second part of this proposition yields time-consistency of $(\Lambda_{t,T})_{t\in\mathcal{S}}$. □

The following lemma considers dynamic convex aggregation functions $(\Lambda_{t,T})_{t\in\mathcal{S}}$ that admit a specific form.

Lemma 7.4.19. *Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex aggregation function with*

$$\Lambda_{t,T}(\bar{x}) := \sum_{i=1}^{n} g_{t,T}((\bar{x}_i - b_{t,T})^+) \quad \textit{for } \bar{x} \in \mathbb{R}^n \tag{7.31}$$

where $b_{t,T} \in \mathbb{R}_+$ and $g_{t,T} : \mathbb{R}_+ \to \mathbb{R}_+$ are convex and strictly increasing functions with $g_{t,T}(0) = 0$. Then $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if $b_{t,T} = b$ for all $t \in \mathcal{S}$ and $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$ for all $\bar{x} \in \mathbb{R}$ and $s, t \in \mathcal{S}$.

Remark. Note that the convex aggregation function $\Lambda_{t,T}$ defined in (7.31) satisfies $f_{\Lambda_{t,T}}(a) = \Lambda_{t,T}(a1_n) = n g_{t,T}((a - b_{t,T})^+)$ for all $a \in \mathbb{R}$ and all $t \in \mathcal{S}$. Therefore, the values $b_{t,T}$ are indeed the b's from the corresponding $f_{\Lambda_{t,T}}$-constancy property.

Proof. If $b_{t,T} = b$ for all $t \in \mathcal{S}$ and $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$ for all $\bar{x} \in \mathbb{R}^n$ and $s, t \in \mathcal{S}$, then $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent due to the second part of Proposition 7.4.18.

To prove the other direction, suppose that $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent and fix $\bar{x} \in \mathbb{R}^n$ and $s, t \in \mathcal{S}$ with $s < t$. The second part of Proposition 7.4.18 yields $b_{t,T} \leq b_{s,T}$. We will show that $b_{t,T} < b_{s,T}$ leads to a contradiction. To this end, suppose that

$b_{t,T} < b_{s,T}$. Equation (7.31) implies for all $u \in \mathcal{S}$ that $f_{\Lambda_{u,T}}(a) = ng_{u,T}((a - b_{u,T})^+)$ for $a \in \mathbb{R}$ and $f^{-1}_{\Lambda_{u,T}}(a) = g^{-1}_{u,T}(a/n) + b_{u,T}$ for $a \in \mathbb{R}_+$. It follows

$$\begin{aligned} f^{-1}_{\Lambda_{u,T}}(\Lambda_{u,T}(\bar{x})) &= g^{-1}_{u,T}(\Lambda_{u,T}(\bar{x})/n) + b_{u,T} \\ &= g^{-1}_{u,T}\left(\sum_{i=1}^{n} g_{u,T}((\bar{x}_i - b_{u,T})^+)/n\right) + b_{u,T} \quad \text{for all } u \in \mathcal{S}. \end{aligned} \tag{7.32}$$

For $\bar{y} \in \mathbb{R}^n$ defined by $\bar{y}_1 = b_{s,T} + 1$, $\bar{y}_2 = \ldots = \bar{y}_n = b_{t,T}$, we get

$$\Lambda_{s,T}(\bar{y}) = \sum_{i=1}^{n} g_{s,T}((\bar{y}_i - b_{s,T})^+) = g_{s,T}(1) > 0 \text{ and } \Lambda_{t,T}(\bar{y}) = g_{t,T}(b_{s,T} + 1 - b_{t,T}) > 0.$$

Since $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent, the second part of Proposition 7.4.18 yields that $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{y})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{y}))$, i.e.,

$$g^{-1}_{s,T}(g_{s,T}(1)/n) + b_{s,T} = g^{-1}_{t,T}(g_{t,T}(b_{s,T} + 1 - b_{t,T})/n) + b_{t,T}.$$

Similarly, for $\bar{z} \in \mathbb{R}^n$ defined by $\bar{z}_1 = b_{s,T} + 1$, $\bar{z}_2 = b_{t,T} + c$, $\bar{z}_3 = \ldots = \bar{z}^n = b_{t,T}$ for $c := (b_{s,T} - b_{t,T})/2 > 0$, we obtain

$$\Lambda_{s,T}(\bar{z}) = g_{s,T}(1) > 0 \quad \text{and} \quad \Lambda_{t,T}(\bar{z}) = g_{t,T}(b_{s,T} + 1 - b_{t,T}) + g_{t,T}(c) > 0.$$

The second part of Proposition 7.4.18 gives $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{z})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{z}))$, i.e.,

$$g^{-1}_{s,T}(g_{s,T}(1)/n) + b_{s,T} = g^{-1}_{t,T}([g_{t,T}(b_{s,T} + 1 - b_{t,T}) + g_{t,T}(c)]/n) + b_{t,T}.$$

Altogether, we obtain

$$g^{-1}_{t,T}(g_{t,T}(b_{s,T} + 1 - b_{t,T})/n) = g^{-1}_{t,T}([g_{t,T}(b_{s,T} + 1 - b_{t,T}) + g_{t,T}(c)]/n).$$

But this is a contradiction because $g^{-1}_{t,T} : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function and $g_{t,T}(c) > 0$.

It remains to prove that $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) = 0$ imply $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = b$. Since $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent, the second part of Proposition 7.4.18 tells us that $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) \leq b$. Suppose that $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) < b$. Then it follows with Equation (7.32) that

$$f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = g^{-1}_{t,T}\left(\sum_{i=1}^{n} g_{t,T}((\bar{x}_i - b)^+)/n\right) + b < b,$$

which is a contradiction to $g^{-1}_{t,T}(\sum_{i=1}^{n} g_{t,T}((\bar{x}_i - b)^+)/n) \geq 0$. $\square$

With the previous results and the convex aggregation functions introduced in Section 5.3 we can easily construct time-consistent dynamic aggregation functions.

Example 7.4.20. Consider Λ_{sum} defined by

$$\Lambda_{\text{sum}}(\bar{x}) := \sum_{i=1}^{n} \bar{x}_i \quad \text{for } \bar{x} \in \mathbb{R}^n.$$

Then $(\Lambda_{t,T}^{\text{sum}})_{t \in \mathcal{S}}$ defined by

$$\Lambda_{t,T}^{\text{sum}}(\bar{x}) := r_t \sum_{i=1}^{n} \bar{x}_i, \quad \text{for } \bar{x} \in \mathbb{R}^n \text{ and } r_t \in \mathbb{R}_+ \backslash \{0\}$$

is a time-consistent dynamic convex aggregation function. In the same way, we can construct the time-consistent dynamic convex aggregation functions $(\Lambda_{t,T}^{\text{loss}})_{t \in \mathcal{S}}$, $(\Lambda_{t,T}^{b,\text{loss}})_{t \in \mathcal{S}}$, $(\Lambda_{t,T}^{\text{exp}})_{t \in \mathcal{S}}$ and $(\Lambda_{t,T}^{\text{plin}})_{t \in \mathcal{S}}$ by

$$\begin{aligned}
\Lambda_{t,T}^{\text{loss}}(\bar{x}) &:= r_t \sum_{i=1}^{n} \bar{x}_i^+ \\
\Lambda_{t,T}^{b,\text{loss}}(\bar{x}) &:= r_t \sum_{i=1}^{n} (\bar{x}_i - b)^+ \quad \text{for } b \in \mathbb{R}_+ \\
\Lambda_{t,T}^{\text{exp}}(\bar{x}) &:= r_t \sum_{i=1}^{n} (e^{\gamma \bar{x}_i^+} - 1) \quad \text{for } \gamma > 0 \\
\Lambda_{t,T}^{\text{plin}}(\bar{x}) &:= r_t \sum_{i=1}^{n} \lambda(\bar{x}_i)
\end{aligned}$$

where $\bar{x} \in \mathbb{R}^n$, $r_t \in \mathbb{R}_+ \backslash \{0\}$ for each $t \in \mathcal{S}$ and

$$\lambda(x) = \begin{cases} 0 & \text{for } x < 0 \\ ax & \text{for } 0 \leq x < c \\ b(x-c) + ac & \text{for } x \geq c \end{cases}$$

with $0 < a < b$ and $c > 0$.

By using the second part of Proposition 7.4.18, we obtain the following two examples.

Example 7.4.21. Define the convex aggregation function $\Lambda_{t,T}^{[\exp]}$ for $t \in \mathcal{S}$ by

$$\Lambda_{t,T}^{[\exp]}(\bar{x}) := e^{r_t \sum_{i=1}^{n} \bar{x}_i^+} - 1 \quad \text{for } \bar{x} \in \mathbb{R}^n \text{ and } r_t \in \mathbb{R}_+ \backslash \{0\}.$$

In this case, we have $f_{\Lambda_{t,T}^{[\exp]}}(a) = e^{r_t n a^+} - 1$ for $a \in \mathbb{R}$ and $f^{-1}_{\Lambda_{t,T}^{[\exp]}}(a) = \log(a+1)/(r_t n)$ for $a \in \mathbb{R}_+$, which implies

$$f^{-1}_{\Lambda_{t,T}^{[\exp]}}(\Lambda_{t,T}^{[\exp]}(\bar{x})) = \frac{\log\left(e^{r_t \sum_{i=1}^{n} \bar{x}_i^+} - 1 + 1\right)}{r_t n} = \frac{\sum_{i=1}^{n} \bar{x}_i^+}{n} \quad \text{for all } \bar{x} \in \mathbb{R}^n.$$

Since all conditions in the second part of Proposition 7.4.18 are satisfied, $(\Lambda_{t,T}^{[\exp]})_{t \in \mathcal{S}}$ is a time-consistent dynamic convex aggregation function according to Proposition 7.4.18.

Example 7.4.22. Fix $k > 1$ and define the convex aggregation function $\Lambda_{t,T}^{[k]}$ for $t \in \mathcal{S}$ by

$$\Lambda_{t,T}^{[k]}(\bar{x}) := \left(r_t \sum_{i=1}^{n} \bar{x}_i^+ \right)^k \quad \text{for } \bar{x} \in \mathbb{R}^n \text{ and } r_t \in \mathbb{R}_+ \backslash \{0\}.$$

In this case, we have $f_{\Lambda_{t,T}^{[k]}}(a) = (r_t n a^+)^k$ for $a \in \mathbb{R}$ and $f^{-1}_{\Lambda_{t,T}^{[k]}}(a) = \sqrt[k]{a/(r_t^k n^k)} = \sqrt[k]{a}/r_t n$ for $a \in \mathbb{R}_+$, which implies

$$f^{-1}_{\Lambda_{t,T}^{[k]}}(\Lambda_{t,T}^{[k]}(\bar{x})) = \frac{\sqrt[k]{(r_t \sum_{i=1}^{n} \bar{x}_i^+)^k}}{r_t n} = \frac{\sum_{i=1}^{n} \bar{x}_i^+}{n} \quad \text{for all } \bar{x} \in \mathbb{R}^n.$$

Again, all conditions in the second part of Proposition 7.4.18 are satisfied. Thus, $(\Lambda_{t,T}^{[k]})_{t \in \mathcal{S}}$ is a time-consistent dynamic convex aggregation function.

Let us recapitulate the results in this section. We have studied time-consistent dynamic convex systemic risk measures $(\rho_{t,T})_{t \in \mathcal{S}}$ as a composition of time-consistent dynamic convex single-firm risk measures $(\rho_{t,T}^0)_{t \in \mathcal{S}}$ and time-consistent dynamic convex aggregation functions $(\Lambda_{t,T})_{t \in \mathcal{S}}$. Here, each $\rho_{t,T}^0$ is a mapping from $\mathcal{R}_{t,T}^{\infty}$ to L_t^{∞}, and each $\Lambda_{t,T}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$. We can compose these mappings since every $\Lambda_{t,T}$ is measurable, and thus induces a mapping from $\mathcal{R}_{t,T}^{\infty,n}$ to $\mathcal{R}_{t,T}^{\infty}$ (see Lemma 7.1.7). In this setting, we have studied how the time-consistency properties of $(\rho_{t,T})_{t \in \mathcal{S}}$, $(\rho_{t,T}^0)_{t \in \mathcal{S}}$ and $(\Lambda_{t,T})_{t \in \mathcal{S}}$, which are based on the strong time-consistency concept, depend on each other. A key result clarifying the relation between these properties is the characterization of time-consistent dynamic convex systemic risk measures in Proposition 7.4.8.

It is important to mention that convex aggregation functions – defined as convex, increasing functions from $\mathbb{R}^n$ to $\mathbb{R}$ – have a rather simple structure. In a certain sense, this definition limits the possibilities to construct dynamic convex aggregation functions that additionally satisfy the time-consistency property from Definition 7.4.3.

Despite this strong notion of time-consistency and the simple setting of functions from $\mathbb{R}^n$ to $\mathbb{R}$, we have found several interesting examples of time-consistent dynamic convex aggregation functions in Subsection 7.4.2. In particular, we refer to Lemma 7.4.19 and Examples 7.4.20-7.4.22. Moreover, an additional example is the trivial time-consistent dynamic convex aggregation function. Here, we fix the (static) convex aggregation function Λ at the beginning of the considered period of time and then this specific aggregation function is used for every following point in time, i.e., $\Lambda_{t,T} = \Lambda$ for all $t \in \mathcal{S}$.

Finally, one could argue that it also makes sense to introduce time-consistent dynamic convex systemic risk measures by considering time-consistent dynamic convex single-firm risk measures and fixing the convex aggregation function. This means that we use the trivial time-consistent dynamic convex aggregation function in order to specify time-consistent dynamic convex systemic risk measures, i.e., $\rho_{t,T} = \rho_{t,T}^0 \circ \Lambda$. Consequently, all dynamics of the corresponding dynamic convex systemic risk measure $(\rho_{t,T})_{t \in \mathcal{S}}$ are determined by the underlying dynamic convex single-firm risk measure.

Appendix

A.1. Appendix to Part I

A.1.1. General results in probability theory

The following version of the strong Markov property, originally proved in Hunt (1956), is needed in Chapter 3.

Theorem A.1.1 (See Theorem 3.5.1 in Petersen (1977))**.** *Consider the filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and suppose that this probability space supports a standard Brownian motion W. Let θ be an optional time, i.e., $\{\theta < t\} \in \mathcal{F}_t$ for all $t \geq 0$, and define the probability space $(\Omega_\theta, \mathcal{A}_\theta, \mathbb{P}_\theta)$ by $\Omega_\theta := \{\theta < \infty\}$, $\mathcal{A}_\theta := \{A \cap \Omega_\theta | A \in \mathcal{A}\}$ and $\mathbb{P}_\theta[A] = \mathbb{P}[A|\Omega_\theta]$. Then*

1. *The process $(W_{t+\theta} - W_\theta)_{t\geq 0}$ is a standard Brownian motion on $(\Omega_\theta, \mathcal{A}_\theta, \mathbb{P}_\theta)$.*
2. *$(W_{t+\theta} - W_\theta)_{t\geq 0}$ is independent of $\mathcal{F}_{\theta+} \cap \Omega_\theta$ in the probability space $(\Omega_\theta, \mathcal{A}_\theta, \mathbb{P}_\theta)$. ($\mathcal{F}_{\theta+} := \{A \in \mathcal{A} | A \cap \{\theta < t\} \in \mathcal{F}_t$ for all $t \geq 0\}$)*

Lemma A.1.2 (See Lemma 3.5.2 in Petersen (1977))**.** *Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and suppose that X and Y are independent random variable and let $A \subset \Omega$ be a measurable set. If Y is independent of $\sigma(A) \vee \sigma(X)$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is measurable, then*

$$\int_{\omega_1 \in A} \int_{\omega_2 \in \Omega} f(X(\omega_1), Y(\omega_2)) d\mathbb{P}(\omega_1) d\mathbb{P}(\omega_2) = \int_A f(X,Y) d\mathbb{P}.$$

Lemma A.1.3. *Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and let θ and τ be $\mathbb{F}$-stopping times. Then every integrable random variable X satisfies*

$$\mathbb{E}[X I_{\{\theta > \tau\}} | \mathcal{F}_{\theta \vee \tau}] = \mathbb{E}[X | \mathcal{F}_{\theta \vee \tau}] I_{\{\theta > \tau\}} = \mathbb{E}[X I_{\{\theta > \tau\}} | \mathcal{F}_\theta].$$

Proof. Consider $A \in \mathcal{F}_{\theta \vee \tau}$. Since $\{\theta > \tau\} \in \mathcal{F}_{\theta \vee \tau}$, we know that $A \cap \{\theta > \tau\} \in \mathcal{F}_{\theta \vee \tau}$. By definition of $\mathcal{F}_{\theta \vee \tau}$, this implies

$$A \cap \{\theta > \tau\} \cap \{\theta \leq t\} = A \cap \{\theta > \tau\} \cap \{\theta \vee \tau \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0,$$

which means that $A \cap \{\theta > \tau\} \in \mathcal{F}_\theta$. For each $B \in \mathcal{B}(\mathbb{R})$, we obtain

$$\begin{aligned} &\{\omega \in \Omega | (\mathbb{E}[X|\mathcal{F}_{\theta\vee\tau}] I_{\{\theta > \tau\}})(\omega) \in B\} \\ &= (\{\theta \leq \tau\} \cap \{\omega \in \Omega | 0 \in B\}) \cup (\{\theta > \tau\} \cap \{\omega \in \Omega | \mathbb{E}[X|\mathcal{F}_{\theta\vee\tau}](\omega) \in B\}) \in \mathcal{F}_\theta \end{aligned}$$

since $\{\theta \leq \tau\} \cap \{\omega \in \Omega | 0 \in B\} \in \mathcal{F}_\theta$ and $\{\omega \in \Omega | \mathbb{E}[X|\mathcal{F}_{\theta\vee\tau}](\omega) \in B\} \in \mathcal{F}_{\theta\vee\tau}$. This means that $\mathbb{E}[X|\mathcal{F}_{\theta\vee\tau}] I_{\{\theta > \tau\}}$ is $\mathcal{F}_\theta$-measurable.

Now, consider $C \in \mathcal{F}_\theta \subset \mathcal{F}_{\theta\vee\tau}$. Then

$$\mathbb{E}[\mathbb{E}[XI_{\{\theta>\tau\}}|\mathcal{F}_\theta]I_C] = \mathbb{E}[XI_{\{\theta>\tau\}}I_C] = \mathbb{E}[\mathbb{E}[XI_{\{\theta>\tau\}}|\mathcal{F}_{\theta\vee\tau}]I_C].$$

Since $\{\theta > \tau\} \in \mathcal{F}_{\theta\vee\tau}$, it follows $\mathbb{E}[\mathbb{E}[XI_{\{\theta>\tau\}}|\mathcal{F}_{\theta\vee\tau}]I_C] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_{\theta\vee\tau}]I_{\{\theta>\tau\}}I_C]$, and thus $\mathbb{E}[XI_{\{\theta>\tau\}}|\mathcal{F}_\theta] = \mathbb{E}[X|\mathcal{F}_{\theta\vee\tau}]I_{\{\theta>\tau\}}$. □

A.1.2. Hitting times

In this subsection we repeat important results for first hitting times in two different approaches. At the beginning, we focus on the standard approach considering a Brownian motion with drift; see, for example, Chapter 3 in Jeanblanc et al. (2009) or Section 2.8 in Karatzas and Shreve (1988).

For the remaining part of this subsection fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ that supports a standard Brownian motion W and a random variable X_0. The process X is defined by

$$X_t := X_0 + \mu t + \sigma W_t \quad \text{for } \mu \in \mathbb{R} \text{ and } \sigma > 0.$$

We are interested in the *first hitting time* T_0^X *of level* 0 for the process X, i.e.,

$$T_0^X := \inf\{t \geq 0 | X_t = 0\}.$$

Theorem A.1.4. *In case of $X_0 = x > 0$ constant, the probability that the first hitting time T_0^X is less than or equal to $t > 0$ is given by*

$$\mathbb{P}[T_0^X \leq t] = \mathbb{P}\left[\min_{s\leq t} X_s \leq 0\right] = \Phi\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right) + e^{-2\mu x/\sigma^2}\Phi\left(\frac{-x+\mu t}{\sigma\sqrt{t}}\right).$$

Yi et al. (2011) generalize this approach by assuming that X_0 is given by a random variable. They obtain the following result.

Theorem A.1.5 (See Proposition 3.1 in Yi et al. (2011))**.** *Suppose that X_0 and W_t are independent for each $t > 0$ and let the density of X_0 be given by*

$$\psi(x; a_0, v_0, \sigma_0) := \begin{cases} \frac{\varphi(x;a_0+v_0,\sigma_0) - e^{-2a_0v_0/\sigma_0^2}\varphi(x;v_0-a_0,\sigma_0)}{\Phi\left(\frac{a_0+v_0}{\sigma_0}\right) - e^{-2a_0v_0/\sigma_0^2}\Phi\left(\frac{v_0-a_0}{\sigma_0}\right)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

with $\sigma_0 > 0$ and $a_0 > |v_0|$ where $\varphi(x, \bar{\mu}, \bar{\sigma})$ denotes the probability density function of the normal distribution $\mathcal{N}(\bar{\mu}, \bar{\sigma}^2)$. Then the probability that the first hitting time T_0^X is lesser than or equal to $t > 0$ is given by

$$\mathbb{P}[T_0^X \leq t] = \mathbb{P}\left[\min_{s\leq t} X_s \leq 0\right] = \frac{A(t) + B(t) - C(t) - D(t)}{\Phi\left(\frac{a_0+v_0}{\sigma_0}\right) - e^{-2a_0v_0/\sigma_0^2}\Phi\left(\frac{v_0-a_0}{\sigma_0}\right)}$$

with

$$A(t) = \Phi_2\left(-\frac{a_0 + v_0 + \mu t}{\sqrt{\sigma_0^2 + \sigma^2 t}}, -\frac{-a_0 - v_0}{\sigma_0}, \rho(t)\right),$$

$$B(t) = \Phi_2\left(-\frac{a_0 + v_0 - 2\mu\sigma_0^2/\sigma^2 - \mu t}{\sqrt{\sigma_0^2 + \sigma^2 t}}, -\frac{-a_0 - v_0 + 2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho(t)\right)$$
$$\cdot\, e^{-2\mu(a_0+v_0)/\sigma^2 + 2\mu^2\sigma_0^2/\sigma^4},$$

$$C(t) = \Phi_2\left(-\frac{v_0 - a_0 + \mu t}{\sqrt{\sigma_0^2 + \sigma^2 t}}, -\frac{-v_0 + a_0}{\sigma_0}, \rho(t)\right) e^{-2a_0v_0/\sigma_0^2},$$

$$D(t) = \Phi_2\left(-\frac{v_0 - a_0 - 2\mu\sigma_0^2/\sigma^2 - \mu t}{\sqrt{\sigma_0^2 + \sigma^2 t}}, -\frac{-v_0 + a_0 + 2\mu\sigma_0^2/\sigma^2}{\sigma_0}, \rho(t)\right)$$
$$\cdot\, e^{-2a_0v_0/\sigma_0^2 - 2\mu(v_0-a_0)/\sigma^2 + 2\mu^2\sigma_0^2/\sigma^4}$$

and $\rho(t) = \frac{-\sigma_0}{\sqrt{\sigma_0^2+\sigma^2 t}}$ *where* $\Phi_2(x_1, x_2, \rho)$ *denotes the* 2*-dimensional normal distribution function with standard normal marginal distributions and correlation coefficient* ρ.

Remark A.1.6. Note that the distribution function Ψ, i.e., $\Psi(x) := \mathbb{P}[X_0 \leq x]$ for $x \in \mathbb{R}$, is given by

$$\Psi(x) = \int_0^x \frac{\varphi(y; a_0 + v_0, \sigma_0) - e^{-2a_0v_0/\sigma_0^2}\varphi(y; v_0 - a_0, \sigma_0)}{\Phi\left(\frac{a_0+v_0}{\sigma_0}\right) - e^{-2a_0v_0/\sigma_0^2}\Phi\left(\frac{v_0-a_0}{\sigma_0}\right)} dy$$
$$= \frac{\Phi\left(\frac{x-a_0-v_0}{\sigma_0}\right) - \Phi\left(\frac{-a_0-v_0}{\sigma_0}\right) - e^{-2a_0v_0/\sigma_0^2}\left(\Phi\left(\frac{x+a_0-v_0}{\sigma_0}\right) - \Phi\left(\frac{a_0-v_0}{\sigma_0}\right)\right)}{\Phi\left(\frac{a_0+v_0}{\sigma_0}\right) - e^{-2a_0v_0/\sigma_0^2}\Phi\left(\frac{v_0-a_0}{\sigma_0}\right)}$$

for $x \geq 0$ and $\Psi(x) = 0$ for $x < 0$.

Proposition A.1.7 (See Example 3.2 in Yi et al. (2011))**.** *Let the assumptions from Theorem A.1.5 be satisfied and suppose that* $\mu/\sigma^2 = v_0/\sigma_0^2$. *Then for every* $t > 0$, *we have*

$$\mathbb{P}[T_0^X \leq t] = \frac{\Phi\left(-\frac{a_0+v_0+\mu t}{\sqrt{\sigma^2 t+\sigma_0^2}}\right) + e^{-2a_0\mu/\sigma^2}\Phi\left(-\frac{a_0-v_0-\mu t}{\sqrt{\sigma^2 T+\sigma_0^2}}\right)}{\Phi\left(\frac{a_0+v_0}{\sigma_0}\right) - e^{-2a_0v_0/\sigma_0^2}\Phi\left(\frac{v_0-a_0}{\sigma_0}\right)}$$
$$- \frac{\Phi\left(-\frac{a_0+v_0}{\sigma_0}\right) + e^{-2a_0\mu/\sigma^2}\Phi\left(\frac{v_0-a_0}{\sigma_0}\right)}{\Phi\left(\frac{a_0+v_0}{\sigma_0}\right) - e^{-2a_0v_0/\sigma_0^2}\Phi\left(\frac{v_0-a_0}{\sigma_0}\right)}.$$

A.2. Appendix to Part II

A.2.1. Functional analysis

The following definitions and results are taken from Dunford and Schwartz (1957), Rockafellar (1974) and Aliprantis and Border (2006).

Definition A.2.1. *Let $\mathcal{X}$ and $\mathcal{V}$ be two linear spaces. A* pairing *between $\mathcal{X}$ and $\mathcal{V}$ is a real-valued bilinear form $\langle \cdot, \cdot \rangle$. A subset $\mathcal{A} \subset \mathcal{V}$* separates points of $\mathcal{X}$ under $\langle \cdot, \cdot \rangle$ *if $\langle x, v \rangle = 0$ for all $v \in \mathcal{A}$ implies $x = 0$. Similarly, a subset $\mathcal{B} \subset \mathcal{X}$* separates points of $\mathcal{V}$ under $\langle \cdot, \cdot \rangle$ *if $\langle x, v \rangle = 0$ for all $x \in \mathcal{B}$ implies $v = 0$.*

We call a topology $\mathfrak{T}$ on $\mathcal{X}$ compatible *with the pairing if it is a locally convex topology such that for all $v \in \mathcal{V}$, the linear functions*

$$\langle \cdot, v \rangle : \mathcal{X} \to \mathbb{R}, \quad x \mapsto \langle x, v \rangle \tag{A.33}$$

are continuous and every continuous linear function on $\mathcal{X}$ admits a representation of this form for some $v \in \mathcal{V}$. Compatible topologies on $\mathcal{V}$ are defined analogously.

We call $\mathcal{X}$ and $\mathcal{V}$ paired spaces, *denoted by $(\mathcal{X}, \mathcal{V})$, if a specific pairing has been chosen, if $\mathcal{X}$ separates points of $\mathcal{V}$ and $\mathcal{V}$ separates points of $\mathcal{X}$ under this pairing and if $\mathcal{X}$ and $\mathcal{V}$ are equipped with compatible topologies.*

If a specific pairing has been chosen and if $\mathcal{V}$ separates points of $\mathcal{X}$ under this pairing, then the $\mathcal{V}$-topology on $\mathcal{X}$, *denoted by $\sigma(\mathcal{X}, \mathcal{V})$, is the weakest (coarsest) topology on $\mathcal{X}$ for which every linear function given in* (A.33) *is continuous.*

The following theorem shows that the topology $\sigma(\mathcal{X}, \mathcal{V})$ is compatible with the pairing $\langle \cdot, \cdot \rangle$.

Theorem A.2.2 (See, e.g., Section V.3 in Dunford and Schwartz (1957))**.** *Consider the paired spaces $(\mathcal{X}, \mathcal{V})$ and let $\langle \cdot, \cdot \rangle$ be the corresponding pairing. Then the following properties hold:*

1. *$(\mathcal{X}, \sigma(\mathcal{X}, \mathcal{V}))$ is a locally convex space.*
2. *The linear functionals on $\mathcal{X}$ which are $\sigma(\mathcal{X}, \mathcal{V})$-continuous are precisely the functionals given by* (A.33).

Theorem A.2.3 (Uniform boundedness principle; see, e.g., Theorem 6.14 in Aliprantis and Border (2006))**.** *Consider a Banach space $\mathcal{X}$ and a nonempty subset $\mathcal{A} \subset \mathcal{L}(\mathcal{X}, \mathbb{R})$. Then $\mathcal{A}$ is norm bounded if and only if it is pointwise bounded, i.e., for each $x \in \mathcal{X}$, there exists $M_x > 0$ such that $|f(x)| \leq M_x$ for each $f \in \mathcal{A}$.*

Theorem A.2.4 (Alaoglu's theorem; see, e.g., Theorem 6.21 in Aliprantis and Border (2006))**.** *Consider a normed vector space $\mathcal{X}$ and let $\mathcal{X}^*$ be the dual space of $(\mathcal{X}, \|\cdot\|)$. Then the closed unit ball of $\mathcal{X}^*$ is $\sigma(\mathcal{X}^*, \mathcal{X})$-compact. Thus, a subset of $\mathcal{X}^*$ is $\sigma(\mathcal{X}^*, \mathcal{X})$-compact if and only if it is $\sigma(\mathcal{X}^*, \mathcal{X})$-closed and norm bounded.*

Theorem A.2.5 (Krein-Šmulian; see, e.g., Theorem V.5.7 in Dunford and Schwartz (1957))**.** *Consider a Banach space $\mathcal{X}$ and let $\mathcal{X}^*$ be the dual space of $(\mathcal{X}, \|\cdot\|)$. Moreover, suppose that $\mathcal{A} \subset \mathcal{X}^*$ is a convex subset of $\mathcal{X}^*$. Then $\mathcal{A}$ is $\sigma(\mathcal{X}^*, \mathcal{X})$-closed if and only if $\mathcal{A} \cap \{l \in \mathcal{X}^* | \|l\|_{\mathcal{X}^*} \leq r\}$ is $\sigma(\mathcal{X}^*, \mathcal{X})$-closed for all $r > 0$.*

A.2.2. Convex optimization

This subsection is based on the definitions and results in Rockafellar (1974). Let us suppose that $(\mathcal{X}, \mathcal{V})$ and $(\mathcal{U}, \mathcal{Y})$ are paired spaces.

Definition A.2.6. *Consider a function* $f : \mathcal{X} \to \overline{\mathbb{R}}$. *The* effective domain *of* f *is defined by* $\operatorname{dom} f := \{x \in \mathcal{X} | f(x) < \infty\}$. f *is called* proper *if* $\operatorname{dom} f \neq \emptyset$ *and* $f(x) > -\infty$ *for all* $x \in \mathcal{X}$. *The* convex hull *of* f, *denoted by* $\operatorname{conv} f$, *is defined as the greatest convex function* h *that satisfies* $h \leq f$. *The function* f *is called* lower semicontinuous (l.s.c.) *if the set* $\{x \in \mathcal{X} | f(x) \leq r\}$ *is closed for all* $r \in \mathbb{R}$. *The greatest l.s.c. function* g *with* $g \leq f$ *is called* l.s.c. hull *of* f *and is denoted by* $\operatorname{lsc} f$. *Finally, the* closure *of* f, *denoted by* $\operatorname{cl} f$, *is defined by*

$$\operatorname{cl} f(x) := \begin{cases} \operatorname{lsc} f(x) & \text{for all } x \in \mathcal{X} \quad \text{if} \quad \operatorname{lsc} f(x) > -\infty \quad \text{for all } x \in \mathcal{X} \\ -\infty & \text{for all } x \in \mathcal{X} \quad \text{if} \quad \operatorname{lsc} f(x) = -\infty \quad \text{for some } x \in \mathcal{X} \end{cases},$$

and we say that f *is* closed *if* $f = \operatorname{cl} f$.

Theorem A.2.7 (See Theorem 4 in Rockafellar (1974))**.** *Consider a convex function* $f : \mathcal{X} \to \overline{\mathbb{R}}$. *Then* $\operatorname{lsc} f$ *and* $\operatorname{cl} f$ *are convex, too. If* $\operatorname{lsc} f(x) \in \mathbb{R}$ *for some* $x \in \mathcal{X}$, *then it follows that* $\operatorname{lsc} f$ *and* f *are proper, and we have* $\operatorname{cl} f = \operatorname{lsc} f$. *If* $\operatorname{lsc} f(x) \in \{-\infty, +\infty\}$ *for all* $x \in \mathcal{X}$, *then* $\operatorname{lsc} f$ *satisfies*

$$\operatorname{lsc} f(x) = \begin{cases} -\infty & \text{if } x \in \operatorname{cl} \operatorname{dom} f \\ +\infty & \text{if } x \notin \operatorname{cl} \operatorname{dom} f \end{cases}.$$

Moreover, in case of $\operatorname{lsc} f(x) \in \{-\infty, +\infty\}$ *for all* $x \in \mathcal{X}$, *we have* $\operatorname{cl} f(x) = \operatorname{lsc} f(x)$, *except in the case where* $x \notin \operatorname{cl} \operatorname{dom} f \neq \emptyset$ *(then* $-\infty = \operatorname{cl} f(x) \neq \operatorname{lsc} f(x) = +\infty$*).*

Definition A.2.8. *Consider a function* $f : \mathcal{X} \to \overline{\mathbb{R}}$. *The* convex conjugate *of* f *is the function* $f^* : \mathcal{V} \to \overline{\mathbb{R}}$ *defined by*

$$f^*(v) := \sup_{x \in \mathcal{X}} \{\langle x, v \rangle - f(x)\}.$$

The convex biconjugate *of* $f : \mathcal{X} \to \overline{\mathbb{R}}$ *is defined as the convex conjugate of* f^*. *The* conjugate in the concave sense *of a function* $g : \mathcal{X} \to \overline{\mathbb{R}}$ *is the function* $g^* : \mathcal{V} \to \overline{\mathbb{R}}$ *defined by*

$$g^*(v) := \inf_{x \in \mathcal{X}} \{\langle x, v \rangle - g(x)\}.$$

Theorem A.2.9 (Duality theorem for conjugate functions; see Theorem 5 in Rockafellar (1974))**.** *Consider an arbitrary function* $f : \mathcal{X} \to \overline{\mathbb{R}}$. *Then the convex conjugate* f^* *is a closed convex function on* $\mathcal{V}$, *and we have* $f^{**} = \operatorname{cl} \operatorname{conv} f$. *Moreover, the mapping* $f \mapsto f^*$ *(called* Fenchel transform*) induces a one-to-one correspondence* $f \mapsto h$ *(with* $h = f^*$ *and* $f = h^*$*) between the closed convex functions on* $\mathcal{X}$ *and the closed convex functions on* $\mathcal{V}$.

Consider a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ and the following optimization problem

(P) Minimize f over $\mathcal{X}$.

Moreover, suppose that

$$f(x) = F(x, 0) \quad \text{for } F : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}. \tag{A.34}$$

Definition A.2.10. *The* Lagrangian *function K on $\mathcal{X} \times \mathcal{Y}$ is defined by*

$$K(x, y) := \inf_{u \in \mathcal{U}} \{F(x, u) + \langle u, y\rangle\},$$

i.e., $K(x, \cdot)$ is the conjugate in the concave sense of $-F(x, \cdot)$.

Remark A.2.11. Rockafellar (1974) points out that if the function $-F(x, \cdot)$ is closed and concave, i.e., $F(x, \cdot)$ is closed and convex for $x \in \mathcal{X}$, then it follows with the duality theorem that $-F(x, \cdot)$ is the conjugate in the concave sense of $K(x, \cdot)$. Hence,

$$F(x, u) = \sup_{y \in \mathcal{Y}} \{K(x, y) - \langle u, y\rangle\} \quad \text{if } F(x, \cdot) \text{ is closed and convex.}$$

Theorem A.2.12 (See Theorem 6 in Rockafellar (1974))**.** *Let K be the Lagrangian. Then for each $x \in \mathcal{X}$, the function $K(x, \cdot)$ is closed and concave. If additionally $F(x, \cdot)$ is closed and convex, then*

$$f(x) = \sup_{y \in \mathcal{Y}} K(x, y). \tag{A.35}$$

On the other hand, suppose that K is any $\overline{\mathbb{R}}$-valued function on $\mathcal{X} \times \mathcal{Y}$ that satisfies (A.35)*. If additionally $K(x, \cdot)$ is closed and concave, then K is the Lagrangian with a uniquely determined representation* (A.34) *where $F(x, \cdot)$ is closed and convex. More precisely, $F : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ is given by*

$$F(x, u) = \sup_{y \in \mathcal{Y}} \{K(x, y) - \langle u, y\rangle\}.$$

Finally, if we suppose that $F(x, \cdot)$ is closed and convex, then $K(\cdot, y)$ is convex if and only if $F(\cdot, \cdot)$ is convex.

Let us now define the dual problem to (P):

(D) Maximize g over $\mathcal{Y}$ where $g(y) := \inf_{x \in \mathcal{X}} K(x, y)$ for $y \in \mathcal{Y}$.

Define

$$\varphi(u) := \inf_{x \in \mathcal{X}} F(x, u) \quad \text{for } u \in \mathcal{U}.$$

The following theorem shows the connection between g and φ.

Theorem A.2.13 (See Theorem 7 in Rockafellar (1974))**.** *Consider the function g defined in (D). Then g is closed and concave. Moreover, we have $g = (-\varphi)^*$ and $-g^* = \operatorname{cl}\operatorname{conv}\varphi$. Here, $(\cdot)^*$ denotes the conjugates in the concave sense. It follows that*

$$\sup_{y \in \mathcal{Y}} g(y) = (\operatorname{cl}\operatorname{conv}\varphi)(0),$$

whereas

$$\inf_{x\in\mathcal{X}} f(x) = \varphi(0).$$

In particular, if the function $F(\cdot,\cdot)$ *is convex, then* $-g^* = \operatorname{cl}\varphi$ *and*

$$\sup_{y\in\mathcal{Y}} g(y) = \liminf_{u\to 0} \varphi(u),$$

unless $0 \notin \operatorname{cl}\operatorname{dom}\varphi \neq \emptyset$ *and the function* $\operatorname{lsc}\varphi$ *satisfies* $\operatorname{lsc}\varphi(u) \in \{-\infty,+\infty\}$ *for all* $u \in \mathcal{U}$. *(In this case, we have* $\liminf_{u\to 0}\varphi(u) = \inf_{x\in\mathcal{X}} f(x) = \varphi(0) = +\infty$ *and* $g(y) = -\infty$ *for all* $y \in \mathcal{Y}$ *such that* $\sup_{y\in\mathcal{Y}} g(y) = -\infty$*.)*

The next remark emphasizes the importance of the previous theorems.

Remark A.2.14. Suppose that the function $F(x,\cdot)$ is closed and convex for all $x \in \mathcal{X}$ and let $F(\cdot,\cdot)$ be convex. Then it follows from Theorem A.2.13 in conjunction with Theorem A.2.12 that

$$\varphi(0) = (\operatorname{cl}\operatorname{conv}\varphi)(0) = \operatorname{cl}\varphi(0)$$

implies

$$\inf_{x\in\mathcal{X}}\sup_{y\in\mathcal{Y}} K(x,y) = \inf_{x\in\mathcal{X}} f(x) = \sup_{y\in\mathcal{Y}} g(y) = \sup_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}} K(x,y).$$

Thus, if $\varphi(0) = \operatorname{cl}\varphi(0)$, then we are allowed to exchange infimum and supremum.

A.2.3. Differential calculus

For the following notions of differentiability we refer the reader to Kurdila and Zabarankin (2005) and Aliprantis and Border (2006).

Definition A.2.15. *Let* $\mathcal{X}$ *be a normed vector space. Then we say that the mapping* $\upsilon : \mathcal{X} \to \overline{\mathbb{R}}$ *is* differentiable at $x_0 \in \mathcal{X}$ in the direction $x \in \mathcal{X}$ *if the limit*

$$d^+\upsilon(x_0)(x) := \lim_{u\downarrow 0} \frac{\upsilon(x_0+ux)-\upsilon(x_0)}{u}$$

exists. In this case, $d^+\upsilon(x_0)(x)$ *is called* directional derivative of υ at x_0 in the direction x.

Definition A.2.16. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be normed vector spaces. A mapping* $\Upsilon : \mathcal{X} \to \mathcal{Y}$ *is called* Gâteaux differentiable at $x_0 \in \mathcal{X}$ *if the mapping* $D_G\Upsilon(x_0) : \mathcal{X} \to \mathcal{Y}$ *satisfies the following properties:*

1. *For all* $x \in \mathcal{X}$,

   $$D_G\Upsilon(x_0)(x) := \lim_{u\downarrow 0} \frac{\Upsilon(x_0+ux)-\Upsilon(x_0)}{u}$$

   exists.
2. *The mapping* $D_G\Upsilon(x_0)$ *is linear.*

3. *The mapping $D_G\Upsilon(x_0)$ is bounded, i.e., for every $x \in \mathcal{X}$, there exists $M > 0$ such that $\|D_G\Upsilon(x_0)(x)\|_{\mathcal{Y}} \leq M\|x\|_{\mathcal{X}}$.*

If Υ is Gâteaux differentiable at $x_0 \in \mathcal{X}$, then the mapping $D_G\Upsilon(x_0)$ is called Gâteaux derivative of Υ at x_0.

Remark A.2.17. Note every linear mapping $\Upsilon : \mathcal{X} \to \mathcal{Y}$ between normed vector spaces is continuous if and only if it is bounded; see, for instance, Lemma II.3.4 in Dunford and Schwartz (1957). We denote the *set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$* by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. This space can be equipped with the *operator norm topology.* This topology is defined by the so called *operator norm* given by

$$\|\Upsilon\| := \sup_{x \in \mathcal{X}\setminus\{0\}} \frac{\|\Upsilon(x)\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} \quad \text{for } \Upsilon \in \mathcal{L}(\mathcal{X}, \mathcal{Y}).$$

For the following condition let us consider a subset $\mathcal{A}$ of a Banach space $\mathcal{X}$, a subset $\mathcal{B}$ of a topological space $\mathcal{Y}$ and a function $J : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$.

Condition A.2.18 (See Hypotheses D1 in Bernhard and Rapaport (1995))**.** Let $a \in \mathcal{A}$ and $h \in \mathcal{A}$ be fixed.

1. $\mathcal{B}$ is compact.
2. For all $b \in \mathcal{B}$, the function $(t, b) \mapsto J(a + th, b)$ is u.s.c. at $(0, b)$.
3. For all $b \in \mathcal{B}$ and all t in a right neighborhood of 0, there exists a bounded directional derivative

$$d_1^+ J(a + th, b)(h) := \lim_{u \downarrow 0} \frac{J(a + (t + u)h, b) - J(a + th, b)}{u}.$$

4. For all $b \in \mathcal{B}$, the function $(t, b) \mapsto d_1^+ J(a + th, b)(h)$ is u.s.c. at $(0, b)$.

Theorem A.2.19 (See Theorem D1 in Bernhard and Rapaport (1995))**.** *Let $\mathcal{A}$ be a subset of a Banach space $\mathcal{X}$ and $\mathcal{B}$ be a subset of a topological space $\mathcal{Y}$ and consider the function $J : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$. If Condition A.2.18 is satisfied, then the function $\bar{J}$ defined by*

$$\bar{J}(a') := \sup_{b \in \mathcal{B}} J(a', b) \quad \textit{for } a' \in \mathcal{A}$$

has a directional derivative at $a \in \mathcal{A}$ in the direction $h \in \mathcal{A}$, which is given by

$$d^+ \bar{J}(a)(h) = \max_{b \in \mathcal{B}^{\#}(a)} d_1^+ J(a, b)(h)$$

where $\mathcal{B}^{\#}(a) := \{b \in \mathcal{B} | J(a, b) = \bar{J}(a)\}$.

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Erklärung

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